

# THE WIENER-ASKEY POLYNOMIAL CHAOS FOR STOCHASTIC DIFFERENTIAL EQUATIONS\*

DONGBIN XIU <sup>†</sup> AND GEORGE EM KARNIADAKIS <sup>‡</sup>

**Abstract.** We present a new method for solving stochastic differential equations based on Galerkin projections and extensions of Wiener’s polynomial chaos. Specifically, we represent the stochastic processes with an optimum trial basis from the Askey family of orthogonal polynomials that reduces the dimensionality of the system and leads to exponential convergence of the error. Several continuous and discrete processes are treated, and numerical examples show substantial speed-up compared to Monte-Carlo simulations for low dimensional stochastic inputs.

**Key words.** Polynomial chaos, Askey-scheme, orthogonal polynomials, stochastic differential equations, spectral methods, Galerkin projection

**AMS subject classifications.** 65C20, 65C30

**1. Introduction.** Wiener first defined ‘Homogeneous Chaos’ as the span of Hermite polynomial functionals of a Gaussian process [19]; Polynomial Chaos is defined as the member of that set. According to the Cameron-Martin theorem [3], the Fourier-Hermite series converge to any  $L^2$  functional in the  $L^2$  sense. In the context of stochastic processes, this implies that the homogeneous chaos expansion converges to any processes with *finite* second-order moments. Therefore, such an expansion provides a means of representing a stochastic process with Hermite orthogonal polynomials. Other names such as ‘Wiener chaos’, ‘Wiener-Hermite chaos’, etc., have also been used in the literature. In this paper, we will use the term Hermite-Chaos.

While Hermite-Chaos is useful in the analysis of stochastic processes, efforts have also been made to apply it to model uncertainty in physical applications. In this case, the continuous integral form of the Hermite-Chaos is written in the discrete form of infinite summation which is further truncated. Ghanem & Spanos [9] combined the Hermite-Chaos expansion with finite element method to model uncertainty encountered in various problems of solid mechanics, e.g., [7], [8], [9], etc. In [20], the polynomial chaos was applied to modeling uncertainty in fluid dynamics applications. The algorithm was implemented in the context of spectral/*hp* element method and various benchmark tests were conducted to demonstrate convergence in prototype flows.

Although for any arbitrary random process with finite second-order moments, the Hermite-Chaos expansion converges in accord with Cameron-Martin theorem [3], it has been demonstrated that the convergence rate is optimal for Gaussian processes, in fact the rate is exponential [15]. This can be understood from the fact that the weighting function of Hermite polynomials is the *same* as the probability density function of the Gaussian random variables. For other types of processes the convergence rate may be substantially slower. In this case, other types of orthogonal polynomials, instead of Hermite polynomials, could be used to construct the chaos expansion. In an early work by Ogura [16], a chaos expansion based on Charlier polynomials was

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proposed to represent the Poisson processes, following the theory of ‘discrete chaos’ by Wiener [19].

An important class of orthogonal polynomials are the members of the so-called Askey-scheme of polynomials [1]. This scheme classifies the hypergeometric orthogonal polynomials that satisfy some type of differential or difference equation and indicates the limit relations between them. Hermite polynomials are a subset of the Askey-scheme. Each subset of the orthogonal polynomials in the Askey-scheme has different weighting function in their orthogonality relationship. It has been realized that some of these weighting functions are identical to the probability function of certain random distributions. For example,

- Hermite polynomials are associated with the Gaussian distribution.
- Laguerre polynomials with the Gamma distribution.
- Jacobi polynomials with the Beta distribution.
- Charlier polynomials with the Poisson distribution.
- Meixner polynomials with the negative Binomial distribution.
- Krawtchouk polynomials with the Binomial distribution, and
- Hahn polynomials with the Hypergeometric distribution.

This finding opens the possibility of representing stochastic processes with different orthogonal polynomials according to the property of the processes.

The close connection between stochastic processes and orthogonal polynomials has long been recognized. Despite of the role of Hermite polynomials in the integration theory of Brownian motion ([19] and [11]), many birth-and-death models were related to specific orthogonal polynomials. The so-called Karlin-McGregor representation of the transition probabilities of a birth-and-death process is in terms of orthogonal polynomials [12]. In [16] and [5], the integral relation between the Poisson process and the Charlier polynomials was found. In [17] the role of the orthogonal polynomials from Askey-scheme in the theory of Markov processes was studied and the connection between the Krawtchouk polynomials and the binomial process was established.

In this paper, we extend the work by Ghanem & Spanos for Hermite-Chaos expansion [9] and Ogura for Charlier-Chaos expansion [16]. We propose an Askey-scheme based polynomial chaos expansion for stochastic processes which includes all the orthogonal polynomials in the above list. We demonstrate numerically the optimal (exponential) convergence rate of each Wiener-Askey polynomial chaos expansion for their corresponding stochastic processes by solving a stochastic ordinary differential equation, for which the exact solutions can be obtained. It is also shown that if for a certain process the optimal Wiener-Askey polynomial chaos expansion is not employed, the solution also converges but the rate is clearly slower. This approach will provide a guideline for representing stochastic processes in physical applications properly.

In practical applications, one often does not know the analytical form of the distribution of the process, or if known, it may not be one of the basic distributions, e.g. Gaussian, Poisson, etc.. In this case, one can choose a set of Wiener-Askey polynomial chaos expansion and conduct a numerical projection procedure to represent the process. This issue will be addressed in the present paper as well.

This paper is organized as follows: In the next section we review the theory of the Askey-scheme of hypergeometric orthogonal polynomials, and in section 3 we review the theory of the original Wiener’s polynomial chaos. In section 4 we present the framework of Wiener-Askey polynomial chaos expansion for stochastic processes. In section 5 we present numerical solutions of a stochastic ordinary differential equation

with different Wiener-Askey chaos expansions. The choice of the particular Wiener-Askey chaos is based on the distribution of the random input, and we demonstrate the exponential convergence rate with the appropriately chosen Wiener-Askey basis. In section 6, we address the issue of representing an arbitrary random distribution and we show that although the Wiener-Askey polynomial chaos converges in general, the exponential convergence is not realized if the optimal type of Wiener-Askey chaos is not chosen. We conclude the paper with a discussion on possible extensions and applications to more complicated problems. An appendix of the definitions and properties of the orthogonal polynomials discussed in this paper is included for completeness.

**2. The Askey-scheme of Hypergeometric Orthogonal Polynomials.** The theory of orthogonal polynomials is relatively mature and several books have been devoted to their study (e.g., [18], [2], [4], etc.). However, more recent work has shown that an important class of orthogonal polynomials belong to the Askey-scheme of hypergeometric polynomials [1]. In this section, we briefly review the theory of hypergeometric orthogonal polynomials. We adopt the notation of [14] and [17].

**2.1. The Generalized Hypergeometric Series.** We first introduce the *Pochhammer symbol*  $(a)_n$  defined by

$$(a)_n = \begin{cases} 1, & \text{if } n = 0, \\ a(a+1)\cdots(a+n-1), & \text{if } n = 1, 2, 3, \dots \end{cases} \quad (2.1)$$

In terms of Gamma function, we have

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n > 0. \quad (2.2)$$

The *generalized hypergeometric series*  ${}_rF_s$  is defined by

$${}_rF_s(a_1, \dots, a_r; b_1, \dots, b_s; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!}, \quad (2.3)$$

where  $b_i \neq 0, -1, -2, \dots$  for  $i = \{1, \dots, s\}$  to ensure the denominator factors in the terms of the series are never zero. Clearly, the ordering of the numerator parameters and of the denominator parameters are immaterial. The radius of convergence  $\rho$  of the hypergeometric series is

$$\rho = \begin{cases} \infty & \text{if } r < s + 1, \\ 1 & \text{if } r = s + 1, \\ 0 & \text{if } r > s + 1. \end{cases} \quad (2.4)$$

Some elementary cases of the hypergeometric series are:

- Exponential series  ${}_0F_0$ ,
- Binomial series  ${}_1F_0$ ,
- Gauss hypergeometric series  ${}_2F_1$ .

If one of the numerator parameters  $a_i, i = 1, \dots, r$  is a negative integer, say  $a_1 = -n$ , the hypergeometric series (2.3) terminates at the  $n^{\text{th}}$ -term and becomes a polynomial in  $z$ ,

$${}_rF_s(-n, \dots, a_r; b_1, \dots, b_s; z) = \sum_{k=0}^n \frac{(-n)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!}. \quad (2.5)$$

**2.2. Properties of the Orthogonal Polynomials.** A system of polynomials  $\{Q_n(x), n \in \mathcal{N}\}$  where  $Q_n(x)$  is a polynomial of exact degree  $n$  and  $\mathcal{N} = \{0, 1, 2, \dots\}$  or  $\mathcal{N} = \{0, 1, \dots, N\}$  for a finite nonnegative integer  $N$ , is an orthogonal system of polynomials with respect to some real positive measure  $\phi$  if the following orthogonality relations are satisfied

$$\int_S Q_n(x)Q_m(x)d\phi(x) = h_n^2\delta_{nm}, \quad n, m \in \mathcal{N}, \quad (2.6)$$

where  $S$  is the support of the measure  $\phi$  and the  $h_n$  are nonzero constants. The system is called orthonormal if  $h_n = 1$ .

The measure  $\phi$  often has a density  $w(x)$  or weights  $w(i)$  at points  $x_i$  in the discrete case. The relations (2.6) then become

$$\int_S Q_n(x)Q_m(x)w(x)dx = h_n^2\delta_{nm}, \quad n, m \in \mathcal{N}, \quad (2.7)$$

in the continuous case, or

$$\sum_{i=0}^M Q_n(x_i)Q_m(x_i)w(x_i) = h_n^2\delta_{nm}, \quad n, m \in \mathcal{N}, \quad (2.8)$$

in the discrete case where it is possible that  $M = \infty$ .

The density  $w(x)$ , or weights  $w(i)$  in the discrete case, is also commonly referred as the *weighting function* in the theory of orthogonal polynomials. It will be shown later that the weighting functions for some orthogonal polynomials are identical to certain probability functions. For example, the weighting function for the Hermite polynomials is the same as probability density function of the Gaussian random variables. This fact plays an important role in representing stochastic processes with orthogonal polynomials.

All orthogonal polynomials  $\{Q_n(x)\}$  satisfy a *three-term recurrence relation*

$$-xQ_n(x) = A_nQ_{n+1}(x) - (A_n + C_n)Q_n(x) + C_nQ_{n-1}(x), \quad n \geq 1, \quad (2.9)$$

where  $A_n, C_n \neq 0$  and  $C_n/A_{n-1} > 0$ . Together with  $Q_{-1}(x) = 0$  and  $Q_0(x) = 1$ , all  $Q_n(x)$  can be determined by the recurrence relation.

It is well known that continuous orthogonal polynomials satisfy the second-order differential equation

$$s(x)y'' + \tau(x)y' + \lambda y = 0, \quad (2.10)$$

where  $s(x)$  and  $\tau(x)$  are polynomials of at most second and first degree, respectively, and

$$\lambda = \lambda_n = -n\tau' - \frac{1}{2}n(n-1)s'' \quad (2.11)$$

are the eigenvalues of the differential equation; the orthogonal polynomials  $y(x) = y_n(x)$  are the eigenfunctions.

In the discrete case, we introduce the forward and backward difference operator, respectively

$$\Delta f(x) = f(x+1) - f(x) \quad \text{and} \quad \nabla f(x) = f(x) - f(x-1). \quad (2.12)$$

The *difference equation* corresponding to the differential equation (2.10) is

$$s(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda y(x) = 0. \quad (2.13)$$

Again  $s(x)$  and  $\tau(x)$  are polynomials of at most second and first degree, respectively;  $\lambda = \lambda_n$  are eigenvalues of the difference equation, and the orthogonal polynomials  $y(x) = y_n(x)$  are the eigenfunctions.

All orthogonal polynomials can be obtained by repeatedly applying the differential operator as follows

$$Q_n(x) = \frac{1}{w(x)} \frac{d^n}{dx^n} [w(x)s^n(x)]. \quad (2.14)$$

In the discrete case, the differential operator ( $d/dx$ ) is replaced by the backward difference operator  $\nabla$ . A constant factor can be introduced for normalization. Equation (2.14) is referred as the *generalized Rodriguez formula*, named after J. Rodriguez who first discovered the specific formula for Legendre polynomials.

**2.3. The Askey-scheme.** The Askey-scheme, which can be represented as a tree structure shown in figure 2.1, classifies the hypergeometric orthogonal polynomials and indicates the limit relations between them. The ‘tree’ starts with the Wilson polynomials and the Racah polynomials on the top. They both belong to the class  ${}_4F_3$  of the hypergeometric orthogonal polynomials (2.5). The Wilson polynomials are continuous polynomials and the Racah polynomials are discrete. The lines connecting different polynomials denote the limit transition relationships between them, which imply that polynomials at the lower end of the lines can be obtained by taking the limit of one parameter from their counterparts on the upper end. For example, the limit relation between Jacobi polynomials  $P_n^{(\alpha,\beta)}(x)$  and Hermite polynomials  $H_n(x)$  is

$$\lim_{\alpha \rightarrow \infty} \alpha^{-\frac{1}{2}n} P_n^{(\alpha,\alpha)} \left( \frac{x}{\sqrt{\alpha}} \right) = \frac{H_n(x)}{2^n n!},$$

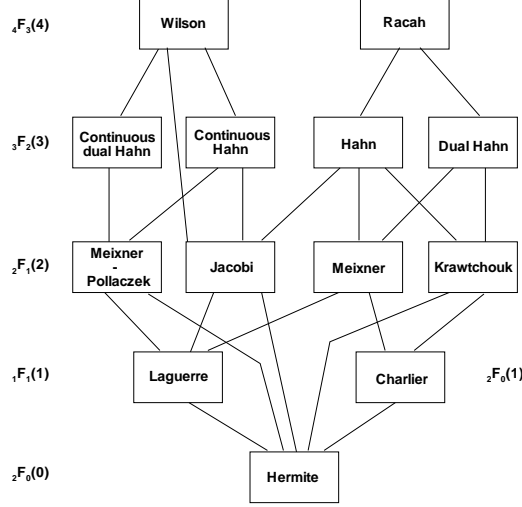
and between Meixner polynomials  $M_n(x; \beta, c)$  and Charlier polynomials  $C_n(x; a)$  is

$$\lim_{\beta \rightarrow \infty} M_n \left( x; \beta, \frac{a}{a + \beta} \right) = C_n(x; a).$$

For a detailed account of the limit relations of Askey-scheme, the interested reader should consult [14] and [17].

The orthogonal polynomials associated with the Wiener-Askey polynomials chaos include: Hermite, Laguerre, Jacobi, Charlier, Meixner, Krawtchouk and Hahn polynomials. A survey with their definitions and properties can be found in the appendix of this paper.

**3. The Original Wiener Polynomial Chaos.** The Homogeneous Chaos expansion was first proposed by Wiener [19]; it employs the Hermite polynomials in terms of Gaussian random variables. According to the theorem by Cameron & Martin [3], it can approximate any functionals in  $L_2(C)$  and converges in the  $L_2(C)$  sense. Therefore, Hermite-Chaos provides a means for expanding second-order random processes in terms of orthogonal polynomials. Second-order random processes are processes with finite variance, and this applies to most physical processes. Thus,

FIG. 2.1. *The Askey scheme of orthogonal polynomials*

a general second-order random process  $X(\theta)$ , viewed as a function of  $\theta$  as the random event, can be represented in the form

$$\begin{aligned}
X(\theta) &= a_0 H_0 \\
&+ \sum_{i_1=1}^{\infty} a_{i_1} H_1(\xi_{i_1}(\theta)) \\
&+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} a_{i_1 i_2} H_2(\xi_{i_1}(\theta), \xi_{i_2}(\theta)) \\
&+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} a_{i_1 i_2 i_3} H_3(\xi_{i_1}(\theta), \xi_{i_2}(\theta), \xi_{i_3}(\theta)) \\
&+ \dots, \tag{3.1}
\end{aligned}$$

where  $H_n(\xi_{i_1}, \dots, \xi_{i_n})$  denotes the Hermite-Chaos of order  $n$  in the variables  $(\xi_{i_1}, \dots, \xi_{i_n})$ , where the  $H_n$  are Hermite polynomials in terms of the standard Gaussian variables  $\xi$  with zero mean and unit variance. Here  $\xi$  denotes the vector consisting of  $n$  independent Gaussian variables  $(\xi_{i_1}, \dots, \xi_{i_n})$ . The above equation is the discrete version of the original Wiener polynomial chaos expansion, where the continuous integrals are replaced by summations. The general expression of the polynomials is given by

$$H_n(\xi_{i_1}, \dots, \xi_{i_n}) = e^{\frac{1}{2}\xi^T \xi} (-1)^n \frac{\partial^n}{\partial \xi_{i_1} \dots \partial \xi_{i_n}} e^{-\frac{1}{2}\xi^T \xi}. \tag{3.2}$$

For notational convenience, equation (3.1) can be rewritten as

$$X(\theta) = \sum_{j=0}^{\infty} \hat{a}_j \Psi_j(\boldsymbol{\xi}), \quad (3.3)$$

where there is a one-to-one correspondence between the functions  $H_n(\xi_{i_1}, \dots, \xi_{i_n})$  and  $\Psi_j(\boldsymbol{\xi})$ . The polynomial basis  $\{\Psi_j\}$  of Hermite-Chaos forms a complete orthogonal basis, i.e.,

$$\langle \Psi_i \Psi_j \rangle = \langle \Psi_i^2 \rangle \delta_{ij}, \quad (3.4)$$

where  $\delta_{ij}$  is the Kronecker delta and  $\langle \cdot, \cdot \rangle$  denotes the ensemble average. This is the inner product in the Hilbert space determined by the support of the Gaussian variables

$$\langle f(\boldsymbol{\xi})g(\boldsymbol{\xi}) \rangle = \int f(\boldsymbol{\xi})g(\boldsymbol{\xi})W(\boldsymbol{\xi})d\boldsymbol{\xi} \quad (3.5)$$

with weighting function

$$W(\boldsymbol{\xi}) = \frac{1}{\sqrt{(2\pi)^n}} e^{-\frac{1}{2}\boldsymbol{\xi}^T \boldsymbol{\xi}}. \quad (3.6)$$

What distinguishes the Hermite-Chaos expansion from other possible expansions is that the basis polynomials are Hermite polynomials in terms of Gaussian variables and are orthogonal with respect to the weighting function  $W(\boldsymbol{\xi})$  that has the form of  $n$ -dimensional independent Gaussian probability density function.

**4. The Wiener-Askey Polynomial Chaos.** The Hermite-Chaos expansion has been proved to be effective in solving stochastic differential equations with Gaussian inputs as well as certain types of non-Gaussian inputs [9], [8], [7], [20]; this can be justified by the Cameron-Martin theorem [3]. However, for general non-Gaussian random inputs, the optimal exponential convergence rate will not be realized. In some cases the convergence rate is in fact severely deteriorated.

In order to deal with more general random inputs, we introduce the Wiener-Askey polynomial chaos expansion as a generalization of the original Wiener-Chaos expansion. The expansion basis is the complete polynomial basis from the Askey-scheme (see section 2.3). Similar to section 3, we represent the general second-order random process  $X(\theta)$  as

$$\begin{aligned} X(\theta) &= a_0 I_0 \\ &+ \sum_{i_1=1}^{\infty} c_{i_1} I_1(\zeta_{i_1}(\theta)) \\ &+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} c_{i_1 i_2} I_2(\zeta_{i_1}(\theta), \zeta_{i_2}(\theta)) \\ &+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} c_{i_1 i_2 i_3} I_3(\zeta_{i_1}(\theta), \zeta_{i_2}(\theta), \zeta_{i_3}(\theta)) \\ &+ \dots, \end{aligned} \quad (4.1)$$



where  $I_n(\zeta_{i_1}, \dots, \zeta_{i_n})$  denotes the Wiener-Askey polynomial chaos of order  $n$  in terms of the random vector  $\zeta = (\zeta_{i_1}, \dots, \zeta_{i_n})$ . In the Wiener-Askey chaos expansion, the polynomials  $I_n$  are not restricted to Hermite polynomials but rather can be all types of the orthogonal polynomials from the Askey-scheme in figure 2.1. Again for notational convenience, we rewrite equation (4.1) as

$$X(\theta) = \sum_{j=0}^{\infty} \hat{c}_j \Phi_j(\zeta), \quad (4.2)$$

where there is a one-to-one correspondence between the functions  $I_n(\zeta_{i_1}, \dots, \zeta_{i_n})$  and  $\Phi_j(\zeta)$ . Since each type of polynomials from the Askey-scheme form a complete basis in the Hilbert space determined by their corresponding support, we can expect each type of Wiener-Askey expansion to converge to any  $L_2$  functional in the  $L_2$  sense in the corresponding Hilbert functional space as a generalized result of Cameron-Martin theorem ([3] and [16]). The orthogonality relation of the Wiener-Askey polynomial chaos takes the form

$$\langle \Phi_i \Phi_j \rangle = \langle \Phi_i^2 \rangle \delta_{ij}, \quad (4.3)$$

where  $\delta_{ij}$  is the Kronecker delta and  $\langle \cdot, \cdot \rangle$  denotes the ensemble average which is the inner product in the Hilbert space of the variables  $\zeta$

$$\langle f(\zeta)g(\zeta) \rangle = \int f(\zeta)g(\zeta)W(\zeta)d\zeta, \quad (4.4)$$

or

$$\langle f(\zeta)g(\zeta) \rangle = \sum_{\zeta} f(\zeta)g(\zeta)W(\zeta) \quad (4.5)$$

in the discrete case. Here  $W(\zeta)$  is the weighting function corresponding to the Wiener-Askey polynomials chaos basis  $\{\Phi_i\}$ : see the appendix for detailed formulas.

As pointed out in appendix, some types of orthogonal polynomials from the Askey-scheme have weighting functions the same as the probability function of certain types of random distributions. In practice, we then choose the type of independent variables  $\zeta$  in the polynomials  $\{\Phi_i(\zeta)\}$  according to the type of random distributions as shown in table 4.1. It is clear that the original Wiener polynomial chaos corresponds

	Random variables $\zeta$	Wiener-Askey chaos $\{\Phi(\zeta)\}$	Support
Continuous	Gaussian	Hermite-Chaos	$(-\infty, \infty)$
	Gamma	Laguerre-Chaos	$[0, \infty)$
	Beta	Jacobi-Chaos	$[a, b]$
	Uniform	Legendre-Chaos	$[a, b]$
Discrete	Poisson	Charlier-Chaos	$\{0, 1, 2, \dots\}$
	Binomial	Krawtchouk-Chaos	$\{0, 1, \dots, N\}$
	Negative Binomial	Meixner-Chaos	$\{0, 1, 2, \dots\}$
	Hypergeometric	Hahn-Chaos	$\{0, 1, \dots, N\}$

TABLE 4.1

*The correspondence of the type of Wiener-Askey polynomial chaos and their underlying random variables ( $N \geq 0$  is a finite integer).*

to the Hermite-Chaos and is a subset of the Wiener-Askey polynomial chaos. The

Hermite-, Laguerre- and Jacobi-Chaos are continuous chaos, while Charlier-, Meixner-, Krawtchouk- and Hahn-Chaos are discrete chaos. It is worthy mentioning that the Legendre polynomials, which is a special case of the Jacobi polynomials with parameters  $\alpha = \beta = 0$  (section A.1.3), corresponds to an important distribution — the *uniform distribution*. Due to the importance of the uniform distribution, we list it separately in the table and term the corresponding chaos expansion as the Legendre-Chaos.

**5. Applications of Wiener-Askey Polynomial Chaos.** In this section we apply the Wiener-Askey polynomial chaos to solution of stochastic differential equations. We first introduce the general procedure of applying the Wiener-Askey polynomial chaos, then we solve a specific stochastic ordinary differential equation with different types of random inputs. We demonstrate the convergence rates of Wiener-Askey expansion by comparing the numerical results with the corresponding exact solution.

**5.1. General Procedure.** Let us consider the stochastic differential equation

$$\mathcal{L}(\mathbf{x}, t, \theta; u) = f(\mathbf{x}, t; \theta), \quad (5.1)$$

where  $u := u(\mathbf{x}, t; \theta)$  is the solution and  $f(\mathbf{x}, t; \theta)$  is the source term. Operator  $\mathcal{L}$  generally involves differentiations in space/time and can be nonlinear. Appropriate initial and boundary conditions are assumed. The existence of random parameter  $\theta$  is due to the introduction of uncertainty into the system via boundary conditions, initial conditions, material properties, etc. The solution  $u$ , which is regarded as a random process, can be expanded by the Wiener-Askey polynomial chaos as

$$u(\mathbf{x}, t; \theta) = \sum_{i=0}^P u_i(\mathbf{x}, t) \Phi_i(\boldsymbol{\zeta}(\theta)). \quad (5.2)$$

Note here the infinite summation has been truncated at the finite term  $P$ . The above representation can be considered as a spectral expansion in the random dimension  $\theta$ , and the random trial basis  $\{\Phi_i\}$  is the Askey-scheme based orthogonal polynomials discussed in section 4. The total number of expansion terms is  $(P + 1)$ , and is determined by the dimension ( $n$ ) of random variable  $\boldsymbol{\zeta}$  and the highest order ( $p$ ) of the polynomials  $\{\Phi_i\}$ ;

$$(P + 1) = \frac{(n + p)!}{n!p!} \quad (5.3)$$

Upon substituting equation (5.2) into the governing equation (5.1), we obtain

$$\mathcal{L} \left( \mathbf{x}, t, \theta; \sum_{i=0}^P u_i \Phi_i \right) = f(\mathbf{x}, t; \theta). \quad (5.4)$$

A Galerkin projection of the above equation onto each polynomial basis  $\{\Phi_i\}$  is then conducted in order to ensure the error is orthogonal to the functional space spanned by the finite-dimensional basis  $\{\Phi_i\}$ ,

$$\left\langle \mathcal{L} \left( \mathbf{x}, t, \theta; \sum_{i=0}^P u_i \Phi_i \right), \Phi_k \right\rangle = \langle f, \Phi_k \rangle, \quad k = 0, 1, \dots, P. \quad (5.5)$$

By using the orthogonality of the polynomial basis, we can obtain a set of  $(P + 1)$  coupled equations for each random mode  $u_i(\mathbf{x}, t)$  where  $i = \{0, 1, \dots, P\}$ . It should be noted that by utilizing the Wiener-Askey polynomial chaos expansion (5.2), the randomness is effectively transferred into the basis polynomials. Thus, the governing equations for the expansion coefficients  $u_i$  resulted from equation (5.5) are *deterministic*. Discretizations in space  $\mathbf{x}$  and time  $t$  can be carried out by any conventional deterministic techniques, e.g., Runge-Kutta solvers in time and the spectral/hp element method in space for high accurate solution in complex geometry [13].

**5.2. Stochastic Ordinary Differential Equation.** We consider the ordinary differential equation

$$\frac{dy(t)}{dt} = -ky, \quad y(0) = \hat{y}, \quad (5.6)$$

where the decay rate coefficient  $k$  is considered to be a random variable  $k(\theta)$  with certain distribution and mean value  $\bar{k}$ . The probability function is  $f(k)$  for the continuous case or  $f(k_i)$  for the discrete case. The *deterministic* solution is

$$y(t) = y_0 e^{-\bar{k}t} \quad (5.7)$$

and the *mean* of *stochastic* solution is

$$\bar{y}(t) = \hat{y} \int_S e^{-kt} f(k) dk \quad \text{or} \quad \bar{y}(t) = \hat{y} \sum_i e^{-k_i t} f(k_i) \quad (5.8)$$

corresponding to the continuous and discrete distributions, respectively. The integration and summation are taken within the support defined by the corresponding distribution.

By applying the Wiener-Askey polynomial chaos expansion (4.2) to the solution  $y$  and random input  $k$

$$y(t) = \sum_{i=0}^P y_i(t) \Phi_i, \quad k = \sum_{i=0}^P k_i \Phi_i \quad (5.9)$$

and substituting the expansions into the governing equation, we obtain

$$\sum_{i=0}^P \frac{dy_i(t)}{dt} \Phi_i = - \sum_{i=0}^P \sum_{j=0}^P \Phi_i \Phi_j k_i y_j(t). \quad (5.10)$$

We then project the above equation onto the random space spanned by the orthogonal polynomial basis  $\{\Phi_i\}$  by taking the inner product of the equation with each basis. By taking  $\langle \cdot, \Phi_l \rangle$  and utilizing the orthogonality condition (4.3), we obtain the following set of equations:

$$\frac{dy_l(t)}{dt} = - \frac{1}{\langle \Phi_l^2 \rangle} \sum_{i=0}^P \sum_{j=0}^P e_{ijl} k_i y_j(t), \quad l = 0, 1, \dots, P, \quad (5.11)$$

where  $e_{ijl} = \langle \Phi_i \Phi_j \Phi_l \rangle$ . Note that the coefficients are smooth and thus any standard ODE solver can be employed here. In the following the standard second-order Runge-Kutta scheme is used.

**5.3. Numerical Results.** In this section we present numerical results of the stochastic ordinary differential equation by the Wiener-Askey polynomial chaos expansion. For the purpose of benchmarking, we will arbitrarily assume the type of distributions of the decay parameter  $k$  and employ the corresponding Wiener-Askey chaos expansion, although in practice there is certainly more favorable assumptions about  $k$  depending on the specific physical background. We define the two error measures for the mean and variance of the solution

$$\varepsilon_{\text{mean}}(t) = \left| \frac{\bar{y}(t) - \bar{y}_{\text{exact}}(t)}{\bar{y}_{\text{exact}}(t)} \right|, \quad \varepsilon_{\text{var}}(t) = \left| \frac{\sigma(t) - \sigma_{\text{exact}}(t)}{\sigma_{\text{exact}}(t)} \right|, \quad (5.12)$$

where  $\bar{y}(t) = E[y(t)]$  is the mean value of  $y(t)$  and  $\sigma(t) = E[(y(t) - \bar{y}(t))^2]$  is the variance of the solution. The initial condition is fixed to be  $\hat{y} = 1$  and the integration is performed up to  $t = 1$  (nondimensional time units).

**5.3.1. Gaussian Distribution and Hermite-Chaos.** In this section the distribution of  $k$  is assumed to be a Gaussian random variable with probability density function

$$f(k) = \frac{1}{\sqrt{2\pi}} e^{-k^2/2} \quad (5.13)$$

which has zero mean value ( $\bar{k} = 0$ ) and unit variance ( $\sigma_k^2 = 1$ ). The exact stochastic mean solution is

$$\bar{y}(t) = \hat{y} e^{t^2/2}. \quad (5.14)$$

The Hermite-Chaos from the Wiener-Askey polynomial chaos family is employed as a natural choice due to the fact that the random input is Gaussian. Figure 5.1 shows

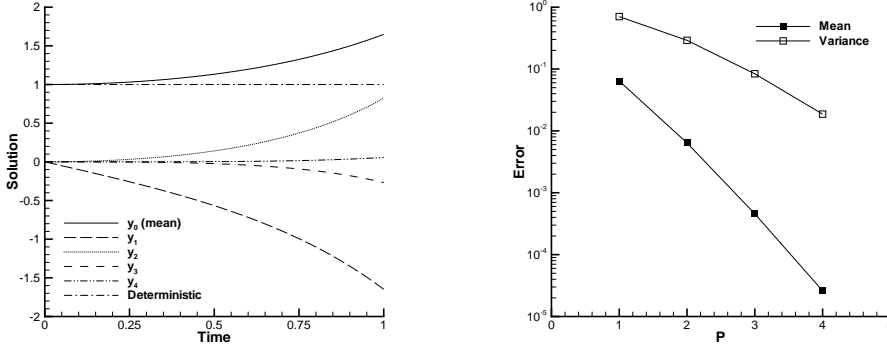


FIG. 5.1. Solution with Gaussian random input by 4th-order Hermite-Chaos; Left: Solution of each random mode, Right: Error convergence of the mean and the variance.

the solution by the Hermite-Chaos expansion. The convergence of errors of the mean and variance as the number of expansion terms increases is shown on semi-log plot, and it is seen that the *exponential* convergence rate is achieved. It is also noticed that the *deterministic* solution remains constant as the mean value of  $k$  is zero; however the mean of the stochastic solution (random mode with index 0,  $y_0$ ) is nonzero and grows with time.

**5.3.2. Gamma Distribution and Laguerre-Chaos.** In this section we assume the distribution of the decay parameter  $k$  is the Gamma distribution with PDF of the form

$$f(k) = \frac{e^{-k} k^\alpha}{\Gamma(\alpha + 1)}, \quad 0 \leq k < \infty, \alpha > -1. \quad (5.15)$$

The mean and variance of  $k$  are:  $\mu_k = \bar{k} = \alpha + 1$  and  $\sigma_k^2 = \alpha + 1$ , respectively. The mean of stochastic solution is

$$\bar{y}(t) = \hat{y} \frac{1}{(1+t)^{\alpha+1}}. \quad (5.16)$$

The special case of  $\alpha = 0$  corresponds to another important distribution: the *exponential* distribution. Because the random input has a Gamma distribution, we employ the

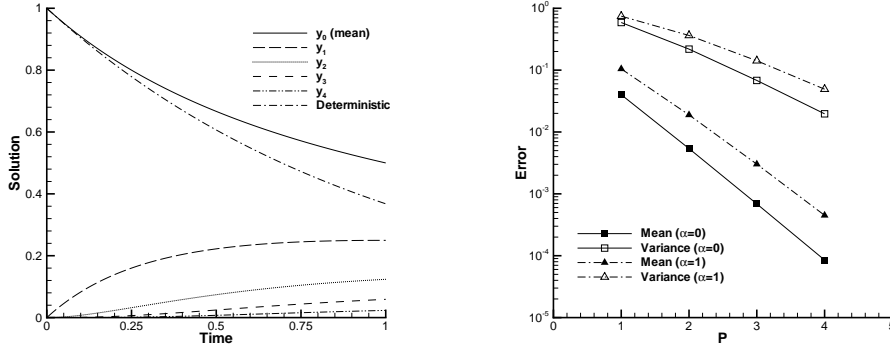


FIG. 5.2. Solution with Gamma random input by 4th-order Laguerre-Chaos; Left: Solution of each mode ( $\alpha = 0$ : exponential distribution), Right: Error convergence of the mean and the variance with different  $\alpha$ .

Laguerre-Chaos as the specific Wiener-Askey chaos (see table 4.1). Figure 5.2 shows the evolution of each solution mode over time, together with the convergence of the errors of the mean and the variance with different values of parameter  $\alpha$ . The special case of exponential distribution is included ( $\alpha = 0$ ). Again the mean of stochastic solution and deterministic solution show significant difference. As  $\alpha$  becomes larger, the spread of the Gamma distribution is larger and this leads to larger errors with fixed number of Laguerre-Chaos expansion. However, the exponential convergence rate is still realized.

**5.3.3. Beta Distribution and Jacobi-Chaos.** We now assume the distribution of the random variable  $k$  to be the Beta distribution with probability density function of the form

$$f(k; \alpha, \beta) = \frac{(1-k)^\alpha (1+k)^\beta}{2^{\alpha+\beta+1} B(\alpha+1, \beta+1)}, \quad -1 < k < 1, \alpha, \beta > -1, \quad (5.17)$$

where  $B(\alpha, \beta)$  is the Beta function defined as  $B(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$ . We then employ the Jacobi-Chaos expansion which has the weighting function in the form of the Beta distribution. An important special case is  $\alpha = \beta = 0$  when the distribution

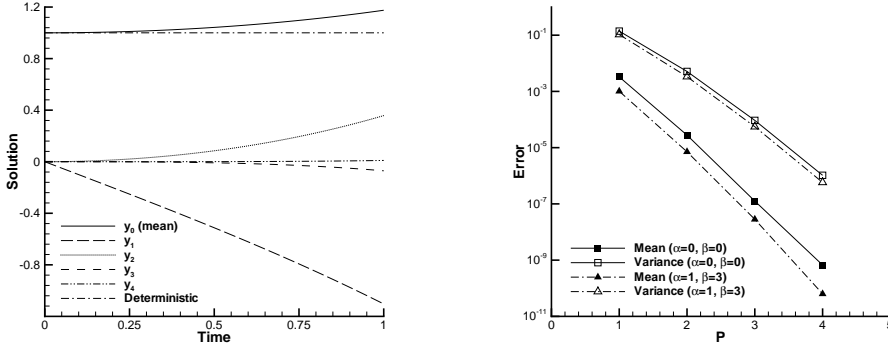


FIG. 5.3. Solution with Beta random input by 4th-order Jacobi-Chaos; Left: Solution of each mode ( $\alpha = \beta = 0$ : Legendre-Chaos), Right: Error convergence of the mean and the variance with different  $\alpha$  and  $\beta$ .

becomes the uniform distribution and the corresponding Jacobi-Chaos becomes the Legendre-Chaos.

Figure 5.3 shows the solution by the Jacobi-Chaos. On the left is the evolution of all random modes of the Legendre-Chaos ( $\alpha = \beta = 0$ ) with uniformly distributed random input. In this case,  $k$  has zero mean value and the deterministic solution remains constant, but the mean of stochastic solution grows over time. The convergence of errors of the mean and the variance of the solution with respect to the order of Jacobi-Chaos expansion is shown on the semi-log scale, and the exponential convergence rate is obtained with different sets of parameter values  $\alpha$  and  $\beta$ .

**5.3.4. Poisson Distribution and Charlier-Chaos.** We now assume the distribution of the decay parameter  $k$  to be Poisson of the form

$$f(k; \lambda) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \dots, \lambda > 0. \quad (5.18)$$

The mean and variance of  $k$  are:  $\mu_k = \bar{k} = \lambda$  and  $\sigma_k^2 = \lambda$ , respectively. The analytic solution of the mean stochastic solution is

$$\bar{y}(t) = \hat{y} e^{-\lambda + \lambda e^{-t}}. \quad (5.19)$$

The Charlier-Chaos expansion is employed to represent the solution process and the results with fourth-order expansion are shown in figure 5.4. Once again we see the noticeable difference between the deterministic solution and the mean of stochastic solution. Exponential convergence rate is obtained for different values of parameter  $\lambda$ .

**5.3.5. Binomial Distribution and Krawtchouk-Chaos.** In this section the distribution of the random input  $k$  is assumed to be binomial

$$f(k; p, N) = \binom{N}{k} p^k (1-p)^{N-k}, \quad 0 \leq p \leq 1, k = 0, 1, \dots, N. \quad (5.20)$$

The exact mean solution of (5.6) is

$$\bar{y}(t) = \hat{y} [1 - (1 - e^{-t}) p]^N. \quad (5.21)$$

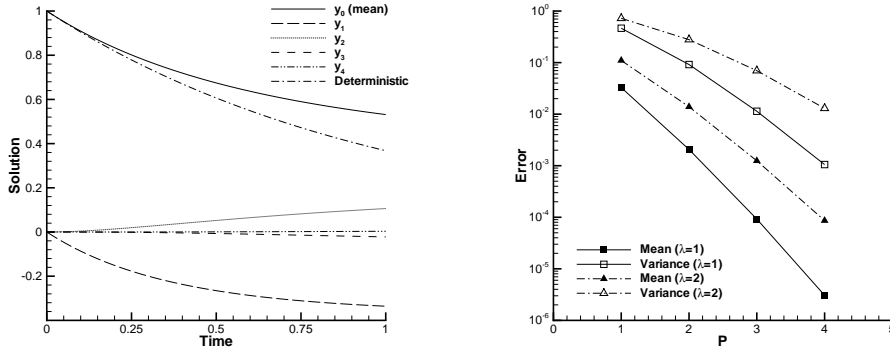


FIG. 5.4. Solution with Poisson random input by 4th-order Charlier-Chaos; Left: Solution of each mode ( $\lambda = 1$ ), Right: Error convergence of the mean and the variance with different  $\lambda$ .

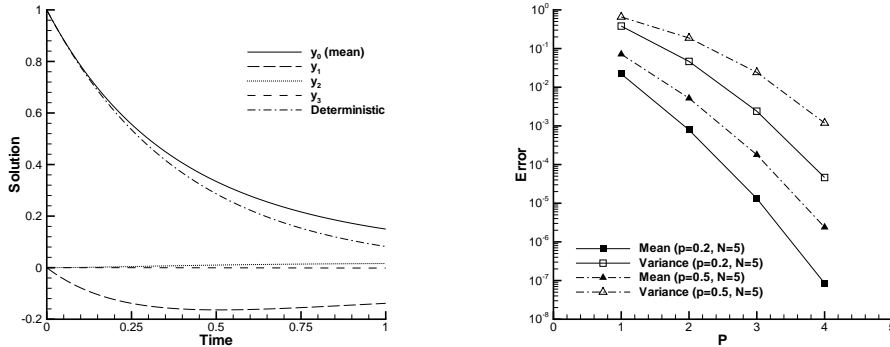


FIG. 5.5. Solution with binomial random input by 4th-order Krawtchouk-Chaos; Left: Solution of each mode ( $p = 0.5, N = 5$ ), Right: Error convergence of the mean and the variance with different  $p$  and  $N$ .

Figure 5.5 shows the solution with 4th-order Krawtchouk-Chaos. With different parameter sets, Krawtchouk-Chaos expansion correctly approximates the exact solution, and the convergence rate with respect to the order of expansion is exponential.

**5.3.6. Negative Binomial Distribution and Meixner-Chaos.** In this section we assume the distribution of the random input of  $k$  is the negative binomial distribution

$$f(k; \beta, c) = \frac{(\beta)_k}{k!} (1-c)^\beta c^k, \quad 0 \leq c \leq 1, \beta > 0, k = 0, 1, \dots \quad (5.22)$$

In case of  $\beta$  being integer, it is often called the Pascal distribution. The exact mean solution of (5.6) is

$$\bar{y}(t) = \hat{y} \left( \frac{1 - ce^{-t}}{1 - c} \right)^{-\beta}. \quad (5.23)$$

The Meixner-Chaos is chosen since the random input is negative binomial (see table 4.1). Figure 5.6 shows the solution with 4th-order Meixner-Chaos. Exponential

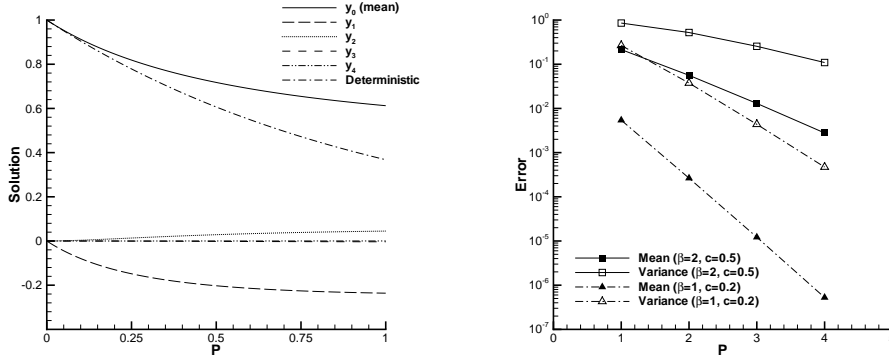


FIG. 5.6. Solution with negative binomial random input by 4th-order Meixner-Chaos; Left: Solution of each mode ( $\beta = 1, c = 0.5$ ), Right: Error convergence of the mean and the variance with different  $\beta$  and  $c$ .

convergence rate is observed by the Meixner-Chaos approximation with different sets of parameter values.

**5.3.7. Hypergeometric Distribution and Hahn-Chaos.** We now assume the distribution of the random input  $k$  is hypergeometric

$$f(k; \alpha, \beta, N) = \frac{\binom{\alpha}{k} \binom{\beta}{N-k}}{\binom{\alpha+\beta}{N}}, \quad k = 0, 1, \dots, N, \quad \alpha, \beta > N. \quad (5.24)$$

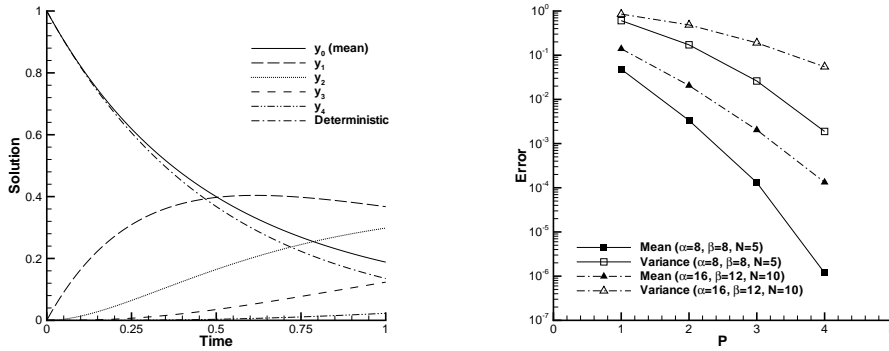


FIG. 5.7. Solution with hypergeometric random input by 4th-order Hahn-Chaos; Left: Solution of each mode ( $\alpha = \beta = 5, N = 4$ ), Right: Error convergence of the mean and the variance with different  $\alpha, \beta$  and  $N$ .

In this case, the optimal Wiener-Askey polynomial chaos is the Hahn-Chaos (table 4.1). Figure 5.7 shows the solution by 4th-order Hahn-Chaos. It can be seen from the semi-log plot of the errors of the mean and variance of the solution that exponential convergence rate is obtained with respect to the order of Hahn-Chaos expansion for different sets of parameter values.



**5.4. Efficiency of Wiener-Askey Chaos Expansion.** We have demonstrated the exponential convergence of the Wiener-Askey polynomial chaos expansion. From the results above, we notice that it normally takes an expansion order  $P = 2 \sim 4$  for the error of the mean solution to reach the order of  $O(10^{-3})$ . Equation (5.11) shows that the Wiener-Askey chaos expansion with highest order of  $P$  results in a set of  $(P + 1)$  coupled ODEs. Thus, the computational cost is slightly more than  $(P + 1)$  times of that of a single realization of the deterministic integration. On the other hand, if the Monte-Carlo simulation is used, it normally requires  $O(10^4) \sim O(10^5)$  number of realizations to reduce the error of the *mean* solution to  $O(10^{-3})$ . For example, if  $k$  is an exponentially distributed random variable, the error convergence of the mean solution of the Monte-Carlo simulation is shown in table 5.1.

$N$	$1 \times 10^2$	$1 \times 10^3$	$1 \times 10^4$	$1 \times 10^5$
$\varepsilon_{\text{mean}}$	$4.0 \times 10^{-2}$	$1.1 \times 10^{-2}$	$5.1 \times 10^{-3}$	$6.5 \times 10^{-4}$

TABLE 5.1

*Error convergence of the mean solution by Monte-Carlo simulation:  $N$  is the number of realizations and  $\varepsilon_{\text{mean}}$  is the error of mean solution defined in (5.12); Random input has exponential distribution.*

Monte-Carlo simulations with other types of random inputs as discussed in this paper have also been conducted and the results are similar. The actual numerical values of the errors with given number of realizations may vary depending on the property of random number generators used, but the order of magnitude should be the same. Techniques such as variance reduction are not used. Although such techniques, if applicable, can greatly speed up Monte-Carlo simulation by an order or more depending on the specific problem, the advantage of Wiener-Askey polynomial chaos expansion is obvious. For the ordinary differential equation discussed in this paper, speed-up of order  $O(10^3) \sim O(10^4)$  compared with straight Monte-Carlo simulations can be expected. However, for more complicated problems where there exist multi-dimensional random inputs, the multi-dimensional Wiener-Askey chaos is needed. The total number of expansion terms increases fast for large dimensional problems (see equation(5.3)). Thus the efficiency of the chaos expansion can be reduced.

**6. Representation of Arbitrary Random Inputs.** As demonstrated above, with appropriately chosen Wiener-Askey polynomial chaos expansion according to the type of the random input, optimal exponential convergence rate of the chaos expansion can be realized. In practice, we often encounter distributions of random inputs not belonging to the basic types of distributions listed in table 4.1, or even when they do belong to certain basic types, the correspondence is not explicitly known. In this case, we need to project the input process onto the Wiener-Askey polynomial chaos basis directly in order to solve the differential equation.

Let us assume in the stochastic ODE of equation (5.6) that the distribution of the decay parameter  $k$  is known in the form of probability function  $f(k)$ . The representation of  $k$  by the Wiener-Askey polynomial chaos expansion takes the form

$$k = \sum_{i=0}^P k_i \Phi_i, \quad k_i = \frac{\langle k \Phi_i \rangle}{\langle \Phi_i^2 \rangle}, \quad (6.1)$$

where the operation  $\langle \cdot, \cdot \rangle$  denotes the inner product in the Hilbert space spanned

by the Wiener-Askey chaos basis  $\{\Phi_i\}$ , i.e.,

$$k_i = \frac{1}{\langle \Phi_i^2 \rangle} \int k \Phi_i(\zeta) g(\zeta) d\zeta, \quad \text{or} \quad k_i = \frac{1}{\langle \Phi_i^2 \rangle} \sum_j k \Phi_i(\zeta_j) g(\zeta_j), \quad (6.2)$$

where  $g(x)$  and  $g(x_i)$  is the probability function of the random variable  $\zeta$  in the Wiener-Askey polynomial chaos for continuous and discrete cases, respectively. The underlying assumption here is that the random variable  $\zeta$  is fully dependent on the target random variable  $k$ . We notice that the above equations are mathematically *meaningless* due to the fact that the support of  $k$  and  $\zeta$  are likely to be different. In other words, the random variables  $k$  and  $\zeta$  could belong to two different probability spaces  $(\Omega, \mathcal{A}, P)$  with different event space  $\Omega$ ,  $\sigma$ -algebra  $\mathcal{A}$  and probability measure  $P$ .

**6.1. Analytical Approach.** In order to conduct the above projection, we need to transform the fully correlated random variables  $k$  and  $\zeta$  to the same probability space. Under the theory of probability this is always possible. In practice, it is convenient to transform them to the uniformly distributed probability space  $u \in U(0, 1)$ . In fact, the inverse procedure is an important technique for random number generation where one first generates the uniformly distributed numbers as the seeds and then performs the inverse transformation according to the desired distribution function. Without loss of generality, we discuss in detail the case when  $k$  and  $\zeta$  are continuous random variables.

Let us assume that the random variable  $u$  is uniformly distributed in  $(0, 1)$  and the probability density functions for  $k$  and  $\zeta$  are  $f(k)$  and  $g(\zeta)$ , respectively. A transformation of variables in probability space shows that

$$du = f(k)dk = dF(k), \quad du = g(\zeta)d\zeta = dG(\zeta), \quad (6.3)$$

where  $F$  and  $G$  are the distribution function of  $k$  and  $\zeta$ , respectively,

$$F(k) = \int_{-\infty}^k f(t)dt, \quad G(\zeta) = \int_{-\infty}^{\zeta} g(t)dt. \quad (6.4)$$

If we require the random variables  $k$  and  $\zeta$  to be transformed to the *same* uniformly distributed random variable  $u$ , we obtain

$$u = F(k) = G(\zeta). \quad (6.5)$$

After inverting the above equations, we obtain

$$k = F^{-1}(u) \equiv h(u), \quad \zeta = G^{-1}(u) \equiv l(u). \quad (6.6)$$

Now that we have effectively transformed the two different random variables  $k$  and  $\zeta$  to the same probability space defined by  $u \in U(0, 1)$ , the projection (6.2) can be performed, i.e.,

$$\begin{aligned} k_i &= \frac{1}{\langle \Phi_i^2 \rangle} \int k \Phi_i(\zeta) g(\zeta) d\zeta \\ &= \frac{1}{\langle \Phi_i^2 \rangle} \int_0^1 h(u) \Phi_i(l(u)) du. \end{aligned} \quad (6.7)$$

In general the above integral can not be integrated analytically. However, it can be efficiently evaluated with the Gauss quadrature in the closed domain  $[0, 1]$  with sufficient accuracy. The analytical forms of the inversion relations (6.6) are known for some basic distributions: Gaussian, exponential, Beta, etc [6].

The above procedure works equally well for the discrete distributions, where the inversion procedure is slightly modified and the integral in (6.7) is replaced by summation.

**6.2. Numerical Approach.** The procedure described above requires the distribution functions  $F(k)$  and  $G(\zeta)$  be known and the inverse functions  $F^{-1}$  and  $G^{-1}$  exist and be known as well. In practice, these conditions are not always satisfied. Often we only know the probability function  $f(k)$  for a specific problem. The probability function  $g(\zeta)$  is known from the choice of Wiener-Askey polynomial chaos but the inversion is not always known either. In this case, we can perform the projection (6.2) directly by the Monte-Carlo integration, where a large ensemble of random numbers  $k$  and  $\zeta$  are generated. The requirement of  $k$  and  $\zeta$  being transformed to the same probability space  $u \in U(0, 1)$  by equation (6.5) implies that each pair of  $k$  and  $\zeta$  has to be generated from the *same* seed of uniformly generated random number  $u \in U(0, 1)$ .

**6.3. Results.** In this section we present numerical examples of representing an arbitrarily given random distribution. More specifically, we present results of using Hermite-Chaos expansion for some non-Gaussian random variables. Although in theory, Hermite-Chaos converges and it has been successfully applied to some non-Gaussian processes ([8], [20]), we demonstrate numerically that optimal exponential convergence rate is not realized.

**6.3.1. Approximation of Gamma distribution by Hermite-Chaos.** Let us assume that the decay parameter  $k$  in the ODE (5.6) is a random variable with Gamma distribution (5.15). We consider the specific case of  $\alpha = 0$ . In this case  $k$  is a random variable with exponential distribution with PDF of the form

$$f(k) = e^{-k}, \quad k > 0. \quad (6.8)$$

The inverse of its distribution function  $F(k)$  (equation (6.6)) is known as

$$h(u) \equiv F^{-1}(u) = -\ln(1 - u), \quad u \in U(0, 1). \quad (6.9)$$

We then use Hermite-Chaos to represent  $k$  instead of the optimal Laguerre-Chaos. The random variable  $\zeta$  in equation (6.7) is a standard Gaussian variable with PDF  $g(\zeta) = \frac{1}{\sqrt{2\pi}}e^{-\zeta^2/2}$ . The inverse of the Gaussian distribution  $G(\zeta)$  is known as

$$l(u) \equiv G^{-1}(u) = \text{sign} \left( u - \frac{1}{2} \right) \left( t - \frac{c_0 + c_1 t + c_2 t^2}{1 + d_1 t + d_2 t^2 + d_3 t^3} \right), \quad (6.10)$$

where

$$t = \sqrt{-\ln [\min(u, 1 - u)]^2}$$

and

$$\begin{aligned} c_0 &= 2.515517, & c_1 &= 0.802853, & c_2 &= 0.010328, \\ d_1 &= 1.432788, & d_2 &= 0.189269, & d_3 &= 0.001308. \end{aligned}$$

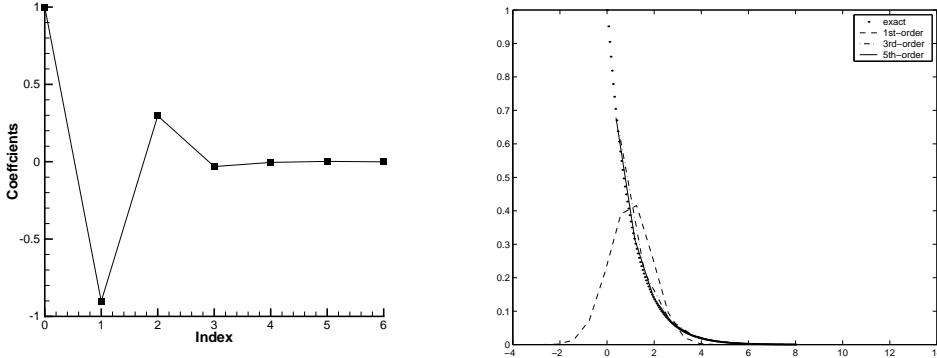


FIG. 6.1. Approximation of exponential distribution with Hermite-Chaos; Left: The expansion coefficients, Right: The PDF of different orders of approximations.

The formula is from Hastings [10] and the numeric values of the constants have absolute error less than  $4.5 \times 10^{-4}$  (also see [6]).

In figure 6.1 we show the result of the approximation of the exponential distribution by the Hermite-Chaos. The expansion coefficients  $k_i$  are shown on the left, and we see the major contributions of the Hermite-Chaos approximation are from the first three terms. The PDF of different orders of the approximations are shown on the right, together with the exact PDF of the exponential distribution. We notice that the third-order approximation gives fairly good result and fifth-order Hermite-Chaos is very close to the exact distribution. The Hermite-Chaos does not approximate the PDF well at  $x \sim 0$  where the PDF reaches its peak at 1. In order to capture this rather sharp region, more Hermite-Chaos terms are needed.

The above result is the representation of the random input  $k$  for the ODE of equation (5.6). If the optimal Wiener-Askey chaos is chosen, in this case the Laguerre-Chaos, only one term is needed to represent  $k$  *exactly*. We can expect if the Hermite-Chaos is used to solve the differential equation in this case, the solution would not retain the exponential convergence as realized by the Laguerre-Chaos.

In figure 6.2 the errors of mean solution defined by equation (5.12) with Laguerre-Chaos and Hermite-Chaos to the ODE of equation (5.6) are shown. The random input of  $k$  has exponential distribution which implies that the Laguerre-Chaos is the optimal Wiener-Askey polynomial chaos. It is seen from the result that the exponential convergence rate is not obtained by the Hermite-Chaos as opposed to the Laguerre-Chaos.

**6.3.2. Approximation of Beta distribution by Hermite-Chaos.** We now assume the distribution of  $k$  is Beta distribution; see equation (5.17). We return to the more conventional definition of Beta distribution in the domain  $[0, 1]$

$$f(k) = \frac{1}{B(\alpha + 1, \beta + 1)} k^\alpha (1 - k)^\beta, \quad \alpha, \beta > -1, 0 \leq k \leq 1. \quad (6.11)$$

Figure 6.3 shows the PDF of first-, third- and fifth-order Hermite-Chaos approximations to the Beta random variable. The special case of  $\alpha = \beta = 0$  is the important uniform distribution. It can be seen that the Hermite-Chaos approximation converges to the exact solution as the number of expansion terms increases. Oscillations are ob-

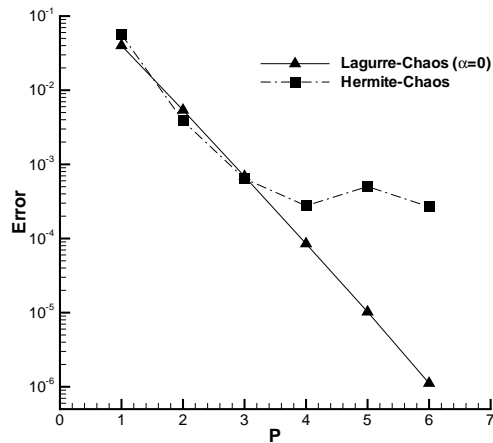


FIG. 6.2. Error convergence of the mean solution of the Laguerre-Chaos and Hermite-Chaos to stochastic ODE with random input of the exponential distribution

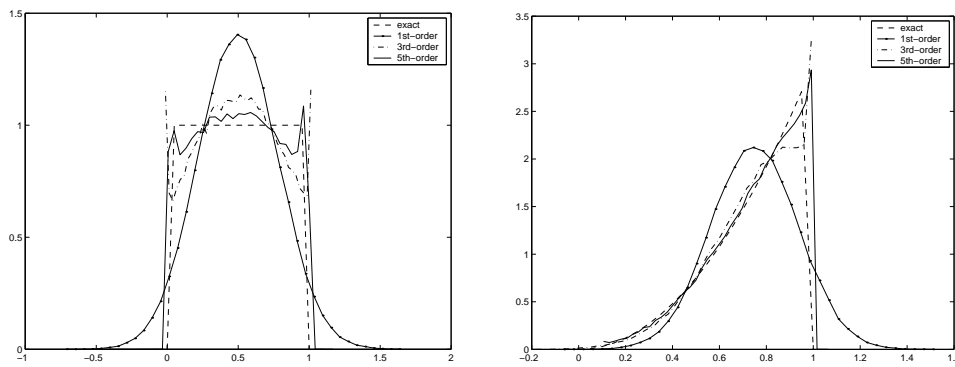


FIG. 6.3. PDF of approximations of Beta distributions by Hermite-Chaos; Left:  $\alpha = \beta = 0$ , the uniform distribution, Right:  $\alpha = 2$ ,  $\beta = 0$ .

served near the corners of the square. This is in analogy with the Gibb's phenomena which occurs when Fourier expansions are used to approximate functions with sharp corners. Since all Wiener-Askey polynomial chaos expansions can be considered as spectral expansions in the random dimension, the oscillations here can be regarded as the *stochastic Gibb's phenomena*. For uniform distribution, Hermite-Chaos does not work very well due to the stochastic Gibb's phenomena even when more higher-order terms are added. On the other hand, the first-order Jacobi-Chaos expansion is already *exact*. In addition to the exponential convergence, the proper Wiener-Askey basis leads to dramatic lowering of dimensionality of the problem.

**7. Conclusion.** We have proposed a Wiener-Askey polynomial chaos expansion to represent stochastic processes and further model the uncertainty in practical applications. The Wiener-Askey polynomial chaos can be regarded as the generalization of the homogeneous chaos first proposed by Wiener in 1938. The original Wiener expansion employs the Hermite polynomials in terms of Gaussian random variables. In

the Wiener-Askey chaos expansion, the basis polynomials are those from the Askey-scheme of hypergeometric orthogonal polynomials, and the underlying variables are random variables chosen according to the weighting function of the polynomials. A general guideline of choosing the optimal Wiener-Askey polynomial chaos according to the random inputs is given. By solving a stochastic ordinary differential equation, we demonstrate numerically that the Wiener-Askey polynomial chaos exhibits exponential convergence rate. For any given type of random input, the Wiener-Askey polynomial chaos converges in general, although the exponential rate is not retained if the optimal chaos is not chosen. The Wiener-Askey polynomial chaos proposed in the present paper can deal with general random inputs more effectively than the original Wiener-Hermite chaos. It can be extended to more complex stochastic systems governed by partial differential equations without any fundamental difficulties.

### Appendix A. Some Important Orthogonal Polynomials in Askey-scheme.

In this section we briefly review the definitions and properties of some important orthogonal polynomials from Askey scheme, which are discussed in this paper for the Wiener-Askey polynomial chaos.

#### A.1. Continuous Polynomials.

##### A.1.1. Hermite Polynomial $H_n(x)$ and Gaussian Distribution.

**Definition:**

$$H_n(x) = (2x)^n {}_2F_0\left(-\frac{n}{2}, -\frac{n-1}{2}; ; -\frac{1}{x^2}\right). \quad (\text{A.1})$$

**Orthogonality:**

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = 2^n n! \delta_{mn}. \quad (\text{A.2})$$

**Recurrence relation:**

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0. \quad (\text{A.3})$$

**Rodriguez formula:**

$$e^{-x^2} H_n(x) = (-1)^n \frac{d^n}{dx^n} \left( e^{-x^2} \right). \quad (\text{A.4})$$

The weighting function is  $w(x) = e^{-x^2}$  from the orthogonality condition (A.2). After rescaling  $x$  by  $\sqrt{2}$ , the weighting function is the same as the probability density function of a standard *Gaussian* random variable with zero mean and unit variance.

##### A.1.2. Laguerre Polynomial $L_n^{(\alpha)}(x)$ and Gamma Distribution.

**Definition:**

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1(-n; \alpha+1; x). \quad (\text{A.5})$$

**Orthogonality:**

$$\int_0^{\infty} e^{-x} x^\alpha L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) dx = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{mn}, \quad \alpha > -1. \quad (\text{A.6})$$

**Recurrence relation:**

$$(n+1)L_{n+1}^{(\alpha)}(x) - (2n+\alpha+1-x)L_n^{(\alpha)}(x) + (n+\alpha)L_{n-1}^{(\alpha)}(x) = 0. \quad (\text{A.7})$$

**Rodriguez formula:**

$$e^{-x}x^\alpha L_n^{(\alpha)}(x) = \frac{1}{n!} \frac{d^n}{dx^n} (e^{-x}x^{n+\alpha}). \quad (\text{A.8})$$

Recall that the *Gamma* distribution has the probability density function

$$f(x) = \frac{x^\alpha e^{-x/\beta}}{\beta^{\alpha+1}\Gamma(\alpha+1)}, \quad \alpha > -1, \beta > 0. \quad (\text{A.9})$$

Despite of the scale parameter  $\beta$  and a constant factor  $\Gamma(\alpha+1)$ , it is the same as the weighting function of Laguerre polynomial.

### A.1.3. Jacobi Polynomial $P_n^{(\alpha,\beta)}(x)$ and Beta Distribution.

**Definition:**

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1 \left( -n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2} \right). \quad (\text{A.10})$$

**Orthogonality:**

$$\int_{-1}^1 (1-x)^\alpha (1+x)^\beta P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) dx = h_n^2 \delta_{mn}, \quad \alpha > -1, \beta > -1, \quad (\text{A.11})$$

where

$$h_n^2 = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}.$$

**Recurrence relation:**

$$\begin{aligned} xP_n^{(\alpha,\beta)}(x) &= \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} P_{n+1}^{(\alpha,\beta)}(x) \\ &+ \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} P_n^{(\alpha,\beta)}(x) \\ &+ \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} P_{n-1}^{(\alpha,\beta)}(x). \end{aligned} \quad (\text{A.12})$$

**Rodriguez formula:**

$$(1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [(1-x)^{n+\alpha} (1+x)^{n+\beta}]. \quad (\text{A.13})$$

The *Beta* distribution has the probability density function

$$f(x) = \frac{(x-a)^\beta (b-x)^\alpha}{(b-a)^{\alpha+\beta+1} B(\alpha+1, \beta+1)}, \quad a \leq x \leq b, \quad (\text{A.14})$$

where  $B(p, q)$  is the *Beta function* defined as

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \quad (\text{A.15})$$

It is clear that despite of a constant factor the weighting function of Jacobi polynomial  $w(x) = (1-x)^\alpha (1+x)^\beta$  from (A.11) is the same as the probability density function of Beta distribution defined in domain  $[-1, 1]$ . When  $\alpha = \beta = 0$ , the Jacobi polynomials become the Legendre polynomials and the weighting function is a constant which corresponds to the important *uniform distribution*.

## A.2. Discrete Polynomials.

### A.2.1. Charlier Polynomial $C_n(x; a)$ and Poisson Distribution.

**Definition:**

$$C_n(x; a) = {}_2F_0 \left( -n, -x; ; -\frac{1}{a} \right). \quad (\text{A.16})$$

**Orthogonality:**

$$\sum_{x=0}^{\infty} \frac{a^x}{x!} C_m(x; a) C_n(x; a) = a^{-n} e^a n! \delta_{mn}, \quad a > 0. \quad (\text{A.17})$$

**Recurrence relation:**

$$-x C_n(x; a) = a C_{n+1}(x; a) - (n+a) C_n(x; a) + n C_{n-1}(x; a). \quad (\text{A.18})$$

**Rodriguez formula:**

$$\frac{a^x}{x!} C_n(x; a) = \nabla^n \left( \frac{a^x}{x!} \right), \quad (\text{A.19})$$

where  $\nabla$  is the backward difference operator (2.12).

The probability function of *Poisson* distribution is

$$f(x; a) = e^{-a} \frac{a^x}{x!}, \quad k = 0, 1, 2, \dots \quad (\text{A.20})$$

Despite of a constant factor  $e^{-a}$ , it is the same as the weighting function of Charlier polynomials.

### A.2.2. Krawtchouk Polynomial $K_n(x; p, N)$ and Binomial Distribution.

**Definition:**

$$K_n(x; p, N) = {}_2F_1 \left( -n, -x; -N; \frac{1}{p} \right), \quad n = 0, 1, \dots, N. \quad (\text{A.21})$$

**Orthogonality:**

$$\sum_{x=0}^N \binom{N}{x} p^x (1-p)^{N-x} K_m(x; p, N) K_n(x; p, N) = \frac{(-1)^n n!}{(-N)_n} \left( \frac{1-p}{p} \right)^n \delta_{mn}, \quad 0 < p < 1. \quad (\text{A.22})$$

**Recurrence relation:**

$$\begin{aligned} -x K(x; p, N) &= p(N-n) K_{n+1}(x; p, N) - [p(N-n) + n(1-p)] K_n(x; p, N) \\ &\quad + n(1-p) K_{n-1}(x; p, N). \end{aligned} \quad (\text{A.23})$$

**Rodriguez formula:**

$$\binom{N}{x} \left( \frac{p}{1-p} \right)^x K_n(x; p, N) = \nabla^n \left[ \binom{N-n}{x} \left( \frac{p}{1-p} \right)^x \right]. \quad (\text{A.24})$$

Clearly, the weighting function from (A.22) is the probability function of the *binomial* distribution.



**A.2.3. Meixner Polynomial  $M_n(x; \beta, c)$  and Negative Binomial Distribution.**

**Definition:**

$$M_n(x; \beta, c) = {}_2F_1 \left( -n, -x; \beta; 1 - \frac{1}{c} \right). \quad (\text{A.25})$$

**Orthogonality:**

$$\sum_{x=0}^{\infty} \frac{(\beta)_x}{x!} c^x M_m(x; \beta, c) M_n(x; \beta, c) = \frac{c^{-n} n!}{(\beta)_n (1-c)^\beta} \delta_{mn}, \quad \beta > 0, \quad 0 < c < 1. \quad (\text{A.26})$$

**Recurrence relation:**

$$(c-1)xM_n(x; \beta, c) = c(n+\beta)M_{n+1}(x; \beta, c) - [n+(n+\beta)c]M_n(x; \beta, c) + nM_{n-1}(x; \beta, c). \quad (\text{A.27})$$

**Rodriguez formula:**

$$\frac{(\beta)_x c^x}{x!} M_n(x; \beta, c) = \nabla^n \left[ \frac{(\beta+n)_x c^x}{x!} \right]. \quad (\text{A.28})$$

The weighting function is

$$f(x) = \frac{(\beta)_x}{x!} (1-c)^\beta c^x, \quad 0 < p < 1, \quad \beta > 0, \quad x = 0, 1, 2, \dots \quad (\text{A.29})$$

It can be verified that it is the probability function of *negative binomial* distribution. In the case of  $\beta$  being integer, it is often called the *Pascal* distribution.

**A.2.4. Hahn Polynomial  $Q_n(x; \alpha, \beta, N)$  and Hypergeometric Distribution.**

**Definition:**

$$Q_n(x; \alpha, \beta, N) = {}_3F_2(-n, n+\alpha+\beta+1, -x; \alpha+1, -N; 1), \quad n = 0, 1, \dots, N. \quad (\text{A.30})$$

**Orthogonality:** For  $\alpha > -1$  and  $\beta > -1$  or for  $\alpha < -N$  and  $\beta < -N$ ,

$$\sum_{x=0}^N \binom{\alpha+x}{x} \binom{\beta+N-x}{N-x} Q_m(x; \alpha, \beta, N) Q_n(x; \alpha, \beta, N) = h_n^2 \delta_{mn}, \quad (\text{A.31})$$

where

$$h_n^2 = \frac{(-1)^n (n+\alpha+\beta+1)_{N+1} (\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n (-N)_n N!}.$$

**Recurrence relation:**

$$-xQ_n(x) = A_n Q_{n+1}(x) - (A_n + C_n) Q_n(x) + C_n Q_{n-1}(x), \quad (\text{A.32})$$

where

$$Q_n(x) := Q_n(x; \alpha, \beta, N)$$

and

$$\begin{cases} A_n = \frac{(n+\alpha+\beta+1)(n+\alpha+1)(N-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} \\ C_n = \frac{n(n+\alpha+\beta+N+1)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}. \end{cases}$$

**Rodriguez formula:**

$$w(x; \alpha, \beta, N) Q_n(x; \alpha, \beta, N) = \frac{(-1)^n (\beta+1)_n}{(-N)_n} \nabla^n [w(x; \alpha+n, \beta+n, N-n)], \quad (\text{A.33})$$

where

$$w(x; \alpha, \beta, N) = \binom{\alpha+x}{x} \binom{\beta+N-x}{N-x}.$$

If we set  $\alpha = -\tilde{\alpha} - 1$  and  $\beta = -\tilde{\beta} - 1$ , we obtain

$$\tilde{w}(x) = \frac{1}{\binom{N-\tilde{\alpha}-\tilde{\beta}-1}{N}} \frac{\binom{\tilde{\alpha}}{x} \binom{\tilde{\beta}}{N-x}}{\binom{\tilde{\alpha}+\tilde{\beta}}{N}}.$$

Apart from the constant factor  $1/\binom{N-\tilde{\alpha}-\tilde{\beta}-1}{N}$ , this is the definition of *hypergeometric* distribution.

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