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14. ABSTRACT
In this paper, a new fluid theory is given in the guiding-center and gyrotropic approximation which is derivable from the Vlasov-Maxwell equations. The theory includes the effect of wave-particle interactions for the weakly turbulent, weakly inhomogeneous, nonuniformly magnetized plasma, and it is applicable to a variety of space and laboratory plasmas. It is assumed that the turbulence in random and electrostatic, and that the velocity-space Fokker-Planck operator can be used to calculate the correlation functions that describe the wave-particle interactions. Conservation laws are derived that relate the low-order velocity moments of the particle distributions to the turbulence. The theory is based on the work of Hubbard [Proc. R. Soc. London, Ser. A **260**, 114 (1961)] and Ichimaru and Rosenbluth [Phys. Fluids **13**, 2778 (1970)]. In the work presented here, the idea is proposed that the fluid equations can be solved (1) by using measurements of the low-order velocity moments to specify the initial and boundary conditions.

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Gyrotropic guiding-center fluid theory for turbulent inhomogeneous magnetized plasma

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In this paper, a new fluid theory is given in the guiding-center and gyrotropic approximation which is derivable from the Vlasov-Maxwell equations. The theory includes the effect of wave-particle interactions for the weakly turbulent, weakly inhomogeneous, nonuniformly magnetized plasma, and it is applicable to a variety of space and laboratory plasmas. It is assumed that the turbulence is random and electrostatic, and that the velocity-space Fokker-Planck operator can be used to calculate the correlation functions that describe the wave-particle interactions. Conservation laws are derived that relate the low-order velocity moments of the particle distributions to the turbulence. The theory is based on the work of Hubbard [Proc. R. Soc. London, Ser. A **260**, 114 (1961)] and Ichimaru and Rosenbluth [Phys. Fluids **13**, 2778 (1970)]. In the work presented here, the idea is proposed that the fluid equations can be solved (1) by using measurements of the turbulence to specify the electric-field fluctuations; and (2) by using measurements of the low-order velocity moments to specify the initial and boundary conditions. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2220006]

I. INTRODUCTION

In the literature, there have been two traditional approaches to plasma turbulence: one is to view the plasma as an initial-value problem; and the other, as plasma where the turbulence is fully developed and in a saturated, stationary state. In the first approach, the plasma is initially in contact with a source of free energy and, as a result, one or more collective modes of the plasma are driven unstable and grow exponentially with time. The theoretical problem here is to solve for the time evolution of the system of wave-kinetic and plasma-kinetic equations. Quasilinear theory,[1–3] more general weak turbulence theories,[4,5] and the renormalized, resonance broadening theory[6–8] are successful examples of this approach. However, if the plasma remains in contact with sources and sinks of free energy, the particle and field distributions will often be driven to a turbulent, steady or quasisteady state. The plasma may be either weakly or strongly turbulent. In this situation, theories have been developed to describe the particle and field distributions for the fully developed turbulent plasma. The main theoretical problem here is to find the correlation functions that determine the anomalous transport and relaxation processes that operate. Over the years, treatments along these lines have produced important advances[9–11] and have been discussed and extended.[12]

The literature on anomalous transport is vast. For early work, see Refs. 13–15 and for a review of some applications to the earth's ionosphere, see Ref. 16 and the citations therein. For more recent work on anomalous transport in axisymmetric, toroidal laboratory plasmas using the drift-kinetic-equation approach, see Ref. 17, and for a recent brief review of applications to some astrophysical plasmas, see Ref. 18. Particle simulations have also been used to study anomalous transport in the earth's magnetosphere.[19–21] For a discussion of a different approach to plasma turbulence using renormalization-group methods and some applications to fully developed intermittent turbulence in space plasmas, see Refs. 22 and 23.

In this paper, we present a new method for incorporating plasma turbulence into the fluid equations which is applicable to a variety of space and laboratory plasmas. The fluid equations are given in the guiding-center and gyrotropic approximation for weakly inhomogeneous, nonuniformly magnetized plasma where the particles are transported in one spatial dimension (the distance along the magnetic field) but the turbulence is two-dimensional. In deriving them, we start with the Vlasov-Maxwell equations.[24] We assume that the turbulence is random and electrostatic, and that the velocity-space Fokker-Planck operator can be used to calculate the correlation functions that describe the wave-particle interactions. The method we present is based on the work of Hubbard,[10] as well as Ichimaru and Rosenbluth,[11] hereafter called I and R. Since the probability distribution that governs the turbulence may not be Gaussian, the Fokker-Planck operator is an approximation which neglects terms in the expansion of the distribution function in powers of $\Delta\mathbf{v}$ higher than order 2. The friction and diffusion coefficients depend on the dielectric screening function and the spectral density of the longitudinal electric-field fluctuations for the turbulent plasma. Another assumption inherent in the Fokker-Planck method is that the turbulence is sufficiently weak so that the fluctuating electric field has a small effect on the unperturbed particle orbits. We also assume that the dielectric screening function and spectral density vary weakly in space and slowly in time compared to the strong spatial and fast temporal variation of the correlation function for the electric-field fluctuations. In this way, the plasma may be treated as locally homogeneous and stationary. This concept is similar to that given in Ref. 25: the separation of space and time scales between (1) the turbulence and the fluid quantities and (2) the use of a weak turbulence theory to treat the interaction between the evolving turbulence and the plasma particles. For an interesting and immensely useful way of incorporating kinetic effects (Landau damping) into the fluid equations using gyrokinetic theory, see Refs. 26 and 27.

If we could find a renormalized solution of the wave-kinetic and plasma-kinetic equations, the problem would be solved. The renormalized propagator could then be used to find the renormalized dielectric screening function and the renormalized spectral density for the longitudinal electric-field fluctuations. However, the development of a renormalized solution for turbulent, inhomogeneous, nonuniformly magnetized plasma is indeed a formidable problem. The idea that we present here is to bypass this difficult problem by solving the fluid equations, where measurements are used to specify the turbulent, electric-field fluctuations and where measurements of the low-order velocity moments of the particle distributions are also used to specify the initial and boundary conditions.

The Birkeland current system of the earth's magnetosphere[28] is a system of upward and downward magnetic field-aligned electrical currents that flow between the magnetosphere and the ionosphere at high geomagnetic latitudes. The plasma in the earth's magnetosphere is driven unstable by its continuous interaction with the solar wind and is far from equilibrium. The primary motivation for the work reported in this paper is to develop a fluid theory applicable to the Birkeland current system where the plasma is weakly inhomogeneous, the geomagnetic field is nonuniform, and electrostatic plasma turbulence is known to occur.[28]

In Secs. II–VI, we give the fluid theory in the guiding-center and gyrotropic approximation in the presence of random electrostatic turbulence where the particles are transported in one spatial dimension (the distance along the magnetic field) but the turbulence is two-dimensional. In Secs. VII and VIII, we give the fluid equations for quiescent (nonturbulent), drifting, bi-Maxwellian, electron-ion plasma in a nonuniform **B** field and show that they may be solved to give the expected result for equilibrium, electron-ion plasma in a uniform **B** field. In Sec. IX, we summarize and discuss the results.

II. THE ENSEMBLE-AVERAGED VLASOV-MAXWELL EQUATIONS FOR ELECTROSTATIC TURBULENCE

The ensemble-averaged Vlasov-Maxwell equations for electrostatic turbulence may be determined from Appendix A by neglecting $\delta\mathbf{B}$ and choosing a time-independent model for $\langle\mathbf{B}\rangle$: $\delta\mathbf{B}\to\mathbf{0}$, $\langle\mathbf{B}\rangle=\mathbf{B}(\mathbf{r})$. This yields

$$\left\{\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{r}}+\frac{q_\alpha}{m_\alpha}\left[\langle\mathbf{E}\rangle+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right]\cdot\frac{\partial}{\partial\mathbf{v}}\right\}\langle f_\alpha\rangle=C_\alpha, \quad (1)$$

$$C_\alpha=-\frac{q_\alpha}{m_\alpha}\left\langle\delta\mathbf{E}\cdot\frac{\partial}{\partial\mathbf{v}}\delta f_\alpha\right\rangle, \quad (2)$$

$$\frac{\partial}{\partial\mathbf{r}}\cdot\langle\mathbf{E}\rangle=4\pi\rho, \quad \frac{\partial}{\partial\mathbf{r}}\times\langle\mathbf{E}\rangle=0. \quad (3)$$

Here, we can write $\langle\mathbf{E}\rangle=-\partial\phi/\partial\mathbf{r}$, so we obtain Poisson's equation

$$\frac{\partial^2}{\partial\mathbf{r}^2}\phi=-4\pi\rho. \quad (4)$$

These equations are given in Gaussian units and are valid in the standard six-dimensional phase space discussed in Appendix A.

The conservation relations for C_α for electrostatic turbulence are obtained from (A6)–(A8) as

$$\int d^3v C_\alpha=0, \quad (5)$$

$$\sum_\alpha\int d^3v m_\alpha\mathbf{v}C_\alpha-\frac{\partial}{\partial\mathbf{r}}\cdot\frac{1}{4\pi}\langle\delta\mathbf{E}\delta\mathbf{E}\rangle+\frac{\partial}{\partial\mathbf{r}}\cdot\frac{1}{8\pi}\langle\delta\mathbf{E}^2\rangle\mathbf{I}=0, \quad (6)$$

$$\sum_\alpha\int d^3v\frac{1}{2}m_\alpha v^2C_\alpha+\frac{\partial}{\partial t}\frac{1}{8\pi}\langle\delta\mathbf{E}^2\rangle=0. \quad (7)$$

III. THE KINETIC EQUATIONS IN THE GUIDING-CENTER AND GYROTROPIC APPROXIMATION

For problems where **B** is a weakly varying function of **r**, it is reasonable to seek approximate solutions of the kinetic equations in the guiding-center coordinate system which are gyrotropic at each point along the **B**-field flux tube. For the case where the magnetic field is strong and a weakly varying function of position, and the ensemble-averaged electric field is weak and a weakly varying function of position and a slowly varying function of time, the guiding-center approximation[29,30] can be made. If $l_{B\perp}(l_{B\parallel})$ denotes the length scale for the variation of **B** in a direction perpendicular (parallel) to **B**, then by a weakly varying **B** we mean one where $l_{B\perp}\gg a_{\alpha\perp}(l_{B\parallel}\gg a_{\alpha\parallel})$. The lengths $a_{\alpha\perp}$ and $a_{\alpha\parallel}$ are defined as

$$a_{\alpha\perp}=|v_\perp/\Omega_\alpha|, \quad a_{\alpha\parallel}=|v_\parallel/\Omega_\alpha|, \quad (8)$$

where $\Omega_\alpha=q_\alpha B/m_\alpha c$ is the gyrofrequency of the particle including the sign of q_α. The length scales are defined in the standard way. For example, $l_{B\parallel}=|(B^{-1}dB/ds)^{-1}|$, where s denotes the distance along **B**, with a similar definition for $l_{B\perp}$. Similar conditions for the variation of $\langle\mathbf{E}\rangle$ can be given.[30] If $l_{E\perp}(l_{E\parallel})$ denotes the length scale for the variation of $\langle\mathbf{E}\rangle$ in a direction perpendicular (parallel) to $\langle\mathbf{E}\rangle$, then by a weakly varying $\langle\mathbf{E}\rangle$ we mean one where $l_{E\perp}\gg a_{\alpha\perp}(l_{E\parallel}\gg a_{\alpha\parallel})$. A slowly varying function of time is one that varies on a time scale, τ, where $\tau \gg \tau_\alpha = |2\pi/\Omega_\alpha|$.

The self-consistent equations of motion for the particles in the guiding-center approximation have been found by Sivukhin.[30] Using Eq. (12.14) of Sivukhin and our notation in (1), we have

$$\left\{\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha}\left[\langle \mathbf{E}\rangle + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right]\cdot\frac{\partial}{\partial \mathbf{v}}\right\}\langle f_\alpha\rangle = \left\{\frac{\partial}{\partial t} + \dot{\mathbf{R}}\cdot\frac{\partial}{\partial \mathbf{r}} + \frac{\dot{P}_\parallel}{m_\alpha}\frac{\partial}{\partial v_\parallel} + \frac{\dot{P}_\perp}{m_\alpha}\frac{\partial}{\partial v_\perp}\right\}\langle f_\alpha\rangle, \quad (9)$$

where $\dot{\mathbf{R}}$, $\dot{P}_\parallel$, and $\dot{P}_\perp$ are given by (6.10)–(6.12) of Sivukhin. If we neglect all drifts perpendicular to **B** ($\mathbf{E}\times\mathbf{B}$, gradient **B**, and curvature **B**), we note that (6.10)–(6.12) of Sivukhin are

$$\dot{\mathbf{R}} = v_\parallel \mathbf{b}, \quad (10)$$

$$\dot{P}_\parallel = q_\alpha\langle \mathbf{E}\rangle\cdot\mathbf{b} - [\mathbf{b}\cdot(\partial B/\partial \mathbf{r})/2B]m_\alpha v_\perp^2, \quad (11)$$

$$\dot{P}_\perp = [\mathbf{b}\cdot(\partial B/\partial \mathbf{r})/2B]m_\alpha v_\parallel v_\perp, \quad (12)$$

where $\mathbf{b}=\mathbf{B}/B$. We now replace the f_α dependence in C_α by the gyrophase average of f_α, and take the gyrophase average of (1) to obtain

$$\left\{\frac{\partial}{\partial t} + v_\parallel\frac{\partial}{\partial s} + \frac{q_\alpha}{m_\alpha}E_\parallel\frac{\partial}{\partial v_\parallel} - \frac{1}{2B}\frac{dB}{ds}\left[v_\perp^2\frac{\partial}{\partial v_\parallel} - v_\parallel v_\perp\frac{\partial}{\partial v_\perp}\right]\right\}\bar{f}_\alpha = \bar{C}_\alpha, \quad (13)$$

$$\bar{f}_\alpha = \bar{f}_\alpha(s,t,v_\perp,v_\parallel) = (2\pi)^{-1}\int_0^{2\pi} d\varphi\langle f_\alpha\rangle, \quad (14)$$

$$\bar{C}_\alpha = \bar{C}_\alpha(s,t,v_\perp,v_\parallel) = (2\pi)^{-1}\int_0^{2\pi} d\varphi C_\alpha, \quad (15)$$

where s denotes the distance along **B** and φ is the gyrophase angle, both of which are expressed in the guiding-center coordinate system. We emphasize here that these equations are valid only for situations where transport along **B** dominates transport perpendicular to **B**. The bar symbol denotes the gyrophase average, which is also calculated in the guiding-center coordinate system. The guiding-center coordinate system[30] is one where the velocity-space coordinates slowly change their orientation as s varies so that the v_z axis is always parallel or antiparallel to **B**. In general, the **B** field is curved and varies slowly in space as described above. We interpret $\bar{f}_\alpha(s,t,v_\perp,v_\parallel)ds v_\perp dv_\perp dv_\parallel$ as $(2\pi)^{-1}$ times the average number of particles per unit area of type α for which the coordinates of the guiding center of the motion lie between s and $s+ds$, while the velocities $v_\perp$ and $v_\parallel$ lie between $v_\perp$ and $v_\perp+dv_\perp$ and $v_\parallel$ and $v_\parallel+dv_\parallel$, respectively. Poisson's equation becomes

$$\partial^2\phi/\partial s^2 = -4\pi\rho, \quad (16)$$

where $\phi=\phi(s,t)$ and

$$\rho = \rho(s,t) = \sum_\alpha q_\alpha \int d^3v \bar{f}_\alpha(s,t,v_\perp,v_\parallel). \quad (17)$$

Here, $\bar{f}_\alpha$ is normalized so that $\int d^3v\bar{f}_\alpha = n_\alpha$, where n_α is the number density and $\langle E_\parallel\rangle = E_\parallel = -\partial\phi/\partial s$.

When the model for **B** is specified and when $\bar{C}_\alpha$ is given, then (13) and (16) are the system of kinetic equations in the guiding-center and gyrotropic approximation that describes the problem, subject to appropriate initial and boundary conditions. When $dB/ds \neq 0$, a direct numerical solution or some orthonormal expansion method of solving (13) and (16) can be considered. For a uniform **B** field, $dB/ds=0$, and the left-hand side of (13) does not contain $v_\perp$, but the right-hand side does. For this case, we note that an orthonormal expansion method on the $v_\perp$ variable needs to be considered. However, in the remainder of this paper, we pursue the multiconstituent, multimoment fluid equations as a method of solution for the problem.

The conservation relations for $\bar{C}_\alpha$ in the guiding-center and gyrotropic approximation are found from (5)–(7) and are

$$\int d^3v \bar{C}_\alpha = 0, \quad (18)$$

$$\sum_\alpha \int d^3v m_\alpha v_\parallel \bar{C}_\alpha + \frac{\partial}{\partial s}\frac{1}{8\pi}[\langle \delta E_\perp^2\rangle - \langle \delta E_\parallel^2\rangle] = 0, \quad (19)$$

$$\sum_\alpha \int d^3v \frac{1}{2}m_\alpha(v_\perp^2 + v_\parallel^2)\bar{C}_\alpha + \frac{\partial}{\partial t}\frac{1}{8\pi}[\langle \delta E_\perp^2\rangle + \langle \delta E_\parallel^2\rangle] = 0. \quad (20)$$

Here, cylindrical coordinates are implied, so $\int d^3v = 2\pi\int_0^\infty dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_\parallel$. In obtaining these results, we have used the fact that $\delta E_\perp$ and $\delta E_\parallel$ are functions of s and t and denote the perpendicular and parallel parts of the fluctuating electric field, respectively.

IV. THE MULTICONSTITUENT, MULTIMOMENT FLUID EQUATIONS IN THE GUIDING-CENTER AND GYROTROPIC APPROXIMATION

We wish to calculate the multiconstituent, multimoment fluid equations in the guiding-center and gyrotropic approximation. To do this, we multiply (13) by $v_\perp^n v_\parallel^l$ and integrate over velocity space.[31] We use the following notation:

$$(v_\perp^n, v_\parallel^l; \bar{f}_\alpha) = (n,l;\bar{f}_\alpha) = 2\pi\int_0^\infty dv_\perp \int_{-\infty}^{+\infty} dv_\parallel v_\perp^{1+n} v_\parallel^l \bar{f}_\alpha(s,t,v_\perp,v_\parallel), \quad (21)$$

$$(v_\perp^n, v_\parallel^l; \bar{C}_\alpha) = (n,l;\bar{C}_\alpha) = 2\pi\int_0^\infty dv_\perp \int_{-\infty}^{+\infty} dv_\parallel v_\perp^{1+n} v_\parallel^l \bar{C}_\alpha(s,t;v_\perp,v_\parallel), \quad (22)$$

$$\dot{A} = -B^{-1}dB/ds = A^{-1}dA/ds, \tag{23}$$

where A is the cross-sectional area of the flux tube. Here, n and l are positive integers or zero. The velocity moments of (13) are

$$\frac{\partial}{\partial t}(n,l;\bar{f}_\alpha) + \frac{\partial}{\partial s}(n,l+1;\bar{f}_\alpha) + \dot{A}(1+n/2)(n,l+1;\bar{f}_\alpha) + l\{(q_\alpha/m_\alpha)\partial\phi/\partial s(n,l-1;\bar{f}_\alpha) - (\dot{A}/2) \times(n+2,l-1;\bar{f}_\alpha)\} = (n,l;\bar{C}_\alpha). \tag{24}$$

The lowest four velocity moments $[(n,l)=(0,0),(0,1),(0,2),(2,0)]$ and Poisson's equation are

$$\frac{\partial}{\partial t}n_\alpha + B\frac{\partial}{\partial s}(n_\alpha u_\alpha/B) = 0, \tag{25}$$

$$\frac{\partial}{\partial t}m_\alpha n_\alpha u_\alpha + 2\frac{\partial}{\partial s}n_\alpha w_{\alpha\parallel} - \frac{2}{B}\frac{dB}{ds}n_\alpha\left(w_{\alpha\parallel} - \frac{w_{\alpha\perp}}{2}\right) + q_\alpha n_\alpha\frac{\partial\phi}{\partial s} = \dot{M}_{\alpha\parallel}, \tag{26}$$

$$\frac{\partial}{\partial t}n_\alpha w_{\alpha\parallel} + \frac{\partial}{\partial s}n_\alpha q_{\alpha\parallel} - \frac{1}{B}\frac{dB}{ds}n_\alpha(q_{\alpha\parallel} - q_{\alpha\perp}) + q_\alpha n_\alpha u_\alpha\frac{\partial\phi}{\partial s} = \dot{W}_{\alpha\parallel}, \tag{27}$$

$$\frac{\partial}{\partial t}n_\alpha w_{\alpha\perp} + B^2\frac{\partial}{\partial s}(n_\alpha q_{\alpha\perp}/B^2) = \dot{W}_{\alpha\perp}, \tag{28}$$

$$\frac{\partial^2\phi}{\partial s^2} = -4\pi\sum_\beta q_\beta n_\beta. \tag{29}$$

Here, we have defined the velocity moments using the following notation: $n_\alpha=(0,0;\bar{f}_\alpha)$, $n_\alpha u_\alpha=(0,1;\bar{f}_\alpha)$, $n_\alpha w_{\alpha\parallel}=(m_\alpha/2)(0,2;\bar{f}_\alpha)$, $n_\alpha w_{\alpha\perp}=(m_\alpha/2)(2,0;\bar{f}_\alpha)$, $n_\alpha q_{\alpha\parallel}=(m_\alpha/2)\times(0,3;\bar{f}_\alpha)$, and $n_\alpha q_{\alpha\perp}=(m_\alpha/2)(2,1;\bar{f}_\alpha)$. For particles of type α, n_α is the number density, u_α is the parallel drift velocity, $w_{\alpha\parallel}$ is the total parallel energy per particle, $w_{\alpha\perp}$ is the total perpendicular energy per particle, $q_{\alpha\parallel}$ is the total parallel energy flux per particle, and $q_{\alpha\perp}$ is the total perpendicular energy flux per particle. By total, we mean the sum of the drift and random parts. Note that $n_\alpha q_{\alpha\parallel}(n_\alpha q_{\alpha\perp})$ is the total parallel (perpendicular) energy flux and $n_\alpha w_{\alpha\parallel}(n_\alpha w_{\alpha\perp})$ is the total parallel (perpendicular) energy density. The definitions for $q_{\alpha\parallel}$ and $q_{\alpha\perp}$ should not be confused with the heat flux per particle, also denoted by $q_\parallel$ and $q_\perp$ in Braginskii,[32] but defined differently here. Also, we have changed the notation used for the energy fluxes per particle from what was used in Ref. 31. In our new notation, the lowercase symbols are used for per particle quantities and uppercase symbols are reserved for per unit volume quantities.

We have also introduced the following notation: $\dot{M}_{\alpha\parallel}=m_\alpha(0,1;\bar{C}_\alpha)$, $\dot{W}_{\alpha\parallel}=(m_\alpha/2)(0,2;\bar{C}_\alpha)$, and $\dot{W}_{\alpha\perp}=(m_\alpha/2)\times(2,0;\bar{C}_\alpha)$. Here, $\dot{M}_{\alpha\parallel}$ is the rate of transfer of momentum per unit volume for particles of type α due to wave-particle interactions, and $\dot{W}_{\alpha\parallel}(\dot{W}_{\alpha\perp})$ is the rate of transfer of parallel (perpendicular) energy per unit volume for particles of type α due to wave-particle interactions. The transfer rates are functions of s and t. The $\dot{M}_{\alpha\parallel}$ are related to the anomalous (turbulent) resistivity for the problem, and $\dot{W}_{\alpha\parallel}$ and $\dot{W}_{\alpha\perp}$ are the anomalous (turbulent) parallel and perpendicular heating or cooling rates per unit volume for particles of type α. If the momentum (or energy) transfer rate is positive, then momentum (or energy) is gained by the particles. If it is negative, then momentum (or energy) is lost by the particles. These quantities are not independent but are related to the turbulent fluctuations through the conservation relations given by (19) and (20).

Equations (25)–(29) are a set of $4N+1$ equations for the $6N+1$ unknowns: $n_\alpha, u_\alpha, w_{\alpha\parallel}, w_{\alpha\perp}, q_{\alpha\parallel}, q_{\alpha\perp}$, and ϕ, where N is the number of plasma constituents. Instead of using this set, we may introduce the parallel and perpendicular temperatures in energy units[32] defined as

$$n_\alpha T_{\alpha\parallel}/2 = p_{\alpha\parallel}/2 = (m_\alpha/2)\int d^3v(v_\parallel - u_\alpha)^2\bar{f}_\alpha, \tag{30}$$

$$n_\alpha T_\perp = p_{\alpha\perp} = (m_\alpha/2)\int d^3v v_\perp^2\bar{f}_\alpha. \tag{31}$$

It follows that $w_{\alpha\parallel}=T_{\alpha\parallel}/2+m_\alpha u_\alpha^2/2$ and $w_{\alpha\perp}=T_{\alpha\perp}$. Substituting for $w_{\alpha\parallel}$ and $w_{\alpha\perp}$ in (26)–(28) gives

$$\frac{\partial}{\partial t}n_\alpha + B\frac{\partial}{\partial s}(n_\alpha u_\alpha/B) = 0, \tag{32}$$

$$\frac{\partial}{\partial t}m_\alpha n_\alpha u_\alpha + \frac{\partial}{\partial s}n_\alpha(T_{\alpha\parallel} + m_\alpha u_\alpha^2) - \frac{1}{B}\frac{dB}{ds}n_\alpha \times(T_{\alpha\parallel} + m_\alpha u_\alpha^2 - T_{\alpha\perp}) + q_\alpha n_\alpha\frac{\partial\phi}{\partial s} = \dot{M}_{\alpha\parallel}, \tag{33}$$

$$\frac{1}{2}\frac{\partial}{\partial t}n_\alpha(T_{\alpha\parallel} + m_\alpha u_\alpha^2) + \frac{\partial}{\partial s}n_\alpha q_{\alpha\parallel} - \frac{1}{B}\frac{\partial B}{\partial s}n_\alpha(q_{\alpha\parallel} - q_{\alpha\perp}) + q_\alpha n_\alpha u_\alpha\frac{\partial\phi}{\partial s} = \dot{W}_{\alpha\parallel}, \tag{34}$$

$$\frac{\partial}{\partial t}n_\alpha T_{\alpha\perp} + B^2\frac{\partial}{\partial s}(n_\alpha q_{\alpha\perp}/B^2) = \dot{W}_{\alpha\perp}, \tag{35}$$

$$\frac{\partial^2\phi}{\partial s^2} = -4\pi\sum_\beta q_\beta n_\beta. \tag{36}$$

Here (32)–(36) are a set of $4N+1$ equations for the $6N+1$ unknowns: n_α, u_α, $T_{\alpha\parallel}$, $T_{\alpha\perp}$, $q_{\alpha\parallel}$, $q_{\alpha\perp}$, and ϕ.

The fluid equations for turbulent, inhomogeneous, nonuniformly magnetized plasma in the guiding-center and gyrotropic approximation are given in two representations by (25)–(29) and by (32)–(36). They are the basis of our fluid theory. For both sets of equations, $2N$ closure conditions are needed. When the model for **B** is given and when $\dot{M}_{\alpha\parallel}, \dot{W}_{\alpha\parallel}$ and $\dot{W}_{\alpha\perp}$ are specified, then either set of fluid equations may be solved subject to appropriate initial and boundary conditions to give the weak spatial and slow temporal evolution of the turbulent system. Examples of how this is done for quiescent plasma are given in Secs. VII and VIII.

It is interesting to compare the above equations to the standard approach for a uniform **B** field. We see that (32) is the one-dimensional equivalent of (1.11) of Braginskii.[32] We also see that (33) is the one-dimensional equivalent of the left-hand side of Eq. (1.12) of Braginskii, where no drifts perpendicular to **B** or off-diagonal elements of the pressure tensor appear. When we add (34) and (35) for a uniform **B** field, we see that the sum is also the one-dimensional equivalent of the left-hand side of Eq. (1.13) of Braginskii.

V. CONSERVATION LAWS IN THE GUIDING-CENTER AND GYROTROPIC APPROXIMATION

The conservation laws for the number density, charge density, momentum density, and total energy density for electrostatic turbulence in the guiding-center and gyrotropic approximation may be written from the results in Secs. III and IV. The conservation of number density is

$$\frac{\partial}{\partial t}n_\alpha + B\frac{\partial}{\partial s}(n_\alpha u_\alpha/B) = 0. \tag{37}$$

The conservation law for the charge density follows from (37) and is

$$\frac{\partial}{\partial t}\rho + B\frac{\partial}{\partial s}(j_\parallel/B) = 0, \tag{38}$$

where we have defined $j_\parallel = \sum_\alpha q_\alpha n_\alpha u_\alpha$. From (19) and (33), we obtain the conservation law for momentum density

$$\frac{\partial}{\partial t}\sum_\alpha m_\alpha n_\alpha u_\alpha + \frac{\partial}{\partial s}\sum_\alpha n_\alpha(T_{\alpha\parallel} + m_\alpha u_\alpha^2) - \frac{1}{B}\frac{dB}{ds} \times\sum_\alpha n_\alpha(T_{\alpha\parallel} + m_\alpha u_\alpha^2 - T_{\alpha\perp}) + \frac{\partial}{\partial s}\frac{1}{8\pi}[\langle \delta E_\perp^2\rangle - \langle \delta E_\parallel^2\rangle] = \rho E_\parallel. \tag{39}$$

In order to obtain the conservation law for the total energy density, we add (34) and (35) and use (20) to obtain

$$\frac{\partial}{\partial t}\sum_\alpha n_\alpha\left(\frac{T_{\alpha\parallel}}{2} + \frac{m_\alpha}{2}u_\alpha^2 + T_{\alpha\perp}\right) + \frac{\partial}{\partial s}\sum_\alpha n_\alpha(q_{\alpha\parallel} + q_{\alpha\perp}) - \frac{1}{B}\frac{dB}{ds}\sum_\alpha n_\alpha(q_{\alpha\parallel} + q_{\alpha\perp}) + \frac{\partial}{\partial t}\frac{1}{8\pi}[\langle \delta E_\perp^2\rangle + \langle \delta E_\parallel^2\rangle] = j_\parallel E_\parallel. \tag{40}$$

For **B** equal to a constant, (37), (39), and (40) reduce to the one-dimensional forms of (A9)–(A11) for electrostatic turbulence, given in Appendix A. Also note that (37), (39), and (40) reduce to the form given by Davidson,[5] Chap. 8, if **B** is uniform, the plasma is homogeneous and one-dimensional, and the turbulence obeys the quasilinear assumptions.

It is interesting to examine the integrated, total energy-density conservation equation for steady-state conditions. Integrating (40) from s_1 to s_2, we obtain

$$\sum_\alpha n_\alpha (q_{\alpha\parallel} + q_{\alpha\perp})|_{s_1}^{s_2} - \int_{s_1}^{s_2} ds\frac{1}{B}\frac{dB}{ds}\sum_\alpha n_\alpha(q_{\alpha\parallel} + q_{\alpha\perp}) = \int_{s_1}^{s_2} ds j_\parallel E_\parallel. \tag{41}$$

Let us assume that both $q_{\alpha\parallel}$ and $q_{\alpha\perp}$ are positive (outgoing particles) on the flux tube from s_1 to s_2. If $B^{-1}dB/ds$ is negative, and if $j_\parallel E_\parallel$ is positive on the flux tube, $j_\parallel E_\parallel$ provides the rate of change of energy density to drive the two terms on the left-hand side of (41). Further, for the case of a uniform **B** field, then

$$\sum_\alpha n_\alpha (q_{\alpha\parallel} + q_{\alpha\perp})|_{s_1}^{s_2} = \int_{s_1}^{s_2} ds j_\parallel E_\parallel. \tag{42}$$

This means that the total energy flux exiting the flux tube at s_2 is greater than that entering at s_1, if $j_\parallel E_\parallel$ is positive on the flux tube.

The conservation laws for turbulent, inhomogeneous, nonuniformly magnetized plasma in the guiding-center and gyrotropic approximation are given by (37)–(40). Since all drifts perpendicular to **B** have been neglected, the particles are responding in one spatial dimension, s, but the electrostatic turbulence is two-dimensional, as $\langle \delta \mathbf{E}^2\rangle = \langle \delta E_\perp^2\rangle + \langle \delta E_\parallel^2\rangle$.

VI. CALCULATION OF THE MOMENTUM AND ENERGY TRANSFER RATES IN THE GUIDING-CENTER AND GYROTROPIC APPROXIMATION

The wave-particle transfer rates per unit volume were defined in Sec. IV as

$$\begin{bmatrix} 0 \\ \dot{M}_{\alpha\parallel} \\ \dot{W}_{\alpha\parallel} \\ \dot{W}_{\alpha\perp} \end{bmatrix} = \int d^3v \begin{bmatrix} 1 \\ m_\alpha v_\parallel \\ m_\alpha v_\parallel^2/2 \\ m_\alpha v_\perp^2/2 \end{bmatrix} \bar{C}_\alpha. \tag{43}$$

We now assume that the fluctuating electric field is random (Markovian), that the length and time scales for the one-particle distribution functions and the two-particle correlation functions separate, and that C_α is given by a velocity-space Fokker-Planck operator,[33,12]

$$C_\alpha = -\frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{F}_\alpha^f + \mathbf{F}_\alpha^p)f_\alpha + \frac{1}{2}\frac{\partial}{\partial \mathbf{v}}\frac{\partial}{\partial \mathbf{v}}:\mathbf{D}_\alpha^f f_\alpha, \tag{44}$$

where $\mathbf{F}_\alpha^f$, $\mathbf{F}_\alpha^p$, and $\mathbf{D}_\alpha^f$ are functionals of $\tilde{\varepsilon}$ and $\langle|\delta\tilde{E}^2|\rangle$, and where $\tilde{\varepsilon}$ and $\langle|\delta\tilde{E}^2|\rangle$ are the dielectric screening function and spectral density of the longitudinal electric-field fluctuations for the turbulent plasma, respectively. By separation of length and time scales, we mean that there is a length l and time τ such that $l_1 \gg l \gg l_2 \sim \lambda_{De}$ and $\tau_1 \gg \tau \gg \tau_2$, where λ_{De} is the electron Debye length and τ_2 is discussed in Appendix B. Here, l_1 and τ_1 are the characteristic length and time scales for the one-particle distribution functions and their moments, respectively. It is important to point out here that these scaling assumptions render (44) for C_α approximate: the Fokker-

Planck method is valid only for large length- and time-scale separations.

In the guiding-center and gyrotropic approximation, we consider a Cartesian, velocity-space coordinate system which slowly changes its orientation as s varies along **B** so that the v_z axis is always parallel or antiparallel to **B**. See Sec. III. Thus, $v_\parallel^2 = v_z^2$ and $v_\perp^2 = v_x^2 + v_y^2$. Using integration by parts and the asymptotic properties of f_α for large **v**, we see from (43) and (44) that

$$\dot{M}_{\alpha\parallel} = m_\alpha \int d^3v (2\pi)^{-1} \int_0^{2\pi} d\varphi (F_z^f + F_z^p) f_\alpha, \tag{45}$$

$$\dot{W}_{\alpha\parallel} = m_\alpha \int d^3v (2\pi)^{-1} \int_0^{2\pi} d\varphi \{ v_z (F_z^f + F_z^p) + (1/2) D_{zz}^f \} f_\alpha, \tag{46}$$

$$\dot{W}_{\alpha\perp} = m_\alpha \int d^3v (2\pi)^{-1} \int_0^{2\pi} d\varphi \{ v_x (F_x^f + F_x^p) + v_y (F_y^f + F_y^p) + (1/2)(D_{xx}^f + D_{yy}^f) \} f_\alpha . \tag{47}$$

In (45)–(47), there is an explicit dependence on f_α as shown, and an implicit dependence on f_α in $\mathbf{F}_\alpha^f$, $\mathbf{F}_\alpha^p$, and $\mathbf{D}_\alpha^f$ through the Vlasov-Maxwell hierarchy of kinetic equations. As discussed in Sec. III, we note that the guiding-center and gyrotropic approximation is the one obtained by replacing both dependences by $\bar{f}_\alpha$. In Appendix B, we calculate $\mathbf{F}_\alpha^f$, $\mathbf{F}_\alpha^p$, and $\mathbf{D}_\alpha^f$ in the guiding-center and gyrotropic approximation. We obtain $\dot{W}_{\alpha\parallel}$ and $\dot{W}_{\alpha\perp}$ as generalizations of Eqs. (64) and (63) of Ichimaru and Rosenbluth.[11] Using the I and R formalism, we may also obtain an expression for $\dot{M}_{\alpha\parallel}$. From Appendix B, we obtain

$$\dot{M}_{\alpha\parallel}(s,t) = \left(\frac{q_\alpha^2 \pi}{m_\alpha}\right) \int d^3v \bar{f}_\alpha(s,t,v_\perp,v_\parallel) \int \frac{d^3k}{(2\pi)^3} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\omega \Bigg\{ \left(\frac{k_\perp^2}{2k^2}\right)\left(\frac{k_\parallel}{\Omega_\alpha}\right) [J_{n-1}^2(\xi_\alpha) - J_{n+1}^2(\xi_\alpha)] \langle |\delta\tilde{E}^2|(s,t;\mathbf{k},\omega)\rangle + \left(\frac{k_\parallel^2}{k^2}\right) k_\parallel J_n^2(\xi_\alpha) \frac{\partial}{\partial\omega} \langle |\delta\tilde{E}^2|(s,t;\mathbf{k},\omega)\rangle + \left(\frac{k_\parallel}{k^2}\right) J_n^2(\xi_\alpha) 4 m_\alpha \, \mathrm{Im}[\tilde{\varepsilon}(s,t;\mathbf{k},\omega)^{-1}] \Bigg\} \delta(n\Omega_\alpha + k_\parallel v_\parallel - \omega), \tag{48}$$

$$\dot{W}_{\alpha\parallel}(s,t) = \left(\frac{q_\alpha^2 \pi}{m_\alpha}\right) \int d^3v \bar{f}_\alpha(s,t,v_\perp,v_\parallel) \int \frac{d^3k}{(2\pi)^3} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\omega \Bigg\{ \left(\frac{k_\perp^2}{2k^2}\right)\left(\frac{k_\parallel v_\parallel}{\Omega_\alpha}\right) [J_{n-1}^2(\xi_\alpha) - J_{n+1}^2(\xi_\alpha)] \langle |\delta\tilde{E}^2|(s,t;\mathbf{k},\omega)\rangle + \left(\frac{k_\parallel^2}{k^2}\right) J_n^2(\xi_\alpha) \times \left[\langle |\delta\tilde{E}^2|(s,t;\mathbf{k},\omega)\rangle + k_\parallel v_\parallel \frac{\partial}{\partial\omega} \langle |\delta\tilde{E}^2|(s,t;\mathbf{k},\omega)\rangle \right] + \left(\frac{k_\parallel v_\parallel}{k^2}\right) J_n^2(\xi_\alpha) 4 m_\alpha \, \mathrm{Im}[\tilde{\varepsilon}(s,t;\mathbf{k},\omega)^{-1}] \Bigg\} \delta(n\Omega_\alpha + k_\parallel v_\parallel - \omega), \tag{49}$$

$$\dot{W}_{\alpha\perp}(s,t) = \left(\frac{q_\alpha^2 \pi}{m_\alpha}\right) \int d^3v \bar{f}_\alpha(s,t,v_\perp,v_\parallel) \int \frac{d^3k}{(2\pi)^3} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\omega \Bigg\{ \left(\frac{k_\perp^2}{2k^2}\right) n [J_{n-1}^2(\xi_\alpha) - J_{n+1}^2(\xi_\alpha)] \langle |\delta\tilde{E}^2|(s,t;\mathbf{k},\omega)\rangle + \left(\frac{k_\parallel^2}{k^2}\right) n\Omega_\alpha J_n^2(\xi_\alpha) \frac{\partial}{\partial\omega} \langle |\delta\tilde{E}^2|(s,t;\mathbf{k},\omega)\rangle + \left(\frac{1}{k^2}\right) n\Omega_\alpha J_n^2(\xi_\alpha) 4 m_\alpha \, \mathrm{Im}[\tilde{\varepsilon}(s,t;\mathbf{k},\omega)^{-1}] \Bigg\} \delta(n\Omega_\alpha + k_\parallel v_\parallel - \omega), \tag{50}$$

where $\xi_\alpha = k_\perp v_\perp / \Omega_\alpha$ and $\Omega_\alpha = q_\alpha B / m_\alpha c$ to include the sign of q_α. Also, $J_n(\xi)$ is the usual Bessel function of order n, $k_\perp (k_\parallel)$ is the perpendicular (parallel) part of **k**, and $k^2 = k_\perp^2 + k_\parallel^2$. The other symbols have their usual meaning. As discussed in Appendix B, the transfer rates are weak functions of s and slow functions of t, as are the low-order velocity moments of the one-particle distribution functions. Here, we reiterate from Appendix B that the following symmetries hold: $\langle |\delta\tilde{E}^2|(s,t;\mathbf{k},\omega)\rangle = \langle |\delta\tilde{E}^2|(s,t;-\mathbf{k},-\omega)\rangle$; and $\tilde{\varepsilon}(s,t;\mathbf{k},\omega)^* = \tilde{\varepsilon}(s,t;-\mathbf{k},-\omega)$.

In deriving (48)–(50), we have neglected spatial diffusion across **B** and assumed that velocity-space transport along **B** dominates spatial transport along **B**. In order to estimate the magnitude of the spatial diffusion processes to see if they are negligible for a given problem, the formulas given by Ichimaru and Tange[34] and Tange[35] may be used. We also note that, since **B** is a weak function of s, so are Ω_α.

The momentum and energy transfer rates per unit volume for turbulent, inhomogeneous, nonuniformly magnetized plasma in the guiding-center and gyrotropic approximation are given by (48)–(50). They are generalizations of the formulas given by I and R.

VII. THE FLUID EQUATIONS FOR QUIESCENT, DRIFTING, BI-MAXWELLIAN, ELECTRON-ION PLASMA IN A NONUNIFORM B FIELD

In order to find the fluid equations for quiescent, drifting, bi-Maxwellian, electron-ion plasma in a nonuniform **B** field in the guiding-center and gyrotropic approximation, we must evaluate the momentum and energy transfer rates due to wave-particle interactions. Let us assume that $\bar{f}_\alpha$ is a drifting bi-Maxwellian, and that the relative drift between the electrons and ions is sufficiently small so that the plasma remains stable (quiescent) as it evolves in space and time. The drifting bi-Maxwellian distributions are

$$\bar{f}_\alpha(s,t,v_\perp,v_\parallel)_{qu} = n_\alpha\left(\frac{1}{2\pi v_{\alpha\perp}^2}\right)\left(\frac{1}{2\pi v_{\alpha\parallel}^2}\right)^{1/2} \times\exp\left[-\frac{v_\perp^2}{2v_{\alpha\perp}^2}-\frac{(v_\parallel-u_\alpha)^2}{2v_{\alpha\parallel}^2}\right], \quad (51)$$

where n_α, u_α, $v_{\alpha\parallel}$, and $v_{\alpha\perp}$ depend on s and t, and we have introduced the thermal velocities for the particle distributions which are given by $v_{\alpha\parallel}^2=T_{\alpha\parallel}/m_\alpha$ and $v_{\alpha\perp}^2=T_{\alpha\perp}/m_\alpha$. We note that $\bar{f}_\alpha$ has been normalized to n_α. Substituting (51) into (48)–(50), we see that the velocity-space integrals are known.[36] We obtain

$$\dot{M}_{\alpha\parallel}(s,t) = n_\alpha\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{q_\alpha^2}{m_\alpha v_{\alpha\parallel}^3}\right)\int\frac{d^3k}{(2\pi)^3}\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\omega\frac{k_\parallel}{k^2|k_\parallel|}\left\{\left[n\Omega_\alpha\frac{T_{\alpha\parallel}}{T_{\alpha\perp}}+(\omega-n\Omega_\alpha-k_\parallel u_\alpha)\right]\langle|\delta E^2|(s,t;\mathbf{k},\omega)\rangle_{\rm qu} + 4T_{\alpha\parallel}\,{\rm Im}[\varepsilon(s,t;\mathbf{k},\omega)_{\rm qu}^{-1}]\right\}\Lambda_n(\beta_\alpha)\exp\left[-\frac{(\omega-n\Omega_\alpha-k_\parallel u_\alpha)^2}{2k_\parallel^2 v_{\alpha\parallel}^2}\right], \quad (52)$$

$$\dot{W}_{\alpha\parallel}(s,t) = n_\alpha\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{q_\alpha^2}{m_\alpha v_{\alpha\parallel}^3}\right)\int\frac{d^3k}{(2\pi)^3}\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\omega\left(\frac{\omega-n\Omega_\alpha}{k^2|k_\parallel|}\right)\left\{\left[n\Omega_\alpha\frac{T_{\alpha\parallel}}{T_{\alpha\perp}}+(\omega-n\Omega_\alpha-k_\parallel u_\alpha)\right]\langle|\delta E^2|(s,t;\mathbf{k},\omega)\rangle_{\rm qu} + 4T_{\alpha\parallel}\,{\rm Im}[\varepsilon(s,t;\mathbf{k},\omega)_{\rm qu}^{-1}]\right\}\Lambda_n(\beta_\alpha)\exp\left[-\frac{(\omega-n\Omega_\alpha-k_\parallel u_\alpha)^2}{2k_\parallel^2 v_{\alpha\parallel}^2}\right], \quad (53)$$

$$\dot{W}_{\alpha\perp}(s,t) = n_\alpha\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{q_\alpha^2}{m_\alpha v_{\alpha\parallel}^3}\right)\int\frac{d^3k}{(2\pi)^3}\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\omega\left(\frac{n\Omega_\alpha}{k^2|k_\parallel|}\right)\left\{\left[n\Omega_\alpha\frac{T_{\alpha\parallel}}{T_{\alpha\perp}}+(\omega-n\Omega_\alpha-k_\parallel u_\alpha)\right]\langle|\delta E^2|(s,t;\mathbf{k},\omega)\rangle_{\rm qu} + 4T_{\alpha\parallel}\,{\rm Im}[\varepsilon(s,t;\mathbf{k},\omega)_{\rm qu}^{-1}]\right\}\Lambda_n(\beta_\alpha)\exp\left[-\frac{(\omega-n\Omega_\alpha-k_\parallel u_\alpha)^2}{2k_\parallel^2 v_{\alpha\parallel}^2}\right]. \quad (54)$$

Here, we have defined $\Lambda_n(x)=\exp(-x)I_n(x)$, $\beta_\alpha=k_\perp^2 v_{\alpha\perp}^2/\Omega_\alpha^2$, and $I_n(x)$ is the Bessel function of imaginary argument of order n. The subscript "qu" refers to quiescent plasma. In obtaining (52)–(54), we have used some well-known integrals[36] involving $J_n^2(\xi)$ and the recursion properties[36] of $I_n(\xi)$.

In what follows, we give explicit expressions for $\varepsilon_{\rm qu}$ and $\langle|\delta E^2|\rangle_{\rm qu}$ as weak functions of s and slow functions of t by making use of the separation of spatial and temporal scales between the one-particle distribution functions and the two-particle correlation functions. The expression for the quiescent dielectric screening function for this case is given by

$$\varepsilon(s,t;\mathbf{k},\omega)_{\rm qu} = 1+\sum_{\alpha=e,i}\frac{k_\alpha^2}{k^2}\left\{1+\sum_{n=-\infty}^{+\infty}\left[1+\left(\frac{T_{\alpha\parallel}}{T_{\alpha\perp}}\right)\frac{n\Omega_\alpha}{\omega-n\Omega_\alpha-k_\parallel u_\alpha}\right] \times\left[W\left(\frac{\omega-n\Omega_\alpha-k_\parallel u_\alpha}{|k_\parallel|v_{\alpha\parallel}}\right)-1\right]\Lambda_n(\beta_\alpha)\right\}, \quad (55)$$

where $W(z)$ is the plasma dispersion function[12] given by

$$W(z) = 1 - z\exp(-z^2/2)\int_0^z dy\exp(y^2/2) + i(\pi/2)^{1/2}z \times\exp(-z^2/2), \tag{56}$$

and where we have defined $k_\alpha^2 = 4\pi n_\alpha q_\alpha^2 / T_{\alpha\|}$. The spectral density of the longitudinal electric-field fluctuations can be calculated using the superposition technique for the dressed test particles.[37,38] For a drifting bi-Maxwellian with $|q_e| = q_i = q$, the spectral density is

$$\langle|\delta E^2|(s,t;\mathbf{k},\omega)\rangle_{\text{qu}} = \frac{16\pi^2 q^2}{k^2|\varepsilon(s,t;\mathbf{k},\omega)_{\text{qu}}|^2}\sum_{\alpha=e,i} S_\alpha^0(s,t;\mathbf{k},\omega)_{\text{qu}}, \tag{57}$$

$$S_\alpha^0(s,t;\mathbf{k},\omega)_{\text{qu}} = \frac{n_\alpha}{(2\pi)^{1/2}|k_\||v_{\alpha\|}}\sum_{n=-\infty}^{+\infty}\Lambda_n(\beta_\alpha) \times\exp\left[-\frac{(\omega - n\Omega_\alpha - k_\| u_\alpha)^2}{2k_\|^2 v_{\alpha\|}^2}\right]. \tag{58}$$

Since ε_{qu} and $\langle|\delta E^2|\rangle_{\text{qu}}$ are known functions of n_α, u_α, $T_{\alpha\|}$, and $T_{\alpha\perp}$, then $\dot{M}_{\alpha\|}$, $\dot{W}_{\alpha\|}$, and $\dot{W}_{\alpha\perp}$ are also known functions of n_α, u_α, $T_{\alpha\|}$, and $T_{\alpha\perp}$, which are, in turn, functions of s and t.

As is the case with all moment equation descriptions, closure conditions must be specified. For electron-ion plasma with drifting, bi-Maxwellian distributions, the $2N(=4)$ closure conditions are found by taking the appropriate velocity moments of (51). The expressions for the perpendicular and parallel total energy fluxes per particle in terms of the low-order velocity moments are

$$q_{\alpha\perp} = u_\alpha T_{\alpha\perp}, \tag{59}$$

$$q_{\alpha\|} = u_\alpha\{(3/2)T_{\alpha\|} + (1/2)m_\alpha u_\alpha^2\}. \tag{60}$$

The $4N+1(=9)$ fluid equations given by (32)–(36) and the $2N(=4)$ closure equations given by (59), (60), and (52)–(54) for the momentum and energy transfer rates when supplemented by (55)–(58) represent a closed set of self-consistent equations for quiescent, drifting, bi-Maxwellian, electron-ion plasma in a nonuniform **B** field in the guiding-center and gyrotropic approximation. There are $6N+1(=13)$ equations for the $6N+1(=13)$ low-order velocity moments and ϕ. The equations may be solved subject to appropriate initial and boundary conditions to give the weak spatial and slow temporal evolution of the system.

We also note that, since the magnetized Balescu-Lenard collision operators for the multiconstituent plasma can be recast into Fokker-Planck forms (see Ref. 12), the effect of charged-particle collisions is contained in the expressions for the transfer rates given by (52)–(58) for quiescent, magnetized, bi-Maxwellian plasma. For an explicit solution of the dielectric function for the unmagnetized, Balescu-Lenard-Poisson equations, see Refs. 39 and 40. When gravity is included in the theory, (32)–(36) and (52)–(60) may be applied to the earth's topside ionosphere for quiescent conditions where electron- and ion-neutral particle collisions are negligible compared to charged-particle collisions. Such an application is currently under study.

VIII. THE SOLUTION OF THE FLUID EQUATIONS FOR EQUILIBRIUM ELECTRON-ION PLASMA IN A UNIFORM B FIELD

In this section, we give the solution of the fluid equations for equilibrium, electron-ion plasma in a uniform **B** field and obtain the expected result. We consider the special case when the plasma is in thermal equilibrium and the **B** field is uniform, i.e., when $n_e = n_i = n, u_e = u_i = 0, T_{e\|} = T_{e\perp} = T_{i\|} = T_{i\perp} = T$. From (51), we see that

$$\bar{f}_\alpha(s,t,v_\perp,v_\|)_{\text{qu}} \to \bar{f}_\alpha(v_\perp,v_\|)_{eq} = n\left(\frac{1}{2\pi v_\alpha^2}\right)^{3/2}\exp\left[-\left(\frac{v_\perp^2 + v_\|^2}{2v_\alpha^2}\right)\right], \tag{61}$$

and that (52)–(58) pass to their equilibrium forms. Here, $v_\alpha^2 = T/m_\alpha$. In particular, the equilibrium forms of ε_{qu} and $\langle|\delta E^2|\rangle_{\text{qu}}$ are

$$\varepsilon(\mathbf{k},\omega)_{eq} = 1 + \sum_{\alpha=e,i}\frac{k_\alpha^2}{k^2}\left\{1 + \sum_{n=-\infty}^{n=+\infty}\left(\frac{\omega}{\omega - n\Omega_\alpha}\right)\Lambda_n(\beta_\alpha) \times\left[W\left(\frac{\omega - n\Omega_\alpha}{|k_\||v_\alpha}\right) - 1\right]\right\}, \tag{62}$$

$$\langle|\delta E^2|(\mathbf{k},\omega)\rangle_{eq} = \frac{16\pi^2 q^2 n}{k^2|\varepsilon(\mathbf{k},\omega)|_{eq}^2} \times\sum_{\alpha=e,i}\frac{1}{(2\pi)^{1/2}|k_\||v_\alpha}\sum_{n=-\infty}^{n=+\infty}\Lambda_n(\beta_\alpha) \times\exp\left[-\frac{(\omega - n\Omega_\alpha)^2}{2k_\|^2 v_\alpha^2}\right]. \tag{63}$$

Using (56) for the definition of the W function, we find from (62) and (63) that

$$\langle|\delta E^2|(\mathbf{k},\omega)\rangle_{\text{eq}} = \left(\frac{4T}{\omega}\right)\frac{\text{Im}\,\varepsilon(\mathbf{k},\omega)_{\text{eq}}}{|\varepsilon(\mathbf{k},\omega)_{\text{eq}}|^2} \equiv -\left(\frac{4T}{\omega}\right)\text{Im}\left[\frac{1}{\varepsilon(\mathbf{k},\omega)_{\text{eq}}}\right], \tag{64}$$

which is the relationship that is required by the fluctuation-dissipation theorem.[41,37,38] As a consequence of (64), we find from (52)–(54) that $\dot{M}_{\alpha\|} = \dot{W}_{\alpha\|} = \dot{W}_{\alpha\perp} = 0$. From (32)–(36), we see that the equilibrium solution for the fluid equations is

$$n_e = n_i = n \text{ (a constant)}, \tag{65}$$

$$T_{e\|} = T_{e\perp} = T_{i\|} = T_{i\perp} = T \text{ (a constant)}, \tag{66}$$

$$u_e = u_i = 0, \tag{67}$$

$$E_\| = 0, \tag{68}$$

$$q_{e\parallel} = q_{e\perp} = q_{i\parallel} = q_{i\perp} = 0. \tag{69}$$

The solution is the expected result. It is important to note that the fluid equations obey the fluctuation-dissipation theorem in the limit as the plasma approaches thermal equilibrium. We also note that this result is independent of the closure assumptions as $q_{e\parallel}=q_{e\perp}=q_{i\parallel}=q_{i\perp}=0$ by symmetry, if $u_e=u_i=0$.

IX. SUMMARY AND DISCUSSION

This work was motivated by the success of the I and R method in treating temperature relaxation problems for homogeneous plasmas in a uniform magnetic field.[11] We have generalized their work in order to obtain a fluid theory applicable to space and laboratory plasmas, when random electrostatic turbulence is present, the magnetic field is nonuniform, the spatial gradients are weak, the guiding-center and gyrotropic approximation is generally valid, and transport parallel to **B** dominates transport perpendicular to **B**.

The main results of this paper are summarized as follows:

(1) We have derived a set of kinetic equations in the guiding-center and gyrotropic approximation where all the drifts perpendicular to **B** are neglected. These equations are valid for a weakly inhomogeneous magnetic field, provided that the conditions on **B** and $\langle \mathbf{E} \rangle$ discussed in Sec. III are satisfied, that the turbulence is electrostatic, and that the $\langle \mathbf{E}_\perp \rangle \times \mathbf{B}$ drift is small compared to the electron and ion drifts along **B**. See (13)–(15) and the self-consistent Poisson's equation given by (16). The kinetic equations include the effect of wave-particle interactions due to the turbulence.

(2) Two sets of multiconstituent, multimoment fluid equations valid in the guiding-center and gyrotropic approximation are given by (25)–(29) and (32)–(36). The fluid equations are new and include anomalous transport of momentum and energy due to the electrostatic turbulence.

(3) Conservation laws are also given in the guiding-center and gyrotropic approximation that relate the spectral density of the longitudinal electric-field fluctuations to the low-order velocity moments of the one-particle distribution functions. See (37)–(40). The conservation laws are also new.

(4) If we assume that the turbulent electric field is random (Markovian), that the length and time scales between the one-particle distribution functions and the two-particle correlation functions separate, and that the correlation functions are given by the velocity-space Fokker-Planck operator, then explicit expressions for the wave-particle transfer rates in the guiding-center and gyrotropic approximation can be found. We have assumed that the turbulence is sufficiently weak so that the fluctuating electric field has a small effect on the unperturbed particle orbits. The transfer rates are functionals of $\tilde{\varepsilon}$ and $\langle |\delta\tilde{E}^2| \rangle$, the dielectric screening function, and the spectral density of the longitudinal electric-field fluctuations for the turbulent plasma, respectively. Here, we have generalized the work of I and R by introducing the tensorial spectral density of the longitudinal electric-field fluctuations in the guiding-center and gyrotropic approximation. See (B10). The transfer rates are given by (48)–(50). If large-amplitude coherent structures are present in the plasma so that the length and time scales do not separate, then the method presented here may not apply.

(5) For quiescent, drifting, bi-Maxwellian, electron-ion plasma in a nonuniform **B** field, we give explicit expressions for the momentum and energy transfer rates. See (52)–(58). In the limit as the plasma approaches thermal equilibrium, the Fokker-Planck expressions for the correlation functions produce momentum and energy transfer rates that are zero, in agreement with the fluctuation-dissipation theorem. The solution of the fluid equations for thermal equilibrium yields the expected result. See (65)–(69).

The method we present has a formal similarity to the standard theory of anomalous resistivity and anomalous heating.[15] The major difference is that in the work presented here, we *assume* that the correlation functions are given by a velocity-space Fokker-Planck operator. We are currently comparing the two approaches to the problem.

For nonturbulent plasmas, $\tilde{\varepsilon} \rightarrow \varepsilon_{\text{qu}}$, and explicit expressions for the wave-particle transfer rates were given for quiescent, drifting, bi-Maxwellian, electron-ion plasma. Since the magnetized Balescu-Lenard collision operators for a multiconstituent plasma can be recast into Fokker-Planck forms (see Ref. 12), the effect of charged-particle collisions is contained in the expressions for $\dot{M}_{\alpha\parallel}$, $\dot{W}_{\alpha\parallel}$, and $\dot{W}_{\alpha\perp}$ given by (52)–(58) for the quiescent plasma. However, if the plasma is turbulent, then we need either a theoretical or an experimental determination of $\tilde{\varepsilon}$ and $\langle |\delta\tilde{E}^2| \rangle$. As we mentioned in the Introduction, if a solution for the renormalized propagator for the turbulent plasma could be found, then it could be used to find the renormalized dielectric screening function and the renormalized spectral density for the longitudinal electric-field fluctuations. These formulas could then be used in (48)–(50).

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APPENDIX A: THE VLASOV-MAXWELL EQUATIONS

In this appendix, we assemble some well-known results from plasma theory for the reader's convenience. The material is discussed in Tidman and Krall,[24] Chap. 1, and we use their notation in this paper.

We wish to average the Vlasov-Maxwell equations for a plasma by splitting the one-particle distribution functions and field quantities into an ensemble-averaged part and a fluctuating part: $f_\alpha = \langle f_\alpha \rangle + \delta f_\alpha$, $\mathbf{E} = \langle \mathbf{E} \rangle + \delta\mathbf{E}$, $\mathbf{B} = \langle \mathbf{B} \rangle + \delta\mathbf{B}$. Here, the ensemble average of a quantity such as f_α is denoted by $\langle f_\alpha \rangle$ and its fluctuation about the average value by δf_α. Averaging the Vlasov-Maxwell equations in the standard way, we obtain

$$\left\{ \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha}\left[\langle \mathbf{E} \rangle + \frac{1}{c}\mathbf{v} \times \langle \mathbf{B} \rangle \right] \cdot \frac{\partial}{\partial \mathbf{v}} \right\} \langle f_\alpha \rangle = C_\alpha, \tag{A1}$$

$$C_\alpha = -\frac{q_\alpha}{m_\alpha} \left\langle \left(\delta\mathbf{E} + \frac{1}{c}\mathbf{v} \times \delta\mathbf{B} \right) \cdot \frac{\partial}{\partial \mathbf{v}} \delta f_\alpha \right\rangle, \tag{A2}$$

$$\frac{\partial}{\partial \mathbf{r}} \cdot \langle \mathbf{E} \rangle = 4\pi\rho, \quad \frac{\partial}{\partial \mathbf{r}} \times \langle \mathbf{E} \rangle + \frac{1}{c}\frac{\partial}{\partial t}\langle \mathbf{B} \rangle = 0, \tag{A3}$$

$$\frac{\partial}{\partial \mathbf{r}} \times \langle \mathbf{B} \rangle - \frac{1}{c}\frac{\partial}{\partial t}\langle \mathbf{E} \rangle = \frac{4\pi}{c}\mathbf{j}, \quad \frac{\partial}{\partial \mathbf{r}} \cdot \langle \mathbf{B} \rangle = 0. \tag{A4}$$

The charge density and current density are defined as

$$\rho = \sum_\beta q_\beta \int d^3v \langle f_\beta \rangle, \quad \mathbf{j} = \sum_\beta q_\beta \int d^3v\, \mathbf{v} \langle f_\beta \rangle. \tag{A5}$$

Here, $\langle f_\alpha \rangle = \langle f_\alpha(\mathbf{r}, \mathbf{v}, t) \rangle$ denotes the one-particle distribution function for the species α in the standard six-dimensional phase space Γ, where $\Gamma = (\mathbf{r}, \mathbf{v})$ and where $\langle f_\alpha \rangle d\Gamma$ has the standard interpretation. C_α are the wave-particle correlation functions for the problem. It is important to note that, for quiescent plasma, the C_α contain the effect of charged-particle collisions, and for turbulent plasma, the effect of turbulent fluctuations. Gaussian units are used, and $\langle f_\alpha \rangle$ are normalized so that $\int d^3v \langle f_\alpha \rangle = n_\alpha$, where n_α is the number density. The other quantities have their usual meaning. The fluctuations and the correlation functions determined from them satisfy the familiar Vlasov-Maxwell hierarchy of kinetic equations, which we do not write here.

The conservation relations for C_α may also be determined. They are

$$\int d^3v\, C_\alpha = 0, \tag{A6}$$

$$\sum_\alpha \int d^3v\, m_\alpha \mathbf{v} C_\alpha + \frac{\partial}{\partial t}\frac{1}{4\pi c}\langle \delta\mathbf{E} \times \delta\mathbf{B} \rangle - \frac{\partial}{\partial \mathbf{r}} \cdot \frac{1}{4\pi}\langle \delta\mathbf{E}\,\delta\mathbf{E} + \delta\mathbf{B}\,\delta\mathbf{B} \rangle + \frac{\partial}{\partial \mathbf{r}} \cdot \frac{1}{8\pi}\langle \delta\mathbf{E}^2 + \delta\mathbf{B}^2 \rangle \mathbf{I} = 0, \tag{A7}$$

$$\sum_\alpha \int d^3v \frac{1}{2} m_\alpha v^2 C_\alpha + \frac{\partial}{\partial t}\frac{1}{8\pi}\langle \delta\mathbf{E}^2 + \delta\mathbf{B}^2 \rangle + \frac{\partial}{\partial \mathbf{r}} \cdot \frac{c}{4\pi}\langle \delta\mathbf{E} \times \delta\mathbf{B} \rangle = 0. \tag{A8}$$

Here, we have used standard vector and tensor dyadic notation, and $\mathbf{I}$ is the unit tensor dyadic. In obtaining these results, Maxwell's equations, standard vector and tensor identities, integration by parts, and the asymptotic properties of $\langle f_\alpha \rangle$ for large $\mathbf{v}$ have been used. See Ref. 24.

In order to obtain the conservation laws for the number density, the momentum density, and the total energy density, we multiply (A1) by 1, $m_\alpha \mathbf{v}$, and $(1/2)m_\alpha v^2$, respectively, and integrate over velocity space. We then substitute for the velocity moments of C_α using (A6)–(A8). This gives the three conservation laws for the low-order velocity moments of $\langle f_\alpha \rangle$, $\langle \mathbf{E} \rangle$, and $\langle \mathbf{B} \rangle$ in terms of the fluctuating electric and magnetic fields. We obtain

$$\frac{\partial}{\partial t} n_\alpha + \frac{\partial}{\partial \mathbf{r}} \cdot n_\alpha \mathbf{u}_\alpha = 0, \tag{A9}$$

$$\frac{\partial}{\partial t}\left\{ \sum_\alpha m_\alpha n_\alpha \mathbf{u}_\alpha + \frac{1}{4\pi c}\langle \delta\mathbf{E} \times \delta\mathbf{B} \rangle \right\} + \frac{\partial}{\partial \mathbf{r}} \cdot \left\{ \sum_\alpha \int d^3v\, m_\alpha \mathbf{v}\mathbf{v} \langle f_\alpha \rangle + \frac{1}{8\pi}\langle \delta\mathbf{E}^2 + \delta\mathbf{B}^2 \rangle \mathbf{I} - \frac{1}{4\pi}\langle \delta\mathbf{E}\,\delta\mathbf{E} + \delta\mathbf{B}\,\delta\mathbf{B} \rangle \right\} = \rho\langle \mathbf{E} \rangle + \frac{1}{c}\mathbf{j} \times \langle \mathbf{B} \rangle, \tag{A10}$$

$$\frac{\partial}{\partial t}\left\{ \sum_\alpha \int d^3v \frac{1}{2} m_\alpha v^2 \langle f_\alpha \rangle + \frac{1}{8\pi}\langle \delta\mathbf{E}^2 + \delta\mathbf{B}^2 \rangle \right\} + \frac{\partial}{\partial \mathbf{r}} \cdot \left\{ \sum_\alpha \int d^3v \frac{1}{2} m_\alpha v^2 \mathbf{v} \langle f_\alpha \rangle + \frac{c}{4\pi}\langle \delta\mathbf{E} \times \delta\mathbf{B} \rangle \right\} = \mathbf{j} \cdot \langle \mathbf{E} \rangle, \tag{A11}$$

where $\mathbf{u}_\alpha$ is the drift velocity defined as $\mathbf{u}_\alpha = n_\alpha^{-1} \int d^3v\, \mathbf{v} \langle f_\alpha \rangle$.

APPENDIX B: CALCULATION OF $\dot{\mathbf{M}}_{\alpha\|}$, $\dot{\mathbf{W}}_{\alpha\|}$, AND $\dot{\mathbf{W}}_{\alpha\perp}$ IN THE GUIDING-CENTER AND GYROTROPIC APPROXIMATION

In this appendix, we give the details of the calculation leading to Eqs. (48)–(50). They may be obtained as generalizations of the work of Hubbard,[10] Ichimaru and Rosenbluth,[11] and Matsuda.[42] Uncertainties in the calculation by I and R have been discussed and clarified in Refs. 43 and 42 and we follow the procedure of Ref. 42 in this appendix.

From I and R, the equation of motion for a particle of charge q and mass m is

$$\frac{d}{dt}\mathbf{v}(t) = \frac{q}{m}\delta\mathbf{E}[\mathbf{r}(t), t] + \frac{q}{mc}\mathbf{v}(t) \times \mathbf{B}, \tag{B1}$$

where $\mathbf{r}(t)$ and $\mathbf{v}(t)$ give the trajectory of the test particle in phase space, $\mathbf{B}$ is the magnetic field, and $\delta\mathbf{E}$ is the fluctuating electric field. We wish to generalize the calculation of I and R by considering $\mathbf{B}$ to be a weak function of position. The meaning of the weak variation of $\mathbf{B}$ is given in Sec. III. In the guiding-center approximation, we consider a velocity-space coordinate system which slowly changes its orientation as s varies along the $\mathbf{B}$-field line so that the v_z axis is always oriented parallel or antiparallel to $\mathbf{B}$. Since $\mathbf{B}$ is changing slowly with s, it may be considered as locally uniform. In this appendix, all subscripts are suppressed in order to simplify the notation. The trajectories of the particles are found by integrating (B1). They are

$$\mathbf{r}(t)=\mathbf{r}(0)+\frac{1}{\Omega}\mathbf{H}(t)\cdot\mathbf{v}(0)+\frac{c}{B}\int_0^t dt'\mathbf{H}(t-t')\cdot\delta\mathbf{E}[\mathbf{r}(t'),t'], \tag{B2}$$

$$\mathbf{v}(t)=\mathbf{B}(t)\cdot\mathbf{v}(0)+\frac{q}{m}\int_0^t dt'\mathbf{B}(t-t')\cdot\delta\mathbf{E}[\mathbf{r}(t'),t'], \tag{B3}$$

where $\Omega=qB/mc$ is the cyclotron frequency including the sign of q. Here, $\mathbf{H}$ and $\mathbf{B}$ are dimensionless tensors which characterize the helical motion of a charged particle in a magnetic field and are expressed in Cartesian coordinates (x,y,z) as

$$\mathbf{H}(t)=\begin{bmatrix}+\sin\Omega t & 1-\cos\Omega t & 0\\ -(1-\cos\Omega t) & \sin\Omega t & 0\\ 0 & 0 & \Omega t\end{bmatrix}, \tag{B4}$$

$$\mathbf{B}(t)=\left(\frac{1}{\Omega}\right)\frac{d}{dt}\mathbf{H}(t)=\begin{bmatrix}+\cos\Omega t & \sin\Omega t & 0\\ -\sin\Omega t & \cos\Omega t & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{B5}$$

The tensor $\mathbf{B}(t)$ should not be confused with the vector $\mathbf{B}$, which denotes the magnetic field. Following I and R, the increments $\Delta\mathbf{r}(\tau)$ and $\Delta\mathbf{v}(\tau)$ of the particle coordinates during the time interval τ are defined as $\Delta\mathbf{r}(\tau)=\mathbf{r}(\tau)-\mathbf{r}(0)$ and $\Delta\mathbf{v}(\tau)=\mathbf{v}(\tau)-\mathbf{v}(0)$. They may be expressed in terms of the fluctuating electric field by making a Taylor expansion of $\Delta\mathbf{r}(\tau)$ and $\Delta\mathbf{v}(\tau)$ in powers of the fluctuating electric field, and then truncating the expansion to the second order. These formulas are given in I and R. For $\Delta\mathbf{v}(\tau)$, they obtain

$$\Delta\mathbf{v}(\tau)=[\mathbf{B}(\tau)-\mathbf{I}]\cdot\mathbf{v}+\frac{q}{m}\int_0^\tau d\tau'\mathbf{B}(\tau')\cdot\delta\mathbf{E}[\mathbf{r_0}(\tau-\tau'),\tau-\tau']+\frac{qc}{mB}\int_0^\tau d\tau'\mathbf{B}(\tau')\cdot\left\{\left(\int_0^{\tau-\tau'}d\tau''\mathbf{H}(\tau'')\cdot\delta\mathbf{E}[\mathbf{r_0}(\tau-\tau'-\tau''),\tau-\tau'-\tau'']\right)\cdot\frac{\partial}{\partial\mathbf{r_0}}\delta\mathbf{E}[\mathbf{r_0}(\tau-\tau'),\tau-\tau']\right\}+\cdots, \tag{B6}$$

where $\mathbf{r}_0(t)=\mathbf{r}(0)+(1/\Omega)\mathbf{H}(t)\cdot\mathbf{v}$. In making this approximation, we have assumed that the turbulence is sufficiently weak so that the fluctuating electric field has a small effect on the unperturbed particle orbits denoted by $\mathbf{r}_0(t)$.

The diffusion tensor in velocity space is defined as an ensemble average of the dyadic $\Delta\mathbf{v}(\tau)\Delta\mathbf{v}(\tau)$, as

$$\mathbf{D}(\mathbf{v})=\frac{\langle\Delta\mathbf{v}(\tau)\Delta\mathbf{v}(\tau)\rangle}{\tau}. \tag{B7}$$

The time τ must be chosen so as to satisfy the condition

$$\tau_1\gg\tau\gg\tau_2, \tag{B8}$$

where τ_1 is defined in Sec. VI and τ_2 is discussed below. Since Hubbard[10] has shown that the self-field contribution to $\mathbf{D}$ is small compared to the dominant, second-order-in-$\delta\mathbf{E}$ contribution, and since $\langle\delta\mathbf{E}\rangle=0$, we may expand $\Delta\mathbf{v}\Delta\mathbf{v}$ using (B6), take the ensemble average, and keep only the second-order terms. The dominant second-order contribution is

$$\mathbf{D}(\mathbf{v})=\frac{q^2}{m^2\tau}\int_0^\tau d\tau'\int_{-\infty}^{+\infty}dt\langle\mathbf{B}(\tau')\cdot\delta\mathbf{E}[\mathbf{r_0}(\tau-\tau'),\tau-\tau']\mathbf{B}(\tau'+t)\cdot\delta\mathbf{E}[\mathbf{r_0}(\tau-\tau'-t),\tau-\tau'-t]\rangle, \tag{B9}$$

where the condition (B8) has been used to extend the range of the t integration from $-\infty$ to $+\infty$. This is (13) of I and R. The other two second-order terms are given in Ref. 42. Using the procedure to evaluate the τ' integration that we describe below, which is used throughout this appendix, it is possible to show that the other two second-order terms vanish. Therefore, we are left with (B9) for $\mathbf{D}$. The procedure we use to evaluate the τ' integration is the same as that used in Ref. 42 and is described as follows. When no magnetic field is present, Hubbard noted that correlations of the fluctuating electric field persist only for times on the order of $\tau'\leqslant\omega_p^{-1}$. When a magnetic field is present, Matsuda[42] has assumed that the major contribution to the τ' integration comes from times where $0\leqslant\tau'\sim$ a few times τ_2, where $\tau_2(=\tau_{ac})$ is the autocorrelation time for the turbulence and where $\tau_{ac}\ll 2\pi/\Omega$. We accept this assumption and will evaluate below the τ' integration in (B9), and the analogous integration in $\mathbf{F}^f$ also given below, by integrating on τ' and then taking the limit as $\Omega\tau/2\pi\to 0$.

We now generalize the I and R calculation to include a weakly varying, fluctuating electric field in space and a slowly varying, fluctuating electric field in time by introducing the tensorial spectral density of the electric-field fluctuations

$$\langle\delta\tilde{\mathbf{E}}(\mathbf{r},t)\delta\tilde{\mathbf{E}}(\mathbf{r}',t')\rangle=\int\frac{d^3k}{(2\pi)^3}\int_{-\infty}^{+\infty}d\omega\left\langle\delta\tilde{\mathbf{E}}\delta\tilde{\mathbf{E}}^*\left(\frac{\mathbf{r}+\mathbf{r}'}{2},\frac{t+t'}{2};\mathbf{k},\omega\right)\right\rangle\times\exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-i\omega(t-t')]. \tag{B10}$$

Here $\langle\delta\tilde{\mathbf{E}}(\mathbf{r},t)\delta\tilde{\mathbf{E}}(\mathbf{r}',t')\rangle$ is the two-point, two-time correlation function for the turbulent electric field. It depends not only on $\mathbf{r}-\mathbf{r}'$ and $t-t'$, but on the average spatial $(\mathbf{r}+\mathbf{r}')/2$, and average temporal $(t+t')/2$ coordinates. In order to remind us that we are describing correlation functions for the turbulent plasma, we use the tilde notation. Now, we wish to impose the same restrictions on the length and time scales of the average spatial and the average temporal variations of $\langle \delta\tilde{\mathbf{E}}\delta\tilde{\mathbf{E}}^*[(\mathbf{r}+\mathbf{r}')/2,(t+t')/2;\mathbf{k},\omega]\rangle$ as we did in Sec. III for the variation of $\langle \mathbf{E}\rangle$. Therefore, we assume that the length scale for the variation of $\langle \delta\tilde{\mathbf{E}}\delta\tilde{\mathbf{E}}^*\rangle$ with $(\mathbf{r}+\mathbf{r}')/2$ in the direction perpendicular and parallel to $\mathbf{B}$ is large compared to $a_\perp$ and $a_\parallel$, respectively. We also assume that the time scale for the variation of $\langle \delta\tilde{\mathbf{E}}\delta\tilde{\mathbf{E}}^*\rangle$ with $(t+t')/2$ is large compared to $|2\pi/\Omega|$. In the guiding-center coordinate system we use, which was discussed in Sec. III, $(\mathbf{r}+\mathbf{r}')/2\to s$ and $(t+t')/2 \to t$. Here, s and t are the weak spatial and slow temporal coordinates, and $\mathbf{r}-\mathbf{r}'$ and $t-t'$ are the strong spatial and fast temporal coordinates, where s is the distance along the magnetic field line. Again, it is important to point out that the scaling assumptions leading to (B10) for the spectral density are valid only for a large separation of the length and time scales.

When the fluctuations are electrostatic, the tensor of the spectral density of the electric-field fluctuations has a simplified form,

$$\langle \delta\tilde{\mathbf{E}}\delta\tilde{\mathbf{E}}^*(s,t;\mathbf{k},\omega)\rangle = (\mathbf{k}\mathbf{k}/k^2)\langle|\delta\tilde{E}^2|(s,t;\mathbf{k},\omega)\rangle, \tag{B11}$$

where $\langle|\delta\tilde{E}^2|(s,t;\mathbf{k},\omega)\rangle$ is the spectral density of the turbulent, longitudinal electric-field fluctuations and has the symmetry $\langle|\delta\tilde{E}^2|(s,t;\mathbf{k},\omega)\rangle=\langle|\delta\tilde{E}^2|(s,t;-\mathbf{k},-\omega)\rangle$. Using (B10), we find that the generalized diffusion tensor is

$$\mathbf{D} = \frac{q^2}{m^2\tau}\int \frac{d^3k}{(2\pi)^3}\int_{-\infty}^{+\infty} d\omega\langle|\delta\tilde{E}^2|(s,t;\mathbf{k},\omega)\rangle \times\int_{-\infty}^{+\infty} dt\int_0^\tau d\tau'\left(\frac{1}{k^2}\right)[\mathbf{B}(\tau')\cdot\mathbf{k}][\mathbf{B}(\tau'+t)\cdot\mathbf{k}] \times\exp[i\mathbf{k}\cdot\Delta\mathbf{r}_0(t)-i\omega t], \tag{B12}$$

$$\Delta\mathbf{r}_0(t) = \mathbf{r}_0(\tau-\tau')-\mathbf{r}_0(\tau-\tau'-t) = \Omega^{-1}[\mathbf{H}(\tau-\tau')-\mathbf{H}(\tau-\tau'-t)]\cdot\mathbf{v}. \tag{B13}$$

Here, the slow time dependence of $\langle|\delta\tilde{E}^2|\rangle$ has been excluded from the integration over the fast time scale. Using the identity[36]

$$\exp(iz\sin\psi) = \sum_{n=-\infty}^{+\infty} J_n(z)\exp(in\psi), \tag{B14}$$

we find that the expanded expression for $\mathbf{D}$ is

$$\mathbf{D} = \frac{q^2}{m^2}\int \frac{d^3k}{(2\pi)^3}\sum_{n=-\infty}^{n=+\infty}\sum_{l=-\infty}^{l=+\infty}\int_{-\infty}^{+\infty} d\omega\langle|\delta\tilde{E}^2(s,t;\mathbf{k},\omega)|\rangle \times\int_{-\infty}^{+\infty} dt\frac{1}{\tau}\int_0^\tau d\tau'\frac{1}{k^2}[\mathbf{B}(\tau')\cdot\mathbf{k}][\mathbf{B}(\tau'+t)\cdot\mathbf{k}] \times J_n(\xi)J_l(\xi)\exp\{i(n-l)[\varphi-\psi-\Omega(\tau-\tau')]\} \times\exp\{i(n\Omega+k_\parallel v_\parallel-\omega)t\}. \tag{B15}$$

In writing (B15), we have chosen a Cartesian, velocity-space coordinate system where $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, and $\hat{\mathbf{e}}_z$ are appropriate unit vectors and the magnetic field is oriented either parallel or antiparallel to $\hat{\mathbf{e}}_z$. Here, $\varphi(\psi)$ is the angle that the component of $\mathbf{v}(\mathbf{k})$ in the $\hat{\mathbf{e}}_x-\hat{\mathbf{e}}_y$ plane makes with the $\hat{\mathbf{e}}_x$ axis. Also, $\xi=k_\perp v_\perp/\Omega$, where $v_\perp(k_\perp)$ is the component of $\mathbf{v}(\mathbf{k})$ in the $\hat{\mathbf{e}}_x-\hat{\mathbf{e}}_y$ plane, and $v_\parallel(k_\parallel)$ is the component of $\mathbf{v}(\mathbf{k})$ either parallel or antiparallel to the magnetic field. Using (B5) and integration by parts, we see that the τ' integration can be carried out and the limit as $\Omega\tau/2\pi\to 0$ taken, as described above, to give

$$\mathbf{D} = \frac{q^2}{m^2}\int \frac{d^3k}{(2\pi)^3}\int_{-\infty}^{+\infty} d\omega\langle|\delta\tilde{E}^2|(s,t;\mathbf{k},\omega)\rangle \times\int_{-\infty}^{+\infty} dt\frac{1}{k^2}[\mathbf{k}\mathbf{B}(t)\cdot\mathbf{k}]\exp[-i\mathbf{k}\cdot\Omega^{-1}\mathbf{H}(-t)\cdot\mathbf{v}-i\omega t]. \tag{B16}$$

This is Eq. (17) of Ref. 42 when $\langle|\delta\tilde{E}^2|\rangle$ is independent of s and t. It is the generalized $\mathbf{D}$ when $\langle|\delta\tilde{E}^2|\rangle$ has a weak spatial and a slow temporal dependence before the gyrophase average is calculated. From Sec. VI, we see that since f_α is replaced by $\bar{f}_\alpha$, we need only to calculate the gyrophase average of $\mathbf{D}$ in order to evaluate $\dot{W}_{\alpha\parallel}$ and $\dot{W}_{\alpha\perp}$. Therefore,

$$\bar{\mathbf{D}} = \frac{q^2}{m^2}\int \frac{d^3k}{(2\pi)^3}\int_{-\infty}^{+\infty} d\omega\langle|\delta\tilde{E}^2|(s,t;\mathbf{k},\omega)\rangle\int_{-\infty}^{+\infty} dt\left(\frac{1}{2k^2}\right) \times\begin{bmatrix} k_\perp^2\cos\Omega t & -k_\perp^2\sin\Omega t & 0 \\ k_\perp^2\sin\Omega t & +k_\perp^2\cos\Omega t & 0 \\ 0 & 0 & 2k_\parallel^2 \end{bmatrix} \times\sum_{n=-\infty}^{+\infty} J_n^2(\xi)\exp[i(n\Omega+k_\parallel v_\parallel-\omega)t]. \tag{B17}$$

Noting that only the even part on t survives the inversion on $\mathbf{k}$, ω, and n, we obtain

$$\bar{\mathbf{D}} = \begin{bmatrix} \bar{D}_\perp & 0 & 0 \\ 0 & \bar{D}_\perp & 0 \\ 0 & 0 & \bar{D}_\parallel \end{bmatrix}, \tag{B18}$$

where

$$\bar{D}_\parallel = \frac{2\pi q^2}{m^2}\int \frac{d^3k}{(2\pi)^3}\int_{-\infty}^{+\infty} d\omega\sum_{n=-\infty}^{+\infty}\frac{k_\parallel^2}{k^2}J_n^2(\xi) \times\langle|\delta\tilde{E}^2|(s,t;\mathbf{k},\omega)\rangle\delta(n\Omega_\alpha+k_\parallel v_\parallel-\omega), \tag{B19}$$

$$\bar{D}_\perp = \frac{2\pi q^2}{m^2}\int \frac{d^3k}{(2\pi)^3}\int_{-\infty}^{+\infty} d\omega\sum_{n=-\infty}^{+\infty}\frac{k_\perp^2}{4k^2}[J_{n-1}^2(\xi)+J_{n+1}^2(\xi)] \times\langle|\delta\tilde{E}^2|(s,t;\mathbf{k},\omega)\rangle\delta(n\Omega+k_\parallel v_\parallel-\omega). \tag{B20}$$

Equations (B19) and (B20) are the generalizations of Eqs.

(26) and (25) of I and R to include the weak spatial and slow temporal variation of $\langle|\delta\widetilde{E}^2|\rangle$.

In order to find the generalized friction coefficient, again we follow I and R. It is defined as

$$\mathbf{F}(\mathbf{v}) = \frac{\langle \Delta \mathbf{v}(\tau)\rangle}{\tau}, \tag{B21}$$

where τ is a time interval satisfying (B8). As shown by Hubbard,[10] there are two contributions to $\mathbf{F}$: a part due to the correlated effect of the electric-field fluctuations on the particle orbit, $\mathbf{F}^f$, and a polarization part due to the self-field of a test particle in a plasma, $\mathbf{F}^p$,

$$\mathbf{F} = \mathbf{F}^f + \mathbf{F}^p. \tag{B22}$$

The formulas given in I and R may be generalized in the same way as we did to obtain (B16) for a weakly varying $\mathbf{B}$ field with position, and a weakly and slowly varying $\langle|\delta\widetilde{E}^2|\rangle$ with position and time, respectively. In the calculation of $\mathbf{F}^f$, we note that the first two terms in (B6) do not contribute because their ensemble average is zero, and that the contribution comes from the third term. Following procedures similar to the calculation of $\mathbf{D}$, we obtain

$$\mathbf{F}^f = i\frac{qc}{2mB}\int \frac{d^3k}{(2\pi)^3}\int_{-\infty}^{+\infty} d\omega \langle|\delta\widetilde{E}^2|(s,t;\mathbf{k},\omega)\rangle \times \int_{-\infty}^{+\infty} dt\left(\frac{\mathbf{k}}{k^2}\right)\mathbf{k}\cdot\mathbf{H}(t)\cdot\mathbf{k} \times \exp[-i\mathbf{k}\cdot\Omega^{-1}\mathbf{H}(-t)\cdot\mathbf{v} - i\omega t]. \tag{B23}$$

We have used the above-described procedure to evaluate the τ' integration and (B8) to extend the t integration, first from 0 to ∞, and then from $-\infty$ to $+\infty$ by taking half the value. Equation (B23) is the same as Eq. (24) of Ref. 42 when $\langle|\delta\widetilde{E}^2|\rangle$ is independent of s and t and is the generalized $\mathbf{F}^f$ before the gyrophase average of $\mathbf{v}\cdot\mathbf{F}^f$ is calculated. It is also possible to show that

$$\partial/\partial\mathbf{v}\cdot\mathbf{D} = 2\mathbf{F}^f, \tag{B24}$$

where $\mathbf{D}$ is given by (B16).

The second term in (B22) is due to the polarization of the plasma and has been given when there is no magnetic field present by Hubbard.[10] When a magnetic field is present, we obtain

$$\mathbf{F}^p = \frac{2q^2}{m}\int \frac{d^3k}{(2\pi)^3}\int_{-\infty}^{+\infty} d\omega\, \mathrm{Im}[\widetilde{\varepsilon}(s,t;\mathbf{k},\omega)^{-1}] \times \int_{-\infty}^{+\infty} dt\left(\frac{\mathbf{k}}{k^2}\right)\exp[-i\mathbf{k}\cdot\Omega^{-1}\mathbf{H}(t)\cdot\mathbf{v} + i\omega t]. \tag{B25}$$

Here, $\widetilde{\varepsilon}$ is the dielectric screening function for the turbulent plasma. It varies weakly with s and slowly with t as the properties of the plasma medium vary. We also note that $\widetilde{\varepsilon}(s,t;\mathbf{k},\omega)^* = \widetilde{\varepsilon}(s,t;-\mathbf{k},-\omega)$, and (B25) is the same as (30) of I and R and (27) of Ref. 42 when $\widetilde{\varepsilon}$ is independent of s and t.

From Sec. VI, we see that in order to find $\dot{W}_\parallel$ and $\dot{W}_\perp$, we need to find the gyrophase average of

$$\mathbf{v}\cdot(\mathbf{F}^f + \mathbf{F}^p)\bar{f} = \{\mathbf{v}\cdot(\mathbf{F}^f + \mathbf{F}^p)\}_\parallel \bar{f} + \{\mathbf{v}\cdot(\mathbf{F}^f + \mathbf{F}^p)\}_\perp \bar{f}. \tag{B26}$$

We proceed as in the calculation for $\bar{\mathbf{D}}$ to obtain

$$(2\pi)^{-1}\int_0^{2\pi} d\varphi m(\mathbf{v}\cdot\mathbf{F}^f)_\parallel = \frac{q^2\pi}{m}\int \frac{d^3k}{(2\pi)^3}\int_{-\infty}^{+\infty} d\omega\Bigg\{\left(\frac{k_\parallel v_\parallel}{\Omega}\right)\left(\frac{k_\perp^2}{2k^2}\right) \times \sum_{n=-\infty}^{+\infty}[J_{n-1}^2(\xi) - J_{n+1}^2(\xi)]\langle|\delta\widetilde{E}^2|(s,t;\mathbf{k},\omega)\rangle + k_\parallel v_\parallel\left(\frac{k_\parallel^2}{k^2}\right)\sum_{n=-\infty}^{+\infty} J_n^2(\xi)\frac{\partial}{\partial\omega}\langle|\delta\widetilde{E}^2|(s,t;\mathbf{k},\omega)\rangle\Bigg\} \times \delta(n\Omega + k_\parallel v_\parallel - \omega), \tag{B27}$$

$$(2\pi)^{-1}\int_0^{2\pi} d\varphi m(\mathbf{v}\cdot\mathbf{F}^f)_\perp = \frac{q^2\pi}{m}\int \frac{d^3k}{(2\pi)^3}\int_{-\infty}^{+\infty} d\omega\Bigg\{-\left(\frac{k_\perp^2}{2k^2}\right) \times \sum_{n=-\infty}^{+\infty}[J_{n-1}^2(\xi) + J_{n+1}^2(\xi)]\langle|\delta\widetilde{E}^2|(s,t;\mathbf{k},\omega)\rangle + \left(\frac{k_\perp^2}{2k^2}\right) \times \sum_{n=-\infty}^{+\infty} n[J_{n-1}^2(\xi) - J_{n+1}^2(\xi)]\langle|\delta\widetilde{E}^2|(s,t;\mathbf{k},\omega)\rangle + \left(\frac{k_\parallel^2}{k^2}\right) \times \sum_{n=-\infty}^{+\infty} n\Omega J_n^2(\xi)\frac{\partial}{\partial\omega}\langle|\delta\widetilde{E}^2|(s,t;\mathbf{k},\omega)\rangle\Bigg\}\delta(n\Omega + k_\parallel v_\parallel - \omega). \tag{B28}$$

A similar calculation for $\mathbf{F}^p$ yields

$$(2\pi)^{-1}\int_0^{2\pi} d\varphi m(\mathbf{v}\cdot\mathbf{F}^p)_\parallel = \frac{q^2\pi}{m}\int \frac{d^3k}{(2\pi)^3}\int_{-\infty}^{+\infty} d\omega\left(\frac{k_\parallel v_\parallel}{k^2}\right)4m\,\mathrm{Im}[\widetilde{\varepsilon}(s,t;\mathbf{k},\omega)^{-1}] \times \sum_{n=-\infty}^{+\infty} J_n^2(\xi)\delta(n\Omega + k_\parallel v_\parallel - \omega), \tag{B29}$$

$$(2\pi)^{-1}\int_0^{2\pi} d\varphi m(\mathbf{v}\cdot\mathbf{F}^p)_\perp = \frac{q^2\pi}{m}\int \frac{d^3k}{(2\pi)^3}\int_{-\infty}^{+\infty} d\omega\left(\frac{4m}{k^2}\right)\mathrm{Im}[\widetilde{\varepsilon}(s,t;\mathbf{k},\omega)^{-1}] \times \sum_{n=-\infty}^{+\infty} n\Omega J_n^2(\xi)\delta(n\Omega + k_\parallel v_\parallel - \omega). \tag{B30}$$

Combining (B27)–(B30) with (B19) and (B20) allows us to write expressions for $\dot{W}_{\alpha\parallel}$ and $\dot{W}_{\alpha\perp}$ in the guiding-center and gyrotropic approximation. They are given by (49) and (50). We note that $\bar{D}_{\perp}$ and the first term on the right-hand side of (B28) cancel, and that $\bar{D}_{\perp}$ does not appear in (50).

In order to find $\dot{M}_{\alpha\parallel}$, we need to evaluate the gyrophase average of F_z^f and F_z^p. Using similar procedures, we obtain

$$\bar{F}_z^f = \frac{q^2\pi}{m}\int \frac{d^3k}{(2\pi)^3}\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\omega\left\{\left(\frac{k_\perp^2}{2k^2}\right)\left(\frac{k_\parallel}{\Omega}\right)[J_{n-1}^2(\xi) - J_{n+1}^2(\xi)]\langle|\delta\tilde{E}^2|(s,t;\mathbf{k},\omega)\rangle + \left(\frac{k_\parallel^2}{k^2}\right)k_\parallel J_n^2(\xi)\frac{\partial}{\partial\omega} \times\langle|\delta\tilde{E}^2|(s,t;\mathbf{k},\omega)\rangle\right\}\delta(n\Omega + k_\parallel v_\parallel - \omega), \tag{B31}$$

$$\bar{F}_z^p = \frac{q^2\pi}{m}\int \frac{d^3k}{(2\pi)^3}\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\omega\left(\frac{k_\parallel}{k^2}\right)J_n^2(\xi)4m \times \mathrm{Im}[\bar{\varepsilon}(s,t;\mathbf{k},\omega)^{-1}]\delta(n\Omega + k_\parallel v_\parallel - \omega). \tag{B32}$$

Combining the two, we obtain (48).

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