An Extension of the Argument Principle and Nyquist Criterion to Systems with Unbounded Generators

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Abstract

The Nyquist Stability Criterion is generalized to systems where the (open-loop) system has infinite-dimensional input/output spaces and a (possibly) unbounded infinitesimal generator. This is done through use of the *perturbation determinant* and an extension of the Argument Principle to infinitesimal generators with trace-class resolvent.

I. INTRODUCTION

The Nyquist criterion is of particular interest in system analysis as it offers a simple visual test to determine the stability of a closed-loop system for a family of feedback gains [1] [2]. Extensions of the Nyquist stability criterion exist for certain classes of distributed [3] and time periodic [4] systems. [3] considers distributed systems in which the open-loop $G(s)$ belongs to the algebra of matrix-valued meromorphic functions of *finite* Euclidean dimension, and the Nyquist analysis is carried out by performing a coprime factorization on $G(s)$.

To motivate the discussion in this paper, let us first consider a finite-dimensional (multi-input multi-output) LTI system $G(s)$ placed in feedback with a constant gain γI. In analyzing the closed-loop stability of such a system, we are concerned with the eigenvalues in $\mathbb{C}^+$ of the closed-loop A-matrix $A^{\rm cl}$. If s is an eigenvalue of $A^{\rm cl}$, then it satisfies $\det[sI - A^{\rm cl}] = 0$. Now to check whether the equation $\det[sI - A^{\rm cl}] = 0$ has solutions inside $\mathbb{C}^+$, one can apply the argument principle to $\det[I + \gamma G(s)]$ as s traverses some path $\mathfrak{D}$ enclosing $\mathbb{C}^+$. More concretely, let us assume that we are given a state-space realization of the open-loop system. Then using

$$\det[I + \gamma G(s)] \;=\; \frac{\det[sI - A^{\rm cl}]}{\det[sI - A]}, \tag{1}$$

if one knows the number of unstable open-loop poles one can determine the number of unstable closed-loop poles by looking at the plot of $\det[I + \gamma G(s)]\big|_{s\in\mathfrak{D}}$.

But in the case of distributed systems the open- and closed-loop *infinitesimal generators*, $\mathcal{A}$ and $\mathcal{A}^{\rm cl}$, may be operators on an infinite-dimensional Hilbert space $\mathcal{X}$ and can be *unbounded*. Hence it is not clear how to define the characteristic functions $\det[s\mathcal{I} - \mathcal{A}]$ and $\det[s\mathcal{I} - \mathcal{A}^{\rm cl}]$. In this paper we find an analog of equation (1) applicable to unbounded $\mathcal{A}$ and $\mathcal{A}^{\rm cl}$ and use operator theoretic arguments to relate the plot of $\det[\mathcal{I} + \gamma\mathcal{G}(s)]\big|_{s\in\mathfrak{D}}$ to the unstable modes of the open-loop and closed-loop systems.

Now if the multiplicity of each of the eigenvalues of $\mathcal{A}$ is finite it can be shown that $\det[\mathcal{I} + \gamma\mathcal{G}(s)]$ is still a meromorphic function of s on $\mathbb{C}$, and one may be tempted to use the methods of [3] to analyze closed-loop stability. But if the open-loop system has distributed input and output spaces, then [3] requires the coprime factorization of an *infinite-dimensional* operator. In addition, one often deals with systems of Partial Differential Equations (PDEs) in which the state-space representation is the natural representation and it is more convenient to deal directly with the operators $\mathcal{A}$ and $\mathcal{A}^{\rm cl}$ rather than $\mathcal{G}(s)$ [see example in Section V].

Our presentation is organized as follows: We lay out the problem setup in Section II and describe the general conditions for stability of distributed systems in Section III. Section IV contains the main contributions of the paper in which the Argument Principle and the Nyquist Stability Criterion are extended to a class of distributed systems. The theory is applied to a simple example in Section V. Proofs and technical details have been placed in the Appendix.

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Notation

$\Sigma(T)$ is the spectrum of T, and $\rho(T)$ its resolvent set. $\sigma_n(T)$ is the n^{th} singular-value of T. $\mathscr{B}(\mathcal{X})$ denotes the bounded operators on the Hilbert space $\mathcal{X}$, $\mathscr{B}_\infty(\mathcal{X})$ the compact operators on $\mathcal{X}$, and $\mathscr{B}_1(\mathcal{X})$ the nuclear (trace-class) operators on $\mathcal{X}$, i.e. operators T that have the property $\sum_{n=1}^{\infty} \sigma_n(T) < \infty$; $\mathscr{B}_1(\mathcal{X}) \subset \mathscr{B}_\infty(\mathcal{X}) \subset \mathscr{B}(\mathcal{X})$. $\mathrm{tr}[T]$ denotes the trace of T and $\det[T]$ its determinant. $\mathbb{C}^+$ and $\mathbb{C}^-$ denote the *closed* right-half and the *open* left-half of the complex plane, respectively, and $j := \sqrt{-1}$. $C(z_0; \mathfrak{P})$ is the number of counter-clockwise encirclements of the point $z_0 \in \mathbb{C}$ by the closed path $\mathfrak{P}$.

II. Problem Setup

Consider the open-loop system $\mathbf{S}^{\mathrm{o}}$

$$\begin{aligned} [\partial_t \psi](t) &= [\mathcal{A}\,\psi](t) \;+\; [\mathcal{B}\,u](t), \\ y(t) &= [\mathcal{C}\,\psi](t), \end{aligned} \tag{2}$$

where $t \in [0, \infty)$ with the following assumptions. The (possibly unbounded) operator $\mathcal{A}$ is defined on a dense domain $\mathscr{D}$ of the Hilbert space $\mathcal{X}$ and is closed. $\mathcal{B}$ and $\mathcal{C}$ are bounded operators on $\mathcal{X}$. At any given point t in time, u, y and ψ belong to the space $\mathcal{X}$ and are the distributed input, output, and state of the system, respectively. We will refer to $\mathcal{A}$ as the *infinitesimal generator* of the system. We may also refer to $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ as the *system operators*. The open-loop system $\mathbf{S}^{\mathrm{o}}_\theta$ has temporal impulse response $\mathcal{G}(t) := \mathcal{C}\, e^{\mathcal{A}t}\, \mathcal{B}$, and transfer function

$$\mathcal{G}(s) \;:=\; \mathcal{C}\,(s\mathcal{I} - \mathcal{A})^{-1}\,\mathcal{B}. \tag{3}$$

Next we place the system $\mathbf{S}^{\mathrm{o}}$ in feedback with a bounded operator $\gamma\mathcal{F}$, $\|\mathcal{F}\| = 1$, $\gamma \in \mathbb{C}$. This forms the closed-loop system shown in Figure 1 (left) with infinitesimal generator $\mathcal{A}^{\mathrm{cl}} := \mathcal{A} - \mathcal{B}\,\gamma\mathcal{F}\,\mathcal{C}$. We separate the function $\mathcal{F}$ from the gain γ as in Figure 1 (right) to form the closed-loop system $\mathbf{S}^{\mathrm{cl}}$, and it is our aim here to determine the stability of $\mathbf{S}^{\mathrm{cl}}$ as the feedback gain γ varies in $\mathbb{C}$.

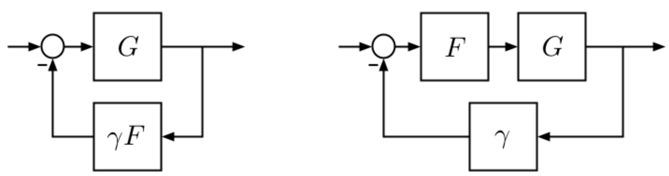

Fig. 1. Left: The spatially periodic closed-loop system as the feedback interconnection of a spatially invariant system G and a spatially periodic multiplication operator F. Right: The closed-loop system $\mathbf{S}^{\mathrm{cl}}$ in the standard form for Nyquist stability analysis.

We also make the following assumptions:

Assumption ($*$): There exists at least one point $s \in \rho(\mathcal{A})$ such that $(s\mathcal{I} - \mathcal{A})^{-1} \in \mathscr{B}_1(\mathcal{X})$,
Assumption ($**$): $\rho(\mathcal{A})$ contains a right sector of the complex plane $|\arg(z - \alpha)| \leq \frac{\pi}{2} + \varphi$, $\varphi > 0$, $\alpha \in \mathbb{R}$.

III. Stability of Distributed Linear Systems

A semigroup $e^{\mathcal{A}t}$ on a Hilbert space is called exponentially stable if there exist constants $M \geq 1$ and $\beta > 0$ such that $\|e^{\mathcal{A}t}\| \leq Me^{-\beta t}$ for $t \geq 0$. It is well-known [5] [6] that if $\mathcal{A}$ is an ifinite-dimensional operator, then in general $\Sigma(\mathcal{A}) \subset \mathbb{C}^-$ is not sufficient to guarantee the exponential decay of $\|e^{\mathcal{A}t}\|$. In this paper we focus on systems which *do* satisfy the so-called *spectrum-determined growth condition*, i.e., systems for which $\Sigma(\mathcal{A}) \subset \mathbb{C}^-$ *does* imply exponential decay of the semigroup. Examples of such semigroups are numerous and include analytic semigroups [7] [8].

Thus to guarantee the exponential stability of $\mathbf{S}^{\mathrm{cl}}$, it is necessary and sufficient to show that $\mathcal{A}^{\mathrm{cl}} = \mathcal{A} - \mathcal{B}\,\gamma\mathcal{F}\,\mathcal{C}$ has spectrum only inside $\mathbb{C}^-$. In the next section we aim to develop a graphical method of checking whether or not $\Sigma(\mathcal{A}^{\mathrm{cl}}) \subset \mathbb{C}^-$. Also, henceforth in this paper wherever we use the term stability we mean exponential stability.

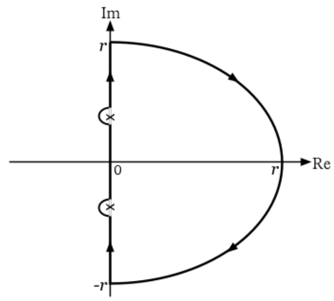

Fig. 2. The closed contour $\mathfrak{D}$ traversed in the clockwise direction taken as the Nyquist path as $r \to \infty$. The indentations are arbitrarily made to avoid the eigenvalues of $\mathcal{A}$ (i.e., open-loop modes) on the imaginary axis.

IV. The Nyquist Stability Criterion for Distibuted Systems

A. The Determinant Method

As discussed in the Introduction, we aim to use operator theoretic arguments to relate the plot of $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}(s)]\big|_{s\in\mathfrak{D}}$ to the unstable modes of the open-loop and closed-loop systems. But first it has to be clarified what is meant by $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}(s)]$ for the infinite-dimensional operator $\mathcal{I} + \gamma\mathcal{F}\mathcal{G}(s)$.

From Assumption $(*)$ we know that $\mathcal{G}(s) \in \mathscr{B}_1(\mathfrak{X})$ for some $s \in \rho(\mathcal{A})$. Then it is simple to show that $\mathcal{G}(s) \in \mathscr{B}_1(\mathfrak{X})$ *for every* $s \in \rho(\mathcal{A})$ [9]. Also $\mathcal{F} \in \mathscr{B}(\mathfrak{X})$ implies $\mathcal{F}\mathcal{G}(s) \in \mathscr{B}_1(\mathfrak{X})$. One can now define [10] [9]

$$\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}(s)] \;:=\; \prod_{n\in\mathbb{Z}} \big(1 + \gamma\lambda_n(s)\big),$$

where $\{\lambda_n(s)\}_{n\in\mathbb{Z}}$ are the eigenvalues of $\mathcal{G}(s)$.

On the other hand, the boundedness of the operators $\mathcal{B}$, $\mathcal{C}$, $\mathcal{F}$ together with Assumption $(*)$ imply that (a) $\mathcal{A}$ and $\mathcal{A}^{\text{cl}} = \mathcal{A} - \gamma\mathcal{B}\mathcal{F}\mathcal{C}$ are defined on the same dense domain $\mathscr{D}$, (b) $\rho(\mathcal{A}) \cap \rho(\mathcal{A}^{\text{cl}})$ is not empty, (c) for all $s \in \rho(\mathcal{A})$ we have $\gamma\mathcal{B}\mathcal{F}\mathcal{C}(s\mathcal{I} - \mathcal{A})^{-1} \in \mathscr{B}_1(\mathfrak{X})$. This allows us to introduce the *perturbation determinant* [11]

$$\begin{aligned}\Delta_{\mathcal{A}^{\text{cl}}/\mathcal{A}}(s) \;&:=\; \det[(s\mathcal{I} - \mathcal{A}^{\text{cl}})(s\mathcal{I} - \mathcal{A})^{-1}] \\ &=\; \det[\mathcal{I} + \gamma\mathcal{B}\mathcal{F}\mathcal{C}(s\mathcal{I} - \mathcal{A})^{-1}] \;=\; \det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}(s)]\end{aligned}$$

which is analytic in $\rho(\mathcal{A}) \cap \rho(\mathcal{A}^{\text{cl}})$ [see Lemma A1]. In fact $\Delta_{\mathcal{A}^{\text{cl}}/\mathcal{A}}(s)$ is the equivalent of the fraction in (1) for systems with unbounded infinitesimal generators. We are now ready to state an extended form of the argument principle for such systems. The following theorem makes use of the formula [11]

$$\frac{d}{ds} \ln \Delta_{\mathcal{A}^{\text{cl}}/\mathcal{A}}(s) \;=\; \operatorname{tr}[(s\mathcal{I} - \mathcal{A}^{\text{cl}})^{-1} - (s\mathcal{I} - \mathcal{A})^{-1}] \quad \text{for all} \quad s \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}^{\text{cl}}) \tag{4}$$

to relate $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}(s)]\big|_{s\in\mathfrak{D}}$ to the eigenvalues of $\mathcal{A}$ and $\mathcal{A}^{\text{cl}}$ inside $\mathfrak{D}$.

Theorem 1: If $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}(s)] \neq 0$ for all $s \in \mathfrak{D}$,

$$\begin{aligned}C\Big(0; \det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}(s)]\big|_{s\in\mathfrak{D}}\Big) \;&=\; \operatorname{tr}\bigg[\frac{1}{2\pi j}\int_{\mathfrak{D}} (s\mathcal{I} - \mathcal{A}^{\text{cl}})^{-1} ds\bigg] \;-\; \operatorname{tr}\bigg[\frac{1}{2\pi j}\int_{\mathfrak{D}} (s\mathcal{I} - \mathcal{A})^{-1} ds\bigg] \\ &=\; -\,(\text{number of eigenvalues of } \mathcal{A}^{\text{cl}} \text{ in } \mathbb{C}^+) \\ &\qquad\qquad +\; (\text{number of eigenvalues of } \mathcal{A} \text{ in } \mathbb{C}^+),\end{aligned}$$

where $\mathfrak{D}$ is the Nyquist path shown in Figure 2 that does not pass through any eigenvalues of $\mathcal{A}$.

Proof: See Appendix. ■

Remark 1: Theorem 1 relies on the fact that under Assumption $(*)$ both $(s\mathcal{I} - \mathcal{A})^{-1}$ and $(s\mathcal{I} - \mathcal{A}^{\text{cl}})^{-1}$ are compact operators, which means the infinitesimal generators $\mathcal{A}$ and $\mathcal{A}^{\text{cl}}$ have discrete spectrum (i.e., their spectrum consists entirely of isolated eigenvalues with finite multiplicity). Then $\mathcal{P} = -\frac{1}{2\pi j}\int_{\mathfrak{D}} (s\mathcal{I} - \mathcal{A})^{-1} ds$ is the group-projection [12] [9] corresponding to the eigenvalues of $\mathcal{A}$ inside $\mathfrak{D}$, and $\operatorname{tr}[\mathcal{P}]$ gives the total number of such eigenvalues [13]. Similarly $\operatorname{tr}[-\frac{1}{2\pi j}\int_{\mathfrak{D}} (s\mathcal{I} - \mathcal{A}^{\text{cl}})^{-1} ds]$ gives the total number of eigenvalues of $\mathcal{A}^{\text{cl}}$ in $\mathfrak{D}$. ■

As a direct consequence of Theorem 1 we have the following.

Theorem 2: Assume p_+ denotes the number of eigenvalues of $\mathcal{A}$ inside $\mathbb{C}^+$. For $\mathfrak{D}$ taken as above, the closed-loop system is stable iff

(a) $\det[\mathcal{I} + \gamma \mathcal{F}\mathcal{G}(s)] \;\neq\; 0, \;\; \forall\, s \in \mathfrak{D}$,

and

(b) $C\Big(0; \det[\mathcal{I} + \gamma \mathcal{F}\mathcal{G}(s)]\big|_{s\in\mathfrak{D}}\Big) \;=\; p_+$.

■

B. The Eigenloci Method

The setback with the method described in the previous paragraph is that to show $\Sigma_{\text{p}}(\mathcal{A}^{\text{cl}}) \subset \mathbb{C}^-$, $\mathcal{A}^{\text{cl}} = \mathcal{A} - \mathcal{B}\gamma\mathcal{F}\mathcal{C}$, for different values of γ, one has to plot $\det[\mathcal{I} + \gamma \mathcal{F}\mathcal{G}(s)]\big|_{s\in\mathfrak{D}}$ for each γ. Note that this includes having to calculate the determinant of an infinite dimensional matrix. This motivates the following eigenloci approach to Nyquist stability analysis, which is very similar to that performed in [4] for the case of time-periodic systems.

Let $\big\{\lambda_n(s)\big\}_{n\in\mathbb{Z}}$ constitute the eigenvalues of $\mathcal{F}\mathcal{G}(s)$. Then

$$\angle\ \det[\mathcal{I} + \gamma \mathcal{F}\mathcal{G}(s)] \;=\; \angle \prod_{n\in\mathbb{Z}} \big(1 + \gamma\lambda_n(s)\big). \tag{5}$$

Recall that $\mathcal{F}\mathcal{G}(s) \in \mathscr{B}_1(\mathfrak{X})$ for every $s \in \rho(\mathcal{A})$. This, in particular, means that $\mathcal{F}\mathcal{G}(s)$ is a compact operator and thus its eigenvalues $\lambda_n(s)$ accumulate at the origin as $|n| \to \infty$ [14]. As a matter of fact one can make a much stronger statement.

Lemma 3: The eigenvalues $\lambda_n(s)$, $s \in \mathfrak{D}$, converge to the origin uniformly on $\mathfrak{D}$.

Proof: See Appendix. ■

Take the positive integer N_ϵ to be such that $|\lambda_n(s)| < \epsilon$, $s \in \mathfrak{D}$, for all $|n| > N_\epsilon$. Let us rewrite (5) as

$$\begin{aligned}\angle\ \det[\mathcal{I} + \gamma \mathcal{F}\mathcal{G}(s)] \;&=\; \angle \prod_{|n|\le N_\epsilon} \big(1 + \gamma\lambda_n(s)\big) \;+\; \angle \prod_{|n|> N_\epsilon} \big(1 + \gamma\lambda_n(s)\big)\\ &=\; \sum_{|n|\le N_\epsilon} \angle \big(1 + \gamma\lambda_n(s)\big) \;+\; \sum_{|n|> N_\epsilon} \angle \big(1 + \gamma\lambda_n(s)\big).\end{aligned} \tag{6}$$

It is clear that if $|\gamma| < \frac{1}{\epsilon}$ then for $|n| > N_\epsilon$ we have $|\gamma\lambda_n(s)| < 1$, and $1 + \gamma\lambda_n(s)$ can never circle the origin as s travels around $\mathfrak{D}$. Thus for $|\gamma| < \frac{1}{\epsilon}$ the final sum in (6) will not contribute to the encirclements of the origin, and hence we lose nothing by considering only the first N_ϵ eigenvalues. There still remain some minor technicalities.

First, let D_ϵ denote the disk $|s| < \epsilon$ in the complex plane. Then said truncation may result in some eigenloci (parts of which reside inside D_ϵ) not forming closed loops. But notice that these can be arbitrarily closed inside D_ϵ, as this does not affect the encirclements [4].

The second issue is that for some values of $s \in \mathfrak{D}$, $\mathcal{F}\mathcal{G}(s)$ may have multiple eigenvalues, and hence there is ambiguity in how the eigenloci of the Nyquist diagram should be indexed. But this poses no problem as far as counting the encirclements is concerned, and it is always possible to find such an indexing; for a detailed treatment see [3].

Let us denote by $\big\{\lambda_n\big\}_{n\in\mathbb{Z}}$ the indexed eigenloci that make up the generalized Nyquist diagram. [To avoid confusion we stress the notation: $\lambda_n(s)$ is the n^{th} *eigenvalue* of $\mathcal{F}\mathcal{G}(s)$ for a given point $s \in \mathfrak{D}$, whereas λ_n is the n^{th} *eigenlocus* traced out by $\lambda_n(s)$ as s travels once around $\mathfrak{D}$.] From (6) and the above discussion it follows that

$$C\Big(0; \det[\mathcal{I} + \gamma \mathcal{F}\mathcal{G}(s)]\big|_{s\in\mathfrak{D}}\Big) \;=\; \sum_{|n|\le N_\epsilon} C\Big(-\frac{1}{\gamma}; \lambda_n\Big)$$

which together with Theorem 2 gives the following.

Theorem 4: Assume p_+ denotes the number of eigenvalues of $\mathcal{A}$ inside $\mathbb{C}^+$. For $\mathfrak{D}$ and N_ϵ as defined previously, the closed-loop system is stable for $|\gamma| < \frac{1}{\epsilon}$ iff

(a) $-\dfrac{1}{\gamma} \notin \{\lambda_n\}_{|n|\leq N_\epsilon}$,

and

(b) $\sum_{|n|\leq N_\epsilon} C\Big(-\dfrac{1}{\gamma};\lambda_n(s)\Big) = p_+$.

■

V. An Illustrative Example

Consider the system defined on the interval $x \in [0,2\pi]$ and governed by the Partial Differential Equation (PDE)

$$\partial_t\,\psi(t,x) \;=\; \partial_x^2\,\psi(t,x) \,-\, \gamma\cos(x)\,\psi(t,x) \,+\, \psi(t,x),$$

with $\gamma \in \mathbb{C}$ and periodic boundary conditions

$$\psi(t,0) \;=\; \psi(t,2\pi), \quad \partial_x\psi(t,0) \;=\; \partial_x\psi(t,2\pi).$$

Let us rewrite this system in the form of a PDE described by

$$\begin{aligned} \partial_t\,\psi(t,x) &= \partial_x^2\,\psi(t,x) \,+\, \psi(t,x) \,+\, u(t,x),\\ y(t,x) &= \psi(t,x), \end{aligned} \tag{7}$$

placed in feedback with the function

$$\gamma F(x) \;=\; \gamma\cos(x).$$

The problem is now in the general from discussed in Section II and can be considered as a differential equation on $\mathcal{X} = L^2[0,2\pi]$; $A = \partial_x^2 + 1$ and is defined on the dense domain

$$\mathscr{D} \;=\; \{\phi \in L^2[0,2\pi] \mid \phi, \frac{d\phi}{dx} \text{ absolutely continuous}, \frac{d^2\phi}{dx^2} \in L^2[0,2\pi], \phi(0)=\phi(2\pi), \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(2\pi)\},$$

B and C are the identity operator, and $F = \cos(x)$.

We take an extra step and use a similarity transformation to put the problem in an *equivalent* form that is more familiar to us from multivariable linear systems theory. Let $\mathscr{F}$ be the transformation that takes the function $\phi(x) \in L^2[0,2\pi]$, $\phi(x) = \sum_{n\in\mathbb{Z}} \phi_n\, e^{jnx}$, to its Fourier series coefficients $\text{col}[\cdots,\phi_{-1},\phi_0,\phi_1,\cdots] \in \ell^2$. Then it is simple to show that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{F}$ have the following (bi-infinite) matrix representations

$$\mathcal{A} = \begin{bmatrix} \ddots & & \\ & -n^2+1 & \\ & & \ddots \end{bmatrix}, \quad \mathcal{B} = \mathcal{C} = \begin{bmatrix} \ddots & & \\ & 1 & \\ & & \ddots \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} \ddots & \ddots & & & \\ \ddots & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & \end{bmatrix}.$$

Since $\Sigma(A) = \{-n^2+1, n \in \mathbb{Z}\}$, then Assumption $(**)$ is satisfied. For any $s \notin \Sigma(A)$ we have $(s-\mathcal{A})^{-1} = \text{diag}\{\cdots,\frac{1}{s+n^2-1},\cdots\}$. Thus $\sum_{n\in\mathbb{N}} \sigma_n\big((s\mathcal{I}-\mathcal{A})^{-1}\big) = \sum_{n\in\mathbb{Z}} |\frac{1}{s+n^2-1}| < \infty$. Hence $(s\mathcal{I}-\mathcal{A})^{-1} \in \mathscr{B}_1(\ell^2)$, and Assumption $(*)$ is satisfied.

Notice that the open-loop system is unstable. Next we demonstrate that by plotting the eigenloci one can read off from this plot the stability of the closed-loop system for any value of $\gamma \in \mathbb{C}$.

$\lambda = 0, 0, 1$ are the eigenvalues of $\mathcal{A}$ inside $\mathfrak{D}$, hence $p_+ = 3$, and we need three counter-clockwise encirclements of $-1/\gamma$ to achieve closed-loop stability. As can be seen in Figure 3(b) and its blown-up version (c), one possible choice would be to take $-1/\gamma$ to be purely imaginary and $-0.2j \leq -1/\gamma \leq 0.2j$. Clearly such $-1/\gamma$ is encircled three times by the eigenloci.

VI. Conclusions

We develop an extension of the Argument Principle and the Nyquist Stability Criterion that is applicable to systems with infinitesimal generators that are unbounded operators with discrete spectrum and whose resolvent operator is trace-class. This theory can be used to verify the stability of spatially extended systems and those governed by partial differential equations, as demonstrated in an example.

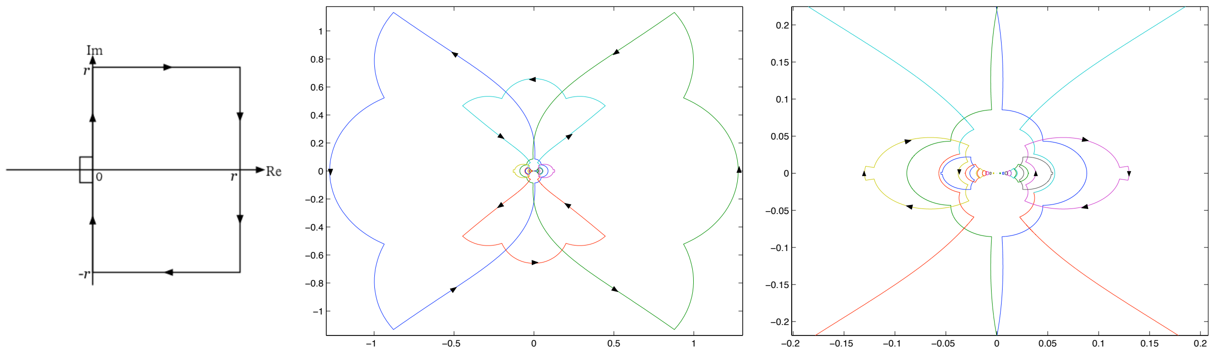

Fig. 3. The Nyquist path $\mathfrak{D}$; The Nyquist plot; Blown-up version of the center part of the Nyquist plot.

VII. APPENDIX

To prove Theorem 1 we need the following lemma.

Lemma A1: For $s \in \rho(\mathcal{A})$, $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}(s)]$ is analytic in both γ and s.

Proof: For $s \in \rho(\mathcal{A})$, $\gamma\mathcal{F}\mathcal{G}(s) \in \mathscr{B}_1(\mathfrak{X})$. Also $\gamma\mathcal{F}\mathcal{G}(s) = \gamma\mathcal{F}\mathcal{C}(s\mathcal{I} - \mathcal{A})^{-1}\mathcal{B}$ is clearly analytic in both γ and s for $s \in \rho(\mathcal{A})$. Then it follows from [9, p163] that $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}(s)]$ too is analytic in both γ and s for $s \in \rho(\mathcal{A})$. ∎

Proof of Theorem 1: Consider any point s in $\mathfrak{D}$. Since $\mathfrak{D}$ does not pass through any eigenvalues of $\mathcal{A}$, $s \in \rho(\mathcal{A})$ and thus $\gamma\mathcal{F}\mathcal{C}(s\mathcal{I} - \mathcal{A})^{-1}\mathcal{B} \in \mathscr{B}_1(\mathfrak{X})$ by Assumption $(*)$. Then from [10], $\big(\mathcal{I} + \gamma\mathcal{F}\mathcal{C}(s\mathcal{I} - \mathcal{A})^{-1}\mathcal{B}\big)^{-1}$ exists and belongs to $\mathscr{B}(\mathfrak{X})$ iff $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{C}(s\mathcal{I} - \mathcal{A})^{-1}\mathcal{B}] \neq 0$, which is satisfied by assumption. Applying an operator version of the matrix inversion lemma to $\big(\mathcal{I} + \gamma\mathcal{F}\mathcal{C}(s\mathcal{I} - \mathcal{A})^{-1}\mathcal{B}\big)^{-1}$, we conclude that $(s\mathcal{I} - \mathcal{A}^{\text{cl}})^{-1} = (s\mathcal{I} - \mathcal{A} + \mathcal{B}\gamma\mathcal{F}\mathcal{C})^{-1} \in \mathscr{B}(\mathfrak{X})$ and thus $s \in \rho(\mathcal{A}^{\text{cl}})$. Therefore $\mathfrak{D}$ is contained inside $\rho(\mathcal{A}) \cap \rho(\mathcal{A}^{\text{cl}})$.

Let the path $\mathfrak{C}$ be that traversed by $\det[\mathcal{I}+\gamma\mathcal{F}\mathcal{G}(s)]$ as s travels once around $\mathfrak{D}$. By Lemma A1, $\det[\mathcal{I}+\gamma\mathcal{F}\mathcal{G}(s)]$ is analytic in s, and if $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}(s)] \neq 0$ on $\mathfrak{D}$ we have

$$\begin{aligned} C\Big(0; \det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}(s)]\big|_{s\in\mathfrak{D}}\Big) &= \frac{1}{2\pi j}\int_{\mathfrak{C}} \frac{dz}{z} \\ &= \frac{1}{2\pi j}\int_{\mathfrak{D}} \frac{\frac{d}{ds}\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}(s)]}{\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}(s)]}\, ds \\ &= \frac{1}{2\pi j}\int_{\mathfrak{D}} \frac{d}{ds}\ln \Delta_{\mathcal{A}^{\text{cl}}/\mathcal{A}}(s)\, ds \\ &= \frac{1}{2\pi j}\int_{\mathfrak{D}} \operatorname{tr}[(s\mathcal{I} - \mathcal{A}^{\text{cl}})^{-1} - (s\mathcal{I} - \mathcal{A})^{-1}]\, ds, \end{aligned} \tag{A1}$$

where we have used (4) in the last equality. Notice that because

$$(s\mathcal{I} - \mathcal{A}^{\text{cl}})^{-1} - (s\mathcal{I} - \mathcal{A})^{-1} \;=\; -\gamma(s\mathcal{I} - \mathcal{A})^{-1}\mathcal{B}\mathcal{F}\mathcal{C}(s\mathcal{I} - \mathcal{A}^{\text{cl}})^{-1} \;\in\; \mathscr{B}_1(\mathfrak{X}) \tag{A2}$$

for all $s \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}^{\text{cl}})$, the right-hand side of (A1) makes sense and is finite. (A2) also gives that $(s\mathcal{I} - \mathcal{A}^{\text{cl}})^{-1} \in \mathscr{B}_1(\mathfrak{X})$.

On the other hand, since $(s\mathcal{I} - \mathcal{A})^{-1}$ and $(s\mathcal{I} - \mathcal{A}^{\text{cl}})^{-1}$ both belong to $\mathscr{B}_1(\mathfrak{X}) \subset \mathscr{B}_\infty(\mathfrak{X})$, their spectra consist entirely of isolated eigenvalues with no finite accumulation point [13, p187]. By Assumption $(**)$ the real part of the essential spectrum of $\mathcal{A}$ is $-\infty$. Now from (A2) we have that $(s\mathcal{I} - \mathcal{A}^{\text{cl}})^{-1} - (s\mathcal{I} - \mathcal{A})^{-1} \in \mathscr{B}_\infty(\mathfrak{X})$ and thus $\mathcal{A}^{\text{cl}}$ has the same essential spectrum as $\mathcal{A}$ [13, p244]. Therefore the path $\mathfrak{D}$ encloses a finite number of the eigenvalues of $\mathcal{A}$ and $\mathcal{A}^{\text{cl}}$. Thus in

$$\frac{1}{2\pi j}\int_{\mathfrak{D}} (s\mathcal{I} - \mathcal{A}^{\text{cl}})^{-1} ds \;-\; \frac{1}{2\pi j}\int_{\mathfrak{D}} (s\mathcal{I} - \mathcal{A})^{-1} ds$$

each term is a finite-dimensional projection [9, p11, p15]. Taking the trace, from [13] it follows that

$$\mathrm{tr}\left[\frac{1}{2\pi j}\int_{\mathfrak{D}}(s\mathcal{I}-\mathcal{A}^{\mathrm{cl}})^{-1}ds\right] \;-\; \mathrm{tr}\left[\frac{1}{2\pi j}\int_{\mathfrak{D}}(s\mathcal{I}-\mathcal{A})^{-1}ds\right] \tag{A3}$$

is equal to the number of eigenvalues of $\mathcal{A}$ in $\mathfrak{D}$ minus the number of eigenvalues of $\mathcal{A}^{\mathrm{cl}}$ in $\mathfrak{D}$, where $\mathfrak{D}$ is the (clockwise) Nyquist path and is taken arbitrarily large to enclose $\mathbb{C}^+$. Finally (A3) and (A1) together give the required result. ■

Proof of Lemma 3: For $s \in \mathfrak{D} \subset \rho(\mathcal{A})$, $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}(s)]$ is analytic in both γ and s by Lemma A1. The proof now proceeds exactly as in [4, p140] and is omitted. ■

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