

LAYER BY LAYER RECONSTRUCTION METHODS FOR THE
EARTH RESISTIVITY FROM DIRECT CURRENT MEASUREMENTS

Bernard C. Levy*

Laboratory for Information and Decision Systems
and the Department of Electrical Engineering and
Computer Science, M.I.T., Cambridge, MA. 02139

ABSTRACT

Several methods for reconstructing the resistivity profile of a layered, laterally homogeneous earth from direct current measurements are described. These methods recover the resistivity of the earth layer by layer in a recursive way, and require a very small amount of computational effort. They are obtained by transforming the inverse resistivity problem into an equivalent inverse scattering problem, and by applying efficient signal processing algorithms such as the Schur, fast Cholesky or Levinson recursions to the transformed problem. These algorithms operate on a layer stripping or layer accumulation principle, and are shown to be related to previous reconstruction techniques of Pekeris, Koefoed, Kunetz and Rocroi, and others.

*This work was supported by the National Science Foundation under Grant ECS-83-12921 and by the Air Force Office of Scientific Research under Grant AFOSR-82-0135B.

Report Documentation Page

Form Approved
OMB No. 0704-0188

Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.

1. REPORT DATE MAY 1985	2. REPORT TYPE	3. DATES COVERED 00-05-1985 to 00-05-1985			
4. TITLE AND SUBTITLE Layer by Layer Reconstruction Methods for the Earth Resistivity from Direct Current Measurements		5a. CONTRACT NUMBER			
		5b. GRANT NUMBER			
		5c. PROGRAM ELEMENT NUMBER			
6. AUTHOR(S)		5d. PROJECT NUMBER			
		5e. TASK NUMBER			
		5f. WORK UNIT NUMBER			
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Massachusetts Institute of Technology, Laboratory for Information and Decision Systems, 77 Massachusetts Avenue, Cambridge, MA, 02139-4307		8. PERFORMING ORGANIZATION REPORT NUMBER			
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)		10. SPONSOR/MONITOR'S ACRONYM(S)			
		11. SPONSOR/MONITOR'S REPORT NUMBER(S)			
12. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution unlimited					
13. SUPPLEMENTARY NOTES					
14. ABSTRACT					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES 39	19a. NAME OF RESPONSIBLE PERSON
a. REPORT unclassified	b. ABSTRACT unclassified	c. THIS PAGE unclassified			

1. Introduction

The problem of reconstructing the resistivity of a layered earth model from direct current measurements has been the object of sustained interest over the years. In this problem, some direct current is injected inside the earth by two current electrodes, and two voltage electrodes are used to measure potential variations on the surface of the earth. The goal is to reconstruct the resistivity profile as a function of depth from the potential measurements on the surface of the earth. The existence of a solution for this problem was established by Slichter [1] and Langer [2], whose solution was however impractical from a computational point of view. In 1940, Pekeris [3] obtained some recursions for reconstructing the earth layer by layer. This reconstruction method was subsequently refined and developed more fully by Koefoed [4], [5]. More recently, Coen and Yu [6] used a transformation procedure of Weidelt [7] to formulate the inverse resistivity problem of the earth in such a way that the Gelfand-Levitan method of inverse scattering theory could be applied to this problem. Another layer by layer reconstruction method was also proposed by Kunetz and Rocroi [8], and it will be shown below that their algorithm can be identified with the Levinson recursions of linear prediction [9], which in fact arise in a large number of signal processing situations.

The objective of this paper is to give a unified account of layer by layer reconstruction techniques for the earth resistivity, which includes

previous reconstruction methods, and new ones as well. The common framework that will be used for this presentation is that of inverse scattering, and in this context, we will show that layerwise reconstruction procedures can be viewed as differential inverse scattering methods of the type discussed by Bube and Burridge [10], and Bruckstein, Levy and Kailath [11] (see also [12] - [14]). Since the equation describing the potential of the earth is elliptic, the inverse resistivity problem of the earth is not an inverse scattering problem. However, by writing the equations for the earth's potential as two coupled first-order equations, we will be able to introduce a matrix whose elements can be viewed as obtained by analytic continuation of the elements of the scattering matrix associated to a true scattering system. In this context, we show that the transformation of Weidelt [7] and Coen and Yu [6] has for effect to map solutions of the potential equation into solutions of a wave equation whose scattering matrix is the one mentioned above.

The advantage of formulating the inverse resistivity problem as an inverse scattering problem is that the relation between layerwise reconstruction methods (which are also called layer stripping techniques) for this class of problems, and efficient signal processing algorithms such as the Schur, fast Cholesky and Levinson recursions has been the object of close scrutiny [10], [11], [15]. We will show for example that the recursions obtained by Pekeris [3] and Koefoed [4], [5] for reconstructing the earth resistivity are just a modification of the Schur algorithm [16], [17] which is now widely used in linear estimation theory, or network synthesis [18]. The continuous

parameter version of this algorithm, which takes the form of a Riccati equation, will also be related to the work of Langer [2] and Slichter [1]. In addition, it will be shown how efficient algorithms such as the fast Cholesky or Levinson recursions can be used to recover the earth's resistivity. The solution based on the Levinson recursions turns out to be identical to the method proposed by Kunetz and Rocroi [8].

This paper is organized as follows. In Section 2, the inverse resistivity problem is described and its relation with an equivalent inverse scattering problem is examined. Section 3 describes the solution of the inverse resistivity problem via the Schur algorithm and its relation to the work of Pekeris, Koefoed and Langer. In Section 4, it is shown that after applying a transformation similar to that of Weidelt [7], the fast Cholesky and Levinson recursions can be used to recover the resistivity of the earth. The relation of these methods with those of Coen and Yu, and Kunetz and Rocroi is also discussed. In Section 5 we describe how the given data, which is usually the apparent resistivity for the Schlumberger electrode configuration [5], can be used to compute the functions used to perform the inversion with the Schur algorithm or the fast Cholesky and Levinson recursions, which are respectively Slichter's kernel function [1], [5] and a certain fictitious current source profile obtained by Maxwell's method of images ([8], [5], Chapter 10). Finally, Section 6 contains some conclusions and some suggestions for further research.

2. Problem Formulation

The inverse resistivity problem of the earth is formulated here as in [1] - [6]. It is assumed that the earth conductivity $\sigma(z)$ varies with depth only, and in a first stage we consider the idealized problem where some direct current flows inside the earth through a single electrode, and where the potential $\phi(0,r)$, where r denotes the radial distance to the current electrode, is measured on the surface of the earth. The objective is to reconstruct $\sigma(z)$ from $\phi(0,r)$. In a second stage, we will examine the more realistic situation where the Schlumberger electrode configuration is used to measure the apparent resistivity

$$\rho_a(r) = -\frac{2\pi}{I} r^2 \frac{\partial}{\partial r} \phi(0,r), \quad (2.1)$$

where $\phi(0,r)$ is the potential obtained for a single current electrode, and I is the current supplied by the source. The objective in this case will be to reconstruct $\sigma(z)$ from $\rho_a(r)$.

When a single current electrode is used, by selecting this electrode as the origin of cylindrical coordinates (z,r,θ) , the current equations are

$$\underline{j}(z,r) = \sigma(z) \nabla\phi(z,r) \quad (2.2)$$

$$\underline{\nabla} \cdot \underline{j}(z,r) = 0 \quad (2.3)$$

where $\phi(z,r)$ is the potential, and

$$\underline{j}(z,r) = \begin{bmatrix} j^z(z,r) \\ j^r(z,r) \end{bmatrix} \quad (2.4)$$

is the decomposition of the current density in its vertical and lateral components. The boundary conditions for these equations are

$$j^z(0,r) = -\frac{I}{2\pi} \frac{\delta(r)}{r} \quad (2.5)$$

where I is the total current supplied, and $\phi(z,r) \rightarrow 0$ and $j(z,r) \rightarrow 0$ as $R = (z^2 + r^2)^{1/2} \rightarrow \infty$. By combining (2.2) and (2.3) we obtain the potential equation

$$\Delta\phi(z,r) + \frac{d}{dz} \ln \sigma(z) \frac{\partial}{\partial z} \phi(z,r) = 0 \quad (2.6)$$

where $\Delta = \text{Laplacian} = \frac{\partial^2}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}$, which is the equation usually used to analyze the inverse resistivity problem. However, instead of focusing our attention on equation (2.6), we will use here the first-order equations (2.2) - (2.3) to formulate the inverse resistivity problem.

Let

$$H_n[f(r)] = \int_0^\infty f(r) J_n(\lambda r) r dr \quad (2.7)$$

be the Hankel transform of order n of a function $f(\cdot)$, and denote

$$\hat{\phi}(z,\lambda) = H_0[\phi(z,r)] \quad (2.8a)$$

$$\hat{j}^z(z,\lambda) = H_0[j^z(z,r)] \quad (2.8b)$$

$$\hat{j}^r(z,\lambda) = H_1[j^r(z,r)] \quad (2.8c)$$

By using the property

$$H_0\left[\frac{1}{r} f(r) + \frac{d}{dr} f(r)\right] = \lambda H_1[f(r)] \quad (2.9)$$

$$H_1 \left[\frac{d}{dr} f(r) \right] = -\lambda H_0 [f(r)] \quad (2.10)$$

of Hankel transforms, the equations (2.2) - (2.3) become

$$\hat{j}^x(z, \lambda) = -\lambda \sigma(z) \hat{\phi}(z, \lambda) \quad (2.11a)$$

$$\hat{j}^z(z, \lambda) = \sigma(z) \frac{d}{dz} \hat{\phi}(z, \lambda) \quad (2.11b)$$

$$\lambda \hat{j}^x(z, \lambda) + \frac{d}{dz} \hat{j}^z(z, \lambda) = 0 \quad (2.12)$$

Eliminating \hat{j}^x , and denoting

$$\hat{\psi}(z, \lambda) \triangleq -\frac{1}{\lambda} \hat{j}^z(z, \lambda) = -\frac{\sigma(z)}{\lambda} \frac{d}{dz} \hat{\phi}(z, \lambda) \quad (2.13)$$

this gives

$$\frac{d}{dz} \begin{bmatrix} \hat{\phi}(z, \lambda) \\ \hat{\psi}(z, \lambda) \end{bmatrix} = \begin{bmatrix} 0 & -\lambda \sigma^{-1}(z) \\ -\lambda \sigma(z) & 0 \end{bmatrix} \begin{bmatrix} \hat{\phi}(z, \lambda) \\ \hat{\psi}(z, \lambda) \end{bmatrix} \quad (2.14)$$

which is the analog of the telegrapher's equation

$$\frac{d}{dz} \begin{bmatrix} \hat{v}(z, \lambda) \\ \hat{i}(z, \lambda) \end{bmatrix} = \begin{bmatrix} 0 & -j\lambda Z(z) \\ -j\lambda Z^{-1}(z) & 0 \end{bmatrix} \begin{bmatrix} \hat{v}(z, \lambda) \\ \hat{i}(z, \lambda) \end{bmatrix} \quad (2.15)$$

satisfied by the voltage and current along a nonuniform transmission line, which was the starting point of the inverse scattering problem considered by Bube and Burrige [10], and Bruckstein, Levy and Kailath [11]. Note however that there is an important difference between (2.14) and (2.15): λ is replaced by $j\lambda$. This difference arises from the fact that the equation satisfied by ϕ is elliptic, whereas the equation satisfied by the voltage along a nonuniform transmission line is hyperbolic.

This difference prevents us from formulating the inverse resistivity problem as an inverse scattering problem, since inverse scattering theory applies only to hyperbolic operators. However, by using a mapping technique originally introduced by Weidelt [7] (see also Coen and Yu [6]), we will show below that the inverse resistivity problem can be transformed into an equivalent inverse scattering problem.

The initial conditions for the differential system (2.14) are $\hat{\phi}(0, \lambda)$, which is obtained by taking the Hankel transform of the observed potential $\phi(0, r)$ on the surface of the earth, and

$$\hat{\psi}(0, \lambda) = \frac{I}{2\pi\lambda} \quad . \quad (2.16)$$

Following [11], we introduce the normalized variables

$$M(z, \lambda) = \sigma^{1/2}(z) \hat{\phi}(z, \lambda) \quad (2.17a)$$

$$N(z, \lambda) = \sigma^{-1/2}(z) \hat{\psi}(z, \lambda) \quad (2.17b)$$

so that

$$\frac{M(z, \lambda)}{N(z, \lambda)} = \sigma(z) \frac{\hat{\phi}(z, \lambda)}{\hat{\psi}(z, \lambda)} \quad . \quad (2.18)$$

Then, if the down and upgoing waves are defined as

$$D(z, \lambda) = \frac{1}{2} (M(z, \lambda) + N(z, \lambda)) \quad (2.19a)$$

$$U(z, \lambda) = \frac{1}{2} (M(z, \lambda) - N(z, \lambda)), \quad (2.19b)$$

the system (2.14) can be rewritten as

$$\frac{d}{dz} \begin{bmatrix} D(z, \lambda) \\ U(z, \lambda) \end{bmatrix} = \begin{bmatrix} -\lambda & -k(z) \\ -k(z) & \lambda \end{bmatrix} \begin{bmatrix} D(z, \lambda) \\ U(z, \lambda) \end{bmatrix} \quad (2.20)$$

where $k(z) \triangleq -\frac{1}{2} \frac{d}{dz} \ln \sigma(z)$ is the reflectivity function. By discretizing (2.20), we obtain the elementary filter sections described in Fig. 1. These sections show that the waves $D(z, \lambda)$ and $U(z, \lambda)$ propagate in opposite downward and upward directions, and for a layer of thickness Δ at depth z , $D(z, \lambda)$ and $U(z, \lambda)$ are attenuated by a factor $\exp(-\lambda \Delta)$ and are partially reflected in the proportion $k(z) \Delta$.

We have that $k(z) \equiv 0$ for $z < 0$, and we assume that $k(\cdot)$ is summable and has compact support, so that there exists $L > 0$ such that $k(z) \equiv 0$ for $z > L$. In this case the two-component system (2.20) can be viewed as perturbed form of the free system

$$\frac{d}{dz} \begin{bmatrix} D_0(z, \lambda) \\ U_0(z, \lambda) \end{bmatrix} = \begin{bmatrix} -\lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} D_0(z, \lambda) \\ U_0(z, \lambda) \end{bmatrix} \quad (2.21)$$

where the perturbation $k(\cdot)$ is small, so that for $z < 0$ and $z > L$, the solutions of (2.20) are identical to those of (2.21), i.e.

$$\begin{aligned} D(z, \lambda) &= D_L(\lambda) e^{-\lambda z} \\ U(z, \lambda) &= U_L(\lambda) e^{\lambda z} \end{aligned} \quad (2.22)$$

for $z < 0$, and

$$\begin{aligned} D(z, \lambda) &= D_R(\lambda) e^{-\lambda z} \\ U(z, \lambda) &= U_R(\lambda) e^{\lambda z} \end{aligned} \quad (2.23)$$

for $z > L$. By linearity, we have

$$\begin{bmatrix} D_R(\lambda) \\ U_L(\lambda) \end{bmatrix} = S(\lambda) \begin{bmatrix} D_L(\lambda) \\ U_R(\lambda) \end{bmatrix}, \quad (2.24)$$

where

$$S(\lambda) = \begin{bmatrix} T_L(\lambda) & R_R(\lambda) \\ R_L(\lambda) & T_R(\lambda) \end{bmatrix} \quad (2.25)$$

is the scattering matrix of (2.20). As shown in Fig. 2, this matrix relates the incoming and outgoing waves for the aggregate medium obtained by composing the elementary layers described in Fig. 1.

The motivation for calling $S(\lambda)$ a scattering matrix is that by setting $\lambda = jk$ with k real in (2.20), we obtain a two-component scattering system of the type discussed in [19], [20], [11], [15]. The scattering matrix of such a system is $S(jk)$. It has the property that

$$S^H(jk)S(jk) = I \quad (2.26)$$

for k real, where the superscript H denotes the Hermitian transpose, and it obeys the reciprocity relation

$$T_L(\lambda) = T_R(\lambda) = T(\lambda) \quad (2.27)$$

for all λ . In addition, since $k(\cdot)$ is summable, the two-component wave system (2.20) with $\lambda = jk$ has no bound states ([19], Chapter 1), and $T(\lambda)$ is analytic in the right half-plane. Similarly, $k(z) \equiv 0$ for $z < 0$ implies that $R_L(\lambda)$ is analytic in the right half-plane, and the assumption that $k(\cdot)$ has compact support implies that $R_R(\lambda)$ is meromorphic in the right half-plane [21].

Thus, $R_L(\lambda)$ and $T(\lambda)$ with $\text{Re } \lambda \geq 0$ can be viewed as obtained by analytic continuation of $R_L(jk)$ and $T(jk)$, whereas the assumption that $k(z)$ has compact support is necessary to guarantee the existence of $R_R(\lambda)$ in the right half-plane (note that it may not be defined at some points). However, since our analysis below will focus exclusively on $R_L(\lambda)$, this assumption may easily be removed.

For the problem considered here, the earth is probed from its surface, so that $U_R(\lambda) = 0$ in (2.24) and the reflection coefficient $R_L(\lambda)$ can be expressed as

$$R_L(\lambda) = \frac{U_L(\lambda)}{D_L(\lambda)} = \frac{\sigma(0)\hat{\phi}(0,\lambda) - \hat{\psi}(0,\lambda)}{\sigma(0)\hat{\phi}(0,\lambda) + \hat{\psi}(0,\lambda)} \quad (2.28)$$

where $\hat{\phi}(0,\lambda)$ is the transform of the observed potential, and $\hat{\psi}(0,\lambda)$ is given by (2.16). In the resistivity prospecting literature $R_L(\lambda)$ is known as the modified kernel function ([5], p. 202). It is related to Slichter's kernel function $K(\lambda)$ by

$$R_L(\lambda) = \frac{K(\lambda) - 1}{K(\lambda) + 1} \quad , \quad (2.29)$$

where $K(\lambda)$ is the normalized impedance of the resistive medium extending over $[0, \infty)$, i.e.

$$\begin{aligned} K(\lambda) &= \frac{M(0,\lambda)}{N(0,\lambda)} = \sigma(0) \frac{\hat{\phi}(0,\lambda)}{\hat{\psi}(0,\lambda)} \\ &= - \frac{\lambda \hat{\phi}(0,\lambda)}{\left. \frac{d}{dz} \hat{\phi}(z,\lambda) \right|_{z=0}} \quad . \end{aligned} \quad (2.30)$$

Since $K(\lambda)$ and $R_L(\lambda)$ are entirely specified by the given data, the inverse resistivity problem can be posed as follows: given $K(\lambda)$ or $R_L(\lambda)$ for λ real and positive, we want to reconstruct $k(z)$ and $\sigma(z)$. In theory, this can be done by using the fact that $R_L(\lambda)$ is analytic in the right half-plane to obtain its value on the imaginary axis, and then by using any of the inverse scattering techniques described in [10], [11] to recover $k(z)$ from $R_L(jk)$. However, this basic scheme can be implemented in a variety of ways, which we will now discuss and compare.

3. The Schur Algorithm and Related Methods

The reconstruction techniques which will be examined in this section can all be viewed as variants of an algorithm introduced by Schur [16] in 1917 (see also Akhiezer [17]) to test the boundedness of an analytic function inside the unit circle, and which solves the inverse scattering problem for a discrete medium [18]. This algorithm was subsequently extended to continuous two-component wave systems in [11], [15], and we will now adapt this version of Schur's algorithm to the system (2,20).

We denote by

$$R_L(z, \lambda) = \frac{U(z, \lambda)}{D(z, \lambda)} \quad (3.1)$$

the reflection coefficient associated to the section of the resistive medium extending over $[z, \infty)$. By using the differential equation (2.20) for $D(z, \lambda)$ and $U(z, \lambda)$, we find that $R_L(z, \lambda)$ satisfies the Riccati equation

$$\frac{d}{dz} R_L(z, \lambda) = 2\lambda R_L(z, \lambda) + k(z) (R_L^2(z, \lambda) - 1), \quad (3.2)$$

with the initial condition

$$R_L(0, \lambda) = R_L(\lambda) \text{ (given) } . \quad (3.3)$$

The equation (3.2) depends on $k(z)$, and therefore if we want to use it to reconstruct $k(\cdot)$, we need to express $k(z)$ as a function of $R_L(z, \lambda)$. To do so, note that since $R_L(z, \lambda)$ is the analytic continuation of $R_L(z, jk)$ with k real, where $R_L(z, jk) \rightarrow 0$ as $k \rightarrow \infty$ [11], it can be expanded as

$$R_L(z, \lambda) = \sum_{i=1}^{\infty} r_i(z) \lambda^{-i} \quad (3.4)$$

so that after substitution in (3.2), we obtain

$$k(z) = 2r_1(z) = \lim_{\lambda \rightarrow \infty} 2\lambda R_L(z, \lambda) \quad (3.5)$$

By substituting (3.5) inside (3.2), $R_L(z, \lambda)$ can be propagated recursively, and in the process we reconstruct $k(z)$ for all z . This gives also $\sigma(z)$ by noting that

$$\sigma(z) = \sigma(0) \exp \left(-2 \int_0^z k(s) ds \right) \quad (3.6)$$

where $\sigma(0)$ is the conductivity on the surface of the earth.

The recursions (3.2), (3.5) constitute the Schur algorithm. This reconstruction procedure can be viewed as the continuous parameter version of the method proposed by Pekeris in [3] (see also [5], Chapter 10). To see this, assume that the conductivity function $\sigma(z)$ is approximated by a piecewise constant function, so that the earth can be viewed as constituted of N homogeneous layers of thickness t_i and conductivity σ_i , $1 \leq i \leq N$. Then, by discretizing (3.2) and assuming that the thickness of every layer is sufficiently small so that terms of order t_i^2 can be neglected, we obtain

$$R_{i+1}(\lambda) = e^{2\lambda t_i} \frac{(R_i(\lambda) - k_i)}{(1 - k_i R_i(\lambda))} \quad (3.7)$$

with $R_0(\lambda) = R_L(\lambda)$, where

$$R_i(\lambda) \triangleq R_L \left(\sum_{j=1}^i t_j, \lambda \right) \quad (3.8)$$

is the reflection coefficient obtained by stripping away the i first layers of the resistive medium, and by assuming that the current electrode is located on top of the $i+1$ th layer. From (3.5), we find also that

$$k_i = \frac{\sigma_i - \sigma_{i+1}}{\sigma_i + \sigma_{i+1}} = \lim_{\lambda \rightarrow \infty} e^{2\lambda t_i} R_i(\lambda) \quad (3.9)$$

or equivalently

$$\ln R_i(\lambda) \approx \ln k_i - 2\lambda t_i \quad (3.10)$$

as $\lambda \rightarrow \infty$, which is precisely the formula used by Pekeris to reconstruct t_i and k_i . From (3.10), we see that as $\lambda \rightarrow \infty$, the function $R_i(\lambda)$ can be approximated by a line whose slope is $-2t_i$ and whose intersection with the vertical axis is $\ln k_i$. Consequently, by combining (3.7) and (3.10), $R_i(\lambda)$, t_i and k_i can be computed recursively for $1 \leq i \leq N$, and (3.9) can be used with the initial condition $\sigma_0 = \sigma(0)$ to obtain σ_i for all i .

Instead of propagating the reflection coefficient $R_L(z, \lambda)$, we could choose to propagate the normalized impedance, i.e. Slichter's kernel

$$K(z, \lambda) = \frac{M(z, \lambda)}{N(z, \lambda)} \quad (3.11)$$

which is related to $R_L(z, \lambda)$ by the relation

$$R_L(z, \lambda) = \frac{K(z, \lambda) - 1}{K(z, \lambda) + 1} \quad (3.12)$$

By noting that

$$\frac{d}{dz} \begin{bmatrix} M(z, \lambda) \\ N(z, \lambda) \end{bmatrix} = \begin{bmatrix} -k(z) & \lambda \\ -\lambda & k(z) \end{bmatrix} \begin{bmatrix} M(z, \lambda) \\ N(z, \lambda) \end{bmatrix} \quad (3.13)$$

we find that $K(z, \lambda)$ satisfies the Riccati equation

$$\frac{d}{dz} K(z, \lambda) = -2k(z)K(z, \lambda) + \lambda(K^2(z, \lambda) - 1) \quad (3.14)$$

with the initial condition $K(0, \lambda) = K(\lambda)$ (given), which was first derived by Langer [2]. By expanding $K(z, \lambda)$ as

$$K(z, \lambda) = \sum_{i=0}^{\infty} K_i(z) \lambda^{-i} \quad (3.15)$$

we can identify $K_0(z) = 1$, and

$$k(z) = K_1(z) = \lim_{\lambda \rightarrow \infty} \lambda (K(z, \lambda) - 1). \quad (3.16)$$

By combining (3.14) and (3.16), we can therefore propagate $K(z, \lambda)$ recursively, and in the process reconstruct $k(z)$. The difference between this reconstruction method and that of Langer [2] is that Langer did not recognize that the inversion could be performed recursively. Instead, he showed that $k(0) = K_1(0)$ and that all derivatives $k^{(i)}(0)$ can be expressed in function of $K_j(0)$ for $j \leq i+1$, which by using the Taylor series expansion

$$k(z) = \sum_{i=0}^{\infty} k^{(i)}(0) \frac{z^i}{i!} \quad (3.17)$$

implies that $k(z)$ can be reconstructed from

$$K(\lambda) = 1 + \sum_{i=1}^{\infty} K_i(0) \lambda^{-i} \quad (3.18)$$

which is the given data.

An even better inversion procedure which can be used to reconstruct $\sigma(z)$ directly (instead of $k(z)$) is to consider the unnormalized impedance

$$Z(z, \lambda) = \sigma^{-1}(z) K(z, \lambda) = \frac{\hat{\phi}(z, \lambda)}{\hat{\psi}(z, \lambda)} \quad (3.19)$$

which was called the "resistivity transform" by Koefoed ([4], [5], Chapter 3).

Then $Z(z, \lambda)$ satisfies the Riccati equation

$$\frac{d}{dz} Z(z, \lambda) = \lambda (\sigma(z) Z^2(z, \lambda) - \sigma^{-1}(z)), \quad (3.20)$$

and by using the expansion (3.15) with the observation that $K_0(z) = 1$,

we find that

$$\sigma^{-1}(z) = \lim_{\lambda \rightarrow \infty} Z(z, \lambda) \quad (3.21)$$

so that $\sigma(z)$ can be recovered by propagating (3.20) and (3.21) recursively. This reconstruction procedure is the continuous analogue of the recursions

$$Z_{i+1}(\lambda) = \frac{Z_i(\lambda) - \sigma_i^{-1} \tanh(\lambda t_i)}{1 - \sigma_i \tanh(\lambda t_i) Z_i(\lambda)} \quad (3.22)$$

with

$$\sigma_i^{-1} = \lim_{\lambda \rightarrow \infty} Z_i(\lambda) \quad (3.23)$$

which were obtained by Koefoed [4], [5] to reconstruct a discrete resistive medium constituted of N horizontal homogeneous layers of thickness t_i and conductivity σ_i , $1 \leq i \leq N$, where $Z_i(\lambda)$ is the impedance obtained by removing the i first layers of the medium. A straightforward discretization of (3.20) - (3.21) can in fact be used to obtain the recursions (3.22) - (3.23).

4. Fast Cholesky and Levinson Recursions

The inversion procedures that were described above use either equations (3.2), (3.5) to reconstruct $k(z)$, or equations (3.20) - (3.21) to reconstruct $\sigma(z)$. Since these methods are variants of the Schur algorithm considered in [11], they suffer from the same limitations. The most significant of these is that we need to take the limit of $2\lambda R_L(z, \lambda)$ or of $Z(z, \lambda)$ as $\lambda \rightarrow \infty$, which is not a very reliable numerical operation. To eliminate this difficulty, we will now show that the problem can be formulated in a way such that efficient signal processing algorithms such as the fast Cholesky or Levinson recursions [9], [22] - [23] can be used.

To do so, we will use the method of Weidelt [7] (see also [6]) to convert the inverse resistivity problem into an equivalent inverse scattering problem. The key step is to view the functions $\lambda \hat{\phi}(z, \lambda)$, $\lambda \hat{\psi}(z, \lambda)$, $\lambda D(z, \lambda)$ and $\lambda U(z, \lambda)$ as Laplace transforms of some functions $\check{\phi}(z, t)$, $\check{\psi}(z, t)$, $\check{D}(z, t)$ and $\check{U}(z, t)$, so that if

$$L[f(t)] \triangleq \int_0^{\infty} f(t) \exp(-\lambda t) dt \quad (4.1)$$

denotes the Laplace transform of a function $f(\cdot)$, we have

$$\lambda \hat{\phi}(z, \lambda) = L[\check{\phi}(z, t)], \quad \lambda \hat{\psi}(z, \lambda) = L[\check{\psi}(z, t)] \quad (4.2a)$$

$$\lambda D(z, \lambda) = L[\check{D}(z, t)], \quad \lambda U(z, \lambda) = L[\check{U}(z, t)] . \quad (4.2b)$$

By multiplying (2.14) and (2.20) by λ , and taking inverse Laplace transforms, we obtain

$$\frac{\partial}{\partial z} \begin{bmatrix} \check{\phi}(z,t) \\ \check{\psi}(z,t) \end{bmatrix} = \begin{bmatrix} 0 & -\sigma^{-1}(z) \frac{\partial}{\partial t} \\ -\sigma(z) \frac{\partial}{\partial t} & 0 \end{bmatrix} \begin{bmatrix} \check{\phi}(z,t) \\ \check{\psi}(z,t) \end{bmatrix} \quad (4.3)$$

and

$$\frac{\partial}{\partial z} \begin{bmatrix} \check{D}(z,t) \\ \check{U}(z,t) \end{bmatrix} = \begin{bmatrix} -\frac{\partial}{\partial t} & -k(z) \\ -k(z) & \frac{\partial}{\partial t} \end{bmatrix} \begin{bmatrix} \check{D}(z,t) \\ \check{U}(z,t) \end{bmatrix} \quad (4.4)$$

which are respectively the telegrapher and two-component wave equations considered in [11].

We can then apply all the inversion techniques described in [10], [11] to reconstruct $k(z)$ or $\sigma(z)$. However, before doing so, it is useful to interpret the relations (4.2). We note first that if $\phi(z,r)$ is the potential of the earth at depth z , then $\phi(z,r)$ is related to $\check{\phi}(z,t)$ by

$$\begin{aligned} \phi(z,r) &= \int_0^\infty \hat{\phi}(z,\lambda) J_0(\lambda r) \lambda d\lambda \\ &= \int_0^\infty \check{\phi}(z,t) \frac{dt}{(t^2 + r^2)^{1/2}}, \end{aligned} \quad (4.5)$$

where we have used the identity

$$\int_0^\infty \exp -\lambda t J_0(\lambda r) d\lambda = \frac{1}{(t^2 + r^2)^{1/2}} \quad (4.6)$$

The transformation (4.5) has an interesting property: it maps solutions of the potential equation (2.6) into solutions of

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} \right) \check{\phi}(z,t) + \frac{d}{dz} \ln \sigma(z) \frac{\partial}{\partial z} \check{\phi}(z,t) = 0 \quad (4.7)$$

which is an hyperbolic equation. To see this, note that $G_0(r,t) = (t^2+r^2)^{-1/2}$ is the Green's function of the Laplacian, i.e.

$$\Delta G_0(t,r) = - 2\delta(t) \frac{\delta(r)}{r} . \quad (4.8)$$

Then, the operator

$$\tilde{\Delta} \triangleq \left(\frac{\partial^2}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} \right) + \frac{d}{dz} \ln \sigma(z) \frac{\partial}{\partial z} \quad (4.9)$$

can be applied to both sides of (4.5), and by using the identity (4.8) and integrating by parts, we obtain

$$\begin{aligned} \tilde{\Delta} \phi(z,r) &= \int_0^\infty (\tilde{W} \phi(z,t)) G_0(t,r) dt \\ &\quad - 2\phi(z,0) \frac{\delta(r)}{r} - \frac{\partial}{\partial t} \nabla \phi(z,0) \frac{1}{r} \end{aligned} \quad (4.10)$$

where \tilde{W} denotes the perturbed wave operator

$$\tilde{W} \triangleq \left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} \right) + \frac{d}{dz} \ln \sigma(z) \frac{\partial}{\partial z} . \quad (4.11)$$

The identity (4.10) shows that the solutions of $\tilde{\Delta} \phi(z,r) = 0$ are mapped into solutions of

$$\tilde{W} \nabla \phi(z,t) = 0 , \quad (4.12)$$

with the initial conditions

$$\nabla \phi(z,0) = 0, \quad \frac{\partial}{\partial t} \nabla \phi(z,0) = 0 \quad (4.13)$$

which correspond to the fact that the system (4.12) is originally at rest.

Fast Cholesky Recursions

Then, we can use either the fast Cholesky or Levinson recursions to solve the inverse scattering problem associated to the system (4.4). However,

as shown in [11], [15], in order to apply the fast Cholesky recursions, the probing waves $\check{D}(0,t)$ and $\check{U}(0,t)$ on the surface of the medium must have a very specific form, i.e.

$$\check{D}(0,t) = \delta(t) + d(0,t)l(t) \quad (4.14a)$$

$$\check{U}(0,t) = u(0,t)l(t) \quad (4.14b)$$

where $l(t)$ is the unit step function, and where $d(0,t)$ and $u(0,t)$ are smooth functions. Thus, $\check{D}(0,t)$ must contain a leading impulse which acts as a tag indicating the wavefront of the probing wave.

For the problem considered here, the potential inside the earth can be expressed as

$$\phi(z,r) = \frac{I}{2\pi\sigma(0)} \left(\frac{1}{(z^2 + r^2)^{1/2}} + f(z,r) \right) \quad (4.15)$$

where the first term in (4.15) is the potential of an homogeneous earth with conductivity $\sigma(0)$, and where $f(z,r)$ is the perturbation away from this reference potential which is due to inhomogeneities in the earth's resistivity. Then, if $\hat{f}(z,\lambda) \triangleq H_0[f(z,r)]$,

$$\hat{\phi}(0,\lambda) = \frac{I}{2\pi\sigma(0)} \left(\frac{1}{\lambda} + \hat{f}(0,\lambda) \right) \quad (4.16)$$

so that by using the expression (2.18) for $\hat{\psi}(0,\lambda)$ and denoting $h(\lambda) = \frac{1}{2} \hat{f}(0,\lambda)$, we obtain

$$\hat{D}(0,\lambda) = \frac{I}{2\pi\sigma^{1/2}(0)} \left(\frac{1}{\lambda} + h(\lambda) \right) \quad (4.17a)$$

$$\hat{U}(0,\lambda) = \frac{I}{2\pi\sigma^{1/2}(0)} h(\lambda) \quad (4.17b)$$

Consequently, if $\lambda h(\lambda) = L[\check{h}(t)]$, we can write

$$\check{D}(0,t) = \frac{I}{2\pi\sigma^{1/2}(0)} (\delta(t) + \check{h}(t)) \quad (4.18a)$$

$$\check{U}(0,t) = \frac{I}{2\pi\sigma^{1/2}(0)} \check{h}(t) \quad (4.18b)$$

so that modulo multiplication by $I/2\pi\sigma^{1/2}(0)$, the waves $\check{D}(0,t)$ and $\check{U}(0,t)$ have the form (4.14). The relation $d(0,t) = u(0,t) = \check{h}(t)$ for these waves indicates also that the earth's surface can be modeled as a perfect reflector. This corresponds to the fact that the air above the surface of the earth acts like a perfect insulator.

Then, a consequence of the special form (4.14) of the probing waves is that the waves inside the scattering medium described by (4.4) must have the form

$$\check{D}(z,t) = \delta(t-z) + d(z,t)l(t-z) \quad (4.19a)$$

$$\check{U}(z,t) = u(z,t)l(t-z) \quad (4.19b)$$

By substituting (4.19) inside (4.4), and identifying coefficients of the impulse $\delta(t-z)$ on both sides of (4.4), we find that

$$\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)d(z,t) = -k(z)u(z,t) \quad (4.20a)$$

$$\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial t}\right)u(z,t) = -k(z)d(z,t) \quad (4.20b)$$

with

$$k(z) = 2u(z, z_+) \quad (4.21)$$

After discretization, the recursions (4.20) - (4.21) constitute the fast Cholesky recursions [11], [15]. The initial data for these recursions is $d(0,t) = u(0,t) = \check{h}(t)$. The relations (4.20) - (4.21) can be viewed as using a layer-stripping principle to identify the parameters of the scattering medium. Thus, assume that the waves $d(z,t)$ and $u(z,t)$ at depth z have

been computed. The reflectivity function $k(z)$ is obtained from (4.21) and is used in (4.20) to compute the waves $d(z+\Delta, t)$ and $u(z+\Delta, t)$ at depth $z+\Delta$, as shown in Fig. 3. The effect of the recursions (4.20) - (4.21) is therefore to identify and strip away the layer $[z, z+\Delta)$. Note that the Schur recursions of Section 3 operated according to a similar principle.

The main feature of the fast Cholesky recursions is that they are quite efficient: let L be the maximum depth over which we want to reconstruct the medium, and let $\Delta = L/N$ be the step-size which is used to discretize the fast Cholesky recursions. Then, by observing that $\checkmark h(t)$ needs only to be known for $0 \leq t \leq 2L$, where $2L$ is the two-way travel time to depth L , and computing $d(z,t)$ and $u(z,t)$ at depth z only for $0 \leq t \leq 2L - z$, we find [11] that only $O(N^2)$ operations are required to recover $k(z)$ for $0 \leq z \leq L$. In addition, it was shown in [24] that this algorithm is numerically stable.

Levinson Recursions

An alternate approach is to formulate the inverse scattering problem in terms of integral equations. Consider the Marchenko integral equations

$$m_{11}(z, t) + \int_{-z}^t \checkmark h(t-\tau) m_{11}(z, \tau) d\tau + \int_{-t}^z \checkmark h(t+\tau) m_{21}(z, \tau) d\tau = 0 \quad (4.22a)$$

$$\begin{aligned} \checkmark h(z+t) + m_{21}(z, t) + \int_{-z}^t \checkmark h(t+\tau) m_{11}(z, \tau) d\tau \\ + \int_{-z}^t \checkmark h(t-\tau) m_{21}(z, \tau) d\tau = 0 \end{aligned} \quad (4.22b)$$

with $-z \leq t \leq z$. Then, it is shown in [11] that the reflectivity function $k(z)$ is given by

$$k(z) = -2m_{21}(z, z^-), \quad (4.23)$$

and that $m_{11}(z, \cdot)$ and $m_{21}(z, \cdot)$ can be propagated for increasing values of z by using the Levinson recursions

$$\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial t} \right) m_{11}(z, t) = -k(z)m_{21}(z, t) \quad (4.24a)$$

$$\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial t} \right) m_{21}(z, t) = -k(z)m_{11}(z, t) \quad (4.24b)$$

which are obtained by exploiting the Toeplitz and Hankel structure of the kernels appearing in (4.22). The initial conditions for these equations are

$$m_{11}(0,0) = m_{21}(0,0) = 0, \quad (4.25)$$

and in the propagation of (4.24) we use the boundary conditions $m_{11}(z, -z) = 0$ and (4.23), where

$$\begin{aligned} m_{21}(z, z^-) = & -h(2z) - \int_{-z}^z h(z+\tau)m_{11}(z, \tau) d\tau \\ & - \int_{-z}^z h(z-\tau)m_{21}(z, \tau) d\tau. \end{aligned} \quad (4.26)$$

After discretization, the Levinson recursions can be propagated as shown in Fig. 4. The complexity of these recursions is identical to that of the fast Cholesky equations, i.e. they require $O(N^2)$ operations to reconstruct $k(z)$ for $0 \leq z \leq L$, where N is the number of subintervals which are used to discretize the interval $[0, L]$. An interesting property of the fast Cholesky and Levinson recursions (4.20) and (4.24) is that they have the same form.

However, the support of the functions $m_{11}(z,t)$ and $m_{21}(z,t)$ is $-z \leq t \leq z$, whereas the support of the waves $d(z,t)$ and $u(z,t)$ is $z \geq t$. In some sense, the Levinson recursions can be viewed as being the complement of the fast Cholesky recursions: they rely on a layer accumulation principle where at depth z we identify a new layer and accumulate it to the part $[0,z]$ of the medium which has already been identified, whereas at each step the Schur and fast Cholesky recursions identify and strip away the same layer from the part $[z,\infty)$ of the medium which is yet to be identified. An additional difference between the fast Cholesky and Levinson recursions is that the fast Cholesky recursions correspond to an initial value problem where all the information about the medium is contained in the initial conditions $d(0,t)$ and $u(0,t)$, while for the Levinson recursions the identification of a new layer requires at every step the evaluation of the integral (4.26) where the information about the medium is contained in $\bigvee h(t)$.

It turns out that the above reconstruction procedure for the earth's resistivity is not new, and appears in disguised form in Kunetz and Rocroi [8] for the case of a discrete medium with layers of equal thickness. However, Kunetz and Rocroi did not identify the recursions that they obtained as the Levinson recursions.

The previous reconstruction procedure can also be related to that of Coen and Yu [6] by noting that if

$$A(z,t) = m_{11}(z,t) + m_{21}(z,t) \quad , \quad (4.27)$$

then $A(z,t)$ satisfies the integral equation (see [11])

$$\bigvee h(z+t) + A(z,t) + \int_{-z}^z (\bigvee h(t-\tau) + \bigvee h(t+\tau))A(z,\tau) d\tau = 0 \quad (4.28)$$

with $-z \leq t \leq z$, which is the equation used by Coen and Yu. Then

$$\left(\frac{\sigma(z)}{\sigma(0)}\right)^{1/2} = 1 + \int_{-z}^z A(z,t) dt \quad (4.29)$$

so that $\sigma(z)$ can be reconstructed directly from $A(\cdot, \cdot)$. The advantage of this method over the procedure described above is that since $\sigma(\cdot)$ is smoother than $k(\cdot)$, it is easier to reconstruct. However, Coen and Yu were unaware of the existence of a fast algorithm to solve the integral equation (4.28).

5. Interpretation and Computation of the Inversion Data

The inversion procedures described above rely on $R_L(\lambda)$, or equivalently on Slichter's kernel $K(\lambda)$, and on the function $\check{h}(t)$ to reconstruct the earth's resistivity. But the given data is the apparent resistivity $\rho_a(r)$ obtained from the Schlumberger electrode configuration. The problem of computing $K(\lambda)$ from $\rho_a(r)$ was solved by Ghosh [25] who used the expression

$$K(\lambda) = \sigma(0) \int_0^{\infty} \frac{1}{r} \rho_a(r) J_1(\lambda r) dr \quad (5.1)$$

which is obtained by combining (2.16) and (2.30), so that

$$K(\lambda) = \frac{2\pi\sigma(0)}{I} \lambda \hat{\phi}(0, \lambda) \quad , \quad (5.2)$$

and by using the identity (2.10) of Hankel transforms and the definition (2.1) of the apparent resistivity. Then, by substituting

$$r = e^x, \lambda = e^{-y} \quad (5.3)$$

inside (5.1) and denoting

$$\tilde{\rho}_a(x) = \rho_a(e^x), \tilde{K}(y) = K(e^{-y}) \quad , \quad (5.4)$$

we obtain the convolution integral

$$\tilde{K}(y) = \int_0^{\infty} \tilde{\rho}_a(x) J_1(\exp(-(y-x))) dx \quad (5.5)$$

which can be implemented by discrete convolution techniques [26].

The problem of computing $\int h(t)$ from $\rho_a(r)$ is more difficult. The first step is to obtain a physical interpretation of $\int h(t)$. From (4.15), we find that

$$\phi(0,r) = \frac{I}{2\pi\sigma(0)} \left(\frac{1}{r} + 2 \int_0^{\infty} \int h(t) \frac{dt}{(t^2 + r^2)^{1/2}} \right) \quad (5.6)$$

The first term in this expression is the potential associated to a homogeneous earth with conductivity $\sigma(0)$, and the second term describes the effect of inhomogeneities in the earth's resistivity. However, to describe the potential on the surface of the earth, instead of assuming that the earth's resistivity is inhomogeneous and that a single current source is located at the origin of coordinates, by Maxwell's method of images ([27], [5], p. 197) we can assume that the earth is homogeneous with conductivity $\sigma(0)$, but that some additional fictitious current sources have been added on the vertical axis. In this case, if $\int h(t)dt$ is the strength, relative to the strength I of the actual current source, of a source located at depth t along an infinitesimal segment of length dt , the potential created at the point $(0,r)$ on the surface of the earth is

$$\frac{I}{2\pi\sigma(0)} \frac{\int h(t)dt}{(t^2 + r^2)^{1/2}} \quad (5.7)$$

Note that in order to guarantee that the vertical component of the current density created by the fictitious sources is zero on the surface of the earth, the function $\int h(t)$ must be symmetric with respect to the origin, i.e. sources must be located above the surface of the earth as well as below. By superposition,

the function $\check{h}(t)$ appearing in (5.6) can therefore be viewed as the fictitious current source profile equivalent to the inhomogeneous conductivity profile $\sigma(z)$.

The function $\check{h}(t)$ was the starting point of the inversion method of Kunetz and Rocroi [8]. However, it is not as easy to compute this function from the potential $\phi(0,r)$ or the apparent resistivity $\rho_a(r)$ as it appears. To see why this is so, note from (4.16) that in order to obtain $\check{h}(t)$ from $\phi(0,r)$, we need first to compute the Hankel transform $\hat{\phi}(0,\lambda)$ followed by an inverse Laplace transform. But inverse Laplace transforms are hard to implement. Instead, it is preferable to discretize the integral equation (5.6) and to solve the resulting system of linear equations. In terms of $\rho_a(r)$, we find from (2.1) that

$$\rho_a(r) = \frac{1}{\sigma(0)} \left(1 + 2r^3 \int_0^\infty \check{h}(t) \frac{dt}{(t^2 + r^2)^{3/2}} \right) \quad (5.8)$$

(see [8], [5], Chapter 10), which can also be discretized and inverted.

An alternate method of computing $\check{h}(t)$, which was proposed by Kunetz and Rocroi [8], is to denote by $H(k)$ the Fourier transform of $\check{h}(t)$ and to introduce the spectral density function

$$W(k) = 1 + H(k) + H(-k) \quad . \quad (5.9)$$

Then, by observing from (4.18) that the left reflection coefficient of the two-component scattering system is

$$R_L(jk) = H(k)/(1 + H(k)) \quad (5.10)$$

and is bounded by one, i.e. $|R_L(jk)| \leq 1$, we can conclude that the spectral density $W(k)$ is positive, i.e. $W(k) \geq 0$ for all k . By using the integral equation (5.8), we also find that

$$\rho_a(r) = \frac{2r^3}{\sigma(0)\pi} \int_0^\infty W(k) K_1(kr) k dk \quad (5.11)$$

where $K_1(\cdot)$ is the modified Bessel function of order one. The problem of inverting the modified Hankel transform (5.11) is analogous to that of inverting a Laplace transform, but by using the positivity of $W(k)$, Kunetz and Rocroi were able to formulate the inversion of (5.11) for a discrete set of sampled values of r as a quadratic programming problem. Then, given the reconstructed $W(k)$, $\check{h}(|t|)$ is the inverse Fourier transform of $W(k) - 1$.

6. Conclusion

In this paper, we have considered the problem of reconstructing the resistivity profile of a layered earth probed by direct current from potential measurements on the surface of the earth. It was shown that this problem could be transformed into an equivalent inverse scattering problem, to which efficient signal processing algorithms such as the Schur, fast Cholesky and Levinson recursions can be applied. These algorithms reconstruct the resistivity of the earth layer by layer in a recursive way, and require only a small number of operations. In this context, it was shown that the recursions obtained by Pekeris [3] and Koefoed [4], [5] for recovering the resistivity of the earth were identical to the discrete Schur recursions, and that the reconstruction method of Kunetz and Rocroi [8] was actually based on the Levinson recursions.

One difficulty associated with these reconstruction methods is that they do not operate directly on the given data, which is the apparent resistivity of the earth, but on Slichter's kernel $K(\lambda)$, or on the fictitious current source profile $\int h(t)$ equivalent to the inhomogeneous conductivity profile $\sigma(z)$. Efficient convolution techniques exist to compute Slichter's kernel from the apparent resistivity, but the problem of computing $\int h(t)$ is more difficult. No efficient method of obtaining $\int h(t)$ exists, short of brute force discretization of the integral equation satisfied by $\int h(t)$. This problem deserves therefore further attention. Another

topic of research, which is currently under investigation, is the study of the numerical behavior of the algorithms described above when they operate on synthetic or real data. The fast Cholesky and Levinson recursions are known to be stable, but the addition of noise, or imperfections in the data due to bandlimitations, can degrade the performance of these algorithms. It would therefore be desirable to develop inversion techniques which can incorporate a priori information on the resistivity profile and on the noise level.

Acknowledgements

The author wishes to thank Dr. Andrew E. Yagle for several interesting conversations, and Dr. Stephen Lang, from Schlumberger-Doll Research Center, for his encouragements and for providing several key references at the early stage of this work.

REFERENCES

- [1] L.B. Slichter, "The Interpretation of the Resistivity Prospecting Method for Horizontal Structures," Physics, Vol. 4, pp. 307-322, Sept. 1933.
- [2] R.E. Langer, "An Inverse Problem in Difference Equations," Bull. Amer. Math. Soc., Vol. 39, pp. 814-820, Oct. 1933.
- [3] C.L. Pekeris, "Direct Method of Interpretation in Resistivity Prospecting," Geophysics, Vol. 5, No. 1, pp. 31-42, Jan. 1940.
- [4] O. Koefoed, "A Fast Method for Determining the Layer Distribution from the Raised Kernel Function in Geoelectrical Sounding," Geophys. Prospecting, Vol. 18, pp. 564-570, 1970.
- [5] O. Koefoed, Geosounding Principles, 1. Resistivity Sounding Measurements, Methods in Geochemistry and Geophysics, 14A, Elsevier, Amsterdam, The Netherlands, 1979.
- [6] S. Coen and M. W-H. Yu, "The Inverse Problem of the Direct Current Conductivity Profile of a Layered Earth," Geophysics, Vol. 46, No. 12, pp. 1702-1713, Dec. 1981.
- [7] P. Weidelt, "The Inverse Problem of Geomagnetic Induction," Zeitschrift für Geophysik, Vol. 38, pp. 257-289, 1972.
- [8] G. Kunetz and J.P. Rocroi, "Traitement Automatique des Sondages Electriques," Geophys. Prospecting, Vol. 18, pp. 157-198, 1970.
- [9] N. Levinson, "The Wiener rms (root mean square) Error Criterion in Filter Design and Prediction," J. Math. Phys., Vol. 25, pp. 261-278, Jan. 1947.
- [10] K.P. Bube and R. Burridge, "The One-Dimensional Inverse Problem of Reflection Seismology," SIAM Review, Vol. 25, No. 4, pp. 497-559, Oct. 1983.
- [11] A.M. Bruckstein, B.C. Levy and T. Kailath, "Differential Methods in Inverse Scattering," Technical Report LIDS-P-1313, Laboratory for Information and Decision Systems, MIT, Cambridge, MA., August 1983, to appear in SIAM J. Applied Math.

- [12] P. Dewilde, J.T. Fokkema and I. Widya, "Inverse Scattering and Linear Prediction, The Time Continuous Case," in Stochastic Systems: The Mathematics of Filtering and Identification and Applications, M. Hazewinkel and J.C. Willems eds., pp. 351-382, D. Reidel Publ. Co., Dordrecht, The Netherlands, 1981.
- [13] W.W. Symes, "Stable Solution of the Inverse Reflection Problem for a Smoothly Stratified Elastic Medium," SIAM J. Math. Anal., Vol. 12, No. 3, pp. 421-453, May 1981.
- [14] F. Santosa and H. Schwetlick, "The Inversion of Acoustical Impedance Profile by Methods of Characteristics," Wave Motion, Vol. 4, pp. 99-110, 1982.
- [15] A.E. Yagle and B.C. Levy, "The Schur Algorithm and its Applications," Technical Report LIDS-P-1362, Laboratory for Information and Decision Systems, MIT, Cambridge, MA., Feb. 1984.
- [16] I. Schur, "Über Potenzreihen, die in Innern des Einheitskreises Beschränkt Sind," J. für die Reine und Angewandte Mathematik, Vol. 147, Berlin, pp. 205-232, 1917.
- [17] N.I. Akhiezer, The Classical Moment Problem, Hafner Publ. Co., New York, 1965.
- [18] P. Dewilde and H. Dym, "Lossless Inverse Scattering for Digital Filters: Theory and Applications," IEEE Trans. Informat. Theory, Vol. IT-30, No. 4, July 1984.
- [19] M.J. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981.
- [20] V.E. Zakharov and P.B. Shabat, "Exact Theory for Two-Dimensional Self-Focusing and One-Dimensional Self-Modulation of Waves in Nonlinear Media," Soviet Phys. JETP, Vol. 34, pp. 62-69, 1972.
- [21] P. Deift and E. Trubowitz, "Inverse Scattering on the Line," Comm. Pure, Applied Math., Vol. 32, No. 2, pp. 121-251, March 1979.
- [22] M. Morf, "Fast Algorithms for Multivariable Systems," Ph.D. Dissertation, Dept. of Electrical Engineering, Stanford Univ., Stanford, CA., Aug. 1974.
- [23] J. Rissanen, "Algorithms for Triangular Decomposition of Block Hankel and Toeplitz Matrices with Application to Factoring Positive Matrix Polynomials," Math. Comput., Vol. 27, pp. 147-154, Jan. 1973.
- [24] A. Bultheel, "Towards an Error Analysis of Fast Toeplitz Factorization," Technical Report TW44, Applied Mathematics and Programming Division, Katholieke Universiteit Leuven, Belgium, May 1979.

- [25] D.P. Ghosh, "The Application of Linear Filter Theory to the Direct Interpretation of Geoelectrical Resistivity Sounding Measurements," Geophys. Prospecting, Vol. 19, pp. 192-217, 1971.
- [26] A.V. Oppenheim and R.W. Shafer, Digital Signal Processing, Prentice-Hall, Englewood Cliffs, N.J., 1975.
- [27] D.O. Ehrenburg and R.J. Watson, "Mathematical Theory of Electrical Flow in Stratified Media with Horizontal, Homogeneous and Isotropic Layers," Trans. Amer. Inst. Mining Metall Eng., Vol. 97 (Geophys. Prospecting), pp. 423-438, 1932.

FIGURE CAPTIONS

- Fig. 1: Elementary filter sections associated to the two-component system.
- Fig. 2: The aggregate medium obtained by composing the elementary filter sections.
- Fig. 3: (a) Propagation of $d(z,t)$; and b) propagation of $u(z,t)$ via the fast Cholesky recursions.
- Fig. 4: (a) Propagation of $m_{11}(z,t)$; and b) propagation of $m_{21}(z,t)$ with the Levinson recursions.

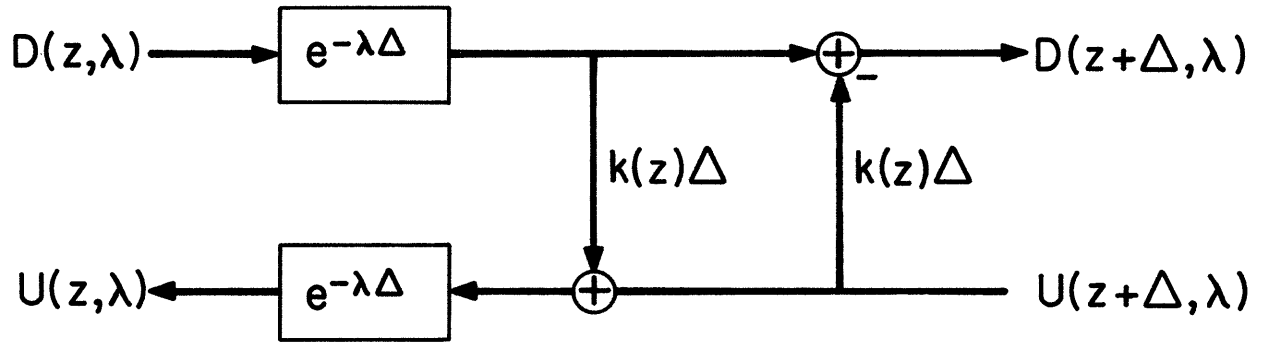


FIG. 1

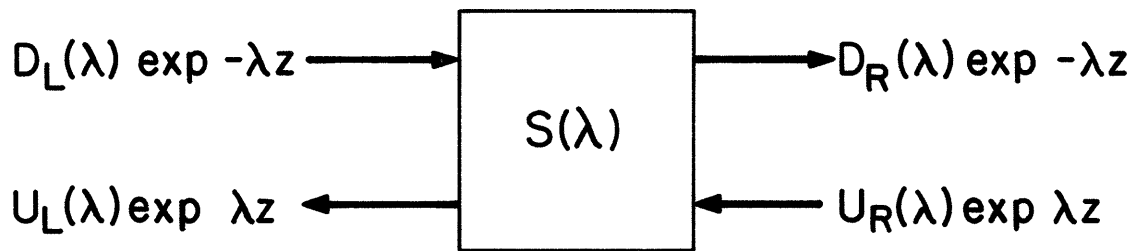
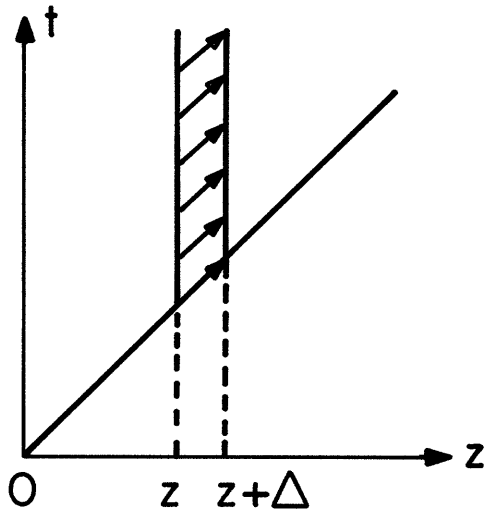
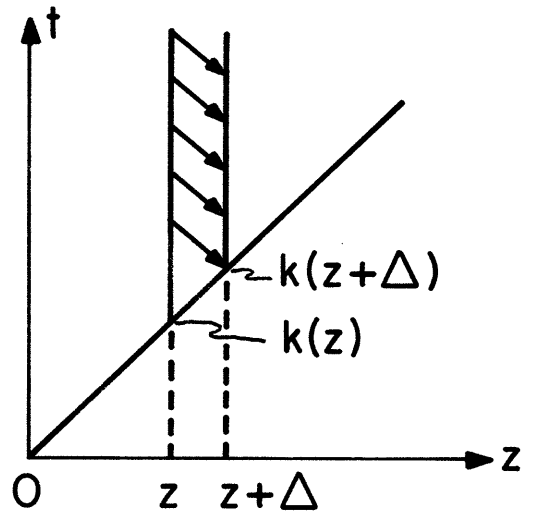


FIG. 2

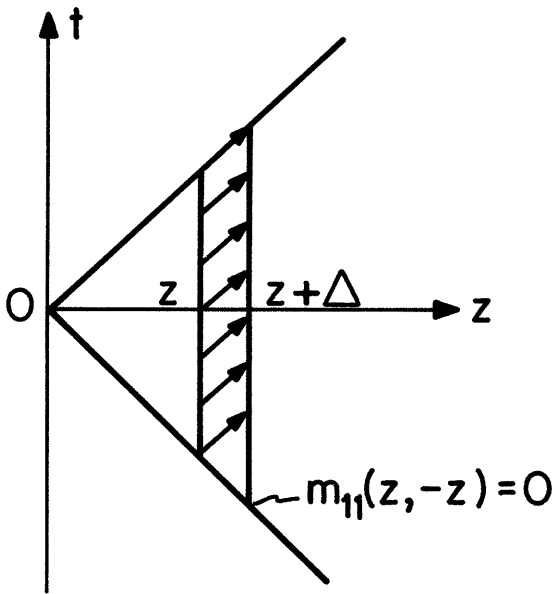


(a)

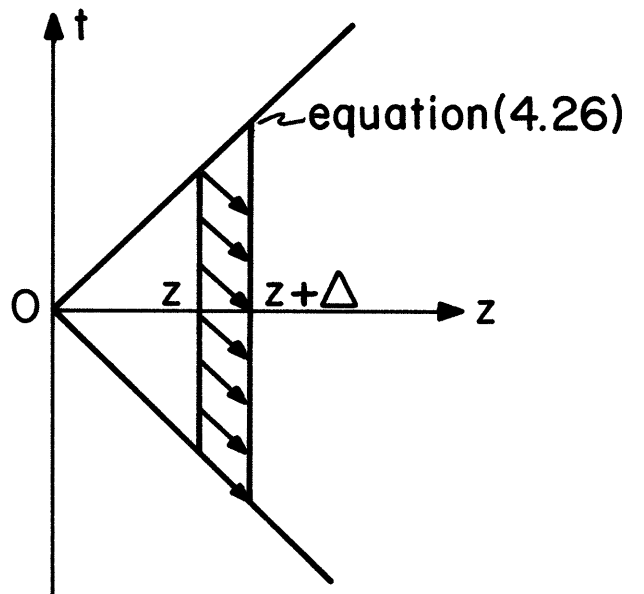


(b)

FIG. 3



(a)



(b)

FIG. 4