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TRAVELING WAVES ON TWO- AND THREE-DIMENSIONAL PERIODIC ARRAYS OF LOSSLESS ACOUSTIC MONOPOLES, ELECTRIC DIPOLES, AND MAGNETODIELECTRIC SPHERES

Robert A. Shore
Arthur D. Yaghjian

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AIR FORCE RESEARCH LABORATORY
Sensors Directorate
Electromagnetics Technology Division
80 Scott Drive
Hanscom AFB MA 01731-2909

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//signature//
ROBERT A. SHORE
Program Manager

//signature//
DAVID D. CURTIS
Acting Chief, Antenna Technology Branch

//signature//
MICHAEL N. ALEXANDER
Technical Advisor
Electromagnetic Technology Division

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14. ABSTRACT
Traveling waves on 2D and 3D infinite periodic arrays of small lossless acoustic monopoles, electric dipoles, and magnetodielectric spheres are investigated. The waves are assumed to propagate along the axis of the arrays. The focus is on obtaining the kd–βd equations (diagrams) characterizing the traveling waves. Initial forms of the kd–βd equations are obtained by summing the fields scattered from all the array elements incident on a reference element of the array located at the origin of the coordinate system. These initial forms of the kd–βd equations are, however, useless for computational purposes since the summations converge extremely slowly. The core of the investigation is, therefore, to convert the slowly convergent summations to forms that can be used for efficient calculation of the kd–βd curves. These conversions are performed by first using either the Poisson summation formula or a method based on the use of Floquet mode expansions. Expressions for the efficient summation of Schlőmilch series are then used to obtain the final forms of the kd–βd equations. Computer programs have been written to obtain the kd–βd diagrams for all the arrays treated, and extensive numerical results are presented and discussed. For closely spaced 3D arrays of electric dipoles perpendicular to the array axis, and for closely spaced 3D arrays of magnetodielectric spheres, it is possible to regard the arrays as media characterized by a bulk or effective permittivity and permeability. Expressions have been obtained for these bulk parameters utilizing quantities readily available from our solutions of the kd–βd equations. These expressions are more accurate than the Clausius-Mossotti bulk parameter expressions. Exact computable expressions for the fields of 3D acoustic monopole, electric dipole, and magnetodielectric sphere arrays that are finite in the direction of the array axis, illuminated by a plane wave parallel to the array axis, are obtained from the analyses performed to obtain the kd–βd curves for the corresponding infinite arrays.

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1 INTRODUCTION

The subject of this report is traveling waves on two-dimensional (2D) and three-dimensional (3D) periodic arrays of lossless scatterers. Our investigation of these arrays is motivated in part by the recent theoretical demonstration by Holloway *et al.* that a doubly negative (DNG) material (a material with negative permittivity and permeability) can be formed by embedding an array of spherical particles in a background matrix [1]. The work of Holloway *et al.* is based on mixing formulas, obtained by Lewin [2], related to the Clausius-Mossotti mixing formulas. In contrast, our work, which corroborates the findings of Holloway *et al.*, is based on an analysis of the k–β equations for traveling waves on periodic arrays and has the advantages of not only yielding the k–β diagrams for all the arrays studied, but also yielding expressions for the effective (bulk) permittivity and permeability of 3D arrays that are more accurate than the Clausius-Mossotti type formulas over a larger range of separation of the array elements. The work described here builds on and extends our earlier investigations of traveling waves on linear [one-dimensional (1D)] periodic arrays of acoustic monopoles [3], electric dipoles [4], and magnetodielectric spheres [5]-[7], using a spherical-wave source scattering-matrix formulation. Although conceptually our treatment of traveling waves on 2D and 3D arrays is identical with our treatment of traveling waves on linear arrays, mathematically it is considerably more complicated because of the necessity of converting to rapidly convergent forms the double or triple summations that play a central role in the analysis.

The class of problems we consider can be described as follows. We have a periodic array of identical elements each characterized by a scattering coefficient that relates the field scattered from the element to the field incident on the element. As in our previous related work, it is assumed that only the fields of the lowest order spherical multipoles (acoustic monopoles, electromagnetic dipoles) are significant in analyzing scattering from the array elements. In the case of a 2D array, the array can be thought of as a linear array whose "elements" are equispaced columns of elements normal to the array axis, and in the case of a 3D array it can be helpful to regard the array as a linear array whose "elements" are equispaced planes of elements normal to the array axis. The spacing of elements in the direction parallel to the array axis is denoted by d and in the direction or directions normal to the array axis by h. Our interest is in lossless traveling waves (with real propagation constants β) that can be supported by the array in the direction parallel to the array axis.[1] The focus of our attention is the so-called k–β equation (or diagram) — in our work more properly referred to as the kd–βd equation (or diagram) — that relates the traveling wave electrical (or acoustical) separation distance βd of the array elements in the direction parallel to the array axis, to the corresponding free-space electrical (or acoustical) separation distance kd, where $k = \omega/c$ is the free-space wavenumber with $\omega > 0$ the angular frequency and c the free-space speed of light. In this report we do not treat traveling waves in directions other than parallel to the array axis. Although this report centers on traveling waves supported by infinite periodic arrays, for closely-spaced 3D arrays of electric dipoles perpendicular to the array axis and for closely spaced 3D arrays of magnetodielectric spheres, the solution of the kd–βd equation

[1] Alù and Engheta [8] have recently used analytic continuation arguments to determine the complex propagation constants for attenuated traveling waves (leaky and absorptive) on 1D infinite arrays of scattering elements.

can be used to obtain an effective permittivity and permeability of the array (see the end of Section 5 and the second subsection of Section 9), which in turn can be used as the basis for an approximate treatment of the exciting of traveling waves in partially finite 3D arrays (arrays that are finite in the direction of the array axis and infinite in the directions transverse to the array axis). Additionally, the analyses we have performed to obtain the kd–βd equations for infinite periodic 3D arrays of acoustic monopoles, electric dipoles perpendicular to the array axis, and magnetodielectric spheres with the electric and magnetic dipoles oriented perpendicular to the array axis, can be used to obtain exact computable expressions for the fields of partially finite periodic arrays of these elements when the arrays are illuminated by a plane wave propagating in a direction parallel to the array axis (see Section 11); that is, with the propagation vector of the plane wave normal to the interface between the array and free space.

Some basic properties of the kd–βd diagram may be noted here. In [9, ch. 7] it is shown that the dependence of the kd–βd diagram on βd is periodic in βd with a period of 2π. In Appendix A it is proven that if a periodic array of reciprocal elements supports a traveling wave with propagation constant β it also supports a corresponding traveling wave with propagation constant $-\beta$. Therefore, for periodic arrays of reciprocal elements, as are all the arrays considered in this report, kd is an even function of βd. It follows that for βd in the interval $\pi < \beta d < 2\pi$

$$kd(\beta d) = kd(2\pi - \beta d), \quad \pi < \beta d < 2\pi \tag{1.1}$$

where we have written kd as a function of βd. Hence we need only consider βd in the interval

$$0 < \beta d \le \pi \, . \tag{1.2}$$

In [3] it was shown that for a general infinite linear periodic array of lossless passive electrically small scatterers

$$kd \le \beta d \tag{1.3}$$

which coupled with (1.2) gives

$$kd \;\le \beta d \;\le \pi \, . \tag{1.4}$$

Thus the wavelengths of the lossless traveling waves are equal to or less than the free-space wavelength, and the traveling waves are slow waves compared with the free space wave. The proof of (1.4) for linear (1D) arrays given in [3] is easily seen to be valid for traveling waves on 2D arrays as well.[2] However, (1.3) is not valid in general for traveling waves on 3D arrays of lossless scatterers, nor is it necessarily true that $kd \le \pi$. Nevertheless, for 3D arrays we can, without loss of generality, still limit our consideration of traveling waves to those for which $0 < \beta d \le \pi$. However, for traveling waves on 3D periodic arrays, kd can be greater than π and both fast and slow waves can be supported. It is worth noting that in the analysis of 2D

[2] The essence of the proof is as follows. A linear or 2D periodic array with separation d between adjacent elements (for a 2D array the elements are periodic line sources) supporting a lossless traveling wave in the direction of the array axis with real propagation constant β can be regarded as a phased array with a phase shift of βd between adjacent elements. If it is assumed that $\beta d < kd$ the array will radiate power into space in the direction $\theta = \cos^{-1}(\beta d / kd)$ measured from the array axis. From considerations of conservation of power this result is inconsistent with the assumption that the traveling wave is lossless.

arrays we give in this report, if it is assumed that $\beta d < kd$ then we are led to the conclusion that the scattering coefficients of the array elements cannot satisfy certain basic relations that have been shown from fundamental principles of reciprocity and power conservation in [3] and [4] to be necessarily obeyed by the scattering coefficients of small lossless scatterers. This contradiction thus serves as an alternate proof of (1.3) for the particular 2D arrays we consider.

Although there is no upper limit on the transverse inter-element spacing h for either 2D or 3D periodic arrays, the expressions we give for the rapidly convergent summations in the kd–βd equations are valid only for $kh < 2\pi$, that is, for h less than a wavelength. This restriction on the size of h is not an essential limitation of either the transverse element separation or of the analyses we perform. It is, rather, a matter of our not wanting to unnecessarily complicate the form of the rapidly convergent expressions we give by making them independent of the range of kh since in most practical applications the transverse element spacing can be expected to be less than a wavelength. As examples of how the range of kh can be extended, in Sections 2 and 3 dealing with 2D and 3D arrays of acoustic monopoles we derive rapidly convergent expressions for the range $2\pi < kh < 4\pi$. These examples can serve as models for a reader interested in extending the range of kh for the kd–βd equations of other arrays. Also of interest are the limiting values of the kd–βd equations as $kh \to 2\pi$ since some of the individual terms of the rapidly convergent expressions in the kd–βd equations are singular at $kh = 2\pi$. Closer analysis shows, however, that the singularities of the various terms in the kd–βd equation cancel one another and hence the kd–βd equations remain non-singular at $kh = 2\pi$.

For all the arrays considered in this report, an initial form of the kd–βd equation is obtained very simply by assuming a traveling wave excitation of the array and summing the acoustic, electric, or electromagnetic fields incident on a reference element from all the other elements of the array. (The field incident on a reference element from all the other elements of the array is called the "interaction field" in the literature [10, ch. 12]). This form of the kd–βd equation consists of summations of an infinite number of terms of the form $\exp(\mathrm{i}kr)/(kr)$, $\exp(\mathrm{i}kr)/(kr)^2$, or $\exp(\mathrm{i}kr)/(kr)^3$. With only one exception, none of these summations can be expressed in closed form. Furthermore, the summations converge so slowly as to make the initial forms of the kd–βd equations useless for calculation purposes. Accordingly, it is necessary to obtain rapidly convergent expressions for all the slowly convergent summations that are encountered.[3] To obtain rapidly convergent expressions we make use of the Poisson summation formula or two different methods based on the use of Floquet mode expansions, leading to a second form of the kd–βd equation that invariably includes series of the form $\sum_j a_j Z_n(jx)$ known as Schlömilch series [11, chap. XIX], [12, secs. 7.10.3, 7.15], where a_j can be a trigonometric function and Z_n is a Bessel function of order n. While the Schlömilch

[3]Some of the summations over the array elements in columns and planes transverse to the array axis encountered in the report are not absolutely convergent. For certain values of the parameter kh they may not even be conditionally convergent. However, these summations can always be made absolutely convergent by the stratagem of adding a small positive imaginary part to the free-space propagation constant k. Physically, this is equivalent to assuming that the contributions of array elements at great distances from the reference element attenuate to zero. The analyses can then be performed rigorously and the small imaginary part allowed to go to zero at the end. In this report we will not enact this stratagem explicitly in our analyses but will proceed formally as if all the summations we treat are absolutely convergent.

series involving modified Bessel functions K_n converge very rapidly because of the exponential decay of these functions, the Schlömilch series involving Hankel functions (or equivalently ordinary Bessel functions, J_n, and Neumann functions, Y_n) converge very slowly. Thus the second form of the kd–βd equation would leave us not all that much better off than we were with the initial form were it not for the fact that rapidly convergent expressions are available for all the Schlömilch series involving Hankel functions that we encounter. For ease of reference we have collected in Appendix B the rapidly convergent expressions of Schlömilch series that we use in the report. Additionally, in Appendix C we have assembled a number of Bessel function relations that we use frequently, and in Appendix D we have collected several miscellaneous summation formulas that we employ often.

The outline of the report is as follows. There are two main parts of the report. The first part, Sections 2 through 11, is devoted to the analysis, and the second part, Section 12, to the numerical results. At the expense of some duplication of material we have tried to make the analysis sections of the report more or less self-contained so that the reader can skip to whatever sections are of particular interest. In Sections 2 and 3 we derive the kd–βd equations for 2D and 3D arrays of acoustic monopoles, respectively, and present two methods, the Poisson summation method and the Floquet mode method, used for converting slowly convergent summations to rapidly convergent forms. The results of these sections also serve as the basis for an alternate form of the Floquet mode method, using the Hertz vector potential to obtain the coefficients in the Floquet mode expansion, that plays an important role in the analysis sections of the report dealing with electric dipole and magnetodielectric sphere array elements.

In Sections 4 and 5 we derive the kd–βd equations for 2D and 3D periodic arrays of electric dipoles (short perfectly electrically conducting wires) oriented perpendicular to the array axis, while in Sections 6 and 7 we derive the kd–βd equations for 2D and 3D periodic arrays of electric dipoles oriented parallel to the array axis. Sections 8 and 9 are devoted to the analysis of traveling waves on 2D and 3D periodic arrays, respectively, of magnetodielectric spheres. Two distinct polarizations of the elements are treated in Sections 4 and 8, one polarization in which the electric dipoles are in the plane of the 2D array, and the other in which they are perpendicular to the array plane.

If the magnetodielectric sphere elements of a 3D array are sufficiently close together, the array can be regarded as a medium with an effective permittivity and permeability that determine the propagation characteristics of a traveling wave supported by the array. In the second subsection of Section 9 we show how the solution to the kd–βd equation for a traveling wave can be used to obtain the effective permittivity and permeability. We also describe a second method, based on the Clausius-Mossotti relation and independent of the kd–βd equation, for obtaining the effective permittivity and permeability.

In Section 10 we consider 2D and 3D magnetodielectric sphere arrays with electric or magnetic dipoles oriented parallel to the array axis, and show that the treatment of these arrays is identical to that of the 2D and 3D arrays of electric dipoles oriented parallel to the array axis considered in Sections 6 and 7.

In Section 11 we show that the analyses we have performed to obtain the kd–βd equations for infinite periodic 3D arrays of acoustic monopoles, electric dipoles perpendicular to the array axis, and magnetodielectric spheres with the electric and magnetic dipoles oriented perpendicular to the array axis, can be used to obtain expressions for the fields of partially finite periodic arrays of these elements (arrays that are finite in the direction of the array axis and are of infinite extent in the directions transverse to the array axis), when the arrays are illuminated by a plane wave propagating in a direction parallel to the array axis; that is, with the propagation vector of the plane wave normal to the interface between the array and free space.

The Numerical Results Section 12 is devoted to presenting and discussing numerically computed kd–βd diagrams for 2D and 3D acoustic monopole arrays, 2D and 3D periodic arrays of short electric dipoles, and 2D and 3D periodic arrays of magnetodielectric spheres. For the sake of comparison, we also compute the corresponding kd–βd diagrams for 1D periodic arrays of some scatterers from their transcendental equations given in previous reports [4], [5]. Once the kd–βd diagram is found for a 3D infinite periodic array, we use the formulas derived in Section 9.2 (referred to herein as the Shore-Yaghjian formulas) for determining the effective (bulk) permittivity and permeability of the array from the parameters in the transcendental equation. In addition, these bulk parameters are also determined from the Clausius-Mossotti relations, which, in general, are not as accurate as those determined from the Shore-Yaghjian formulas. In Subsection 12.1 we show plots of the family of kd–βd curves determined by different values of the phase ψ of the scattering coefficient for 2D and 3D arrays of acoustic monopoles and electric (magnetic) dipoles. (The 1D, 2D, and 3D family of kd–βd curves for magnetic dipoles are identical to those for electric dipoles.) In Subsections 12.2–12.5, kd–βd diagrams and effective permittivity and permeability curves (for 3D arrays) are given for representative scatterers, namely, for short perfectly electrically conducting (PEC) wires, for PEC spheres, for diamond spheres, for silver nanospheres, and for magnetodielectric spheres.

In Appendix A it is proved that a reciprocal (lossy or lossless) waveguide (uniform or periodic) that supports a traveling wave with propagation constant β also supports a corresponding traveling wave with propagation constant $-\beta$; that is, all reciprocal waveguides are bidirectional. In Appendix B we give rapidly convergent expressions for all the Schlömilch series encountered in the analysis part of the report. In Appendix C we list Bessel function relations that we use frequently, and in Appendix D we list important summation formulas. For ease of reference, in Appendix E we list all the rapidly convergent forms of the kd–βd equations derived in Sections 2 to 9.

Although, apart from Section 10, we do not explicitly consider periodic arrays of magnetic dipoles, it should be noted that the kd–βd equations for 2D and 3D arrays of magnetic dipoles are identical with the kd–βd equations obtained for 2D and 3D arrays of electric dipoles. Thus, for example, the kd–βd equation for a 2D or 3D array of magnetodielectric spheres in a frequency range where the magnetic dipole scattering coefficient is much larger than the electric dipole coefficient, can be obtained very accurately by solving the transcendental equation for the corresponding array of electric dipoles oriented in the direction of the magnetodielectric sphere magnetic dipoles, but with the phase of the normalized (see Footnote 6 on page 90) Mie magnetic dipole scattering coefficient replacing the phase of the electric dipole scattering coefficient in the transcendental equation.

Although in Sections 8, 9, and 10 we refer to the array elements as "magnetodielectric spheres", in fact the analyses performed are equally applicable to any array elements that can be modeled by a pair of crossed electric and magnetic dipoles perpendicular to the array axis. The electric and magnetic dipoles of the individual array elements are assumed to be uncoupled (as indeed they are for spheres whose permittivity and permeability are radially symmetric) so that an incident electric field at the element center in the direction of the electric dipole excites only the electric dipole field, and an incident magnetic field at the element center in the direction of the magnetic dipole excites only the magnetic dipole field. The electric and magnetic dipoles of different array elements are coupled, however, because the field scattered from an electric dipole has a component of the magnetic field parallel to the magnetic dipoles of the array elements, and the field scattered from a magnetic dipole has a component of the electric field parallel to the electric dipoles of the array elements.[4]

Since Tretyakov and his co-workers have also devoted considerable effort to studying traveling waves on periodic structures [13]-[16], it is important to point out similarities and differences between their work and ours. Similarly to us, Tretyakov *et al.* focus on the kd–βd equation (which they call the eigenvalue equation) supported by periodic arrays of scatterers, and like us they assume that scattering from the array elements is adequately described by considering the elements to be dipoles. The first step for both of us in obtaining the kd–βd equation is to assume a traveling wave excitation of the array. Whereas we obtain our initial form of the kd–βd equation from the expression for the field incident on a reference array element from all the other elements of the array (the interaction field), Tretyakov *et al.* obtain their eigenvalue equation from the expression for the dipole moment induced in the reference dipole by the interaction field. Another significant difference between their work and ours is how the interaction field is evaluated. To obtain the interaction field from a plane (normal to the array axis) of array elements, Tretyakov *et al.* use an approximation technique in which they divide the plane into a small circular region ("hole") of radius R_0 centered on the array axis, and the region outside this circle. The contributions of dipoles inside the circular region to the interaction field are considered individually, while the contributions of the dipoles outside the circular region are obtained by replacing the dipoles by a homogeneous polarization sheet with an average dipole moment per unit area from which an equivalent averaged current density is obtained by multiplication by $-\mathrm{i}\omega$. The contribution to the interaction field of this equivalent current sheet is then obtained from the standard integral expression for the electric field radiated by a current distribution. For transverse element separations $kh < 1$, Tretyakov *et al.* take $R_0 = h/1.438$, a value obtained from static considerations, so that only the contributions of the dipoles on the array axis are considered individually, the remainder of the array dipoles being treated as equivalent current sheets. In contrast with this approximate method, our approach is to obtain the interaction field by rigorously evaluating all the summations using either the Poisson summation formula or a Floquet mode expansion method, the results of which are then combined with Schlömilch series expressions to convert the initial numerically intractable summations to rapidly convergent forms. Our resulting kd–βd equation is valid for transverse element separations $kh < 2\pi$ rather than $kh < 1$ for the approximation method used by Tretyakov

[4]The only restrictive assumption for spherical scatterers (with radially symmetric permittivity and permeability) are that they are either small enough, or the frequency is such, that all scattered multipoles of higher order than dipoles are negligible. Then with respect to the center of each sphere, orthogonality relations demand that only the incident Bessel-function dipolar fields couple to the scattered Hankel-function scattered fields. Moreover, only the incident Bessel-function electric (magnetic) dipole has a non-zero electric (magnetic) field at the center of each sphere. The fields of all higher-order Bessel-function multipoles are zero at the center of each sphere.

et al., and can, as noted above, be extended if desired for any transverse element separations. Several important advantages also accrue to our approach for obtaining the kd–βd equation centering on the expression for the field incident on the reference element rather than on the polarization induced in the reference element. One important advantage is that the frequency dependence for all frequencies of the scattering coefficients of array sphere elements is incorporated in the Mie scattering coefficients, whereas Tretyakov *et al.* use an approximate expression for the polarizability which is valid for $ka << 1$ where a is the sphere radius. A second advantage of focusing on fields rather than polarization is that it facilitates consideration of coupling between electric and magnetic dipoles. As we have noted, in periodic arrays of magnetodielectric spheres the electric and magnetic dipoles are coupled because the field scattered from an electric (magnetic) dipole has a component of the magnetic (electric) field parallel to the magnetic (electric) dipoles of the array elements. This coupling is fully taken into account in our framework and analysis but is neglected by Tretyakov *et al.* Finally, the scattering matrix framework that underlies our work makes it easy to generalize our approach to scattered modes higher than dipole modes, whereas the polarization-centered approach of Tretyakov *et al.* would require intensive reworking to incorporate higher order modes.

2 2D ACOUSTIC MONOPOLE ARRAYS

In this section we investigate traveling waves on 2D periodic arrays of isotropic acoustic monopoles. The major steps of the procedure we will use here — calculating the field at a reference element due to all the other elements in the array, deriving the kd–βd equation by assuming a traveling wave excitation of the array, and converting slowly convergent summations to rapidly convergent ones to obtain a form of the kd–βd equation suitable for calculation purposes — are the same ones that we will use for all the different arrays considered in the report. The z axis of a Cartesian coordinate system is taken here to be the array axis and equispaced columns of acoustic monopoles parallel to the x axis are located at $z = nd,\ n = 0,\ \pm 1,\ \pm 2,\ \cdots$. In each column the monopoles are located at $x = mh,\ m = 0,\ \pm 1,\ \pm 2,\ \cdots$. We assume an excitation of the array such that all monopoles in any column of the array are excited identically. The pressure, p_0^0 (the subscript 0 is used here and throughout the report to indicate an incident pressure or field), incident on the monopole at the location $x = 0, y = 0, z = 0$, from all the other monopoles in the array is then given by

$$p_0^0 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} b_n \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr_{mn}}}{kr_{mn}} + b_0 \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr_{m0}}}{kr_{m0}} \tag{2.1}$$

where r_{mn} is the distance from the $(m,n)^{th}$ monopole to the $(0,0)$ monopole[5],

$$r_{mn} = \sqrt{(mh)^2 + (nd)^2} = h\sqrt{m^2 + (nd/h)^2} \tag{2.2}$$

and $k = \omega/c = 2\pi/\lambda$ is the free-space acoustic propagation constant where λ is the acoustic wavelength, ω the frequency, $\omega > 0$, and c is the acoustic wave speed. An implicit harmonic

[5] As long as our analysis of 2D arrays remains in the plane of the array, we will often refer to points by their two relevant coordinates (here x and z) rather than their full three coordinates.

time dependence, $e^{-i\omega t}$, is assumed here and throughout the report. In the "self-column" (the column containing the reference (0,0) monopole), $n = 0$ and

$$r_{m0} = h|m| \ . \tag{2.3}$$

The constants b_n are related to the pressure incident on any monopole in the n^{th} column, p_0^n, by the equation

$$b_n = S p_0^n \tag{2.4}$$

where S is the scattering coefficient of an acoustic monopole.

We now assume that the array is excited by a traveling wave in the z direction with real (lossless) propagation constant β. Then the constants b_n in (2.1) are identical apart from a phase shift given by

$$b_n = b_0 \, e^{in\beta d} \tag{2.5}$$

and thus (2.1) becomes

$$p_0^0 = b_0 \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{in\beta d} \sum_{m=-\infty}^{\infty} \frac{e^{ikh\sqrt{m^2+(nd/h)^2}}}{kh\sqrt{m^2+(nd/h)^2}} + b_0 \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{e^{ikh|m|}}{kh|m|} \ . \tag{2.6}$$

Since

$$b_0 = S \, p_0^0 \tag{2.7}$$

we then have, multiplying through by kh,

$$kh = S \left\{ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{in\beta d} \sum_{m=-\infty}^{\infty} \frac{e^{ikh\sqrt{m^2+(nd/h)^2}}}{\sqrt{m^2+(nd/h)^2}} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{e^{ikh|m|}}{|m|} \right\} \ . \tag{2.8}$$

Equation (2.8) is the kd–βd equation that determines the normalized traveling wave propagation constant βd in terms of $kh, d/h$, and the acoustic monopole scattering coefficient S. It has been obtained very simply by substituting the traveling-wave phase shift (2.5) and the scattering equation (2.7) into the expression (2.1) for the field at the reference element due to all the other elements in the array. If the double summation over m and n and the self-column summation over m in (2.8) converged rapidly we could use (2.8) to calculate kd–βd diagrams for 2D acoustic monopole arrays and our work for these arrays would be done. Unfortunately, however, these summations converge very slowly and hence we must find ways of transforming these summations to rapidly convergent forms. Since, as will be seen below, the self-column sum can be directly evaluated in closed form, we focus on the double summation in (2.8).

One important method for converting slowly convergent summations to rapidly convergent forms, and by far the simplest when it can be used for the summations encountered in our analysis of traveling waves on periodic arrays, is the Poisson summation formula [17, pp. 315-318]. In its one-dimensional form it is given by

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{m=-\infty}^{\infty} \hat{f}(m) \tag{2.9}$$

where

$$\hat{f}(m) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-\mathrm{i}2\pi m x}\, \mathrm{d}x. \tag{2.10}$$

In (2.8) let

$$I(n) = \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+(nd/h)^2}}}{\sqrt{(m^2+(nd/h)^2}}\ , \quad n \neq 0\ . \tag{2.11}$$

In applying the Poisson summation formula

$$f(x) = \frac{\mathrm{e}^{\mathrm{i}kh\sqrt{x^2+(nd/h)^2}}}{\sqrt{x^2+nd/h)^2}}\ ,\ n \neq 0\ . \tag{2.12}$$

Then, making use of tabulated integrals [18, eqs. 3.876(1), 3.876(2)], we obtain

$$\hat{f}(m) = \left\{ \begin{array}{l} \mathrm{i}\pi H_0^{(1)}\left(|n|(d/h)\ \sqrt{(kh)^2-(2\pi m)^2}\right)\ ,\ kh > 2\pi|m|\ ,\ n \neq 0 \\ \\ 2K_0\left(|n|(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right)\ ,\ kh < 2\pi|m|\ ,\ n \neq 0 \end{array} \right\} \tag{2.13}$$

with

$$I(n) = \sum_{m=-\infty}^{\infty} \hat{f}(m) = \hat{f}(0) + 2\sum_{m=1}^{\infty} \hat{f}(m)\ . \tag{2.14}$$

In (2.13) $H_0^{(1)}$ and K_0 are the Hankel function of the first kind of order zero and the modified Bessel function of order zero, respectively, and we have made use of the relation (C.1) between $H_0^{(1)}$ and K_0. If $h/\lambda < 1$ then $kh < 2\pi$ so that

$$\hat{f}(m) = \left\{ \begin{array}{l} \mathrm{i}\pi H_0^{(1)}(|n|kd))\ ,\ m = 0\ ,\ n \neq 0 \\ \\ 2K_0\left(|n|(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right)\ ,\ |m| \geq 1\ ,\ n \neq 0 \end{array} \right\} \tag{2.15}$$

$$I(n) = \mathrm{i}\pi H_0^{(1)}(|n|kd) + 4\sum_{m=1}^{\infty} K_0\left(|n|(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right) \tag{2.16}$$

and (2.8) becomes

$$kh = S\left\{2\sum_{n=1}^{\infty}\cos(n\beta d)\left[\mathrm{i}\pi H_0^{(1)}(nkd) + 4\sum_{m=1}^{\infty} K_0\left(n(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right)\right] + 2\sum_{m=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}khm}}{m}\right\}\ . \tag{2.17}$$

In (2.17) the slowly convergent Schlömilch series

$$\sum_{n=1}^{\infty}\cos(n\beta d) H_0^{(1)}(nkd) = \sum_{n=1}^{\infty}\cos(n\beta d)\left[J_0(nkd) + \mathrm{i}\, Y_0(nkd)\right] \tag{2.18}$$

can be efficiently evaluated using (B.1a) and (B.2a). The series

$$\sum_{n=1}^{\infty} \cos(n\beta d) \sum_{m=1}^{\infty} K_0 \left(n(d/h) \sqrt{(2\pi m)^2 - (kh)^2} \right) \tag{2.19}$$

converges extremely rapidly because of the exponential decay of K_0. For example, for $n = m = 2$, $d/h > 0.5$, and $0 < kh < 2\pi$, $K_0(n(d/h) \sqrt{(2\pi m)^2 - (kh)^2}) < 7 \times 10^{-6}$. The series can thus be truncated keeping only a very small number of terms. The self-column sum in (2.17) can be evaluated in closed form using (D.1).

In obtaining a convenient form of the kd–βd equation (2.17) to be used for calculations, it is helpful to write (2.17) as

$$kh = S\{\Re + \mathrm{i}\Im\} \tag{2.20}$$

where $\Re$, the real part of the expression within the brackets of (2.17), is given by

$$\Re = -2\pi \sum_{n=1}^{\infty} \cos(n\beta d) Y_0(nkd) + 8 \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{m=1}^{\infty} K_0 \left(n(d/h) \sqrt{(2\pi m)^2 - (kh)^2} \right)$$
$$-2 \ln \left[2 \sin \left(\frac{kh}{2} \right) \right] \tag{2.21}$$

and $\Im$, the imaginary part of the expression within the brackets of (2.17), is given by

$$\Im = 2\pi \sum_{n=1}^{\infty} \cos(n\beta d) J_0(nkd) + \pi - kh \tag{2.22}$$

so that, with (B.1a),

$$\Im = -kh\ . \tag{2.23}$$

If we write the scattering coefficient S as

$$S = |S| \mathrm{e}^{\mathrm{i}\psi} \tag{2.24}$$

and then equate imaginary parts in (2.20) we obtain the relation

$$|S| = \sin \psi\ . \tag{2.25}$$

This relation was derived in [3] using reciprocity and power conservation relations, and has been shown here to also be a necessary condition for a 2D array of lossless acoustic monopoles to support a traveling wave. The derivation of (2.25) here thus serves as an important check on the correctness of our analysis. It is worth noting that if $\beta d < kd$ then, from (B.1b), $\sum \cos(n\beta d) J_0(nkd) \neq -1/2$ and hence $\Im \neq -kh$ so that (2.25) would not hold. This is not possible for an array of acoustic monopoles. Hence $\beta d > kd$. This is a particular instance of the general result (1.4) noted in the Introduction which holds for 2D arrays as well as for linear arrays. Substituting (2.25) in (2.20) and equating real parts we then obtain the form of the kd–βd equation that is used to calculate βd as a function of kh, d/h, and the phase ψ of the scattering coefficient,

$$kh \cos \psi - \Re \sin \psi = 0 \tag{2.26}$$

with $\Re$ given by (2.21), the Neumann function sum evaluated using (B.2), the modified Bessel function sum truncated in accordance with the remark following (2.19), and $kh < 2\pi$. Equation (2.26) is easily solved numerically for βd given values of kd, kh, and ψ, using, for example, a simple search procedure with secant algorithm refinement.

The key step in our analysis of traveling waves on 2D periodic arrays of lossless acoustic monopoles was the use of the Poisson summation formula, (2.9) and (2.10), to convert the slowly convergent double summation in (2.8) to a rapidly convergent form. The use of the Poisson summation formula depended here on it being possible to evaluate the Fourier transform of the function $f(x)$ given by (2.12). If an expression had not been available for this Fourier transform we would not have been able to use the Poisson summation formula. Although we were fortunate here to be able to find an expression for the desired Fourier transform, it will be seen when we come to the analysis of traveling waves on 2D and 3D periodic arrays of electric dipoles and magnetodielectric spheres that often the Fourier transforms necessary to apply the Poisson summation formula are not available. Hence it is extremely important to have methods other than the Poisson summation formula for converting the slowly convergent summations that we encounter in our analyses to rapidly convergent forms. Accordingly we will now present an alternate method which we will refer to as the Floquet mode method.

The Floquet mode method proceeds here as follows. We begin by letting $p^0(x,y,z)$ be the pressure radiated by the monopoles in the $n=0$ column at a general point in space (x,y,z), $\sqrt{y^2+z^2}>0$. Because of the rotational symmetry of the column of monopoles, this pressure is the same for all points for which $\rho=\sqrt{y^2+z^2}$ has the same value. Accordingly

$$p^0(x,y,z) = p^0(x,\rho) = b_0 \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}k\sqrt{(mh-x)^2+\rho^2}}}{k\sqrt{(mh-x)^2+\rho^2}},\ \rho>0\ . \tag{2.27}$$

Note that we have here moved out of the plane of the 2D monopole array, something that was not necessary in applying the Poisson summation formula. Then

$$p^0(0,|n|d) = b_0 \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+(nd/h)^2}}}{kh\sqrt{m^2+(nd/h)^2}},\ n\neq 0\ . \tag{2.28}$$

Hence the sum $I(n)$ in (2.11) that we are trying to convert to a rapidly convergent form is given by

$$I(n) = \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+(nd/h)^2}}}{\sqrt{m^2+(nd/h)^2}} = \frac{kh}{b_0}\, p^0(0,|n|d),\ n\neq 0\ . \tag{2.29}$$

Now $p^0(x,\rho)$ can be expressed in terms of cylindrical waves by [20, sec. 6.6], [3, Appendix]

$$p^0(x,\rho) = \int_{-\infty}^{\infty} B(k_x)\, H_0^{(1)}(k_\rho\rho)\, \mathrm{e}^{\mathrm{i}k_x x}\, \mathrm{d}k_x,\ k_\rho = \sqrt{k^2-k_x^2} \tag{2.30}$$

where k_ρ is positive real (positive imaginary) according as $k^2 > (<)\ k_x^2$. Because of the periodicity of the array in the x direction,

$$p^0(x+h,\rho) = p^0(x,\rho)\ . \tag{2.31}$$

It follows from taking the inverse Fourier transform of (2.30) inserted into (2.31) that

$$\mathrm{e}^{\mathrm{i}k_x h} = 1 \tag{2.32}$$

and hence

$$k_x h = 2\pi m, \ m = 0, \pm 1, \pm 2, \cdots \tag{2.33}$$

so that

$$p^0(x, \rho) = \sum_{m=-\infty}^{\infty} B_m \, H_0^{(1)}(k_m \rho) \, \mathrm{e}^{\mathrm{i}(2\pi/h)mx} \tag{2.34}$$

where

$$k_m = \sqrt{k^2 - (2\pi m/h)^2} \tag{2.35}$$

with k_m positive real (positive imaginary) according as $(kh)^2 > (<) \ (2\pi m)^2$. By inversion

$$B_m H_0^{(1)}(k_m \rho) = \frac{1}{h} \int\limits_{-h/2}^{h/2} p^0(x, \rho) \, \mathrm{e}^{-\mathrm{i}(2\pi/h)mx} \, \mathrm{d}x \ . \tag{2.36}$$

Then

$$I(n) = \frac{kh}{b_0} p^0(0, |n|d) = \frac{kh}{b_0} \sum_{m=-\infty}^{\infty} B_m \, H_0^{(1)}(k_m |n| d) \ . \tag{2.37}$$

The question now is how do we find the unknown coefficients B_m? We know from (2.36) and (2.27) that

$$B_m H_0^{(1)}(k_m \rho) = \frac{b_0}{kh} \sum_{m'=-\infty}^{\infty} \int\limits_{-h/2}^{h/2} \frac{\mathrm{e}^{\mathrm{i}k\sqrt{(m'h - x)^2 + \rho^2}}}{\sqrt{(m'h - x)^2 + \rho^2}} \, \mathrm{e}^{-\mathrm{i}(2\pi/h)mx} \, \mathrm{d}x \ . \tag{2.38}$$

Since B_m is independent of ρ, for $\rho << 1$ the LHS of (2.38) behaves as [see (C.4)]

$$B_m H_0^{(1)}(k_m \rho) \overset{\rho << 1}{\sim} \frac{2\mathrm{i}}{\pi} B_m \ln \rho \ . \tag{2.39}$$

Hence the RHS of (2.38) must also have a $\ln \rho$ singularity as $\rho \to 0$. By equating $(2\mathrm{i}/\pi) B_m$ with the coefficient of the $\ln \rho$ singularity of the RHS of (2.38) we can then obtain B_m. In investigating the singularity of the RHS of (2.38) as $\rho \to 0$ we note that we can ignore all terms in the summation over m' for which $m' \neq 0$ since these terms are not singular as $\rho \to 0$. For $m' = 0$ it is simple to show, using [18, eqs. 2.271(4), 2.272(3)], that the behavior of the RHS of (2.38) for $\rho << 1$ is given by

$$\frac{2b_0}{kh} \int\limits_0^{h/2} \frac{\mathrm{e}^{\mathrm{i}k\sqrt{\rho^2 + x^2}}}{\sqrt{\rho^2 + x^2}} \, \cos(2\pi/h)mx \, \mathrm{d}x \overset{\rho << 1}{\sim} -\frac{2b_0}{kh} \ln \rho \ . \tag{2.40}$$

But then, equating the coefficients of the $\ln \rho$ singularity of the LHS and the RHS of (2.38) as $\rho \to 0$ we obtain

$$\frac{2\mathrm{i}}{\pi} B_m = -\frac{2b_0}{kh} \tag{2.41}$$

or

$$B_m = \frac{\mathrm{i}\pi b_0}{kh} \ . \tag{2.42}$$

Substituting (2.42) in (2.37), assuming $0 < kh < 2\pi$, and using the relation (C.1) between $H_0^{(1)}$ and K_0 we then obtain

$$I(n) = \mathrm{i}\pi H_0^{(1)}(k|n|d) + 4\sum_{m=1}^{\infty} K_0\left(|n|(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2}\right) \tag{2.43}$$

which is identical to the result (2.16) obtained using the Poisson summation formula.

The rapidly convergent expressions (2.14)-(2.17), (D.1), and (2.21), we have derived are valid for adjacent element separations in the direction transverse to the array axis satisfying the condition $0 < kh < 2\pi$ (that is, for h less than a wavelength). The important point we wish to make now is that this condition is not an essential limitation of either the transverse element separation or of the analysis we have performed. The condition is, rather, a matter of our not wanting to complicate the form of the rapidly convergent expressions we give by making them independent of the range of kh in view of the fact that in most practical applications the tranverse element spacing is less than a wavelength. To demonstrate this, let us assume for now that

$$2\pi < \ kh < \ 4\pi \ . \tag{2.44}$$

If (2.44) holds then (2.15) becomes

$$\hat{f}(m) = \left\{ \begin{array}{l} \mathrm{i}\pi H_0^{(1)}(|n|kd)) \ , \ m = 0 \ , \ n \neq 0 \\ \\ \mathrm{i}\pi H_0^{(1)} \left(|n|(d/h)\ \sqrt{(kh)^2 - (2\pi m)^2}\right) \ , \ |m| = 1 \ , \ n \neq 0 \\ \\ 2K_0 \left(|n|(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2}\right) \ , \ |m| > 1 \ , \ n \neq 0 \end{array} \right\} \tag{2.45}$$

and (2.17) becomes

$$\begin{aligned} kh = S\Bigg\{ 2\sum_{n=1}^{\infty} \cos(n\beta d)\Bigg[& \mathrm{i}\pi H_0^{(1)}(nkd) + 2\mathrm{i}\pi H_0^{(1)}\left(n(d/h)\ \sqrt{(kh)^2 - (2\pi)^2}\right) \\ & + 4\ \sum_{m=2}^{\infty} K_0\left(n(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2}\right)\Bigg] + 2\sum_{m=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}khm}}{m}\Bigg\} \end{aligned} \tag{2.46}$$

for $2\pi < kh < 4\pi$. To evaluate the self-column sum $2\,\Sigma \exp(\mathrm{i}khm)/m$ in (2.46) when $2\pi < kh < 4\pi$ we let

$$kh = kh' + 2\pi,\ 0 < kh' < 2\pi \ . \tag{2.47}$$

Then

$$2\sum_{m=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}khm}}{m} = 2\sum_{m=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}(kh' + 2\pi)m}}{m} = 2\sum_{m=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh'm}}{m} \ . \tag{2.48}$$

The RHS of (D.1) then equals

$$
\begin{aligned}
&-2\ln\left[2\sin\left(\frac{kh-2\pi}{2}\right)\right]+\mathrm{i}[\pi-(kh-2\pi)] \\
&=-2\ln\left[-2\sin\left(\frac{kh}{2}\right)\right]+\mathrm{i}(3\pi-kh)\ . \qquad (2.49)
\end{aligned}
$$

Equations (2.21) and (2.22) then become

$$
\begin{aligned}
\Re = -2\pi\sum_{n=1}^{\infty}\cos(n\beta d)\left[Y_0(nkd)+2Y_0\left(n(d/h)\ \sqrt{(kh)^2-(2\pi)^2}\right)\right] \\
+\,8\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=2}^{\infty}K_0\left(n(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right)-\ 2\ln\left[-2\sin\left(\frac{kh}{2}\right)\right] \qquad (2.50)
\end{aligned}
$$

and

$$
\Im = 2\pi\sum_{n=1}^{\infty}\cos(n\beta d)\ \left[J_0(nkd)+2J_0\left(n(d/h)\ \sqrt{(kh)^2-(2\pi)^2}\right)\right]+(3\pi-kh) \qquad (2.51)
$$

for $2\pi <\ kh < 4\ \pi$. Now

$$
\sum_{n=1}^{\infty}\cos(n\beta d)J_0\left(n(d/h)\ \sqrt{(kh)^2-(2\pi)^2}\right)=\sum_{n=1}^{\infty}\cos(n\beta d)J_0\left(nkd\ \sqrt{1-(2\pi/kh)^2}\right) \qquad (2.52)
$$

so that

$$
\beta d\ \geq kd >\ kd\ \sqrt{1-(2\pi/kh)^2} \qquad (2.53)
$$

Hence, from (B.1) with $(d/h)\sqrt{(kh)^2-(2\pi)^2}$ substituted for kd,

$$
\sum_{n=1}^{\infty}\cos(n\beta d)J_0\left(n(d/h)\ \sqrt{(kh)^2-(2\pi)^2}\right)=-\frac{1}{2}\ . \qquad (2.54)
$$

But then, substituting (B.1) and (2.54) in (2.51), we obtain

$$
\Im = -2\pi\left(\frac{1}{2}+2\ \frac{1}{2}\right)+3\pi-kh=-kh \qquad (2.55)
$$

just as before in (2.23) so that (2.25), the necessary condition for a lossless traveling wave to be supported by the array, holds as it did when we assumed that $0 <\ kh <\ 2\pi$. In (2.50) the sum

$$
\sum_{n=1}^{\infty}\cos(n\beta d)Y_0\left(n(d/h)\ \sqrt{(kh)^2-(2\pi)^2}\right) \qquad (2.56)
$$

can be efficiently evaluated using (B.2) with $(d/h)\ \sqrt{(kh)^2-(2\pi)^2}$ substituted for kd. We have thus extended the analysis for the range $0 <\ kh <\ 2\pi$ to the range $2\pi <\ kh <\ 4\pi$. We could generalize this analysis for kh in the range $2\pi M <\ kh <\ 2\pi(M+1)$ for arbitrary positive integers M. Little is gained, however, by our attempting to give expressions for the rapidly convergent summations for arbitrary kh since, as we have said, the resulting expressions would be considerably more involved and, in practical applications, the range $0 < kh < 2\pi$ is the one with the most importance by far. If a computer program is written to solve the kd–βd equation using the expressions we give for $0 < kh < 2\pi$, it is a relatively simple matter to modify the program to allow for kh being in a different range. These remarks apply equally well to all the arrays treated in this report.

So far we have not mentioned the case when $kh = 2M\pi$, M a positive integer. Consider now as an example the kd–βd equation (2.26) with $\Re$ given by (2.21), and let kh approach 2π. For $m = 1$, $K_0\left(n(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2}\right)$ is singular as $kh \to 2\pi$. Using the Schlömilch series expression (B.4) we obtain

$$8\sum_{n=1}^{\infty}\cos(n\beta d)\ K_0\left(n(d/h)\ \sqrt{(2\pi)^2-(kh)^2}\right) \overset{kh\to 2\pi}{\sim} 4\gamma + 4\ln\frac{1}{\sqrt{4\pi}}\frac{d}{h} + 2\ln\epsilon + \frac{4\pi}{\beta d}$$
$$+\,4\pi\left[\sum_{l=1}^{\infty}\left(\frac{1}{(2l\pi-\beta d)^2}-\frac{1}{2l\pi}\right)+\sum_{l=1}^{\infty}\left(\frac{1}{(2l\pi+\beta d)^2}-\frac{1}{2l\pi}\right)\right] \tag{2.57}$$

where γ is the Euler constant and $\epsilon = 2\pi - kh,\ 0 < \epsilon << 1$. But

$$-\,2\ln\left[2\sin\left(\frac{kh}{2}\right)\right] = -2\ln 2 - \ 2\ln\sin\left(\frac{kh}{2}\right) = -2\ln 2 - 2\sin\left(\frac{2\pi-\epsilon}{2}\right) \overset{\epsilon\to 0}{\sim} -2\ln\epsilon \tag{2.58}$$

canceling the logarithmic singularity of (2.57). It follows from (2.21) that in the $kd - \beta d$ equation (2.26)

$$\lim_{kh\to 2\pi}\Re = -2\pi\sum_{n=1}^{\infty}\cos(n\beta d)Y_0(nkd) + 8\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=2}^{\infty}K_0\left(2\pi n(d/h)\ \sqrt{m^2-1}\right)$$
$$+\,4\gamma + 4\ln\frac{1}{\sqrt{4\pi}}\frac{d}{h} + \frac{4\pi}{\beta d} + \ 4\pi\left[\sum_{l=1}^{\infty}\left(\frac{1}{(2l\pi-\beta d)^2}-\frac{1}{2l\pi}\right)+\sum_{l=1}^{\infty}\left(\frac{1}{(2l\pi+\beta d)^2}-\frac{1}{2l\pi}\right)\right]. \tag{2.59}$$

3 3D ACOUSTIC MONOPOLE ARRAYS

In this section we investigate traveling waves supported by 3D periodic arrays of lossless acoustic monopoles. The procedure used closely follows that used in Section 2 for 2D acoustic monopole arrays. The z axis of a Cartesian coordinate system is taken to be the array axis and equispaced planes of acoustic monopoles parallel to the xy plane are located at $z = nd,\ n = 0,\ \pm1,\ \pm2\ \cdots$. In each plane the monopoles are located at $(x, y) = (mh, lh),\ m, l = 0,\ \pm1,\ \pm2,\ \cdots$. We assume an excitation of the array such that all monopoles in any plane of the array are excited identically. The pressure, p_0^0, incident on the monopole at the location $x = 0,\ y = 0,\ z = 0$ from all the other monopoles in the array is given by

$$p_0^0 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} b_n \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr_{mln}}}{kr_{mln}} + b_0 \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr_{ml0}}}{kr_{ml0}} \tag{3.1}$$

where r_{mln} is the distance from the $(m, l, n)^{th}$ monopole to the $(0, 0, 0)$ monopole,

$$r_{mln} = \sqrt{(m^2 + l^2)h^2 + (nd)^2} = h\sqrt{m^2 + l^2 + (nd/h)^2} \tag{3.2}$$

and $k = \omega/c = 2\pi/\lambda$ is the acoustic propagation constant, λ is the acoustic wavelength, and c is the acoustic wave speed. In the "self-plane" (the plane containing the reference (0,0,0) monopole), $n = 0$ and

$$r_{ml0} = h\sqrt{m^2 + l^2}\ . \tag{3.3}$$

The constants b_n are related to the pressure incident on any monopole in the n^{th} plane, p_0^n, by the equation

$$b_n = Sp_0^n \tag{3.4}$$

where S is the scattering coefficient of an acoustic monopole.

We now assume that the array is excited by a traveling wave in the z direction with real propagation constant β. Then the constants b_n in (3.1) are identical apart from a phase shift given by

$$b_n = b_0\, \mathrm{e}^{\mathrm{i}n\beta d} \tag{3.5}$$

and

$$p_0^0 = b_0 \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+l^2+(nd/h)^2}}}{kh\sqrt{m^2+l^2+(nd/h)^2}} + b_0 \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+l^2}}}{kh\sqrt{m^2+l^2}}\ . \tag{3.6}$$

Since

$$b_0 = Sp_0^0 \tag{3.7}$$

we then have, multiplying through by kh,

$$kh = S\left\{ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+l^2+(nd/h)^2}}}{\sqrt{m^2+l^2+(nd/h)^2}} + \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+l^2}}}{\sqrt{m^2+l^2}} \right\}\ . \tag{3.8}$$

Equation (3.8) is the kd–βd equation that determines the normalized traveling wave propagation constant βd in terms of kh, d/h, and the acoustic monopole scattering coefficient S. If the triple summation over n, m, and l, and the self-plane summation over m and l converged rapidly we could use (3.8) to calculate kd–βd diagrams for 3D acoustic monopole arrays. Unfortunately, however, like the corresponding summations in the 2D acoustic monopole case, the summations converge very slowly and hence must be transformed to rapidly convergent forms if (3.8) is to become practical for purposes of calculating βd as a function of kh, d/h, and S. As will be seen below [see (3.44)] the self-plane double summation over m

and l in (3.8) can be regarded as a special case of the quantity within the brackets in (2.8). Accordingly here we focus on the triple summation in (3.8).

As we did in Section 2 we will use two different methods for converting the slowly convergent summation to a rapidly convergent form, the Poisson summation formula method and the Floquet mode method. The simpler of these two methods when it can be used is the Poisson summation formula. In its two-dimensional form, it is given by [17, pp. 315-318]

$$\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f(m,l) = \sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \hat{f}(m,l) \tag{3.9}$$

where

$$\hat{f}(m,l) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, \mathrm{e}^{-\mathrm{i}2\pi(mx+ly)}\, \mathrm{d}x\mathrm{d}y\ . \tag{3.10}$$

In (3.8) let

$$I(n) = \sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+l^2+(nd/h)^2}}}{\sqrt{m^2+l^2+(nd/h)^2}},\quad n\neq 0\ . \tag{3.11}$$

In applying the Poisson summation formula

$$f(x,y) = \frac{\mathrm{e}^{\mathrm{i}kh\sqrt{x^2+y^2+(nd/h)^2}}}{\sqrt{x^2+y^2+(nd/h)^2}},\quad n\neq 0\ . \tag{3.12}$$

To calculate the double Fourier transform of $f(x,y)$ the integral over x and y is converted to an integral over the polar coordinates $\rho = \sqrt{x^2+y^2}$ and $\phi = \tan^{-1}(y/x)$ and the ϕ integration performed using (C.7) to obtain

$$\hat{f}(m,l) = 2\pi\int_0^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\sqrt{\rho^2+(nd/h)^2}}}{\sqrt{\rho^2+(nd/h)^2}} J_0(2\pi\sqrt{m^2+l^2}\,\rho)\,\rho\,\mathrm{d}\rho\ ,\ n\neq 0\ . \tag{3.13}$$

The ρ integration can then be performed making use of tabulated integrals [18, eqs. 6.737 (5,6)] yielding

$$\hat{f}(m,l) = \left\{\begin{array}{l} \mathrm{i}\sqrt{2}\pi^{3/2}\,\dfrac{\sqrt{|n|d/h}}{\sqrt{kh}}\, H^{(1)}_{-1/2}(|n|kd)\ ,\ (m,l)=(0,0)\ ,\ n\neq 0 \\[2ex] 2^{3/2}\sqrt{\pi}\,\dfrac{\sqrt{|n|d/h}}{[(2\pi)^2(m^2+l^2)-(kh)^2]^{1/4}}\, K_{1/2}\left(|n|(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}\right)\ , \\[2ex] \qquad\qquad\qquad\qquad\qquad\qquad (m,l)\neq(0,0)\ ,\ n\neq 0 \end{array}\right\} \tag{3.14}$$

assuming that $0 < kh < 2\pi$. Substituting explicit expressions for the spherical Bessel functions [18, eqs. 8.469 (3,6)]

$$H^{(1)}_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \mathrm{e}^{\mathrm{i}z}\ ,\quad K_{1/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} \mathrm{e}^{-z} \tag{3.15}$$

we then obtain

$$\hat{f}(m,l) = \left\{ \begin{array}{ll} \dfrac{2\pi \mathrm{i}}{kh}\, \mathrm{e}^{\mathrm{i}|n|kd} \;, \; (m,l) = (0,0) \;, \; n \neq 0 & \\ & \\ 2\pi \, \dfrac{\mathrm{e}^{-|n|(d/h)\, \sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}} \;, \; (m,l) \neq (0,0) \;, \; n \neq 0 & \end{array} \right\} \tag{3.16}$$

with

$$I(n) = \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \hat{f}(m,l) \tag{3.17}$$

and (3.8) becomes

$$kh = S \Bigg\{ 2 \sum_{n=1}^{\infty} \cos(n\beta d) \left[\frac{2\pi \mathrm{i}}{kh}\, \mathrm{e}^{\mathrm{i}nkd} + 2\pi \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \; \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty} \frac{\mathrm{e}^{-n(d/h)\, \sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}} \right]$$

$$+ \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \; \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+l^2}}}{\sqrt{m^2+l^2}} \Bigg\} \tag{3.18}$$

for $0 < kh < 2\pi$.

From (D.5)

$$2 \sum_{n=1}^{\infty} \cos(n\beta d)\, \frac{2\pi \mathrm{i}}{kh}\, \mathrm{e}^{\mathrm{i}nkd} = -\frac{2\pi}{kh}\, \frac{\sin kd}{\cos \beta d - \cos kd} - \mathrm{i}\frac{2\pi}{kh} \,. \tag{3.19}$$

The sum

$$\sum_{n=1}^{\infty} \cos(n\beta d) \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \; \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty} \frac{\mathrm{e}^{-n(d/h)\, \sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}} \tag{3.20}$$

converges very rapidly because of the negative exponential so that it is necessary to include only a few terms in the sum, for example n from 1 to 2 and m, l from -2 to 2, for sufficient accuracy. Alternately an approximation to the sum can be obtained by first performing the summation over n from 1 to ∞ in closed form using (D.4) and then including only terms in the summation over m and l from -1 to 1. When this is done we obtain

$$\sum_{n=1}^{\infty} \cos(n\beta d) \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \; \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty} \frac{\mathrm{e}^{-n(d/h)\, \sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}$$

$$\approx 4 \left(\frac{1}{r_1} \frac{\mathrm{e}^{-(d/h)r_1} \cos \beta d - \mathrm{e}^{-2(d/h)r_1}}{1 - 2\cos \beta d\, \mathrm{e}^{-(d/h)r_1} + \mathrm{e}^{-2(d/h)r_1}} + \frac{1}{r_2} \frac{\mathrm{e}^{-(d/h)r_2} \cos \beta d - \mathrm{e}^{-2(d/h)r_2}}{1 - 2\cos \beta d\, \mathrm{e}^{-(d/h)r_2} + \mathrm{e}^{-2(d/h)r_2}} \right) \tag{3.21}$$

where $r_1 = \sqrt{(2\pi)^2 - (kh)^2}$, and $r_2 = \sqrt{8\pi^2 - (kh)^2}$.

Before considering the self-plane summation in (3.8) or (3.18) we will demonstrate the use of the Floquet mode method to convert $I(n)$ in (3.11) to a rapidly convergent form. Following the Floquet mode treatment in Section 2 we begin by letting $p^0(x,y,z)$ be the pressure radiated by the monopoles in the $n=0$ plane at a general point in space (x,y,z), $z \neq 0$, so that

$$p^0(x,y,z) = b_0 \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr(m,l,x,y,z)}}{kr(m,l,x,y,z)} \tag{3.22}$$

where $r(m,l,x,y,z)$ is the distance from the $(m,l,0)$ monopole in the $n=0$ plane to the point (x,y,z),

$$r(m,l,x,y,z) = \sqrt{(mh-x)^2+(lh-y)^2+z^2}, \quad z \neq 0 \; . \tag{3.23}$$

Then

$$p^0(0,0,|n|d) = b_0 \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+l^2+(n(d/h))^2}}}{kh\sqrt{m^2+l^2+(nd/h)^2}}, \; n \neq 0 \; . \tag{3.24}$$

Hence the sum $I(n)$ in (3.11) that we are trying to convert to a rapidly convergent form is given by

$$I(n) = \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+l^2+(nd/h)^2}}}{\sqrt{m^2+l^2+(nd/h)^2}} = \frac{kh}{b_0}\, p^0(0,0,|n|d), \; n \neq 0 \; . \tag{3.25}$$

Now $p^0(x,y,|z|)$ can be expressed in terms of a plane-wave spectrum by

$$p^0(x,y,|z|) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(k_x,k_y)\, \mathrm{e}^{\mathrm{i}(k_x x + k_y y + k_z|z|)}\, \mathrm{d}k_x \mathrm{d}k_y, \; k_z = \sqrt{k^2-k_x^2-k_y^2} \tag{3.26}$$

where k_z is positive real (positive imaginary) according as $k^2 > (<)\; k_x^2+k_y^2$. Because of the periodicity of the array in the x and y directions,

$$p^0(x+h,y,|z|) = p^0(x,y,|z|) \; , \quad p^0(x,y+h,|z|) = p^0(x,y,|z|) \; . \tag{3.27}$$

It follows from taking the inverse transform of (3.26) inserted into (3.27) that

$$\mathrm{e}^{\mathrm{i}k_x h} = 1 \; , \quad \mathrm{e}^{\mathrm{i}k_y h} = 1 \tag{3.28}$$

and hence

$$k_x h = 2\pi m, \; m = 0, \pm 1, \pm 2, \cdots \; , \quad k_y h = 2\pi l, \; l = 0, \pm 1, \pm 2, \cdots \tag{3.29}$$

so that

$$p^0(x,y,|z|) = \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} B_{ml}\, \mathrm{e}^{\mathrm{i}(2\pi/h)(mx+ly)}\; \mathrm{e}^{\mathrm{i}k_{ml}|z|} \tag{3.30}$$

where

$$k_{ml} = \sqrt{k^2-(2\pi m/h)^2-(2\pi l/h)^2} \tag{3.31}$$

and k_{ml} is positive real (positive imaginary) according as $(kh)^2 > (<)\ (2\pi)^2(m^2+l^2)$. By inverting (3.30)

$$B_{ml}\,\mathrm{e}^{\mathrm{i}k_{ml}|z|} = \frac{1}{h^2}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} p^0(x,y,|z|)\,\mathrm{e}^{-\mathrm{i}(2\pi/h)(mx+ly)}\,\mathrm{d}x\mathrm{d}y\ . \tag{3.32}$$

Then

$$I(n) = \frac{kh}{b_0}p^0(0,0,|n|d) = \frac{kh}{b_0}\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} B_{ml}\,\mathrm{e}^{\mathrm{i}k_{ml}|n|d}\ . \tag{3.33}$$

It remains to find the unknown coefficients B_{ml}. Now from (3.32), (3.22), and (3.23)

$$B_{ml}\,\mathrm{e}^{\mathrm{i}k_{ml}|z|}$$

$$= \frac{b_0}{kh^2}\sum_{m'=-\infty}^{\infty}\sum_{l'=-\infty}^{\infty}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\frac{\mathrm{e}^{\mathrm{i}k\sqrt{(m'h-x)^2+(l'h-y)^2+z^2}}}{\sqrt{(m'h-x)^2+(l'h-y)^2+z^2}}\,\mathrm{e}^{-\mathrm{i}(2\pi/h)(mx+ly)}\,\mathrm{d}x\mathrm{d}y\ . \tag{3.34}$$

Since B_{ml} is independent of z, if the LHS of (3.34) is expanded for small $|z|$

$$B_{ml}\,\mathrm{e}^{\mathrm{i}k_{ml}|z|} \overset{|z|<<1}{\sim} B_{ml}(1+\mathrm{i}k_{ml}|z|)\ . \tag{3.35}$$

We can then obtain an expression for B_{ml} by investigating the behavior of the RHS of (3.34) for $|z|<<1$ and equating coefficients of $|z|$. [Note that this method for obtaining the coefficients of the Floquet mode expansion differs from that used in Section 2 in the 2D case where we obtained the unknown coefficients by equating coefficients of a $\ln\rho$ singularity for small ρ; see (2.38)-(2.42).]

Consider first the terms in the double summation for which $(m',l')\neq(0,0)$. Letting

$$A^2 = (m'h-x)^2+(l'h-y)^2 \tag{3.36}$$

and assuming that $z^2 << A^2$ we find that

$$\frac{\mathrm{e}^{\mathrm{i}k\sqrt{(m'h-x)^2+(l'h-y)^2+z^2}}}{\sqrt{(m'h-x)^2+(l'h-y)^2+z^2}} = \frac{\mathrm{e}^{\mathrm{i}k\sqrt{A^2+z^2}}}{\sqrt{A^2+z^2}} \approx \frac{\mathrm{e}^{\mathrm{i}kA}}{A}\left[1+\left(\frac{\mathrm{i}k}{2A}-\frac{1}{2A^2}z^2\right)+\cdots\right] \tag{3.37}$$

so that the terms in the double summation for which $(m',l')\neq(0,0)$ do not contain a term in $|z|$ for $|z|<<1$. Hence a term in $|z|$ must come from the $(m',l')=(0,0)$ term.

The $(m',l')=(0,0)$ term, converted to polar coordinates $\rho=\sqrt{x^2+y^2}$, $\phi=\tan^{-1}(y/x)$, is approximately

$$\frac{b_0}{kh^2}\int_0^{h/2}\int_0^{2\pi}\frac{\mathrm{e}^{\mathrm{i}k\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}}\,\mathrm{e}^{-\mathrm{i}(2\pi/h)(m\cos\phi+l\sin\phi)\rho}\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \tag{3.38}$$

We can obtain a term in $|z|$ for $|z| << 1$ only from the portion of the integral in the vicinity of $\rho = 0$. Expanding the trigonometric exponential, (3.38) is approximately equal to

$$\frac{b_0}{kh^2} \int_0^{h/2} \int_0^{2\pi} \frac{e^{ik\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}} \Big[1 - i\,(2\pi/h)(m\cos\phi + l\sin\phi)\rho$$

$$- \frac{1}{2}\,(2\pi/h)^2(m^2\cos^2\phi + l^2\sin^2\phi + 2ml\cos\phi\sin\phi)\rho^2 \Big]\ \rho\ d\rho\ d\phi \tag{3.39}$$

from which, performing the ϕ integration, we obtain

$$\frac{\pi b_0}{kh^2} \int_0^{h/2} \frac{e^{ik\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}} \left[2 - \frac{1}{2}\,(2\pi/h)^2\ \ (m^2+l^2)\rho^2 \right] \rho\ d\rho\ . \tag{3.40}$$

The ρ integration can be performed exactly by making the change of variables $u = \sqrt{\rho^2 + z^2}$. The integral is then evaluated at the lower end of the range of integration $u = |z|$ (there is no contribution to a term in $|z|$ from the upper end, $u = \sqrt{(h/2)^2 + z^2}$), and the exponential $\exp(ik|z|)$ expanded. Collecting terms in $|z|$ we then find that the RHS of (3.34) when $(m', l') = (0, 0)$ behaves as

$$-\frac{2\pi b_0}{kh^2}\ |z| \tag{3.41}$$

for $|z| << 1$. (It is found that there is no contribution to the term in $|z|$ from the trigonometric exponential.) But then, equating the coefficients of $|z|$ in (3.35) and (3.41),

$$B_{ml} = \frac{2\pi i b_0}{kh^2 k_{ml}}\ ,\ \ k_{ml} = \sqrt{k^2 - (2\pi m/h)^2 - (2\pi l/h)^2} \tag{3.42}$$

and hence from (3.33)

$$I(n) = 2\pi i \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{e^{i|n|(d/h)\ \sqrt{(kh)^2-(2\pi)^2(m^2+l^2)}}}{\sqrt{(kh)^2-(2\pi)^2(m^2+l^2)}}$$

$$= \frac{2\pi i}{kh} e^{i|n|kd} + 2\pi \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty} \frac{e^{-n(d/h)\ \sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}} \tag{3.43}$$

an expression identical to that obtained using the Poisson summation formula [see (3.16) and (3.17)]. Although the Floquet mode method is more complicated than the Poisson summation formula method, it does have the advantage here of yielding the same final result as the Poisson method without going through an intermediate spherical Bessel function representation, and of course does not depend on the availability of integrals to evaluate a Fourier transform.

Next we consider the self-plane sum in (3.8)

$$\sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty} \frac{\mathrm{e}^{ikh\sqrt{m^2+l^2}}}{\sqrt{m^2+l^2}} \tag{3.44}$$

which we can write as

$$\sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty} \frac{\mathrm{e}^{ikh\sqrt{m^2+l^2}}}{\sqrt{m^2+l^2}} = 2\sum_{l=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{ikh\sqrt{m^2+l^2}}}{\sqrt{m^2+l^2}} + 2\sum_{m=1}^{\infty}\frac{\mathrm{e}^{ikhm}}{m} \; . \tag{3.45}$$

Comparing (3.45) with (2.8) we see that the double sum over l and m in the RHS of (3.45) is identical with the double sum over n and m in (2.8) if n is replaced by l, βd is taken equal to 0, and $d = h$. Thus, referring to the treatment of the double sum over n and m in Section 2 in (2.9)-(2.17)

$$2\sum_{l=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{ikh\sqrt{m^2+l^2}}}{\sqrt{m^2+l^2}} = 2\sum_{l=1}^{\infty}\left[\mathrm{i}\pi H_0^{(1)}(lkh) + 4\sum_{m=1}^{\infty} K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\right)\right] \; . \tag{3.46}$$

The slowly convergent Schlömilch series

$$\sum_{l=1}^{\infty} H_0^{(1)}(lkh) \tag{3.47}$$

can be evaluated using (B.11) and (B.12). The series

$$\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\right) \tag{3.48}$$

converges extremely rapidly because of the exponential decay of K_0 (for example, for $l = m = 2$, and $kh < 2\pi$, $K_0(l\sqrt{(2\pi m)^2-(kh)^2}) < 9.5\times 10^{-11}$) so that only a few terms of the series need be included.

The self-column sum over m in (3.45) is identical with that in (2.8) and hence from (D.1)

$$2\sum_{m=1}^{\infty}\frac{\mathrm{e}^{ikhm}}{m} = -2\ln\left[2\sin\left(\frac{kh}{2}\right)\right] + \mathrm{i}(\pi - kh), \; 0 < kh < 2\pi. \tag{3.49}$$

As we did in Section 2 [see (2.20)], it is useful in obtaining a convenient form of the kd–βd equation (3.18) for calculation purposes to first write the equation as

$$kh = S\{\Re + \mathrm{i}\Im\} \tag{3.50}$$

where, from (3.18), (3.19), (3.46), and (3.49), $\Re$, the real part of the expression within the brackets of (3.18) is given by

$$\Re = -\frac{2\pi}{kh}\,\frac{\sin kd}{\cos\beta d - \cos kd} + 4\pi\sum_{n=1}^{\infty}\cos(n\beta d) \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty} \frac{\mathrm{e}^{-n(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}$$

$$- 2\pi \sum_{l=1}^{\infty} Y_0(lkh) + 8 \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} K_0\left(l\sqrt{(2\pi m)^2 - (kh)^2}\right) - 2\ln\left[2\sin\left(\frac{kh}{2}\right)\right] \tag{3.51}$$

and $\Im$, the imaginary part of the expression within the brackets of (3.18), is given by

$$\Im = -\frac{2\pi}{kh} + 2\pi \sum_{l=1}^{\infty} J_0(lkh) + \pi - kh = -kh \tag{3.52}$$

using (B.11). Equations (2.24)-(2.26) then hold here exactly as they did for the 2D array treated in Section 2. Once again we thus obtain as a useful check on the correctness of our analysis the relation $|S| = \sin\psi$ where ψ is the argument of the acoustic monopole scattering coefficient S, a relation derived independently in [3] from reciprocity and power conservation relations, and the final form of the kd–βd equation is again

$$kh\cos\psi - \Re\sin\psi = 0 \tag{3.53}$$

with $\Re$ here given by (3.51) and $kh < 2\pi$. Equation (3.53) is easily solved numerically for βd given values of kd, kh, and ψ, using, for example, a simple search procedure with secant algorithm refinement. In calculating $\Re$ the sum of the exponentials is either truncated in accordance with the remark following (3.20) or evaluated using (3.21), the Neumann function sum is evaluated using (B.12), and the modified Bessel function sum is truncated in accordance with the remark following (3.48).

The rapidly convergent expressions (3.18), (3.46), and (3.49) we have derived are valid for adjacent element separations in the directions normal to the array axis satisfying the condition $0 < kh < 2\pi$. As we showed in Section 2 [see (2.44)-(2.56)] this condition is not an essential limitation of either the transverse element separation or of the analysis we have performed. Rather it is a matter of our not wanting to complicate the rapidly convergent expressions we give by making them independent of the range of kh since in most practical applications the transverse element spacings can be expected to be less than a wavelength. To demonstrate this here, let us assume for now that

$$2\pi < kh < 4\pi \,. \tag{3.54}$$

If (3.54 holds then (3.16) becomes

$$\hat{f}(m,l) = \left\{ \begin{array}{ll} 2\pi\mathrm{i}\, \dfrac{\mathrm{e}^{\mathrm{i}|n|(d/h)\sqrt{(kh)^2-(2\pi)^2(m^2+l^2)}}}{\sqrt{(kh)^2-(2\pi)^2(m^2+l^2)}} \,, & (m,l) = (0,0),(\pm1,0),(0,\pm1) \,,\ n \neq 0 \\ \\ 2\pi\, \dfrac{\mathrm{e}^{-|n|(d/h)\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}} \,, & (m,l) \neq (0,0),(\pm1,0),(0,\pm1) \,,\ n \neq 0 \end{array} \right\} \tag{3.55}$$

and (3.18) becomes

$$kh = S\Bigg\{ 2\sum_{n=1}^{\infty} \cos(n\beta d)\Bigg[\frac{2\pi\mathrm{i}}{kh}\,\mathrm{e}^{\mathrm{i}nkd} + 8\pi\mathrm{i}\,\frac{\mathrm{e}^{\mathrm{i}n(d/h)\sqrt{(kh)^2-(2\pi)^2}}}{\sqrt{(kh)^2-(2\pi)^2}}$$

$$+\,2\pi \sum_{\substack{m=-\infty \\ |m|+|l|}}^{\infty} \sum_{\substack{l=-\infty \\ > 1}}^{\infty} \frac{\mathrm{e}^{-n(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}\Bigg] + \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \sum_{\substack{l=-\infty \\ \neq (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+l^2}}}{\sqrt{m^2+l^2}}\Bigg\}. \tag{3.56}$$

In (3.56), from (3.19)

$$2\sum_{n=1}^{\infty} \cos(n\beta d)\,\frac{2\pi \mathrm{i}}{kh}\,\mathrm{e}^{\mathrm{i}nkd} = -\frac{2\pi}{kh}\,\frac{\sin kd}{\cos\beta d - \cos kd} - \mathrm{i}\frac{2\pi}{kh} \tag{3.57}$$

and

$$2\sum_{n=1}^{\infty} \cos(n\beta d)\;8\pi \mathrm{i}\;\frac{\mathrm{e}^{\mathrm{i}n(d/h)\,\sqrt{(kh)^2-(2\pi)^2}}}{\sqrt{(kh)^2-(2\pi)^2}}$$

$$= -\frac{8\pi}{\sqrt{(kh)^2-(2\pi)^2}}\;\frac{\sin\left((d/h)\,\sqrt{(kh)^2-(2\pi)^2}\right)}{\cos\beta d - \cos\left((d/h)\,\sqrt{(kh)^2-(2\pi)^2}\right)} - \frac{8\pi \mathrm{i}}{\sqrt{(kh)^2-(2\pi)^2}}\,. \tag{3.58}$$

The self-plane double sum given for $kh < 2\pi$ by (3.46) now becomes

$$2\sum_{l=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+l^2}}}{\sqrt{m^2+l^2}} = \sum_{l=1}^{\infty}\Bigg[\mathrm{i}\pi H_0^{(1)}(lkh) + 2\mathrm{i}\pi H_0^{(1)}\left(l\sqrt{(kh)^2-(2\pi)^2}\right)$$

$$+\,4\sum_{m=2}^{\infty} K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\right)\Bigg] \tag{3.59}$$

and the self-column sum given by (3.49) for $kh < 2\pi$ becomes [see (2.49)]

$$2\sum_{m=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}khm}}{m} = -2\ln - \left[2\sin\left(\frac{kh}{2}\right)\right] + \mathrm{i}(3\pi - kh),\;\; 0 < kh < 2\pi. \tag{3.60}$$

From (3.56)-(3.60) we see that $\Im$, the imaginary part of the quantity within the brackets of (3.56), is

$$\Im = -\frac{2\pi}{kh} - \frac{8\pi}{\sqrt{(kh)^2-(2\pi)^2}} + 2\pi\sum_{l=1}^{\infty} J_0(lkh) + 4\pi\sum_{l=1}^{\infty} J_0\left(l\sqrt{(kh)^2-(2\pi)^2}\right)$$

$$+\,3\pi - kh$$

$$= -\frac{2\pi}{kh} - \frac{8\pi}{\sqrt{(kh)^2-(2\pi)^2}} + 2\pi\left[-\frac{1}{2} + \frac{1}{kh} + 2\,\frac{1}{\sqrt{(kh)^2-(2\pi)^2}}\right]$$

$$+\,4\pi\left[-\frac{1}{2} + \frac{1}{\sqrt{(kh)^2-(2\pi)^2}}\right] + 3\pi - kh = -kh \tag{3.61}$$

where we have made use of the Schlömilch series formula (B.11) and (B.11) with $\sqrt{(kh)^2-(2\pi)^2}$ substituted for kh to sum the Bessel function series. As noted above in connection with

(3.52), the equality $\Im = -kh$ serves as an important check on the correctness of our analysis because it implies (2.25), a necessary condition for a lossless traveling wave to be supported by the array. From (3.56)-(3.60) we find that $\Re$, the real part of the quantity within the brackets of (3.56), is

$$\Re = -\frac{2\pi}{kh}\,\frac{\sin kd}{\cos\beta d - \cos kd} - \frac{8\pi}{\sqrt{(kh)^2-(2\pi)^2}}\frac{\sin\left((d/h)\,\sqrt{(kh)^2-(2\pi)^2}\right)}{\cos\beta d - \cos\left((d/h)\,\sqrt{(kh)^2-(2\pi)^2}\right)}$$

$$+\, 4\pi\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{\substack{m=-\infty\\ |m|+|l|}}^{\infty}\ \sum_{\substack{l=-\infty\\ >\ 1}}^{\infty}\frac{\mathrm{e}^{-n(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}$$

$$-\, 2\pi\sum_{l=1}^{\infty}Y_0(lkh) - 4\pi\sum_{l=1}^{\infty}Y_0\left(l\sqrt{(kh)^2-(2\pi)^2}\right) + 8\sum_{l=1}^{\infty}\sum_{m=2}^{\infty}K_0(\left(l\sqrt{(2\pi)^2-(kh)^2}\right)$$

$$-\, 2\ln\left[-2\sin\left(\frac{kh}{2}\right)\right],\ 2\pi\ <\ kh\ <\ 4\pi\ . \qquad (3.62)$$

In calculating $\Re$ the sum of the exponentials and the sum of the modified Bessel functions converge very rapidly, and the Neumann function sums are evaluated very efficiently using (B.12), and (B.12) with $\sqrt{(kh)^2-(2\pi)^2}$ substituted for kh.

In concluding, let us investigate the limit of the kd–βd equation (3.53) as $kh \to 2\pi$ from below. The double sum of the modified Bessel functions in (3.51)

$$8\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\right) \qquad (3.63)$$

has two singularities for $m = 1$. Using the Schlömilch series summation formula (B.4) with $d/h = 1$ and $\beta d = 0$ we find that

$$8\sum_{l=1}^{\infty}K_0\left(l\sqrt{(2\pi)^2-(kh)^2}\right) \overset{kh\to 2\pi}{\sim} 4\gamma + 4\ln\frac{1}{\sqrt{4\pi}} + 2\ln\epsilon + 2\frac{\sqrt{\pi}}{\epsilon} \qquad (3.64)$$

where γ is the Euler constant and we have let $kh = 2\pi - \epsilon,\ 0 < \epsilon << 1$. The logarithmic singularity exactly cancels the logarithmic singularity of

$$-2\ln\left[2\sin\left(\frac{kh}{2}\right)\right] \qquad (3.65)$$

in (3.51) as $kh \to 2\pi$ [see (2.58)]. Next let us consider the singularity of

$$4\pi\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{\substack{m=-\infty\\ (m,l)}}^{\infty}\ \sum_{\substack{l=-\infty\\ \neq\ (0,0)}}^{\infty}\frac{\mathrm{e}^{-n(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}} \qquad (3.66)$$

in (3.51). The four terms of the double summation over m and l for which $(m,l) = (\pm1,0),(0,\pm1)$ are singular as $kh \to 2\pi$, each of these terms behaving as

$$\frac{1}{\sqrt{4\pi\epsilon}} \tag{3.67}$$

for $\epsilon = 2\pi - kh << 1$. Since from (D.5)

$$\sum_{n=1}^{\infty} \cos(n\beta d) = -\frac{1}{2} \tag{3.68}$$

$$4\pi \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \sum_{\substack{l=-\infty \\ \neq (0,0)}}^{\infty} \frac{\mathrm{e}^{-n(d/h)\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}$$

$$\overset{\epsilon\to 0}{\sim} 2\sum_{n=1}^{\infty} \cos(n\beta d) \sum_{\substack{m=-\infty \\ |m|+|l|}}^{\infty} \sum_{\substack{l=-\infty \\ > 1}}^{\infty} \frac{\mathrm{e}^{-2\pi n(d/h)\sqrt{m^2+l^2-1}}}{\sqrt{m^2+l^2-1}} - 4\frac{\sqrt{\pi}}{\sqrt{\epsilon}}\,. \tag{3.69}$$

The sum $\sum Y_0(lkh)$ in (3.51) can be evaluated using the Schlömilch series summation formula (B.12). We then find that

$$-2\pi\sum_{l=1}^{\infty} Y_0(lkh) \overset{kh\to 2\pi}{\sim} 2\gamma + 2\ln\frac{1}{2} - 2 + 2\sum_{l=2}^{\infty}\left(\frac{1}{\sqrt{l^2-1}} - \frac{1}{l}\right) + 2\frac{\sqrt{\pi}}{\sqrt{\epsilon}} \tag{3.70}$$

where we have again let $kh = 2\pi - \epsilon$. Thus the $1/\sqrt{\epsilon}$ singularities of the sum of the modified Bessel functions in (3.64), the sum of the exponentials in (3.69), and the sum of the Neumann functions in (3.70) cancel each other. It follows from (3.51) that in the kd–βd equation (3.53)

$$\begin{aligned}\lim_{kh\to 2\pi} \Re = &-\frac{\sin kd}{\cos\beta d - \cos kd} \\ &+ 2\sum_{n=1}^{\infty} \cos(n\beta d) \sum_{\substack{m=-\infty \\ |m|+|l|}}^{\infty} \sum_{\substack{l=-\infty \\ > 1}}^{\infty} \frac{\mathrm{e}^{-2\pi n(d/h)\sqrt{m^2+l^2-1}}}{\sqrt{m^2+l^2-1}} \\ &+ 4\ln\frac{1}{\sqrt{4\pi}} + 6\gamma + 2\ln\frac{1}{2} - 2 + 2\sum_{l=2}^{\infty}\left(\frac{1}{\sqrt{l^2-1}} - \frac{1}{l}\right)\,.\end{aligned} \tag{3.71}$$

From (3.21) it follows that

$$\begin{aligned}&2\sum_{n=1}^{\infty} \cos(n\beta d) \sum_{\substack{m=-\infty \\ |m|+|l|}}^{\infty} \sum_{\substack{l=-\infty \\ > 1}}^{\infty} \frac{\mathrm{e}^{-2\pi n(d/h)\sqrt{m^2+l^2-1}}}{\sqrt{m^2+l^2-1}} \\ &\approx \frac{4}{\pi}\frac{\mathrm{e}^{-2\pi d/h)}\cos\beta d - \mathrm{e}^{-4\pi(d/h)}}{1 - 2\cos\beta d\,\mathrm{e}^{-2\pi d/h} + \mathrm{e}^{-4\pi d/h}}\,.\end{aligned} \tag{3.72}$$

4 2D ELECTRIC DIPOLE ARRAYS, DIPOLES ORIENTED PERPENDICULAR TO THE ARRAY AXIS

In this section we consider traveling waves supported by 2D periodic arrays of lossless short electric dipoles oriented perpendicular to the array axis. The major steps of the procedure we will follow — calculating the field at a reference element due to all the other elements in the array, deriving the kd–βd equation by assuming a traveling wave excitation of the array, and converting slowly convergent summations to rapidly convergent ones to obtain a form of the kd–βd equation suitable for calculation purposes — are the same ones we have used in treating 2D and 3D periodic arrays of lossless acoustic monopoles in Sections 2 and 3, respectively. The details of the procedure are more complicated, however, because the field radiated by an electric dipole contains $\mathrm{e}^{\mathrm{i}kr}/(kr)^2$ and $\mathrm{e}^{\mathrm{i}kr}/(kr)^3$ terms in addition to an $\mathrm{e}^{\mathrm{i}kr}/(kr)$ term. There are two polarizations of the electric dipoles to be considered, one where the dipoles are in the array plane determined by the dipole centers and array axis, and the other where the dipoles are perpendicular to the array plane. These two polarizations are treated in Subsections 4.1 and 4.2, respectively. In 4.1, as will be seen below when we come to the kd–βd equation (4.15), the Poisson summation formula cannot be used to convert the slowly convergent summations to rapidly convergent ones because all the integrals needed to evaluate the transforms are not available. The Floquet mode method of Section 2 can be used, however, and in addition an alternate method for obtaining the coefficients in the Floquet mode expansion, based on the Hertz vector potential, will be introduced. In 4.2 the Poisson summation formula can be used to convert the slowly convergent summations to rapidly convergent ones. As a check we will also use the Floquet mode method to derive the rapidly convergent expressions.

4.1 ELECTRIC DIPOLES IN THE ARRAY PLANE

It is more convenient here to take the x axis, rather than the z axis, of a Cartesian coordinate system to be the array axis, with the electric dipoles oriented in the z direction, because the field of a small electric dipole is expressed most simply in a spherical polar coordinate system with the z axis aligned with the dipole direction. Equispaced columns of electric dipoles are located at $x = nd,\ n = 0,\ \pm 1,\ \pm 2,\ \cdots$. In each column the dipoles are centered at $z = mh,\ m = 0,\ \pm 1,\ \pm 2,\ \cdots$, so that the dipoles lie in the xz plane, the plane of the array. We assume an excitation of the array with the electric field parallel to the z axis and such that all the dipoles in any column of the array are excited identically. Let $\mathbf{E}_0^0$ be the electric field incident on the electric dipole at the location $x = 0, y = 0, z = 0$ from all the other dipoles in the array. As will be seen [see (4.9)] this field has a z component only. Let $\mathbf{E}_0^{0mn}$ be the electric field incident on the reference dipole from the electric dipole at the location $(z, x) = (mh, nd)$ so that

$$\mathbf{E}_0^0 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{m=-\infty}^{\infty} \mathbf{E}_0^{0mn} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \mathbf{E}_0^{0m0} \ . \tag{4.1}$$

From [4, eq. (40)]

$$\mathbf{E}_0^{0mn} = b_n \frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}} \left[\frac{-2\mathrm{i}}{kr_{mn0}} \left(1 + \frac{\mathrm{i}}{kr_{mn0}}\right) \cos\theta_{mn0}\, \hat{\mathbf{r}}_{mn0} \right.$$

$$\left. - \left(1 + \frac{\mathrm{i}}{kr_{mn0}} - \frac{1}{(kr_{mn0})^2}\right) \sin\theta_{mn0}\, \hat{\boldsymbol{\theta}}_{mn0} \right] . \tag{4.2}$$

The quantities in (4.2) are defined with reference to a local spherical polar coordinate system with origin at $(z, x) = (mh, nd)$ (in turn defined with reference to a local Cartesian coordinate system with the same origin whose axes are parallel to those of the global Cartesian coordinate system). The distance from the (m, n) dipole to the $(0, 0)$ dipole, r_{mn0}, is given by

$$r_{mn0} = \sqrt{(mh)^2 + (nd)^2} = h\sqrt{m^2 + (nd/h)^2} \tag{4.3}$$

and the unit vector in the direction from the (m, n) dipole to the $(0, 0)$ dipole, $\hat{\mathbf{r}}_{mn0}$, is

$$\hat{\mathbf{r}}_{mn0} = \mathbf{r}_{mn0}/r_{mn0}, \quad \mathbf{r}_{mn0} = -mh\, \hat{\mathbf{z}} - nd\, \hat{\mathbf{x}} \tag{4.4}$$

so that

$$\cos\theta_{mn0} = \hat{\mathbf{r}}_{mn0} \cdot \hat{\mathbf{z}} = -\frac{mh}{r_{mn0}} \tag{4.5}$$

$$\sin\theta_{mn0} = \sqrt{1 - \cos^2\theta_{mn0}} = \frac{|n|d}{r_{mn0}} \tag{4.6}$$

and

$$\hat{\boldsymbol{\theta}}_{mn0} = \pm\, \cos\theta_{mn0}\, \hat{\mathbf{x}} - \sin\theta_{mn0}\, \hat{\mathbf{z}} = \mp \frac{mh}{r_{mn0}}\, \hat{\mathbf{x}} - \frac{|nd|}{r_{mn0}}\, \hat{\mathbf{z}}\,, \quad n \lessgtr 0\,. \tag{4.7}$$

The free-space propagation constant $k = 2\pi/\lambda = \omega/c$ where λ is the wavelength, c is the speed of light, and ω is the frequency, $\omega > 0$. The corresponding quantities in the self-column summation of (4.1) are obtained from the quantities given by (4.3)-(4.7) by setting $n = 0$. The constants b_n are related to the z component of the electric field incident on any dipole in the nth column by the scattering equation [4, eq. (59)]

$$b_n = SE_{0z}^{0n} \tag{4.8}$$

where S is the normalized dipole scattering coefficient of a short electric dipole. “Normalized” here means that b_n is the coefficient of $\exp(\mathrm{i}kr)/(kr)$ in the transverse component of the outgoing electric field in response to the incident field $E_{0z}^{0n}\, \hat{\mathbf{z}}$ at the center of the z directed electric dipole. When $\mathbf{E}_0^{0mn}$ and $\mathbf{E}_0^{0m0}$ are summed over a column from $m = -\infty$ to ∞ in (4.1) the x components of the electric field incident on the reference $(0, 0)$ electric dipole vanish and the z components add. Then

$$\mathbf{E}_0^0 = E_{0z}^0\, \hat{\mathbf{z}} \tag{4.9}$$

with

$$E_{0z}^0 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} b_n \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}} \left[\frac{-2\mathrm{i}}{kr_{mn0}} \left(1 + \frac{\mathrm{i}}{kr_{mn0}}\right) \frac{(mh)^2}{r_{mn0}^2} \right.$$

$$+\left(1+\frac{\mathrm{i}}{kr_{mn0}}-\frac{1}{(kr_{mn0})^2}\right)\frac{(nd)^2}{r_{mn0}^2}\Bigg]+4b_0\sum_{m=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kr_{m00}}}{kr_{m00}}\frac{-\mathrm{i}}{kr_{m00}}\left(1+\frac{\mathrm{i}}{kr_{m00}}\right) \tag{4.10}$$

where r_{mn0} is given by (4.3) and

$$r_{m00} = mh\ . \tag{4.11}$$

We now assume that the array is excited by a traveling wave in the x direction with real propagation constant β. Then the constants b_n in (4.10) are identical apart from a phase shift given by

$$b_n = b_0\,\mathrm{e}^{\mathrm{i}n\beta d} \tag{4.12}$$

and

$$E_{0z}^0 = b_0\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\mathrm{e}^{\mathrm{i}n\beta d}\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}}\Bigg[\frac{-2\mathrm{i}}{kr_{mn0}}\left(1+\frac{\mathrm{i}}{kr_{mn0}}\right)\frac{(mh)^2}{r_{mn0}^2}$$

$$+\left(1+\frac{\mathrm{i}}{kr_{mn0}}-\frac{1}{(kr_{mn0})^2}\right)\frac{(nd)^2}{r_{mn0}^2}\Bigg]+4b_0\sum_{m=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kr_{m00}}}{kr_{m00}}\frac{-\mathrm{i}}{kr_{m00}}\left(1+\frac{\mathrm{i}}{kr_{m00}}\right)\ . \tag{4.13}$$

Since from (4.8)

$$b_0 = SE_{0z}^0 \tag{4.14}$$

it follows by substituting (4.14) in (4.13) and multiplying by $(kh)^3$ that

$$(kh)^3 = S\Bigg\{\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\mathrm{e}^{\mathrm{i}n\beta d}\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}}\Bigg[\frac{-2\mathrm{i}}{\rho_{mn}}\left(kh+\frac{\mathrm{i}}{\rho_{mn}}\right)\frac{m^2}{\rho_{mn}^2}$$

$$+\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mn}}-\frac{1}{\rho_{mn}^2}\right)\frac{(nd/h)^2}{\rho_{mn}^2}\Bigg]+4\sum_{m=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{m0}}}{\rho_{m0}}\frac{-\mathrm{i}}{\rho_{m0}}\left(kh+\frac{\mathrm{i}}{\rho_{m0}}\right)\Bigg\} \tag{4.15}$$

where we have let

$$\rho_{mn} = \sqrt{m^2+(nd/h)^2} \tag{4.16}$$

with

$$\rho_{m0} = m\ . \tag{4.17}$$

Equation (4.15) is the kd–βd equation that determines the normalized traveling wave propagation constant βd in terms of kh, d/h, and the normalized electric dipole scattering coefficient S.

To use the Poisson summation formula method to convert the slowly convergent double summation in (4.15) to a rapidly convergent form, we would need expressions for the integrals of functions like $\int_0^\infty \cos(p\sqrt{x^2+a^2})x^2/(x^2+a^2)^2\,\mathrm{d}x$, $\int_0^\infty \sin(p\sqrt{x^2+a^2}x^2/(x^2+a^2)^{5/2}\,\mathrm{d}x$, $\int_0^\infty \cos(p\sqrt{x^2+a^2})/(x^2+a^2)^{5/2}\,\mathrm{d}x$, etc. Unfortunately these integrals do not appear to be tabulated so that we are unable to use the Poisson summation formula method here. Accordingly we will use the Floquet mode method.

The Floquet mode method proceeds here similarly to how it was used in Section 2, (2.27)-(2.43). We begin by letting $E_z^0(z,\rho)$ be the z component of the electric field radiated by all the electric dipoles in the $n=0$ column at a general point in space (z,ρ), $\rho > 0$. (Note that because of symmetry, this field is the same for all points (z,x,y) such that $z^2+y^2=\rho^2$.) We establish a local spherical polar coordinate system with origin at the dipole located at $(z,x)=(mh,0)$ and with $\theta(m,z,\rho)$ the polar angle from the z axis to the vector $\mathbf{r}(m,z,\rho)$ from $(mh,0)$ to a field point (z,ρ). The distance $r(m,z,\rho)$ from $(mh,0)$ to (z,ρ) is given by

$$r(m,z,\rho) = \sqrt{(z-mh)^2+\rho^2} \tag{4.18}$$

$$\cos\theta(m,z,\rho) = \frac{z-mh}{r(m,z,\rho)} \tag{4.19}$$

$$\sin\theta(m,z,\rho) = \frac{\rho}{r(m,z,\rho)} \tag{4.20}$$

$$[\cos\theta(m,z,\rho)\,\hat{\mathbf{r}}(m,z,\rho)]_z = \cos^2\theta(m,z,\rho) = \frac{(z-mh)^2}{r^2(m,z,\rho)} \tag{4.21}$$

and

$$\left[\sin\theta(m,z,\rho)\,\hat{\boldsymbol{\theta}}(m,z,\rho)\right]_z = -\sin^2\theta(m,z,\rho) = -\frac{\rho^2}{r^2(m,z,\rho)}\ . \tag{4.22}$$

Then, referring to (4.2)

$$E_z^0(z,\rho) = b_0 \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr(m,z,\rho)}}{kr(m,z,\rho)} \left[\frac{-2\mathrm{i}}{kr(m,z,\rho)} \left(1+\frac{\mathrm{i}}{kr(m,z,\rho)}\right) \frac{(z-mh)^2}{r^2(m,z,\rho)} \right.$$
$$\left. + \left(1+\frac{\mathrm{i}}{kr(m,z,\rho)} - \frac{1}{(kr(m,z,\rho))^2}\right) \frac{\rho^2}{r^2(m,z,\rho)} \right], \quad \rho > 0\ . \tag{4.23}$$

For $\rho = |n|d$

$$E_z^0(0,|n|d) = b_0 \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr(m,0,|n|d)}}{kr(m,0,|n|d)} \left[\frac{-2\mathrm{i}}{kr(m,0,|n|d)} \left(1+\frac{\mathrm{i}}{kr(m,0,|n|d)}\right) \frac{(mh)^2}{r^2(m,0,|n|d)} \right.$$
$$\left. + \left(1+\frac{\mathrm{i}}{kr(m,0,|n|d)} - \frac{1}{(kr(m,0,|n|d))^2}\right) \frac{(nd)^2}{r^2(m,0,|n|d)} \right] \tag{4.24}$$

with

$$r(m,0,|n|d) = \sqrt{(mh)^2+(nd)^2}\ . \tag{4.25}$$

Since

$$r(m,0,|n|d) = h\rho_{mn} \tag{4.26}$$

with ρ_{mn} defined by (4.16),

$$E_z^0(z,|n|d) = b_0 \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{kh\rho_{mn}} \left[\frac{-2\mathrm{i}}{kh\rho_{mn}} \left(1+\frac{\mathrm{i}}{kh\rho_{mn}}\right) \frac{m^2}{\rho_{mn}^2} \right.$$

$$+\left(1+\frac{\mathrm{i}}{kh\rho_{mn}}-\frac{1}{(kh)^2\rho_{mn}^2}\right)\frac{(nd/h)^2}{\rho_{mn}^2}\Bigg] \ . \tag{4.27}$$

Hence in (4.15)

$$\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}}\left[\frac{-2\mathrm{i}}{\rho_{mn}}\left(kh+\frac{\mathrm{i}}{\rho_{mn}}\right)\frac{m^2}{\rho_{mn}^2}+\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mn}}-\frac{1}{\rho_{mn}^2}\right)\frac{(nd/h)^2}{\rho_{mn}^2}\right]=\frac{(kh)^3}{b_0}E_z^0(0,|n|d) \ . \tag{4.28}$$

We proceed to find an expression for $E_z^0(z,\rho)$ from which we can then obtain $E_z^0(0,|n|d)$.

Now $E_z^0(z,\rho)$ can be expressed in terms of cylindrical waves by [20, sec. 6.6]

$$E_z^0(z,\rho)=\int_{-\infty}^{\infty}B(k_z)H_0^{(1)}(k_\rho\rho)\,\mathrm{e}^{\mathrm{i}k_z z}\,\mathrm{d}k_z,\ k_\rho=\sqrt{k^2-k_z^2} \tag{4.29}$$

where k_ρ is positive real (positive imaginary) according as $k^2>(<)\ k_z^2$. Because of the periodicity of the array in the z direction,

$$E_z^0(z+h,\rho)=E_z^0(z,\rho) \ . \tag{4.30}$$

It follows by inverting (4.29) and inserting into (4.30) that

$$\mathrm{e}^{\mathrm{i}k_z h}=1 \tag{4.31}$$

and hence

$$k_z h=2\pi m,\ m=0,\pm1,\pm2,\cdots \tag{4.32}$$

so that

$$E_z^0(z,\rho)=\sum_{m=-\infty}^{\infty}B_m H_0^{(1)}(k_m\rho)\,\mathrm{e}^{\mathrm{i}(2\pi/h)mz} \tag{4.33}$$

where

$$k_m=\sqrt{k^2-(2\pi m/h)^2} \tag{4.34}$$

with k_m positive real (positive imaginary) according as $(kh)^2>(<)\ (2\pi m)^2$. By inversion

$$B_m H_0^{(1)}(k_m\rho)=\frac{1}{h}\int_{-h/2}^{h/2}E_z^0(z,\rho)\,\mathrm{e}^{-\mathrm{i}(2\pi/h)mz}\,\mathrm{d}z \ . \tag{4.35}$$

Then

$$\frac{(kh)^3}{b_0}E_z^0(0,|n|d)=\frac{(kh)^3}{b_0}\sum_{m=-\infty}^{\infty}B_m H_0^{(1)}(k_m|n|d) \ . \tag{4.36}$$

From (4.35) and (4.23)

$$B_m H_0^{(1)}(k_m\rho)=\frac{b_0}{kh}\sum_{m'=-\infty}^{\infty}\int_{-h/2}^{h/2}\frac{\mathrm{e}^{\mathrm{i}kr(m',z,\rho)}}{r(m',z,\rho)}\Bigg[\frac{-2\mathrm{i}}{kr(m',z,\rho)}\left(1+\frac{\mathrm{i}}{kr(m',z,\rho)}\right)\frac{(z-m'h)^2}{r^2(m',z,\rho)}$$

$$+\left(1+\frac{\mathrm{i}}{kr(m',z,\rho)}-\frac{1}{(kr(m',z,\rho))^2}\right)\frac{\rho^2}{r^2(m',z,\rho)}\Bigg]\,\mathrm{e}^{-\mathrm{i}(2\pi/h)mz}\,\mathrm{d}z \tag{4.37}$$

with

$$r(m',z,\rho)=\sqrt{(z-m'h)^2+\rho^2}\ . \tag{4.38}$$

Since B_m is independent of ρ, for $\rho << 1$ the LHS of (4.37) behaves as [see (C.4)]

$$B_m H_0^{(1)}(k_m\rho) \overset{\rho<<1}{\sim} \frac{2\mathrm{i}}{\pi}B_m \ln\rho\ . \tag{4.39}$$

Hence the RHS of (4.37) must also have a $\ln\rho$ singularity as $\rho \to 0$. In investigating the singularity of the RHS of (4.37) as $\rho \to 0$ we note that we can ignore all terms in the summation over m' for which $m' \neq 0$ since these terms are not singular as $\rho \to 0$. We must therefore consider the behavior for $\rho << 1$ of

$$\frac{2b_0}{kh}\int_0^{h/2}\frac{\mathrm{e}^{\mathrm{i}kr(0,z,\rho)}}{r(0,z,\rho)}\Bigg[\frac{-2\mathrm{i}}{kr(0,z,\rho)}\left(1+\frac{\mathrm{i}}{kr(0,z,\rho)}\right)\frac{z^2}{r^2(0,z,\rho)}$$

$$+\left(1+\frac{\mathrm{i}}{kr(0,z,\rho)}-\frac{1}{(kr(0,z,\rho))^2}\right)\frac{\rho^2}{r^2(0,z,\rho)}\Bigg]\,\cos(2\pi/h)mz\,\mathrm{d}z \tag{4.40}$$

where

$$r(0,z,\rho)=\sqrt{z^2+\rho^2}\ . \tag{4.41}$$

Any logarithmic singularity of (4.40) as $\rho \to 0$ must come from the vicinity of $z = 0$. To obtain the logarithmic singularity we expand both $\exp(\mathrm{i}kr(0,z,\rho))$ and $\cos(2\pi/h)mz$ in power series, systematically integrate all the resulting indefinite integrals using integrals tabulated in [18, eqs. 2.17, 2.26], evaluate the integrals at the lower end of the range of integration, $z = 0$, and collect terms in $\ln\rho$. (There is no contribution to the logarithmic singularity from the upper end of the interval of integration, $z = h/2$.) The procedure is laborious but straightforward. We find that it is necessary to include terms through $-[kr(0,z,\rho)]^2/2$ in the expansion of the exponential and terms through $-(2\pi m/h)^2z^2/2$ in the expansion of the cosine to include all contributions to the $\ln\rho$ singularity. When this is done we find that the RHS of (4.37) behaves as

$$-\frac{2b_0}{kh}\left[1-\left(\frac{2\pi m}{kh}\right)^2\right]\ln\rho \tag{4.42}$$

for $\rho << 1$. But then, equating the coefficient of the $\ln\rho$ singularity of the LHS of (4.37) given by (4.39) with the coefficient of the $\ln\rho$ singularity of the RHS of (4.37) given by (4.42) and solving for B_m, we obtain

$$B_m=\frac{\mathrm{i}\pi b_0}{kh}\left[1-\left(\frac{2\pi m}{kh}\right)^2\right]\ . \tag{4.43}$$

Hence, in (4.15) from (4.28), (4.36), and (4.43),

$$\sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \left[\frac{-2\mathrm{i}}{\rho_{mn}} \left(kh + \frac{\mathrm{i}}{\rho_{mn}} \right) \frac{m^2}{\rho_{mn}^2} + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} - \frac{1}{\rho_{mn}^2} \right) \frac{(nd/h)^2}{\rho_{mn}^2} \right]$$

$$= \mathrm{i}\pi \sum_{m=-\infty}^{\infty} \left[(kh)^2 - (2\pi m)^2 \right] \; H_0^{(1)}(k_m |n| d)$$

$$= \mathrm{i}\pi (kh)^2 H_0^{(1)}(k|n|d) - 4 \sum_{m=1}^{\infty} \left[(2\pi m)^2 - (kh)^2 \right] \; K_0 \left(|n| (d/h) \, \sqrt{(2\pi m)^2 - (kh)^2} \right) \tag{4.44}$$

where we have assumed that $h/\lambda < 1$ ($kh < 2\pi$) and made use of the relationship (C.1) between $H_0^{(1)}$ and K_0. Substituting (4.44) in (4.15) and replacing ρ_{m0} by m [see (4.17)] we obtain

$$(kh)^3 = S \Bigg\{ 2 \sum_{n=1}^{\infty} \cos(n\beta d) \Bigg[\mathrm{i}\pi (kh)^2 H_0^{(1)}(nkd)$$

$$- 4 \sum_{m=1}^{\infty} \left[(2\pi m)^2 - (kh)^2 \right] K_0 \left(n (d/h) \, \sqrt{(2\pi m)^2 - (kh)^2} \; \right) \Bigg]$$

$$+ 4 \sum_{m=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}khm}}{m} \frac{-\mathrm{i}}{m} \left(kh + \frac{\mathrm{i}}{m} \right) \Bigg\} \tag{4.45}$$

for $kh < 2\pi$. The slowly convergent Schlömilch series

$$\sum_{n=1}^{\infty} \cos(n\beta d) H_0^{(1)}(nkd) \tag{4.46}$$

in (4.45) can be efficiently evaluated using (B.1) and (B.2). The sum

$$\sum_{n=1}^{\infty} \cos(n\beta d) \sum_{m=1}^{\infty} \left[(2\pi m)^2 - (kh)^2 \right] K_0 \left(n (d/h) \, \sqrt{(2\pi m)^2 - (kh)^2} \; \right) \tag{4.47}$$

converges very rapidly because of the exponential decay of K_0. For example, for $n = 2$, $m = 3$, $d/h > 0.5$, and $0 < kh < 2\pi$, $K_0 \left(n (d/h) \, \sqrt{(2\pi m)^2 - (kh)^2} \right) \left[(2\pi m)^2 - (kh)^2 \right] < 1.8 \times 10^{-6}$. The series can thus be truncated keeping only a very few terms.

Before considering the self-column sum in (4.45) we will present an alternate method, based on the Hertz vector potential, for obtaining the coefficients B_m in the Floquet mode expansion (4.33). The method is very useful in its own right and will serve here as an important check on the correctness of the expression (4.43) that we have derived for the coefficients B_m in the Floquet mode expansion. We begin by noting [21, secs. 14-5,14-7] that the electric field of a small z directed dipole at the origin of a Cartesian coordinate system is given by

$$C \, \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \left(\frac{\mathrm{e}^{\mathrm{i}kr}}{kr} \hat{\mathbf{z}} \right) = C \left(\boldsymbol{\nabla} \boldsymbol{\nabla} \cdot - \boldsymbol{\nabla}^2 \right) \frac{\mathrm{e}^{\mathrm{i}kr}}{kr} \hat{\mathbf{z}} \, . \tag{4.48}$$

where C is a proportionality constant. Since (4.2) with $n = 0$ and the triple subscripts dropped also gives the field radiated by a small z directed electric dipole located at the origin, the value of C can be found easily by expanding (4.48) in spherical coordinates (using, for example, [22, Appendix 1, eqs. 119, 161]) and equating the $1/(kr)$ term of the θ component of the field with the corresponding $1/(kr)$ term of (4.2). We thereby obtain

$$C = \frac{b_0}{k^2} \, . \tag{4.49}$$

Since the z component of an electric field satisfies the scalar wave equation, then using the expressions [22, Appendix 1, eqs. 58, 57] for $\nabla\nabla\cdot$ and ∇^2 in Cartesian coordinates we see that the z component of the electric dipole field is given by

$$C\left(\frac{\partial^2}{\partial z^2} + k^2\right)\frac{\mathrm{e}^{\mathrm{i}kr}}{kr} \, . \tag{4.50}$$

Now from (2.34) and (2.42) the field radiated by the acoustic monopoles located in the column $x = 0$, $y = 0$ at $z = 0, \ \pm h, \ \pm 2h, \ \cdots$, each of which radiates a field equal to $\mathrm{e}^{\mathrm{i}kr}/(kr)$ is (allowing for the different choice of coordinate axes in this section as compared with Section 2)

$$\sum_{m=-\infty}^{\infty} B_m^0 H_0^{(1)}(k_m\rho)\, \mathrm{e}^{\mathrm{i}(2\pi/h)mz}, \ k_m = \sqrt{k^2 - (2\pi m/h)^2} \tag{4.51}$$

with

$$B_m^0 = \frac{\mathrm{i}\pi}{kh} \tag{4.52}$$

and k_m positive real or positive imaginary. Hence from (4.50) the z component of the electric field radiated by a periodic linear array of z directed electric dipoles on the z axis ($x = 0$) is equal to

$$\begin{aligned}
&C\sum_{m=-\infty}^{\infty} B_m^0 \left(\frac{\partial^2}{\partial z^2} + k^2\right)\left[H_0^{(1)}(k_m\rho)\, \mathrm{e}^{\mathrm{i}(2\pi/h)mz}\right] \\
&= C\sum_{m=-\infty}^{\infty} B_m^0 \left[-\left(\frac{2\pi m}{h}\right)^2 + k^2\right] H_0^{(1)}(k_m\rho)\, \mathrm{e}^{\mathrm{i}(2\pi/h)mz} \\
&= C\sum_{m=-\infty}^{\infty} B_m^0 k_m^2 H_0^{(1)}(k_m\rho)\, \mathrm{e}^{\mathrm{i}(2\pi/h)mz} \, .
\end{aligned} \tag{4.53}$$

But the same field is also given by the Floquet mode expansion (4.33). Hence, equating (4.33) with (4.53), we see that the coefficients B_m in (4.33) are equal to

$$B_m = CB_m^0 k_m^2 = \frac{b_0}{k^2}\frac{\mathrm{i}\pi}{kh}\left[k^2 - \left(\frac{2\pi m}{h}\right)^2\right] = \frac{\mathrm{i}\pi b_0}{kh}\left[1 - \left(\frac{2\pi m}{kh}\right)^2\right] \tag{4.54}$$

identical with the expression (4.43) obtained by the very different procedure in (4.37)-(4.42) of equating coefficients of a logarithmic singularity. While the second procedure for obtaining the coefficients of the Floquet mode expansion is considerably more elegant than the first, it is very useful for checking purposes to have two independent methods at our disposal.

It remains to consider the self-column sum in (4.15)

$$4\sum_{m=1}^{\infty} \frac{e^{ikhm}}{m}\frac{-i}{m}\left(kh+\frac{i}{m}\right) . \tag{4.55}$$

The sum is readily evaluated using the summation formulas (D.7) and (D.8). We then obtain

$$4\sum_{m=1}^{\infty} \frac{e^{ikhm}}{m}\frac{-i}{m}\left(kh+\frac{i}{m}\right) = 4\ kh\ \mathrm{Cl}_2(kh) + 4\ \mathrm{Cl}_3(kh) + i\pi(kh)^2 - i\frac{2}{3}(kh)^3 \tag{4.56}$$

for $0 < kh < 2\pi$, where the Clausen functions Cl_2 and Cl_3 are defined and approximated by equations (D.8).

Similarly to what we have done in our treatments of acoustic monopole arrays, it is useful for calculation purposes to write the kd–βd equation (4.15) in the form

$$(kh)^3 = S\{\Re + i\Im\} \tag{4.57}$$

where, from (4.45) and (4.56), $\Re$, the real part of the expression within the brackets of (4.15) with the original summations replaced by the rapidly convergent expressions we have derived, is given by

$$\begin{aligned}\Re = -2\pi(kh)^2\sum_{n=1}^{\infty}\cos(n\beta d)Y_0(nkd)\\ - 8\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=1}^{\infty}\left[(2\pi m)^2-(kh)^2\right]K_0\left(n(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right)\\ + 4(kh)\ \mathrm{Cl}_2(kh) + 4\ \mathrm{Cl}_3(kh)\end{aligned} \tag{4.58}$$

and $\Im$, the imaginary part of the expression within the brackets of (4.15), is given by

$$\Im = 2\pi(kh)^2\sum_{n=1}^{\infty}\cos(n\beta d)J_0(nkd) + \pi(kh)^2 - \frac{2}{3}(kh)^3 = -\frac{2}{3}(kh)^3 \tag{4.59}$$

using (B.1a). If we write the scattering coefficient S as

$$S = |S|e^{i\psi} \tag{4.60}$$

and equate imaginary parts in (4.57) we obtain the relation

$$|S| = \frac{3}{2}\sin\psi\ . \tag{4.61}$$

This relation was derived in [4] using reciprocity and power conservation relations, and has here been shown here to also be a necessary condition for a 2D array of lossless short electric dipoles to support a traveling wave. The derivation of (4.61) thus serves as an important check on our analysis. It is worth noting that if $\beta d < kd$ then, from (B.1b),

$\sum \cos(n\beta d) J_0(nkd) \neq -1/2$ and hence $\Im \neq -2/3(kh)^3$ so that (4.61) would not hold. This is not possible for an array of short lossless dipole scatterers. Hence $\beta d > kd$. This is a particular instance of the general result (1.4) noted in the Introduction which holds for 2D arrays as well as for linear arrays. Substituting (4.60) in (4.57) and equating real parts we obtain the form of the kd–βd equation that is used to calculate βd as a function of kh, d/h, and the phase ψ of the scattering coefficient

$$\frac{2}{3}(kh)^3 \cos\psi - \Re \sin\psi = 0 \tag{4.62}$$

with $\Re$ given by (4.58), (B.2) used to evaluate the Neumann function sum, the modified Bessel function sum truncated in accordance with the remark following (4.47), and $kh < 2\pi$. It is easy to solve (4.62) numerically for βd using, for example, a simple search procedure with secant algorithm refinement.

The expression for $\Re$ given in (4.58) is valid for $kh < 2\pi$. Since for $m = 1$, $K_0\left(n(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2}\right)$ is singular as kh approaches 2π, the equation cannot be used as is to calculate $\Re$ at $kh = 2\pi$. However, referring to (2.57) we see that when $m = 1$

$$-8\sum_{n=1}^{\infty} \cos(n\beta d) \left[(2\pi m)^2 - (kh)^2\right] K_0\left(n(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2}\right) \overset{kh\to 2\pi}{\longrightarrow} 0 \tag{4.63}$$

so that

$$\Re \overset{kh\to 2\pi}{\sim} -2\pi(2\pi)^2 \sum_{n=1}^{\infty} \cos(n\beta d) Y_0(nkd)$$

$$-8(2\pi)^2 \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{m=2}^{\infty} (m^2-1) K_0\left(2\pi n(d/h)\ \sqrt{m^2-1}\right) + 4(1.20205...) \tag{4.64}$$

using (D.9) and (D.10).

4.2 ELECTRIC DIPOLES PERPENDICULAR TO THE ARRAY PLANE

As in 4.1 the x axis of a Cartesian coordinate system is taken to be the array axis, with the dipoles oriented in the z direction. Here, however, the dipoles are centered at $y = mh, m = 0, \pm 1, \pm 2, \cdots$, so that the dipoles are perpendicular to the xy plane, the plane of the array. We assume an excitation of the array with the electric field parallel to the z axis and such that all the dipoles in any row of the array are excited identically. Let $\mathbf{E}_0^0$ be the electric field incident on the electric dipole at the location $x = 0, y = 0, z = 0$ from all the other dipoles in the array. As will be seen shortly, this field has a z component only. Let $\mathbf{E}_0^{0mn}$ be the electric field incident on the reference dipole from the electric dipole at the location $(y, x) = (mh, nd)$ so that

$$\mathbf{E}_0^0 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{m=-\infty}^{\infty} \mathbf{E}_0^{0mn} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \mathbf{E}_0^{0m0}\ . \tag{4.65}$$

From [4, eq. (40)]

$$\mathbf{E}_0^{0mn} = b_n \frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}} \left[\frac{-2\mathrm{i}}{kr_{mn0}} \left(1 + \frac{\mathrm{i}}{kr_{mn0}} \right) \cos\theta_{mn0}\, \hat{\mathbf{r}}_{mn0} \right.$$

$$\left. - \left(1 + \frac{\mathrm{i}}{kr_{mn0}} - \frac{1}{(kr_{mn0})^2} \right) \sin\theta_{mn0}\, \hat{\boldsymbol{\theta}}_{mn0} \right] . \tag{4.66}$$

The quantities in (4.66) are defined with reference to a local spherical polar coordinate system with origin at $(z, y, x) = (0, mh, nd)$ (in turn defined with reference to a local Cartesian coordinate system with the same origin whose axes are parallel to those of the global Cartesian coordinate system). The distance from the (m, n) dipole to the $(0, 0)$ dipole, r_{mn0}, is given by

$$r_{mn0} = \sqrt{(mh)^2 + (nd)^2} = h\sqrt{m^2 + (nd/h)^2} \tag{4.67}$$

and the unit vector in the direction from the (m, n) dipole to the $(0, 0)$ dipole, $\hat{\mathbf{r}}_{mn0}$, is

$$\hat{\mathbf{r}}_{mn0} = \mathbf{r}_{mn0}/r_{mn0}, \quad \mathbf{r}_{mn0} = -mh\, \hat{\mathbf{y}} - nd\, \hat{\mathbf{x}} \tag{4.68}$$

so that

$$\cos\theta_{mn0} = \hat{\mathbf{r}}_{mn0} \cdot \hat{\mathbf{z}} = 0 \tag{4.69}$$

$$\sin\theta_{mn0} = 1 \tag{4.70}$$

and

$$\hat{\boldsymbol{\theta}}_{mn0} = -\sin\theta_{mn0}\, \hat{\mathbf{z}} = -\hat{\mathbf{z}}. \tag{4.71}$$

The coefficients b_n are related to the z component of the electric field incident on any dipole in the nth row by the scattering equation (4.8). Substituting (4.69)-(4.71) in (4.66) we see that $\mathbf{E}_0^{0mn}$ has a z component only, and from (4.65) we then obtain

$$E_{0z}^0 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} b_n \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}} \left(1 + \frac{\mathrm{i}}{kr_{mn0}} - \frac{1}{(kr_{mn0})^2} \right)$$

$$+ 2b_0 \sum_{m=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr_{m00}}}{kr_{m00}} \left(1 + \frac{\mathrm{i}}{kr_{m00}} - \frac{1}{(kr_{m00})^2} \right) \tag{4.72}$$

where r_{mn0} is given by (4.67) and

$$r_{m00} = mh\ . \tag{4.73}$$

We now assume that the array is excited by a traveling wave in the x direction with real propagation constant β. Then the constants b_n in (4.72) are identical apart from a phase shift given by

$$b_n = b_0\, \mathrm{e}^{\mathrm{i}n\beta d} \tag{4.74}$$

and

$$E_{0z}^0 = b_0 \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}} \left(1 + \frac{\mathrm{i}}{kr_{mn0}} - \frac{1}{(kr_{mn0})^2} \right)$$

$$+ 2b_0 \sum_{m=1}^{\infty} \frac{\mathrm{e}^{ikr_{m00}}}{kr_{m00}} \frac{\mathrm{i}}{kr_{m00}} \left(1 + \frac{\mathrm{i}}{kr_{m00}} - \frac{1}{(kr_{m00})^2}\right) . \tag{4.75}$$

Since from (4.8)

$$b_0 = SE^0_{0z} \tag{4.76}$$

it follows by substituting (4.76) in (4.75) and multiplying by $(kh)^3$ that

$$\begin{aligned}(kh)^3 = S\Bigg\{ &\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} - \frac{1}{\rho_{mn}^2}\right) \\ &+ 2\sum_{m=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{m0}}}{\rho_{m0}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{m0}} - \frac{1}{\rho_{m0}^2}\right) \Bigg\}\end{aligned} \tag{4.77}$$

where we have let

$$\rho_{mn} = \sqrt{m^2 + (nd/h)^2} \tag{4.78}$$

so that

$$\rho_{m0} = m\ . \tag{4.79}$$

Equation (4.77) is the kd–βd equation that determines the normalized traveling wave propagation constant βd in terms of kh, d/h, and the normalized electric dipole scattering coefficient S.

To convert the slowly convergent summation

$$\sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} - \frac{1}{\rho_{mn}^2}\right) \tag{4.80}$$

in (4.77) to a rapidly convergent form, we write this sum as

$$(kh)^2 \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} + \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kh\rho_{mn}} \left(\frac{\mathrm{i}kh}{\rho_{mn}^2} - \frac{1}{\rho_{mn}^3}\right) \tag{4.81}$$

and proceed to evaluate these two sums using the Poisson summation formula. The first of these sums is equal to $(kh)^2 I(n)$ where $I(n)$ is given by (2.11) in Section 2 dealing with 2D arrays of acoustic monopoles. From (2.16) we then have

$$(kh)^2 \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} = \mathrm{i}\pi (kh)^2 H_0^{(1)}(|n|kd) + 4(kh)^2 \sum_{m=1}^{\infty} K_0\left(|n|(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2}\right) . \tag{4.82}$$

The second sum in (4.81)

$$\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kh\rho_{mn}} \left(\frac{\mathrm{i}kh}{\rho_{mn}^2} - \frac{1}{\rho_{mn}^3}\right) \tag{4.83}$$

can be written as

$$\mathrm{i}kh \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kh\sqrt{m^2+(nd/h)^2}} \left(\frac{1}{m^2 + (nd/h)^2} + \frac{1}{(-\mathrm{i}kh)[m^2 + (nd/h)^2]^{3/2}}\right) . \tag{4.84}$$

The sum over m can be treated with the Poisson summation formula (2.9) and (2.10) with

$$f(x) = \mathrm{e}^{\mathrm{i}kh\sqrt{x^2+(nd/h)^2}} \left(\frac{1}{\sqrt{x^2+(nd/h)^2}} + \frac{1}{(-\mathrm{i}kh)[x^2+(nd/h)^2]^{3/2}} \right) . \tag{4.85}$$

Since $f(x)$ is an even function of x,

$$\hat{f}(m) = 2\int_0^\infty f(x) \cos 2\pi m x \, \mathrm{d}x . \tag{4.86}$$

We can evaluate the cosine transform using the formula [18, eq. 3.914(4)]

$$\int_0^\infty \left(\frac{1}{\beta(x^2+\gamma^2)^{3/2}} + \frac{1}{x^2+\gamma^2} \right) \mathrm{e}^{-\beta\sqrt{x^2+\gamma^2}} \cos bx \, \mathrm{d}x = \frac{1}{\beta\gamma}\sqrt{\beta^2+b^2}\, K_1\left(\gamma\sqrt{\beta^2+b^2}\right) \tag{4.87}$$

with $\beta = -\mathrm{i}kh$, $\gamma = |n|d/h$, and $b = 2\pi m$ so that

$$\begin{aligned}\hat{f}(m) &= 2\int_0^\infty \mathrm{e}^{\mathrm{i}kh\sqrt{x^2+(nd/h)^2}} \left(\frac{1}{x^2+(nd/h)^2} + \frac{1}{(-\mathrm{i}kh)[x^2+(nd/h)^2]^{3/2}} \right) \cos 2\pi m x \, \mathrm{d}x \\ &= \frac{2\mathrm{i}}{kh(|n|d/h)} \sqrt{(2\pi m)^2-(kh)^2}\, K_1\left((|n|(d/h))\, \sqrt{(2\pi m)^2-(kh)^2} \right) .\end{aligned} \tag{4.88}$$

Hence

$$\begin{aligned}&\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kh\sqrt{m^2+(nd/h)^2}} \left(\frac{1}{m^2+(nd/h)^2} + \frac{1}{(-\mathrm{i}kh)[m^2+(nd/h)^2]^{3/2}} \right) \\ &= \frac{2\mathrm{i}}{kh(|n|d/h)} \sum_{m=-\infty}^{\infty} \sqrt{(2\pi m)^2-(kh)^2}\, K_1\left(|n|(d/h)\, \sqrt{(2\pi m)^2-(kh)^2} \right) \\ &= -\frac{\pi}{(|n|d/h)} H_1^{(1)}(|n|kd) + \frac{4\mathrm{i}}{|n|kd} \sum_{m=1}^{\infty} \sqrt{(2\pi m)^2-(kh)^2}\, K_1\left(|n|(d/h)\, \sqrt{(2\pi m)^2-(kh)^2} \right) \\ &= -\frac{\pi}{2} kh \left[H_0^{(1)}(|n|kd) + H_2^{(1)}(|n|kd) \right] \\ &+ \frac{2\mathrm{i}}{kh} \sum_{m=1}^{\infty} \left[(2\pi m)^2 - (kh)^2 \right] \left[K_2\left(|n|(d/h)\, \sqrt{(2\pi m)^2-(kh)^2} \right) - K_0\left(|n|(d/h)\, \sqrt{(2\pi m)^2-(kh)^2} \right) \right]\end{aligned} \tag{4.89}$$

where we have used the Bessel function relationships (C.2)-(C.9) and assumed that $0 < kh < 2\pi$. Combining (4.80)-(4.84) and (4.89) we have shown that

$$\sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} - \frac{1}{\rho_{mn}^2} \right) = \frac{\mathrm{i}\pi(kh)^2}{2} \left[H_0^{(1)}(|n|kd) - H_2^{(1)}(|n|kd) \right]$$

$$+\,2\sum_{m=1}^{\infty}\Bigg[\left[(2\pi m)^2+(kh)^2\right]K_0\left(|n|(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right)$$

$$-\ \left[(2\pi m)^2-(kh)^2\right]K_2\left(|n|(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right)\Bigg]\ . \tag{4.90}$$

[An alternate derivation of (4.90) using the Floquet mode expansion method is given in (4.107)-(4.127).] Then in (4.77)

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d}\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}}\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mn}}-\frac{1}{\rho_{mn}^2}\right)$$

$$=2\sum_{n=1}^{\infty}\cos(n\beta d)\Bigg(\frac{\mathrm{i}\pi(kh)^2}{2}\left[H_0^{(1)}(nkd)-H_2^{(1)}(nkd)\right]$$

$$+\,2\sum_{m=1}^{\infty}\Bigg[\left[(2\pi m)^2+(kh)^2\right]K_0\left(n(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right)$$

$$-\ \left[(2\pi m)^2-(kh)^2\right]K_2\left(n(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right)\Bigg]\Bigg)\ . \tag{4.91}$$

In (4.91) the slowly convergent Schlömilch series

$$\sum_{n=1}^{\infty}\cos(n\beta d)H_0^{(1)}(nkd) \tag{4.92}$$

can be efficiently evaluated using (B.1) and (B.2), and the slowly convergent Schlömilch series

$$\sum_{n=1}^{\infty}\cos(n\beta d)H_2^{(1)}(nkd) \tag{4.93}$$

can be efficiently evaluated using (B.8)-(B.10). The series with the modified Bessel functions K_0 and K_2 converge very rapidly because of the exponential decay of these functions.

It remains to consider the self-column sum in (4.77)

$$2\sum_{m=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}khm}}{m}\left((kh)^2+\frac{\mathrm{i}kh}{m}-\frac{1}{m^2}\right)\ . \tag{4.94}$$

For $0<kh<2\pi$ the sum is readily evaluated using the summation formulas (D.13) and the approximations (D.8). We then obtain

$$2\sum_{m=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}khm}}{m}\left((kh)^2+\frac{\mathrm{i}kh}{m}-\frac{1}{m^2}\right)$$

$$=-\,2\left((kh)^2\ln\left[2\sin\left(\frac{kh}{2}\right)\right]+kh\mathrm{Cl}_2(kh)+\mathrm{Cl}_3(kh)\right)+\mathrm{i}\left[\frac{\pi}{2}(kh)^2-\frac{2}{3}(kh)^3\right] \tag{4.95}$$

for $0 < kh < 2\pi$. The Clausen functions Cl_2 and Cl_3 in (4.95) are defined and approximated by equations (D.8). Substituting (4.90) and (4.95) in the $kd - \beta d$ equation (4.77) we can then write the $kd - \beta d$ equation in the form

$$(kh)^3 = S\{\Re + \mathrm{i}\Im\} \tag{4.96}$$

where $\Re$, the real part of the quantity within the brackets of (4.77) with the original summations replaced by the rapidly convergent expressions we have derived, is given by

$$\begin{aligned}\Re = &-\pi(kh)^2\left[\sum_{n=1}^{\infty}\cos(n\beta d)Y_0(nkd) - \sum_{n=1}^{\infty}\cos(n\beta d)Y_2(nkd)\right]\\ &+ 4\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=1}^{\infty}\Bigg[\left[(2\pi m)^2 + (kh)^2\right]K_0\left(n(d/h)\sqrt{(2\pi m)^2 - (kh)^2}\right)\\ &- \left[(2\pi m)^2 - (kh)^2\right]K_2\left(n(d/h)\sqrt{(2\pi m)^2 - (kh)^2}\right)\Bigg]\\ &- 2\left((kh)^2\ln\left[2\sin\left(\frac{kh}{2}\right)\right] + kh\mathrm{Cl}_2(kh) + \mathrm{Cl}_3(kh)\right)\end{aligned} \tag{4.97}$$

and $\Im$, the imaginary part of the quantity within the brackets of (4.77), is given by

$$\Im = \pi(kh)^2\left[\sum_{n=1}^{\infty}\cos(n\beta d)J_0(nkd) - \sum_{n=1}^{\infty}\cos(n\beta d)J_2(nkd)\right] + \frac{\pi}{2}(kh)^2 - \frac{2}{3}(kh)^3\,. \tag{4.98}$$

In the expression (4.97) for $\Re$, the sum $\sum\cos(n\beta d)Y_0(nkd)$ can be evaluated using (B.2). The sum $\sum\cos(n\beta d)Y_2(nkd)$ can be evaluated very efficiently by using (B.9) and (B.10). The K_0 and K_2 series in (4.97) converge extremely rapidly because of the exponential decay of $K_0(z)$ and $K_2(z)$.

In the expression (4.98) for $\Im$, the sum $\sum\cos(n\beta d)J_0(nkd)$ = -1/2 [see (B.1)] and the sum $\sum\cos(n\beta d)J_2(nkd) = 0$ [see (B.8a)]. Hence

$$\Im = -\frac{2}{3}(kh)^3 \tag{4.99}$$

which, together with (4.96), has been shown in Subsection 4.1 [see (4.59)-(4.61)] to imply that

$$|S| = \frac{3}{2}\sin\psi \tag{4.100}$$

where ψ is the phase of the scattering coefficient S, a relationship derived independently in [4] from reciprocity and power conservation principles, and thereby serving as an important check here. It is worth noting that if $\beta d < kd$ then, from (B.1b), $\sum\cos(n\beta d)J_0(nkd) \neq -1/2$ and from (B.8b), $\sum\cos(n\beta d)J_2(nkd) \neq 0$ and hence $\Im \neq -2/3(kh)^3$ so that (4.99) would not hold. This is not possible for an array of short lossless dipole scatterers. Hence $\beta d > kd$. This is a particular instance of the general result (1.4) noted in the Introduction which holds for 2D arrays as well as for linear arrays. The kd–βd equation (4.96) for traveling waves supported by 2D arrays of short electric dipoles perpendicular to the array axis and to the array plane then becomes

$$\frac{2}{3}(kh)^3 \cos\psi - \Re \sin\psi = 0 \tag{4.101}$$

with $\Re$ given by (4.97) and $kh < 2\pi$. Equation (4.101) can be easily solved numerically for βd given values of kd, kh, and ψ, using, for example, a simple search procedure with secant algorithm refinement.

Since some of the terms of (4.97) become singular as kh approaches 2π the equation cannot be used to calculate $\Re$ at $kh = 2\pi$. It is therefore worthwhile to obtain the limit of $\Re$ given by (4.97) as $kh \to 2\pi$ from below. From (2.57)

$$\begin{aligned} 4\sum_{n=1}^{\infty} \cos(n\beta d) \left[(2\pi)^2 + (kh)^2\right] K_0\left(n(d/h)\ \sqrt{(2\pi)^2 - (kh)^2}\right) \\ \overset{kh\to 2\pi}{\sim} (2\pi)^2 \left(4\gamma + 4\ln\frac{1}{\sqrt{4\pi}}\frac{d}{h} + \frac{4\pi}{\beta d}\right) + \ (2\pi)^2\, 2\ln\epsilon \\ + (2\pi)^2(4\pi)\left[\sum_{l=1}^{\infty}\left(\frac{1}{(2l\pi - \beta d)^2} - \frac{1}{2l\pi}\right) + \sum_{l=1}^{\infty}\left(\frac{1}{(2l\pi + \beta d)^2} - \frac{1}{2l\pi}\right)\right] \end{aligned} \tag{4.102}$$

where γ is the Euler constant and $\epsilon = 2\pi - kh,\ 0 < \epsilon << 1$. From (2.58) we see that the logarithmic singularity is exactly canceled by the logarithmic singularity of

$$-\ 2(kh)^2 \ln\left[2\sin\left(\frac{kh}{2}\right)\right] \tag{4.103}$$

at $kh = 2\pi$. Also as $kh \to 2\pi$, for $m = 1$

$$\begin{aligned} \left[(2\pi m)^2 - (kh)^2\right] K_2\left(n(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2}\right) \overset{kh\to 2\pi}{\sim} 2\left[(2\pi)^2 - (kh)^2\right] \frac{1}{(nd/h)^2\left[(2\pi)^2 - (kh)^2\right]} \\ = \frac{2}{(nd/h)^2} \end{aligned} \tag{4.104}$$

using the small argument form of the modified Bessel function K_2, (C.6), so that

$$\begin{aligned} -\ 4\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=1}^{\infty}\left[(2\pi m)^2 - (kh)^2\right] K_2\left(n(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2}\right) \\ \overset{kh\to 2\pi}{\to} -\frac{8}{(d/h)^2}\left[\frac{\pi^2}{6} - \frac{\pi\beta d}{2} + \frac{(\beta d)^2}{4}\right] \end{aligned} \tag{4.105}$$

where we have made use of the summation formula (D.13c). Hence as $kh \to 2\pi$

$$\begin{aligned} \Re \overset{kh\to 2\pi}{\to} -\pi(2\pi)^2\left[\sum_{n=1}^{\infty}\cos(n\beta d) Y_0(nkd) - \sum_{n=1}^{\infty}\cos(n\beta d) Y_2(nkd)\right] \\ + 4(2\pi)^2\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=2}^{\infty}\left[(m^2+1)K_0\left(2\pi n(d/h)\ \sqrt{m^2-1}\right)\right. \end{aligned}$$

$$
- (m^2-1)K_2\left(2\pi n(d/h)\,\sqrt{m^2-1}\right)\Bigg] + (2\pi)^2\left(4\gamma + 4\ln\frac{1}{\sqrt{4\pi}}\frac{d}{h} + \frac{4\pi}{\beta d}\right)
$$

$$
+ (2\pi)^2(4\pi)\left[\sum_{l=1}^{\infty}\left(\frac{1}{(2l\pi-\beta d)^2} - \frac{1}{2l\pi}\right) + \sum_{l=1}^{\infty}\left(\frac{1}{(2l\pi+\beta d)^2} - \frac{1}{2l\pi}\right)\right]
$$

$$
- \frac{8}{(d/h)^2}\left[\frac{\pi^2}{6} - \frac{\pi\beta d}{2} + \frac{(\beta d)^2}{4}\right] - 2\,\mathrm{Cl}_3(2\pi) \tag{4.106}
$$

where we have set $\mathrm{Cl}_2(2\pi) = 0$ [see (D.9)] and $\mathrm{Cl}_3(2\pi)$ is given by (D.10).

We close this subsection by giving an alternate derivation of (4.90) using the Floquet mode expansion method, proceeding similarly to how it was used in 4.1. We begin by letting $E_z^0(y,z,x)$ be the z component of the electric field radiated by all the electric dipoles in the $n=0$ row (x = 0, z = 0) at a general point in space (y,z,x), $\sqrt{z^2+x^2} > 0$. Note that the field is not radially symmetric the way it was in 4.1 where the dipoles were in the array plane. However, for conciseness whenever a quantity is a function of $\rho = \sqrt{z^2+x^2}$, we will replace the two coordinates z and x by ρ. We establish a local spherical polar coordinate system with origin at the dipole located at $(y,z,x) = (mh,0,0)$ and with $\theta(m,y,z,x)$ the polar angle from the z axis to the vector $\mathbf{r}(m,y,z,x) = (y-mh)\,\hat{\mathbf{y}} + x\,\hat{\mathbf{x}} + z\,\hat{\mathbf{z}}$ from $(mh,0,0)$ to a field point (y,z,x). The distance $r(m,y,\rho)$ from $(mh,0,0)$ to (y,z,x) is given by

$$
r(m,y,\rho) = \sqrt{(y-mh)^2+\rho^2} \tag{4.107}
$$

$$
\cos\theta(m,y,z,x) = \frac{\mathbf{r}(m,y,z,x)}{r(m,y,\rho)}\cdot\hat{\mathbf{z}} = \frac{z}{r(m,y,\rho)} \tag{4.108}
$$

$$
\sin\theta(m,y,z,x) = \sqrt{1-\cos^2\theta(m,y,z,x)} = \frac{\sqrt{(y-mh)^2+x^2}}{r^2(m,y,\rho)} \tag{4.109}
$$

$$
[\cos\theta(m,y,z,x)\,\hat{\mathbf{r}}(m,y,z,x)]_z = \cos^2\theta(m,y,z,x) = \frac{z^2}{r^2(m,y,\rho)} \tag{4.110}
$$

and

$$
\Big[\sin\theta(m,y,z,x\,\hat{\boldsymbol{\theta}}(m,y,z,x)\Big]_z = -\sin^2\theta(m,y,z,x) = -\frac{(y-mh)^2+x^2}{r^2(m,y,\rho)}\,. \tag{4.111}
$$

Then, referring to (4.2)

$$
E_z^0(y,z,x) = b_0\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kr(m,y,\rho)}}{kr(m,y,\rho)}\Bigg[\frac{-2\mathrm{i}}{kr(m,y,\rho)}\left(1+\frac{\mathrm{i}}{kr(m,y,\rho)}\right)\frac{z^2}{r^2(m,y,\rho)}
$$

$$
+\left(1+\frac{\mathrm{i}}{kr(m,y,\rho)} - \frac{1}{(kr(m,y,\rho))^2}\right)\frac{(y-mh)^2+x^2}{r^2(m,y,\rho)}\Bigg]\,,\quad \rho>0\,. \tag{4.112}
$$

For $\rho = |n|d$

$$
E_z^0(0,z,x) = b_0\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kr(m,0,|n|d)}}{kr(m,0,|n|d)}\Bigg[\frac{-2\mathrm{i}}{kr(m,0,|n|d)}\left(1+\frac{\mathrm{i}}{kr(m,0,|n|d)}\right)\frac{z^2}{r^2(m,0,|n|d)}
$$

$$+\left(1+\frac{\mathrm{i}}{kr(m,0,|n|d)}-\frac{1}{(kr(m,0,|n|d))^2}\right)\frac{(mh)^2+x^2}{r^2(m,0,|n|d)}\Bigg] \tag{4.113}$$

with

$$r(m,0,|n|d)=\sqrt{(mh)^2+(nd)^2}\ . \tag{4.114}$$

Since

$$r(m,0,|n|d)=h\rho_{mn} \tag{4.115}$$

with

$$\rho_{mn}=\sqrt{m^2+(nd/h)^2} \tag{4.116}$$

$$E_z^0(0,z,x)=b_0\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{kh\rho_{mn}}\Bigg[\frac{-2\mathrm{i}}{kh\rho_{mn}}\left(1+\frac{\mathrm{i}}{kh\rho_{mn}}\right)\frac{z^2}{h^2\rho_{mn}^2}$$

$$+\left(1+\frac{\mathrm{i}}{kh\rho_{mn}}-\frac{1}{(kh)^2\rho_{mn}^2}\right)\frac{m^2+(nd/h)^2}{\rho_{mn}^2}\Bigg]\ . \tag{4.117}$$

and

$$E_z^0(0,0,|n|d)=b_0\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{kh\rho_{mn}}\left(1+\frac{\mathrm{i}}{kh\rho_{mn}}-\frac{1}{(kh)^2\rho_{mn}^2}\right)\ . \tag{4.118}$$

Hence in (4.77)

$$\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}}\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mn}}-\frac{1}{\rho_{mn}^2}\right)=\frac{(kh)^3}{b_0}E_z^0(0,0,|n|d)\ . \tag{4.119}$$

We proceed to find an expression for $E_z^0(0,z,x)$ from which we can then obtain $E_z^0(0,0,|n|d)$.

The electric field radiated by a small z directed electric dipole at the origin of a Cartesian coordinate system is given by (4.50) which we repeat here

$$C\left(\frac{\partial^2}{\partial z^2}+k^2\right)\frac{\mathrm{e}^{\mathrm{i}kr}}{kr} \tag{4.120}$$

with

$$C=\frac{b_0}{k^2}\ . \tag{4.121}$$

Now from (2.34) and (2.42) the field radiated by the acoustic monopoles located in the row $x=0,\ z=0$ at $y=0,\ \pm h,\ \pm 2h,\ \cdots$, each of which radiates a field equal to $\mathrm{e}^{\mathrm{i}kr}/(kr)$ is (allowing for the different choice of coordinate axes in this section as compared with Section 2)

$$\sum_{m=-\infty}^{\infty}B_m^0H_0^{(1)}(k_m\rho)\ \mathrm{e}^{\mathrm{i}(2\pi/h)my},\ k_m=\sqrt{k^2-(2\pi m/h)^2} \tag{4.122}$$

where $\rho=\sqrt{x^2+z^2}$, k_m is positive real or positive imaginary, and

$$B_m^0=\frac{\mathrm{i}\pi}{kh}\ . \tag{4.123}$$

Hence the z component of the electric field radiated by a periodic linear array of z directed electric dipoles on the y axis ($x = 0$) is equal to

$$C \sum_{m=-\infty}^{\infty} B_m^0 \left(\frac{\partial^2}{\partial z^2} + k^2\right) H_0^{(1)}(k_m\sqrt{x^2+z^2})\, \mathrm{e}^{\mathrm{i}(2\pi/h)my} \,. \tag{4.124}$$

Performing the differentiation we obtain

$$\begin{aligned} E_z^0(y,z,x) = C B_m^0 \sum_{m=-\infty}^{\infty} \Bigg[&\left(k^2 - \frac{k_m^2}{2}\frac{z^2+x^2}{x^2+z^2}\right) H_0^{(1)}(k_m\sqrt{x^2+z^2}) \\ &+ \frac{k_m^2}{2}\frac{z^2-x^2}{x^2+z^2}\, H_2^{(1)}(k_m\sqrt{x^2+z^2})\Bigg] \mathrm{e}^{\mathrm{i}(2\pi/h)my} \end{aligned} \tag{4.125}$$

so that

$$\begin{aligned} E_z^0(0,0,|n|d) &= C B_m^0 \sum_{m=-\infty}^{\infty} \left[\left(k^2 - \frac{k_m^2}{2}\right) H_0^{(1)}(k_m|n|d) - \frac{k_m^2}{2} H_2^{(1)}(k_m|n|d)\right] \\ &= \frac{1}{2}\frac{C B_m^0}{h^2} \sum_{m=-\infty}^{\infty} \Bigg[\left[(kh)^2 + (2\pi m)^2\right] H_0^{(1)}\left(|n|(d/h)\ \sqrt{(kh)^2-(2\pi m)^2}\right) \\ &\quad - \left[(kh)^2 - (2\pi m)^2\right] H_2^{(1)}\left(|n|(d/h)\ \sqrt{(kh)^2-(2\pi m)^2}\right)\Bigg] \\ &= \frac{1}{2}\frac{\mathrm{i}\pi b_0}{(kh)^3}(kh)^2 \left[H_0^{(1)}(|n|kd) - H_2^{(1)}(|n|kd)\right] \\ &\quad + \frac{2b_0}{(kh)^3} \sum_{m=1}^{\infty} \Bigg[\left[(2\pi m)^2 + (kh)^2\right] K_0\left(|n|(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right) \\ &\quad - \left[(2\pi m)^2 - (kh)^2\right] K_2\left(|n|(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right)\Bigg] \end{aligned} \tag{4.126}$$

where we have made use of the Bessel function relations (C.1) and (C.3). But then, referring to (4.119),

$$\begin{aligned} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} - \frac{1}{\rho_{mn}^2}\right) &= \frac{(kh)^3}{b_0} E_z^0(0,0,|n|d) \\ &= \frac{\mathrm{i}\pi(kh)^2}{2}\left[H_0^{(1)}(|n|kd) - H_2^{(1)}(|n|kd)\right] \\ &\quad + 2\sum_{m=1}^{\infty} \Bigg[\left[(2\pi m)^2 + (kh)^2\right] K_0\left(|n|(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right) \\ &\quad - \left[(2\pi m)^2 - (kh)^2\right] K_2\left(|n|(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right)\Bigg] \end{aligned} \tag{4.127}$$

in agreement with (4.90), obtained using the Poisson summation formula.

5 3D ELECTRIC DIPOLE ARRAYS, DIPOLES ORIENTED PERPENDICULAR TO THE ARRAY AXIS

In this section we consider traveling waves supported by 3D periodic arrays of short lossless electric dipoles oriented perpendicular to the array axis. We follow the same major steps of the procedure — calculating the field at a reference element due to all the other elements in the array, deriving the kd–βd equation by assuming a traveling wave excitation of the array, and converting slowly convergent summations to rapidly convergent ones to obtain a form of the kd–βd equation suitable for calculation purposes — used in treating 2D periodic arrays of lossless electric dipoles in Section 4. As in Subsection 4.1, the Poisson summation formula method cannot be used to convert the slowly convergent summations we encounter to rapidly convergent forms because expressions for the necessary integrals are not available. We can, however, accomplish this with the same two forms of the Floquet mode method, one based on the asymptotic analysis of an integral and the other based on the Hertz vector potential, that we used in Subsection 4.1.

As in Section 4 it is again more convenient to take the x axis, rather than the z axis, of a Cartesian coordinate system to be the array axis, with the electric dipoles oriented in the z direction because the field of a small electric dipole is expressed most simply in a spherical polar coordinate system with the z axis aligned with the dipole direction. Equispaced planes of electric dipoles are located at $x = nd,\ n = 0,\ \pm1,\ \pm2,\ \cdots$. In each plane the dipoles are centered at $y = lh,\ z = mh,\ l, m = 0,\ \pm1,\ \pm2,\ \cdots$. We assume an excitation of the array with the electric field parallel to the z axis and such that all the dipoles in any plane of the array are excited identically. Let $\mathbf{E}_0^0$ be the electric field incident on the electric dipole at the location $(x, y, z) = (0, 0, 0)$ from all the other dipoles in the array. As will be seen [see (5.14)] this field has a z component only. Let $\mathbf{E}_0^{0lmn}$ be the electric field incident on the reference dipole from the electric dipole at the location $(x, y, z) = (nd, lh, mh)$ so that

$$\mathbf{E}_0^0 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathbf{E}_0^{0lmn} + \sum_{\substack{l=-\infty \\ (l,m)}}^{\infty} \ \ \sum_{\substack{m=-\infty \\ \neq\ (0,0)}}^{\infty} \mathbf{E}_0^{0lm0} \ . \tag{5.1}$$

From [4, eq. (40)]

$$\mathbf{E}_0^{0lmn} = b_n \frac{\mathrm{e}^{\mathrm{i}kr_{lmn0}}}{kr_{lmn0}} \left[\frac{-2\mathrm{i}}{kr_{lmn0}} \left(1 + \frac{\mathrm{i}}{kr_{lmn0}}\right) \cos\theta_{lmn0}\, \hat{\mathbf{r}}_{lmn0} \right.$$
$$\left. - \left(1 + \frac{\mathrm{i}}{kr_{lmn0}} - \frac{1}{(kr_{lmn0})^2}\right) \sin\theta_{lmn0}\, \hat{\boldsymbol{\theta}}_{lmn0} \right] . \tag{5.2}$$

The quantities in (5.2) are defined with reference to a local spherical polar coordinate system with origin at $(x, y, z) = (nd, lh, mh)$ (in turn defined with reference to a local Cartesian coordinate system with the same origin whose axes are parallel to those of the global Cartesian coordinate system). The distance from the (n, l, m) dipole to the $(0, 0, 0)$ dipole, r_{lmn0}, is given by

$$r_{lmn0} = \sqrt{(lh)^2 + (mh)^2 + (nd)^2} = h\sqrt{l^2 + m^2 + (nd/h)^2} \tag{5.3}$$

and the unit vector in the direction from the (n, l, m) dipole to the $(0,0,0)$ dipole, $\hat{\mathbf{r}}_{lmn0}$, is

$$\hat{\mathbf{r}}_{lmn0} = \mathbf{r}_{lmn0}/r_{lmn0}, \quad \mathbf{r}_{lmn0} = -nd\,\hat{\mathbf{x}} - lh\,\hat{\mathbf{y}} - mh\,\hat{\mathbf{z}} \tag{5.4}$$

so that

$$\cos\theta_{lmn0} = \hat{\mathbf{r}}_{lmn0}\cdot\hat{\mathbf{z}} = -\frac{mh}{r_{lmn0}} \tag{5.5}$$

$$\sin\theta_{lmn0} = \sqrt{1-\cos^2\theta_{lmn0}} = \frac{\sqrt{(nd)^2+(lh)^2}}{r_{lmn0}} \tag{5.6}$$

$$\phi_{lmn0} = \tan^{-1}\left(\frac{\hat{\mathbf{r}}_{lmn0}\cdot\hat{\mathbf{y}}}{\hat{\mathbf{r}}_{lmn0}\cdot\hat{\mathbf{x}}}\right) = \tan^{-1}\left(\frac{-lh}{-nd}\right) \tag{5.7}$$

$$\cos\phi_{lmn0} = \frac{-nd}{\sqrt{(nd)^2+(lh)^2}} \tag{5.8}$$

$$\sin\phi_{lmn0} = \frac{-lh}{\sqrt{(nd)^2+(lh)^2}} \tag{5.9}$$

$$\hat{\boldsymbol{\theta}}_{lmn0} = \cos\theta_{lmn0}\cos\phi_{lmn0}\,\hat{\mathbf{x}} + \cos\theta_{lmn0}\sin\phi_{lmn0}\,\hat{\mathbf{y}} - \sin\theta_{lmn0}\,\hat{\mathbf{z}} \tag{5.10}$$

$$\cos\theta_{lmn0}\,\hat{\mathbf{r}}_{lmn0} = \frac{mh\,(nd\,\hat{\mathbf{x}} + lh\,\hat{\mathbf{y}} + mh\,\hat{\mathbf{z}})}{r^2_{lmn0}} \tag{5.11}$$

and

$$\sin\theta_{lmn0}\,\hat{\boldsymbol{\theta}}_{lmn0} = \frac{(mh)(nd)\,\hat{\mathbf{x}} + (mh)(lh)\,\hat{\mathbf{y}} - \left[(nd)^2+(lh)^2\right]\,\hat{\mathbf{z}}}{r^2_{lmn0}}\,. \tag{5.12}$$

The corresponding quantities in the self-plane summation of (5.1) are obtained from the quantities given by (5.3)-(5.12) by setting $n = 0$. The constants b_n are related to the z component of the electric field incident on any dipole in the nth plane by the scattering equation [4, eq. (59)]

$$b_n = SE^{0n}_{0z} \tag{5.13}$$

where S is the normalized dipole scattering coefficient of a short electric dipole. As in Section 4, "normalized" here means that b_n is the coefficient of $\exp(\mathrm{i}kr)/(kr)$ in the transverse component of the outgoing electric field in response to the incident field $E^{0n}_{0z}\hat{\mathbf{z}}$ at the center of the z directed electric dipole. We note that the x and y components of $\cos\theta_{lmn0}\,\hat{\mathbf{r}}_{lmn0}$ and $\sin\theta_{lmn0}\,\hat{\boldsymbol{\theta}}_{lmn0}$ are odd functions of l and m and so vanish when summed from $-\infty$ to ∞ over l and m. The z components, however, are even functions of l and m and so add when summed from $l, m = -\infty$ to ∞ over a plane. Thus

$$\mathbf{E}^0_0 = E^0_{0z}\hat{\mathbf{z}} \tag{5.14}$$

with

$$E^0_{0z} = \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} b_n \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr_{lmn0}}}{kr_{lmn0}}\left[\frac{-2\mathrm{i}}{kr_{lmn0}}\left(1+\frac{\mathrm{i}}{kr_{lmn0}}\right)\frac{(mh)^2}{r^2_{lmn0}}\right.$$
$$\left. + \left(1+\frac{\mathrm{i}}{kr_{lmn0}} - \frac{1}{(kr_{lmn0})^2}\right)\frac{(lh)^2+(nd)^2}{r^2_{lmn0}}\right]$$

$$+\, b_0 \sum_{\substack{l=-\infty \\ (l,m)}}^{\infty} \; \sum_{\substack{m=-\infty \\ \neq\; (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr_{lm00}}}{kr_{lm00}} \left[\frac{-2\mathrm{i}}{kr_{lm00}} \left(1+\frac{\mathrm{i}}{kr_{lm00}}\right) \frac{(mh)^2}{r_{lm00}^2} + \left(1+\frac{\mathrm{i}}{kr_{lm00}} - \frac{1}{(kr_{lm00})^2}\right) \frac{(lh)^2}{r_{lm00}^2} \right] \tag{5.15}$$

where r_{lmn0} is given by (5.3) and

$$r_{lm00} = h\sqrt{l^2+m^2}\,. \tag{5.16}$$

We now assume that the array is excited by a traveling wave in the x direction with real propagation constant β. Then the constants b_n in (5.15) are identical apart from a phase shift given by

$$b_n = b_0\, \mathrm{e}^{\mathrm{i}n\beta d} \tag{5.17}$$

and

$$\begin{aligned} E_{0z}^0 = b_0 \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr_{lmn0}}}{kr_{lmn0}} \Bigg[\frac{-2\mathrm{i}}{kr_{lmn0}} \left(1+\frac{\mathrm{i}}{kr_{lmn0}}\right) \frac{(mh)^2}{r_{lmn0}^2} \\ + \left(1+\frac{\mathrm{i}}{kr_{lmn0}} - \frac{1}{(kr_{lmn0})^2}\right) \frac{(lh)^2+(nd)^2}{r_{lmn0}^2} \Bigg] \\ + b_0 \sum_{\substack{l=-\infty \\ (l,m)}}^{\infty} \; \sum_{\substack{m=-\infty \\ \neq\; (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr_{lm00}}}{kr_{lm00}} \left[\frac{-2\mathrm{i}}{kr_{lm00}} \left(1+\frac{\mathrm{i}}{kr_{lm00}}\right) \frac{(mh)^2}{r_{lm00}^2} + \left(1+\frac{\mathrm{i}}{kr_{lm00}} - \frac{1}{(kr_{lm00})^2}\right) \frac{(lh)^2}{r_{lm00}^2} \right]. \end{aligned} \tag{5.18}$$

Since from (5.13)

$$b_0 = SE_{0z}^0 \tag{5.19}$$

it follows by substituting (5.19) in (5.18) and multiplying by $(kh)^3$ that

$$\begin{aligned} (kh)^3 = S\Bigg\{ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{lmn}}}{\rho_{lmn}} \Bigg[\frac{-2\mathrm{i}}{\rho_{lmn}} \left(kh+\frac{\mathrm{i}}{\rho_{lmn}}\right) \frac{m^2}{\rho_{lmn}^2} \\ + \left((kh)^2+\frac{\mathrm{i}kh}{\rho_{lmn}} - \frac{1}{\rho_{lmn}^2}\right) \frac{l^2+(nd/h)^2}{\rho_{lmn}^2} \Bigg] \\ + \sum_{\substack{l=-\infty \\ (l,m)}}^{\infty} \; \sum_{\substack{m=-\infty \\ \neq\; (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{lm0}}}{\rho_{lm0}} \left[\frac{-2\mathrm{i}}{\rho_{lm0}} \left(kh+\frac{\mathrm{i}}{\rho_{lm0}}\right) \frac{m^2}{\rho_{lm0}^2} + \left((kh)^2+\frac{\mathrm{i}kh}{\rho_{lm0}} - \frac{1}{\rho_{lm0}^2}\right) \frac{l^2}{\rho_{lm0}^2} \right] \Bigg\} \end{aligned} \tag{5.20}$$

where we have let

$$\rho_{lmn} = \sqrt{l^2+m^2+(nd/h)^2} \tag{5.21}$$

with

$$\rho_{lm0} = \sqrt{l^2+m^2}\,. \tag{5.22}$$

Equation (5.20) is the kd–βd equation that determines the normalized traveling wave propagation constant βd in terms of kh, d/h, and the normalized electric dipole scattering coefficient S.

The two-dimensional Poisson summation formula method cannot be used here, the way it was used in Section 3, to help convert the slowly convergent triple summation in (5.20) to a rapidly convergent form, because we are unable to calculate the double Fourier transform of functions like $f(x,y) = \exp[\mathrm{i}kh\sqrt{x^2+y^2+(nd/h)^2}]/[x^2+y^2+(nd/h)^2]^2 x^2$, and $f(x,y) = \exp[\mathrm{i}kh\sqrt{x^2+y^2+(nd/h)^2}]/[x^2+y^2+(nd/h)^2]^{5/2}[(y^2+(nd/h)^2]$. Accordingly we will use the Floquet mode method, proceeding similarly to the way we used it in Section 4. We let $E_z^0(P)$, be the z component of the electric field radiated by all the electric dipoles in the $n=0$ plane at a general point in space $P=(x,y,z)$, $x \neq 0$. (Note that, because of symmetry, $E_z^0(-x,y,z) = E_z^0(x,y,z)$.) We establish a local spherical polar coordinate system with origin at the dipole located at $(x,y,z) = (0,lh,mh)$ and with $\theta(l,m,P)$ the polar angle from the z axis to the vector $\mathbf{r}(l,m,P)$ from $(0,lh,mh)$ to the field point P. The distance $r(l,m,P)$ from $(0,lh,mh)$ to P is

$$r(l,m,P) = \sqrt{x^2+(y-lh)^2+(z-mh)^2} \tag{5.23}$$

and the unit vector $\hat{\mathbf{r}}(l,m,P$ is

$$\hat{\mathbf{r}}(l,m,P) = \frac{\mathbf{r}(l,m,P)}{r(l,m,P)} = \frac{x\,\hat{\mathbf{x}} + (y-lh)\,\hat{\mathbf{y}} + (z-mh)\,\hat{\mathbf{z}}}{r(l,m,P)} \tag{5.24}$$

so that

$$\cos\theta(l,m,P) = \hat{\mathbf{r}}(l,m,P)\cdot\hat{\mathbf{z}} = \frac{z-mh}{r(l,m,P)} \tag{5.25}$$

$$\sin\theta(l,m,P) = \sqrt{1-\cos^2\theta(l,m,P)} = \frac{\sqrt{x^2+(y-lh)^2}}{r(l,m,P)} \tag{5.26}$$

$$[\cos\theta(l,m,P)\,\hat{\mathbf{r}}(l,m,P)]_z = \cos^2\theta(l,m,P) = \frac{(z-mh)^2}{r^2(l,m,P)} \tag{5.27}$$

$$\left[\hat{\boldsymbol{\theta}}(l,m,P)\right]_z = -\sin\theta(l,m,P) = -\frac{\sqrt{x^2+(y-lh)^2}}{r(l,m,P)} \tag{5.28}$$

$$\left[\sin\theta(l,m,P)\,\hat{\boldsymbol{\theta}}(l,m,P)\right]_z = -\sin^2\theta(l,m,P) = -\frac{x^2+(y-lh)^2}{r^2(l,m,P)} \tag{5.29}$$

and hence, referring to (5.2),

$$E_z^0(|x|,y,z) = b_0 \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr(l,m,P)}}{kr(l,m,P)} \left[\frac{-2\mathrm{i}}{kr(l,m,P)}\left(1+\frac{\mathrm{i}}{kr(l,m,P)}\right)\frac{(z-mh)^2}{r^2(l,m,P)}\right.$$
$$\left. + \left(1+\frac{\mathrm{i}}{kr(l,m,P)} - \frac{1}{(kr(l,m,P))^2}\right)\frac{x^2+(y-lh)^2}{r^2(l,m,P)}\right]. \tag{5.30}$$

For $P = P_0 = (|n|d,0,0)$

$$E_z^0(|n|d,0,0) = b_0 \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr(l,m,P_0)}}{kr(l,m,P_0)} \left[\frac{-2\mathrm{i}}{kr(l,m,P_0)}\left(1+\frac{\mathrm{i}}{kr(l,m,P_0)}\right)\frac{(mh)^2}{r^2(l,m,P_0)}\right.$$

$$+\left(1+\frac{\mathrm{i}}{kr(l,m,P_0)}-\frac{1}{(kr(l,m,P_0))^2}\right)\frac{(nd)^2+(lh)^2}{r^2(l,m,P_0)}\Bigg] \tag{5.31}$$

where

$$r(l,m,P_0)=\sqrt{(lh)^2+(mh)^2+(nd)^2}\,. \tag{5.32}$$

But, referring to (5.21),

$$r(l,m,P_0)=h\rho_{lmn} \tag{5.33}$$

and thus

$$E_z^0(|n|d,0,0)=b_0\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{lmn}}}{kh\rho_{lmn}}\left[\frac{-2\mathrm{i}}{kh\rho_{lmn}}\left(1+\frac{\mathrm{i}}{kh\rho_{lmn}}\right)\frac{m^2}{\rho_{lmn}^2}\right.$$

$$\left.+\left(1+\frac{\mathrm{i}}{kh\rho_{lmn}}-\frac{1}{(kh)^2\rho_{lmn}^2}\right)\frac{l^2+(nd/h)^2}{\rho_{lmn}^2}\right]. \tag{5.34}$$

Hence, in (5.20),

$$\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{lmn}}}{\rho_{lmn}}\left[\frac{-2\mathrm{i}}{\rho_{lmn}}\left(kh+\frac{\mathrm{i}}{\rho_{lmn}}\right)\frac{m^2}{\rho_{lmn}^2}\right.$$

$$\left.+\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{lmn}}-\frac{1}{\rho_{lmn}^2}\right)\frac{l^2+(nd/h)^2}{\rho_{lmn}^2}\right]=\frac{(kh)^3}{b_0}E_z^0(|n|d,0,0)\,. \tag{5.35}$$

Now $E_z^0(|x|,y,z)$ can be expressed in terms of a plane wave spectrum by

$$E_z^0(|x|,y,z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}B(k_y,k_z)\,\mathrm{e}^{\mathrm{i}(k_yy+k_zz+k_x|x|)}\,\mathrm{d}k_y\mathrm{d}k_z,\;k_x=\sqrt{k^2-k_y^2-k_z^2} \tag{5.36}$$

where k_x is positive real (positive imaginary) according as $k^2>(<)\;k_y^2+k_z^2$. Because of the periodicity of the array in the y and z directions,

$$E_z^0(|x|,y+h,z)=E_z^0(|x|,y,z)\,,\quad E_z^0(|x|,y,z+h)=E^0(|x|,y,z)\,. \tag{5.37}$$

It follows from taking the inverse transform of (5.36) inserted into (5.37) that

$$\mathrm{e}^{\mathrm{i}k_yh}=1\,,\quad \mathrm{e}^{\mathrm{i}k_zh}=1 \tag{5.38}$$

and hence

$$k_yh=2\pi l,\;l=0,\pm1,\pm2,\cdots\,,\quad k_zh=2\pi m,\;m=0,\pm1,\pm2,\cdots \tag{5.39}$$

so that

$$E_z^0(|x|,y,z)=\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}B_{lm}\,\mathrm{e}^{\mathrm{i}(2\pi/h)(ly+mz)}\,\mathrm{e}^{\mathrm{i}k_{lm}|x|} \tag{5.40}$$

where

$$k_{lm}=\sqrt{k^2-(2\pi l/h)^2-(2\pi m/h)^2} \tag{5.41}$$

with k_{lm} positive real (positive imaginary) according as $(kh)^2 > (<)\ (2\pi)^2(l^2+m^2)$. It remains to find the unknown Floquet mode expansion coefficients B_{lm}. As in Section 4.1 we will employ two different methods for obtaining the coefficients, one based on the asymptotic analysis of an integral, and the other on the Hertz vector potential. We begin with the integral method.

By inverting (5.40)

$$B_{lm}e^{ik_{lm}|x|} = \frac{1}{h^2}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} E_z^0(|x|,y,z)e^{-i(2\pi/h)(ly+mz)}\,dydz \tag{5.42}$$

so that with (5.30)

$$B_{lm}e^{ik_{lm}|x|} =$$

$$\frac{b_0}{h^2}\sum_{l'=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\frac{e^{ikr(l',m',P)}}{kr(l',m',P)}\left[\frac{-2i}{kr(l',m',P)}\left(1+\frac{i}{kr(l',m',P)}\right)\frac{(z-m'h)^2}{r^2(l',m',P)}\right.$$

$$\left.+\left(1+\frac{i}{kr(l',m',P)}-\frac{1}{(kr(l',m',P))^2}\right)\frac{(y-l'h)^2+x^2}{r^2(l',m',P)}\right]e^{-i(2\pi/h)(ly+mz)}\,dydz \tag{5.43}$$

where

$$r(l',m',P) = \sqrt{x^2+(y-l'h)^2+(z-m'h)^2}\ . \tag{5.44}$$

Since B_{lm} is independent of x, if the LHS of (5.43) is expanded for small $|x|$

$$B_{lm}e^{ik_{lm}|x|} \overset{|x|<<1}{\sim} B_{lm}(1+ik_{lm}|x|)\ . \tag{5.45}$$

We can then obtain an expression for B_{lm} by investigating the behavior of the RHS of (5.43) for $|x| << 1$ and equating coefficients of $|x|$.

First we show that the terms in the double summation in (5.43) for which $(l',m') \neq (0,0)$ cannot contribute a term in $|x|$ for $|x| << 1$. For, letting

$$A^2 = (y-l'h)^2+(z-m'h)^2 \tag{5.46}$$

so that

$$r(l',m',P) = \sqrt{A^2+x^2} \tag{5.47}$$

and assuming that $x^2 << A^2$,

$$\frac{e^{ik\sqrt{(l'h-y)^2+(m'h-z)^2+x^2}}}{\sqrt{(l'h-y)^2+(m'h-z)^2+x^2}} = \frac{e^{ik\sqrt{A^2+x^2}}}{\sqrt{A^2+x^2}} \approx \frac{e^{ikA}}{A}\left[1+\left(\frac{ik}{2A}-\frac{1}{2A^2}x^2\right)+\cdots\right] \tag{5.48}$$

containing no term in $|x|$. Also,

$$\frac{2i}{kr(l',m',P)}\left(1+\frac{i}{kr(l',m',P)}\right)\frac{(z-m'h)^2}{r^2(l',m',P)} = \frac{2i}{k\sqrt{A^2+x^2}}\left(1+\frac{i}{k\sqrt{A^2+x^2}}\right)\frac{(z-m'h)^2}{A^2+x^2}$$

$$\approx \frac{2\mathrm{i}}{kA}\left(1-\frac{x^2}{2A^2}\right)\left[1+\frac{\mathrm{i}}{kA}\left(1-\frac{x^2}{2A^2}\right)\right]\frac{(z-m'h)^2}{A^2}\left(1-\frac{x^2}{A^2}\right) \tag{5.49}$$

containing no term in $|x|$. Similarly there is no term in $|x|$ in the expansion of

$$\left(1+\frac{\mathrm{i}}{kr(l',m',P)}-\frac{1}{(kr(l',m',P))^2}\right)\frac{(y-l'h)^2+x^2}{r^2(l',m',P)} \tag{5.50}$$

for $x^2 << A^2$. Hence a term in $|x|$ in the RHS of (5.43) for $|x| << 1$ can come only from the $(l',m')=(0,0)$ term

$$\frac{b_0}{h^2}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\frac{\mathrm{e}^{\mathrm{i}kr(0,0,P)}}{kr(0,0,P)}\left[\frac{-2\mathrm{i}}{kr(0,0,P)}\left(1+\frac{\mathrm{i}}{kr(0,0,P)}\right)\frac{z^2}{r^2(0,0,P)}\right.$$

$$\left.+\left(1+\frac{\mathrm{i}}{kr(0,0,P)}-\frac{1}{(kr(0,0,P))^2}\right)\frac{y^2+x^2}{r^2(0,0,P)}\right]\mathrm{e}^{-\mathrm{i}(2\pi/h)(ly+mz)}\,\mathrm{d}y\mathrm{d}z \tag{5.51}$$

where

$$r(0,0,P)=\sqrt{x^2+y^2+z^2}\,. \tag{5.52}$$

In cylindrical polar coordinates $\rho=\sqrt{y^2+z^2}$, $\phi=\tan^{-1}(y/z)$, the $(l',m')=(0,0)$ term is approximately

$$\frac{b_0}{kh^2}\int_0^{h/2}\int_0^{2\pi}\frac{\mathrm{e}^{\mathrm{i}k\sqrt{\rho^2+x^2}}}{\sqrt{\rho^2+x^2}}\left[\frac{-2\mathrm{i}}{k\sqrt{\rho^2+x^2}}\left(1+\frac{\mathrm{i}}{k\sqrt{\rho^2+x^2}}\right)\frac{\rho^2\cos^2\phi}{\rho^2+x^2}\right.$$

$$\left.+\left(1+\frac{\mathrm{i}}{k\sqrt{\rho^2+x^2}}-\frac{1}{k^2(\rho^2+x^2)}\right)\frac{\rho^2\sin^2\phi+x^2}{\rho^2+x^2}\right]\mathrm{e}^{-\mathrm{i}(2\pi/h)(l\sin\phi+m\cos\phi)\rho}\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,. \tag{5.53}$$

We can obtain a term in $|x|$ for $|x| << 1$ only in the vicinity of $\rho=0$. We expand the trigonometric exponential in (5.53) in a power series in ρ, and note that terms containing odd powers of $\sin\phi$ and $\cos\phi$ integrate to 0 over the interval $\phi=[0,2\pi]$, to obtain

$$\mathrm{e}^{-\mathrm{i}(2\pi/h)(l\sin\phi+m\cos\phi)\rho}\approx 1-\frac{1}{2}(2\pi/h)^2\ \left(l^2\sin^2\phi+m^2\cos^2\phi\right)\rho^2+\cdots\,. \tag{5.54}$$

We then substitute (5.54) in (5.53), perform the ϕ integrations using [18, eq. 3.62(3)], systematically obtain all the resulting indefinite integrals by making the change of variables

$$u=\sqrt{\rho^2+x^2},\ du=\frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2+x^2}} \tag{5.55}$$

and using integrals tabulated in [18, eqs. 2.324, 2.325], evaluate the integrals at the lower range of integration, $u=|x|$, and collect terms in $|x|$. (There is no contribution to terms in $|x|$ from the upper end of the interval of integration $u=\sqrt{(h/2)^2+x^2}$.) Terms higher than

ρ^2 in (5.54) are found not to contribute any terms in $|x|$. When this is done we find that the RHS of (5.43) behaves as

$$-\frac{2\pi b_0}{kh^2}\left[1-\left(\frac{2\pi m}{kh}\right)^2\right]|x| \tag{5.56}$$

for $|x| << 1$. But then, equating coefficients of $|x|$ in (5.45) and (5.56) we obtain the coefficients of the Floquet mode expansion (5.40)

$$B_{lm} = \frac{2\pi \mathrm{i} b_0}{kh^2 k_{lm}}\left[1-\left(\frac{2\pi m}{kh}\right)^2\right] \tag{5.57}$$

with k_{lm} given by (5.41).

We now give an alternate derivation of the Floquet mode expansion coefficients based on the Hertz vector potential, following the procedure used in (4.48)-(4.54). The starting point is the expression (4.48) for the electric field of a small z directed electric dipole at the origin of a Cartesian coordinate system yielding the expression (4.50) for the z component of the electric dipole field which we repeat here:

$$C\left(\frac{\partial^2}{\partial z^2}+k^2\right)\frac{\mathrm{e}^{\mathrm{i}kr}}{kr} \tag{5.58}$$

with $C = b_0/k^2$ from (4.49). Now from (3.30) and (3.42) the field radiated by the acoustic monopoles located in the plane $x = 0$ at the locations $(y, z) = (lh, mh)$, $l, m = 0, \pm h, \pm 2h, \cdots$, each of which radiates a field equal to $\mathrm{e}^{\mathrm{i}kr}/(kr)$, is (allowing for the different choice of coordinate axes in this section as compared with Section 3)

$$\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} B^0_{lm}\,\mathrm{e}^{\mathrm{i}(2\pi/h)(ly+mz)}\,\mathrm{e}^{\mathrm{i}k_{lm}|x|} \tag{5.59}$$

where

$$B^0_{lm} = \frac{2\pi\mathrm{i}}{kh^2 k_{lm}} \tag{5.60}$$

and

$$k_{lm} = \sqrt{k^2-(2\pi l/h)^2-(2\pi m/h)^2}\,. \tag{5.61}$$

Hence from (5.58) the z component of the electric field radiated by the plane $x = 0$ of z directed electric dipoles is equal to

$$C\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} B^0_{lm}\left(\frac{\partial^2}{\partial z^2}+k^2\right)\mathrm{e}^{\mathrm{i}(2\pi/h)(ly+mz)}\,\mathrm{e}^{\mathrm{i}k_{lm}|x|}$$

$$= C\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} B^0_{lm}\left[k^2-\left(\frac{2\pi m}{h}\right)^2\right]\mathrm{e}^{\mathrm{i}(2\pi/h)(ly+mz)}\,\mathrm{e}^{\mathrm{i}k_{lm}|x|}\,. \tag{5.62}$$

Since the same field is also given by the Floquet mode expansion (5.40), by equating (5.40) with (5.62) we obtain the coefficients B_{lm} in (5.40)

$$B_{lm} = CB^0_{lm}\left[k^2 - \left(\frac{2\pi m}{h}\right)^2\right] = \frac{b_0}{k^2}\frac{2\pi \mathrm{i}}{kh^2 k_{lm}}\left[k^2 - \left(\frac{2\pi m}{h}\right)^2\right]$$

$$= \frac{2\pi \mathrm{i} b_0}{kh^2 k_{lm}}\left[1 - \left(\frac{2\pi m}{kh}\right)^2\right] \tag{5.63}$$

so that, comparing (5.63) with (5.57), the two methods of obtaining the coefficients of the Floquet mode expansion yield the same result.

Now, referring to (5.35),(5.40), and (5.57) or (5.63),

$$\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{lmn}}}{\rho_{lmn}}\left[\frac{-2\mathrm{i}}{\rho_{lmn}}\left(kh + \frac{\mathrm{i}}{\rho_{lmn}}\right)\frac{m^2}{\rho^2_{lmn}}\right.$$

$$\left. + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{lmn}} - \frac{1}{\rho^2_{lmn}}\right)\frac{l^2 + (nd/h)^2}{\rho^2_{lmn}}\right]$$

$$= 2\pi \mathrm{i}\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[(kh)^2 - (2\pi m)^2\right]\frac{\mathrm{e}^{\mathrm{i}|n|(d/h)\sqrt{(kh)^2 - (2\pi)^2(l^2+m^2)}}}{\sqrt{(kh)^2 - (2\pi)^2(l^2+m^2)}}$$

$$= 2\pi \mathrm{i}\, kh\, \mathrm{e}^{\mathrm{i}|n|kd} - 2\pi \sum_{\substack{l=-\infty \\ (l,m)}}^{\infty} \ \sum_{\substack{m=-\infty \\ \neq\ (0,0)}}^{\infty}\left[(2\pi m)^2 - (kh)^2\right]\frac{\mathrm{e}^{-|n|(d/h)\sqrt{(2\pi)^2(l^2+m^2) - (kh)^2}}}{\sqrt{(2\pi)^2(l^2+m^2) - (kh)^2}} \tag{5.64}$$

where we have assumed that $0 < kh < 2\pi$. This is a remarkable conversion of a complicated and very slowly converging summation to a simple and rapidly converging summation. It follows that in (5.20)

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\mathrm{e}^{\mathrm{i}n\beta d}\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{lmn}}}{\rho_{lmn}}\left[\frac{-2\mathrm{i}}{\rho_{lmn}}\left(kh + \frac{\mathrm{i}}{\rho_{lmn}}\right)\frac{m^2}{\rho^2_{lmn}}\right.$$

$$\left. + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{lmn}} - \frac{1}{\rho^2_{lmn}}\right)\frac{l^2 + (nd/h)^2}{\rho^2_{lmn}}\right]$$

$$= 4\pi \mathrm{i}\, kh \sum_{n=1}^{\infty}\cos(n\beta d)\,\mathrm{e}^{\mathrm{i}nkd}$$

$$- 4\pi\sum_{n=1}^{\infty}\cos(n\beta d) \quad \sum_{\substack{l=-\infty \\ (l,m)}}^{\infty} \ \sum_{\substack{m=-\infty \\ \neq\ (0,0)}}^{\infty}\left[(2\pi m)^2 - (kh)^2\right]\frac{\mathrm{e}^{-n(d/h)\sqrt{(2\pi)^2(l^2+m^2) - (kh)^2}}}{\sqrt{(2\pi)^2(l^2+m^2) - (kh)^2}}$$

$$= -2\pi \mathrm{i}\, kh - 2\pi\, kh\,\frac{\sin kd}{\cos\beta d - \cos kd}$$

$$- 4\pi \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{\substack{l=-\infty \\ (l,m)}}^{\infty} \;\; {}_{\neq} \sum_{\substack{m=-\infty \\ (0,0)}}^{\infty} \left[(2\pi m)^2 - (kh)^2\right] \frac{\mathrm{e}^{-n(d/h)\sqrt{(2\pi)^2(l^2+m^2)-(kh)^2}}}{\sqrt{(2\pi)^2(l^2+m^2)-(kh)^2}} \tag{5.65}$$

where we have made use of (D.5). The sum

$$\sum_{n=1}^{\infty} \cos(n\beta d) \sum_{\substack{l=-\infty \\ (l,m)}}^{\infty} \;\; {}_{\neq} \sum_{\substack{m=-\infty \\ (0,0)}}^{\infty} \left[(2\pi m)^2 - (kh)^2\right] \frac{\mathrm{e}^{-n(d/h)\sqrt{(2\pi)^2(l^2+m^2)-(kh)^2}}}{\sqrt{(2\pi)^2(l^2+m^2)-(kh)^2}} \tag{5.66}$$

converges very rapidly because of the negative exponentials so that only a very few terms are needed in the sum, for example n from 1 to 2 and l, m from -2 to 2, for sufficient accuracy.

Now we consider the self-plane summation in (5.20)

$$\sum_{\substack{l=-\infty \\ (l,m)}}^{\infty} \;\; {}_{\neq} \sum_{\substack{m=-\infty \\ (0,0)}}^{\infty} \frac{\mathrm{e}^{ikh\rho_{lm0}}}{\rho_{lm0}} \left[\frac{-2\mathrm{i}}{\rho_{lm0}} \left(kh + \frac{\mathrm{i}}{\rho_{lm0}} \right) \frac{m^2}{\rho_{lm0}^2} + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{lm0}} - \frac{1}{\rho_{lm0}^2} \right) \frac{l^2}{\rho_{lm0}^2} \right] \tag{5.67}$$

where $\rho_{lm0} = \sqrt{l^2 + m^2}$. Referring to the $kd - \beta d$ equation (4.15) in the analysis of traveling waves on 2D arrays of z directed electric dipoles perpendicular to the array axis and in the array plane we see that the evaluation of the self-plane summation here is identical to the 2D problem with n replaced by l, β set equal to 0, and d replaced by h. Accordingly, referring to (4.45) and (4.56), the self-plane double summation is equal to

$$2 \sum_{l=1}^{\infty} \left[\mathrm{i}\pi (kh)^2 H_0^{(1)}(lkh) - 4 \sum_{m=1}^{\infty} \left[(2\pi m)^2 - (kh)^2\right] K_0\left(l\sqrt{(2\pi m)^2 - (kh)^2} \right) \right]$$

$$+ 4\, kh\, \mathrm{Cl}_2(kh) + 4\, \mathrm{Cl}_3(kh) + \mathrm{i}\pi (kh)^2 - \mathrm{i}\frac{2}{3}(kh)^3, \; 0 < kh < 2\pi \tag{5.68}$$

with the Clausen functions Cl_2 and Cl_3 defined and approximated by equations (D.8). In (5.68) the slowly convergent Schlömilch series

$$\sum_{l=1}^{\infty} H_0^{(1)}(lkh) \tag{5.69}$$

can be efficiently evaluated using the expressions (B.11) and (B.12). The series

$$\sum_{m=1}^{\infty} \left[(2\pi m)^2 - (kh)^2\right] K_0\left(l\sqrt{(2\pi m)^2 - (kh)^2} \right) \tag{5.70}$$

converges extremely rapidly because of the exponential decay of K_0. For example, for $l = 2$, $m = 2$, and $0 < kh < 2\pi$, $[(2\pi m)^2 - (kh)^2]\, K_0(l\sqrt{(2\pi m)^2 - (kh)^2}) < 1.2 \times 10^{-8}$. The series can thus be truncated keeping only a very few terms.

Similarly to what we have done in our treatment of 2D electric dipole arrays in Section 4, it is useful for calculation purposes to write the kd–βd equation (5.20) in the form

$$(kh)^3 = S\{\Re + \mathrm{i}\Im\} \tag{5.71}$$

where, from (5.64) and (5.68), $\Re$, the real part of the expression within the brackets of (5.20) with the original summations replaced by the rapidly convergent expressions we have derived, is given by

$$\Re = -2\pi\ kh\ \frac{\sin kd}{\cos\beta d - \cos kd}$$
$$- 4\pi \sum_{n=1}^{\infty} \cos(n\beta d) \quad \sum_{\substack{l=-\infty \\ (l,m)}}^{\infty} \ \ \sum_{\substack{m=-\infty \\ \neq\ (0,0)}}^{\infty} \left[(2\pi m)^2 - (kh)^2\right] \frac{\mathrm{e}^{-n(d/h)\ \sqrt{(2\pi)^2(l^2+m^2)-(kh)^2}}}{\sqrt{(2\pi)^2(l^2+m^2)-(kh)^2}}$$
$$- 2\sum_{l=1}^{\infty}\left[\pi(kh)^2\, Y_0(lkh) +\ 4\sum_{m=1}^{\infty}\left[(2\pi m)^2-(kh)^2\right] K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\ \right)\right]$$
$$+ 4\ kh\ \mathrm{Cl}_2(kh) + 4\ \mathrm{Cl}_3(kh) \tag{5.72}$$

with the Clausen functions Cl_2 and Cl_3 defined and approximated by (D.8), and $\Im$, the imaginary part of the expression within the brackets of (5.20), is given by

$$\Im = -2\pi\ kh + 2\pi(kh)^2\sum_{l=1}^{\infty} J_0(lkh) + \pi(kh)^2 - \frac{2}{3}(kh)^3 = -\frac{2}{3}(kh)^3 \tag{5.73}$$

using (B.11). As shown in Section 4 [see (4.59)-(4.61)] (5.73), together with (5.71), implies that the magnitude, $|S|$, and the phase, ψ, of the scattering coefficient, S, satisfy the relation

$$|S| = \frac{3}{2}\sin\psi\ . \tag{5.74}$$

This relation, as we noted in Section 4, was derived in [4] using reciprocity and power conservation relations, and has been shown here to also be a necessary condition for a 3D array of lossless short electric dipoles to support a traveling wave. The derivation of (5.74) thus serves as an important check on our analysis. Substituting (5.74) in (5.71) and equating real parts we obtain the form of the kd–βd equation that is used to calculate βd as a function of kh, d/h, and the phase ψ of the scattering coefficient

$$\frac{2}{3}(kh)^3\cos\psi - \Re\sin\psi = 0 \tag{5.75}$$

with $\Re$ given by (5.72) and $kh < 2\pi$. It is easy to solve (5.75) numerically for βd given values of kd, kh, and ψ, using, for example, a simple search procedure with secant algorithm refinement. In calculating $\Re$, the sum of exponentials is truncated in accordance with the remark following (5.66), the Neumann function sum is evaluated using (B.12), and the modified Bessel function sum is truncated in accordance with the remark following (5.70). Alternately an approximate closed form expression for the sum of exponentials can be obtained by first performing the summation over n from 1 to ∞ in closed form using (D.4) and then including only terms in the summation over l and m from -1 to 1. When this is done we obtain

$$\sum_{n=1}^{\infty} \cos(n\beta d) \quad \sum_{\substack{l=-\infty \\ (l,m)}}^{\infty} \ \ \sum_{\substack{m=-\infty \\ \neq\ (0,0)}}^{\infty} \left[(2\pi m)^2 - (kh)^2\right] \frac{\mathrm{e}^{-n(d/h)\ \sqrt{(2\pi)^2(l^2+m^2)-(kh)^2}}}{\sqrt{(2\pi)^2(l^2+m^2)-(kh)^2}}$$

$$
\approx 2\,\frac{(2\pi)^2-2(kh)^2}{\sqrt{(2\pi)^2-(kh)^2}}\;\frac{\mathrm{e}^{-(d/h)r_1}\;\cos\beta d-\mathrm{e}^{-2(d/h)r_1}}{1-2\cos\beta d\,\mathrm{e}^{-(d/h)r_1}+\mathrm{e}^{-2(d/h)r_1}}
$$

$$
+\,4\,\frac{(2\pi)^2-(kh)^2}{\sqrt{8\pi^2-(kh)^2}}\;\frac{\mathrm{e}^{-(d/h)r_2}\;\cos\beta d-\mathrm{e}^{-2(d/h)r_2}}{1-2\cos\beta d\,\mathrm{e}^{-(d/h)r_2}+\mathrm{e}^{-2(d/h)r_2}} \tag{5.76}
$$

where $r_1=\sqrt{(2\pi)^2-(kh)^2}$, and $r_2=\sqrt{8\pi^2-(kh)^2}$.

Since some of the terms in (5.72) become singular as kh approaches 2π, this equation cannot be used to calculate $\Re$ at $kh=2\pi$. It is therefore worthwhile to obtain the limit of $\Re$ given by (5.72) as $kh\to 2\pi$ from below. Consider first the singularity of

$$
-\,4\pi\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{\substack{l=-\infty\\(l,m)}}^{\infty}\;\sum_{\substack{m=-\infty\\ \neq\;(0,0)}}^{\infty}\left[(2\pi m)^2-(kh)^2\right]\frac{\mathrm{e}^{-n(d/h)\;\sqrt{(2\pi)^2(l^2+m^2)-(kh)^2}}}{\sqrt{(2\pi)^2(l^2+m^2)-(kh)^2}}\;. \tag{5.77}
$$

The two terms of the double summation over l and m for which $(l,m)=(\pm1,0)$ are singular as $kh\to 2\pi$, each of these terms behaving as

$$
-\frac{(2\pi)^2}{\sqrt{4\pi\epsilon}} \tag{5.78}
$$

for $\epsilon=2\pi-kh<<1$. Since from (D.5)

$$
\sum_{n=1}^{\infty}\cos(n\beta d)=-\frac{1}{2} \tag{5.79}
$$

$$
-4\pi\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{\substack{l=-\infty\\(l,m)}}^{\infty}\;\sum_{\substack{m=-\infty\\ \neq\;(0,0)}}^{\infty}\left[(2\pi m)^2-(kh)^2\right]\frac{\mathrm{e}^{-n(d/h)\;\sqrt{(2\pi)^2(l^2+m^2)-(kh)^2}}}{\sqrt{(2\pi)^2(l^2+m^2)-(kh)^2}}
$$

$$
\overset{\epsilon\to0}{\sim}\;-4\pi(2\pi)\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{\substack{l=-\infty\\|l|+|m|\;>}}^{\infty}\;\sum_{\substack{m=-\infty\\1}}^{\infty}(m^2-1)\frac{\mathrm{e}^{-2\pi n(d/h)\;\sqrt{l^2+m^2-1}}}{\sqrt{l^2+m^2-1}}
$$

$$
-\frac{\sqrt{4\pi}\,(2\pi)^2}{\sqrt{\epsilon}}\;. \tag{5.80}
$$

But using the Schlömilch summation formula (B.12) for the sum of the Neumann functions in (5.72) we have

$$
-\,2\sum_{l=1}^{\infty}\pi(kh)^2\,Y_0(lkh)\;\overset{kh\to2\pi}{\sim}\;(2\pi)^2\left[2\gamma-2+2\ln\frac{1}{2}+2\sum_{l=2}^{\infty}\left(\frac{1}{\sqrt{l^2-1}}-\frac{1}{l}\right)\right]+\frac{2(2\pi)^2\sqrt{\pi}}{\sqrt{\epsilon}} \tag{5.81}
$$

where γ is the Euler constant. Thus the $1/\sqrt{\epsilon}$ singularity in (5.81) exactly cancels the corresponding singularity from the sum of the negative exponentials in (5.80). As we have seen above [see (4.63)] the singularity of $K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\right)$ for $m=1$ at $kh=2\pi$

is eliminated by the multiplicative factor of $(2\pi m)^2 - (kh)^2$ so that $\Re$ is not singular at $kh = 2\pi$. Its limiting value is given by

$$\lim_{kh\to 2\pi} \Re = -(2\pi)^2 \frac{\sin kd}{\cos\beta d - \cos kd}$$

$$- 4\pi(2\pi)\sum_{n=1}^{\infty}\cos(n\beta d) \sum_{\substack{l=-\infty \\ |l|+|m|}}^{\infty} \;\; > \sum_{\substack{m=-\infty \\ 1}}^{\infty}(m^2-1)\frac{\mathrm{e}^{-2\pi n(d/h)\sqrt{l^2+m^2-1}}}{\sqrt{l^2+m^2-1}}$$

$$+ 2(2\pi)^2\left[\gamma - 1 + \ln\frac{1}{2} + \sum_{l=2}^{\infty}\left(\frac{1}{\sqrt{l^2-1}} - \frac{1}{l}\right)\right]$$

$$- 8(2\pi)^2 \sum_{m=2}^{\infty}(m^2-1)K_0\left(2\pi l\sqrt{m^2-1}\right) + 4\,\mathrm{Cl}_3(2\pi) \tag{5.82}$$

where we have set $\mathrm{Cl}_2(2\pi) = 0$ [see (D.9)]. The value of $\mathrm{Cl}_3(2\pi)$ is given by (D.10). The sum of negative exponentials in (5.82) is very small and can be neglected.

In closing this section we note that if the array of dipoles are close together then the array can be regarded macroscopically as an anisotropic medium with effective or bulk relative permeability $\mu_\mathrm{r}^\mathrm{eff} = 1$ and effective relative permittivity $\epsilon_\mathrm{r}^\mathrm{eff}$ that determine the propagation constant of a traveling wave in the direction of the array axis perpendicular to the orientation of the electric dipoles of the array. The effective relative permeability and permittivity satisfy the equation

$$\frac{\beta d}{kd} = \sqrt{\mu_\mathrm{r}^\mathrm{eff}\epsilon_\mathrm{r}^\mathrm{eff}} = \sqrt{\epsilon_\mathrm{r}^\mathrm{eff}} \tag{5.83}$$

so that

$$\epsilon_\mathrm{r}^\mathrm{eff} = \left(\frac{\beta d}{kd}\right)^2 \tag{5.84}$$

where βd is the solution of the kd–βd equation (5.75). For more details the reader is referred to Subsection 9.2, (9.109), where the effective relative permittivity of the dipole array of this section of the report is obtained as the special case of the effective relative permittivity of a 3D array of combined electric and magnetic dipoles perpendicular to the array axis, when the magnetic dipoles are absent.

6 2D ELECTRIC DIPOLE ARRAYS, DIPOLES ORIENTED PARALLEL TO THE ARRAY AXIS

In this section we consider traveling waves supported by 2D periodic arrays of lossless short electric dipoles with the dipoles oriented parallel to the array axis rather than perpendicular to it as in Section 4. We follow the same major steps of the procedure — calculating the field at a reference element due to all the other elements in the array, deriving the kd–βd equation by assuming a traveling wave excitation of the array, and converting slowly convergent summations to rapidly convergent ones to obtain a form of the kd–βd equation suitable for calculation purposes — used in treating 2D and 3D periodic arrays of lossless electric dipoles oriented perpendicular to the array axis in Sections 4 and 5. As in Sections 4.1 and 5 the Floquet mode expansion method will be used to convert slowly convergent summations to rapidly convergent ones, with two different procedures used to obtain the Floquet mode expansion coefficients — a procedure based on the asymptotic analysis of an integral, and a Hertz vector potential procedure.

Although it may appear that it would be more convenient here to choose the array axis to be the z axis rather than the x axis as in Section 4, the use of the Floquet mode method to convert the slowly convergent summations of the kd–βd equation to rapidly convergent ones is simpler if we continue to use the x axis as the array axis, because the field radiated by a column of dipoles then splits naturally into a term with no ϕ variation and a term with $\cos 2\phi$ variation. Equispaced columns of x directed electric dipoles are located at $x = nd,\ n = 0,\ \pm 1,\ \pm 2,\ \cdots$. In each column the dipoles are centered at $z = mh,\ m = 0,\ \pm 1,\ \pm 2,\ \cdots$. We assume an excitation of the array with the electric field parallel to the x axis and such that all the dipoles in any column of the array are excited identically. Because of the symmetry of the dipole locations and excitations the electric field incident on the dipole at the location $y = 0, z = 0, x = 0$ from all the other dipoles in the array has an x component only which we denote by E^0_{0x}. Let E^{0mn}_{0x} be the x component of the electric field incident on the reference dipole from the electric dipole at the location $(z, x) = (mh, nd)$ so that

$$E^0_{0x} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{m=-\infty}^{\infty} E^{0mn}_{0x} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} E^{0m0}_{0x} . \tag{6.1}$$

We proceed to obtain expressions for E^{0mn}_{0x} and E^{0m0}_{0x}. Now the electric field of a short electric dipole located at the origin of a Cartesian coordinate system with the z axis in the direction of the dipole is [4, eq. (40)]

$$\mathbf{E}(\mathbf{r}) = \frac{\mathrm{e}^{\mathrm{i}kr}}{kr}\left[\frac{-2\mathrm{i}}{kr}\left(1+\frac{\mathrm{i}}{kr}\right)\cos\theta\,\hat{\mathbf{r}} - \left(1+\frac{\mathrm{i}}{kr}-\frac{1}{(kr)^2}\right)\sin\theta\,\hat{\boldsymbol{\theta}}\right] . \tag{6.2}$$

What we want to do is to obtain an expression for the field of a short electric dipole at the origin of a Cartesian coordinate system whose x axis is in the direction of the electric dipole. If we let α be the polar angle measured from the x axis then

$$\mathbf{E}(\mathbf{r}) = \frac{\mathrm{e}^{\mathrm{i}kr}}{kr}\left[\frac{-2\mathrm{i}}{kr}\left(1+\frac{\mathrm{i}}{kr}\right)\cos\alpha\,\hat{\mathbf{r}} - \left(1+\frac{\mathrm{i}}{kr}-\frac{1}{(kr)^2}\right)\sin\alpha\,\hat{\boldsymbol{\alpha}}\right] . \tag{6.3}$$

But [23, sec. 4-6]

$$\hat{\boldsymbol{\alpha}} = -\frac{1}{\sqrt{1-\sin^2\theta\cos^2\phi}}(\cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}) \tag{6.4}$$

with

$$\cos\alpha = \sin\theta\ \cos\phi \tag{6.5a}$$

and

$$\sin\alpha = \sqrt{1 - \sin^2\theta\ \cos^2\phi} \tag{6.5b}$$

so that in a Cartesian coordinate system with the x axis in the direction of the electric dipole, the field of a short electric dipole at the origin is

$$\mathbf{E}(\mathbf{r}) = \frac{\mathrm{e}^{\mathrm{i}kr}}{kr}\left[\frac{-2\mathrm{i}}{kr}\left(1+\frac{\mathrm{i}}{kr}\right)\sin\theta\cos\phi\,\hat{\mathbf{r}} + \left(1+\frac{\mathrm{i}}{kr}-\frac{1}{(kr)^2}\right)\left(\cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}\right)\right] \tag{6.6}$$

and the x component of the field is

$$E_x(\mathbf{r}) = \frac{\mathrm{e}^{\mathrm{i}kr}}{kr}\left[\frac{-2\mathrm{i}}{kr}\left(1+\frac{\mathrm{i}}{kr}\right)\sin^2\theta\cos^2\phi + \left(1+\frac{\mathrm{i}}{kr}-\frac{1}{(kr)^2}\right)\left(\cos^2\theta\cos^2\phi + \sin^2\phi\right)\right]. \tag{6.7}$$

Hence in (6.1)

$$E_{0x}^{0mn} = b_n\frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}}\left[\frac{-2\mathrm{i}}{kr_{mn0}}\left(1+\frac{\mathrm{i}}{kr_{mn0}}\right)\sin^2\theta_{mn0}\cos^2\phi_{mn0}\right.$$

$$\left. + \left(1+\frac{\mathrm{i}}{kr_{mn0}}-\frac{1}{(kr_{mn0})^2}\right)\left(\cos^2\theta_{mn0}\cos^2\phi_{mn0} + \sin^2\phi_{mn0}\right)\right]. \tag{6.8}$$

The quantities in (6.8) are defined with reference to a local spherical polar coordinate system with origin at $(z,x) = (mh, nd)$ (in turn defined with reference to a local Cartesian coordinate system with the same origin whose axes are parallel to those of the global Cartesian coordinate system). The distance from the (m,n) dipole to the $(0,0)$ dipole, r_{mn0}, is given by

$$r_{mn0} = \sqrt{(mh)^2 + (nd)^2} = h\sqrt{m^2 + (nd/h)^2} \tag{6.9}$$

and the unit vector in the direction from the (m,n) dipole to the $(0,0)$ dipole, $\hat{\mathbf{r}}_{mn0}$, is

$$\hat{\mathbf{r}}_{mn0} = \mathbf{r}_{mn0}/r_{mn0}, \quad \mathbf{r}_{mn0} = -mh\,\hat{\mathbf{z}} - nd\,\hat{\mathbf{x}} \tag{6.10}$$

so that

$$\cos\theta_{mn0} = \hat{\mathbf{r}}_{mn0}\cdot\hat{\mathbf{z}} = -\frac{mh}{r_{mn0}} \tag{6.11}$$

$$\sin\theta_{mn0} = \sqrt{1-\cos^2\theta_{mn0}} = \frac{|n|d}{r_{mn0}} \tag{6.12}$$

$$\phi_{mn0} = \tan^{-1}\frac{\hat{\mathbf{r}}_{mn0}\cdot\hat{\mathbf{y}}}{\hat{\mathbf{r}}_{mn0}\cdot\hat{\mathbf{x}}} = \tan^{-1}\frac{0}{-nd} = \begin{cases} 0 & : \quad n<0 \\ \pi & : \quad n>0 \end{cases} \tag{6.13}$$

$$\cos^2\phi_{mn0} = 1 \tag{6.14}$$

and

$$\sin^2\phi_{mn0} = 0\ . \tag{6.15}$$

The constants b_n in (6.8) are related to the x component of the electric field incident on any dipole in the nth column by the scattering equation [4, eq. (59)]

$$b_n = SE_{0x}^{0n} \tag{6.16}$$

where S is the normalized dipole scattering coefficient of a short electric dipole. "Normalized" means that b_n is the coefficient of $\exp(\mathrm{i}kr)/(kr)$ in the transverse component of the outgoing electric field in response to the incident field $E_{0x}^{0n}\,\hat{\mathbf{x}}$ at the center of the x directed electric dipole. If we let

$$\rho_{mn} = \sqrt{m^2 + (nd/h)^2} \tag{6.17}$$

then

$$r_{mn0} = h\rho_{mn} \tag{6.18}$$

$$E_{0x}^{0mn} = b_n \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{kh\rho_{mn}} \left[\frac{-2\mathrm{i}}{kh\rho_{mn}} \left(1 + \frac{\mathrm{i}}{kh\rho_{mn}}\right) \frac{(nd/h)^2}{\rho_{mn}^2} + \left(1 + \frac{\mathrm{i}}{kh\rho_{mn}} - \frac{1}{(kh)^2\rho_{mn}^2}\right) \frac{m^2}{\rho_{mn}^2} \right] \tag{6.19}$$

and in the self-column, $n = 0$,

$$E_{0x}^{0m0} = b_0 \frac{\mathrm{e}^{\mathrm{i}kh|m|}}{kh|m|} \left(1 + \frac{\mathrm{i}}{kh|m|} - \frac{1}{(kh)^2 m^2}\right) . \tag{6.20}$$

Substituting (6.19) and (6.20) in (6.1) we obtain the electric field incident on the dipole at $(z, x) = (0, 0)$ scattered from all the other dipoles in the array

$$E_{0x}^{0} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} b_n \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{kh\rho_{mn}} \left[\frac{-2\mathrm{i}}{kh\rho_{mn}} \left(1 + \frac{\mathrm{i}}{kh\rho_{mn}}\right) \frac{(nd/h)^2}{\rho_{mn}^2} \right.$$

$$\left. + \left(1 + \frac{\mathrm{i}}{kh\rho_{mn}} - \frac{1}{(kh)^2\rho_{mn}^2}\right) \frac{m^2}{\rho_{mn}^2} \right] + b_0 \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh|m|}}{kh|m|} \left(1 + \frac{\mathrm{i}}{kh|m|} - \frac{1}{(kh)^2 m^2}\right) . \tag{6.21}$$

We now assume that the array is excited by a traveling wave in the x direction with real propagation constant β. Then the constants b_n in (6.21) are identical apart from a periodic phase shift

$$b_n = b_0\, \mathrm{e}^{\mathrm{i}n\beta d} \tag{6.22}$$

and

$$E_{0x}^{0} = b_0 \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{kh\rho_{mn}} \left[\frac{-2\mathrm{i}}{kh\rho_{mn}} \left(1 + \frac{\mathrm{i}}{kh\rho_{mn}}\right) \frac{(nd/h)^2}{\rho_{mn}^2} \right.$$

$$\left. + \left(1 + \frac{\mathrm{i}}{kh\rho_{mn}} - \frac{1}{(kh)^2\rho_{mn}^2}\right) \frac{m^2}{\rho_{mn}^2} \right] + 2b_0 \sum_{m=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}khm}}{khm} \left(1 + \frac{\mathrm{i}}{khm} - \frac{1}{(kh)^2 m^2}\right) . \tag{6.23}$$

Since from (6.16)

$$b_0 = SE_{0x}^{0} \tag{6.24}$$

it follows by substituting (6.24) in (6.23) and multiplying by $(kh)^3$ that

$$(kh)^3 = S\Bigg\{ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \Bigg[\frac{-2\mathrm{i}}{\rho_{mn}} \left(kh + \frac{\mathrm{i}}{\rho_{mn}} \right) \frac{(nd/h)^2}{\rho_{mn}^2}$$

$$+ \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} - \frac{1}{\rho_{mn}^2} \right) \frac{m^2}{\rho_{mn}^2} \Bigg] + 2\sum_{m=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}khm}}{m} \left((kh)^2 + \frac{\mathrm{i}kh}{m} - \frac{1}{m^2} \right) \Bigg\} . \tag{6.25}$$

Equation (6.25) is the kd–βd equation that determines the normalized traveling wave propagation constant βd in terms of kh, d/h, and the normalized electric dipole scattering coefficient S. Although (6.25) could have been obtained considerably more simply had we chosen the z axis of the global Cartesian coordinate system to be the array axis, the advantage of choosing the x axis as the array axis is that the Floquet mode procedure to help convert the very slowly convergent double summation in (6.25) to a rapidly convergent one becomes more transparent, and makes use of the representation (6.7) of the field of an x-directed electric dipole.

To begin the Floquet mode procedure we let $E_x^0(x,y,z)$ be the x component of the electric field radiated by all the x directed electric dipoles in the $n = 0$ column at the general field point $(x,y,z), x \neq 0$ and proceed to obtain an expression for $E_x^0(x,y,z)$. We establish a local spherical polar coordinate system with origin at the dipole located at $(x,y,z) = (0,0,mh)$ and with $\theta(m,x,y,z)$ the polar angle from the z axis to the vector $\mathbf{r}(m,x,y,z)$ from $(0,0,mh)$ to the field point (x,y,z). The distance $r(m,x,y,z)$ from $(0,0,mh)$ to (x,y,z) is given by

$$r(m,x,y,z) = \sqrt{x^2 + y^2 + (z-mh)^2} \tag{6.26}$$

the unit vector $\hat{\mathbf{r}}(m,x,y,z)$ in the direction from the dipole at $(0,0,mh)$ to the field point is

$$\hat{\mathbf{r}}(m,x,y,z) = \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + (z-mh)\,\hat{\mathbf{z}}}{r(m,x,y,z)} \tag{6.27}$$

and the trigonometric functions are given by

$$\cos\theta(m,x,y,z) = \hat{\mathbf{r}}(m,x,y,z)\cdot\hat{\mathbf{z}} = \frac{z-mh}{r(m,x,y,z)} \tag{6.28}$$

$$\sin\theta(m,x,y,z) = \frac{x^2+y^2}{r(m,x,y,z)} \tag{6.29}$$

$$\phi(m,x,y,z) = \tan^{-1}\frac{\hat{\mathbf{r}}(mx,y,z)\cdot\hat{\mathbf{y}}}{\hat{\mathbf{r}}(mx,y,z)\cdot\hat{\mathbf{x}}} = \tan^{-1}\frac{y}{x} \tag{6.30}$$

$$\cos\phi(m,x,y,z) = \frac{x}{\sqrt{x^2+y^2}} \tag{6.31}$$

and

$$\sin\phi(m,x,y,z) = \frac{y}{\sqrt{x^2+y^2}} \,. \tag{6.32}$$

Then, referring to (6.7),

$$E_x^0(x,y,z) = b_0 \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr(m,x,y,z)}}{kr(m,x,y,z)} \left[\frac{-2\mathrm{i}}{kr(m,x,y,z)} \left(1 + \frac{\mathrm{i}}{kr(m,x,y,z)}\right) \frac{x^2}{r^2(m,x,y,z)} \right.$$

$$\left. + \left(1 + \frac{\mathrm{i}}{kr(m,x,y,z)} - \frac{1}{(kr(m,x,y,z))^2}\right) \frac{y^2 + (z-mh)^2}{r^2(m,x,y,z)} \right] . \tag{6.33}$$

When $x = nd,\ y = 0,\ z = 0$

$$r(m,nd,0,0) = \sqrt{(mh)^2 + (nd)^2} = h\sqrt{m^2 + (nd/h)^2} = h\rho_{mn} \tag{6.34}$$

[see (6.17)] and in (6.25)

$$\sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \left[\frac{-2\mathrm{i}}{\rho_{mn}} \left(kh + \frac{\mathrm{i}}{\rho_{mn}}\right) \frac{(nd/h)^2}{\rho_{mn}^2} + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} - \frac{1}{\rho_{mn}^2}\right) \frac{m^2}{\rho_{mn}^2} \right]$$

$$= \frac{(kh)^3}{b_0} E_x^0(|n|d,0,0) \ . \tag{6.35}$$

In (6.33) let $\rho = \sqrt{x^2+y^2)},\ x = \rho\cos\phi,\ y = \rho\sin\phi$. Then

$$x^2 = \frac{\rho^2}{2}(1+\cos 2\phi) \tag{6.36a}$$

$$y^2 = \frac{\rho^2}{2}(1-\cos 2\phi) \tag{6.36b}$$

and

$$E_x^0(\rho,\phi,z) = b_0 \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr(m,\rho,z)}}{kr(m,\rho,z)} \left\{ \left(1 + \frac{\mathrm{i}}{kr(m,\rho,z)} - \frac{1}{(kr(m,\rho,z))^2}\right) \frac{(z-mh)^2}{r^2(m,\rho,z)} \right.$$

$$+ \left[\frac{-2\mathrm{i}}{kr(m,\rho,z)} \left(1 + \frac{\mathrm{i}}{kr(m,\rho,z)}\right) + \left(1 + \frac{\mathrm{i}}{kr(m,\rho,z)} - \frac{1}{(kr(m,\rho,z))^2}\right) \right] \frac{\rho^2}{2r^2(m,\rho,z)}$$

$$\left. + \left[\frac{-2\mathrm{i}}{kr(m,\rho,z)} \left(1 + \frac{\mathrm{i}}{kr(m,\rho,z)}\right) - \left(1 + \frac{\mathrm{i}}{kr(m,\rho,z)} - \frac{1}{(kr(m,\rho,z))^2}\right) \right] \frac{\rho^2}{2r^2(m,\rho,z)} \cos 2\phi \right\}$$

$$= E_{x1}^0(\rho,z) + E_{x2}^0(\rho,\phi,z) \tag{6.37}$$

where

$$E_{x1}^0(\rho,z) = b_0 \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr(m,\rho,z)}}{kr(m,\rho,z)} \left\{ \left(1 + \frac{\mathrm{i}}{kr(m,\rho,z)} - \frac{1}{(kr(m,\rho,z))^2}\right) \frac{(z-mh)^2}{r^2(m,\rho,z)} \right.$$

$$\left. + \left[\frac{-2\mathrm{i}}{kr(m,\rho,z)} \left(1 + \frac{\mathrm{i}}{kr(m,\rho,z)}\right) + \left(1 + \frac{\mathrm{i}}{kr(m,\rho,z)} - \frac{1}{(kr(m,\rho,z))^2}\right) \right] \frac{\rho^2}{2r^2(m,\rho,z)} \right\} \tag{6.38a}$$

and

$$E^0_{x2}(\rho,\phi,z) = b_0 \sum_{m=-\infty}^{\infty} \frac{e^{ikr(m,\rho,z)}}{kr(m,\rho,z)} \left[\frac{-2i}{kr(m,\rho,z)} \left(1 + \frac{i}{kr(m,\rho,z)}\right) - \left(1 + \frac{i}{kr(m,\rho,z)} - \frac{1}{(kr(m,\rho,z))^2}\right) \right] \frac{\rho^2}{2r^2(m,\rho,z)} \cos 2\phi \ . \tag{6.38b}$$

Now $E^0_{x1}(\rho,z)$ and $E^0_{x2}(\rho,\phi,z)$ can be expressed in terms of cylindrical waves by [20, sec. 6.6]

$$E^0_{x1}(\rho,z) = \int_{-\infty}^{\infty} B_1(k_z) H_0^{(1)}(k_\rho \rho) e^{ik_z z} dk_z, \ k_\rho = \sqrt{k^2 - k_z^2} \tag{6.39a}$$

and

$$E^0_{x2}(\rho,z) = \cos 2\phi \int_{-\infty}^{\infty} B_2(k_z) H_2^{(1)}(k_\rho \rho) e^{ik_z z} dk_z, \ k_\rho = \sqrt{k^2 - k_z^2} \tag{6.39b}$$

where $H_0^{(1)}$ and $H_2^{(1)}$ are the Hankel functions of the first kind of order zero and two, respectively, and k_ρ is positive real (positive imaginary) according as $k^2 > (<)\ k_z^2$. Because of the periodicity of $E^0_{x1}(\rho,z)$ and $E^0_{x2}(\rho,\phi,z)$ in z with period h, it follows that [see (4.30)-(4.34)]

$$E^0_{x1}(\rho,z) = \sum_{m=-\infty}^{\infty} B_{1m} H_0^{(1)}(k_m \rho)\ e^{i(2\pi/h)mz} \tag{6.40a}$$

and

$$E^0_{x2}(\rho,\phi,z) = \cos 2\phi \sum_{m=-\infty}^{\infty} B_{2m} H_2^{(1)}(k_m \rho)\ e^{i(2\pi/h)mz} \tag{6.40b}$$

where

$$k_m = \sqrt{k^2 - (2\pi m/h)^2} \tag{6.41}$$

with k_m positive real (positive imaginary) according as $(kh)^2 > (<)\ (2\pi m)^2$. By inversion

$$B_{1m} H_0^{(1)}(k_m \rho) = \frac{1}{h} \int_{-h/2}^{h/2} E^0_{x1}(\rho,z)\ e^{-i(2\pi/h)mz}\ dz \tag{6.42a}$$

and

$$B_{2m} \cos 2\phi\ H_2^{(1)}(k_m \rho) = \frac{1}{h} \int_{-h/2}^{h/2} E^0_{x2}(\rho,\phi,z)\ e^{-i(2\pi/h)mz}\ dz \ . \tag{6.42b}$$

Then in the RHS of (6.35)

$$\frac{(kh)^3}{b_0} E^0_x(|n|d,0,0) = \frac{(kh)^3}{b_0} \left[\sum_{m=-\infty}^{\infty} B_{1m} H_0^{(1)}(k_m |n| d) + \sum_{m=-\infty}^{\infty} B_{2m} H_2^{(1)}(k_m |n| d) \right] \ . \tag{6.43}$$

It remains to obtain expressions for the unknown Floquet mode expansion coefficients B_{1m} and B_{2m}. As we have done in Sections 4.1 and 5, two independent procedures will be used, one based on the asymptotic behavior of an integral, and the other based on the Hertz vector potential, thereby providing an important check on the validity of the expressions obtained.

We begin with the use of the integral method to obtain expressions for B_{1m}. From (6.42a) and (6.38a)

$$B_{1m}H_0^1(k_m\rho) = \frac{b_0}{kh}\sum_{m'=-\infty}^{\infty}\int_{-h/2}^{h/2}\frac{\mathrm{e}^{\mathrm{i}kr(m',\rho,z)}}{r(m',\rho,z)}\left\{\left(1+\frac{\mathrm{i}}{kr(m',\rho,z)}-\frac{1}{(kr(m',\rho,z))^2}\right)\frac{(z-m'h)^2}{r^2(m',\rho,z)}\right.$$
$$+\left[\frac{-2\mathrm{i}}{kr(m',\rho,z)}\left(1+\frac{\mathrm{i}}{kr(m',\rho,z)}\right)+\left(1+\frac{\mathrm{i}}{kr(m',\rho,z)}\right.\right.$$
$$\left.\left.\left.-\frac{1}{(kr(m',\rho,z))^2}\right)\right]\frac{\rho^2}{2r^2(m',\rho,z)}\right\}\mathrm{e}^{-\mathrm{i}(2\pi/h)mz}\mathrm{d}z \qquad (6.44)$$

with

$$r(m',\rho,z) = \sqrt{\rho^2+(z-m'h)^2}\ . \qquad (6.45)$$

Since B_{1m} is independent of ρ, for $\rho << 1$ the LHS of (6.44) behaves as [see (C.4)]

$$B_{1m}H_0^{(1)}(k_m\rho) \overset{\rho<<1}{\sim} \frac{2\mathrm{i}}{\pi}B_{1m}\ln\rho\ . \qquad (6.46)$$

Hence the RHS of (6.44) must also have a $\ln\rho$ singularity as $\rho \to 0$. By equating $(2\mathrm{i}/\pi)B_m$ with the coefficient of the $\ln\rho$ singularity of the RHS of (6.44) we can then obtain B_m. In investigating the singularity of the RHS of (6.44) as $\rho \to 0$ we note that we can ignore all terms in the summation over m' for which $m' \neq 0$ since these terms are not singular as $\rho \to 0$. We must therefore consider the behavior for $\rho << 1$ of

$$\frac{2b_0}{kh}\int_0^{h/2}\frac{\mathrm{e}^{\mathrm{i}kr(0,\rho,z)}}{r(0,\rho,z)}\left\{\left(1+\frac{\mathrm{i}}{kr(0,\rho,z)}-\frac{1}{(kr(0,\rho,z))^2}\right)\frac{z^2}{r^2(0,\rho,z)}+\left[\frac{-2\mathrm{i}}{kr(0,\rho,z)}\left(1+\frac{\mathrm{i}}{kr(0,\rho,z)}\right)\right.\right.$$
$$\left.\left.+\left(1+\frac{\mathrm{i}}{kr(0,\rho,z)}-\frac{1}{(kr(0,\rho,z))^2}\right)\right]\frac{\rho^2}{2r^2(0,\rho,z)}\right\}\cos(2\pi/h)mz\ \mathrm{d}z \qquad (6.47)$$

with

$$r(0,\rho,z) = \sqrt{\rho^2+z^2}\ . \qquad (6.48)$$

Any logarithmic singularity of (6.47) as $\rho \to 0$ must come from the vicinity of $z = 0$. To obtain the logarithmic singularity we expand both $\exp[\mathrm{i}kr(0,\rho,z)]$ and $\cos(2\pi/h)mz$ in power series in z, systematically obtain all the resulting indefinite integrals using integrals tabulated in [18, eqs. 2.17, 2.26], evaluate the integrals at the lower end of the range of integration, $z = 0$, and collect terms in $\ln\rho$. (There is no contribution to the logarithmic singularity from the upper end of the interval of integration, $z = h/2$.) It is found that it is necessary to include terms through $-[kr(0,z,\rho)]^2/2$ in the expansion of the exponential and terms through $-(2\pi m/h)^2 z^2/2$ in the expansion of the cosine to include all contributions to the $\ln\rho$ singularity. [Interestingly it is found that there is no contribution to the $\ln\rho$ singularity from

$$\frac{2b_0}{kh}\int_0^{h/2}\frac{\mathrm{e}^{ikr(0,\rho,z)}}{r(0,\rho,z)}\left\{\left[\frac{-2\mathrm{i}}{kr(0,\rho,z)}\left(1+\frac{\mathrm{i}}{kr(0,\rho,z)}\right)\right.\right.$$

$$\left.\left.+\left(1+\frac{\mathrm{i}}{kr(0,\rho,z)}-\frac{1}{(kr(0,\rho,z))^2}\right)\right]\frac{\rho^2}{2r^2(0,\rho,z)}\right\}\cos(2\pi/h)mz\,\mathrm{d}z \tag{6.49}$$

in (6.47).] When this is done we find that the RHS of (6.47) behaves as

$$-\frac{b_0}{kh}\left[1+\left(\frac{2\pi m}{kh}\right)^2\right]\ln\rho \tag{6.50}$$

for $\rho << 1$. But then, equating the coefficient of the $\ln\rho$ singularity of the LHS of (6.44) given by (6.46) with the coefficient of the $\ln\rho$ singularity of the RHS of (6.44) given by (6.50) and solving for B_{1m} we obtain

$$B_{1m}=\frac{\mathrm{i}\pi}{2}\frac{b_0}{kh}\left[1+\left(\frac{2\pi m}{kh}\right)^2\right]\ . \tag{6.51}$$

Next we use the integral method to obtain expressions for the Floquet mode expansion coefficients B_{2m} in (6.42b). From (6.42b) and (6.38b)

$$B_{2m}H_2^{(1)}(k_m\rho)=\frac{b_0}{kh}\sum_{m'=-\infty}^{\infty}\int_{-h/2}^{h/2}\frac{\mathrm{e}^{ikr(m',\rho,z)}}{kr(m',\rho,z)}\left[\frac{-2\mathrm{i}}{kr(m',\rho,z)}\left(1+\frac{\mathrm{i}}{kr(m',\rho,z)}\right)\right.$$

$$\left.-\left(1+\frac{\mathrm{i}}{kr(m',\rho,z)}-\frac{1}{(kr(m',\rho,z))^2}\right)\right]\frac{\rho^2}{2r^2(m',\rho,z)}\ \mathrm{e}^{-\mathrm{i}(2\pi/h)mz}\ \mathrm{d}z\ . \tag{6.52}$$

Since B_{2m} is independent of ρ, for $\rho << 1$ the LHS of (6.52) behaves as [see (C.6)]

$$B_{2m}H_2^{(1)}(k_m\rho)\overset{\rho<<1}{\sim}-B_{2m}\frac{\mathrm{i}}{\pi}\frac{1}{(k_m\rho/2)^2}\ . \tag{6.53}$$

Hence the RHS of (6.52) must also have a $1/\rho^2$ singularity as $\rho\to 0$. In investigating the singularity of the RHS of (6.52) as $\rho\to 0$ we can again ignore all terms in the summation over m' for which $m'\neq 0$ since these terms are not singular as $\rho\to 0$. We must therefore consider the behavior for $\rho << 1$ of

$$\frac{2b_0}{kh}\int_0^{h/2}\frac{\mathrm{e}^{ikr(0,\rho,z)}}{kr(0,\rho,z)}\left[\frac{-2\mathrm{i}}{kr(0,\rho,z)}\left(1+\frac{\mathrm{i}}{kr(0,\rho,z)}\right)\right.$$

$$- \left(1+\frac{\mathrm{i}}{kr(0,\rho,z)}-\frac{1}{(kr(0,\rho,z))^2}\right)\Bigg]\frac{\rho^2}{2r^2(0,\rho,z)}\ \cos(2\pi/h)mz\ \mathrm{d}z \qquad (6.54)$$

where

$$r(0,\rho,z)=\sqrt{\rho^2+z^2}\,. \qquad (6.55)$$

To obtain the coefficient of the $1/\rho^2$ singularity we expand both $\exp[\mathrm{i}kr(0,\rho,z)]$ and $\cos(2\pi/h)mz$ in power series in z, systematically integrate all the resulting indefinite integrals using integrals tabulated in [18, eqs. 2.17, 2.26], evaluate the integrals, and collect terms in $1/\rho^2$. It is found that the only contribution to the $1/\rho^2$ singularity comes from the leading term in the expansion of both the exponential and cosine. Interestingly the contribution to the $1/\rho^2$ singularity comes from evaluating the relevant indefinite integral at the upper end of the interval of integration, $z=h/2$, and then letting $\rho\to 0$. The end result is that the RHS of (6.52) behaves as

$$\frac{2b_0}{kh}\frac{1}{(k\rho)^2} \qquad (6.56)$$

for $\rho << 1$. But then, equating the coefficients of the $1/\rho^2$ singularity in the LHS and RHS of (6.52) given by (6.53) and (6.56), respectively, we obtain

$$B_{2m}=\frac{\mathrm{i}\pi}{2}\frac{b_0}{kh}\left[1-\left(\frac{2\pi m}{kh}\right)^2\right]\,. \qquad (6.57)$$

We now give an alternate derivation of the Floquet mode expansion coefficients based on the Hertz vector potential, following the procedure used in (4.48)-(4.54). We begin with the expression [21, secs. 14-5,14-7] for the electric field of a small x directed electric dipole at the origin of a Cartesian coordinate system

$$C\,\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{kr}\hat{\mathbf{x}}\right)\,. \qquad (6.58)$$

The value of the proportionality constant C can be found similarly to the way it was found in Section 4 [see (4.49)] by expanding (6.58) in spherical coordinates (using, for example, [22, Appendix 1, eqs. 117, 161]) and equating the $1/(kr)$ term of the ϕ component with the corresponding $1/(kr)$ term of (6.6) with a multiplicative factor of b_0 added as in (6.8). We thus obtain

$$C=\frac{b_0}{k^2}. \qquad (6.59)$$

Alternately, we can obtain this value of C immediately from (4.49) since it makes no difference whether we equate two expressions for the field of a z directed electric dipole at the origin as was done in Section 4, or two expressions for the field of an x directed electric dipole at the origin as here. From (6.58) the ρ and ϕ components of the electric field radiated by the x directed electric dipole are given by the ρ and ϕ components of

$$C\,\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\frac{\mathrm{e}^{\mathrm{i}kr}}{kr}\left(\cos\phi\,\hat{\boldsymbol{\rho}}-\sin\phi\,\hat{\boldsymbol{\phi}}\right),\ r=\sqrt{\rho^2+z^2}\,. \qquad (6.60)$$

Using the expression [22, Appendix 1, eq. 101] for $\boldsymbol{\nabla}\times\boldsymbol{\nabla}$ in cylindrical coordinates we then find that the ρ component is given by

$$-\cos\phi\left(\frac{\partial^2}{\partial z^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\frac{\mathrm{e}^{\mathrm{i}kr}}{kr} \tag{6.61}$$

and the ϕ component is given by

$$\sin\phi\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial\rho^2}\right)\frac{\mathrm{e}^{\mathrm{i}kr}}{kr}\ . \tag{6.62}$$

Now from (2.34) and (2.42) the field radiated by the acoustic monopoles located in the column $x=0,\ y=0$ at $z=0,\ \pm h,\ \pm 2h,\ \cdots$, each of which radiates a field equal to $\mathrm{e}^{\mathrm{i}kr}/(kr)$ is (allowing for the different choice of coordinate axes in this section as compared with Section 2)

$$\sum_{m=-\infty}^{\infty} B_m^0 H_0^{(1)}(k_m\rho)\ \mathrm{e}^{\mathrm{i}(2\pi/h)mz},\ k_m=\sqrt{k^2-(2\pi m/h)^2} \tag{6.63}$$

with

$$B_m^0=\frac{\mathrm{i}\pi}{kh} \tag{6.64}$$

and k_m positive real or positive imaginary. Then the ρ component of the electric field radiated by a column of x directed electric dipoles for which $x=0$ is given by

$$\begin{aligned}
&-C\cos\phi\sum_{m=-\infty}^{\infty} B_m^0\left[-\left(\frac{2\pi m}{h}\right)^2 H_0^{(1)}(k_m\rho)+\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}H_0^{(1)}(k_m\rho)\right]\ \mathrm{e}^{\mathrm{i}(2\pi/h)mz}\\
&=-C\cos\phi\sum_{m=-\infty}^{\infty} B_m^0\left[-\left(\frac{2\pi m}{h}\right)^2 H_0^{(1)}(k_m\rho)-\frac{k_m^2}{k_m\rho}H_1^{(1)}(k_m\rho)\right]\ \mathrm{e}^{\mathrm{i}(2\pi/h)mz}\\
&=-C\cos\phi\sum_{m=-\infty}^{\infty} B_m^0\left\{-\left(\frac{2\pi m}{h}\right)^2 H_0^{(1)}(k_m\rho)-\frac{k_m^2}{2}\left[H_0^{(1)}(k_m\rho)+H_2^{(1)}(k_m\rho)\right]\right\}\ \mathrm{e}^{\mathrm{i}(2\pi/h)mz}\\
&=-C\cos\phi\sum_{m=-\infty}^{\infty} B_m^0\left\{\left[-\left(\frac{2\pi m}{h}\right)^2-\frac{k_m^2}{2}\right] H_0^{(1)}(k_m\rho)-\frac{k_m^2}{2}H_2^{(1)}(k_m\rho)\right\}\ \mathrm{e}^{\mathrm{i}(2\pi/h)mz}\\
&=\frac{1}{2}C\cos\phi\sum_{m=-\infty}^{\infty} B_m^0\left\{\left[k^2+\left(\frac{2\pi m}{h}\right)^2\right] H_0^{(1)}(k_m\rho)+\left[k^2-\left(\frac{2\pi m}{h}\right)^2\right] H_2^{(1)}(k_m\rho)\right\}\ \mathrm{e}^{\mathrm{i}(2\pi/h)mz}\ .
\end{aligned} \tag{6.65}$$

Similarly the ϕ component of the electric field radiated by a column of x directed electric dipoles for which $x=0$ is given by

$$C\sin\phi\sum_{m=-\infty}^{\infty} B_m^0\left[-\left(\frac{2\pi m}{h}\right)^2 H_0^{(1)}(k_m\rho)+\frac{(k_m\rho)^2}{\rho^2}\frac{\mathrm{d}^2}{\mathrm{d}(k_m\rho)^2}H_0^{(1)}(k_m\rho)\right]\ \mathrm{e}^{\mathrm{i}(2\pi/h)mz}$$

$$= C \sin\phi \sum_{m=-\infty}^{\infty} B_m^0 \Bigg[-\left(\frac{2\pi m}{h}\right)^2 H_0^{(1)}(k_m\rho) - \frac{k_m\rho}{\rho^2}\frac{\mathrm{d}}{\mathrm{d}(k_m\rho)} H_0^{(1)}(k_m\rho)$$

$$- \frac{(k_m\rho)^2}{\rho^2} H_0^{(1)}(k_m\rho)\Bigg] \mathrm{e}^{\mathrm{i}(2\pi/h)mz}$$

$$= C \sin\phi \sum_{m=-\infty}^{\infty} B_m^0 \left\{ \left[-\left(\frac{2\pi m}{h}\right)^2 - k_m^2\right] H_0^{(1)}(k_m\rho) + \frac{k_m^2}{k_m\rho} H_1^{(1)}(k_m\rho)\right\} \mathrm{e}^{\mathrm{i}(2\pi/h)mz}$$

$$= C \sin\phi \sum_{m=-\infty}^{\infty} B_m^0 \left\{ -k^2 H_0^{(1)}(k_m\rho) + \frac{k_m^2}{2}\left[H_0^{(1)}(k_m\rho) + H_2^{(1)}(k_m\rho)\right]\right\} \mathrm{e}^{\mathrm{i}(2\pi/h)mz}$$

$$= C \sin\phi \sum_{m=-\infty}^{\infty} B_m^0 \Bigg\{ \left[-k^2 + \frac{k^2}{2} - \frac{1}{2}\left(\frac{2\pi m}{h}\right)^2\right] H_0^{(1)}(k_m\rho)$$

$$+ \frac{1}{2}\left[k^2 - \left(\frac{2\pi m}{h}\right)^2\right] H_2^{(1)}(k_m\rho)\Bigg\} \mathrm{e}^{\mathrm{i}(2\pi/h)mz}$$

$$= \frac{1}{2} C \sin\phi \sum_{m=-\infty}^{\infty} B_m^0 \Bigg\{ - \left[k^2 + \left(\frac{2\pi m}{h}\right)^2\right] H_0^{(1)}(k_m\rho)$$

$$+ \left[k^2 - \left(\frac{2\pi m}{h}\right)^2\right] H_2^{(1)}(k_m\rho)\Bigg\} \mathrm{e}^{\mathrm{i}(2\pi/h)mz} \,. \tag{6.66}$$

The first of these equalities is obtained by making use of the differential equation (C.10) satisfied by $H_0^{(1)}(z)$. The x component of the electric field radiated by the column of x directed electric dipoles for which $x = 0$ is then given by $E_\rho \cos\phi - E_\phi \cos\phi$ which, from (6.65) and (6.66), equals

$$\frac{1}{2} C \cos^2\phi \sum_{m=-\infty}^{\infty} B_m^0 \Bigg\{ \left[k^2 + \left(\frac{2\pi m}{h}\right)^2\right] H_0^{(1)}(k_m\rho)$$

$$+ \left[k^2 - \left(\frac{2\pi m}{h}\right)^2\right] H_2^{(1)}(k_m\rho)\Bigg\} \mathrm{e}^{\mathrm{i}(2\pi/h)mz}$$

$$= -\frac{1}{2} C \sin^2\phi \sum_{m=-\infty}^{\infty} B_m^0 \Bigg\{ - \left[k^2 + \left(\frac{2\pi m}{h}\right)^2\right] H_0^{(1)}(k_m\rho)$$

$$+ \left[k^2 - \left(\frac{2\pi m}{h}\right)^2\right] H_2^{(1)}(k_m\rho)\Bigg\} \mathrm{e}^{\mathrm{i}(2\pi/h)mz}$$

$$= \frac{1}{2} C \sum_{m=-\infty}^{\infty} B_m^0 \Bigg\{ \left[k^2 + \left(\frac{2\pi m}{h}\right)^2\right] H_0^{(1)}(k_m\rho)$$

$$+\ \cos 2\phi \left[k^2 - \left(\frac{2\pi m}{h}\right)^2\right] H_2^{(1)}(k_m\rho)\Bigg\}\, \mathrm{e}^{\mathrm{i}(2\pi/h)mz} \tag{6.67}$$

with C given by (6.59). Since the same field is also given by the sum of the two Floquet mode expansions (6.40a) and (6.40b), the coefficients B_{1m} and B_{2m} in (6.40) are found by equating the sum of (6.40a) and (6.40b) with (6.67). Thus

$$B_{1m} = \frac{1}{2}\frac{b_0}{k^2}B_m^0\left[1+\left(\frac{2\pi m}{kh}\right)^2\right] = \frac{\mathrm{i}\pi}{2}\frac{b_0}{kh}\left[1+\left(\frac{2\pi m}{kh}\right)^2\right] \tag{6.68}$$

in agreement with (6.51) and

$$B_{2m} = \frac{1}{2}\frac{b}{k^2}B_m^0\left[1-\left(\frac{2\pi m}{kh}\right)^2\right] = \frac{\mathrm{i}\pi}{2}\frac{b}{kh}\left[1-\left(\frac{2\pi m}{kh}\right)^2\right] \tag{6.69}$$

in agreement with (6.57).

Now that we have obtained expressions, using two independent methods, for the Floquet mode expansion coefficients B_{1m} and B_{2m} in (6.43) we can return to the kd–βd equation (6.25). From (6.35), (6.43), (6.51) or (6.68), and (6.57) or (6.69) we obtain

$$\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}}\left[\frac{-2\mathrm{i}}{\rho_{mn}}\left(kh+\frac{\mathrm{i}}{\rho_{mn}}\right)\frac{(nd/h)^2}{\rho_{mn}^2} + \left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mn}}-\frac{1}{\rho_{mn}^2}\right)\frac{m^2}{\rho_{mn}^2}\right]$$

$$= \frac{\mathrm{i}\pi}{2}\sum_{m=-\infty}^{\infty}\Big(\left[(kh)^2+(2\pi m)^2\right]H_0^{(1)}(k_m|n|d) + \left[(kh)^2-(2\pi m)^2\right]H_2^{(2)}(k_m|n|d)\Big)$$

$$= \frac{\mathrm{i}\pi}{2}(kh)^2\left[H_0^{(1)}(|n|kd)+H_2^{(1)}(|n|kd)\right]$$

$$+\ 2\sum_{m=1}^{\infty}\Bigg[\left[(2\pi m)^2+(kh)^2\right]K_0\left(|n|(d/h)\sqrt{(2\pi m)^2-(kh)^2}\right)$$

$$+\left[(2\pi m)^2-(kh)^2\right]K_2\left(|n|(d/h)\sqrt{(2\pi m)^2-(kh)^2}\right)\Bigg] \tag{6.70}$$

where we have made use of the Bessel function relations (C.1) and (C.3) and assumed that $kh < 2\pi$. Then in (6.25)

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\mathrm{e}^{\mathrm{i}n\beta d}\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}}\left[\frac{-2\mathrm{i}}{\rho_{mn}}\left(kh+\frac{\mathrm{i}}{\rho_{mn}}\right)\frac{(nd/h)^2}{\rho_{mn}^2}\right.$$

$$\left.+\ \left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mn}}-\frac{1}{\rho_{mn}^2}\right)\frac{m^2}{\rho_{mn}^2}\right]$$

$$= 2\sum_{n=1}^{\infty}\cos(n\beta d)\Bigg\{\frac{\mathrm{i}\pi}{2}(kh)^2\left[H_0^{(1)}(|n|kd)+H_2^{(1)}(|n|kd)\right]$$

$$+ 2\sum_{m=1}^{\infty}\left[\left[(2\pi m)^2+(kh)^2\right]K_0\left(|n|(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right)\right.$$

$$\left.\left.+\left[(2\pi m)^2-(kh)^2\right]K_2\left(|n|(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right)\right]\right\}. \tag{6.71}$$

The self-column sum in (6.25)

$$2\sum_{m=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}khm}}{m}\left((kh)^2+\frac{\mathrm{i}kh}{m}-\frac{1}{m^2}\right). \tag{6.72}$$

has been evaluated earlier in Section 4. From (4.95)

$$2\sum_{m=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}khm}}{m}\left((kh)^2+\frac{\mathrm{i}kh}{m}-\frac{1}{m^2}\right)$$

$$= -2\left((kh)^2\ln\left[2\sin\left(\frac{kh}{2}\right)\right]+kh\ \mathrm{Cl}_2(kh)+\mathrm{Cl}_3(kh)\right)+\mathrm{i}\left[\frac{\pi}{2}(kh)^2-\frac{2}{3}(kh)^3\right] \tag{6.73}$$

for $0 < kh < 2\pi$. Substituting (6.71) and (6.73) in the kd–βd equation (6.25) we can then write the kd–βd equation in the form

$$(kh)^3 = S\left\{\Re+\mathrm{i}\Im\right\} \tag{6.74}$$

where $\Re$, the real part of the quantity within the brackets of (6.25) with the original summations replaced by the new expressions we have derived, is given by

$$\Re = -\pi(kh)^2\left[\sum_{n=1}^{\infty}\cos(n\beta d)Y_0(nkd)+\sum_{n=1}^{\infty}\cos(n\beta d)Y_2(nkd)\right]$$

$$+ 4\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=1}^{\infty}\left[\left[(2\pi m)^2+(kh)^2\right]K_0\left(n(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right)\right.$$

$$\left.+\ \left[(2\pi m)^2-(kh)^2\right]K_2\left(n(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right)\right]$$

$$-\ 2\left((kh)^2\ln\left[2\sin\left(\frac{kh}{2}\right)\right]+kh\mathrm{Cl}_2(kh)+\mathrm{Cl}_3(kh)\right) \tag{6.75}$$

and $\Im$, the imaginary part of the quantity within the brackets of (6.25), is

$$\Im = \pi(kh)^2\sum_{n=1}^{\infty}\cos(n\beta d)J_0(nkd)+\pi(kh)^2\sum_{n=1}^{\infty}\cos(n\beta d)J_2(nkd)+\frac{\pi}{2}(kh)^2-\frac{2}{3}(kh)^3. \tag{6.76}$$

In the expression (6.75) for $\Re$, the sum $\sum\cos(n\beta d)Y_0(nkd)$ can be evaluated using (B.2), and the sum $\sum\cos(n\beta d)Y_2(nkd)$ can be evaluated very efficiently by using (B.9)-(B.10). The

K_0 and K_2 series in (6.75) converge extremely rapidly because of the exponential decay of $K_0(z)$ and $K_2(z)$.

In the expression (6.76) for $\Im$, $\sum \cos(n\beta d) J_0(nkd) = -1/2$ [see (B.1)] and $\sum \cos(n\beta d) J_2(nkd) = 0$ [see (B.8a)]. Hence

$$\Im = -\frac{2}{3}(kh)^3 \tag{6.77}$$

which, together with (6.74), has been shown in Section 4 [see (4.59)-(4.61)] to imply that

$$|S| = \frac{3}{2}\sin\psi \tag{6.78}$$

where ψ is the phase of the scattering coefficient S, a relationship derived independently in [4] from reciprocity and power conservation principles, and thereby serving as an important check here. It is worth noting that if $\beta d < kd$ then, from (B.1b), $\sum \cos(n\beta d) J_0(nkd) \neq -1/2$ and from (B.8b), $\sum \cos(n\beta d) J_2(nkd) \neq 0$ and hence $\Im \neq -2/3(kh)^3$ so that (6.78) would not hold. This is not possible for an array of short lossless dipole scatterers. Hence $\beta d > kd$. This is a particular instance of the general result (1.4) noted in the Introduction which holds for 2D arrays as well as for linear arrays. The kd–βd equation (6.74) for traveling waves supported by 2D arrays of short electric dipoles parallel to the array axis then becomes

$$\frac{2}{3}(kh)^3 \cos\psi - \Re \sin\psi = 0\ . \tag{6.79}$$

with $\Re$ given by (6.75) and $kh < 2\pi$. Equation (6.79) can be easily solved numerically for βd given values of kd, kh, and ψ, using, for example, a simple search procedure with secant algorithm refinement.

Since some of the terms of (6.75) become singular as kh approaches 2π this equation cannot be used to calculate $\Re$ when $kh = 2\pi$. It is therefore worthwhile to obtain the limit of $\Re$ given by (6.75) as $kh \to 2\pi$. Comparing (6.75) with the expression for $\Re$ given by (4.97) in Section 4.2 we see that the two expressions are identical apart from two sign changes. Accordingly the analysis of the limiting behavior of $\Re$ given in (4.102)-(4.106) is immediately applicable and making the appropriate two sign changes in (4.106) we obtain

$$\begin{aligned}
\Re \overset{kh\to 2\pi}{\rightarrow} & -\pi(2\pi)^2 \left[\sum_{n=1}^{\infty} \cos(n\beta d) Y_0(nkd) + \sum_{n=1}^{\infty} \cos(n\beta d) Y_2(nkd)\right] \\
& + 4(2\pi)^2 \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{m=2}^{\infty} \left[(m^2+1) K_0\left(2\pi n (d/h)\sqrt{m^2-1}\right)\right. \\
& \left. + (m^2-1) K_2\left(2\pi n (d/h)\sqrt{m^2-1}\right)\right] + (2\pi)^2 \left(4\gamma + 4\ln\frac{1}{\sqrt{4\pi}}\frac{d}{h} + \frac{4\pi}{\beta d}\right) \\
& + (2\pi)^2 (4\pi) \left[\sum_{l=1}^{\infty}\left(\frac{1}{(2l\pi - \beta d)^2} - \frac{1}{2l\pi}\right) + \sum_{l=1}^{\infty}\left(\frac{1}{(2l\pi + \beta d)^2} - \frac{1}{2l\pi}\right)\right] \\
& - \frac{8}{(d/h)^2}\left[\frac{\pi^2}{6} - \frac{\pi\beta d}{2} + \frac{(\beta d)^2}{4}\right] - 2\,\mathrm{Cl}_3(2\pi)
\end{aligned} \tag{6.80}$$

where the Clausen function $\mathrm{Cl}_3(2\pi)$ is given by (D.10).

7 3D ELECTRIC DIPOLE ARRAYS, DIPOLES ORIENTED PARALLEL TO THE ARRAY AXIS

In this section we consider traveling waves supported by 3D periodic arrays of lossless short electric dipoles with the dipoles oriented parallel to the array axis, following the same major steps of the procedure — calculating the field at a reference element due to all the other elements in the array, deriving the kd–βd equation by assuming a traveling wave excitation of the array, and converting slowly convergent summations to rapidly convergent ones to obtain a form of the kd–βd equation suitable for calculation purposes — that we have used in treating 2D periodic arrays of lossless electric dipoles oriented parallel to the array axis in Section 6. The Floquet mode expansion method will be used to convert the slowly convergent summations over the non-self planes to rapidly convergent ones, with two different procedures used to obtain the Floquet mode expansion coefficients, an integral procedure and a Hertz vector potential procedure, while the Poisson summation formula will be used to convert the slowly convergent summation over the self-plane to a rapidly convergent form.

The array axis is chosen to be the z axis rather than the x axis as in Section 6. Equispaced planes of z directed electric dipoles are located at $z = nd,\ n = 0,\ \pm 1,\ \pm 2,\ \cdots$. In each plane the dipoles are centered at $x = mh,\ y = lh,\ m, l = 0,\ \pm 1,\ \pm 2,\ \cdots$. We assume an excitation of the array with the electric field parallel to the z axis and such that all the dipoles in any plane of the array are excited identically. Because of the symmetry of the dipole locations and excitations the electric field incident on the dipole at the location $x = 0, y = 0, z = 0$ from all the other dipoles in the array has an z component only which we denote by E^0_{0z}. Let E^{0mln}_{0z} be the z component of the electric field incident on the reference dipole from the electric dipole at the location $(x, y, z) = (mh, lh, nd)$ so that

$$E^0_{0z} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} E^{0mln}_{0z} + \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \ \neq \ \sum_{\substack{l=-\infty \\ (0,0)}}^{\infty} E^{0ml0}_{0z} \ . \tag{7.1}$$

We proceed to obtain expressions for E^{0mln}_{0z} and E^{0ml0}_{0z}. Since the electric field of a short electric dipole located at the origin of a Cartesian coordinate system with the z axis in the direction of the dipole is [4, eq. (40)]

$$\mathbf{E}(\mathbf{r}) = \frac{\mathrm{e}^{\mathrm{i}kr}}{kr}\left[\frac{-2\mathrm{i}}{kr}\left(1 + \frac{\mathrm{i}}{kr}\right)\cos\theta\,\hat{\mathbf{r}} - \left(1 + \frac{\mathrm{i}}{kr} - \frac{1}{(kr)^2}\right)\sin\theta\,\hat{\boldsymbol{\theta}}\right] \tag{7.2}$$

it follows that the electric field $\mathbf{E}^{0mln}_0$ incident on the reference dipole from the electric dipole at the location $(x, y, z) = (mh, lh, nd)$ is

$$\mathbf{E}^{0mln}_0 = b_n \frac{\mathrm{e}^{\mathrm{i}kr_{mln0}}}{kr_{mln0}}\left[\frac{-2\mathrm{i}}{kr_{mln0}}\left(1 + \frac{\mathrm{i}}{kr_{mln0}}\right)\cos\theta_{mln0}\,\hat{\mathbf{r}}_{mln0}\right.$$

$$\left. - \left(1 + \frac{\mathrm{i}}{kr_{mln0}} - \frac{1}{(kr_{mln0})^2}\right)\sin\theta_{mln0}\,\hat{\boldsymbol{\theta}}_{mln0}\right] . \tag{7.3}$$

The quantities in (7.3) are defined with reference to a local spherical polar coordinate system with origin at $(x, y, z) = (mh, lh, nd)$ (in turn defined with reference to a local Cartesian coordinate system with the same origin whose axes are parallel to those of the global Cartesian coordinate system). The distance from the (m, l, n) dipole to the $(0, 0, 0)$ dipole, r_{mln0}, is given by

$$r_{mln0} = \sqrt{(mh)^2 + (lh)^2 + (nd)^2} = h\sqrt{m^2 + l^2 + (nd/h)^2} \tag{7.4}$$

and the unit vector in the direction from the (m, l, n) dipole to the $(0, 0, 0)$ dipole, $\hat{\mathbf{r}}_{mln0}$, is

$$\hat{\mathbf{r}}_{mln0} = \mathbf{r}_{mln0}/r_{mln0}, \quad \mathbf{r}_{mln0} = -mh\,\hat{\mathbf{x}} - lh\,\hat{\mathbf{y}} - nd\,\hat{\mathbf{z}} \tag{7.5}$$

so that

$$\cos\theta_{mln0} = \hat{\mathbf{r}}_{mln0} \cdot \hat{\mathbf{z}} = -\frac{nd}{r_{mln0}} \tag{7.6}$$

$$\sin\theta_{mln0} = \sqrt{1 - \cos^2\theta_{mln0}} = \frac{\sqrt{(mh)^2 + (lh)^2}}{r_{mln0}} \tag{7.7}$$

$$\phi_{mln0} = \tan^{-1}\left(\frac{\hat{\mathbf{r}}_{mln0} \cdot \hat{\mathbf{y}}}{\hat{\mathbf{r}}_{mln0} \cdot \hat{\mathbf{x}}}\right) = \tan^{-1}\left(\frac{-lh}{-mh}\right) \tag{7.8}$$

$$\cos\phi_{mln0} = \frac{-m}{\sqrt{m^2 + l^2}} \tag{7.9}$$

$$\sin\phi_{mln0} = \frac{-l}{\sqrt{m^2 + l^2}} \tag{7.10}$$

$$\hat{\boldsymbol{\theta}}_{mln0} = \cos\theta_{mln0}\cos\phi_{mln0}\,\hat{\mathbf{x}} + \cos\theta_{mln0}\sin\phi_{mln0}\,\hat{\mathbf{y}} - \sin\theta_{mln0}\,\hat{\mathbf{z}} \tag{7.11}$$

$$\cos\theta_{mln0}\,\hat{\mathbf{r}}_{mln0} = \frac{nd\,(mh\,\hat{\mathbf{x}} + lh\,\hat{\mathbf{y}} + nd\,\hat{\mathbf{z}})}{r_{mln0}^2} \tag{7.12}$$

and

$$\sin\theta_{mln0}\,\hat{\boldsymbol{\theta}}_{mln0} = \frac{(nd)(mh)\,\hat{\mathbf{x}} + (nd)(lh)\,\hat{\mathbf{y}} - [(mh)^2 + (lh)^2]\,\hat{\mathbf{z}}}{r_{mln0}^2}\,. \tag{7.13}$$

The constants b_n are related to the z component of the electric field incident on any dipole in the nth plane by the scattering equation [4, eq. (59)]

$$b_n = SE_{0z}^{0n} \tag{7.14}$$

where S is the normalized dipole scattering coefficient of a short electric dipole. "Normalized" means that b_n is the coefficient of $\exp(\mathrm{i}kr)/(kr)$ in the transverse component of the outgoing electric field in response to the incident field $E_{0z}^{0n}\,\hat{\mathbf{z}}$ at the center of the z directed electric dipole. When summed over m and l the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components of $\mathbf{E}_0^{0mln}$ vanish because they are odd functions of m and l. Hence for $n \neq 0$

$$E_{0z}^{0mln} = b_n \frac{\mathrm{e}^{\mathrm{i}kr_{mln0}}}{kr_{mln0}}\left[\frac{-2\mathrm{i}}{kr_{mln0}}\left(1 + \frac{\mathrm{i}}{kr_{mln0}}\right)\frac{(nd)^2}{r_{mln0}^2} + \left(1 + \frac{\mathrm{i}}{kr_{mln0}} - \frac{1}{(kr_{mln0})^2}\right)\frac{(mh)^2 + (lh)^2}{r_{mln0}^2}\right] \tag{7.15}$$

while for $n = 0$, the self-plane,

$$E_{0z}^{0ml0} = b_0 \frac{\mathrm{e}^{\mathrm{i}kr_{ml00}}}{kr_{ml00}} \left(1 + \frac{\mathrm{i}}{kr_{ml00}} - \frac{1}{(kr_{ml00})^2}\right) \tag{7.16}$$

where from (7.4)

$$r_{ml00} = h\sqrt{m^2 + l^2}\ . \tag{7.17}$$

Substituting (7.15) and (7.16) in (7.1) we obtain the expression for the total field incident on the reference $(0, 0, 0)$ dipole

$$\begin{aligned} E_{0z}^{0} = & \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} b_n \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{kh\rho_{mln}} \left[\frac{-2\mathrm{i}}{kh\rho_{mln}} \left(1 + \frac{\mathrm{i}}{kh\rho_{mln}}\right) \frac{(nd/h)^2}{\rho_{mln}^2} \right. \\ & \left. + \left(1 + \frac{\mathrm{i}}{kh\rho_{mln}} - \frac{1}{(kh)^2\rho_{mln}^2}\right) \frac{m^2 + l^2}{\rho_{mln}^2} \right] \\ & + b_0 \sum_{m=-\infty}^{\infty} \sum_{\substack{l=-\infty \\ (m,l) \neq (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{ml0}}}{kh\rho_{ml0}} \left(1 + \frac{\mathrm{i}}{kh\rho_{ml0}} - \frac{1}{(kh)^2\rho_{ml0}^2}\right) \end{aligned} \tag{7.18}$$

where we have let

$$\rho_{mln} = \sqrt{m^2 + l^2 + (nd/h)^2} \tag{7.19a}$$

so that

$$\rho_{ml0} = \sqrt{m^2 + l^2}\ . \tag{7.19b}$$

We now assume that the array is excited by a traveling wave in the z direction with real propagation constant β. Then the constants b_n in (7.18) are identical apart from a phase shift given by

$$b_n = b_0\, \mathrm{e}^{\mathrm{i}n\beta d}\ . \tag{7.20}$$

Substituting (7.20) in (7.18), using [from (7.14)] $b_0 = SE_{0z}^0$, and multiplying by $(kh)^3$ we obtain

$$\begin{aligned} (kh)^3 = S \Bigg\{ & \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{\rho_{mln}} \left[\frac{-2\mathrm{i}}{\rho_{mln}} \left(kh + \frac{\mathrm{i}}{\rho_{mln}}\right) \frac{(nd/h)^2}{\rho_{mln}^2} \right. \\ & \left. + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mln}} - \frac{1}{\rho_{mln}^2}\right) \frac{m^2 + l^2}{\rho_{mln}^2} \right] \\ & + \sum_{m=-\infty}^{\infty} \sum_{\substack{l=-\infty \\ (m,l) \neq (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{ml0}}}{\rho_{ml0}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{ml0}} - \frac{1}{\rho_{ml0}^2}\right) \Bigg\}\ . \end{aligned} \tag{7.21}$$

Equation (7.21) is the kd–βd equation that determines the normalized traveling wave propagation constant βd in terms of kh, d/h, and the normalized electric dipole scattering coefficient S.

We now use the Floquet mode method to help transform the slowly convergent triple summation in (7.21) to a rapidly convergent form, proceeding similarly to the way we used it in Section 5. We let $E_z^0(P)$ be the z component of the electric field radiated by all the electric dipoles in the $n = 0$ plane at a general point in space $P = (x, y, z)$, $z \neq 0$. (Note that, because of symmetry, $E_z^0(x, y, -z) = E_z^0(x, y, z)$.) We establish a local spherical polar coordinate system with origin at the dipole located at $(x, y, z) = (mh, lh, 0)$ and with $\theta(m, l, P)$ the polar angle from the z axis to the vector $\mathbf{r}(m, l, P)$ from $(mh, lh, 0)$ to the field point P. The distance $r(m, l, P)$ from $(mh, lh, 0)$ to P is

$$r(m, l, P) = \sqrt{(x - mh)^2 + (y - lh)^2 + z^2} \tag{7.22}$$

and the unit vector $\hat{\mathbf{r}}(m, l, P)$ is

$$\hat{\mathbf{r}}(m, l, P) = \frac{\mathbf{r}(m, l, P)}{r(m, l, P)} = \frac{(x - mh)\,\hat{\mathbf{x}} + (y - lh)\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{r(m, l, P)} \tag{7.23}$$

so that

$$\cos\theta(m, l, P) = \hat{\mathbf{r}}(l, m, P) \cdot \hat{\mathbf{z}} = \frac{z}{r(m, l, P)} \tag{7.24}$$

$$\sin\theta(m, l, P) = \sqrt{1 - \cos^2\theta(m, l, P)} = \frac{\sqrt{(x - mh)^2 + (y - lh)^2}}{r(m, l, P)} \tag{7.25}$$

$$[\cos\theta(m, l, P)\,\hat{\mathbf{r}}(m, l, P)]_z = \cos^2\theta(m, l, P) = \frac{z^2}{r^2(m, l, P)} \tag{7.26}$$

$$\left[\hat{\boldsymbol{\theta}}(m, l, P)\right]_z = -\sin\theta(m, l, P) = -\frac{\sqrt{(x - mh)^2 + (y - lh)^2}}{r(m, l, P)} \tag{7.27}$$

$$\left[\sin\theta(m, l, P)\,\hat{\boldsymbol{\theta}}(m, l, P)\right]_z = -\sin^2\theta(m, l, P) = -\frac{(x - mh)^2 + (y - lh)^2}{r^2(m, l, P)} \tag{7.28}$$

and hence, referring to (7.2),

$$\begin{aligned} E_z^0(x, y, |z|) = b_0 \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr(m, l, P)}}{kr(m, l, P)} \Bigg[& \frac{-2\mathrm{i}}{kr(m, l, P)} \left(1 + \frac{\mathrm{i}}{kr(m, l, P)}\right) \frac{z^2}{r^2(m, l, P)} \\ & + \left(1 + \frac{\mathrm{i}}{kr(m, l, P)} - \frac{1}{(kr(m, l, P))^2}\right) \frac{(x - mh)^2 + (y - lh)^2}{r^2(m, l, P)} \Bigg]. \end{aligned} \tag{7.29}$$

For $P = P_0 = (0, 0, |n|d)$

$$\begin{aligned} E_z^0(0, 0, |n|d) = b_0 \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr(m, l, P_0)}}{kr(m, l, P_0)} \Bigg[& \frac{-2\mathrm{i}}{kr(m, l, P_0)} \left(1 + \frac{\mathrm{i}}{kr(m, l, P_0)}\right) \frac{(nd)^2}{r^2(m, l, P_0)} \\ & + \left(1 + \frac{\mathrm{i}}{kr(m, l, P_0)} - \frac{1}{(kr(m, l, P_0))^2}\right) \frac{(mh)^2 + (lh)^2}{r^2(l, m, P_0)} \Bigg] \end{aligned} \tag{7.30}$$

where

$$r(m,l,P_0) = \sqrt{(m^2+l^2)h^2+(nd)^2}\ . \tag{7.31}$$

But, referring to (7.19a),

$$r(m,l,P_0) = h\rho_{mln} \tag{7.32}$$

and thus

$$E_z^0(0,0,|n|d) = b_0 \sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{kh\rho_{mln}}\left[\frac{-2\mathrm{i}}{kh\rho_{mln}}\left(1+\frac{\mathrm{i}}{kh\rho_{mln}}\right)\frac{(nd/h)^2}{\rho_{mln}^2}\right.$$
$$\left.+\left(1+\frac{\mathrm{i}}{kh\rho_{mln}}-\frac{1}{(kh)^2\rho_{mln}^2}\right)\frac{m^2+l^2}{\rho_{mln}^2}\right]\ . \tag{7.33}$$

Hence, in (7.21),

$$\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{\rho_{mln}}\left[\frac{-2\mathrm{i}}{\rho_{mln}}\left(kh+\frac{\mathrm{i}}{\rho_{mln}}\right)\frac{(nd/h)^2}{\rho_{mln}^2}\right.$$
$$\left.+\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mln}}-\frac{1}{\rho_{mln}^2}\right)\frac{m^2+l^2}{\rho_{mln}^2}\right] = \frac{(kh)^3}{b}E_z^0(0,0,|n|d)\ . \tag{7.34}$$

Now $E_z^0(x,y,|z|)$ can be expressed in terms of a plane wave spectrum by

$$E_z^0(x,y,|z|) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(k_x,k_y)\mathrm{e}^{\mathrm{i}(k_x x+k_y y+k_z|z|)}\,\mathrm{d}k_x\mathrm{d}k_y,\ k_z=\sqrt{k^2-k_x^2-k_y^2} \tag{7.35}$$

where k_z is positive real (positive imaginary) according as $k^2 > (<)\ k_x^2+k_z^2$. Because of the periodicity of the array in the x and y directions,

$$E_z^0(x+h,y,|z|) = E_z^0(x,y,|z|)\ ,\quad E_z^0(x,y+h,|z|) = E^0(x,y,|z|)\ . \tag{7.36}$$

It follows by taking the inverse Fourier transform of (7.35) and substituting in (7.36) that

$$\mathrm{e}^{\mathrm{i}k_x h} = 1\ ,\quad \mathrm{e}^{\mathrm{i}k_y h} = 1 \tag{7.37}$$

and hence

$$k_x h = 2\pi m,\ m=0,\pm1,\pm2,\cdots\ ,\quad k_y h = 2\pi l,\ l=0,\pm1,\pm2,\cdots \tag{7.38}$$

so that

$$E_z^0(x,y,|z|) = \sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} B_{ml}\,\mathrm{e}^{\mathrm{i}(2\pi/h)(mx+ly)}\,\mathrm{e}^{\mathrm{i}k_{ml}|z|} \tag{7.39}$$

where

$$k_{ml} = \sqrt{k^2-(2\pi m/h)^2-(2\pi l/h)^2} \tag{7.40}$$

with k_{ml} positive real (positive imaginary) according as $(kh)^2 > (<)\ (2\pi)^2(m^2+l^2)$. It remains to find the unknown Floquet mode expansion coefficients B_{ml}. As in Section 5 we will employ two different methods for obtaining the coefficients, one based on the asymptotic behavior of an integral, and the other on the Hertz vector potential. We begin with the integral method.

By inverting (7.39)

$$B_{ml}\,\mathrm{e}^{\mathrm{i}k_{ml}|z|} = \frac{1}{h^2}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} E_z^0(x,y,|z|)\,\mathrm{e}^{-\mathrm{i}(2\pi/h)(mx+ly)}\,\mathrm{d}x\mathrm{d}y \tag{7.41}$$

so that with (7.29)

$$\begin{aligned}
&B_{ml}\mathrm{e}^{\mathrm{i}k_{ml}|z|}\\
&= \frac{b_0}{h^2}\sum_{m'=-\infty}^{\infty}\sum_{l'=-\infty}^{\infty}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\frac{\mathrm{e}^{\mathrm{i}kr(m',l',P)}}{kr(m',l',P)}\left[\frac{-2\mathrm{i}}{kr(m',l',P)}\left(1+\frac{\mathrm{i}}{kr(m',l',P)}\right)\frac{z^2}{r^2(m',l',P)}\right.\\
&\left.+\left(1+\frac{\mathrm{i}}{kr(m',l',P)}-\frac{1}{(kr(m',l',P))^2}\right)\frac{(x-m'h)^2+(y-l'h)^2}{r^2(m',l',P)}\right]\mathrm{e}^{-\mathrm{i}(2\pi/h)(mx+ly)}\,\mathrm{d}x\mathrm{d}y
\end{aligned} \tag{7.42}$$

where

$$r(m',l',P) = \sqrt{(x-m'h)^2+(y-l'h)^2+z^2}\;. \tag{7.43}$$

Since B_{ml} is independent of z, if the LHS of (7.42) is expanded for small $|z|$

$$B_{ml}\mathrm{e}^{\mathrm{i}k_{ml}|z|} \overset{|z|<<1}{\sim} B_{ml}(1+\mathrm{i}k_{ml}|z|)\;. \tag{7.44}$$

We can then obtain an expression for B_{ml} by investigating the behavior of the RHS of (7.42) for $|z| << 1$ and equating coefficients of $|z|$.

First we show that the terms in the double summation in (7.42) for which $(m',l') \neq (0,0)$ cannot contribute a term in $|z|$ for $|z| << 1$. For, letting

$$A^2 = (x-m'h)^2+(y-l'h)^2 \tag{7.45}$$

so that

$$r(m',l',P) = \sqrt{A^2+z^2} \tag{7.46}$$

and assuming that $z^2 << A^2$,

$$\frac{\mathrm{e}^{\mathrm{i}k\sqrt{(m'h-x)^2+(l'h-y)^2+z^2}}}{\sqrt{(m'h-x)^2+(l'h-y)^2+z^2}} = \frac{\mathrm{e}^{\mathrm{i}k\sqrt{A^2+z^2}}}{\sqrt{A^2+z^2}} \approx \frac{\mathrm{e}^{\mathrm{i}kA}}{A}\left[1+\left(\frac{\mathrm{i}k}{2A}-\frac{1}{2A^2}z^2\right)+\cdots\right] \tag{7.47}$$

containing no term in $|z|$. Also,

$$\begin{aligned}
\frac{2\mathrm{i}}{kr(m',l',P)}\left(1+\frac{\mathrm{i}}{kr(m',l',P)}\right)\frac{z^2}{r^2(m',l',P)} &= \frac{2\mathrm{i}}{k\sqrt{A^2+z^2}}\left(1+\frac{\mathrm{i}}{k\sqrt{A^2+z^2}}\right)\frac{z^2}{A^2+z^2}\\
&\approx \frac{2\mathrm{i}}{kA}\left(1-\frac{z^2}{2A^2}\right)\left[1+\frac{\mathrm{i}}{kA}\left(1-\frac{z^2}{2A^2}\right)\right]\frac{z^2}{A^2}\left(1-\frac{z^2}{A^2}\right)
\end{aligned} \tag{7.48}$$

containing no term in $|z|$. Similarly there is no term in $|z|$ in the expansion of

$$\left(1 + \frac{\mathrm{i}}{kr(m',l',P)} - \frac{1}{(kr(m',l',P))^2}\right) \frac{(x-m'h)^2 + (y-l'h)^2}{r^2(m',l',P)} \tag{7.49}$$

for $z^2 << A^2$. Hence a term in $|z|$ in the RHS of (7.42) for $|z| << 1$ can come only from the $(m',l') = (0,0)$ term

$$\frac{b_0}{h^2} \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} \frac{\mathrm{e}^{\mathrm{i}kr(0,0,P)}}{kr(0,0,P)} \left[\frac{-2\mathrm{i}}{kr(0,0,P)} \left(1 + \frac{\mathrm{i}}{kr(0,0,P)}\right) \frac{z^2}{r^2(0,0,P)} \right.$$
$$\left. + \left(1 + \frac{\mathrm{i}}{kr(0,0,P)} - \frac{1}{(kr(0,0,P))^2}\right) \frac{x^2+y^2}{r^2(0,0,P)} \right] \mathrm{e}^{-\mathrm{i}(2\pi/h)(mx+ly)}\, \mathrm{d}x\mathrm{d}y \tag{7.50}$$

where

$$r(0,0,P) = \sqrt{x^2+y^2+z^2}\,. \tag{7.51}$$

In cylindrical polar coordinates $\rho = \sqrt{x^2+y^2}$, $\phi = \tan^{-1}(y/x)$, the $(m',l') = (0,0)$ term is approximately

$$\frac{b_0}{kh^2} \int_0^{h/2} \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}k\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}} \left[\frac{-2\mathrm{i}}{k\sqrt{\rho^2+z^2}} \left(1 + \frac{\mathrm{i}}{k\sqrt{\rho^2+z^2}}\right) \frac{z^2}{\rho^2+z^2} \right.$$
$$\left. + \left(1 + \frac{\mathrm{i}}{k\sqrt{\rho^2+z^2}} - \frac{1}{k^2(\rho^2+z^2)}\right) \frac{\rho^2}{\rho^2+z^2} \right] \mathrm{e}^{-\mathrm{i}(2\pi/h)(m\cos\phi + l\sin\phi)\rho}\, \rho\, \mathrm{d}\rho\, \mathrm{d}\phi\,. \tag{7.52}$$

We can obtain a term in $|z|$ for $|z| << 1$ only in the vicinity of $\rho = 0$. We expand the trigonometric exponential in (7.52) in a power series in ρ, and note that terms containing odd powers of $\cos\phi$ and $\sin\phi$ integrate to 0 over the interval $\phi = [0, 2\pi]$, to obtain

$$\mathrm{e}^{-\mathrm{i}(2\pi/h)(m\cos\phi + l\sin\phi)\rho} \approx 1 - \frac{1}{2}\,(2\pi/h)^2\, \left(m^2\cos^2\phi + l^2\sin^2\phi\right)\rho^2 + \cdots\,. \tag{7.53}$$

We then substitute (7.53) in (7.52), perform the ϕ integration, systematically integrate all the resulting indefinite integrals by making the change of variables

$$u = \sqrt{\rho^2+z^2},\ du = \frac{\rho\, \mathrm{d}\rho}{\sqrt{\rho^2+z^2}} \tag{7.54}$$

and using integrals tabulated in [18, eqs. 2.324,2.325], evaluate the integrals at the lower range of integration, $u = |z|$, and collect terms in $|z|$. (There is no contribution to terms in $|z|$ from the upper end of the interval of integration $u = \sqrt{(h/2)^2+z^2}$.) It is found that there is no contribution to terms in $|z|$ from the constant term in the expansion of the trigonometric exponential in (7.53), nor is there any contribution from terms higher than ρ^2 in this expansion. The end result is that the RHS of (7.42) behaves as

$$-8\pi^3 \frac{1}{(kh)^3} \frac{b_0}{h} (m^2+l^2)\, |z| \tag{7.55}$$

for $|z| << 1$. But then, equating coefficients of $|z|$ in (7.44) and (7.55) we obtain the coefficients of the Floquet mode expansion (7.39)

$$B_{ml} = \mathrm{i}8\pi^3 \frac{b_0}{(kh)^3} \frac{m^2 + l^2}{\sqrt{(kh)^2 - (2\pi)^2(m^2 + l^2)}} \,. \tag{7.56}$$

We now give an alternate derivation of the Floquet mode expansion coefficients based on the Hertz vector potential, following the procedure used in (5.58)-(5.63). The starting point is the expression (4.48) for the electric field of a small z directed electric dipole at the origin of a Cartesian coordinate system yielding the expression (4.50) for the z component of the electric dipole field which we repeat here:

$$C\left(\frac{\partial^2}{\partial z^2} + k^2\right)\frac{\mathrm{e}^{\mathrm{i}kr}}{kr} \tag{7.57}$$

with $C = b_0/k^2$ from (4.49). Now from (3.30) and (3.42) the field radiated by the acoustic monopoles located in the plane $z = 0$ at the locations $(x, y) = (mh, lh)$, $m, l = 0, \pm h, \pm 2h, \cdots$, each of which radiates a field equal to $\mathrm{e}^{\mathrm{i}kr}/(kr)$, is

$$\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} B^0_{ml}\, \mathrm{e}^{\mathrm{i}(2\pi/h)(mx+ly)}\, \mathrm{e}^{\mathrm{i}k_{ml}|z|} \tag{7.58}$$

where

$$B^0_{ml} = \frac{2\pi\mathrm{i}}{kh^2 k_{ml}} \tag{7.59}$$

and

$$k_{ml} = \sqrt{k^2 - (2\pi m/h)^2 - (2\pi l/h)^2} \tag{7.60}$$

where k_{ml} is positive real or positive imaginary. Hence from (7.57) and (7.58) the z component of the electric field radiated by the plane $z = 0$ of z directed electric dipoles is equal to

$$\begin{aligned} &C\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} B^0_{ml}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\mathrm{e}^{\mathrm{i}(2\pi/h)(mx+ly)}\, \mathrm{e}^{\mathrm{i}k_{ml}|z|} \\ &= C\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} B^0_{ml}(k^2 - k^2_{ml})\, \mathrm{e}^{\mathrm{i}(2\pi/h)(mx+ly)}\, \mathrm{e}^{\mathrm{i}k_{ml}|z|} \,. \end{aligned} \tag{7.61}$$

Since the same field is given by the Floquet mode expansion (7.39), by equating (7.39) with (7.61) we obtain the coefficients B_{ml}

$$B_{ml} = CB^0_{ml}(k^2 - k^2_{ml}) = \frac{b_0}{k^2}\frac{2\pi\mathrm{i}}{kh^2 k_{ml}}\,(k^2 - k^2_{ml}) = \mathrm{i}8\pi^3 \frac{b_0}{(kh)^3} \frac{m^2 + l^2}{\sqrt{(kh)^2 - (2\pi)^2(m^2 + l^2)}} \tag{7.62}$$

so that, comparing (7.62) with (7.56), we see that the two methods of obtaining the coefficients of the Floquet mode expansion yield the same result.

Now, referring to (7.34), (7.39), and (7.56) or (7.62),

$$\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{\rho_{mln}}\left[\frac{-2\mathrm{i}}{\rho_{mln}}\left(kh+\frac{\mathrm{i}}{\rho_{mln}}\right)\frac{(nd/h)^2}{\rho_{mln}^2}+\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mln}}-\frac{1}{\rho_{mln}^2}\right)\frac{m^2+l^2}{\rho_{mln}^2}\right]$$

$$=\mathrm{i}8\pi^3\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{(m^2+l^2)\,\mathrm{e}^{\mathrm{i}|n|(d/h)\,\sqrt{(kh)^2-(2\pi)^2(m^2+l^2)}}}{\sqrt{(kh)^2-(2\pi)^2(m^2+l^2)}}$$

$$=8\pi^3\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{(m^2+l^2)\,\mathrm{e}^{-|n|(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}} \tag{7.63}$$

for $0<kh<2\pi$. We have thus effected a rather extraordinary conversion of a very slowly convergent double summation of a complex quantity to a very rapidly convergent double summation of a real quantity. It is not at all obvious that the imaginary part of the original summation in (7.34) is zero. It follows by substituting (7.63) in (7.21) that

$$\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\mathrm{e}^{\mathrm{i}n\beta d}\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{\rho_{mln}}\left[-\frac{2\mathrm{i}}{\rho_{mln}}\left(kh+\frac{\mathrm{i}}{\rho_{mln}}\right)\frac{(nd/h)^2}{\rho_{mln}^2}\right.$$

$$\left.-\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mln}}-\frac{1}{\rho_{mln}^2}\right)\frac{m^2+l^2}{\rho_{mln}^2}\right]$$

$$=16\pi^3\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{(m^2+l^2)\,\mathrm{e}^{-n(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}\,. \tag{7.64}$$

We now turn our attention to the self-plane double sum in the kd–βd equation (7.21)

$$\sum_{\substack{m=-\infty\\(m,l)}}^{\infty}\ \sum_{\substack{l=-\infty\\\neq\ (0,0)}}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+l^2}}}{\sqrt{m^2+l^2}}\left((kh)^2+\frac{\mathrm{i}kh}{\sqrt{m^2+l^2}}-\frac{1}{m^2+l^2}\right)$$

$$=2\sum_{l=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+l^2}}}{\sqrt{m^2+l^2}}\left((kh)^2+\frac{\mathrm{i}kh}{\sqrt{m^2+l^2}}-\frac{1}{m^2+l^2}\right)$$

$$+\,2\sum_{m=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}khm}}{m}\left((kh)^2+\frac{\mathrm{i}kh}{m}-\frac{1}{m^2}\right)$$

$$=2(kh)^2\sum_{l=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\sqrt{m^2+l^2}}}{\sqrt{m^2+l^2}}$$

$$+\,2\mathrm{i}kh\sum_{l=1}^{\infty}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+l^2}}\left(\frac{1}{m^2+l^2}+\frac{1}{(-\mathrm{i}kh)(m^2+l^2)^{3/2}}\right)$$

$$+\,2\sum_{m=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}khm}}{m}\left((kh)^2+\frac{\mathrm{i}kh}{m}-\frac{1}{m^2}\right)\,. \tag{7.65}$$

We consider each of these three sums in turn. The inner (m) sum of the first double summation is treated in Section 2 dealing with the 2D acoustic monopole array. From (2.29) and (2.43) with nd/h replaced by l we obtain

$$2(kh)^2\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\sqrt{m^2+l^2}}}{\sqrt{m^2+l^2}}=2\pi\mathrm{i}(kh)^2H_0^{(1)}(lkh)+8(kh)^2\sum_{m=1}^{\infty}K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\right) \tag{7.66}$$

when $0<kh<2\pi$. Thus

$$2(kh)^2\sum_{l=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\sqrt{m^2+l^2}}}{\sqrt{m^2+l^2}}=2\pi\mathrm{i}(kh)^2\sum_{l=1}^{\infty}H_0^{(1)}(lkh)$$

$$+\,8(kh)^2\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\right)\,. \tag{7.67}$$

For the inner (m) sum of the second double summation in (7.65) we refer to (4.89) with l substituted for $|n|d/h$ to obtain

$$\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+l^2}}\left(\frac{1}{m^2+l^2}+\frac{1}{(-\mathrm{i}kh)(m^2+l^2)^{3/2}}\right)$$

$$=-\frac{\pi}{2}kh\left[H_0^{(1)}(lkh)+H_2^{(1)}(lkh)\right]$$

$$+\,\frac{2\mathrm{i}}{kh}\sum_{m=1}^{\infty}\left[(2\pi m)^2-(kh)^2\right]\,\left[K_2\left(l\sqrt{(2\pi m)^2-(kh)^2}\right)-K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\right)\right] \tag{7.68}$$

for $0<kh<2\pi$. Thus

$$2\mathrm{i}kh\sum_{l=1}^{\infty}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+l^2}}\left(\frac{1}{m^2+l^2}+\frac{1}{(-\mathrm{i}kh)(m^2+l^2)^{3/2}}\right)$$

$$=-\pi\mathrm{i}(kh)^2\sum_{l=1}^{\infty}\left[H_0^{(1)}(lkh)+H_2^{(1)}(lkh)\right]$$

$$-4\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\left[(2\pi m)^2-(kh)^2\right]\,\left[K_2\left(l\sqrt{(2\pi m)^2-(kh)^2}\right)-K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\right)\right]\,. \tag{7.69}$$

The third sum in (7.65) has been evaluated in Section 4.2, (4.95):

$$2\sum_{m=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}khm}}{m}\left((kh)^2+\frac{\mathrm{i}kh}{m}-\frac{1}{m^2}\right)$$

$$= -2\left((kh)^2 \ln\left[2\sin\left(\frac{kh}{2}\right)\right] + kh\mathrm{Cl}_2(kh) + \mathrm{Cl}_3(kh)\right) + \mathrm{i}\left[\frac{\pi}{2}(kh)^2 - \frac{2}{3}(kh)^3\right] \tag{7.70}$$

with the Clausen functions $\mathrm{Cl}_2(kh)$ and $\mathrm{Cl}_3(kh)$ defined and approximated in (D.8) and $0 < kh < 2\pi$. Combining (7.67), (7.69), and (7.70), we have shown that the self-plane sum

$$\sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \; \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\sqrt{m^2+l^2}}}{\sqrt{m^2+l^2}} \left((kh)^2 + \frac{\mathrm{i}kh}{\sqrt{m^2+l^2}} - \frac{1}{m^2+l^2}\right)$$

$$= \pi\mathrm{i}(kh)^2 \sum_{l=1}^{\infty}\left[H_0^{(1)}(lkh) - H_2^{(1)}(lkh)\right]$$

$$+ 4\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\Bigg(\left[(2\pi m)^2 + (kh)^2\right] K_0\left(l\sqrt{(2\pi m)^2 - (kh)^2}\right)$$

$$- \left[(2\pi m)^2 - (kh)^2\right] K_2\left(l\sqrt{(2\pi m)^2 - (kh)^2}\right)\Bigg)$$

$$-2\left((kh)^2 \ln\left[2\sin\left(\frac{kh}{2}\right)\right] + kh\mathrm{Cl}_2(kh) + \mathrm{Cl}_3(kh)\right) + \mathrm{i}\left[\frac{\pi}{2}(kh)^2 - \frac{2}{3}(kh)^3\right] . \tag{7.71}$$

Substituting (7.64) and (7.71) in the kd–βd equation (7.21) we can then write the kd–βd equation in the form

$$(kh)^3 = S\left\{\Re + \mathrm{i}\Im\right\} \tag{7.72}$$

where $\Re$, the real part of the quantity within the brackets of (7.21) with the original summations replaced by the new expressions we have derived, is

$$\Re = 16\pi^3 \sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{(m^2+l^2)\,\mathrm{e}^{-n(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}$$

$$- \pi(kh)^2\left[\sum_{l=1}^{\infty} Y_0(lkh) - \sum_{l=1}^{\infty} Y_2(lkh)\right]$$

$$+ 4\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\Bigg[\left[(2\pi m)^2 + (kh)^2\right] K_0\left(l\sqrt{(2\pi m)^2 - (kh)^2}\right)$$

$$- \left[(2\pi m)^2 - (kh)^2\right] K_2\left(l\sqrt{(2\pi m)^2 - (kh)^2}\right)\Bigg]$$

$$- 2\left((kh)^2 \ln\left[2\sin\left(\frac{kh}{2}\right)\right] + kh\mathrm{Cl}_2(kh) + \mathrm{Cl}_3(kh)\right) \tag{7.73}$$

and $\Im$, the imaginary part of the quantity within the brackets of (7.21), is

$$\Im = \pi(kh)^2\left[\sum_{l=1}^{\infty} J_0(lkh) - \sum_{l=1}^{\infty} J_2(lkh)\right] + \frac{\pi}{2}(kh)^2 - \frac{2}{3}(kh)^3 = -\frac{2}{3}(kh)^3 \tag{7.74}$$

using (B.11) and (B.13). As has been noted above in Section 4 [see (4.59)-(4.61)], $\Im = -2/3(kh)^3$ together with (7.72) imply that

$$|S| = \frac{3}{2} \sin \psi \tag{7.75}$$

where ψ is the phase of the scattering coefficient S, a relationship that the scattering coefficient S must satisfy based on reciprocity and power conservation principles, and so the derivation here of (7.75) serves as a valuable check on the correctness of our analysis. The kd–βd equation (7.72) for traveling waves supported by 3D arrays of short electric dipoles parallel to the array axis then becomes

$$\frac{2}{3}(kh)^3 \cos\psi - \Re \sin\psi = 0 \tag{7.76}$$

with $\Re$ given by (7.73) and $kh < 2\pi$. Equation (7.76) can be easily solved numerically for βd given values of kd, kh, and ψ, using, for example, a simple search procedure with secant algorithm refinement.

To facilitate the calculation of $\Re$ the following may be noted. The sum

$$\sum_{n=1}^{\infty} \cos(n\beta d) \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{(m^2+l^2)\, \mathrm{e}^{-n(d/h)\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}} \tag{7.77}$$

converges very rapidly because of the negative exponential so that it is necessary to include only a few terms in the sum, for example, n from 1 to 2 and m, l from -2 to 2, for sufficient accuracy. Alternately an approximation to the sum can be obtained by first performing the summation over n from 1 to ∞ in closed form using (D.4) and then including only terms in the summation over m and l from -1 to 1. When this is done we obtain

$$\sum_{n=1}^{\infty} \cos(n\beta d) \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{(m^2+l^2)\, \mathrm{e}^{-n(d/h)\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}$$

$$\approx 4\left(\frac{1}{r_1}\frac{\mathrm{e}^{-(d/h)r_1}\cos\beta d - \mathrm{e}^{-2(d/h)r_1}}{1-2\cos\beta d\, \mathrm{e}^{-(d/h)r_1} + \mathrm{e}^{-2(d/h)r_1}} + \frac{2}{r_2}\frac{\mathrm{e}^{-(d/h)r_2}\cos\beta d - \mathrm{e}^{-2(d/h)r_2}}{1-2\cos\beta d\, \mathrm{e}^{-(d/h)r_2} + \mathrm{e}^{-2(d/h)r_2}}\right). \tag{7.78}$$

where $r_1 = \sqrt{(2\pi)^2 - (kh)^2}$, and $r_2 = \sqrt{8\pi^2 - (kh)^2}$. Accelerated convergence expressions for the Schlömilch series $\sum Y_0(lkh)$ and $Y_2(lkh)$ are given in (B.12) and (B.14), respectively. The modified Bessel function series

$$\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}[(2\pi m)^2 + (kh)^2]\, K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\right) \tag{7.79a}$$

and

$$\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}[(2\pi m)^2 - (kh)^2]\, K_2\left(l\sqrt{(2\pi m)^2-(kh)^2}\right) \tag{7.79b}$$

converge extremely rapidly because of the exponential decay of K_0 and K_1 so that only a few terms of the series need be included.

The expression for $\Re$ given by (7.73) is valid for $kh < 2\pi$. Because some of the terms of $\Re$ are singular as $kh \to 2\pi$, (7.73) cannot be used as is at $kh = 2\pi$. It is therefore worthwhile to obtain the limit of $\Re$ as $kh \to 2\pi$ from below. Consider first the singularity of

$$16\pi^3 \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{(m^2+l^2)\, \mathrm{e}^{-n(d/h)\, \sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}} \tag{7.80}$$

as $kh \to 2\pi$. The four terms of the double summation over m and l for which $(m,l) = (\pm 1, 0), (0, \pm 1)$ are singular as $kh \to 2\pi$, each of these terms behaving as

$$\frac{1}{\sqrt{4\pi\epsilon}} \tag{7.81}$$

for $\epsilon = 2\pi - kh << 1$. Using (3.68) it follows that

$$16\pi^3 \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{(m^2+l^2)\, \mathrm{e}^{-n(d/h)\, \sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}$$

$$\overset{kh\to 2\pi}{\sim} 8\pi^2 \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{\substack{m=-\infty \\ |m|+|l|}}^{\infty} \quad \sum_{\substack{l=-\infty \\ > \; 1}}^{\infty} (m^2+l^2) \frac{\mathrm{e}^{-2\pi n(d/h)\, \sqrt{m^2+l^2-1}}}{\sqrt{m^2+l^2-1}} - \frac{16\pi^{5/2}}{\sqrt{\epsilon}}\,. \tag{7.82}$$

The behavior of the $Y_0(lkh)$ sum as $kh \to 2\pi$ has been considered above. From (3.70) we have

$$-\pi(kh)^2 \sum_{l=1}^{\infty} Y_0(lkh) \overset{kh\to 2\pi}{\sim} 2\pi^2 \left[2\gamma + 2\ln\frac{1}{2} - 2 + 2\sum_{l=2}^{\infty} \left(\frac{1}{\sqrt{l^2-1}} - \frac{1}{l} \right) \right] + \frac{4\pi^{5/2}}{\sqrt{\epsilon}} \tag{7.83}$$

where we have again let $kh = 2\pi - \epsilon$. The behavior of the $Y_2(lkh)$ sum as $kh \to 2\pi$ can be easily found from (B.14)-(B.15). As $kh \to 0$ only the term in the sum of the RHS of (B.14) for $m = 1$ becomes singular since

$$\sinh q_1 = \frac{\sqrt{(2\pi)^2 - (kh)^2}}{kh} \overset{\epsilon\to 0}{\sim} \frac{\sqrt{4\pi\epsilon}}{2\pi}\,. \tag{7.84}$$

Thus in (7.73)

$$\pi(kh)^2 \sum_{l=1}^{\infty} Y_2(lkh) \overset{\epsilon\to 0}{\sim} (2\pi)^2 \sum_{m=2}^{\infty} \frac{\mathrm{e}^{-2q_m}}{\sinh q_m} + \lambda_2 + \frac{4\pi^{5/2}}{\sqrt{\epsilon}} \tag{7.85}$$

where

$$\sinh q_m = \sqrt{m^2-1} \tag{7.86}$$

and

$$\lambda_2 = \frac{1}{3\pi}\,. \tag{7.87}$$

Next we consider the singularities of

$$4\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}[(2\pi m)^2+(kh)^2]\ K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\right)\ . \tag{7.88}$$

From (3.64)

$$4\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}[(2\pi m)^2+(kh)^2]\ K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\right)$$

$$\overset{kh\to 2\pi}{\sim}\ 4(2\pi)^2\sum_{l=1}^{\infty}\sum_{m=2}^{\infty}(m^2+1)\ K_0\left(2\pi l\sqrt{m^2-1}\right)+(2\pi)^2\left(4\gamma+4\ln\frac{1}{\sqrt{4\pi}}\right)+2(2\pi)^2\ln\epsilon+\frac{8\pi^{5/2}}{\sqrt{\epsilon}} \tag{7.89}$$

where γ is the Euler constant and we have let $kh = 2\pi-\epsilon,\ 0<\epsilon<<1$. Finally consider

$$-4\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}[(2\pi m)^2-(kh)^2]\ K_2\left(l\sqrt{(2\pi m)^2-(kh)^2}\right)\ . \tag{7.90}$$

Referring to (4.104)

$$-4\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}[(2\pi m)^2-(kh)^2]\ K_2\left(l\sqrt{(2\pi m)^2-(kh)^2}\right)$$

$$\overset{kh\to 2\pi}{\sim}\ -4(2\pi)^2\sum_{l=1}^{\infty}\sum_{m=2}^{\infty}(m^2-1)\ K_2\left(2\pi l\sqrt{m^2-1}\right)-4\sum_{l=1}^{\infty}\frac{2}{l^2}$$

$$=-4(2\pi)^2\sum_{l=1}^{\infty}\sum_{m=2}^{\infty}(m^2-1)\ K_2\left(2\pi l\sqrt{m^2-1}\right)-8(1.64493\cdots) \tag{7.91}$$

using (D.12). The logarithmic singularity of (7.89 is exactly canceled by the logarithmic singularity of

$$-\ 2(kh)^2\ln\left[2\sin\left(\frac{kh}{2}\right)\right] \tag{7.92}$$

at $kh = 2\pi$ [see (2.58)] and the $1/\sqrt{\epsilon}$ singularities of (7.82), (7.83, (7.85), and (7.89) also cancel. Thus $\Re$ given by (7.73) is not singular at $kh = 2\pi$ and

$$\lim_{kh\to 2\pi}\Re = 8\pi^2\sum_{n=1}^{\infty}\cos(n\beta d)\qquad\sum_{\substack{m=-\infty\\|m|+|l|\ >\ 1}}^{\infty}\qquad\sum_{l=-\infty}^{\infty}(m^2+l^2)\ \frac{e^{-2\pi n(d/h)\ \sqrt{m^2+l^2-1}}}{\sqrt{m^2+l^2-1}}$$

$$+\ 2\pi^2\left[2\gamma+2\ln\frac{1}{2}-2+2\sum_{l=2}^{\infty}\left(\frac{1}{\sqrt{l^2-1}}-\frac{1}{l}\right)\right]+\ (2\pi)^2\sum_{m=2}^{\infty}\frac{e^{-2q_m}}{\sinh q_m}+\lambda_2$$

$$+\ 4(2\pi)^2\sum_{l=1}^{\infty}\sum_{m=2}^{\infty}(m^2+1)\ K_0\left(2\pi l\sqrt{m^2-1}\right)+(2\pi)^2\left(4\gamma+4\ln\frac{1}{\sqrt{4\pi}}\right)$$

$$- 4(2\pi)^2 \sum_{l=1}^{\infty} \sum_{m=2}^{\infty} (m^2-1)\; K_2\left(2\pi l \sqrt{m^2-1}\right) - 8(1.64493\cdots) - 2\mathrm{Cl}_3(2\pi) \tag{7.93}$$

with $\sinh q_m$, λ_2, and $\mathrm{Cl}_3(2\pi)$, given by (7.86) (7.87) and (D.10), respectively. Referring to (7.78) we see that

$$8\pi^2 \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{\substack{m=-\infty \\ |m|+|l|>1}}^{\infty} \sum_{l=-\infty}^{\infty} (m^2+l^2) \frac{\mathrm{e}^{-2\pi n (d/h)\sqrt{m^2+l^2-1}}}{\sqrt{m^2+l^2-1}}$$

$$\approx 32\pi \frac{\mathrm{e}^{-2\pi d/h} \cos\beta d - \mathrm{e}^{-4\pi d/h}}{1 - 2\cos\beta d\, \mathrm{e}^{-2\pi d/h} + \mathrm{e}^{-4\pi d/h}}\,. \tag{7.94}$$

In closing this section we note that we shall not define for a three-dimensional array of closely spaced parallel dipoles an effective relative permittivity $\epsilon_{\mathrm{r}}^{\mathrm{eff}}$ (as we did for three-dimensional arrays of closely spaced perpendicular dipoles) that determine the propagation constant of the traveling wave. Although such a defined effective permittivity would indeed determine the propagation constant of the traveling wave, the average polarization within the array would not be given by $\mathbf{P} = (\epsilon_{\mathrm{r}}^{\mathrm{eff}} - 1)\epsilon_0 \mathbf{E}$ because the dipoles are oriented parallel to the direction of propagation and thus perpendicular to the fields in the approximate transverse electromagnetic (TEM) wave that would be supported by the closely spaced array of parallel dipoles.

8 2D MAGNETODIELECTRIC SPHERE ARRAYS

In this section we consider traveling waves supported by 2D periodic arrays of lossless magnetodielectric spheres. It is assumed that the spheres can be modeled by pairs of crossed electric and magnetic dipoles, each of the dipoles perpendicular to the array axis. (It is unnecessary to consider 2D arrays of electric and magnetic dipoles with the electric (magnetic) dipoles in the direction of the array axis and the magnetic (electric) dipoles perpendicular to the array axis, or 2D arrays of electric and magnetic dipoles with all dipoles oriented in the direction of the array axis, because an electric (magnetic) dipole has no radial or longitudinal magnetic (electric) field [20, secs. 8.5, 8.6] and so there is no coupling of the electric dipoles with the magnetic dipoles of such arrays. 2D arrays of electric dipoles oriented in the direction of the array axis have been treated in Section 6.) It is important to note that although we refer in this and the following section of the report to the array elements as "magnetodielectric spheres", the analyses that we perform apply equally well to any array elements that can be modeled as a pair of electric and magnetic dipoles at right angles to each other such that only an incident electric (magnetic) field at the element center in the direction of the electric (magnetic) dipole excites only the electric (magnetic) dipole field. (Orthogonality of the spherical modes, as well as the fields of higher-than-dipole-order incident spherical multipoles being zero at the center of each sphere, ensure these conditions hold for magnetodielectric spheres whose scattering is predominantly dipolar fields.) There are two polarizations of the electric dipoles to be considered, one where the dipoles are in the array plane determined by the sphere centers and array axis, and the other where the dipoles are perpendicular to the array plane. These two polarizations are treated in 8.1 and 8.2, respectively. Since the electric and magnetic dipoles of one polarization are, apart from a sign change, the magnetic and electric dipoles, respectively, of the other polarization, the kd–βd equation for the polarization in which the electric dipoles are perpendicular to the array plane can be obtained very simply from the kd–βd equation for the polarization in which the electric dipoles are in the array plane.

8.1 ELECTRIC DIPOLES IN THE ARRAY PLANE

We choose the array axis to be the z axis of a Cartesian coordinate system with equi-spaced columns of magnetodielectric spheres parallel to the x axis located at $z = nd, n = 0, \pm 1, \pm 2, \cdots$. In each column the spheres are centered at $x = mh, m = 0, \pm 1, \pm 2, \cdots$. The electric and magnetic dipole components of each sphere are oriented in the x and y direction, respectively, so that the electric dipoles lie in the xz plane, the array plane. We assume an excitation of the array with the electric field parallel to the x axis and the magnetic field parallel to the y axis, and such that all the spheres in any column of the array are excited identically. Let $\mathbf{E}_0^0$ and $\mathbf{H}_0^0$ be the electric and magnetic field, respectively, incident on the sphere at the location $x = 0, y = 0, z = 0$ from all the other spheres in the array. As will be seen [see (8.18)] $\mathbf{E}_0^0$ has an x component only, and $\mathbf{H}_0^0$ has a y component only. Let $\mathbf{E}_0^{0mn}$ and $\mathbf{H}_0^{0mn}$ be the electric and magnetic field, respectively, incident on the reference sphere from the sphere at the location $(x, y, z) = (mh, 0, nd)$ so that

$$\mathbf{E}_0^0 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{m=-\infty}^{\infty} \mathbf{E}_0^{0mn} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \mathbf{E}_0^{0m0} \tag{8.1a}$$

$$\mathbf{H}_0^0 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{m=-\infty}^{\infty} \mathbf{H}_0^{0mn} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \mathbf{H}_0^{0m0} \,. \tag{8.1b}$$

From [5, eqs. (32),(33)] with $\sin\phi_{nm} = 0$,

$$\begin{aligned}
\mathbf{E}_0^{0mn} &= b_{-n}\frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}}\left[\frac{-2\mathrm{i}}{kr_{mn0}}\left(1+\frac{\mathrm{i}}{kr_{mn0}}\right)\sin\theta_{mn0}\cos\phi_{mn0}\;\hat{\mathbf{r}}_{mn0}\right. \\
&\quad \left. +\;\left(1+\frac{\mathrm{i}}{kr_{mn0}}-\frac{1}{(kr_{mn0})^2}\right)\cos\theta_{mn0}\cos\phi_{mn0}\;\hat{\boldsymbol{\theta}}_{mn0}\right] \\
&\quad + b_{+n}\frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}}\left(1+\frac{\mathrm{i}}{kr_{mn0}}\right)\cos\phi_{mn0}\;\hat{\boldsymbol{\theta}}_{mn0}
\end{aligned} \tag{8.2a}$$

and

$$\begin{aligned}
\mathbf{H}_0^{mn0} &= Y_0 b_{+n}\frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}}\left(1+\frac{\mathrm{i}}{kr_{mn0}}-\frac{1}{(kr_{mn0})^2}\right)\cos\phi_{mn0}\;\hat{\boldsymbol{\phi}}_{mn0} \\
&\quad + Y_0 b_{-n}\frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}}\left(1+\frac{\mathrm{i}}{kr_{mn0}}\right)(\cos\theta_{mn0}\cos\phi_{mn0}\;\hat{\boldsymbol{\phi}}_{mn0})
\end{aligned} \tag{8.2b}$$

where Y_0 is the free-space admittance. The quantities in (8.2) are defined with reference to a local spherical polar coordinate system with origin at $(x, y, z) = (mh, 0, nd)$ (in turn defined with reference to a local Cartesian coordinate system with the same origin whose axes are parallel to those of the global Cartesian coordinate system). The distance from the (m, n) sphere to the $(0, 0)$ sphere, r_{mn0}, is given by

$$r_{mn0} = \sqrt{(mh)^2 + (nd)^2} = h\sqrt{m^2 + (nd/h)^2} \tag{8.3}$$

and the unit vector in the direction from the (m, n) sphere to the $(0, 0)$ sphere, $\hat{\mathbf{r}}_{mn0}$, is

$$\hat{\mathbf{r}}_{mn0} = \mathbf{r}_{mn0}/r_{mn0}, \quad \mathbf{r}_{mn0} = -mh\,\hat{\mathbf{x}} - nd\,\hat{\mathbf{z}} \tag{8.4}$$

so that

$$\cos\theta_{mn0} = \hat{\mathbf{r}}_{mn0} \cdot \hat{\mathbf{z}} = -\frac{nd}{r_{mn0}} \tag{8.5}$$

and

$$\sin\theta_{mn0} = \sqrt{1 - \cos^2\theta_{mn0}} = \frac{|m|h}{r_{mn0}}\,. \tag{8.6}$$

Since $\hat{\mathbf{r}}_{mn0}$ is also given by

$$\hat{\mathbf{r}}_{mn0} = \sin\theta_{mn0}\cos\phi_{mn0}\,\hat{\mathbf{x}} + \cos\theta_{mn0}\,\hat{\mathbf{z}} \tag{8.7}$$

it follows by substituting (8.5) and (8.6) in (8.7) and comparing with (8.4) that

$$\cos\phi_{mn0} = \begin{cases} -1 & : \quad m > 0 \\ +1 & : \quad m < 0 \end{cases} \quad \text{or} \quad \phi_{mn0} = \begin{cases} \pi & : \quad m > 0 \\ 0 & : \quad m < 0 \end{cases}\,. \tag{8.8}$$

Then

$$\hat{\boldsymbol{\theta}}_{mn0} = \cos\theta_{mn0}\cos\phi_{mn0}\,\hat{\mathbf{x}} - \sin\theta_{mn0}\,\hat{\mathbf{z}} = \pm\frac{nd}{r_{mn0}}\,\hat{\mathbf{x}} - \frac{|m|h}{r_{mn0}}\,\hat{\mathbf{z}}\,, \quad m \gtrless 0 \tag{8.9}$$

$$\hat{\boldsymbol{\phi}}_{mn0} = \cos\phi_{mn0}\,\hat{\mathbf{y}} = \mp\,\hat{\mathbf{y}}, \;\; m \gtrless 0 \tag{8.10}$$

$$\sin\theta_{mn0}\cos\phi_{mn0}\,\hat{\mathbf{r}}_{mn0} = \frac{mh}{r_{mn0}^2}\,(mh\,\hat{\mathbf{x}} + nd\,\hat{\mathbf{z}}) \tag{8.11}$$

$$\cos\theta_{mn0}\cos\phi_{mn0}\,\hat{\boldsymbol{\theta}}_{mn0} = \frac{(nd)^2\,\hat{\mathbf{x}} - (mh)(nd)\,\hat{\mathbf{z}}}{r_{mn0}^2} \tag{8.12}$$

$$\cos\phi_{mn0}\,\hat{\boldsymbol{\theta}}_{mn0} = \frac{-nd\,\hat{\mathbf{x}} + mh\,\hat{\mathbf{z}}}{r_{mn0}} \tag{8.13}$$

$$\cos\phi_{mn0}\,\hat{\boldsymbol{\phi}}_{mn0} = \hat{\mathbf{y}} \tag{8.14}$$

and

$$\cos\theta_{mn0}\cos\phi_{mn0}\,\hat{\boldsymbol{\phi}}_{mn0} = -\frac{nd}{r_{mn0}}\,\hat{\mathbf{y}}\,. \tag{8.15}$$

The constants b_{-n} and b_{+n} are related to the x component of the electric field and the y component of the magnetic field, respectively, incident on any sphere in the nth column by the scattering equations [5, eq. (31)]

$$b_{-n} = S_-\,E_{0x}^{0n} \tag{8.16a}$$

$$b_{+n} = S_+ \, \frac{H_{0y}^{0n}}{Y_0}. \tag{8.16b}$$

where S_- and S_+ are the normalized magnetodielectric sphere electric and magnetic dipole scattering coefficients, respectively. "Normalized" means that b_{-n} (b_{+n}) is the coefficient of $\exp(\mathrm{i}kr)/(kr)$ in the outgoing electric (magnetic) dipole field in response to the incident field $E_{0x}^{0n}\,\hat{\mathbf{x}}$ ($H_{0y}^{0n}/Y_0\,\hat{\mathbf{y}}$) at the center of the x (y) directed electric (magnetic) dipole.[6] Substituting (8.11)-(8.15) in (8.2) we obtain

$$\begin{aligned}
\mathbf{E}_0^{0mn} = b_{-n}\frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}}\Bigg[&\frac{-2\mathrm{i}}{kr_{mn0}}\left(1+\frac{\mathrm{i}}{kr_{mn0}}\right)\frac{mh}{r_{mn0}^2}\,(mh\,\hat{\mathbf{x}}+nd\,\hat{\mathbf{z}}) \\
&+\left(1+\frac{\mathrm{i}}{kr_{mn0}}-\frac{1}{(kr_{mn0})^2}\right)\frac{nd}{r_{mn0}}\,(nd\,\hat{\mathbf{x}}-mh\,\hat{\mathbf{z}})\Bigg] \\
&- b_{+n}\frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}}\left(1+\frac{\mathrm{i}}{kr_{mn0}}\right)\frac{nd\,\hat{\mathbf{x}}-mh\,\hat{\mathbf{z}}}{r_{mn0}}
\end{aligned} \tag{8.17a}$$

and

$$\begin{aligned}
\mathbf{H}_0^{mn0} = Y_0\,b_{+n}&\frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}}\left(1+\frac{\mathrm{i}}{kr_{mn0}}-\frac{1}{(kr_{mn0})^2}\right)\,\hat{\mathbf{y}} \\
&- Y_0\,b_{-n}\frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}}\left(1+\frac{\mathrm{i}}{kr_{mn0}}\right)\frac{nd}{r_{mn0}}\,\hat{\mathbf{y}}\;.
\end{aligned} \tag{8.17b}$$

Note that when we sum over m from $-\infty$ to ∞ the z components of $\mathbf{E}_0^{mn0}$ cancel and we are left with an x component only of the electric field incident on the reference sphere (and of course a y component only of the magnetic field). Thus, substituting (8.17) in (8.1) we obtain

$$\begin{aligned}
E_{0x}^0 = \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\Bigg\{b_{-n}\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{kh\rho_{mn}}\Bigg[&\frac{-2\mathrm{i}}{kh\rho_{mn}}\left(1+\frac{\mathrm{i}}{kh\rho_{mn}}\right)\frac{m^2}{\rho_{mn}^2} \\
+\left(1+\frac{\mathrm{i}}{kh\rho_{mn}}-\frac{1}{(kh)^2\rho_{mn}^2}\right)\frac{(nd/h)^2}{\rho_{mn}^2}\Bigg] - b_{+n}&\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{kh\rho_{mn}}\left(1+\frac{\mathrm{i}}{kh\rho_{mn}}\right)\frac{nd/h)}{\rho_{mn}}\Bigg\}
\end{aligned}$$

[6] As we noted at the beginning of this section of the report, although we refer to the array elements as "magnetodielectric spheres", our analysis applies equally well to any array elements that can be modeled as a pair of crossed electric and magnetic dipoles such that an incident electric (magnetic) field in the direction of the electric (magnetic) dipole excites only the electric (magnetic) dipole field. If the array elements are indeed spheres then S_- and S_+ are the normalized Mie dipole scattering coefficients [5, eqs.(30a,b)],

$$S_- = -\mathrm{i}\frac{3}{2}b_1^{\mathrm{sc}}$$

$$S_+ = -\mathrm{i}\frac{3}{2}a_1^{\mathrm{sc}}$$

where b_1^{sc} and a_1^{sc} are the electric and magnetic Mie dipole scattering coefficients defined in Stratton [20]. If the array elements are not magnetodielectric spheres then S_- and S_+ must be known for the results of this and the following section of the report to be applied.

$$+\,4b_{-0}\sum_{m=1}^{\infty}\frac{e^{ikh\rho_{m0}}}{kh\rho_{m0}}\frac{-i}{kh\rho_{m0}}\left(1+\frac{i}{kh\rho_{m0}}\right) \tag{8.18a}$$

and

$$\frac{H_{0y}^{mn0}}{Y_0}=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left\{ b_{+n}\sum_{m=-\infty}^{\infty}\frac{e^{ikh\rho_{mn}}}{kh\rho_{mn}}\left(1+\frac{i}{kh\rho_{m}}-\frac{1}{(kh)^2\rho_{mn}^2}\right)\right.$$
$$\left.-\,b_{-n}\sum_{m=-\infty}^{\infty}\frac{e^{ikh\rho_{mn}}}{kh\rho_{mn}}\left(1+\frac{i}{kh\rho_{mn}}\right)\frac{nd/h}{\rho_{mn}}\right\}$$
$$+\,2b_{+0}\sum_{m=1}^{\infty}\frac{e^{ikh\rho_{m0}}}{kh\rho_{m0}}\left(1+\frac{i}{kh\rho_{m0}}-\frac{1}{(kh)^2\rho_{m0}^2}\right) \tag{8.18b}$$

where we have let

$$\rho_{mn}=\sqrt{m^2+(nd/h)^2} \tag{8.19}$$

with

$$\rho_{m0}=m\ . \tag{8.20}$$

We now assume that the array is excited by a traveling wave in the z direction with real propagation constant β. Then the constants b_{-n} and b_{+n} in (8.18) equal b_{-0} and b_{+0}, respectively, apart from a phase shift given by

$$b_{-n}=b_{-0}\,e^{in\beta d},\ \ b_{+n}=b_{+0}\,e^{in\beta d}\ . \tag{8.21}$$

Substituting (8.21) in (8.18), using [from (8.16)] $b_{-0}=S_-E_{0x}^0$ and $b_{+0}=S_+H_{0y}^0/Y_0$,and multiplying by $(kh)^3$ we obtain

$$(kh)^3=S_-\left\{\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}e^{in\beta d}\left(\sum_{m=-\infty}^{\infty}\frac{e^{ikh\rho_{mn}}}{\rho_{mn}}\left[\frac{-2i}{\rho_{mn}}\left(kh+\frac{i}{\rho_{mn}}\right)\frac{m^2}{\rho_{mn}^2}\right.\right.\right.$$
$$\left.\left.+\left((kh)^2+\frac{ikh}{\rho_{mn}}-\frac{1}{\rho_{mn}^2}\right)\frac{(nd/h)^2}{\rho_{mn}^2}\right]-q\sum_{m=-\infty}^{\infty}\frac{e^{ikh\rho_{mn}}}{\rho_{mn}}\left((kh)^2+\frac{ikh}{\rho_{mn}}\right)\frac{nd/h}{\rho_{mn}}\right)$$
$$\left.+\,4\sum_{m=1}^{\infty}\frac{e^{ikh\rho_{m0}}}{\rho_{m0}}\frac{-i}{\rho_{m0}}\left(kh+\frac{i}{\rho_{m0}}\right)\right\} \tag{8.22a}$$

and

$$(kh)^3=S_+\left\{\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}e^{in\beta d}\left[\sum_{m=-\infty}^{\infty}\frac{e^{ikh\rho_{mn}}}{\rho_{mn}}\left((kh)^2+\frac{ikh}{\rho_{mn}}-\frac{1}{\rho_{mn}^2}\right)\right.\right.$$
$$\left.-\,\frac{1}{q}\sum_{m=-\infty}^{\infty}\frac{e^{ikh\rho_{mn}}}{\rho_{mn}}\left((kh)^2+\frac{ikh}{\rho_{mn}}\right)\frac{nd/h}{\rho_{mn}}\right]$$
$$\left.+\,2\sum_{m=1}^{\infty}\frac{e^{ikh\rho_{m0}}}{\rho_{m0}}\left((kh)^2+\frac{ikh}{\rho_{m0}}-\frac{1}{\rho_{m0}^2}\right)\right\} \tag{8.22b}$$

where

$$q = \frac{b_{+0}}{b_{-0}} \; . \tag{8.23}$$

(It is noted in [5] that the electromagnetic field scattered from each of the spheres in the array is, apart from the phase factor of the traveling wave, a linear combination of an electromagnetic field proportional to the field of an infinitesimal x directed electric dipole with a normalized coefficient $b_{-,0}$, and an electromagnetic field proportional to the field of an infinitesimal y directed magnetic dipole with normalized coefficient $b_{+,0}$, so that q is the ratio of these two normalized scattered field coefficients.) As will be done below [see (8.44)-(8.52)], by eliminating q from (8.22a) and (8.22b), the kd–βd equation is obtained that determines the normalized traveling wave propagation constant βd in terms of kh, d/h, and the normalized magnetodielectric sphere electric and magnetic dipole scattering coefficients S_- and S_+.

It remains to convert the slowly convergent summations in (8.22) to rapidly convergent forms. We begin with

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \left[\frac{-2\mathrm{i}}{\rho_{mn}} \left(kh + \frac{\mathrm{i}}{\rho_{mn}} \right) \frac{m^2}{\rho_{mn}^2} \right.$$
$$\left. + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} - \frac{1}{\rho_{mn}^2} \right) \frac{(nd/h)^2}{\rho_{mn}^2} \right] \tag{8.24}$$

in (8.22a). This double sum is proportional to the electric field incident on the reference sphere at $(x,z) = (0,0)$ scattered from the x directed electric dipoles in all the columns of the array except for those in the self-column ($z = 0$). We have immediately from (4.15), (4.44), and (4.45), in our analysis of 2D periodic arrays of electric dipoles in the array plane and perpendicular to the array axis, that

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \left[\frac{-2\mathrm{i}}{\rho_{mn}} \left(kh + \frac{\mathrm{i}}{\rho_{mn}} \right) \frac{m^2}{\rho_{mn}^2} \right.$$
$$\left. + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} - \frac{1}{\rho_{mn}^2} \right) \frac{(nd/h)^2}{\rho_{mn}^2} \right]$$
$$= 2 \sum_{n=1}^{\infty} \cos(n\beta d) \left[\mathrm{i}\pi (kh)^2 H_0^{(1)}(nkd) \right.$$
$$\left. - 4 \sum_{m=1}^{\infty} \left[(2\pi m)^2 - (kh)^2 \right] K_0 \left(n(d/h) \sqrt{(2\pi m)^2 - (kh)^2} \right) \right] . \tag{8.25}$$

Next we consider the sum

$$\sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} \right) \frac{nd/h}{\rho_{mn}} \tag{8.26}$$

with $\rho_{mn} = \sqrt{m^2+(nd/h)^2}$. This sum is proportional to the y directed magnetic field incident on the reference sphere scattered from all the x directed electric dipoles in the nth column, $n \neq 0$, or equivalently to the x directed electric field incident on the reference sphere scattered from all the y directed magnetic dipoles in the nth column, $n \neq 0$. We write (8.26) as

$$\frac{nd}{h}(kh)^2 \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kh\sqrt{m^2+(nd/h)^2}} \left(\frac{1}{m^2+(nd/h)^2} + \frac{1}{(-\mathrm{i}kh)\left[m^2+(nd/h)^2\right]^{3/2}} \right) . \quad (8.27)$$

The sum has been evaluated earlier. Referring to (4.89)

$$\sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} \right) \frac{nd/h}{\rho_{mn}}$$

$$= nkd \Bigg\{ -\frac{\pi}{2}(kh)^2 \left[H_0^{(1)}(|n|kd) + H_2^{(1)}(|n|kd) \right]$$

$$+ 2\mathrm{i} \sum_{m=1}^{\infty} [(2\pi m)^2 - (kh)^2] \left[K_2\left(|n|(d/h)\sqrt{(2\pi m)^2-(kh)^2} \right) - K_0\left(|n|(d/h)\sqrt{(2\pi m)^2-(kh)^2} \right) \right] \Bigg\}$$

$$= \mathrm{sgn}(n) \left[-\pi(kh)^2 H_1^{(1)}(|n|kd) + 4\mathrm{i}(kh) \sum_{m=1}^{\infty} K_1\left(|n|(d/h)\sqrt{(2\pi m)^2-(kh)^2} \right) \right] . \quad (8.28)$$

[An alternate derivation of (8.28) using the Floquet mode expansion method is given below in (8.59) - (8.69).] Then in (8.22)

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} \right) \frac{nd/h}{\rho_{mn}}$$

$$= \sum_{n=1}^{\infty} 2\mathrm{i}\sin(n\beta d) \Bigg(-\pi(kh)^2 H_1^{(1)}(nkd)$$

$$+ 4\mathrm{i}(kh) \sum_{m=1}^{\infty} \sqrt{(2\pi m)^2-(kh)^2}\; K_1\left(n(d/h)\sqrt{(2\pi m)^2-(kh)^2} \right) \Bigg) . \quad (8.29)$$

Next we consider the sum in (8.22b)

$$\sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_m} - \frac{1}{\rho_{mn}^2} \right) . \quad (8.30)$$

This sum is proportional to the y directed magnetic field incident on the reference sphere scattered from all the y directed magnetic dipoles in the nth column, $n \neq 0$. (While it might appear at first that (apart from a factor of Y_0) the y directed magnetic field scattered from all the y directed magnetic dipoles in the nth column of the array incident on the reference sphere at $(x, y, z) = (0, 0, 0)$ should equal the x directed electric field scattered from all the x

directed electric dipoles in the nth column incident on the reference sphere, in fact symmetry does not hold because of the two-dimensionality of the problem. The y directed magnetic dipoles are perpendicular to the xz plane in which scattering is calculated, whereas the x directed electric dipoles lie in the xz plane.) From (4.90)

$$\sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} - \frac{1}{\rho_{mn}^2} \right) = \frac{\mathrm{i}\pi(kh)^2}{2} \left[H_0^{(1)}(|n|kd) - H_2^{(1)}(|n|kd) \right]$$

$$+ 2 \sum_{m=1}^{\infty} \left[\left[(2\pi m)^2 + (kh)^2 \right] K_0 \left(|n|(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2} \right) \right.$$

$$\left. - \left[(2\pi m)^2 - (kh)^2 \right] K_2 \left(|n|(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2} \right) \right] \tag{8.31}$$

assuming that $0 < kh < 2\pi$, and thus in (8.22b)

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} - \frac{1}{\rho_{mn}^2} \right)$$

$$= 2 \sum_{n=1}^{\infty} \cos(n\beta d) \left\{ \frac{\mathrm{i}\pi(kh)^2}{2} \left[H_0^{(1)}(nkd) - H_2^{(1)}(nkd) \right] \right.$$

$$+ 2 \sum_{m=1}^{\infty} \left[\left[(2\pi m)^2 + (kh)^2 \right] K_0 \left(n(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2} \right) \right.$$

$$\left. \left. - \left[(2\pi m)^2 - (kh)^2 \right] K_2 \left(n(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2} \right) \right] \right\}. \tag{8.32}$$

[An alternate derivation of (8.31) using the Floquet mode expansion method is given below in (8.70) - (8.81).]

The self-column sum in (8.22a)

$$-4 \sum_{m=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}khm}}{m} \frac{\mathrm{i}}{m} \left(kh + \frac{\mathrm{i}}{m} \right) \tag{8.33}$$

has been treated in Section 4. From (4.56) we have

$$4 \sum_{m=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}khm}}{m} \frac{-\mathrm{i}}{m} \left(kh + \frac{\mathrm{i}}{m} \right) = 4\ kh\ \mathrm{Cl}_2(kh) + 4\ \mathrm{Cl}_3(kh) + \mathrm{i}\pi(kh)^2 - \mathrm{i}\frac{2}{3}(kh)^3 \tag{8.34}$$

with the Clausen functions $\mathrm{Cl}_2(kh)$ and $\mathrm{Cl}_3(kh)$ defined and approximated in (D.8). The self-column sum in (8.22b)

$$2 \sum_{m=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}khm}}{m} \left((kh)^2 + \frac{\mathrm{i}kh}{m} - \frac{1}{m^2} \right) \tag{8.35}$$

has been treated in Section 4. From (4.95)

$$2\sum_{m=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}khm}}{m}\left((kh)^2+\frac{\mathrm{i}kh}{m}-\frac{1}{m^2}\right)$$

$$=-2\left((kh)^2\ln\left[2\sin\left(\frac{kh}{2}\right)\right]+kh\mathrm{Cl}_2(kh)+\mathrm{Cl}_3(kh)\right)+\mathrm{i}\left[\frac{\pi}{2}(kh)^2-\frac{2}{3}(kh)^3\right] \tag{8.36}$$

for $0<kh<2\pi$. Substituting (8.25), (8.29), (8.32), (8.34), and (8.36) in (8.22) we can then write these equations in the form

$$(kh)^3=S_-\left\{\Re_-+\mathrm{i}\Im_-\right\} \tag{8.37a}$$

and

$$(kh)^3=S_+\left\{\Re_++\mathrm{i}\Im_+\right\} \tag{8.37b}$$

where, assuming that q is real, an assumption that is verified shortly below [see 8.52)], $\Re_-$, the real part of the quantity within the brackets of (8.22a) with the original summations replaced by the new expressions we have derived, is

$$\Re_-=-2\pi(kh)^2\sum_{n=1}^{\infty}\cos(n\beta d)Y_0(nkd)$$

$$-8\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=1}^{\infty}\left[(2\pi m)^2-(kh)^2\right]K_0\left(n(d/h)\sqrt{(2\pi m)^2-(kh)^2}\right)$$

$$-q\left[2\pi(kh)^2\sum_{n=1}^{\infty}\sin(n\beta d)Y_1(nkd)\right.$$

$$\left.-8\,kh\sum_{n=1}^{\infty}\sin(n\beta d)\sum_{m=1}^{\infty}\sqrt{(2\pi m)^2-(kh)^2}\,K_1\left(n(d/h)\sqrt{(2\pi m)^2-(kh)^2}\right)\right]$$

$$+4\,kh\,\mathrm{Cl}_2(kh)+4\,\mathrm{Cl}_3(kh)\;; \tag{8.38a}$$

$\Im_-$, the imaginary part of the quantity within the brackets of (8.22a), is

$$\Im_-=2\pi(kh)^2\sum_{n=1}^{\infty}\cos(n\beta d)J_0(nkd)+q\,2\pi(kh)^2\sum_{n=1}^{\infty}\sin(n\beta d)J_1(nkd)$$

$$+\pi(kh)^2-\frac{2}{3}(kh)^3\;; \tag{8.38b}$$

$\Re_+$, the real part of the quantity within the brackets of (8.22b) with the original summations replaced by the rapidly convergent expressions we have derived, is

$$\Re_+=-\pi(kh)^2\left[\sum_{n=1}^{\infty}\cos(n\beta d)Y_0(nkd)-\sum_{n=1}^{\infty}\cos(n\beta d)Y_2(nkd)\right]$$

$$
+ 4\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=1}^{\infty}\Bigg[\left[(2\pi m)^2+(kh)^2\right]K_0\left(n(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right)
$$

$$
- \ \left[(2\pi m)^2-(kh)^2\right]K_2\left(n(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right)\Bigg]
$$

$$
-\ \frac{1}{q}\Bigg[2\pi(kh)^2\sum_{n=1}^{\infty}\sin(n\beta d)Y_1(nkd)
$$

$$
-\ 8\ kh\ \sum_{n=1}^{\infty}\sin(n\beta d)\sum_{m=1}^{\infty}\sqrt{(2\pi m)^2-(kh)^2}\ K_1\left(n(d/h)\ \sqrt{(2\pi m)^2-(kh)^2}\right)\Bigg]
$$

$$
-\ 2\left((kh)^2\ln\left[2\sin\left(\frac{kh}{2}\right)\right]+kh\ \mathrm{Cl}_2(kh)+\mathrm{Cl}_3(kh)\right)\ ; \tag{8.39a}
$$

and $\Im_+$, the imaginary part of the quantity within the brackets of (8.22b), is

$$
\Im_+ = \pi(kh)^2\left[\sum_{n=1}^{\infty}\cos(n\beta d)J_0(nkd)-\sum_{n=1}^{\infty}\cos(n\beta d)J_2(nkd)\right]
$$

$$
-\ \frac{1}{q}\left(-2\pi(kh)^2\sum_{n=1}^{\infty}\sin(n\beta d)J_1(nkd)\right)+\frac{\pi}{2}(kh)^2-\frac{2}{3}(kh)^3\ . \tag{8.39b}
$$

From (B.1a) and (B.5) we see that

$$
\Im_- = -\frac{2}{3}(kh)^3 \tag{8.40}
$$

which, together with (8.37a), by the argument used above in Section 4 [see (4.59)-(4.61)] implies that

$$
|S_-| = \frac{3}{2}\sin\psi_- \tag{8.41}
$$

where ψ_- is the phase of the scattering coefficient S_-. Also from (8.39b), making use of (B.1a), (B.8a), and (B.5),

$$
\Im_+ = -\frac{2}{3}(kh)^3 \tag{8.42}
$$

implying similarly that

$$
|S_+| = \frac{3}{2}\sin\psi_+ \tag{8.43}
$$

where ψ_+ is the phase of the scattering coefficient S_+. The properties of the scattering coefficients (8.41) and (8.43) were derived independently in [4] from reciprocity and power conservation principles, and our obtaining them here thereby serves as an important check on our analysis. It is worth noting that if $\beta d < kd$ then, from (B.1b), $\sum\cos(n\beta d)J_0(nkd)\neq -1/2$ and from (B.8b), $\sum\cos(n\beta d)J_2(nkd)\neq 0$ and hence $\Im_-$ and $\Im_+$ could not equal $-2/3(kh)^3$ so that (8.41) and (8.43) would not hold. This is not possible for an array of short lossless dipole scatterers. Hence $\beta d > kd$. This is a particular instance of the general result (1.4) noted in the Introduction which holds for 2D arrays as well as for linear arrays.

To obtain the kd–βd equation determining βd as a function of kd, d/h, and the scattering coefficients S_- and S_+, we write (8.37) as

$$(kh)^3 = S_-\Big\{\Sigma_1 - q\Sigma_2\Big\} \tag{8.44a}$$

and

$$(kh)^3 = S_+\Big\{\Sigma_3 - \frac{1}{q}\Sigma_2\Big\} \tag{8.44b}$$

where from (8.38), (8.39), (8.40), and (8.42),

$$\begin{aligned}\Sigma_1 = &-2\pi(kh)^2\sum_{n=1}^{\infty}\cos(n\beta d)Y_0(nkd)\\ &- 8\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=1}^{\infty}\left[(2\pi m)^2-(kh)^2\right]K_0\left(n(d/h)\sqrt{(2\pi m)^2-(kh)^2}\right)\\ &+ 4\,kh\,\mathrm{Cl}_2(kh) + 4\,\mathrm{Cl}_3(kh) - \mathrm{i}\frac{2}{3}(kh)^3\end{aligned} \tag{8.45}$$

$$\begin{aligned}\Sigma_2 = &\,2\pi(kh)^2\sum_{n=1}^{\infty}\sin(n\beta d)Y_1(nkd)\\ &- 8(kh)\sum_{n=1}^{\infty}\sin(n\beta d)\sum_{m=1}^{\infty}\sqrt{(2\pi m)^2-(kh)^2}\,K_1\left(n(d/h)\sqrt{(2\pi m)^2-(kh)^2}\right)\end{aligned} \tag{8.46}$$

and

$$\begin{aligned}\Sigma_3 = &-\pi(kh)^2\left[\sum_{n=1}^{\infty}\cos(n\beta d)Y_0(nkd) - \sum_{n=1}^{\infty}\cos(n\beta d)Y_2(nkd)\right]\\ &+ 4\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=1}^{\infty}\Bigg[\left[(2\pi m)^2+(kh)^2\right]K_0\left(n(d/h)\sqrt{(2\pi m)^2-(kh)^2}\right)\\ &\qquad - \left[(2\pi m)^2-(kh)^2\right]K_2\left(n(d/h)\sqrt{(2\pi m)^2-(kh)^2}\right)\Bigg]\\ &- 2(kh)^2\ln\left[2\sin\left(\frac{kh}{2}\right)\right] - 2kh\mathrm{Cl}_2(kh) - 2\mathrm{Cl}_3(kh) - \mathrm{i}\frac{2}{3}(kh)^3\end{aligned} \tag{8.47}$$

with the Clausen functions $\mathrm{Cl}_2(kh)$ and $\mathrm{Cl}_3(kh)$ defined and approximated by (D.8).

It is straightforward to show that the pair of equations (8.44a) and (8.44b) whose solution gives the normalized traveling wave constant βd implies the important relations (8.41) and (8.43) without having to assume that q is real as was done when we obtained (8.41) and (8.43) above. For, from (8.44a),

$$S_- \equiv |S_-|\mathrm{e}^{\mathrm{i}\psi_-} = (kh)^3\,\frac{(\Sigma_{1r}-q_r\Sigma_2) - \mathrm{i}\,(\Sigma_{1\mathrm{i}}-q_\mathrm{i}\Sigma_2)}{(\Sigma_{1r}-q_r\Sigma_2)^2 + (\Sigma_{1\mathrm{i}}-q_\mathrm{i}\Sigma_2)^2} \tag{8.48}$$

where we have let $q = q_r + \mathrm{i}q_\mathrm{i}$, $\Sigma_1 = \Sigma_{1r} + \mathrm{i}\Sigma_{1\mathrm{i}}$, and noted that Σ_2 is real. It is then simple to solve (8.48) for $|S_-|$ and $\sin\psi_-$ and, using the fact that $\Sigma_{1\mathrm{i}} = -2/3\ (kh)^3$, to obtain

$$\frac{\sin\psi_-}{|S_-|} = \frac{2}{3} + q_\mathrm{i}\,\frac{\Sigma_2}{(kh)^2}\ . \tag{8.49}$$

It follows immediately from (8.49) and (8.44b) that

$$\frac{\sin\psi_+}{|S_+|} = \frac{2}{3} - \frac{q_\mathrm{i}}{|q|^2}\,\frac{\Sigma_2}{(kh)^2}\ . \tag{8.50}$$

since the imaginary part of Σ_3 is also equal to $-2/3\ (kh)^3$. Adding (8.49) and (8.50) then gives

$$\left(\frac{\sin\psi_-}{|S_-|} - \frac{2}{3}\right) + |q|^2\left(\frac{\sin\psi_+}{|S_+|} - \frac{2}{3}\right) = 0\ . \tag{8.51}$$

But the quantities within the parentheses of (8.51) depend only on properties of the individual array elements whereas $|q|^2$ varies with the array parameters d and h. It follows that (8.51) implies (8.41) and (8.43). Thus the relations (8.41) and (8.43) must be satisfied if the array is to support a lossless dipolar traveling wave. (The converse is, of course, not true. The relations (8.41 and (8.43) do not guarantee that there will be a solution to equations (8.44a) and (8.44b) that must be satisfied if a lossless traveling wave can be supported by the array.) In [5] we noted that (8.41) and (8.43) are indeed satisfied by the normalized electric and magnetic Mie scattering coefficients defined in Footnote 6. Here we have shown that (8.41) and (8.43) must be satisfied by any element of a 2D periodic array that supports a lossless traveling wave if the element can be modeled by a pair of crossed electric and magnetic dipoles at right angles to each other such that an incident electric (magnetic) field at the element center in the direction of the electric (magnetic) dipole excites only the electric (magnetic) dipole field. Since equations (8.44a) and (8.44b) also hold for 1D periodic arrays [5, eqs.(51a,b)] and 3D periodic arrays [see (9.81a), (9.81b)] of such elements, this conclusion applies to the elements of 1D and 3D periodic arrays as well.[7]

Solving for $-q$ in (8.44a) and (8.44b) and equating the resulting expressions we obtain the kd–βd equation

$$\frac{(kh)^3 - S_-\Sigma_1}{S_-\Sigma_2} = \frac{S_+\Sigma_2}{(kh)^3 - S_+\Sigma_3}\ . \tag{8.52}$$

We note that the kd–βd equation (8.52) is an equation of real quantities, for using the expression for $-q$ given by the LHS of (8.52), the fact that the imaginary part of Σ_1 is $-\frac{2}{3}(kh)^3$, the fact that Σ_2 is real, and (8.41), the imaginary part of the LHS of (8.52) is

$$\mathrm{Im}[-q] = \frac{1}{\Sigma_2}\mathrm{Im}\left[\frac{(kh)^3}{S_-} - \Sigma_1\right] = \frac{(kh)^3}{\Sigma_2}\,\mathrm{Im}\left[\frac{1}{|S_-|\mathrm{e}^{\mathrm{i}\psi_-}} + \frac{2}{3}\right] = \frac{2}{3}\frac{(kh)^3}{\Sigma_2}\,\mathrm{Im}\left[\frac{\mathrm{e}^{-\mathrm{i}\psi_-}}{\sin\psi_-} + 1\right] = 0 \tag{8.53}$$

[7] In the 1D and 3D cases $\Sigma_3 = \Sigma_1$. Although the actual expressions for Σ_1 and Σ_2 in the 1D and 3D cases differ from those in the 2D case the essential features are the same: the imaginary part of Σ_1 equals $-2/3\ (kh)^3$ — $-2/3\ (kd)^3$ in the 1D case — and Σ_2 is real.

and similarly for the RHS of (8.52). It is simple to solve (8.52) numerically for βd given values of kd, kh, S_-, and S_+, using, for example, a simple search procedure with secant algorithm refinement.

To facilitate calculations of Σ_1, Σ_2, and Σ_3, the following may be noted. Rapidly convergent expressions for the slowly convergent Schlömilch series $\sum \cos(n\beta d)\, Y_0(nkd)$ and $\sum \cos(n\beta d)\, Y_2(nkd)$ are given in (B.2a) and (B.9)-(B.10), respectively. All series involving the modified Bessel functions K_0, K_1, and K_2, converge very rapidly because of the exponential decay of these functions so that only a few terms of the series give sufficient accuracy. The convergence of the series $\sum \sin(n\beta d) Y_1(nkd)$ can be greatly accelerated by using (B.6)-(B.7).

The expressions we have given for Σ_1, Σ_2, and Σ_3, in (8.45)-(8.47) are valid for the transverse element separation h in the range $0 < kh < 2\pi$. Since several of the individual terms of these quantities become singular as kh approaches 2π the expressions (8.45)-(8.47) cannot be used to calculate Σ_1, Σ_2, and Σ_3 at $kh = 2\pi$. It is therefore worthwhile to obtain the limiting values of these quantities as $kh \to 2\pi$ from below. First, considering Σ_1 given by (8.45) and comparing it to the expression (4.58) for $\Re$ in our treatment of 2D arrays of electric dipoles perpendicular to the array axis and in the plane of the array, we see that the two expressions are identical apart from the term $-\mathrm{i}(2/3)(kh)^3$ in Σ_1. Hence from (4.64) we have immediately

$$\Sigma_1 \overset{kh\to 2\pi}{\sim} -2\pi(2\pi)^2 \sum_{n=1}^{\infty} \cos(n\beta d) Y_0(nkd)$$

$$- 8(2\pi)^2 \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{m=2}^{\infty} (m^2-1) K_0\left(2\pi n (d/h)\sqrt{m^2-1}\right) + 4(1.20205...) - \mathrm{i}\frac{2}{3}(2\pi)^3 \ . \quad (8.54)$$

Next, considering Σ_2, from the small argument form of the modified Bessel function (C.5) we have for $m = 1$

$$\sqrt{(2\pi m)^2 - (kh)^2}\; K_1\left(n(d/h)\; \sqrt{(2\pi m)^2 - (kh)^2}\right)$$

$$\overset{kh\to 2\pi}{\sim} \sqrt{(2\pi)^2 - (kh)^2} \left[\frac{1}{2} \frac{1}{\frac{1}{2}(nd/h)\sqrt{(2\pi)^2 - (kh)^2}} \right] = \frac{1}{nd/h} \qquad (8.55)$$

so that

$$\lim_{kh\to 2\pi} - 8(kh) \sum_{n=1}^{\infty} \sin(n\beta d) \sqrt{(2\pi)^2 - (kh)^2}\; K_1\left(n(d/h)\; \sqrt{(2\pi)^2 - (kh)^2}\right)$$

$$= - 8(2\pi)\frac{1}{d/h} \sum_{n=1}^{\infty} \frac{\sin(n\beta d)}{n} = -\frac{8\pi}{d/h}(\pi - \beta d) \qquad (8.56)$$

where we have used (D.1). Hence

$$\lim_{kh\to 2\pi} \Sigma_2 = 2\pi(2\pi)^2 \sum_{n=1}^{\infty} \sin(n\beta d) Y_1(nkd) - \frac{8\pi}{d/h}(\pi - \beta d)$$

$$- 8(2\pi)\sum_{n=1}^{\infty}\sin(n\beta d)\sum_{m=2}^{\infty}2\pi\sqrt{m^2-1}\ K_1\left(2\pi n(d/h)\ \sqrt{m^2-1}\right)\ . \tag{8.57}$$

Finally, considering Σ_3 given by (8.47) and comparing it to the expression (4.55) for $\Re$ in our treatment of 2D arrays of electric dipoles perpendicular to both the array axis and the array plane, we see that the two expressions are identical apart from the term $\mathrm{i}(2/3)(kh)^3$ in Σ_3. Hence from (4.106) we have immediately that

$$\begin{aligned}
\Sigma_3 \overset{kh\to 2\pi}{\to} & -\pi(2\pi)^2\left[\sum_{n=1}^{\infty}\cos(n\beta d)Y_0(nkd)-\sum_{n=1}^{\infty}\cos(n\beta d)Y_2(nkd)\right] \\
& + 4(2\pi)^2\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=2}^{\infty}\left[(m^2+1)K_0\left(2\pi n(d/h)\ \sqrt{m^2-1}\right)\right. \\
& \left. -(m^2-1)K_2\left(2\pi n(d/h)\ \sqrt{m^2-1}\right)\right] + (2\pi)^2\left(4\gamma+4\ln\frac{1}{\sqrt{4\pi}}\frac{d}{h}+\frac{4\pi}{\beta d}\right) \\
& + (2\pi)^2(4\pi)\left[\sum_{l=1}^{\infty}\left(\frac{1}{(2l\pi-\beta d)^2}-\frac{1}{2l\pi}\right)+\sum_{l=1}^{\infty}\left(\frac{1}{(2l\pi+\beta d)^2}-\frac{1}{2l\pi}\right)\right] \\
& -\frac{8}{(d/h)^2}\left[\frac{\pi^2}{6}-\frac{\pi\beta d}{2}+\frac{(\beta d)^2}{4}\right]-2\ \mathrm{Cl}_3(2\pi)-\mathrm{i}\frac{2}{3}(2\pi)^3
\end{aligned} \tag{8.58}$$

with $\mathrm{Cl}_3(2\pi)$ given by (D.10).

In concluding this subsection we give alternate derivations, using the Hertz vector potential, of the rapidly convergent expressions (8.28), proportional to the y directed magnetic field incident on the reference sphere scattered from all the x directed electric dipoles in the nth column, $n \neq 0$, and (8.31), proportional to the y directed magnetic field incident on the reference sphere scattered from all the y directed magnetic dipoles in the nth column, $n \neq 0$. We begin with (8.26), the slowly convergent form of (8.28) which we repeat here

$$\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}}\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mn}}\right)\frac{nd/h}{\rho_{mn}},\ \rho_{mn}=\sqrt{m^2+(nd/h)^2}\ . \tag{8.59}$$

The magnetic field radiated by an x directed electric dipole at the origin of a Cartesian coordinate system is proportional to the curl of the Hertz vector potential for an x directed electric dipole [see (4.48)],

$$C\ \left(\boldsymbol{\nabla}\times\frac{\mathrm{e}^{\mathrm{i}kr}}{kr}\hat{\mathbf{x}}\right)\ . \tag{8.60}$$

where C is the proportionality constant whose value we do not need here, so that the y component of the magnetic field is

$$C\ \left(\boldsymbol{\nabla}\times\frac{\mathrm{e}^{\mathrm{i}kr}}{kr}\hat{\mathbf{x}}\right)_y = C\frac{\partial}{\partial z}\frac{\mathrm{e}^{\mathrm{i}kr}}{kr},\ r=\sqrt{x^2+z^2} \tag{8.61}$$

referring to [22, Appendix 1, eq. 51]. Hence the magnetic field radiated by the column of x directed electric dipoles located at $z = 0,\ x = mh,\ m = 0,\ \pm1,\ \pm2,\ \cdots,$ is

$$C\frac{\partial}{\partial z}\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kr_m}}{kr_m} = Cz\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kr_m}}{kr_m^2}\left(\mathrm{i}k-\frac{1}{r_m}\right),\ r_m=\sqrt{(x-mh)^2+z^2}\,. \tag{8.62}$$

Now from (2.34) and (2.42) the field radiated by the acoustic monopoles located in the column $y = 0,\ z = 0$ at $x = 0,\ \pm h,\ \pm 2h,\ \cdots,$ each of which radiates a field equal to $\mathrm{e}^{\mathrm{i}kr}/(kr)$ is

$$\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kr_m}}{kr_m} = \sum_{m=-\infty}^{\infty}B_m^0H_0^{(1)}(k_m\rho)\,\mathrm{e}^{\mathrm{i}(2\pi/h)mx},\ r_m=\sqrt{(x-mh)^2+\rho^2} \tag{8.63}$$

where

$$B_m^0=\frac{\mathrm{i}\pi}{kh} \tag{8.64}$$

and

$$k_m=\sqrt{k^2-(2\pi m/h)^2} \tag{8.65}$$

with k_m positive real (positive imaginary) according as $(kh)^2 > (<)\ (2\pi m)^2$. Letting $z = \rho$ in (8.62) and substituting the RHS of (8.63) for the sum over m in the LHS of (8.62) then yields

$$C\rho\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kr_m}}{kr_m^2}\left(\mathrm{i}k-\frac{1}{r_m}\right) = -C\sum_{m=-\infty}^{\infty}B_m^0k_mH_1^{(1)}(k_m\rho)\,\mathrm{e}^{\mathrm{i}(2\pi/h)mx}\,. \tag{8.66}$$

In particular, for $\rho = |n|d$, and $x = 0$, r_m given in (8.62) is equal to

$$r_m=\sqrt{\rho^2+(x-mh)^2}=\sqrt{(nd)^2+(mh)^2}=h\sqrt{m^2+(nd/h)^2}=h\rho_{mn} \tag{8.67}$$

where ρ_{mn} is defined in (8.19) and given again in (8.59) so that

$$\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{kh^2\rho_{mn}^2}\left(\mathrm{i}kh-\frac{1}{\rho_{mn}}\right)|n|d/h = -\sum_{m=-\infty}^{\infty}B_m^0k_mH_1^{(1)}(k_m|n|d)\,. \tag{8.68}$$

Now the sum we wish to evaluate, (8.59), equals the LHS of (8.68) multiplied by $\mathrm{sgn}(n)(-\mathrm{i}k^2h^3)$. Therefore, with (8.64) and (8.65), and making use of the Bessel function relation (C.2),

$$\begin{aligned}\mathrm{sgn}(n)\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}}\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mn}}\right)\frac{|n|d/h}{\rho_{mn}} &= \mathrm{sgn}(n)(-\mathrm{i}k^2h^3)\sum_{m=-\infty}^{\infty}B_m^0k_mH_1^{(1)}(k_m|n|d)\\ &= -\mathrm{sgn}(n)\Bigg[-\pi(kh)^2H_1^{(1)}(|n|kd)\\ &+4\mathrm{i}\,kh\sum_{m=1}^{\infty}\sqrt{(2\pi m)^2-(kh)^2}\,K_1\left(|n|(d/h)\sqrt{(2\pi m)^2-(kh)^2}\right)\Bigg]\end{aligned} \tag{8.69}$$

identical to the expression (8.28) obtained using the Poisson summation formula.

We end this subsection with an alternate derivation of the rapidly convergent expression obtained in (8.31) for the sum [see (8.30)]

$$\sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} - \frac{1}{\rho_{mn}^2} \right), \; \rho_{mn} = \sqrt{m^2 + (nd/h)^2}\,. \tag{8.70}$$

This is the sum proportional to the y directed magnetic field at $x = 0, y = 0, z = 0$ due to all the y directed magnetic dipoles in the nth column, $z = nd, n \neq 0$. As we did just above in our alternate derivation of the rapidly convergent form of (8.59) we take the x axis to be the array axis and the magnetic dipoles to be located in columns parallel to the z axis at $z = mh, m = 0, \pm h, \pm 2h, \cdots$. The magnetic dipoles themselves are parallel to the y axis just as they are in the original choice of coordinate axes. The magnetic field radiated by a y directed magnetic dipole located at the coordinate system origin is given by

$$C\, \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \left(\frac{\mathrm{e}^{\mathrm{i}kr}}{kr} \hat{\mathbf{y}} \right)\,. \tag{8.71}$$

Since the same field is also given by the b_{+n} term of (8.2b) with $n = 0$ and the triple subscripts dropped, the value of C can be found easily by expanding (8.71) in spherical coordinates (using, for example, [22, Appendix 1, eqs. 118, 161]) and equating the $1/(kr)$ term of the ϕ component of the field with $Y_0 b_{+0} \exp(\mathrm{i}kr)/(kr) \cos\phi$, thus yielding

$$C = \frac{Y_0 b_{+0}}{k^2}\,. \tag{8.72}$$

The ρ and ϕ components of the magnetic field radiated by a y directed magnetic dipole located at the coordinate system origin are given by the ρ and ϕ components of

$$C\, \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \frac{\mathrm{e}^{\mathrm{i}kr}}{kr} \left(\sin\phi\, \hat{\boldsymbol{\rho}} + \cos\phi\, \hat{\boldsymbol{\phi}} \right), \; r = \sqrt{\rho^2 + z^2}\,. \tag{8.73}$$

Using the expression [22, Appendix 1, eq. 101] for $\boldsymbol{\nabla} \times \boldsymbol{\nabla}$ in cylindrical coordinates we then find that the ρ component is given by

$$-C \sin\phi \left(\frac{\partial^2}{\partial z^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \right) \frac{\mathrm{e}^{\mathrm{i}kr}}{kr} \tag{8.74}$$

and the ϕ component is given by

$$-C \cos\phi \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2} \right) \frac{\mathrm{e}^{\mathrm{i}kr}}{kr}\,. \tag{8.75}$$

Since we are concerned with the y directed magnetic field when $\phi = 0$ or π we need pay attention only to the ϕ component of the magnetic field. Applying the operator $-C \cos\phi \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2} \right)$ to both sides of (8.63) giving the field radiated by a column of acoustic monopoles for which

$x = 0$ each of which radiates a field equal to $\mathrm{e}^{\mathrm{i}kr}/(kr)$, we find that the ϕ component of the magnetic field is given by

$$-C\cos\phi \sum_{m=-\infty}^{\infty} B_m^0 \left[-\left(\frac{2\pi m}{h}\right)^2 H_0^{(1)}(k_m\rho) + \frac{(k_m\rho)^2}{\rho^2}\frac{\mathrm{d}^2}{\mathrm{d}(k_m\rho)^2} H_0^{(1)}(k_m\rho) \right] \mathrm{e}^{\mathrm{i}(2\pi/h)mz}$$

$$= -\frac{1}{2}C\cos\phi \sum_{m=-\infty}^{\infty} B_m^0 \left\{ -\left[k^2 + \left(\frac{2\pi m}{h}\right)^2\right] H_0^{(1)}(k_m\rho) + \left[k^2 - \left(\frac{2\pi m}{h}\right)^2\right] H_2^{(1)}(k_m\rho) \right\} \mathrm{e}^{\mathrm{i}(2\pi/h)mz} \tag{8.76}$$

with B_m^0 given by (8.64), k_m given by (8.65), and where we have referred to (6.66) for some of the intermediate steps. The y component of the magnetic field radiated by the column of y directed magnetic dipoles is then given by $H_\phi \cos\phi, \phi = 0, \pi$ or

$$-\frac{1}{2}C \sum_{m=-\infty}^{\infty} B_m^0 \left\{ -\left[k^2 + \left(\frac{2\pi m}{h}\right)^2\right] H_0^{(1)}(k_m\rho) + \left[k^2 - \left(\frac{2\pi m}{h}\right)^2\right] H_2^{(1)}(k_m\rho) \right\} \mathrm{e}^{\mathrm{i}(2\pi/h)mz}$$

$$= -\frac{1}{2}C \sum_{m=-\infty}^{\infty} B_m^0 \left\{ -\left[k^2 + \left(\frac{2\pi m}{h}\right)^2\right] H_0^{(1)}(k_m\rho) + \left[k^2 - \left(\frac{2\pi m}{h}\right)^2\right] \left[\frac{2}{k_m\rho} H_2^{(1)}(k_m\rho) - H_0^{(1)}(k_m\rho)\right] \right\} \mathrm{e}^{\mathrm{i}(2\pi/h)mz}$$

$$= -C \sum_{m=-\infty}^{\infty} B_m^0 \left\{ -k^2 H_0^{(1)}(k_m\rho) + \left[k^2 - \left(\frac{2\pi m}{h}\right)^2\right] \frac{H_1^{(1)}(k_m\rho)}{k_m\rho} \right\} \mathrm{e}^{\mathrm{i}(2\pi/h)mz} \, . \tag{8.77}$$

When $\rho = |n|d$ and $z = 0$,

$$r_m = \sqrt{(nd)^2 + (mh)^2} = h\sqrt{m^2 + (nd/h)^2} = h\rho_{mn} \tag{8.78}$$

with ρ_{mn} given in (8.70) . We then have, referring to (8.2b),

$$Y_0 b_{+0} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{kh\rho_{mn}} \left(1 + \frac{\mathrm{i}}{kh\rho_{mn}} - \frac{1}{(kh)^2\rho_{mn}^2}\right)$$

$$= -Y_0 \frac{b_{+0}}{k^2} \sum_{m=-\infty}^{\infty} B_m^0 \left[-k^2 H_0^{(1)}(k_m|n|d) + \frac{k_m}{|n|d} H_1^{(1)}(k_m|n|d)\right] \tag{8.79}$$

or

$$\sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{kh^3\rho_{mn}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} - \frac{1}{\rho_{mn}^2}\right) = -\sum_{m=-\infty}^{\infty} B_m^0 \left[-k^2 H_0^{(1)}(k_m|n|d) + \frac{k_m}{|n|d} H_1^{(1)}(k_m|n|d)\right] . \tag{8.80}$$

Now the sum we wish to evaluate, (8.70), equals the LHS of (8.80) multiplied by kh^3. Therefore, with (8.64) and (8.65), making use of the Bessel function relations (C.1) and (C.2)-(C.9), and assuming that $0 < kh < 2\pi$,

$$\sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} - \frac{1}{\rho_{mn}^2} \right) = \mathrm{i}\pi (kh)^2 \sum_{m=-\infty}^{\infty} H_0^{(1)} \left(|n|(d/h)\, \sqrt{(kh)^2 - (2\pi m)^2} \right)$$

$$- \mathrm{i}\pi \frac{1}{|n|d/h} \sum_{m=-\infty}^{\infty} \frac{\sqrt{(kh)^2 - (2\pi m)^2}}{|n|d/h} H_1^{(1)} \left(|n|(d/h)\, \sqrt{(kh)^2 - (2\pi m)^2} \right)$$

$$= \frac{\mathrm{i}\pi (kh)^2}{2} \left[H_0^{(1)}(|n|kd) - H_2^{(1)}(|n|kd) \right]$$

$$+ 2 \sum_{m=1}^{\infty} \Bigg[\left[(2\pi m)^2 + (kh)^2 \right] K_0 \left(|n|(d/h)\, \sqrt{(2\pi m)^2 - (kh)^2} \right)$$

$$- \left[(2\pi m)^2 - (kh)^2 \right] K_2 \left(|n|(d/h)\, \sqrt{(2\pi m)^2 - (kh)^2} \right) \Bigg] \tag{8.81}$$

in agreement with the expression (8.31) obtained using the Poisson summation formula.

8.2 ELECTRIC DIPOLES PERPENDICULAR TO THE ARRAY PLANE

As we noted above in the introduction to Section 8, the effective difference between the polarization considered in this subsection and the polarization considered in 8.1 is that the electric and magnetic dipoles of the spheres in the two arrays are interchanged. This implies that the basic equations equivalent to (8.22) for the polarization considered in 8.1, can be obtained for the polarization considered in this subsection simply by replacing the electric and magnetic dipole scattering coefficients S_- and S_+ in (8.22) by S_+ and S_-, respectively, and also by replacing q by $1/q$ since, as noted following (8.23), q is the ratio of the normalized magnetic dipole field coefficient, b_{+0} to the normalized electric dipole field coefficient, b_{-0}. To substantiate this claim we will proceed to derive the equations corresponding to (8.22). It will then be obvious that the kd–βd equation in its final form for the polarization considered here can be obtained directly from the kd–βd equation, (8.52), considered in 8.1, simply by taking Σ_1 and Σ_3 here to equal Σ_3 and Σ_1, respectively, given by (8.47) and (8.45) in 8.1, and taking Σ_2 here to equal Σ_2 given by (8.46) in 8.1.

We proceed by choosing the array axis to be the z axis of a Cartesian coordinate system with equispaced rows of magnetodielectric spheres parallel to the y axis located at $z = nd, n = 0, \pm 1, \pm 2, \cdots$. In each row the spheres are centered at $y = mh, m = 0, \pm 1, \pm 2, \cdots$. The electric and magnetic dipole components of each sphere are oriented in the x and y direction, respectively, so that the electric dipoles are perpendicular to the yz plane, the plane of the array. We assume an excitation of the array with the electric field parallel to the x axis and the magnetic field parallel to the y axis, and such that all the spheres in any row of the array are excited identically. Let $\mathbf{E}_0^0$ and $\mathbf{H}_0^0$ be the electric and magnetic field, respectively, incident on the sphere at the location $x = 0, y = 0, z = 0$ from all the other spheres in the array. As will be seen [see (8.99)] $\mathbf{E}_0^0$ has an x component only, and $\mathbf{H}_0^0$ has a y component only. Let $\mathbf{E}_0^{0mn}$ and $\mathbf{H}_0^{0mn}$be the electric and magnetic field, respectively, incident on the reference sphere from the sphere at the location $(x, y, z) = (0, mh, nd)$ so that

$$\mathbf{E}_0^0 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{m=-\infty}^{\infty} \mathbf{E}_0^{0mn} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \mathbf{E}_0^{0m0} \tag{8.82a}$$

$$\mathbf{H}_0^0 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{m=-\infty}^{\infty} \mathbf{H}_0^{0mn} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \mathbf{H}_0^{0m0} \, . \tag{8.82b}$$

From [5, eqs. (32),(33)]

$$\begin{aligned}\mathbf{E}_0^{0mn} = {} & b_{-n} \frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}} \left[\frac{-2\mathrm{i}}{kr_{mn0}} \left(1 + \frac{\mathrm{i}}{kr_{mn0}} \right) \sin\theta_{mn0} \cos\phi_{mn0} \, \hat{\mathbf{r}}_{mn0} \right. \\ & \left. + \left(1 + \frac{\mathrm{i}}{kr_{mn0}} - \frac{1}{(kr_{mn0})^2} \right) \left(\cos\theta_{mn0} \cos\phi_{mn0} \, \hat{\boldsymbol{\theta}}_{mn0} - \sin\phi_{mn0} \, \hat{\boldsymbol{\phi}}_{mn0} \right) \right] \\ & + b_{+n} \frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}} \left(1 + \frac{\mathrm{i}}{kr_{mn0}} \right) \left(\cos\phi_{mn0} \, \hat{\boldsymbol{\theta}}_{mn0} - \cos\theta_{mn0} \sin\phi_{mn0} \, \hat{\boldsymbol{\phi}}_{mn0} \right)\end{aligned} \tag{8.83a}$$

and

$$\begin{aligned}\mathbf{H}_0^{mn0} = {} & Y_0 b_{+n} \frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}} \left[\frac{-2\mathrm{i}}{kr_{mn0}} \left(1 + \frac{\mathrm{i}}{kr_{mn0}} \right) \sin\theta_{mn0} \sin\phi_{mn0} \, \hat{\mathbf{r}}_{mn0} \right. \\ & + \left(1 + \frac{\mathrm{i}}{kr_{mn0}} - \frac{1}{(kr_{mn0})^2} \right) \left(\cos\theta_{mn0} \sin\phi_{mn0} \, \hat{\boldsymbol{\theta}}_{mn0} + \cos\phi_{mn0} \, \hat{\boldsymbol{\phi}}_{mn0} \right) \\ & + Y_0 b_{-n} \frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}} \left(1 + \frac{\mathrm{i}}{kr_{mn0}} \right) \left(\sin\phi_{mn0} \, \hat{\boldsymbol{\theta}}_{mn0} + \cos\theta_{mn0} \cos\phi_{mn0} \, \hat{\boldsymbol{\phi}}_{mn0} \right)\end{aligned} \tag{8.83b}$$

where Y_0 is the free-space admittance. The quantities in (8.83) are defined with reference to a local spherical polar coordinate system with origin at $(x, y, z) = (0, mh, nd)$ (in turn defined with reference to a local Cartesian coordinate system with the same origin whose axes are parallel to those of the global Cartesian coordinate system). The distance from the (m, n) sphere to the $(0, 0)$ sphere, r_{mn0}, is given by

$$r_{mn0} = \sqrt{(mh)^2 + (nd)^2} = h\sqrt{m^2 + (nd/h)^2} \tag{8.84}$$

and the unit vector in the direction from the (m, n) sphere to the $(0, 0)$ sphere, $\hat{\mathbf{r}}_{mn0}$, is

$$\hat{\mathbf{r}}_{mn0} = \mathbf{r}_{mn0}/r_{mn0}, \quad \mathbf{r}_{mn0} = -mh\,\hat{\mathbf{y}} - nd\,\hat{\mathbf{z}} \tag{8.85}$$

so that

$$\cos\theta_{mn0} = \hat{\mathbf{r}}_{mn0} \cdot \hat{\mathbf{z}} = -\frac{nd}{r_{mn0}} \tag{8.86}$$

and

$$\sin\theta_{mn0} = \sqrt{1-\cos^2\theta_{mn0}} = \frac{|m|h}{r_{mn0}} \; . \tag{8.87}$$

Since $\hat{\mathbf{r}}_{mn0}$ is also given by

$$\hat{\mathbf{r}}_{mn0} = \sin\theta_{mn0}\sin\phi_{mn0}\;\hat{\mathbf{y}} + \cos\theta_{mn0}\;\hat{\mathbf{z}} \tag{8.88}$$

it follows by substituting (8.86) and (8.87) in (8.88) and comparing with (8.85) that

$$\sin\phi_{mn0} = \begin{cases} -1 & : \quad m>0 \\ +1 & : \quad m<0 \end{cases} \quad \text{or} \quad \phi_{mn0} = \begin{cases} -\pi/2 & : \quad m>0 \\ +\pi/2 & : \quad m<0 \end{cases} \; . \tag{8.89}$$

and

$$\cos\phi_{mn0} = 0 \; . \tag{8.90}$$

Then

$$\hat{\boldsymbol{\theta}}_{mn0} = \cos\theta_{mn0}\sin\phi_{mn0}\;\hat{\mathbf{y}} - \sin\theta_{mn0}\;\hat{\mathbf{z}} = \pm\,\frac{nd}{r_{mn0}}\;\hat{\mathbf{y}} - \frac{|m|h}{r_{mn0}}\;\hat{\mathbf{z}}\;, \quad m \gtrless 0 \tag{8.91}$$

$$\hat{\boldsymbol{\phi}}_{mn0} = -\sin\phi_{mn0}\;\hat{\mathbf{x}} = \pm\;\hat{\mathbf{x}}\;, \quad m \gtrless 0 \tag{8.92}$$

$$\sin\phi_{mn0}\;\hat{\boldsymbol{\phi}}_{mn0} = -\hat{\mathbf{x}} \tag{8.93}$$

$$\cos\theta_{mn0}\sin\phi_{mn0}\;\hat{\boldsymbol{\phi}}_{mn0} = \frac{nd}{r_{mn0}}\;\hat{\mathbf{x}} \tag{8.94}$$

$$\sin\theta_{mn0}\sin\phi_{mn0}\;\hat{\mathbf{r}}_{mn0} = \frac{(mh)^2}{r_{mn0}^2}\;\hat{\mathbf{y}} + \frac{(mh)(nd)}{r_{mn0}}\;\hat{\mathbf{z}} \tag{8.95}$$

$$\cos\theta_{mn0}\sin\phi_{mn0}\;\hat{\boldsymbol{\theta}}_{mn0} = \frac{(nd)^2\;\hat{\mathbf{y}} - (mh)(nd)\;\hat{\mathbf{z}}}{r_{mn0}^2} \tag{8.96}$$

and

$$\sin\phi_{mn0}\;\hat{\boldsymbol{\theta}}_{mn0} = \frac{-nd\;\hat{\mathbf{y}} + mh\;\hat{\mathbf{z}}}{r_{mn0}} \; . \tag{8.97}$$

The constants b_{-n} and b_{+n} are related to the x component of the electric field and the y component of the magnetic field, respectively, incident on any sphere in the nth row by the scattering equations (8.16). Substituting (8.93)-(8.97) in (8.83) we obtain

$$\begin{aligned} \mathbf{E}_0^{0mn} &= b_{-n}\frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}}\left(1+\frac{\mathrm{i}}{kr_{mn0}} - \frac{1}{(kr_{mn0})^2}\right)\;\hat{\mathbf{x}} \\ &\quad - b_{+n}\frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}}\left(1+\frac{\mathrm{i}}{kr_{mn0}}\right)\frac{nd}{r_{mn0}}\;\hat{\mathbf{x}} \end{aligned} \tag{8.98a}$$

and

$$\mathbf{H}_0^{mn0} = Y_0\; b_{+n}\frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}}\left(\frac{-2\mathrm{i}}{kr_{mn0}}\left(1+\frac{mi}{kr_{mn0}}\right)\left[\frac{(mh)^2\;\hat{\mathbf{y}} + (mh)(nd)\;\hat{\mathbf{z}}}{r_{mn0}^2}\right]\right.$$

$$+\left(1+\frac{\mathrm{i}}{kr_{mn0}}-\frac{1}{(kr_{mn0})^2}\right)\left[\frac{(nd)^2\,\hat{\mathbf{y}}-(mh)(nd)\,\hat{\mathbf{z}}}{r_{mn0}^2}\right]\Bigg)$$

$$+\,Y_0\,b_{-n}\frac{\mathrm{e}^{\mathrm{i}kr_{mn0}}}{kr_{mn0}}\left(1+\frac{\mathrm{i}}{kr_{mn0}}\right)\frac{-nd\,\hat{\mathbf{y}}+mh\,\hat{\mathbf{z}}}{r_{mn0}}. \tag{8.98b}$$

Note that when summed over m from $-\infty$ to ∞ the z components of H_0^{mn0} cancel and we are left with a y component only of the magnetic field. Substituting (8.98) in (8.82) we obtain

$$E_{0x}^0=\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\left\{b_{-n}\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{kh\rho_{mn}}\left(1+\frac{\mathrm{i}}{kh\rho_{mn}}-\frac{1}{(kh)^2\rho_{mn}^2}\right)\right.$$

$$\left.-\,b_{+n}\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{kh\rho_{mn}}\left(1+\frac{\mathrm{i}}{kh\rho_{mn}}\right)\frac{nd/h)}{\rho_{mn}}\right\}$$

$$+\,2b_{-0}\sum_{m=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{m0}}}{kh\rho_{m0}}\left(1+\frac{\mathrm{i}}{kh\rho_{m0}}-\frac{1}{(kh)^2\rho_{m0}^2}\right) \tag{8.99a}$$

and

$$\frac{H_{0y}^{mn0}}{Y_0}=\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\left\{\,b_{+n}\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{kh\rho_{mn}}\left[\frac{-2\mathrm{i}}{kh\rho_{mn}}\left(1+\frac{\mathrm{i}}{kh\rho_{mn}}\right)\frac{m^2}{\rho_{mn}^2}\right.\right.$$

$$\left.\left.+\left(1+\frac{\mathrm{i}}{kh\rho_{mn}}-\frac{1}{(kh)^2\rho_{mn}^2}\right)\frac{(nd/h)^2}{\rho_{mn}^2}\right]-\,b_{-n}\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{kh\rho_{mn}}\left(1+\frac{\mathrm{i}}{kh\rho_{mn}}\right)\frac{nd/h}{\rho_{mn}}\right\}$$

$$+\,2b_{+0}\sum_{m=1}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{m0}}}{kh\rho_{m0}}\frac{-2\mathrm{i}}{kh\rho_{m0}}\left(1+\frac{\mathrm{i}}{kh\rho_{m0}}\right) \tag{8.99b}$$

where we have let

$$\rho_{mn}=\sqrt{m^2+(nd/h)^2} \tag{8.100}$$

with

$$\rho_{m0}=m\,. \tag{8.101}$$

We now assume that the array is excited by a traveling wave in the z direction with real propagation constant β. Then the constants b_{-n} and b_{+n} in (8.99) are identical apart from a phase shift given by

$$b_{-n}=b_{-0}\mathrm{e}^{\mathrm{i}n\beta d},\quad b_{+n}=b_{+0}\mathrm{e}^{\mathrm{i}n\beta d}\,. \tag{8.102}$$

Substituting (8.102) in (8.99), using [from (8.16)] $b_{-0}=S_-E_{0x}^0$ and $b_{+0}=S_+H_{0y}^0/Y_0$,and multiplying by $(kh)^3$ we obtain

$$(kh)^3=S_-\left\{\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\mathrm{e}^{\mathrm{i}n\beta d}\left[\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}}\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mn}}-\frac{1}{\rho_{mn}^2}\right)\right.\right.$$

$$- q \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} \right) \frac{nd/h)}{\rho_{mn}} \Bigg]$$

$$+ 2 \sum_{m=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{m0}}}{\rho_{m0}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{m0}} - \frac{1}{\rho_{m0}^2} \right) \Bigg\} \tag{8.103a}$$

and

$$(kh)^3 = S_+ \Bigg\{ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \Bigg(\sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\rho_{mn}}}{\rho_{mn}} \Bigg[\frac{-2\mathrm{i}}{\rho_{mn}} \left(kh + \frac{\mathrm{i}kh}{\rho_{mn}} \right) \frac{m^2}{\rho_{mn}^2}$$

$$+ \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} - \frac{1}{\rho_{mn}^2} \right) \frac{(nd/h)^2}{\rho_{mn}^2} \Bigg]$$

$$- \frac{1}{q} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mn}}}{\rho_{mn}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mn}} \right) \frac{nd/h}{\rho_{mn}} \Bigg)$$

$$+ 4 \sum_{m=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{m0}}}{\rho_{m0}} \frac{-\mathrm{i}}{\rho_{m0}} \left(kh + \frac{\mathrm{i}}{\rho_{m0}} \right) \Bigg\} . \tag{8.103b}$$

Comparing these equations with the corresponding equations (8.22) in 8.1, we see, exactly as we claimed at the beginning of this subsection, that (8.103) can be obtained from (8.22) simply by replacing S_-, S_+, and q in (8.22) by S_+, S_-, and $1/q$, respectively. This is as it should be since the role played by the electric dipoles in 8.1 is now played by magnetic dipoles and vice versa. Since (8.22) can be put in the form (8.44) leading to the kd–βd equation (8.52), it follows that we can go directly from (8.103) to the final form of the kd–βd equation here simply by interchanging Σ_1 and Σ_3 in (8.52) thus obtaining

$$\frac{(kh)^3 - S_-\Sigma_1}{S_-\Sigma_2} = \frac{S_+\Sigma_2}{(kh)^3 - S_+\Sigma_1} . \tag{8.104}$$

where Σ_1 equals Σ_3 given by (8.47)

$$\Sigma_1 = -\pi(kh)^2 \left[\sum_{n=1}^{\infty} \cos(n\beta d) Y_0(nkd) - \sum_{n=1}^{\infty} \cos(n\beta d) Y_2(nkd) \right]$$

$$+ 4 \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{m=1}^{\infty} \Bigg[\left[(2\pi m)^2 + (kh)^2\right] K_0 \left(n(d/h) \sqrt{(2\pi m)^2 - (kh)^2} \right)$$

$$- \left[(2\pi m)^2 - (kh)^2\right] K_2 \left(n(d/h) \sqrt{(2\pi m)^2 - (kh)^2} \right) \Bigg]$$

$$-2 \left((kh)^2 \ln \left[2 \sin \left(\frac{kh}{2} \right) \right] + kh \, \mathrm{Cl}_2(kh) + \mathrm{Cl}_3(kh) \right) - \mathrm{i}\frac{2}{3}(kh)^3 \tag{8.105}$$

Σ_2 equals Σ_2 given by (8.46)

$$\Sigma_2 = 2\pi(kh)^2 \sum_{n=1}^{\infty} \sin(n\beta d) Y_1(nkd)$$

$$- 8\; kh \sum_{n=1}^{\infty} \sin(n\beta d) \sum_{m=1}^{\infty} \sqrt{(2\pi m)^2 - (kh)^2}\; K_1 \left(n(d/h)\; \sqrt{(2\pi m)^2 - (kh)^2} \right) \tag{8.106}$$

and Σ_3 equals Σ_1 given by (8.45)

$$\Sigma_3 = -2\pi(kh)^2 \sum_{n=1}^{\infty} \cos(n\beta d) Y_0(nkd)$$

$$- 8 \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{m=1}^{\infty} \left[(2\pi m)^2 - (kh)^2 \right] K_0 \left(n(d/h)\; \sqrt{(2\pi m)^2 - (kh)^2} \right)$$

$$+ 4\; kh\; \mathrm{Cl}_2(kh) + 4\; \mathrm{Cl}_3(kh) - \mathrm{i}\frac{2}{3}(kh)^3 \tag{8.107}$$

with the Clausen functions $\mathrm{Cl}_2(kh)$ and $\mathrm{Cl}_3(kh)$ defined and approximated by (D.8). The proof given in Section 8 [see (8.53)] that the kd–βd equation is an equation of real quantities applies here as well. It is simple to solve (8.104) numerically for βd given values of kd, kh, S_-, and S_+, using, for example, a simple search procedure with secant algorithm refinement.

To facilitate calculations of Σ_1, Σ_2, and Σ_3, the following may be noted. The series $\sum \cos(n\beta d)\, Y_0(nkd)$ is treated in (B.2) and the series $\sum \cos(n\beta d)\, Y_2(nkd)$ is evaluated by (B.9)-(B.10). All series involving the modified Bessel functions K_0, K_1, or K_2 converge very rapidly because of the exponential decay of these functions so that only a few terms of the series give sufficient accuracy. The convergence of the series $\sum \sin(n\beta d) Y_1(nkd)$ can be greatly accelerated by using (B.6)-(B.7). The forms of (8.105)-(8.107) to be used when $kh = 2\pi$ are given by (8.58), (8.57), and (8.54), respectively.

9 3D MAGNETODIELECTRIC SPHERE ARRAYS

This section consists of two subsections. In the first and principal subsection we obtain the kd–βd equation for traveling waves supported by 3D infinite periodic arrays of magnetodielectric spheres. Two forms of the Floquet mode expansion method, one based on the asymptotic analysis of an integral and the other employing the Hertz vector potential, are used to convert the original form of the kd–βd equation containing extremely slowly convergent summations to a form suitable for calculations. If the array elements are sufficiently close together, the array can be regarded as a form of homogeneous medium characterized by an effective permittivity and permeability. In the second subsection we describe two methods for obtaining the effective permittivity and permeability. The first method obtains the permittivity and permeability directly from the solution to the kd–βd equation. The second method, based on the Clausius-Mossotti relation, is completely independent of the kd–βd equation and is more restrictive than the first method.

9.1 kd–βd EQUATION FOR 3D MAGNETODIELECTRIC SPHERE ARRAYS

In this subsection we consider traveling waves supported by 3D periodic arrays of lossless magnetodielectric spheres. It is assumed that the spheres can be modeled by pairs of crossed electric and magnetic dipoles, each of the dipoles perpendicular to the array axis. (It is unnecessary to consider 3D arrays of electric and magnetic dipoles with the electric (magnetic) dipoles in the direction of the array axis and the magnetic (electric) dipoles perpendicular to the array axis, or 3D arrays of electric and magnetic dipoles with all dipoles oriented in the direction of the array axis, because an electric (magnetic) dipole has no radial or longitudinal magnetic (electric) field [20, secs. 8.5, 8.6] and so there is no coupling of the electric dipoles with the magnetic dipoles of such arrays.) As we noted in Section 8, the analysis performed here is equally applicable to any 3D periodic arrays whose elements can be modeled by a pair of crossed electric and magnetic dipoles at right angles to each other such that only an incident electric (magnetic) field at the element center in the direction of the electric (magnetic) dipole excites only the electric (magnetic) dipole field. We choose the array axis to be the z axis of a Cartesian coordinate system with equispaced planes of magnetodielectric spheres normal to the z axis located at $z = nd, n = 0, \pm 1, \pm 2, \cdots$. In each plane the spheres are centered at $x = mh, y = lh, l, m = 0, \pm 1, \pm 2, \cdots$. The electric and magnetic dipole components of each sphere are oriented in the x and y direction, respectively. We assume an excitation of the array with the electric field parallel to the x axis and the magnetic field parallel to the y axis, and such that all the spheres in any column of the array are excited identically. Let $\mathbf{E}_0^0$ and $\mathbf{H}_0^0$ be the electric and magnetic field, respectively, incident on the sphere at the location $x = 0, y = 0, z = 0$ from all the other spheres in the array. As will be seen [see (9.21)] $\mathbf{E}_0^0$ has an x component only, and $\mathbf{H}_0^0$ has a y component only. Let $\mathbf{E}_0^{0mln}$ and $\mathbf{H}_0^{0mln}$ be the electric and magnetic field, respectively, incident on the reference sphere from the sphere at the location $(x, y, z) = (mh, lh, nd)$ so that

$$\mathbf{E}_0^0 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \mathbf{E}_0^{0mln} + \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \; \sum_{\substack{l=-\infty \\ \neq \; (0,0)}}^{\infty} \mathbf{E}_0^{0ml0} \tag{9.1a}$$

$$\mathbf{H}_0^0 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \mathbf{H}_0^{0mln} + \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \; \sum_{\substack{l=-\infty \\ \neq \; (0,0)}}^{\infty} \mathbf{H}_0^{0ml0} \, . \tag{9.1b}$$

From [5, eqs. (32),(33)]

$$\mathbf{E}_0^{0mln} = b_{-n} \frac{\mathrm{e}^{\mathrm{i}kr_{mln0}}}{kr_{mln0}} \left[\frac{-2\mathrm{i}}{kr_{mln0}} \left(1 + \frac{\mathrm{i}}{kr_{mln0}}\right) \sin\theta_{mln0} \cos\phi_{mln0} \; \hat{\mathbf{r}}_{mln0} \right.$$

$$\left. + \left(1 + \frac{\mathrm{i}}{kr_{mln0}} - \frac{1}{(kr_{mln0})^2}\right) \left(\cos\theta_{mln0} \cos\phi_{mln0} \; \hat{\boldsymbol{\theta}}_{mln0} - \sin\phi_{mln0} \; \hat{\boldsymbol{\phi}}_{mln0}\right) \right]$$

$$+ b_{+n} \frac{\mathrm{e}^{\mathrm{i}kr_{mln0}}}{kr_{mln0}} \left(1 + \frac{\mathrm{i}}{kr_{mln0}}\right) \left(\cos\phi_{mln0} \; \hat{\boldsymbol{\theta}}_{mln0} - \cos\theta_{mln0} \sin\phi_{mln0} \; \hat{\boldsymbol{\phi}}_{mln0}\right) \tag{9.2a}$$

and

$$\mathbf{H}_0^{0mln} = Y_0 b_{+n} \frac{\mathrm{e}^{\mathrm{i}kr_{mln0}}}{kr_{mln0}} \left[\frac{-2\mathrm{i}}{kr_{mln0}} \left(1 + \frac{\mathrm{i}}{kr_{mln0}}\right) \sin\theta_{mln0} \sin\phi_{mln0}\, \hat{\mathbf{r}}_{mln0} \right.$$

$$\left. + \left(1 + \frac{\mathrm{i}}{kr_{mln0}} - \frac{1}{(kr_{mln0})^2}\right) \left(\cos\theta_{mln0} \sin\phi_{mln0}\, \hat{\boldsymbol{\theta}}_{mln0} + \cos\phi_{mln0}\, \hat{\boldsymbol{\phi}}_{mln0}\right) \right]$$

$$+ Y_0 b_{-n} \frac{\mathrm{e}^{\mathrm{i}kr_{mln0}}}{kr_{mln0}} \left(1 + \frac{\mathrm{i}}{kr_{mln0}}\right) \left(\sin\phi_{mln0}\, \hat{\boldsymbol{\theta}}_{mln0} + \cos\theta_{mln0} \cos\phi_{mln0}\, \hat{\boldsymbol{\phi}}_{mln0}\right) . \tag{9.2b}$$

The quantities in (9.2) are defined with reference to a local spherical polar coordinate system with origin at $(x, y, z) = (mh, lh, nd)$ (in turn defined with reference to a local Cartesian coordinate system with the same origin whose axes are parallel to those of the global Cartesian coordinate system). The distance from the (m, l, n) sphere to the $(0, 0, 0)$ sphere, r_{mln0}, is given by

$$r_{mln0} = \sqrt{(mh)^2 + (lh)^2 + (nd)^2} = h\sqrt{m^2 + l^2 + (nd/h)^2} \tag{9.3}$$

and the unit vector in the direction from the (m, l, n) sphere to the $(0, 0, 0)$ sphere, $\hat{\mathbf{r}}_{mln0}$, is

$$\hat{\mathbf{r}}_{mln0} = \mathbf{r}_{mln0}/r_{mln0}, \quad \mathbf{r}_{mln0} = -mh\, \hat{\mathbf{x}} - lh\, \hat{\mathbf{y}} - nd\, \hat{\mathbf{z}} \tag{9.4}$$

so that

$$\cos\theta_{mln0} = \hat{\mathbf{r}}_{mln0} \cdot \hat{\mathbf{z}} = -\frac{nd}{r_{mln0}} = \frac{-nd/h}{\sqrt{m^2 + l^2 + (nd/h)^2}} \tag{9.5}$$

and

$$\sin\theta_{mln0} = \sqrt{1 - \cos^2\theta_{mln0}} = \frac{h\sqrt{m^2 + l^2}}{r_{mln0}} = \frac{\sqrt{m^2 + l^2}}{\sqrt{m^2 + l^2 + (nd/h)^2}} . \tag{9.6}$$

Also

$$\phi_{mln0} = \tan^{-1} \frac{\hat{\mathbf{r}}_{mln0} \cdot \hat{\mathbf{y}}}{\hat{\mathbf{r}}_{mln0} \cdot \hat{\mathbf{x}}} = \frac{-lh}{-mh} \tag{9.7}$$

$$\cos\phi_{mln0} = \frac{-mh}{h\sqrt{m^2 + l^2}} = -\frac{m}{\sqrt{m^2 + l^2}} \tag{9.8}$$

and

$$\sin\phi_{mln0} = \frac{-lh}{h\sqrt{m^2 + l^2}} = -\frac{l}{\sqrt{m^2 + l^2}} . \tag{9.9}$$

Then

$$\sin\theta_{mln0} \cos\phi_{mln0}\, \hat{\mathbf{r}}_{mln0} = \frac{1}{m^2 + l^2 + (nd/h)^2} \left[m^2\, \hat{\mathbf{x}} + ml\, \hat{\mathbf{y}} + (nd/h)m\, \hat{\mathbf{z}}\right] \tag{9.10}$$

$$\cos\theta_{mln0} \cos\phi_{mln0}\, \hat{\boldsymbol{\theta}}_{mln0} = \frac{1}{m^2 + l^2 + (nd/h)^2} \left[\frac{(nd/h)^2}{m^2 + l^2} \left(m^2\, \hat{\mathbf{x}} + ml\, \hat{\mathbf{y}}\right) - (nd/h)m\, \hat{\mathbf{z}}\right] \tag{9.11}$$

$$\sin\phi_{mln0}\, \hat{\boldsymbol{\phi}}_{mln0} = -\frac{1}{m^2 + l^2} \left(l^2\, \hat{\mathbf{x}} - ml\, \hat{\mathbf{y}}\right) \tag{9.12}$$

$$\cos\phi_{mln0}\ \hat{\boldsymbol{\theta}}_{mln0} = -\frac{1}{\sqrt{m^2+l^2+(nd/h)^2}}\left[\frac{nd/h}{m^2+l^2}\left(m^2\ \hat{\mathbf{x}} + ml\ \hat{\mathbf{y}}\right) + m\ \hat{\mathbf{z}}\right] \tag{9.13}$$

$$\cos\theta_{mln0}\sin\phi_{mln0}\ \hat{\boldsymbol{\phi}}_{mln0} = \frac{nd/h}{\sqrt{m^2+l^2+(nd/h)^2}\ (m^2+l^2)}\ \left(l^2\ \hat{\mathbf{x}} - ml\ \hat{\mathbf{y}}\right) \tag{9.14}$$

$$\sin\theta_{mln0}\sin\phi_{mln0}\ \hat{\mathbf{r}}_{mln0} = \frac{1}{m^2+l^2+(nd/h)^2}\left[ml\ \hat{\mathbf{x}} + l^2\ \hat{\mathbf{y}} + (nd/h)l\ \hat{\mathbf{z}}\right] \tag{9.15}$$

$$\cos\theta_{mln0}\sin\phi_{mln0}\ \hat{\boldsymbol{\theta}}_{mln0} = \frac{1}{m^2+l^2+(nd/h)^2}\left[\frac{(nd/h)^2}{m^2+l^2}\left(ml\ \hat{\mathbf{x}} + l^2\ \hat{\mathbf{y}}\right) + (nd/h)l\ \hat{\mathbf{z}}\right] \tag{9.16}$$

$$\cos\phi_{mln0}\ \hat{\boldsymbol{\phi}}_{mln0} = \frac{1}{m^2+l^2}\left(ml\ \hat{\mathbf{x}} + m^2\ \hat{\mathbf{y}}\right) \tag{9.17}$$

$$\sin\phi_{mln0}\ \hat{\boldsymbol{\theta}}_{mln0} = -\frac{1}{\sqrt{m^2+l^2+(nd/h)^2}}\left[\frac{nd/h}{m^2+l^2}\left(ml\ \hat{\mathbf{x}} + l^2\hat{\mathbf{y}}\right) - l\ \hat{\mathbf{z}}\right] \tag{9.18}$$

and

$$\cos\theta_{mln0}\cos\phi_{mln0}\ \hat{\boldsymbol{\phi}}_{mln0} = \frac{nd/h}{\sqrt{m^2+l^2+(nd/h)^2}\ (m^2+l^2)}\left(ml\ \hat{\mathbf{x}} - m^2\ \hat{\mathbf{y}}\right]\ . \tag{9.19}$$

The corresponding quantities in the self-plane summation of (9.1) are obtained from the quantities given by (9.3)-(9.19) by setting $n=0$. The constants b_{-n} and b_{+n} are related to the x component of the electric field and the y component of the magnetic field, respectively, incident on any sphere in the nth plane by the scattering equations [5, eq. (31)]

$$b_{-n} = S_-\ E_{0x}^{0n} \tag{9.20a}$$

$$b_{+n} = S_+\ \frac{H_{0y}^{0n}}{Y_0}. \tag{9.20b}$$

where S_- and S_+ are the normalized magnetodielectric sphere electric and magnetic dipole scattering coefficients, respectively. "Normalized" means that b_{-n} (b_{+n}) is the coefficient of $\exp(\mathrm{i}kr)/(kr)$ in the outgoing electric (magnetic) dipole field in response to the incident field $E_{0x}^{0n}\ \hat{\mathbf{x}}$ ($H_{0y}^{0n}/Y_0\ \hat{\mathbf{y}}$) at the center of the x (y) directed electric (magnetic) dipole. We note that when summed over m and l from $-\infty$ to ∞ the $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ components of the electric field vanish, and the $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ components of the magnetic field vanish. Thus for $n \neq 0$,

$$\begin{aligned} E_{0x}^{0n} = b_{-n} \sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{kh\rho_{mln}}\Bigg[&\frac{-2\mathrm{i}}{kh\rho_{mln}}\left(1+\frac{\mathrm{i}}{kh\rho_{mln}}\right)\frac{m^2}{\rho_{mln}^2} \\ &+ \left(1+\frac{\mathrm{i}}{kh\rho_{mln}} - \frac{1}{(kh\rho_{mln})^2}\right)\frac{l^2+(nd/h)^2}{\rho_{mln}^2}\Bigg] \\ &- b_{+n}\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{kh\rho_{mln}}\left(1+\frac{\mathrm{i}}{kh\rho_{mln}}\right)\frac{nd/h}{\rho_{mln}} \end{aligned} \tag{9.21a}$$

and

$$\frac{H_{0y}^{0n}}{Y_0} = b_{+n} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{kh\rho_{mln}} \left[\frac{-2\mathrm{i}}{kh\rho_{mln}} \left(1 + \frac{\mathrm{i}}{kh\rho_{mln}}\right) \frac{l^2}{\rho_{mln}^2} \right.$$

$$\left. + \left(1 + \frac{\mathrm{i}}{kh\rho_{mln}} - \frac{1}{(kh\rho_{mln})^2}\right) \frac{m^2 + (nd/h)^2}{\rho_{mln}^2} \right]$$

$$- \, b_{-n} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{kh\rho_{mln}} \left(1 + \frac{\mathrm{i}}{kh\rho_{mln}}\right) \frac{nd/h}{\rho_{mln}} \tag{9.21b}$$

where we have let

$$\rho_{mln} = \sqrt{m^2 + l^2 + (nd/h)^2}\ . \tag{9.22}$$

For $n = 0$, the self-plane,

$$E_{0x}^{00} = b_{-0} \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \ \neq \sum_{\substack{l=-\infty \\ (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{ml0}}}{kh\rho_{ml0}} \left[\frac{-2\mathrm{i}}{kh\rho_{ml0}} \left(1 + \frac{\mathrm{i}}{kh\rho_{ml0}}\right) \frac{m^2}{\rho_{ml0}^2} \right.$$

$$\left. + \left(1 + \frac{\mathrm{i}}{kh\rho_{ml0}} - \frac{1}{(kh\rho_{ml0})^2}\right) \frac{l^2 + (nd/h)^2}{\rho_{ml0}^2} \right] \tag{9.23a}$$

and

$$\frac{H_{0y}^{00}}{Y_0} = b_{+0} \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \ \neq \sum_{\substack{l=-\infty \\ (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{ml0}}}{kh\rho_{ml0}} \left[\frac{-2\mathrm{i}}{kh\rho_{ml0}} \left(1 + \frac{\mathrm{i}}{kh\rho_{ml0}}\right) \frac{l^2}{\rho_{ml0}^2} \right.$$

$$\left. + \left(1 + \frac{\mathrm{i}}{kh\rho_{ml0}} - \frac{1}{(kh\rho_{ml0})^2}\right) \frac{m^2}{\rho_{ml0}^2} \right] \tag{9.23b}$$

where

$$\rho_{ml0} = \sqrt{m^2 + l^2}\ . \tag{9.24}$$

The total x directed electric field and y directed magnetic field incident on the sphere in the $(m, l) = (0, 0)$ position of the $n = 0$ plane are given by

$$E_{0x}^{0} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} E_{0x}^{0n} + E_{0x}^{00}\ , \quad H_{0y}^{0} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} H_{0y}^{0n} + H_{0y}^{00} \tag{9.25}$$

where from (9.21)-(9.24)

$$E_{0x}^{0} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} b_{-n} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{kh\rho_{mln}} \left[\frac{-2\mathrm{i}}{kh\rho_{mln}} \left(1 + \frac{\mathrm{i}}{kh\rho_{mln}}\right) \frac{m^2}{\rho_{mln}^2} \right.$$

$$\left. + \left(1 + \frac{\mathrm{i}}{kh\rho_{mln}} - \frac{1}{(kh\rho_{mln})^2}\right) \frac{l^2 + (nd/h)^2}{\rho_{mln}^2} \right]$$

$$
- \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} b_{+n} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{kh\rho_{mln}} \left(1 + \frac{\mathrm{i}}{kh\rho_{mln}}\right) \frac{nd/h}{\rho_{mln}}
$$

$$
+ b_{-0} \sum_{m=-\infty}^{\infty} \sum_{\substack{l=-\infty \\ (m,l) \neq (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{ml0}}}{kh\rho_{ml0}} \left[\frac{-2\mathrm{i}}{kh\rho_{ml0}} \left(1 + \frac{\mathrm{i}}{kh\rho_{ml0}}\right) \frac{m^2}{\rho_{ml0}^2} + \left(1 + \frac{\mathrm{i}}{kh\rho_{ml0}} - \frac{1}{(kh\rho_{ml0})^2}\right) \frac{l^2}{\rho_{ml0}^2}\right] \tag{9.26a}
$$

and

$$
\begin{aligned}
\frac{H_{0y}^0}{Y_0} = &\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} b_{+n} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{kh\rho_{mln}} \left[\frac{-2\mathrm{i}}{kh\rho_{mln}} \left(1 + \frac{\mathrm{i}}{kh\rho_{mln}}\right) \frac{l^2}{\rho_{mln}^2}\right. \\
&\left. + \left(1 + \frac{\mathrm{i}}{kh\rho_{mln}} - \frac{1}{(kh\rho_{mln})^2}\right) \frac{m^2 + (nd/h)^2}{\rho_{mln}^2}\right] \\
&- \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} b_{-n} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{kh\rho_{mln}} \left(1 + \frac{\mathrm{i}}{kh\rho_{mln}}\right) \frac{nd/h}{\rho_{mln}}
\end{aligned}
$$

$$
+ b_{+0} \sum_{m=-\infty}^{\infty} \sum_{\substack{l=-\infty \\ (m,l) \neq (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{ml0}}}{kh\rho_{ml0}} \left[\frac{-2\mathrm{i}}{kh\rho_{ml0}} \left(1 + \frac{\mathrm{i}}{kh\rho_{ml0}}\right) \frac{l^2}{\rho_{ml0}^2} + \left(1 + \frac{\mathrm{i}}{kh\rho_{ml0}} - \frac{1}{(kh\rho_{ml0})^2}\right) \frac{m^2}{\rho_{ml0}^2}\right] . \tag{9.26b}
$$

Note that the terms with the coefficients b_{-n} [b_{+n}] in (9.26a) [(9.26b)] represent the x [y] directed electric [magnetic] field incident on the reference sphere from the electric [magnetic] dipoles in all the spheres of the array other than in the self- ($z = 0$) plane, while the term with the coefficient b_{-0} [b_{+0}] represents the x [y] directed electric [magnetic] field incident on the reference sphere from the electric [magnetic] dipoles in all the spheres of the self-plane other than the reference sphere itself. The terms with the coefficient b_{+n} [b_{-n}] in (9.26a) [(9.26b)] – the "cross terms" – represent the electric [magnetic] field incident on the reference sphere from the magnetic [electric] dipoles of all the spheres of the array. There are no cross terms from dipoles in the self-plane.

We now assume that the array is excited by a traveling wave in the z direction with real propagation constant β. Then the constants b_{-n} and b_{+n} in (9.26) are equal to b_{-0} and b_{+0}, respectively, apart from a phase shift given by

$$
b_{-n} = b_{-0}\, \mathrm{e}^{\mathrm{i}n\beta d}, \quad b_{+n} = b_{+0}\, \mathrm{e}^{\mathrm{i}n\beta d} . \tag{9.27}
$$

Substituting (9.27) in (9.26), using [from (9.20)] $b_{-0} = S_- E_{0x}^0$ and $b_{+0} = S_+ H_{0y}^0 / Y_0$, and multiplying by $(kh)^3$ we obtain

$$
\begin{aligned}
(kh)^3 = S_- \Bigg\{ &\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{\rho_{mln}} \left[\frac{-2\mathrm{i}}{\rho_{mln}} \left(kh + \frac{\mathrm{i}}{\rho_{mln}}\right) \frac{m^2}{\rho_{mln}^2}\right. \\
&\left. + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mln}} - \frac{1}{\rho_{mln}^2}\right) \frac{l^2 + (nd/h)^2}{\rho_{mln}^2}\right]
\end{aligned}
$$

$$
- q \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{\rho_{mln}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mln}} \right) \frac{nd/h}{\rho_{mln}}
$$

$$
+ \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \sum_{\substack{l=-\infty \\ \neq (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{ml0}}}{\rho_{ml0}} \left[\frac{-2\mathrm{i}}{\rho_{ml0}} \left(kh + \frac{\mathrm{i}}{\rho_{ml0}} \right) \frac{m^2}{\rho_{ml0}^2} + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{ml0}} - \frac{1}{\rho_{ml0}^2} \right) \frac{l^2}{\rho_{ml0}^2} \right] \Bigg\} \tag{9.28a}
$$

and

$$
(kh)^3 = S_+ \Bigg\{ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{\rho_{mln}} \left[\frac{-2\mathrm{i}}{\rho_{mln}} \left(kh + \frac{\mathrm{i}}{\rho_{mln}} \right) \frac{l^2}{\rho_{mln}^2} \right.
$$

$$
\left. + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mln}} - \frac{1}{\rho_{mln}^2} \right) \frac{m^2 + (nd/h)^2}{\rho_{mln}^2} \right]
$$

$$
- \frac{1}{q} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{\rho_{mln}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mln}} \right) \frac{nd/h}{\rho_{mln}}
$$

$$
+ \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \sum_{\substack{l=-\infty \\ \neq (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{ml0}}}{\rho_{ml0}} \left[\frac{-2\mathrm{i}}{\rho_{ml0}} \left(kh + \frac{\mathrm{i}}{\rho_{ml0}} \right) \frac{l^2}{\rho_{ml0}^2} + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{ml0}} - \frac{1}{\rho_{ml0}^2} \right) \frac{m^2}{\rho_{ml0}^2} \right] \Bigg\} \tag{9.28b}
$$

where

$$
q = \frac{b_{+0}}{b_{-0}} \tag{9.29}
$$

and ρ_{mln} and ρ_{ml0} are given by (9.22) and (9.24), respectively. As was done in Section 8 with the corresponding equations for the 2D magnetodielectric sphere arrays, by eliminating q from (9.28a) and (9.28b) the kd–βd equation is obtained that determines the normalized traveling wave propagation constant βd in terms of kh, d/h, and the normalized magnetodielectric sphere electric and magnetic dipole scattering coefficients S_- and S_+. This will be done below [see (9.81)-(9.84)].

We first note that (9.28a) and (9.28b), without the cross-term sums multiplied by $-q$ and $-1/q$, are uncoupled and are simply the kd–βd equations for 3D arrays of electric and magnetic dipoles transverse to the array axis, respectively, and furthermore, that these two equations are then identical apart from the scattering coefficient, S_- or S_+. (Note that the self-plane sums in (9.28a) and (9.28b) are identical as can be seen by interchanging the indices m and l.) Hence the non-cross-term summations in (9.28a) and (9.28b) are identical with the sums in (5.20) treated in Section 5 dealing with 3D arrays of electric dipoles perpendicular to the array axis. Thus from (5.65) and (5.68) we have

$$
\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{\rho_{mln}} \left[\frac{-2\mathrm{i}}{\rho_{mln}} \left(kh + \frac{\mathrm{i}}{\rho_{mln}} \right) \frac{m^2}{\rho_{mln}^2} \right.
$$

$$
\left. + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mln}} - \frac{1}{\rho_{mln}^2} \right) \frac{l^2 + (nd/h)^2}{\rho_{mln}^2} \right]
$$

$$+ \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{ml0}}}{\rho_{ml0}} \left[\frac{-2\mathrm{i}}{\rho_{ml0}} \left(kh + \frac{\mathrm{i}}{\rho_{ml0}} \right) \frac{m^2}{\rho_{ml0}^2} + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{ml0}} - \frac{1}{\rho_{ml0}^2} \right) \frac{l^2}{\rho_{ml0}^2} \right]$$

$$= -2\pi\mathrm{i}(kh) - 2\pi kh \frac{\sin kd}{\cos\beta d - \cos kd}$$

$$- 4\pi \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty} \left[(2\pi m)^2 - (kh)^2 \right] \frac{\mathrm{e}^{-n(d/h)\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}$$

$$+ 2\pi\mathrm{i}(kh)^2 \sum_{l=1}^{\infty} H_0^{(1)}(lkh) - 8 \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \left[(2\pi m)^2 - (kh)^2 \right] K_0 \left(l\sqrt{(2\pi m)^2 - (kh)^2} \right)$$

$$+ 4\ kh\ \mathrm{Cl}_2(kh) + 4\ \mathrm{Cl}_3(kh) + \mathrm{i}\pi(kh)^2 - \mathrm{i}\frac{2}{3}(kh)^3 \tag{9.30}$$

with the Clausen functions Cl_2 and Cl_3 defined and approximated by equations (D.8), and with $0 < kh < 2\pi$.

Now let us treat the sum in (9.28a) and (9.28b) multiplied by $-q$ and $-1/q$,

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{\rho_{mln}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mln}} \right) \frac{nd/h}{\rho_{mln}}$$

$$= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d}\, \mathrm{sgn}(n) \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{\rho_{mln}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mln}} \right) \frac{|n|d/h}{\rho_{mln}}\ . \tag{9.31}$$

This sum is proportional to the y directed magnetic field incident on the $(0,0,0)$ sphere scattered from the x directed electric dipoles of all the other spheres in the array, or to the x directed electric field incident on the $(0,0,0)$ sphere scattered from the y directed magnetic dipoles of all the other spheres in the array. We use the Floquet mode method to help transform this slowly convergent sum to a rapidly convergent form, proceeding similarly to the way we used it in Section 7. We let $H_y^0(P)$ be the y component of the magnetic field radiated by all the x directed electric dipoles in the $n = 0$ plane at a general point in space $P = (x, y, z),\ z \neq 0$, an expression for which is available from the b_{-n} term of (9.2b). We establish a local spherical polar coordinate system with origin at the sphere located at $(x, y, z) = (mh, lh, 0)$ and with $\theta(m, l, P)$ the polar angle from the z axis to the vector $\mathbf{r}(m, l, P)$ from $(mh, lh, 0)$ to the field point P. The distance $r(m, l, P)$ from $(mh, lh, 0)$ to P is

$$r(m, l, P) = \sqrt{(x - mh)^2 + (y - lh)^2 + z^2} \tag{9.32}$$

and the unit vector $\hat{\mathbf{r}}(m, l, P)$ is

$$\hat{\mathbf{r}}(m, l, P) = \frac{\mathbf{r}(m, l, P)}{r(m, l, P)} = \frac{(x - mh)\ \hat{\mathbf{x}} + (y - lh)\ \hat{\mathbf{y}} + z\ \hat{\mathbf{z}}}{r(m, l, P)} \tag{9.33}$$

so that

$$\cos\theta(m, l, P) = \hat{\mathbf{r}}(l, m, P) \cdot \hat{\mathbf{z}} = \frac{z}{r(m, l, P)} \tag{9.34}$$

$$\sin\theta(m,l,P) = \sqrt{1-\cos^2\theta(m,l,P)} = \frac{\sqrt{(x-mh)^2+(y-lh)^2}}{r(m,l,P)} \tag{9.35}$$

$$\phi(m,l,P) = \tan^{-1}\left(\frac{\mathbf{r}(m,l,P)\cdot\hat{\mathbf{y}}}{\mathbf{r}(m,l,P)\cdot\hat{\mathbf{x}}}\right) = \frac{y-lh}{x-mh} \tag{9.36}$$

$$\cos\phi(m,l,P) = \frac{x-mh}{\sqrt{(x-mh)^2+(y-lh)^2}} \tag{9.37}$$

and

$$\sin\phi(m,l,P) = \frac{y-lh}{\sqrt{(x-mh)^2+(y-lh)^2}}\ . \tag{9.38}$$

Then [see (9.2b), (9.18), and (9.19)]

$$\begin{aligned}[\sin\phi(m,l,P)\ \hat{\boldsymbol{\theta}}(m,l,P)]_y &= \cos\theta(m,l,P)\sin^2(m,l,P)\\ &= \frac{z}{r(m,l,P)}\frac{(y-mh)^2}{(x-mh)^2+(y-lh)^2}\end{aligned} \tag{9.39}$$

and

$$\begin{aligned}[\cos\theta(m,l,P)\cos\phi(m,l,P)\ \hat{\boldsymbol{\phi}}(m,l,P)]_y &= \cos\theta(m,l,P)\cos^2(m,l,P)\\ &= \frac{z}{r(m,l,P)}\frac{(x-mh)^2}{(x-mh)^2+(y-lh)^2}\end{aligned} \tag{9.40}$$

so that

$$[\sin\phi(m,l,P)\ \hat{\boldsymbol{\theta}}(m,l,P)]_y + [\cos\theta(m,l,P)\cos\phi(m,l,P)\ \hat{\boldsymbol{\phi}}(m,l,P)]_y = \frac{z}{r(m,l,P)} \tag{9.41}$$

and hence referring to (9.2b)

$$\frac{H_y^0(P)}{Y_0} = b_{-0}\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kr(m,l,P)}}{kr(m,l,P)}\left(1+\frac{\mathrm{i}}{kr(m,l,P)}\right)\frac{z}{r(m,l,P)}\ . \tag{9.42}$$

For $P_0 = (0,0,nd)$

$$\frac{H_y^0(0,0,nd)}{Y_0} = b_{-0}\ \mathrm{sgn}(n)\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{kh\rho_{mln}}\left(1+\frac{\mathrm{i}}{kh\rho_{mln}}\right)\frac{|n|d/h}{\rho_{mln}} \tag{9.43}$$

where

$$\rho_{mln} = \sqrt{m^2+l^2+(nd/h)^2} \tag{9.44}$$

and so in (9.31)

$$\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{\rho_{mln}}\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mln}}\right)\frac{|n|d/h}{\rho_{mln}} = \frac{(kh)^3}{b}\frac{H_y^0(0,0,|n|d)}{Y_0}\ . \tag{9.45}$$

Now $H_y^0(x,y,z)$ can be expressed in terms of a plane wave spectrum by

$$\frac{H_y^0(x,y,z)}{Y_0} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(k_x,k_y)\mathrm{e}^{\mathrm{i}(k_x x + k_y y + k_z z)}\,\mathrm{d}k_x\mathrm{d}k_y,\ k_z = \sqrt{k^2 - k_x^2 - k_y^2} \qquad (9.46)$$

where k_z is positive real (positive imaginary) according as $k^2 > (<)\ k_x^2 + k_y^2$. Because of the periodicity of the array in the x and y directions,

$$H_y^0(x+h,y,z) = H_y^0(x,y,z)\ ,\ \ H_y^0(x,y+h,z) = H_y^0(x,y,z)\ . \qquad (9.47)$$

It follows by taking the inverse Fourier transform of (9.46) and inserting into (9.47) that

$$\mathrm{e}^{\mathrm{i}k_x h} = 1\ ,\ \ \mathrm{e}^{\mathrm{i}k_y h} = 1 \qquad (9.48)$$

and hence

$$k_x h = 2\pi m,\ m = 0, \pm 1, \pm 2, \cdots\ ,\ \ k_y h = 2\pi l,\ l = 0, \pm 1, \pm 2, \cdots \qquad (9.49)$$

so that

$$\frac{H_y^0(x,y,z)}{Y_0} = \sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} B_{ml}\ \mathrm{e}^{\mathrm{i}(2\pi/h)(mx+ly)}\ \mathrm{e}^{\mathrm{i}k_{ml}z} \qquad (9.50)$$

where

$$k_{ml} = \sqrt{k^2 - (2\pi m/h)^2 - (2\pi l/h)^2} \qquad (9.51)$$

with k_{ml} positive real (positive imaginary) according as $(kh)^2 > (<)\ (2\pi)^2(m^2+l^2)$. It remains to find the unknown Floquet mode expansion coefficients B_{ml}. As in Section 7 we will employ two different methods for obtaining the coefficients, one based on the analysis of an integral, and the other on the Hertz vector potential. We begin with the integral method.

By inverting (9.50)

$$B_{ml}\mathrm{e}^{\mathrm{i}k_{ml}z} = \frac{1}{h^2}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} \frac{H_y^0(x,y,z)}{Y_0}\mathrm{e}^{-\mathrm{i}(2\pi/h)(mx+ly)}\,\mathrm{d}x\mathrm{d}y \qquad (9.52)$$

so that with (9.42)

$$B_{ml}\ \mathrm{e}^{\mathrm{i}k_{ml}z}$$

$$= \frac{b_{-0}}{h^2}\sum_{m'=-\infty}^{\infty}\sum_{l'=-\infty}^{\infty}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} \frac{\mathrm{e}^{\mathrm{i}kr(m',l',P)}}{kr(m',l',P)}\left(1 + \frac{\mathrm{i}}{kr(m',l',P)}\right)\frac{z}{r(m',l',P)}\mathrm{e}^{-\mathrm{i}(2\pi/h)(mx+ly)}\,\mathrm{d}x\mathrm{d}y$$

$$(9.53)$$

where

$$r(m',l',P) = \sqrt{(x-m'h)^2 + (y-l'h)^2 + z^2}\ . \qquad (9.54)$$

Since B_{ml} is independent of z, if the LHS of (9.53) is expanded for small z

$$B_{ml}\mathrm{e}^{\mathrm{i}k_{ml}z} \overset{|z|<<1}{\sim} B_{ml}\left(1 + \mathrm{i}k_{ml}z - \frac{k_{ml}^2}{2}z^2\right)\ . \qquad (9.55)$$

We can then obtain an expression for B_{ml} by investigating the behavior of the RHS of (9.53) for $|z| << 1$ and equating coefficients of z^2. (We focus on coefficients of z^2 rather than on coefficients of z because terms in z in the expansion of the RHS of (9.53) for $|z| << 1$ can come from all terms of the double summation whereas, as we show directly, terms in z^2 can come only from the $(m', l') = (0, 0)$ term of the double summation.)

First we show that the terms in the double summation in (9.53) for which $(m', l') \neq (0, 0)$ cannot contribute a term in z^2 for $|z| << 1$. For, letting

$$A^2 = (x - m'h)^2 + (y - l'h)^2 \tag{9.56}$$

so that

$$r(m', l', P) = \sqrt{A^2 + z^2} \tag{9.57}$$

and assuming that $z^2 << A^2$,

$$\begin{aligned}\frac{\mathrm{e}^{\mathrm{i}kr(m',l',P)}}{kr(m',l',P)}\left(1+\frac{\mathrm{i}}{kr(m',l',P)}\right)\frac{z}{r(m',l',P)} &= \frac{\mathrm{e}^{\mathrm{i}k\sqrt{A^2+z^2}}}{k\sqrt{A^2+z^2}}\left(1+\frac{\mathrm{i}}{k\sqrt{A^2+z^2}}\right)\frac{z}{\sqrt{A^2+z^2}} \\ &\approx z\mathrm{e}^{\mathrm{i}kA}\left(1+\frac{\mathrm{i}kz^2}{2A}\right)\left[1+\frac{\mathrm{i}}{kA}\left(1-\frac{z^2}{2A^2}\right)\right]\frac{1}{kA}\left(1-\frac{z^2}{2A^2}\right)\end{aligned} \tag{9.58}$$

so that there are no terms in z^2 from the terms in the double summation for which $(m', l') \neq (0, 0)$. Thus a term in z^2 can come only from the $(m', l') = (0, 0)$ term

$$\frac{b_{-0}}{h^2}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\frac{\mathrm{e}^{\mathrm{i}kr(0,0,P)}}{kr(0,0,P)}\left(1+\frac{\mathrm{i}}{kr(0,0,P)}\right)\frac{z}{r(0,0,P)}\mathrm{e}^{-\mathrm{i}(2\pi/h)(mx+ly)}\,\mathrm{d}x\mathrm{d}y \tag{9.59}$$

where

$$r(0, 0, P) = \sqrt{x^2 + y^2 + z^2}\ . \tag{9.60}$$

In cylindrical polar coordinates $\rho = \sqrt{x^2 + y^2}$, $\phi = \tan^{-1}(y/x)$, the $(m', l') = (0, 0)$ term is approximately

$$\frac{b}{kh^2}\int_0^{h/2}\int_0^{2\pi}\frac{\mathrm{e}^{\mathrm{i}k\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}}\left(1+\frac{\mathrm{i}}{k\sqrt{\rho^2+z^2}}\right)\frac{z}{\sqrt{\rho^2+z^2}}\mathrm{e}^{-\mathrm{i}(2\pi/h)(m\cos\phi+l\sin\phi)\rho}\,\rho\,\mathrm{d}\rho\mathrm{d}\phi\ . \tag{9.61}$$

We can obtain a term in z^2 for $|z| << 1$ only in the vicinity of $\rho = 0$. We expand the trigonometric exponential in (9.61) in a power series in ρ, and note that terms containing odd powers of $\cos\phi$ and $\sin\phi$ integrate to 0 over the interval $\phi = [0, 2\pi]$, to obtain

$$\mathrm{e}^{-\mathrm{i}(2\pi/h)(m\cos\phi+l\sin\phi)\rho} \approx 1 - \frac{1}{2}(2\pi/h)^2\ \left(m^2\cos^2\phi + l^2\sin^2\phi\right)\rho^2 + \cdots\ . \tag{9.62}$$

We then substitute (9.62) in (9.61), perform the ϕ integration, systematically integrate analytically all the resulting indefinite integrals in ρ by making the change of variables

$$u = \sqrt{\rho^2 + z^2},\ du = \frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2 + z^2}} \tag{9.63}$$

and using integrals tabulated in [18, eqs. 2.324,2.325], evaluate the integrals at the lower range of integration, $u = |z|$, and collect terms in z^2. (There is no contribution to terms in z^2 from the upper end of the interval of integration $u = \sqrt{(h/2)^2 + z^2}$.) It is found that there is no contribution to terms in z^2 from terms higher than ρ^2 in the expansion of the trigonometric exponentials. The end result is that the z^2 term of the expansion of the RHS of (9.52) in a power series in z for $|z| << 1$ is

$$\frac{2\pi \mathrm{i} b_{-0}}{h^2} \, \frac{(kh)^2 - (2\pi)^2(m^2 + l^2)}{(kh)^2} \, z^2 \, . \tag{9.64}$$

But then, equating coefficients of z^2 in (9.55) and (9.64) we obtain the coefficients of the Floquet mode expansion (9.50)

$$B_{ml} = \frac{2\pi \mathrm{i} b_{-0}}{h^2 k_{ml}^2} \, \frac{(kh)^2 - (2\pi)^2(m^2 + l^2)}{(kh)^2} = \frac{2\pi \mathrm{i} b_{-0}}{(kh)^2} \, . \tag{9.65}$$

Before continuing we will give an alternate derivation of the Floquet mode expansion coefficients B_{ml} based on the Hertz vector potential. The magnetic field of an x directed electric dipole at the origin of a Cartesian coordinate system is given by the curl of the Hertz vector potential for the field of an x directed electric dipole [see (4.48)]

$$C \left(\nabla \times \frac{\mathrm{e}^{\mathrm{i}kr}}{kr} \hat{\mathbf{x}} \right) \, . \tag{9.66}$$

The proportionality constant C can be found easily by expanding (9.66) in spherical coordinates (using, for example, [22, Appendix 1, eqs. 117, 151]) and equating the $1/(kr)$ term of the ϕ component of the field with $Y_0 b_{-0} \exp(\mathrm{i}kr)/(kr) \cos\theta \cos\phi$ [see (9.2b)] thereby obtaining

$$C = -\frac{\mathrm{i}Y_0}{k} b_{-0} \, . \tag{9.67}$$

From (9.66) the y component of the magnetic field radiated by an x directed electric dipole at the origin of a Cartesian coordinate system is given by

$$C \left(\nabla \times \frac{\mathrm{e}^{\mathrm{i}kr}}{kr} \hat{\mathbf{x}} \right)_y = C \frac{\partial}{\partial z} \frac{\mathrm{e}^{\mathrm{i}kr}}{kr} \tag{9.68}$$

referring to [22, Appendix 1, eq. 51]. Now from (3.30) and (3.42) the field radiated by the acoustic monopoles located in the plane $z = 0$ at the locations $(x, y) = (mh, lh)$, $m, l = 0, \pm h, \pm 2h, \cdots$, each of which radiates a field equal to $\exp(\mathrm{i}kr)/(kr)$, is

$$\sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} B_{ml}^0 \, \mathrm{e}^{\mathrm{i}(2\pi/h)(mx + ly)} \, \mathrm{e}^{\mathrm{i}k_{ml} z} \tag{9.69}$$

where

$$B_{ml}^0 = \frac{2\pi \mathrm{i}}{k h^2 k_{ml}} \tag{9.70}$$

and

$$k_{ml} = \sqrt{k^2 - (2\pi m/h)^2 - (2\pi l/h)^2} \tag{9.71}$$

with k_{ml} positive real or positive imaginary. Hence from (9.68) the y component of the magnetic field radiated by the plane $z = 0$ of x directed electric dipoles is equal to

$$C \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} B^0_{ml}\ \mathrm{e}^{\mathrm{i}(2\pi/h)(mx+ly)}\ \frac{\mathrm{d}}{\mathrm{d}z}\mathrm{e}^{\mathrm{i}k_{ml}z}$$

$$= C \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} B^0_{ml}\ (\mathrm{i}k_{ml})\ \mathrm{e}^{\mathrm{i}(2\pi/h)(mx+ly)}\ \mathrm{e}^{\mathrm{i}k_{ml}z}\ . \tag{9.72}$$

The same field is also given by the Floquet mode expansion (9.50) multiplied by Y_0. Hence, equating (9.50) with (9.72) we see that the coefficients B_{ml} in (9.50) are given by

$$B_{ml} = \frac{C}{Y_0} B^0_{ml} \mathrm{i}k_{ml} = -\frac{\mathrm{i}}{k}\frac{2\pi \mathrm{i} b_{-0}}{kh^2 k_{ml}} \mathrm{i}k_{ml} = \frac{2\pi \mathrm{i} b_{-0}}{(kh)^2} \tag{9.73}$$

in agreement with (9.65).

Now that we have derived the Floquet mode expansion coefficients B_{ml} by two independent methods, we have from (9.45), (9.50), (9.65) or (9.73), and (9.51),

$$\sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{\rho_{mln}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mln}} \right) \frac{|n|d/h}{\rho_{mln}}$$

$$= \frac{(kh)^3}{b_{-0}} \frac{2\pi \mathrm{i} b_{-0}}{(kh)^2} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}|n|(d/h)\ \sqrt{(kh)^2 - (2\pi)^2(m^2+l^2)}}$$

$$= 2\pi \mathrm{i}(kh)\ \mathrm{e}^{\mathrm{i}|n|kd} + 2\pi \mathrm{i}(kh) \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \ \ \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty} \mathrm{e}^{-|n|(d/h)\ \sqrt{(2\pi)^2(m^2+l^2) - (kh)^2}} \tag{9.74}$$

for $0 < kh < 2\pi$. But then in (9.31)

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d}\ \mathrm{sgn}(n) \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{\rho_{mln}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mln}} \right) \frac{|n|d/h}{\rho_{mln}}$$

$$= 2\pi \mathrm{i}(kh) \sum_{n=1}^{\infty} 2\mathrm{i} \sin(n\beta d) \left(\mathrm{e}^{\mathrm{i}nkd} + \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \ \ \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty} \mathrm{e}^{-n(d/h)\ \sqrt{(2\pi)^2(m^2+l^2) - (kh)^2}} \right) . \tag{9.75}$$

Thus, using (D.6),

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{e}^{\mathrm{i}n\beta d} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mln}}}{\rho_{mln}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mln}} \right) \frac{nd/h}{\rho_{mln}}$$

$$= 2\pi kh \, \frac{\sin\beta d}{\cos\beta d - \cos kd}$$

$$- 4\pi \; kh \, \sum_{n=1}^{\infty} \sin(n\beta d) \quad \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \; \sum_{\substack{l=-\infty \\ \neq \; (0,0)}}^{\infty} \mathrm{e}^{-n(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}} \, . \tag{9.76}$$

Substituting (9.30) and (9.76) in (9.28a) and (9.28b) we can then write these equations in the form

$$(kh)^3 = S_- \left\{\Re_- + \mathrm{i}\Im_-\right\} \tag{9.77a}$$

and

$$(kh)^3 = S_+ \left\{\Re_+ + \mathrm{i}\Im_+\right\} \tag{9.77b}$$

where, assuming that q is real, an assumption that is verified shortly below [see the comment following 9.84)], $\Re_-$, the real part of the quantity within the brackets of (9.28a) with the original summations replaced by the new expressions we have derived, is

$$\Re_- = -2\pi kh \, \frac{\sin kd}{\cos\beta d - \cos kd}$$

$$- 4\pi \sum_{n=1}^{\infty} \cos(n\beta d) \quad \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \; \sum_{\substack{l=-\infty \\ \neq \; (0,0)}}^{\infty} \left[(2\pi m)^2 - (kh)^2\right] \frac{\mathrm{e}^{-n(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}$$

$$- 2\pi(kh)^2 \sum_{l=1}^{\infty} Y_0(lkh) - 8 \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \left[(2\pi m)^2 - (kh)^2\right] K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\,\right)$$

$$- q\Bigg(2\pi kh \, \frac{\sin\beta d}{\cos\beta d - \cos kd}$$

$$- 4\pi \; kh \, \sum_{n=1}^{\infty} \sin(n\beta d) \quad \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \; \sum_{\substack{l=-\infty \\ \neq \; (0,0)}}^{\infty} \mathrm{e}^{-n(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}} \Bigg)$$

$$+ 4 \; kh \; \mathrm{Cl}_2(kh) + 4 \; \mathrm{Cl}_3(kh) \; ; \tag{9.78a}$$

$\Im_-$, the imaginary part of the quantity within the brackets of (9.28a), is

$$\Im_- = -2\pi kh + 2\pi(kh)^2 \sum_{l=1}^{\infty} J_0(lkh) + \pi(kh)^2 - \frac{2}{3}(kh)^3 \; ; \tag{9.78b}$$

$\Re_+$, the real part of the quantity within the brackets of (9.28b) with the original summations replaced by the rapidly convergent expressions we have derived, equals $\Re_-$ with $1/q$ substituted for q; and $\Im_+$, the imaginary part of the quantity within the brackets of (9.28b), equals $\Im_-$. Using (B.11)

$$\Im_- = \Im_+ = -2\pi kh + 2\pi(kh)^2\left(-\frac{1}{2} + \frac{1}{kh}\right) + \pi(kh)^2 - \frac{2}{3}(kh)^3 = -\frac{2}{3}(kh)^3 \tag{9.79}$$

which, together with (9.77), by the argument used above in Section 4 [see (4.59)-(4.61)] implies that

$$|S_-| = \frac{3}{2}\sin\psi_- \tag{9.80a}$$

and

$$|S_+| = \frac{3}{2}\sin\psi_+ \tag{9.80b}$$

where ψ_- and ψ_+ are the phases of the scattering coefficients S_- and S_+, respectively. The properties of the scattering coefficients (9.80) were derived independently in [4] from reciprocity and power conservation principles, and our obtaining them here thereby serves as an important check on the validity of our analysis.

To obtain the kd–βd equation determining βd as a function of kd, d/h, and the scattering coefficients S_- and S_+, we write (9.77) as

$$(kh)^3 = S_-\left\{\Sigma_1 - q\Sigma_2\right\} \tag{9.81a}$$

and

$$(kh)^3 = S_+\left\{\Sigma_1 - \frac{1}{q}\Sigma_2\right\} \tag{9.81b}$$

where, from (9.78) and (9.79),

$$\begin{aligned}\Sigma_1 = {} & -2\pi kh\,\frac{\sin kd}{\cos\beta d - \cos kd} \\ & - 4\pi\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{\substack{m=-\infty \\ (m,l)}}^{\infty}\ \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty}\left[(2\pi m)^2 - (kh)^2\right]\frac{\mathrm{e}^{-n(d/h)\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}} \\ & - 2\pi(kh)^2\sum_{l=1}^{\infty}Y_0(lkh) - 8\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\left[(2\pi m)^2-(kh)^2\right]K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\,\right) \\ & + 4\,kh\,\mathrm{Cl}_2(kh) + 4\,\mathrm{Cl}_3(kh) - \mathrm{i}\frac{2}{3}(kh)^3\end{aligned} \tag{9.82}$$

and

$$\begin{aligned}\Sigma_2 = {} & 2\pi\ kh\,\frac{\sin\beta d}{\cos\beta d - \cos kd} \\ & - 4\pi\ kh\sum_{n=1}^{\infty}\sin(n\beta d)\sum_{\substack{m=-\infty \\ (m,l)}}^{\infty}\ \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty}\mathrm{e}^{-n(d/h)\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}\,.\end{aligned} \tag{9.83}$$

The proof given in Section 8 [see (8.48) to (8.51)] that equations (8.44a) and (8.44b) imply (8.41) and (8.43) applies equally well to (9.81a) and (9.81b) so that, as we noted in Section 8, (9.80a) and (9.80b) must be satisfied by the elements of any 3D periodic array that supports a lossless traveling wave if the elements can be modeled by pairs of crossed electric and magnetic dipoles at right angles such that an incident electric (magnetic) field at the element center in the direction of the electric (magnetic) dipole excites only the electric (magnetic) dipole field. Solving for $-q$ in (9.81a) and (9.81b) and equating the resulting expressions we obtain the kd–βd equation

$$-q = \frac{(kh)^3 - S_-\Sigma_1}{S_-\Sigma_2} = \frac{S_+\Sigma_2}{(kh)^3 - S_+\Sigma_1}\ . \tag{9.84}$$

The proof given in Section 8 [see (8.53)] that the kd–βd equation is an equation of real quantities applies here as well. It is simple to solve (9.84) numerically for βd given values of kd, kh, S_-, and S_+, using, for example, a simple search procedure with secant algorithm refinement.

To facilitate calculations of Σ_1 and Σ_2 we note a rapidly convergent expression for the slowly convergent Schlömilch series $\sum Y_0(lkh)$ is given by (B.12), and that all series involving negative exponentials and the modified Bessel function K_0 (which decays exponentially) converge very rapidly so that only a few terms of these series gives sufficient accuracy. Alternately, approximate closed form expressions for the summations involving negative exponentials can be obtained by first performing the summation over n from 1 to ∞ using (D.4) and then including only terms in the summations over m and l for which $|m| \leq 1$ and $|l| \leq 1$. The sum of negative exponentials in Σ_1 is identical to the sum of negative exponentials in (5.72) and so (5.76) gives the approximate closed form expression for this sum which we repeat here

$$\sum_{n=1}^{\infty} \cos(n\beta d) \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \ \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty} \left[(2\pi m)^2 - (kh)^2\right] \frac{\mathrm{e}^{-n(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}$$

$$\begin{aligned} &\approx 2\,\frac{(2\pi)^2 - 2(kh)^2}{\sqrt{(2\pi)^2-(kh)^2}}\,\frac{\mathrm{e}^{-(d/h)r_1}\,\cos\beta d - \mathrm{e}^{-2(d/h)r_1}}{1 - 2\cos\beta d\,\mathrm{e}^{-(d/h)r_1} + \mathrm{e}^{-2(d/h)r_1}} \\ &\quad + 4\,\frac{(2\pi)^2 - (kh)^2}{\sqrt{8\pi^2-(kh)^2}}\,\frac{\mathrm{e}^{-(d/h)r_2}\,\cos\beta d - \mathrm{e}^{-2(d/h)r_2}}{1 - 2\cos\beta d\,\mathrm{e}^{-(d/h)r_2} + \mathrm{e}^{-2(d/h)r_2}} \end{aligned} \tag{9.85}$$

where $r_1 = \sqrt{(2\pi)^2 - (kh)^2}$, and $r_2 = \sqrt{8\pi^2 - (kh)^2}$. The corresponding approximate closed form expression for the sum of negative exponentials in Σ_2 is

$$\sum_{n=1}^{\infty} \sin(n\beta d) \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \ \sum_{\substack{l=-\infty \\ \neq\ (0,0)}}^{\infty} \mathrm{e}^{-n(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}$$

$$\approx 4\sin\beta d\left(\frac{\mathrm{e}^{-(d/h)r_1}}{1 - 2\cos\beta d\,\mathrm{e}^{-(d/h)r_1} + \mathrm{e}^{-2(d/h)r_1}} + \frac{\mathrm{e}^{-(d/h)r_2}}{1 - 2\cos\beta d\,\mathrm{e}^{-(d/h)r_2} + \mathrm{e}^{-2(d/h)r_2}}\right) \tag{9.86}$$

where $r_1 = \sqrt{(2\pi)^2 - (kh)^2}$, and $r_2 = \sqrt{8\pi^2 - (kh)^2}$.

Since some of the terms in the expression (9.82) for Σ_1 become singular as kh approaches 2π it is worthwhile to obtain the limiting value of Σ_1 as $kh \to 2\pi$. Comparing (9.82) with the expression (5.72) for $\Re$ in our treatment of 3D arrays of electric dipoles perpendicular to the array axis, we see that the two expressions are identical apart from the term $-\mathrm{i}(2/3)(kh)^3$ in (9.82). We thus obtain immediately from (5.82) that

$$\lim_{kh\to 2\pi} \Sigma_1 = -(2\pi)^2 \frac{\sin kd}{\cos\beta d - \cos kd}$$

$$- 4\pi(2\pi) \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{\substack{m=-\infty \\ |m|+|l|}}^{\infty} \sum_{\substack{l=-\infty \\ > \; 1}}^{\infty} (m^2-1) \frac{\mathrm{e}^{-2\pi n(d/h)\sqrt{m^2+l^2-1}}}{\sqrt{m^2+l^2-1}}$$

$$- 2\pi(2\pi)^2 \left[-\frac{1}{\pi}\left(\gamma + \ln\frac{1}{2}\right) - 2\sum_{l=2}^{\infty}\left(\frac{1}{2\pi\sqrt{l-1}} - \frac{1}{2\pi l}\right)\right]$$

$$+ 4(2\pi)^2 \sum_{m=2}^{\infty} (m^2-1) K_0\left(2\pi l\sqrt{m^2-1}\right) + 4\,\mathrm{Cl}_3(2\pi) - \mathrm{i}\frac{2}{3}(2\pi)^3 \qquad (9.87)$$

where γ is the Euler constant, $\mathrm{Cl}_3(2\pi)$ is given by (D.10), and $\mathrm{Cl}_2(2\pi)$ has been set equal to 0 [see (D.9)]. The expression (9.83) for Σ_2 simplifies at $kh = 2\pi$. Using (D.6) we see that the contribution of the terms in the sum of negative exponentials for which $(m, l) = (0, \pm 1), (\pm 1, 0)$,

$$- 4\pi kh \sum_{n=1}^{\infty} \sin(n\beta d) \sum_{\substack{m=-\infty \\ |m|+|l|}}^{\infty} \sum_{\substack{l=-\infty \\ = \; 1}}^{\infty} \mathrm{e}^{-n(d/h)\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}} \overset{kh=2\pi}{=}$$

$$16\pi^2 \frac{\sin\beta d}{\cos\beta d - 1} \,. \qquad (9.88)$$

Hence

$$\Sigma_2 \overset{kh=2\pi}{=} 4\pi^2 \frac{\sin\beta d}{\cos\beta d - \cos kd} + 16\pi^2 \frac{\sin\beta d}{\cos\beta d - 1}$$

$$- 8\pi^2 \sum_{n=1}^{\infty} \sin(n\beta d) \sum_{\substack{m=-\infty \\ |m|+|l|}}^{\infty} \sum_{\substack{l=-\infty \\ > \; 1}}^{\infty} \mathrm{e}^{-2\pi n(d/h)\sqrt{m^2+l^2-1}} \,. \qquad (9.89)$$

Referring to (9.86) we see that

$$- 8\pi^2 \sum_{n=1}^{\infty} \sin(n\beta d) \sum_{\substack{m=-\infty \\ |m|+|l|}}^{\infty} \sum_{\substack{l=-\infty \\ > \; 1}}^{\infty} \mathrm{e}^{-2\pi n(d/h)\sqrt{m^2+l^2-1}} \qquad (9.90)$$

in (9.89) can be approximated by

$$- 32\pi^2 \sin\beta d \, \frac{\mathrm{e}^{-2\pi d/h}}{1 - 2\cos\beta d\, \mathrm{e}^{-2\pi d/h} + \mathrm{e}^{-4\pi d/h}} \,. \qquad (9.91)$$

9.2 EFFECTIVE PERMITTIVITY AND PERMEABILITY OF THE ARRAY

So far in this report, apart from a brief remark at the end of Section 5 relevant to this subsection, we have focused exclusively on obtaining the kd–βd equations for the various arrays considered. If the magnetodielectric sphere elements of a 3D periodic array are sufficiently close together so that

$$\beta d << 1 \tag{9.92}$$

then the array can be regarded macroscopically as a medium (which we will refer to as the array medium) with effective or bulk relative permittivity $\epsilon_{\rm r}^{\rm eff}$ and effective relative permeability $\mu_{\rm r}^{\rm eff}$ that determine the propagation constant of a traveling wave in the direction of the array axis perpendicular to the orientations of the crossed electric and magnetic dipoles by which the spheres are modeled.[8] We will now show how $\epsilon_{\rm r}^{\rm eff}$ and $\mu_{\rm r}^{\rm eff}$ can be obtained from the parameters available to us in solving the kd–βd equation (9.84). To begin with, the propagation constant β can be expressed in terms of $\epsilon_{\rm r}^{\rm eff}$ and $\mu_{\rm r}^{\rm eff}$ by the equation

$$\frac{\beta d}{kd} = \sqrt{\mu_{\rm r}^{\rm eff}\ \epsilon_{\rm r}^{\rm eff}}\ . \tag{9.93}$$

For a 3D periodic array of magnetodielectric spheres with $\epsilon_{\rm r} = \mu_{\rm r}$, the effective permittivity and permeability of the array medium are equal and we obtain immediately from (9.93)

$$\epsilon_{\rm r}^{\rm eff} = \mu_{\rm r}^{\rm eff} = \pm\frac{\beta d}{kd} \tag{9.94}$$

where plus (minus) is taken accordingly as the group velocity is positive (negative).

If $\epsilon_{\rm r} \neq \mu_{\rm r}$ we proceed as follows. The magnetic and dielectric properties of a dipolar medium are characterized by a magnetic polarization or magnetization $\mathbf{M}$ and an electric polarization $\mathbf{P}$ where $\mathbf{M}$ ($\mathbf{P}$) is the magnetic (electric) dipole moment per unit volume of the medium. For the array medium with $\beta d << 1$, we can approximate $\mathbf{M}$ and $\mathbf{P}$ by

$$\mathbf{M} = N\mathbf{m} \tag{9.95a}$$

$$\mathbf{P} = N\mathbf{p} \tag{9.95b}$$

where N is the number of magnetic (electric) dipoles per unit volume contributing to $\mathbf{M}$ ($\mathbf{P}$) and $\mathbf{m}$ ($\mathbf{p}$) is the magnetic (electric) moment of each elementary magnetic (electric) dipole. For a $d \times h \times h$ rectangular lattice,

$$N = \frac{1}{dh^2}\ . \tag{9.96}$$

[8]It should be noted that in general the array medium is anisotropic and that $\epsilon_{\rm r}^{\rm eff}$ and $\mu_{\rm r}^{\rm eff}$ do not determine the propagation of waves traveling in directions other than along the array axis. If the array elements are homogeneous magnetodielectric spheres then the directions of the electric and magnetic dipoles are established by the traveling wave, and as the number of spheres per unit volume becomes large ($kd << 1$ as well as $\beta d << 1$) the array medium becomes increasingly isotropic. We have, however, called attention to the fact that our analyses for arrays of magnetodielectric spheres apply equally well to any array elements that can be modeled by a pair of uncoupled crossed electric and magnetic dipoles. If the directions of the electric and magnetic dipoles of the array elements are fixed independently of the traveling wave, as they are for split-ring resonators for example, then the array medium is anisotropic no matter how closely spaced the elements.

Assuming the magnetization and polarization are related to the average magnetic and electric field, **H** and **E**, by the scalar constitutive equations [20, sec. 1.6], we have

$$\mathbf{M} = (\mu_\mathrm{r}^\mathrm{eff} - 1)\mathbf{H} \tag{9.97a}$$

and

$$\mathbf{P} = (\epsilon_\mathrm{r}^\mathrm{eff} - 1)\,\epsilon_0\mathbf{E} \tag{9.97b}$$

where ϵ_0 is the permittivity of free space. (Since the dipoles are excited by a traveling wave along the array axis there will be a phase variation of **m**, **p**, **H**, and **E** in the direction of the array axis. However we assume that the volume we are considering is both large enough to contain a large number of dipoles and narrow enough so that there is only a small phase variation ($\beta d << 1$) of the local incident fields and electric and magnetic dipoles inside the volume. We can then, if we like, regard **m** (**p**) as the average moment of the magnetic (electric) dipoles in the volume and **H** (**E**) as the average magnetic (electric) field.) As we have noted above [see the remark just preceding (9.21)], the symmetry of the array results in the incident magnetic (electric) field at each magnetic (electric) dipole being in the same direction as the dipole in agreement with (9.97) above. In (9.95) **m** (**p**) is in the same direction as **M** (**P**), and in (9.97) **M** (**P**) is in the same or opposite direction as **H** (**E**) accordingly as $\mu_\mathrm{r}^\mathrm{eff} - 1$ ($\epsilon_\mathrm{r}^\mathrm{eff} - 1$) is positive or negative, respectively. If in (9.95a) and (9.97a) ((9.95b) and (9.97b)) **M**, **m**, **H** (**P**, **p**, **E**) are written as M, m, H (P, p, E) respectively, multiplied by unit vectors, then the unit vectors are identical and cancel. We can therefore replace the vector quantities in (9.95) and (9.97) by their respective scalar quantities and obtain

$$\frac{M}{P} = \frac{m}{p} = \frac{(\mu_\mathrm{r}^\mathrm{eff} - 1)}{(\epsilon_\mathrm{r}^\mathrm{eff} - 1)\epsilon_0}\frac{H}{E}\,. \tag{9.98}$$

If we assume that the ratio of the average magnetic and electric fields can be approximated by the ratio of the magnetic and electric fields of the traveling plane wave (the effective admittance of the array medium), then

$$\frac{H}{E} = \pm\sqrt{\frac{\epsilon^\mathrm{eff}}{\mu^\mathrm{eff}}} = \pm\sqrt{\frac{\epsilon_0\epsilon_\mathrm{r}^\mathrm{eff}}{\mu_0\mu_\mathrm{r}^\mathrm{eff}}} \tag{9.99}$$

where μ_0 is the permeability of free space and the plus (minus) sign corresponds to $\mu_\mathrm{r}^\mathrm{eff}$ and $\epsilon_\mathrm{r}^\mathrm{eff}$ both positive (negative) [24]. Hence

$$\frac{m}{p} = \pm\frac{\mu_\mathrm{r}^\mathrm{eff} - 1}{(\epsilon_\mathrm{r}^\mathrm{eff} - 1)\epsilon_0}\sqrt{\frac{\epsilon_0\,\epsilon_\mathrm{r}^\mathrm{eff}}{\mu_0\,\mu_\mathrm{r}^\mathrm{eff}}} = \pm\frac{\mu_\mathrm{r}^\mathrm{eff} - 1}{\epsilon_\mathrm{r}^\mathrm{eff} - 1}\sqrt{\frac{\epsilon_\mathrm{r}^\mathrm{eff}}{\mu_\mathrm{r}^\mathrm{eff}}}\,c\,. \tag{9.100}$$

The ratio of m to p can be related to the parameter q defined by (9.29) which is known as a result of solving the kd–βd equation (9.84). We do this by comparing the magnetic and electric far field multiplied by b_{+0} and b_{-0}, respectively, with the magnetic far field of a magnetic dipole of moment **m** and the electric far field of an electric dipole of moment **p**. The magnetic far field multiplied by b_{+0} is obtained by taking the $1/kr$ term of the field multiplied b_{+n} in (9.2b) and dropping all the subscripts yielding

$$b_{+0}Y_0\frac{\mathrm{e}^{\mathrm{i}kr}}{kr}\hat{\mathbf{r}} \times (\hat{\mathbf{y}} \times \hat{\mathbf{r}}) \tag{9.101a}$$

while the electric far field multiplied by b_{-0} is obtained by taking the $1/kr$ term of the field multiplied b_{-n} in (9.2a) and dropping all the subscripts yielding

$$b_{-0}\frac{\mathrm{e}^{\mathrm{i}kr}}{kr}\hat{\mathbf{r}}\times(\hat{\mathbf{x}}\times\hat{\mathbf{r}})\;. \tag{9.101b}$$

The magnetic far field of a magnetic dipole of magnetic dipole moment $m\,\hat{\mathbf{m}}$ and the electric far field of an electric dipole of electric dipole moment $p\,\hat{\mathbf{p}}$ are given respectively by [25, secs. 9.2, 9.3]

$$m\frac{k^2}{4\pi}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\hat{\mathbf{r}}\times(\hat{\mathbf{m}}\times\hat{\mathbf{r}}) \tag{9.102a}$$

and

$$p\frac{k^2}{4\pi\epsilon_0}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\hat{\mathbf{r}}\times(\hat{\mathbf{p}}\times\hat{\mathbf{r}})\;. \tag{9.102b}$$

Letting $\hat{\mathbf{m}}=\hat{\mathbf{y}}$ and $\hat{\mathbf{p}}=\hat{\mathbf{x}}$ and equating (9.101a) with (9.102a) and (9.101b) with (9.102b) then yields

$$m=\frac{4\pi Y_0}{k^3}\,b_{+0} \tag{9.103a}$$

and

$$p=\frac{4\pi\epsilon_0}{k^3}\,b_{-0}\;. \tag{9.103b}$$

Thus

$$\frac{m}{p}=\frac{Y_0}{\epsilon_0}\frac{b_{+0}}{b_{-0}}=\frac{1}{\epsilon_0}\sqrt{\frac{\epsilon_0}{\mu_0}}q=\frac{q}{\sqrt{\mu_0\epsilon_0}}=c\,q \tag{9.104}$$

where q is the real number defined by (9.29) and obtained from (9.84).

From (9.100) we then have

$$\frac{\mu_\mathrm{r}^\mathrm{eff}-1}{\epsilon_\mathrm{r}^\mathrm{eff}-1}\sqrt{\frac{\epsilon_\mathrm{r}^\mathrm{eff}}{\mu_\mathrm{r}^\mathrm{eff}}}=\pm\,q\;. \tag{9.105}$$

Equations (9.93) and (9.105) form a pair of simultaneous equations which can be solved for the two unknowns $\epsilon_\mathrm{r}^\mathrm{eff}$ and $\mu_\mathrm{r}^\mathrm{eff}$. Letting

$$R=\frac{\beta d}{kd} \tag{9.106}$$

we obtain

$$\epsilon_\mathrm{r}^\mathrm{eff}=\frac{R(R+q)}{1+R\,q} \tag{9.107}$$

and

$$\mu_\mathrm{r}^\mathrm{eff}=\frac{R(1+R\,q)}{R+q}\;. \tag{9.108}$$

These expressions for the effective permittivity and permeability of the array medium are easily computed from the values of R and q that are found from solving the transcendental equation (9.84) for the kd–βd diagram of the array. If there are only electric dipole scattered fields (no magnetic dipole scattered fields) of each array element, $q = 0$ and (9.107)-(9.108) reduce to

$$\epsilon_{\mathrm{r}}^{\mathrm{eff}} = R^2 \tag{9.109}$$

and

$$\mu_{\mathrm{r}}^{\mathrm{eff}} = 1 \; . \tag{9.110}$$

Similarly, with only magnetic dipole scattered fields (no electric dipole scattered fields), $q = \infty$ and (9.107)-(9.108) give

$$\epsilon_{\mathrm{r}}^{\mathrm{eff}} = 1 \tag{9.111}$$

and

$$\mu_{\mathrm{r}}^{\mathrm{eff}} = R^2 \; . \tag{9.112}$$

Although we will not be concerned in this report with the practical details of exciting traveling waves, it is worth noting here that if the relative permittivity and permeability of the magnetodielectric sphere array elements are equal, then the effective relative permittivity and permeability of the array medium are also equal, and, from (9.99), the effective admittance of the array medium equals the admittance of free space. It is therefore likely that it will be easier to excite a lossless traveling wave in a slab of the array medium than it will be if the relative permittivity and permeability of the magnetodielectric sphere array elements differ appreciably.

An alternative, more restrictive and thus less satisfactory method of obtaining expressions for $\epsilon_{\mathrm{r}}^{\mathrm{eff}}$ and $\mu_{\mathrm{r}}^{\mathrm{eff}}$ when both

$$\beta d << 1 \tag{9.113a}$$

and

$$kd << 1 \tag{9.113b}$$

is to make use of the Clausius-Mossotti relation [9, sec. 8-1], [21, sec. 2-4]. This method, unlike the procedure we have described above, makes no use of the solution to the kd–βd equation (9.84). Since the usual form of the Clausius-Mossotti relation is based on the assumption of a cubic lattice, we shall apply it only to arrays for which the transverse element spacing h equals the spacing d in the direction of the array axis. If the inequalities (9.113) are satisfied, then the array can be regarded macroscopically as a medium with effective relative permittivity $\epsilon_{\mathrm{r}}^{\mathrm{eff}}$ and effective relative permeability $\mu_{\mathrm{r}}^{\mathrm{eff}}$ that determine the propagation constant of a traveling wave in the direction of the array axis perpendicular to the orientations of the crossed electric and magnetic dipoles by which the spheres are modeled. We will focus on obtaining $\epsilon_{\mathrm{r}}^{\mathrm{eff}}$ because, as will be seen, an expression for $\mu_{\mathrm{r}}^{\mathrm{eff}}$ can be obtained almost immediately from the expression for $\epsilon_{\mathrm{r}}^{\mathrm{eff}}$.

We consider the electric polarizability $\mathbf{P}$ of a cubic volume of the array medium, taking the axes of the cube to be parallel to the axes of the cubic array lattice. The polarization of the array medium is given from the Clausius-Mossotti relation [9, eq. 8-1] as

$$\mathbf{P} = N\mathbf{p} = 3\,\frac{\epsilon_{\mathrm{r}}^{\mathrm{eff,CM}} - 1}{\epsilon_{\mathrm{r}}^{\mathrm{eff,CM}} + 2}\,\epsilon_0 \mathbf{E_0} \tag{9.114}$$

where $\mathbf{p}$ is the moment of each electric dipole,[9] N is the number of dipoles per unit volume, and $\mathbf{E_0}$ is the local electric field incident on a dipole from all the other dipoles of the array, both electric and magnetic.For a cubic lattice

$$N = \frac{1}{d^3}\,. \tag{9.118}$$

If, similarly to what we did above in obtaining (9.98), $\mathbf{P}$, $\mathbf{p}$, and $\mathbf{E_0}$ in (9.114) are written as P, p, and E_0, respectively, multiplied by unit vectors, then the unit vectors are identical and cancel. We can therefore replace the vector quantities in (9.114) by their respective scalar quantities and obtain

$$\frac{Np}{\epsilon_0 E_0} = 3\,\frac{\epsilon_{\mathrm{r}}^{\mathrm{eff,CM}} - 1}{\epsilon_{\mathrm{r}}^{\mathrm{eff,CM}} + 2}\,. \tag{9.119}$$

Solving (9.119) for $\epsilon_{\mathrm{r}}^{\mathrm{eff,CM}}$ yields

$$\epsilon_{\mathrm{r}}^{\mathrm{eff,CM}} = \frac{2B+3}{3-B} \tag{9.120}$$

with

$$B = \frac{Np}{\epsilon_0 E_0}\,. \tag{9.121}$$

We can find p by equating the expression for the far field radiated by an electric dipole of moment p with the expression for the far field of an electric dipole excited by an incident field E_0 in the direction of the dipole at the center of the dipole. From [20, sec. 8.5 eq. (30)] the far field radiated by an electric dipole of moment p is

$$E_\theta = -\,\frac{k^2 p}{4\pi\epsilon_0}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\sin\theta \tag{9.122}$$

where θ is the spherical polar angle measured from the direction of the dipole. From the scattering equation (9.20a) giving the coefficient of $\exp(\mathrm{i}kr)/(kr)$ in the electric dipole field

[9] An easy way to derive (9.114) for a cubic lattice is to begin with the constitutive relation $\mathbf{D} = \epsilon\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P}$ so that

$$\mathbf{P} = (\epsilon - \epsilon_0)\mathbf{E} \tag{9.115}$$

where $\mathbf{E}$ is the macroscopic electric field satisfying Maxwell's equations, and $\mathbf{P}$ can be expressed as

$$\mathbf{P} = N\mathbf{p}\,. \tag{9.116}$$

To find the local field applied to one dipole from all the other dipoles, remove that one dipole and consider the free-space cubical cavity formed by surfaces that contain the dipoles adjacent to the one removed dipole. Outside this cubical cavity approximate the average polarization density $\mathbf{P}$ as a continuum of polarization density. For this continuum containing the small cubical cavity, the electric field at the center of the cavity (the point where the center of the removed dipole was located) is the local field $\mathbf{E}_0$ given by (in the limit as the maximum breadth of the cavity becomes infinitesimally small—much smaller than a free-space or traveling-wave wavelength will suffice)

$$\mathbf{E}_0 = \mathbf{E} + \overline{\mathbf{L}}\cdot\mathbf{P}/\epsilon_0 \tag{9.117}$$

where $\overline{\mathbf{L}}$ is the depolarization dyadic for the center of the cube [26]. It is given by $\overline{\mathbf{L}} = \overline{\mathbf{I}}/3$, the same value as inside a spherical cavity [26], [27, sec. 3.3.1]. Thus, $\mathbf{E}_0 = \mathbf{E} + \mathbf{P}/(3\epsilon_0)$, which combines with (9.115) and (9.116) to yield (9.114).

scattered from a dipole in response to the incident field E_0 we see that the far scattered field is, referring to (6.2),

$$E_\theta = -\, S_- E_0 \frac{\mathrm{e}^{\mathrm{i}kr}}{kr} \sin\theta \tag{9.123}$$

where, for a magnetodielectric sphere, S_- is the normalized electric dipole scattering coefficient given by [5, eq. (30a)]

$$S_- = -\,\mathrm{i}\,\frac{3}{2} b_1^{\mathrm{sc}} \tag{9.124}$$

with b_1^{sc} the Mie electric dipole scattering coefficient given by [20, sec. 9.25 eq. (11)]. Equating (9.122) and (9.123) then yields

$$p = -\,\frac{6\pi\mathrm{i}\epsilon_0 E_0 b_1^{\mathrm{sc}}}{k^3} \tag{9.125}$$

and hence from (9.121) and (9.118)

$$B = -\,\frac{6\pi\mathrm{i} b_1^{\mathrm{sc}}}{(kd)^3} \tag{9.126}$$

so that the relative permittivity of the array medium is now known from (9.120).

A similar analysis performed for the magnetic dipoles of the array with the magnetization $\mathbf{M}$, the magnetic dipole moment $\mathbf{m}$, and the incident magnetic field $\mathbf{H_0}$, paralleling $\mathbf{P}$, $\mathbf{p}$, and $\epsilon_0\mathbf{E_0}$, respectively, gives us an expression for the relative permeability of the array medium

$$\mu_{\mathrm{r}}^{\mathrm{eff,CM}} = \frac{2A+3}{3-A} \tag{9.127}$$

with

$$A = -\,\frac{6\pi\mathrm{i} a_1^{\mathrm{sc}}}{(kd)^3} \tag{9.128}$$

where a_1^{sc} is the Mie magnetic dipole scattering coefficient given by [20, sec. 9.5 eq. (10)]. The expressions for $\epsilon_{\mathrm{r}}^{\mathrm{eff}}$ and $\mu_{\mathrm{r}}^{\mathrm{eff}}$ that we have obtained by using the Clausius-Mossotti relation can then be used to obtain an approximate kd–βd equation when the inequalities (9.113) are satisfied,

$$\frac{\beta^{\mathrm{CM}} d}{kd} = \sqrt{\epsilon_{\mathrm{r}}^{\mathrm{eff,CM}}\ \mu_{\mathrm{r}}^{\mathrm{eff,CM}}}\ . \tag{9.129}$$

For our arrays of magnetodielectric spheres, it is obvious that the Clausius-Mossotti relations give values of approximate effective permittivity and permeability, and consequently approximate values for the propagation constant β via (9.129) because all three of these quantities have imaginary parts when B and A are inserted from (9.126) and (9.128), whereas the exact values of β, and thus effective values of $\epsilon_{\mathrm{r}}^{\mathrm{eff}}$ and $\mu_{\mathrm{r}}^{\mathrm{eff}}$ in (9.107) and (9.108) are real. Nonetheless, when the inequalities in (9.113) are satisfied, the real parts of the approximate values of β^{CM}, $\epsilon_{\mathrm{r}}^{\mathrm{eff,CM}}$, and $\mu_{\mathrm{r}}^{\mathrm{eff,CM}}$, will agree closely with the values of β, $\epsilon_{\mathrm{r}}^{\mathrm{eff}}$, and $\mu_{\mathrm{r}}^{\mathrm{eff}}$, respectively. In the figures of this report showing the effective constitutive parameters obtained from numerical results, we plot only the real parts of $\epsilon_{\mathrm{r}}^{\mathrm{eff,CM}}$ and $\mu_{\mathrm{r}}^{\mathrm{eff,CM}}$.

At first sight it appears that the derivation of $\epsilon_{\mathrm{r}}^{\mathrm{eff}}$ and $\mu_{\mathrm{r}}^{\mathrm{eff}}$ in (9.107) and (9.108) is valid under the sole condition in (9.92) that $\beta d << 1$, at least for the direction of propagation of the traveling wave (see Footnote 8). However, this condition in (9.92) is not generally sufficient for the effective admittance equation (9.99) to be an accurate expression for the ratio of the average electric and magnetic fields. In fact, numerical calculations of $\epsilon_{\rm r}^{\rm eff}$ and $\mu_{\rm r}^{\rm eff}$ from (9.107) and (9.108) show that the sign of these effective parameters do not always agree with the group velocity obtained from the kd–βd diagram. In general, (9.99) is an accurate expression for H/E only if $kd << 1$ in addition to $\beta d << 1$. Moreover, even for radially symmetric scatterers, like spheres of homogeneous isotropic material, the electric admittance $\sqrt{\epsilon_0\epsilon_{\rm r}^{\rm eff}/(\mu_0\mu_{\rm r}^{\rm eff})}$ will not determine the reflection and transmission properties for a plane wave incident upon a flat interface of the array medium (except possibly for normal incidence) unless $kd << 1$ as well as $\beta d << 1$, that is, if the spatial dispersion is negligible [15, sec. 5.3]. However, if we ignore this problem of unreliable predictions for oblique plane-wave incidence upon an interface of the array medium, there are at least two reasons to prefer the formulas (9.107)-(9.108) for the effective permittivity and permeability of the array medium to the Clausius-Mossotti formulas, (9.120) and (9.127). First, the formulas (9.107)-(9.108) correctly predict that the values of effective permittivity and permeability are real for lossless scatterers, whereas the effective permittivity and permeability in (9.120) and (9.127) generally have imaginary parts as well as real parts even for lossless scatterers. (Of course, these imaginary parts become small for $\beta d << 1$ and $kd << 1$.) Second, in the important case of magnetodielectric spheres made of materials with $\mu_{\rm r} = \epsilon_{\rm r}$, the admittance equation (9.99) holds exactly, the value of q equals ± 1, and

$$\mu_{\rm r}^{\rm eff} = \epsilon_{\rm r}^{\rm eff} = \pm R = \pm\frac{\beta d}{kd} \tag{9.130}$$

with the $+$ or $-$ sign occurring if the group velocity is positive or negative, respectively. If $kd \gtrsim 1$ or $\beta d \gtrsim 1$, the group velocity does not necessarily determine the direction of the energy flow in the traveling wave with respect to the direction of the phase velocity of the traveling wave [15, sec. 5.3].

10 "LONGITUDINAL TRAVELING WAVES" ON 2D AND 3D MAGNETODIELECTRIC SPHERE ARRAYS

Our treatment of 2D and 3D magnetodielectric sphere arrays in Sections 8 and 9 has been based on the assumption that the spheres can be modeled by pairs of electric and magnetic dipoles, each of the dipoles perpendicular to the array axis along which the traveling wave supported by the array propagates. In this brief section we address the propagation of traveling waves with electric or magnetic dipoles parallel to the direction of the array axis. We have already shown in Sections 6 and 7 that if the traveling wave excites the electric dipoles of the array in the direction parallel to the array axis, then the scattered electric field incident on any element of the array has a component only in the direction of the array axis. In these two previous sections we did not consider the scattered magnetic field because we were concerned with arrays composed of short wires. Here we will show that if the traveling wave excites the electric dipoles of the array in the direction parallel to the array axis, then the scattered magnetic field incident on any element of the array is zero. Hence there is no coupling of the longitudinal electric field excited by the traveling wave with either a transverse electric field or with any magnetic field. Obviously then, there is no coupling of the longitudinal magnetic field excited by a traveling wave with either a transverse magnetic field or with any electric field. Our treatment of longitudinal waves on 2D and 3D magnetodielectric sphere arrays thus reduces to our treatment in Sections 6 and 7 of 2D and 3D arrays of electric dipoles parallel to the array axis. The only difference to be noted is that if (6.79) or (7.76) is used to obtain the kd–βd curves for a 2D or 3D array, respectively, of longitudinally directed magnetic dipoles, then the phase ψ of the scattering coefficient S must be obtained from the Mie magnetic dipole scattering coefficient for the spheres being considered rather than from the Mie electric dipole scattering coefficient.

We now proceed to show that if the traveling wave excites the electric dipoles of a 3D array of magnetodielectric spheres in the direction of propagation parallel to the array axis, then the scattered magnetic field incident on any element of the array is zero. To do this we note first that the magnetic field of a short electric dipole located at the origin of a Cartesian coordinate system with the z axis in the direction of the dipole is [by taking the curl of (7.2)]

$$\mathbf{H}(\mathbf{r}) = -\frac{1}{\mathrm{i}\omega\mu}\frac{\mathrm{e}^{\mathrm{i}kr}}{kr}\left(\mathrm{i}k - \frac{1}{r}\right)\sin\theta\,\hat{\boldsymbol{\phi}} \tag{10.1}$$

The magnetic field $\mathbf{H}_0^{0mln}$ incident on the reference sphere at the origin from the electric dipole at the location $(x, y, z) = (mh, lh, nd)$ is then

$$\mathbf{H}_0^{0mln} = -\frac{b_n}{\mathrm{i}\omega\mu}\frac{\mathrm{e}^{\mathrm{i}kr_{mln0}}}{kr_{mln0}}\left(ik - \frac{1}{r_{mln0}}\right)\sin\theta_{mln0}\,\hat{\boldsymbol{\phi}}_{mln0} \tag{10.2}$$

where from (7.4) and (7.7)

$$r_{mln0} = h\sqrt{m^2 + l^2 + (nd/h)^2} \tag{10.3}$$

and

$$\sin\theta_{mln0} = \frac{h\sqrt{m^2 + l^2}}{r_{mln0}}\ . \tag{10.4}$$

Also

$$\hat{\boldsymbol{\phi}}_{mln0} = -\sin\phi_{mln0}\ \hat{\mathbf{x}} + \cos_{mln0}\ \hat{\mathbf{y}} \tag{10.5}$$

where from (7.9) and (7.10)

$$\cos\phi_{mln0} = \frac{-m}{\sqrt{m^2 + l^2}} \tag{10.6}$$

$$\sin\phi_{mln0} = \frac{-l}{\sqrt{m^2 + l^2}}\ . \tag{10.7}$$

But then when $\mathbf{H}_0^{0mln}$ is summed over m and l from $-\infty$ to ∞, both the x and y components vanish because they are odd functions of m and l. Thus there is no coupling between the electric field scattered from the z-directed electric dipoles and a magnetic field. For a 2D array, $\hat{\boldsymbol{\phi}}$ in (10.1) is either $\hat{\mathbf{x}}$ or $\hat{\mathbf{y}}$, one of the two subscripts m or l is dropped, and the same conclusion holds.

Lastly, we note that for 1D arrays, an electric (magnetic) dipole parallel to the array axis has no magnetic (electric) field along the array axis and thus the traveling waves of longitudinal electric and magnetic dipoles are also uncoupled on 1D arrays. The kd–βd equation for either a 1D array of electric or magnetic longitudinal dipoles is given in both [4] and [6].

11 PARTIALLY FINITE 3D ARRAYS OF ACOUSTIC MONOPOLES, ELECTRIC DIPOLES, AND MAGNETODIELECTRIC SPHERES

In this section we show that the analyses we have performed in Sections 3, 5, and 9 to obtain the kd–βd equations for infinite periodic 3D arrays of acoustic monopoles, electric dipoles oriented perpendicular to the array axis, and magnetodielectric spheres with electric and magnetic dipoles oriented perpendicular to the array axis, can be used to obtain expressions for the fields of partially finite periodic arrays of these elements (arrays that are finite in the direction of the array axis and of infinite extent in the directions transverse to the array axis). The arrays are illuminated by a plane wave propagating in a direction parallel to the array axis; that is, with the propagation vector of the plane wave normal to the interface between free space and the array. The procedure followed for these three arrays is identical and uses a method due to Foldy [28].

11.1 PARTIALLY FINITE 3D ACOUSTIC MONOPOLE ARRAY

We investigate the field excited by a plane wave incident from free space on a 3D periodic array of lossless acoustic monopoles. The array is finite in the direction of the array axis and infinite in the directions transverse to the array axis. The direction of incidence of the illuminating plane wave is parallel to the array axis, normal to the interface between the array and free space. The z axis of a Cartesian coordinate system is taken to be the array axis and N+1 equispaced planes of acoustic monopoles parallel to the xy plane are located at $z = nd,\ n = 0,\ 1,\ 2,\ \cdots,\ N$. In each plane the monopoles are located at $(x, y) = (mh, lh),\ m, l = 0,\ \pm 1,\ \pm 2,\ \cdots$. The incident plane wave is

$$p_{\text{inc}}(z) = \mathrm{e}^{\mathrm{i}kz} \tag{11.1}$$

so that all monopoles in any plane of the array are excited identically. From Section 3 the field at a point on the array axis (not coinciding with an array element) is

$$p(z) = \mathrm{e}^{\mathrm{i}kz} + \sum_{n=0}^{N} b_n \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr_{mlnz}}}{kr_{mlnz}} \tag{11.2}$$

where

$$r_{mlnz} = \sqrt{(m^2 + l^2)h^2 + (nd - z)^2}\ . \tag{11.3}$$

That is, the total field is equal to the incident field plus the sum of the waves scattered from all the elements of the array. The coefficients b_n of the scattered waves are given by

$$b_n = Sp^n(z_n) \tag{11.4}$$

where S is the scattering coefficient of the monopoles, $p^n(z_n)$ is the external field incident on an element in the nth plane, and $z_n = nd$, $n = 0, 1, 2, \cdots, N$, so that from (11.2) and (11.4) the total field at a point on the array axis other than at an array element is given by

$$p(z) = \mathrm{e}^{\mathrm{i}kz} + \sum_{n=0}^{N} S\, p^n(z_n) \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr_{mlnz}}}{kr_{mlnz}} \,. \tag{11.5}$$

Since all the monopoles in any plane of the array normal to the array axis are excited identically, $p^n(z_n)$ is equal to the external field incident on the monopole on the z axis at $z = z_n = nd$. An expression for $p^n(z_n)$ is therefore obtained by summing the contribution of the incident plane wave, $\mathrm{e}^{\mathrm{i}kz_n}$, and the contribution of the fields scattered from all the array elements other than the $(0, 0, z_n)$ element

$$p^n(z_n) = \mathrm{e}^{\mathrm{i}kz_n} + \sum_{\substack{j=0 \\ j\neq n}}^{N} S\, p^j(z_j) \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr_{mljn}}}{kr_{mljn}} + S\, p^n(z_n) \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \;\neq\; \sum_{\substack{l=-\infty \\ (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr_{ml}}}{kr_{ml}} \tag{11.6}$$

where

$$r_{mljn} = r_{ml|j-n|} = \sqrt{(m^2 + l^2)h^2 + [(j-n)d]^2} \tag{11.7}$$

and

$$r_{ml} = \sqrt{(m^2 + l^2)h^2} \,. \tag{11.8}$$

The triple sum in (11.6) is the sum of the scattered waves incident on the $(0, 0, nd)$ element from the elements in all the planes of the array except the nth plane, and the double sum is the sum of the scattered waves incident on the $(0, 0, nd)$ monopole from the monopoles of the nth plane except for the element at $(0, 0, nd)$. (The reason for the notation $p^n(z_n)$ is that (11.6) can be extended to be a function of z, $p^n(z)$, at any point z on the array axis not coincident with an array element simply by replacing nd in (11.7) by z.)

Let

$$\sigma_1(|j-n|d) = \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr_{mljn}}}{kr_{mljn}} \tag{11.9}$$

and

$$\sigma_2 = \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \;\neq\; \sum_{\substack{l=-\infty \\ (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr_{ml}}}{kr_{ml}} \,. \tag{11.10}$$

Then (11.6) can be written as

$$p^n(z_n) = \mathrm{e}^{\mathrm{i}kz_n} + \sum_{\substack{j=0 \\ j\neq n}}^{N} S\, p^j(z_j)\, \sigma_1(|j-n|d) + S\, p^n(z_n)\, \sigma_2 \tag{11.11}$$

and thus we have a system of $N+1$ equations for the $N+1$ unknowns $p^n(z_n)$, $n = 0, 1, 2, \cdots, N$

$$(1 - S\sigma_2)\, p^n(z_n) - S \sum_{\substack{j=0 \\ j\neq n}}^{N} \sigma_1(|j-n|d)\, p^j(z_j) = \mathrm{e}^{\mathrm{i}kz_n},\ n = 0, 1, \cdots,\ N\ . \tag{11.12}$$

From our treatment of the infinite periodic array of acoustic monopoles in Section 3, rapidly convergent expressions are available for σ_1 and σ_2. From (3.11), (3.16), and (3.17) we have

$$(kh)\sigma_1(|n|d) = \frac{2\pi \mathrm{i}}{kh}\mathrm{e}^{\mathrm{i}|n|kd} + 2\pi \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \ \neq \ \sum_{\substack{l=-\infty \\ (0,0)}}^{\infty} \frac{\mathrm{e}^{-|n|(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}} \tag{11.13}$$

and from (3.45), (3.46), and (3.49)

$$(kh)\sigma_2 = 2\sum_{l=1}^{\infty}\left[\mathrm{i}\pi H_0^{(1)}(lkh) + 4\sum_{m=1}^{\infty} K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\right)\right]$$

$$-2\ln\left[2\sin\left(\frac{kh}{2}\right)\right] + \mathrm{i}(\pi - kh),\ 0 < kh < 2\pi. \tag{11.14}$$

Using (B.11) and (B.12)

$$(kh)\sigma_2 = 2\pi \mathrm{i}\left\{\left(-\frac{1}{2} + \frac{1}{kh}\right) +\ \mathrm{i}\left(-\frac{1}{\pi}\left(\gamma + \ln\frac{kh}{4\pi}\right) - 2\sum_{l=1}^{\infty}\left[\frac{1}{\sqrt{(2\pi l)^2-(kh)^2}} - \frac{1}{2\pi l}\right]\right)\right\}$$

$$+\, 8\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\right) - 2\ln\left[2\sin\left(\frac{kh}{2}\right)\right] + \mathrm{i}(\pi - kh)\ ,\ 0 < kh < 2\pi\ . \tag{11.15}$$

It is then straightforward to write a computer program to solve the system of equations (11.12) for the values of $p^n(z_n)$. To calculate the total field at any point on the array axis (other then at elements of the array) we then use (11.2) which can be written as

$$p(z) = \mathrm{e}^{\mathrm{i}kz} + \sum_{n=0}^{N} S\, p^n(z_n)\, \sigma_1(|nd - z|),\ z \neq nd \tag{11.16}$$

with $\sigma_1(|nd - z|)$ given by (11.9) with j and n replaced by n and z/d, respectively. The rapidly convergent expression (11.13) is used for calculating $\sigma_1(|z - nd|)$ with n replaced by $n - z/d$.

Given values of $p(z)$ we can calculate the reflection coefficient of the wave scattered back in the negative z direction for $z < 0$ as well as the transmission coefficient of the wave traveling in the positive z direction for $z > Nd$. Since the amplitude of the plane wave incident on the partially finite array is 1, the reflection coefficient, R, is the complex coefficient of the wave $\mathrm{e}^{-\mathrm{i}kz}$ for $z < 0$ with R obtained from the equation

$$R = p(z)\ \mathrm{e}^{\mathrm{i}kz},\ z < 0\ . \tag{11.17}$$

In practice z should not be chosen too close to the origin in order for the transient scattered waves to die out, say $z/d < -10$. The square of the magnitude of the transmission coefficient, $|T|^2$, is, of course, equal to $1 - |R|^2$, since the array is assumed to be lossless. If the complex transmission coefficient is desired, it can be obtained from the equation

$$T = [p(z) + \mathrm{e}^{\mathrm{i}kz}]\, \mathrm{e}^{-\mathrm{i}kz},\ z > Nd. \tag{11.18}$$

In practice, z should not be chosen too close to the end of the array at $z = Nd$, say $z/d > N + 10$.

11.2 PARTIALLY FINITE 3D ARRAY OF ELECTRIC DIPOLES

We investigate the field excited by a plane wave incident from free space on a 3D periodic array of lossless short electric dipoles. The array is finite in the direction of the array axis and infinite in the directions transverse to the array axis. The direction of incidence of the illuminating plane wave is parallel to the array axis, normal to the interface between the array and free space. The x axis of a Cartesian coordinate system is taken to be the array axis and N+1 equispaced planes parallel to the yz plane of z-directed electric dipoles are located at $x = nd,\ n = 0,\ 1,\ 2,\ \cdots,\ N$. In each plane the dipoles are centered at $(y, z) = (lh, mh),\ l, m = 0,\ \pm 1,\ \pm 2,\ \cdots$. Our procedure parallels that used in our treatment of a finite 3D array of acoustic monopoles. The electric field vector of the incident plane wave illuminating the array from the left is

$$E_{z,\mathrm{inc}}(x) = \mathrm{e}^{\mathrm{i}kx} \tag{11.19}$$

so that all dipoles in any plane of the array are excited identically. As shown in Section 5, the electric field at a point on the array axis has a z-component only. Hence, at a point on the array axis not coinciding with an array element, the total electric field is given by [see (5.20)]

$$\begin{aligned} E_z(x) = \mathrm{e}^{\mathrm{i}kx} + \frac{1}{(kh)^3} \sum_{n=0}^{N} b_n \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{lmnx}}}{\rho_{lmnx}} \Bigg[\frac{-2\mathrm{i}}{\rho_{lmnx}} \left(kh + \frac{\mathrm{i}}{\rho_{lmnx}} \right) \frac{m^2}{\rho_{lmnx}^2} \\ + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{lmnz}} - \frac{1}{\rho_{lmnx}^2} \right) \frac{l^2 + [(nd - x)/h]^2}{\rho_{lmnx}^2} \Bigg] \end{aligned} \tag{11.20}$$

where

$$\rho_{lmnx} = \sqrt{l^2 + m^2 + [(nd - x)/h]^2}\ . \tag{11.21}$$

That is, the total field is equal to the incident field plus the sum of the waves scattered from all the elements of the array. The coefficients b_n of the scattered waves are given by

$$b_n = SE_z^n(x_n) \tag{11.22}$$

where S is the scattering coefficient of the dipoles, $E_z^n(x_n)$ is the external electric field incident on an element in the nth plane, and $x_n = nd, n = 0, 1, 2, \cdots, N$, so that from (11.20) and

(11.22) the total electric field at a point on the array axis not coinciding with an array element is

$$E_z(x) = \mathrm{e}^{\mathrm{i}kx} + \frac{1}{(kh)^3}\left\{ \sum_{n=0}^{N} S\, E_z^n(x_n) \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{lmnz}}}{\rho_{lmnx}} \left[\frac{-2\mathrm{i}}{\rho_{lmnx}} \left(kh + \frac{\mathrm{i}}{\rho_{lmnx}} \right) \frac{m^2}{\rho_{lmnz}^2} \right.\right.$$

$$\left.\left. + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{lmnz}} - \frac{1}{\rho_{lmnx}^2} \right) \frac{l^2 + [(nd-x)/h]^2}{\rho_{lmnx}^2} \right] \right\}. \tag{11.23}$$

Since all the dipoles in any plane of the array of the array normal to the array axis are excited identically, $E_z^n(x_n)$ is equal to the external field incident on the dipole on the x axis at $x = x_n = nd$. An expression for $E_z^n(x_n)$ is therefore obtained by summing the contribution of the incident plane wave, $\mathrm{e}^{\mathrm{i}kx_n}$, and the contribution of the fields scattered from all the array elements other than the $(x_n, 0, 0)$ dipole

$$E_z^n(x_n) = \mathrm{e}^{\mathrm{i}kx_n} + \frac{1}{(kh)^3}\left\{ \sum_{\substack{j=0\\ j\neq n}}^{N} S\, E_z^j(x_j) \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{lmjn}}}{\rho_{lmjn}} \left[\frac{-2\mathrm{i}}{\rho_{lmjn}} \left(kh + \frac{\mathrm{i}}{\rho_{lmjn}} \right) \frac{m^2}{\rho_{lmjn}^2} \right.\right.$$

$$\left. + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{lmjn}} - \frac{1}{\rho_{lmjn}^2} \right) \frac{l^2 + [(j-n)d/h)]^2}{\rho_{lmjn}^2} \right]$$

$$\left. + S\, E_z^n(x_n) \sum_{\substack{l=-\infty\\ (l,m)}}^{\infty} \; \neq \; \sum_{\substack{m=-\infty\\ (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{lm}}}{\rho_{lm}} \left[\frac{-2\mathrm{i}}{\rho_{lm}} \left(kh + \frac{\mathrm{i}}{\rho_{lm}} \right) \frac{m^2}{\rho_{lm}^2} + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{lm}} - \frac{1}{\rho_{lm}^2} \right) \frac{l^2}{\rho_{lm}^2} \right] \right\} \tag{11.24}$$

where

$$\rho_{lmjn} = \rho_{lm|j-n|} = \sqrt{l^2 + m^2 + [(j-n)d/h]^2} \tag{11.25}$$

and

$$\rho_{lm} = \sqrt{l^2 + m^2}\,. \tag{11.26}$$

The triple sum in (11.24) is the sum of the scattered waves from the dipoles in all the planes of the array except the nth plane, and the double sum is the sum of the scattered waves from the dipoles of the nth plane except for the element at $(nd, 0, 0)$. (The reason for the notation $E_z^n(x_n)$ is that (11.24) can be extended to be a function of x, $E_z^n(x)$, at any point x on the array axis not coincident with an array element simply by replacing nd in (11.25) by x.)

Let

$$\sigma_1(|j-n|d) = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{lmjn}}}{\rho_{lmjn}} \left[\frac{-2\mathrm{i}}{\rho_{lmjn}} \left(kh + \frac{\mathrm{i}}{\rho_{lmjn}} \right) \frac{m^2}{\rho_{lmjn}^2} \right.$$

$$\left. + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{lmjn}} - \frac{1}{\rho_{lmjn}^2} \right) \frac{(l^2 + [(j-n)d/h]^2}{\rho_{lmjn}^2} \right] \tag{11.27}$$

and

$$\sigma_2 = \sum_{\substack{l=-\infty \\ (l,m)}}^{\infty} \ \neq \ \sum_{\substack{m=-\infty \\ (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{lm}}}{\rho_{lm}} \left[\frac{-2\mathrm{i}}{\rho_{lm}} \left(kh + \frac{\mathrm{i}}{\rho_{lm}} \right) \frac{m^2}{\rho_{lm}^2} + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{lm}} - \frac{1}{\rho_{lm}^2} \right) \frac{l^2}{\rho_{lm}^2} \right] . \quad (11.28)$$

Then (11.24) can be written as

$$E_z^n(x_n) = \mathrm{e}^{\mathrm{i}kx_n} + \frac{1}{(kh)^3} \left[\sum_{\substack{j=0 \\ j\neq n}}^{N} SE_z^j(x_j)\ \sigma_1(|j-n|d) + SE_z^n(x_n)\ \sigma_2 \right] \quad (11.29)$$

and thus we have a system of $N+1$ equations for the $N+1$ unknowns $E_z^n(x_n)$, $n = 0, 1, 2, \cdots, N$

$$\left[1 - S\sigma_2/(kh)^3\right]\ E_z^n(x_n) - \frac{S}{(kh)^3} \sum_{\substack{j=0 \\ j\neq n}}^{N} \sigma_1(|j-n|d)\ E_z^j(x_j) = \mathrm{e}^{\mathrm{i}kx_n},\ n = 0, 1, \cdots, N\ . \quad (11.30)$$

From Section 5 rapidly convergent expressions are available for σ_1 and σ_2. From (5.64) we have

$$\sigma_1(|n|d) = 2\pi\mathrm{i}\ kh\ \mathrm{e}^{\mathrm{i}|n|kd}$$
$$- 2\pi \sum_{\substack{l=-\infty \\ (l,m)}}^{\infty} \ \neq \ \sum_{\substack{m=-\infty \\ (0,0)}}^{\infty} \left[(2\pi m)^2 - (kh)^2\right] \frac{\mathrm{e}^{-|n|(d/h)\ \sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}} \quad (11.31)$$

and from (5.68)

$$\sigma_2 = 2\sum_{l=1}^{\infty} \left[\mathrm{i}\pi(kh)^2 H_0^{(1)}(lkh) - \ 4 \sum_{m=1}^{\infty} \left[(2\pi m)^2 - (kh)^2\right] K_0\left(l\sqrt{(2\pi m)^2 - (kh)^2}\ \right) \right]$$
$$+ 4\ kh\ \mathrm{Cl}_2(kh) + 4\ \mathrm{Cl}_3(kh) + \mathrm{i}\pi(kh)^2 - \mathrm{i}\frac{2}{3}(kh)^3,\ 0 < kh < 2\pi \quad (11.32)$$

with the Clausen functions Cl_2 and Cl_3 defined and approximated by equations (D.8). Using (B.11) and (B.12)

$$\sigma_2 = 2\pi\mathrm{i}(kh)^2 \left\{ \left(-\frac{1}{2} + \frac{1}{kh} \right) + \mathrm{i} \left(-\frac{1}{\pi} \left(\gamma + \ln\frac{kh}{4\pi} \right) - 2\sum_{l=1}^{\infty} \left[\frac{1}{\sqrt{(2\pi l)^2 - (kh)^2}} - \frac{1}{2\pi l} \right] \right) \right\}$$
$$- 8\ \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \left[(2\pi m)^2 - (kh)^2\right] K_0\left(l\sqrt{(2\pi m)^2 - (kh)^2}\ \right)$$
$$+ 4\ kh\ \mathrm{Cl}_2(kh) + 4\ \mathrm{Cl}_3(kh) + \mathrm{i}\pi(kh)^2 - \mathrm{i}\frac{2}{3}(kh)^3,\ 0 < kh < 2\pi\ . \quad (11.33)$$

It is then straightforward to write a computer program to solve the system of equations (11.30) for the values of $E_z^n(z_n)$. To calculate the field at any point on the array axis (other than at elements of the array) we then use (11.23) which can be written as

$$E_z(x) = \mathrm{e}^{\mathrm{i}kx} + \frac{1}{(kh)^3} \sum_{n=0}^{N} S\ E_z^n(x_n)\ \sigma_1(|nd - x|),\ x \neq nd \tag{11.34}$$

with $\sigma_1(|nd - x|)$ given by (11.27) with j and n replaced by n and x/d, respectively. The rapidly convergent expression (11.31) is used for calculating $\sigma_1(|nd - x|)$ with n replaced by $n - x/d$.

Given values of $E_z(x)$ we can calculate the reflection coefficient of the wave scattered back in the negative x direction for $x < 0$ as well as the transmission coefficient of the wave traveling in the positive x direction for $x > Nd$. Since the amplitude of the plane wave incident on the partially finite array is 1, the reflection coefficient, R, is the complex coefficient of the wave $\mathrm{e}^{-\mathrm{i}kx}$ for $x < 0$ with R obtained from the equation

$$R = E_z(x)\ \mathrm{e}^{\mathrm{i}kx},\ x < 0\ . \tag{11.35}$$

In practice x should not be chosen too close to the origin in order for the transient scattered waves to die out, say $x/d < -10$. The square of the magnitude of the transmission coefficient, $|T|^2$, is, of course, equal to $1 - |R|^2$, since the array is assumed to be lossless. If the complex transmission coefficient is desired, it can be obtained from the equation

$$T = [E_z(x) + \mathrm{e}^{\mathrm{i}kx}]\ \mathrm{e}^{-\mathrm{i}kx},\ x > Nd. \tag{11.36}$$

In practice, x should not be chosen too close to the end of the array at $x = Nd$, say $x/d > N + 10$.

11.3 PARTIALLY FINITE 3D ARRAY OF MAGNETODIELECTRIC SPHERES

We investigate the field excited by a plane wave incident from free space on a 3D periodic array of lossless magnetodielectric spheres. The array is finite in the direction of the array axis and infinite in the directions transverse to the array axis. The direction of incidence of the illuminating plane wave is parallel to the array axis, normal to the interface between the array and free space. As in Section 9 it is assumed that the spheres can be modeled by pairs of crossed electric and magnetic dipoles, each of the dipoles perpendicular to the array axis. The z axis of a Cartesian coordinate system is taken to be the array axis and N+1 equispaced planes parallel to the xy plane of magnetodielectric spheres are located at $x = nd,\ n = 0,\ 1,\ 2,\ \cdots,\ N$. In each plane the spheres are centered at $(x, y) = (mh, lh),\ l, m = 0,\ \pm 1,\ \pm 2,\ \cdots$ with the electric and magnetic dipoles oriented in the x and y direction, respectively. Our procedure is parallel to that used in our treatment of finite 3D acoustic monopole and electric dipole arrays. The electric and magnetic field vectors of the incident plane wave illuminating the array from the left are

$$E_{x,\mathrm{inc}}(z) = \mathrm{e}^{\mathrm{i}kz} \tag{11.37a}$$

$$H_{y,\mathrm{inc}}(z)/Y_0 = \mathrm{e}^{\mathrm{i}kz} \tag{11.37b}$$

so that all spheres in any plane of the array are excited identically. As shown in Section 9, the electric field at a point on the array axis has an x-component only, and the magnetic field has a y-component only. Hence, at a point on the array axis not coinciding with an array element, the electric field is given by [see (9.21)]

$$\begin{aligned} E_x(z) = \mathrm{e}^{\mathrm{i}kz} + \frac{1}{(kh)^3}\Bigg\{ &\sum_{n=0}^{N} b_{-n} \sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mlnz}}}{\rho_{mlnz}} \Bigg[\frac{-2\mathrm{i}}{\rho_{mlnz}} \left(kh + \frac{\mathrm{i}}{\rho_{mlnz}} \right) \frac{m^2}{\rho^2_{mlnz}} \\ &+ \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mlnz}} - \frac{1}{\rho^2_{mlnz}} \right) \frac{l^2 + [(nd-z)/h]^2}{\rho^2_{mlnz}} \Bigg] \\ &- \sum_{n=0}^{N} b_{+n} \sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mlnz}}}{\rho_{mlnz}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mlnz}} \right) \frac{(nd-z)/h}{\rho_{mlnz}} \Bigg\} \end{aligned} \tag{11.38a}$$

and

$$\begin{aligned} \frac{H_y(z)}{Y_0} = \mathrm{e}^{\mathrm{i}kz} + \frac{1}{(kh)^3}\Bigg\{ &\sum_{n=0}^{N} b_{+n} \sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mlnz}}}{\rho_{mlnz}} \Bigg[\frac{-2\mathrm{i}}{\rho_{mlnz}} \left(kh + \frac{\mathrm{i}}{\rho_{mlnz}} \right) \frac{l^2}{\rho^2_{mlnz}} \\ &+ \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mlnz}} - \frac{1}{\rho^2_{mlnz}} \right) \frac{m^2 + [(nd-z)/h]^2}{\rho^2_{mlnz}} \Bigg] \\ &- \sum_{n=0}^{N} b_{-n} \sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mlnz}}}{\rho_{mlnz}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mlnz}} \right) \frac{(nd-z)/h}{\rho_{mlnz}} \Bigg\} \end{aligned} \tag{11.38b}$$

where

$$\rho_{mlnz} = \sqrt{m^2 + l^2 + [(nd-z)/h]^2}\ . \tag{11.39}$$

That is, the total field is equal to the incident field plus the sum of the waves scattered from all the elements of the array. The coefficients b_{-n} and b_{+n} of the scattered waves are given by

$$b_{-n} = S_- E_x^n(z_n) \tag{11.40a}$$

and

$$b_{+n} = S_+ H_y^n(z_n)/Y_0 \tag{11.40b}$$

where S_- and S_+ are the normalized magnetodielectric sphere electric and magnetic dipole scattering coefficients, respectively, $E_x^n(z_n)$ and $H_y^n(z_n)$ are the external electric and magnetic fields, respectively, incident on a sphere in the nth plane, and $z_n = nd, n = 0, 1, 2, \cdots, N$, so that from (11.38) and (11.40) the total electric field at a point on the array axis not coinciding with an array element is

$$E_x(z) = \mathrm{e}^{\mathrm{i}kz} + \frac{1}{(kh)^3}\Bigg\{ \sum_{n=0}^{N} S_- E_x^n(z_n) \sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mlnz}}}{\rho_{mlnz}} \Bigg[\frac{-2\mathrm{i}}{\rho_{mlnz}} \left(kh + \frac{\mathrm{i}}{\rho_{mlnz}} \right) \frac{m^2}{\rho^2_{mlnz}}$$

$$+\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mlnz}}-\frac{1}{\rho_{mlnz}^2}\right)\frac{l^2+[(nd-z)/h]^2}{\rho_{mlnz}^2}\Bigg]$$

$$-\sum_{n=0}^{N}S_+H_y^n(z_n)/Y_0\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mlnz}}}{\rho_{mlnz}}\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mlnz}}\right)\frac{(nd-z)/h}{\rho_{mlnz}}\Bigg\} \tag{11.41a}$$

and the total magnetic field at a point on the array axis not coinciding with an array element is

$$\frac{H_y(z)}{Y_0}=\mathrm{e}^{\mathrm{i}kz}+\frac{1}{(kh)^3}\Bigg\{\sum_{n=0}^{N}S_+H_y^n(z_n)/Y_0\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mlnz}}}{\rho_{mlnz}}\Bigg[\frac{-2\mathrm{i}}{\rho_{mlnz}}\left(kh+\frac{\mathrm{i}}{\rho_{mlnz}}\right)\frac{l^2}{\rho_{mlnz}^2}$$

$$+\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mlnz}}-\frac{1}{\rho_{mlnz}^2}\right)\frac{m^2+[(nd-z)/h]^2}{\rho_{mlnz}^2}\Bigg]$$

$$-\sum_{n=0}^{N}S_-E_x^n(z_n)\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mlnz}}}{\rho_{mlnz}}\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mlnz}}\right)\frac{(nd-z)/h}{\rho_{mlnz}}\Bigg\}. \tag{11.41b}$$

Since all the spheres in any plane of the array normal to the array axis are excited identically, $E_x^n(z_n)$ and $H_y^n(z_n)/Y_0$ are equal, respectively, to the external electric and magnetic fields incident on the sphere on the z axis at $z=z_n=nd$. Expressions for $E_x^n(z_n)$ and $H_y^n(z_n)/Y_0$ are therefore obtained by summing the contribution of the incident plane wave $\mathrm{e}^{\mathrm{i}kz_n}$ and the contribution of the fields scattered from all the array elements other than the $(0,0,z_n)$ sphere

$$E_x^n(z_n)=\mathrm{e}^{\mathrm{i}kz_n}+\frac{1}{(kh)^3}\Bigg\{\sum_{\substack{j=0\\j\neq n}}^{N}S_-E_x^j(z_j)\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mljn}}}{\rho_{mljn}}\Bigg[\frac{-2\mathrm{i}}{\rho_{mljn}}\left(kh+\frac{\mathrm{i}}{\rho_{mljn}}\right)\frac{m^2}{\rho_{mljn}^2}$$

$$+\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mljn}}-\frac{1}{\rho_{mljn}^2}\right)\frac{l^2+[(j-n)d/h]^2}{\rho_{mljn}^2}\Bigg]$$

$$-\sum_{\substack{j=0\\j\neq n}}^{N}S_+H_y^j(z_j)/Y_0\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mljn}}}{\rho_{mljn}}\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{mljn}}\right)\frac{(j-n)d/h}{\rho_{mljn}}$$

$$+S_-E_x^n(z_n)\sum_{\substack{m=-\infty\\(m,l)}}^{\infty}\sum_{\substack{l=-\infty\\\neq(0,0)}}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{ml}}}{\rho_{ml}}\Bigg[\frac{-2\mathrm{i}}{\rho_{ml}}\left(kh+\frac{\mathrm{i}}{\rho_{ml}}\right)\frac{m^2}{\rho_{ml}^2}$$

$$+\left((kh)^2+\frac{\mathrm{i}kh}{\rho_{ml}}-\frac{1}{\rho_{ml}^2}\right)\frac{l^2}{\rho_{ml}^2}\Bigg]\Bigg\} \tag{11.42a}$$

and

$$\frac{H_y^n(z_n)}{Y_0}=\mathrm{e}^{\mathrm{i}kz_n}+\frac{1}{(kh)^3}\Bigg\{\sum_{\substack{j=0\\j\neq n}}^{N}S_+H_y^j(z_j)/Y_0\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kh\rho_{mljn}}}{\rho_{mljn}}\Bigg[\frac{-2\mathrm{i}}{\rho_{mljn}}\left(kh+\frac{\mathrm{i}}{\rho_{mljn}}\right)\frac{l^2}{\rho_{mljn}^2}$$

$$
+ \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mljn}} - \frac{1}{\rho_{mljn}^2}\right) \frac{m^2 + [(j-n)d/h]^2}{\rho_{mljn}^2}\Bigg]
$$

$$
- \sum_{\substack{j=0 \\ j\neq n}}^{N} S_- E_x^j(z_j) \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mljn}}}{\rho_{mljn}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mljn}}\right) \frac{(j-n)d/h}{\rho_{mljn}}
$$

$$
+ S_+ H_y^n(z_n)/Y_0 \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \ \neq \sum_{\substack{l=-\infty \\ (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{ml}}}{\rho_{ml}} \left[\frac{-2\mathrm{i}}{\rho_{ml}} \left(kh + \frac{\mathrm{i}}{\rho_{ml}}\right) \frac{l^2}{\rho_{ml}^2}\right.
$$

$$
\left. + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{ml}} - \frac{1}{\rho_{ml}^2}\right) \frac{m^2}{\rho_{ml}^2}\right]\Bigg\} \tag{11.42b}
$$

where

$$
\rho_{mljn} = \rho_{ml|j-n|} = \sqrt{m^2 + l^2 + [(j-n)d/h]^2} \tag{11.43}
$$

and

$$
\rho_{ml} = \sqrt{m^2 + l^2}\ . \tag{11.44}
$$

The triple sums in (11.42) are the sums of the scattered waves from the electric and magnetic dipoles in all the planes of the array except the nth plane, and the double sums are the sums of the scattered waves from the dipoles of the nth plane except for the element at $(0, 0, nd)$. (The reason for the notation $E_x^n(z_n)$ and $H_y^n(z_n)$ is that (11.42) can be extended to be functions of z, $E_x^n(z)$ and $H_y^n(z_n)$, at any point z on the array axis not coincident with an array element simply by replacing nd in (11.43) by z.)

Let

$$
\sigma_{11}(|j-n|d) = \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mljn}}}{\rho_{mljn}} \left[\frac{-2\mathrm{i}}{\rho_{mljn}} \left(kh + \frac{\mathrm{i}}{\rho_{mljn}}\right) \frac{m^2}{\rho_{mljn}^2}\right.
$$

$$
\left. + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mljn}} - \frac{1}{\rho_{mljn}^2}\right) \frac{l^2 + [(j-n)d/h]^2}{\rho_{mljn}^2}\right] \tag{11.45a}
$$

$$
\sigma_{12}[(j-n)d] = \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{mljn}}}{\rho_{mljn}} \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{mljn}}\right) \frac{(j-n)d/h}{\rho_{mljn}} \tag{11.45b}
$$

and

$$
\sigma_2 = \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \ \neq \sum_{\substack{l=-\infty \\ (0,0)}}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kh\rho_{ml}}}{\rho_{ml}} \left[\frac{-2\mathrm{i}}{\rho_{ml}} \left(kh + \frac{\mathrm{i}}{\rho_{ml}}\right) \frac{l^2}{\rho_{ml}^2}\right.
$$

$$
\left. + \left((kh)^2 + \frac{\mathrm{i}kh}{\rho_{ml}} - \frac{1}{\rho_{ml}^2}\right) \frac{m^2}{\rho_{ml}^2}\right]. \tag{11.46}
$$

Then (11.42a) and (11.42b) can be written respectively as

$$
E_x^n(z_n) = \mathrm{e}^{\mathrm{i}kz_n} + \frac{1}{(kh)^3}\Bigg\{ \sum_{\substack{j=0 \\ j\neq n}}^{N} S_- E_x^j(z_j)\ \sigma_{11}(|j-n|d)
$$

$$- \sum_{\substack{j=0\\ j\neq n}}^{N} S_+ H_y^j(z_j)/Y_0\ \sigma_{12}[(j-n)d] + \ S_- E_x^n(z_n)\ \sigma_2 \Bigg\} \tag{11.47a}$$

and

$$\frac{H_y^n(z_n)}{Y_0} = \mathrm{e}^{\mathrm{i}kz_n} + \frac{1}{(kh)^3}\Bigg\{ \sum_{\substack{j=0\\ j\neq n}}^{N} S_+ H_y^j(z_j)/Y_0\ \sigma_{11}(|j-n|d)$$

$$- \sum_{\substack{j=0\\ j\neq n}}^{N} S_- E_x^j(z_j)\ \sigma_{12}[(j-n)d] + \ S_+ H_y^n(z_n)/Y_0\ \sigma_2 \Bigg\} \tag{11.47b}$$

and thus we have a system of $2(N{+}1)$ equations for the $2(N{+}1)$ unknowns $E_x^n(z_n)$, $H_y^n(z_n)/Y_0$, $n = 0, 1, 2, \cdots, N$

$$\left[1 - S_-\ \sigma_2/(kh)^3\right] E_x^n(z_n) - \frac{S_-}{(kh)^3} \sum_{\substack{j=0\\ j\neq n}}^{N} \sigma_{11}(|j-n|d) E_x^j(z_j)$$

$$+ \frac{S_+}{(kh)^3} \sum_{\substack{j=0\\ j\neq n}}^{N} \sigma_{12}[(j-n)d] H_y^j(z_j)/Y_0 = \mathrm{e}^{\mathrm{i}kz_n} \tag{11.48a}$$

$$\left[1 - S_+\ \sigma_2/(kh)^3\right] H_y^n(z_n)/Y_0 - \frac{S_+}{(kh)^3} \sum_{\substack{j=0\\ j\neq n}}^{N} \sigma_{11}(|j-n|d) H_y^j(z_j)/Y_0$$

$$+ \frac{S_-}{(kh)^3} \sum_{\substack{j=0\\ j\neq n}}^{N} \sigma_{12}[(j-n)d] E_x^j(z_j) = \mathrm{e}^{\mathrm{i}kz_n}\ . \tag{11.48b}$$

Comparing σ_{11} defined by (11.45a) with σ_1 defined by (11.27), we see that $\sigma_{11} = \sigma_1$ and so (11.31) can be used for calculating σ_{11}. Also σ_2 given by (11.46) is identical to σ_2 given by (11.28) (as can be seen by interchanging l and m) so that (11.33) can be used for calculating σ_2 here as well. Finally, from (9.74), we have a rapidly convergent expression for σ_{12},

$$\sigma_{12}(nd) = \mathrm{sgn}(n) \Bigg[2\pi\mathrm{i}(kh)\ \mathrm{e}^{\mathrm{i}|n|kd}$$

$$+ 2\pi\mathrm{i}(kh) \sum_{\substack{m=-\infty\\ (m,l)}}^{\infty} \ \neq \sum_{\substack{l=-\infty\\ (0,0)}}^{\infty} \mathrm{e}^{-|n|(d/h)\ \sqrt{(2\pi)^2(m^2+l^2) - (kh)^2}} \Bigg]\ . \tag{11.49}$$

It is then straightforward to write a computer program to solve the system of equations (11.48) for the values of $E_z^n(z_n)$ and $H_y^n(z_n)/Y_0$. To calculate the field at any point on the array axis (other than at elements of the array) we then use (11.41) which can be written as

$$E_x(z) = \mathrm{e}^{\mathrm{i}kz} + \frac{1}{(kh)^3} \left\{ \sum_{n=0}^{N} S_- E_x^n(z_n)\ \sigma_{11}(|nd - z|) - \sum_{n=0}^{N} S_+ H_y^n(z_n)/Y_0\ \sigma_{12}(nd - z) \right\},\ z \neq nd \tag{11.50a}$$

and

$$\frac{H_y(z)}{Y_0} = \mathrm{e}^{\mathrm{i}kz} + \frac{1}{(kh)^3}\left\{\sum_{n=0}^{N} S_+ H_y^n(z_n)/Y_0\ \sigma_{11}(|nd-z|) - \sum_{n=0}^{N} S_- E_x^n(z_n)\ \sigma_{12}(nd-z)\right\},\ z \neq nd \tag{11.50b}$$

with $\sigma_{11}(|nd-z|)$ and $\sigma_{12}(nd-z)$ given by (11.45) with j and n replaced by n and z/d, respectively. The rapidly convergent expressions (11.31) and (11.49) are used for calculating $\sigma_{11}(|nd-z|)$ and $\sigma_{12}(nd-z)$ with n replaced by $n-x/d$.

Given values of $E_x(z)$ we can calculate the reflection coefficient of the wave scattered back in the negative z direction for $z<0$ as well as the transmission coefficient of the wave traveling in the positive z direction for $z>Nd$. Since the amplitude of the plane wave incident on the partially finite array is 1, the reflection coefficient, R, is the complex coefficient of the wave $\mathrm{e}^{-\mathrm{i}kz}$ for $z<0$ with R obtained from the equation

$$R = E_x(z)\,\mathrm{e}^{\mathrm{i}kz},\ z<0\,. \tag{11.51}$$

In practice z should not be chosen too close to the origin in order for the transient scattered waves to die out, say $z/d < -10$. The square of the magnitude of the transmission coefficient, $|T|^2$, is, of course, equal to $1-|R|^2$, since the array is assumed to be lossless. If the complex transmission coefficient is desired, it can be obtained from the equation

$$T = [E_x(z) + \mathrm{e}^{\mathrm{i}kz}]\,\mathrm{e}^{-\mathrm{i}kz},\ z>Nd. \tag{11.52}$$

In practice, z should not be chosen too close to the end of the array at $z=Nd$, say $z/d > N+10$.

12 NUMERICAL RESULTS

In this section we present the results obtained by numerically solving the transcendental equations derived in Sections 2–10 for the real propagation constants β of traveling waves on 2D and 3D infinite periodic arrays of lossless scatterers whose only significant scattered fields are those of spherical monopoles, in the case of acoustic scattering, and electric and/or magnetic dipoles, in the case of electromagnetic scattering. For the sake of comparison, we also find the propagation constants on the corresponding 1D periodic arrays of some scatterers from their transcendental equations given in previous reports [4], [5]. All these transcendental equations involve only well-known functions or rapidly convergent summations. They are readily solved on a personal computer to efficiently obtain kd–βd curves (diagrams) for the traveling waves.

As explained in the Introduction, traveling waves for $\beta d > \pi$ can be re-expressed as traveling waves with $-\pi < \beta d < 0$. Moreover, for periodic arrays of elements composed of reciprocal material only, it is proven in Appendix A that every traveling wave is bidirectional, that is, for every traveling wave with propagation constant β there exists a corresponding traveling wave with propagation constant $-\beta$. Thus, the kd–βd diagrams in this section require βd to cover only the domain $[0,\pi]$ because all the arrays considered in this section are composed of reciprocal elements.

Also, as explained in the Introduction, unattenuated traveling waves (real β) on 1D and 2D infinite periodic arrays cannot exist for $kd > \beta d$. In other words, all fast waves on 1D and 2D lossless periodic arrays are improper leaky waves. Therefore, all the kd axes on the kd–βd diagrams for the 1D and 2D arrays in this section span the range only from 0 to π. Since unattenuated traveling waves on lossless 3D arrays can exist for $kd > \beta d$, the kd–βd diagrams for the 3D periodic arrays considered in this section may have their kd axes extend to values larger than π. In all kd–βd diagrams, however, the range of kd is restricted to the values of kd at which the quadrupole moments appear (the first spherical multipoles of higher order than the electric and magnetic dipoles), since our analysis assumes that all scattered fields are negligible except the electric and magnetic dipole fields.

The analysis also assumes that the spheres that circumscribe the elements of the periodic arrays do not intersect. For generally shaped scatterers, the analysis also assumes that the scattered electric and magnetic dipoles of any one element are determined only by the values of the incident field (of all the other elements) at the center of that element and, thus, there may be some inaccuracy introduced if the elements are too closely packed, even if their circumscribing spheres do not intersect. *Emphatically, however, for spherical scatterers we have proven (as part of the foregoing analysis and the analysis in the previous reports [4], [5]), using the orthogonality of the spherical harmonics and the field values at the center of the incident spherical waves, that this assumption of the scattered dipole fields being determined solely by the central incident field values holds exactly. Consequently, for spherical scatterers there is no loss of accuracy introduced into our equations by packing the scatterers as tightly as possible as long as the predominant scattering is that of electric and magnetic dipoles.*

Once the kd–βd diagram is found for a 3D infinite periodic array, we use the formulas derived in Section 9.2 (referred to herein as the Shore-Yaghjian formulas) for determining the effective (bulk) permittivity and permeability of the array from the parameters in the transcendental equation. In addition, these bulk parameters are also determined from the Clausius-Mossotti relations, which, in general, are not as accurate as those determined from the Shore-Yaghjian formulas. As a rule of thumb, the bulk parameters are not accurate predictors of propagation characteristics and reflection or transmission coefficients unless both kd and βd are less than about unity. Thus, we show the values of the bulk permittivity and permeability in the following figures for kd no greater than about unity.

All the elements of the 2D and 3D arrays considered in this report are arranged in rectangular and rectangular parallelepiped lattices, respectively. Furthermore, all the 2D and 3D arrays considered in this section on the numerical results have the elements arranged in square and cubic lattices, respectively, that is, $h = d$. For acoustic monopoles, or if only electric (magnetic) dipoles are present in a traveling wave, the transcendental equation for βd depends only on kd and the phase ψ of the scattering coefficient. Although the phase ψ of the scattering coefficient of any given scatterer will generally change with frequency (that is, with kd for a fixed separation distance d), it is revealing in the following subsection to plot the family of kd–βd curves determined by different values of the phase ψ for 2D and 3D arrays of acoustic monopoles and electric (magnetic) dipoles. Corresponding plots for 1D periodic arrays of acoustic monopoles, electric (magnetic) dipoles, and magnetodielectric spheres, were given, respectively, in [3]–[5]. The 1D, 2D, and 3D families of kd–βd curves for magnetic dipoles are identical to those for electric dipoles.

In Subsections 12.2–12.5 kd–βd diagrams and effective permittivity and permeability curves (for 3D arrays) are given for representative scatterers, namely, for short perfectly electrically conducting (PEC) wires, for PEC spheres, for diamond spheres, for silver nanospheres, and for magnetodielectric spheres. In Subsection 12.6 we show plots of the reflection coefficient for partially finite arrays of magnetodielectric spheres.

12.1 FAMILY OF *kd*–*βd* CURVES FOR ACOUSTIC MONOPOLES AND ELECTRIC OR MAGNETIC DIPOLES

The transcendental equation in (2.26) for a 2D square lattice ($h = d$) of lossless acoustic monopole scatterers determines βd, the propagation constant times the separation distance, of the traveling waves in terms of kd and the phase ψ of the scattering coefficient S, which was proven to satisfy $|S| = \sin\psi$ so that ψ is confined to the range $[0, 180]$ degrees. The family of kd–βd curves for different values of ψ is shown in Fig. 1. Similarly, the family of kd–βd curves for a 3D cubic lattice of lossless acoustic monopole scatterers is found from the transcendental equation (3.53) and is given in Fig. 2. The curves in Fig. 1 for the 2D array and those in Fig. 2 for the slow waves of the 3D array agree qualitatively with the corresponding curves for the 1D array in Fig. 3 of [3]. In particular, these curves show that if the scattering from each acoustic monopole is nearly in phase with its incident field (ψ small), a traveling wave exists for nearly all values of kd, but if the scattering from each acoustic monopole is nearly 180 degrees out of phase with its incident field, a traveling wave only exists for very close spacing ($kd \ll 1$).

Numerically solving the transcendental equations in (4.62) and (4.101) for the propagation constants of traveling waves on a 2D square-lattice array of electric (or magnetic) dipole scatterers with dipole moments normal to the propagation direction produces the family of kd–βd curves shown in Fig. 3 if the dipoles are in the plane of the 2D array and in Fig. 4 if the dipoles are perpendicular to the plane of the 2D array. Figure 5 shows the family of kd–βd curves obtained from the transcendental equation (5.75) for traveling waves on a 3D cubic-lattice array of electric (or magnetic) dipole scatterers with dipole moments normal to the direction of propagation. In Figs. 6 and 7 the family of kd–βd curves is given from the transcendental equations (6.79) and (7.76) for electric (or magnetic) dipole scatterers parallel to the direction of propagation on square-lattice 2D and cubic-lattice 3D arrays, respectively. For all these lossless electric (or magnetic) dipole scattering curves, the scattering coefficient was proven to satisfy the relationship $|S| = (3/2)\sin\psi$.

The family of kd–βd curves in Fig. 4 for the dipoles perpendicular to the plane of the 2D array is qualitatively the same as the corresponding family of curves in Fig. 3 of [4] for the 1D array of dipoles normal to the direction of propagation. The family of kd–βd curves in Fig. 3 for dipoles in the plane of the 2D array (and normal to the direction of propagation) and in the slow wave curves of Fig. 5 for dipoles normal to the direction of propagation on a 3D array is quite different from the 1D array curves in Fig. 3 of [4], because, evidently, the coupling of the radial fields between adjacent scatterers is stronger in the 2D and 3D arrays of Figs. 3 and 5. The family of kd–βd curves in Fig. 6 for the 2D array of dipoles parallel to the direction of propagation is qualitatively the same as the corresponding 1D array of parallel dipoles in Fig. 13 of [4], whereas the family of kd–βd curves in Fig. 7 for dipoles parallel to the direction of propagation on a 3D array exists only in a very narrow frequency band for a fixed value of ψ.

12.2 PEC SHORT WIRES AND PEC SPHERES

In this subsection, the kd–βd diagrams and relative permittivity and permeability of perfectly electrically conducting (PEC) short wires and PEC spheres are determined from the relevant transcendental equations derived in the previous sections.

12.2.1 PEC Short Wires

The kd–βd diagram for an infinite linear (that is, 1D) periodic array of short parallel PEC wires (electric dipoles), normal to the propagation axis, with different ratios of wire length to separation distance $(2h/d)$[10] is given in Fig. 4 of [4]. The ratio of the radius of each wire to its length $[\rho/(2h)]$ is equal to 0.1, and the phase ψ of the scattering coefficient S was computed at each kh with the Numerical Electromagnetics Code (NEC) [41]. The corresponding kd–βd diagrams for traveling waves on 2D and 3D short parallel PEC wires (electric dipoles) normal to the propagation axis are computed from the transcendental equations in (4.62), (4.101), and (5.75) and are shown in Figs. 8, 9, and 10, all three of which display the same general variation for the 2D and 3D arrays of parallel wires as for the 1D array in Fig. 4 of [4]. Until the values of kd become significantly greater than 1, the values of β are fairly close to the values of k and thus the traveling waves on these short wires will be loosely coupled to the wires except for values of βd fairly close to π, where the 3D array does not well approximate a homogeneous medium characterized by bulk permittivity. Incidentally, the curves in these short-wire figures do not continue to very small values of kd simply because we didn't have the NEC-determined values of ψ readily available for these small values of kd. Also, the range of the kd axis in the 3D-array Fig. 10 is restricted to π because significant multipoles of higher order than electric dipoles are excited on the wires for $kd > \pi$.

The numerical solution to the transcendental equations in (6.79) and (7.76) showed that there are no unattenuated traveling waves on 2D and 3D arrays ($kd < \pi$) of short PEC wires oriented parallel to the propagation axis–as was also found in the case of 1D arrays of wires parallel to the array axis [4]. It is noted, however, that the constant ψ curves in Figs. 6 and 7 imply that if the short wires were passively loaded to change the values of their scattering coefficients, that is, change the values of ψ, traveling waves could be supported by these longitudinally oriented wires [42].

The effective relative permittivity versus kd of the 3D array of transverse dipoles corresponding to the kd–βd curves of Fig. 10 are computed from the Shore-Yaghjian formula in (9.109) and from the real part[11] of the Clausius-Mossotti relation (9.120) and are shown in Fig. 11. In the region $kd \lesssim 1$, the values of βd are $\lesssim 1$ and thus, as would be expected, the real part of the Clausius-Mossotti relation predicts a relative permittivity that agrees quite

[10] Here the symbol h denotes the half-length of the wires and should not be confused with the 2D- and 3D-array transverse separation distance denoted by the same symbol h, which is set equal to the longitudinal separation distance d along the propagation axis in all the numerical examples of this section.

[11] As explained in Section 9.2, the Clausius-Mossotti relations, unlike the Shore-Yaghjian formulas, erroneously predict imaginary parts for the effective permittivity and permeability of these unattenuated traveling waves on lossless arrays. The imaginary parts are usually much smaller than the real parts, however, if kd and βd are $\lesssim 1$.

well with the Shore-Yaghjian results obtained from the exact transcendental equation. Of course, since there are no appreciable magnetic dipole moments on short wires, the relative permeability of the 3D arrays is equal to unity.

12.2.2 PEC Spheres

The fields scattered by a PEC sphere illuminated by a incident plane wave contains a significant magnetic dipole moment as well as an electric dipole moment even as the frequency (kd) approaches zero. Thus, the transcendental equations for magnetodielectric spheres must be used to find the kd–βd diagrams for the traveling waves on 1D, 2D, and 3D arrays of PEC spheres. Specifically, the kd–βd diagrams for the transverse traveling waves (that is, traveling waves with orthogonal electric and magnetic dipole moments normal to the direction of propagation) on 1D, square-lattice 2D, and cubic-lattice 3D arrays of PEC spheres are determined from equation (52) of [5] or (18) of [7] (for 1D arrays), from (8.52) and (8.104) (for 2D arrays), and from (9.84) (for 3D arrays), and are shown in Figs. 12–15 for values of $a/d = .3$, .4, and .45, where a is the radius of the spheres. The electric and magnetic dipole scattering coefficients, S_- and S_+, required by the transcendental equations are obtained from the electric and magnetic dipole coefficients in the Mie solution; see Section 8 and [5, eqs. (30a,b)]. The resonance of the first quadrupole moment of the PEC sphere occurs at a value of $ka = 2.3$ and thus the 3D curves in Fig. 15 are truncated at the corresponding values of kd.

All the Figs. 12–15 for PEC spheres exhibit kd–βd curves similar to those in [4, fig. 4] and Figs. 8–10 for the short PEC wires. In particular, traveling waves on the PEC spheres do not travel with speeds much less than the free-space speed of light (and thus are not strongly coupled to the spheres) until the values of βd are fairly close to π, where the 3D array does not well approximate a homogeneous media characterized by a bulk permittivity and permeability. We also note that the 3D-array curves in Fig. 15 have regions of negative group velocity for values of $kd > \pi$. However, because $kd > \pi$ in this region, the negative group velocity does not imply that the direction of power flow in the traveling wave is opposite the direction of the phase velocity; see Section 9.2 and [15, sec. 5.3].

The effective relative permittivity and permeability versus kd of the 3D array of orthogonal transverse electric and magnetic dipoles corresponding to the kd–βd curves of Fig. 15 are computed from the Shore-Yaghjian formulas in (9.107)–(9.108) and from the real parts (see Footnote 11) of the Clausius-Mossotti relations in (9.120) and (9.127), and are shown in Fig. 16. In the region $kd \lesssim 1$, the values of βd are $\lesssim 1$ and thus, as would be expected, the real parts of the Clausius-Mossotti relations predict a relative permittivity and permeability that agree quite well with the Shore-Yaghjian formulas obtained from the exact transcendental equation. The values of the effective relative permittivity and permeability in Fig. 16 are commensurate with the facts that for $ka \ll 1$ the electric and magnetic dipole moments in the Mie solution to the PEC sphere are in the ratio of two to one and have opposite signs.

The numerical solution to the transcendental equations in [4, eq. (96)], (6.79), and (7.76) for longitudinal traveling waves (that is, traveling waves with either electric or magnetic dipoles along the direction of propagation) show that none exist on 1D and 2D arrays of PEC spheres and that only an electric dipole longitudinal wave exists on a 3D array of PEC spheres, as shown in Fig. 17, and then only for the case of $a/d = .3$ and values of kd greater than 5.5.

12.3 DIAMOND SPHERES

Diamond has a nearly constant relative permittivity of 5.84 with very low loss at optical frequencies and a relative permeability equal to unity. We shall assume this value of relative permittivity for diamond in all the numerical computations for arrays of diamond spheres of radius a. We shall also assume a single packing ratio of $a/d = .45$. The first resonance of the diamond sphere is a magnetic dipole resonance that occurs at a $ka = 1.25$ even though the electric dipole moment dominates at lower values of ka. (In general, the first resonance of a positive-permittivity sphere is a magnetic dipole and the first resonance of a negative-permeability sphere is an electric dipole.) Therefore, to obtain the kd–βd curves for the transverse waves on 1D, 2D, and 3D arrays of diamond spheres, one must use the magnetodielectric transcendental equations [5, eq. (52)] or [7, eq. (18)] (for 1D arrays), (8.52) and (8.104) (for 2D arrays), and (9.84) (for 3D arrays). The electric and magnetic dipole scattering coefficients, S_- and S_+, required by the transcendental equations are obtained from the electric and magnetic dipole coefficients in the Mie solution; see Section 8 and [5, eqs. (30a,b)]. The numerical solutions to these transcendental equations are plotted in the kd–βd diagrams of Figs. 18–24. The curves in these figures for the slow traveling waves are similar to those for the traveling waves on PEC wires and PEC spheres except for the second branches of the curves produced by the magnetic dipole resonance of the diamond sphere. On these second branches the group velocity is negative (opposite the phase velocity). However, the 3D array of these diamonds would not be meaningfully characterized by negative effective permittivity and permeability in this region of negative group velocity because $kd \gg 1$ in this region. In the region $kd < 1$ the effective relative permeability of the 3D array is approximately equal to unity and the effective relative permittivity computed from both the Shore-Yaghjian formula (9.107) and the real part (see Footnote 11) of the Clausius-Mossotti relation (9.120) is shown in Fig. 24, which shows that the bulk relative permittivity of the 3D array of closely packed diamond spheres in this small kd region is about 2, a value between that of diamond (5.84) and that of free space (1). This is reflected in the fact that, in contrast to the 1D and 2D kd–βd diagrams of Figs. 18 and 20, the 3D kd–βd curve of Fig. 21 has a constant slope of less than one in the low frequency (large wavelength) limit since the slope of the kd–βd line approaching the origin is equal to [see (9.94)]

$$\frac{kd}{\beta d} = \sqrt{\frac{1}{\epsilon_{\mathrm{r}}^{\mathrm{eff}}}} \tag{12.1}$$

which in this case is approximately equal to $1/\sqrt{2} \approx 0.7$.

The range of the values of kd in Fig. 21 for transverse traveling waves on the 3D array of diamond spheres is limited to 4 because the first quadrupole resonance occurs at about this value. In Fig. 22 the range of kd in Fig. 21 is extended to a value of 6. Fig. 23 shows the kd–βd curves for the same 3D array of diamond spheres computed from a finite difference time domain (FDTD) code [43]. A comparison of Figs. 22 and 23 shows good agreement between the kd–βd curves computed from our transcendental equations and from the FDTD code up to a value of $kd \approx 4$ where the resonance of the first quadrupole moment occurs.

The kd–βd curves for longitudinal traveling waves of electric or magnetic dipoles parallel to the direction of propagation are computed for the diamond spheres from the transcendental equations in [4, eq. (96)], (6.79), and (7.76) for 1D, 2D, and 3D arrays, respectively, and are plotted in Figs. 25, 26, and 27. Figures 25 and 26 show that on the 1D and 2D arrays of diamond spheres there are only magnetic dipole longitudinal waves and then only in a narrow bandwidth near the first resonant frequency of the spheres. Figure 27 shows that both electric (for $kd > \pi$) and magnetic dipole longitudinal traveling waves exist on 3D arrays of diamond spheres, again in fairly narrow bandwidths.

12.4 SILVER NANOSPHERES

The kd–βd diagrams for the unattenuated traveling waves on PEC wires, PEC spheres, and diamond spheres have shown that $\beta d \not\gg kd$ and thus the fields of these traveling waves are not strongly coupled or tightly confined to the elements of the arrays. That is, the preponderance of the power in the fields is not concentrated within a distance appreciably less than a wavelength from the scattering elements of the arrays. For use in optical circuitry, it is desirable to produce traveling waves at optical frequencies that are confined to a small fraction of a wavelength from array elements that are also small fractions of a wavelength across. One possibility[12] for producing such traveling waves is to use linear (1D) chains of silver nanospheres [48]–[51], since at optical frequencies silver behaves as a plasma with a negative dielectric constant (relative permittivity). Such plasmas support surface waves at the silver interface which lead to electric dipole resonances of the silver spheres with fields confined to a small fraction of a wavelength from the spheres. Therefore, in this subsection we shall use our transcendental equations to compute the kd–βd diagrams for traveling waves on 1D, 2D, and 3D arrays of silver nanospheres [6]. Unlike spheres with a positive dielectric constant (like diamond spheres) whose first resonance is that of a magnetic dipole, the first resonance of "plasmonic" spheres with negative dielectric constant is that of an electric dipole. Consequently, for silver nanospheres the transcendental equations for purely electric dipoles would produce accurate kd–βd curves through the first dipole resonance. Nonetheless, we used the magnetodielectric transcendental equations in [5, eq. (52)] or [7, eq. (18)] for 1D arrays, (8.52) and (8.104) for 2D arrays, and (9.84) for 3D arrays of silver nanospheres.

The kd–βd diagrams for transverse traveling waves on 1D, 2D, and 3D silver nanospheres are shown in Figs. 28–31. The curves were computed using the following Drude model for relative permittivity, which agrees quite well with the values of relative permittivity measured by Johnson and Christy [44] over the visible range of frequencies where lowest order traveling waves exist:

$$\epsilon_{\rm r} = \frac{\epsilon}{\epsilon_0} = 5.45 - .73\frac{\omega_p^2}{\omega(\omega + {\rm i}\gamma)} \tag{12.2}$$

with the plasma frequency $\omega_p = 1.72 \times 10^{16}$ and the loss parameter $\gamma = 8.35 \times 10^{13}$. To conform to the parameters used in [45], the radius a of the spheres was chosen to be 5

[12] Another possibility for producing tightly confined traveling waves would be to use spheres with high values ($\gtrsim 100$) of permittivity or permeability. However, low-loss materials apparently do not exist that have such high values of permittivity or permeability at optical frequencies.

nanometers and the spheres were embedded in glass with a dielectric constant equal to 2.56 ($k = 1.6 \times \omega/c$). With these parameters the relative permittivity of the spheres in glass can be written from the Drude equation in (12.2) as

$$\epsilon_{\rm r} = \frac{\epsilon}{\epsilon_{\rm glass}} = 2.129 - \frac{.0234}{(ka/1.6)[(ka/1.6) + .00139{\rm i}]}. \tag{12.3}$$

Since our formulation assumes lossless array elements, and the permittivity in (12.2)–(12.3) contains loss (an imaginary part), we inserted just the real part of (12.3) into the transcendental equations used to obtain Figs. 28–31. This approximation should give reasonably accurate values for the real parts of the propagation constants (βd) of the traveling waves because the imaginary part of the permittivity in (12.3) is a small fraction of the real part in the visible range of frequencies where the lowest order traveling waves exist. The electric and magnetic dipole scattering coefficients, S_- and S_+, required by the transcendental equations are obtained from the electric and magnetic dipole coefficients in the Mie solution; see Section 8 and [5, eqs. (30a,b)]. The parameter s in Figs. 28–31 is the free-space distance between the spheres so that $d = s + 2a$, and traveling waves are found in Figs. 28–31 for s equal to 1 nanometer and 4 nanometers, again to conform to the values chosen in [45].

The electric-dipole longitudinal traveling waves on 1D, 2D, and 3D arrays of the same silver nanospheres embedded in glass are computed from [4, eq. (96)], (6.79), and (7.76), respectively, and are shown in Figs. 32, 33, and 34.

The figures for the transverse traveling waves on silver nanospheres in Figs. 28–31 and for the longitudinal traveling waves in Figs. 32–34 show that much of the kd–βd curves of all the traveling waves exist in fairly narrow frequency bands and over much of each of these narrow frequency bands $\beta d \gg kd$. This implies that these plasmonic traveling waves, as expected, can have most of their power confined to within a small fraction of a free-space wavelength from the spheres. The 1D and 2D kd–βd curves in Figs. 28–30 for the transverse traveling waves on the silver nanospheres asymptotically approach the origin along the $kd = \beta d$ line, that is, the "light line." In contrast, the 3D kd–βd curves in Fig. 31 asymptotically approach the origin along lines of constant slope less than one. This is because, similarly to the behavior of the kd–βd curve of the 3D diamond sphere array discussed in Subsection 12.3, in the low frequency (large wavelength) limit the 3D silver nanosphere array behaves as a medium with an effective relative permeability approximately equal to unity and an effective relative permittivity greater than one (see below) with the slope of the kd–βd line approaching the origin given by (12.1).

The kd–βd curves in Figs. 32 and 33 for the slow longitudinal traveling waves on 1D and 2D arrays of silver nanospheres, in contrast to the kd–βd curves for the transverse traveling waves, end abruptly on the $kd = \beta d$ light line.

Concentrating on the kd–βd curves in Figs. 28–31 for the transverse traveling waves on the 1D, 2D, and 3D arrays of silver nanospheres shows that for each curve the group velocity becomes zero and then the curve remains fairly flat to the right of this maximum. This implies that for a traveling wave containing a spectrum of frequencies covering this maximum and the continuation of the kd–βd curve to the right of the maximum, there exists a superposition of a "frozen-mode" field and a spectrum of traveling waves moving with very slow group velocities [46]. It should be noted that the slightly negative group velocities in Fig. 31 for the 3D array occur for βd significantly greater than unity and thus these negative group velocities do not imply meaningful bulk relative permittivities and permeabilities with values both less than zero. Of course, since the magnetic dipole moments of the silver nanospheres are negligible, the bulk relative permeability cannot have a meaningful value other than unity.

The effective relative permeability of the 3D array of glass-embedded silver nanospheres is approximately equal to unity and the effective relative permittivity computed from both the Shore-Yaghjian formula (9.107) and the real part (see Footnote 11) of the Clausius-Mossotti relation (9.120) is shown in Fig. 35. This figure reveals for the more loosely packed 3D array of silver nanospheres ($s = 4$ nm) in the region $[kd < .3, \beta d < 1]$ that the bulk relative permittivity is less than 10 and approaches a constant value of about 1.8 as $kd \to 0$. For the more tightly packed nanospheres ($s = 1$ nm) in the region $[kd < .2, \beta d < 1]$, the bulk relative permittivity is less than 12 and approaches a constant value of about 3.0 as $kd \to 0$.

12.5 MAGNETODIELECTRIC SPHERES

Using expressions derived by Lewin [2] that are closely related to the Clausius-Mossotti relations for spherical scatterers, Holloway *et al.* [1] have shown that 3D arrays of magnetodielectric spheres with equal (and unequal) values of relative permittivity and permeability exhibit frequency bands (near the resonances of these spheres) in which the bulk permittivity and permeability are both negative. We concentrate on the case of equal relative permittivity and permeability because the impedance of a 3D array of such spheres will be close to the impedance of free space. To confirm these predictions of "double negative" (DNG) "metamaterials," we apply the magnetodielectric transcendental equations in [5, eq. (52)] or [7, eq. (18)] (for 1D arrays), (8.52) and (8.104) (for 2D arrays), and (9.84) (for 3D arrays) to obtain the kd–βd diagrams shown in Figs. [5, fig. 15] or [7, fig. (8)], 36, and 37 for the transverse traveling waves on spheres with relative permittivity and permeability equal to 20. In all these figures, the value of $a/d = .45$ is used for the sphere radius to spacing ratio and the range of kd is restricted to values less than the first quadrupole resonance that occurs at $ka = .28$. There is only one figure for the 2D array because the transverse traveling waves with the electric dipole moments perpendicular and parallel to the plane of the 2D array have the same kd–βd diagram if the permittivity and permeability of the spheres have the same value.

The shape of the kd-βd curves in Figs. [5, fig. 15] or [7, fig. (8)] and 36 for the transverse traveling waves on the 1D and 2D arrays of $\mu_{\rm r} = \epsilon_{\rm r} = 20$ magnetodielectric spheres are qualitatively the same but with the first branch of the 1D curve having two values of kd with zero group velocity and the second branch of the 2D curve having one value of kd with zero group velocity. The kd–βd curve for the 3D array, which is of greatest interest as a possible DNG material, crosses the light line and has no values of kd with zero group velocity.

The effective (bulk) relative permittivity and permeability of this 3D metamaterial is computed from the Shore-Yaghjian formulas in (9.107)–(9.108) and from the real parts (see Footnote 11) of the Clausius-Mossotti relations in (9.120) and (9.127), and are shown in Fig. 38. The Clausius-Mossotti curve in Fig. 38 is indistinguishable from the corresponding curve computed from the Lewin formulas in [1, fig. 7]. Both the results of the Shore-Yaghjian formulas and the Clausius-Mossotti relations plotted in Fig. 38 confirm the results obtained in [1] that there exists a frequency band (near the first resonance of the magnetodielectric spheres) in which both $\mu_{\mathrm{r}}^{\mathrm{eff}}$ and $\epsilon_{\mathrm{r}}^{\mathrm{eff}}$ are negative, namely in the approximately 10% fractional bandwidth between kd equal to about .45 and .50. Moreover, βd is also less than 1 over a sizable portion of this DNG frequency band so that the material should behave as a fairly homogeneous medium at these frequencies. The bulk properties of this 3D magnetodielectric array medium are reflected in the slope of the kd–βd curve of Fig. 37) as the origin is approached. The slope of the curve in the low frequency (large wavelength) limit is given by [see (9.94)]

$$\frac{kd}{\beta d} = \frac{1}{\epsilon_{\mathrm{r}}^{\mathrm{eff}}} = \frac{1}{\mu_{\mathrm{r}}^{\mathrm{eff}}}\ . \tag{12.4}$$

Low-loss magnetodielectric material is commercially available at GHz frequencies with $\epsilon_{\mathrm{r}} = 13.8$ and $\mu_{\mathrm{r}} = 11.0$ [47]. Our computations with spheres made from this magnetodielectric material result in the kd–βd diagram of Fig. 39 and the effective permittivity and permeability curves shown in Fig. 40. Again there is a frequency band (between kd approximately equal to .7 and .8) where both $\mu_{\mathrm{r}}^{\mathrm{eff}}$ and $\epsilon_{\mathrm{r}}^{\mathrm{eff}}$ are negative. And, βd is also less than 1 over a sizable portion of this DNG frequency band so that the material should behave as a fairly homogeneous medium at these frequencies.

We also looked for longitudinal traveling waves (electric or magnetic dipoles aligned with the direction of propagation) on the $\mu_{\mathrm{r}} = \epsilon_{\mathrm{r}} = 20$ magnetodielectric sphere separated with $a/d = .45$. None exist on the 2D array in the range of kd up to the first quadrupole resonance, but the 1D and 3D arrays do support electric- or magnetic-dipole longitudinal traveling waves in the extremely narrow frequency bands shown in Figs. 41 and 42. As seen in Fig. 41, the 1D kd–βd longitudinal curve terminates on the $kd = \beta d$ light line, whereas the 3D longitudinal curve in Fig. 42 continues unperturbed through the light line.

12.6 PARTIALLY FINITE MAGNETODIELECTRIC SPHERE ARRAY REFLECTION COEFFICIENTS

In this subsection we show examples of reflection coefficient curves for partially finite arrays of magnetodielectric spheres. Plots for two kinds of spheres are shown: diamond spheres with $\epsilon_{\mathrm{r}} = 5.84, \mu_{\mathrm{r}} = 1$, and spheres composed of low-loss commercially available material with $\epsilon_{\mathrm{r}} = 13.0, \mu_{\mathrm{r}} = 11.8$. The values of the reflection coefficients, R, for the partially finite arrays are obtained from (11.51) and (11.50a). In all cases the value of N is 100 (that is, there are 101 equispaced infinite planes of spheres normal to the array axis), and the ratio of the radius of the spheres, a, to the separation of adjacent sphere centers, d, is 0.45. For both kinds of spheres we show plots for two cases, one where there is no loss, and one with loss inserted into the propagation constant of the incident plane wave when calculating the values of $E_x^n(z_n)$ and $H_y^n(z_n)/Y_0, n = 0, 1, \cdots, N$ from (11.48) that enter into (11.50a). The value of the loss constant, ε, is chosen via the equation

$$\mathrm{e}^{-Nkd\varepsilon} = 10^{-P} \tag{12.5}$$

with $P = 1$. Given the value of ε from (12.5) the values of the incident plane wave, $\mathrm{e}^{\mathrm{i}kz_n}$, at the locations $z = z_n = nd,\ n = 0, 1, \cdots, N$ in the RHS of (11.48) are then multiplied by the respective factors $e^{-\varepsilon nkd}$, $n = 0, 1, \cdots, N$.[13] The purpose of inserting loss is to reduce the reflections at the far end of the partially finite arrays and so make them behave more like semi-infinite arrays with no reflections at the far end. It is then possible to compare the reflection coefficients obtained from the partially finite array equation (11.51) with the reflection coefficients obtained from the calculations of bulk permittivity and permeability given by the Shore-Yaghjian formulas (9.120) and (9.127) using the expression [25, eq. (7.42)]

$$R = \frac{\sqrt{\mu_{\rm r}/\epsilon_{\rm r}} - 1}{\sqrt{\mu_{\rm r}/\epsilon_{\rm r}} + 1}\,. \tag{12.6}$$

In Fig. 43 we show a plot of the magnitude of the reflection coefficient for the partially finite lossless diamond array. The pronounced oscillations of the pattern are the result of reflections between the two ends of the array. Thus, the array behaves somewhat like a Fabry-Perot resonator. Note that the intervals of the plot where the magnitude of the reflection coefficient equals one correspond exactly to the gaps in the kd–βd diagram of Fig. 21, that is, the intervals of kd where no traveling wave exists to convey power from one end of the array to the other.

In Fig. 44 we show the plot of the magnitude of the reflection coefficient for the partially finite diamond array with loss inserted into the incident plane wave, together with a plot of the magnitude of the reflection coefficient obtained from the Shore-Yaghjian bulk parameter equations. The oscillations of Fig. 43 have been considerably reduced. Note that for values of kd less than one there is excellent agreement between the lossy partially finite array reflection coefficient and the Shore-Yaghjian coefficient given by (12.6) apart from a small interval of kd between zero and about 0.1. This is to be expected since the derivation of the Shore-Yaghjian bulk parameter expressions assumes a separation of the array elements sufficiently small so that the array can be regarded as a homogeneous medium. For larger values of kd there is a kind of rough qualitative agreement but the Shore-Yaghjian reflection coefficient values are in general not highly accurate. As kd becomes smaller than .1, the total thickness of the slab with decaying incident field (simulating a loss) becomes smaller than a free-space wavelength and its scattering becomes weaker.

In Fig. 45 we show a plot of the magnitude of the reflection coefficient for the partially finite lossless array of $\epsilon_{\rm r} = 13.0, \mu_{\rm r} = 11.8$ spheres over an extended range of kd. As with the lossless diamond sphere array reflection coefficient curve, the intervals of kd for which the magnitude of the reflection coefficient equals one correspond to intervals of the extended kd–βd diagram of Fig. 46 where no traveling wave exists. The portions of the curve between adjacent spikes correspond to the backward traveling wave branches of the kd–βd diagram.

In Fig. 47 we show a corresponding plot of the magnitude of the reflection coefficient for the partially finite array of $\epsilon_{\rm r} = 13.0, \mu_{\rm r} = 11.8$ spheres with loss inserted into the incident plane wave, together with a plot of the magnitude of the reflection coefficient obtained from the Shore-Yaghjian bulk parameters in (12.6). Here the agreement between the two reflection coefficient curves is surprisingly good even for larger values of kd. There is, however, one

[13] This exponential decay inserted into the incident field on the right-hand sides of (11.48) does not satisfy Maxwell's equations in lossless free space. Nonetheless, this mathematical ansatz serves to reduce the multiple interactions between the leading and trailing interfaces of the partially finite array, thereby producing a reflection coefficient that is nearly equal to that of the leading interface alone.

feature of the partially finite array curve that is not reproduced in the Shore-Yaghjian curve, namely the smaller spike between each pair of principal spikes at the two ends of the backward traveling wave intervals. These smaller spikes occur exactly where the kd–βd curve in Fig. 46 crosses the $kd = \beta d$ light line and, since they occur in the backward traveling wave intervals, correspond to places where the traveling wave has exactly the negative of the incident wave phase dependence on z. Some numerical experimentation shows that these smaller spikes are attributable to reflections from only the first few planes of the partially finite array, and so cannot be found in the reflection coefficient curve obtained from the Shore-Yaghjian bulk parameter expressions. It is also worth noting that for values of $kd >$ 1, a reflection coefficient curve obtained from the Clausius-Mossotti mixing formula bulk parameter expressions (not shown here) does not track the partially finite array curve nearly as well as the curve obtained from the Shore-Yaghjian expressions. This is not surprising in view of the fact that the Clausius-Mossotti bulk parameter expressions are much more tightly bound to the assumption of small array element separations than are the Shore-Yaghjian bulk parameter expressions.

In closing this subsection we note that the reflection coefficient for a partially finite array of magnetodielectric spheres with $\epsilon_{\rm r} = \mu_{\rm r} = 20$ was found to be very close to zero for all values of kd, a result that is consistent with $\epsilon_{\rm r} = \mu_{\rm r}$ in (12.6).

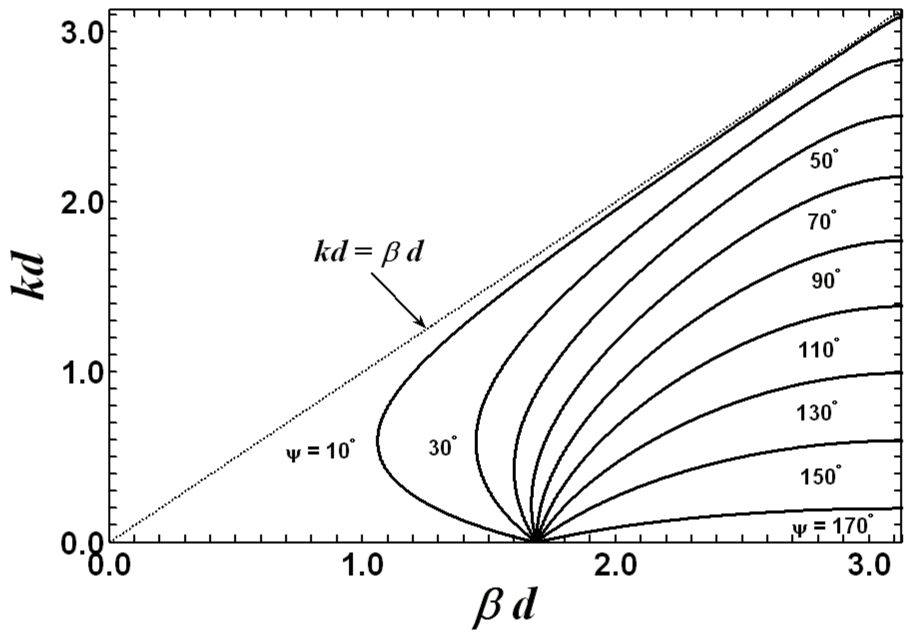

Figure 1: Family of kd–βd curves for 2D acoustic array of monopoles with constant values of the phase ψ of the scattering coefficient S.

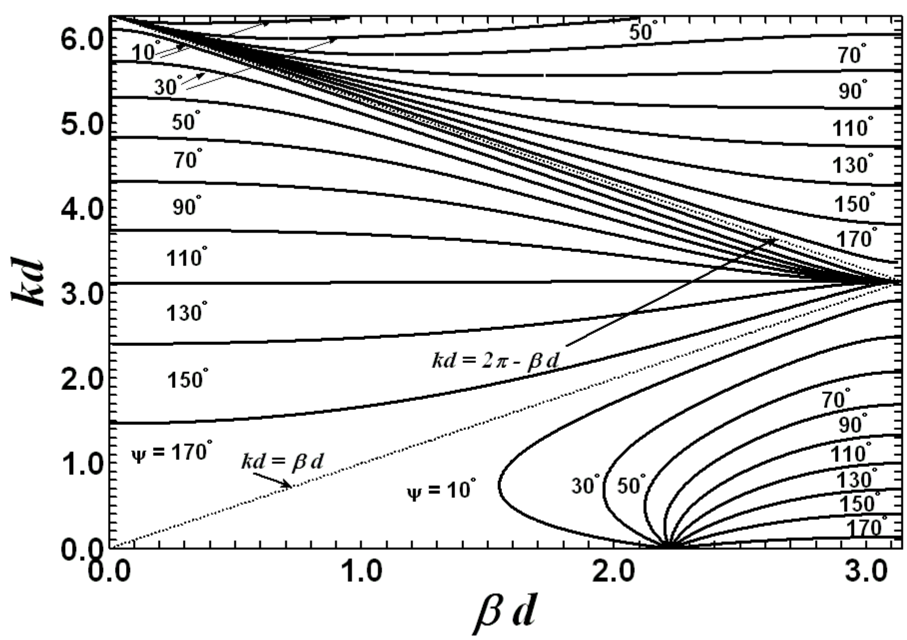

Figure 2: Family of kd–βd curves for 3D acoustic array of monopoles with constant values of the phase ψ of the scattering coefficient S.

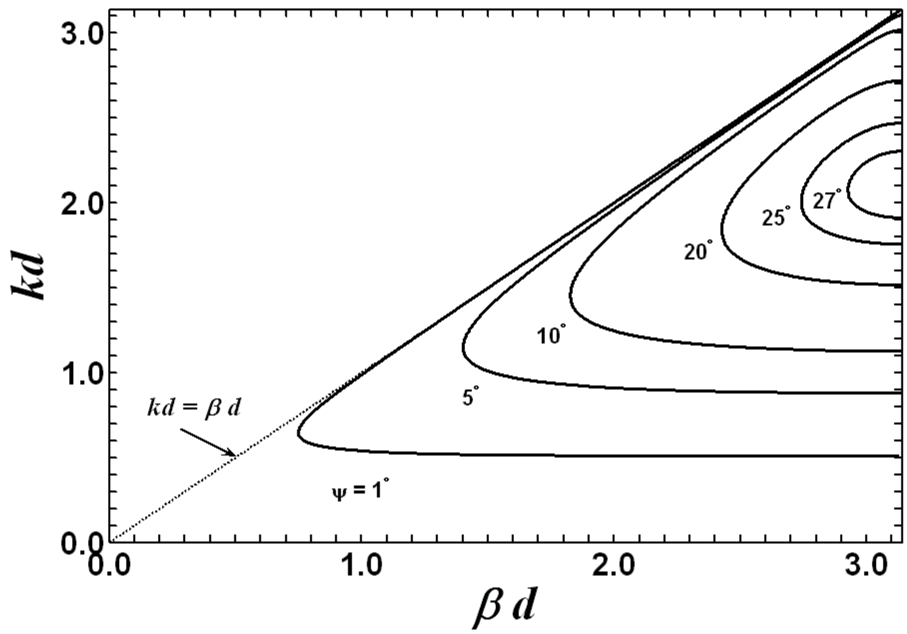

Figure 3: Family of kd–βd curves for 2D array of dipoles (parallel to the array plane) with constant values of the phase ψ of the scattering coefficient S.

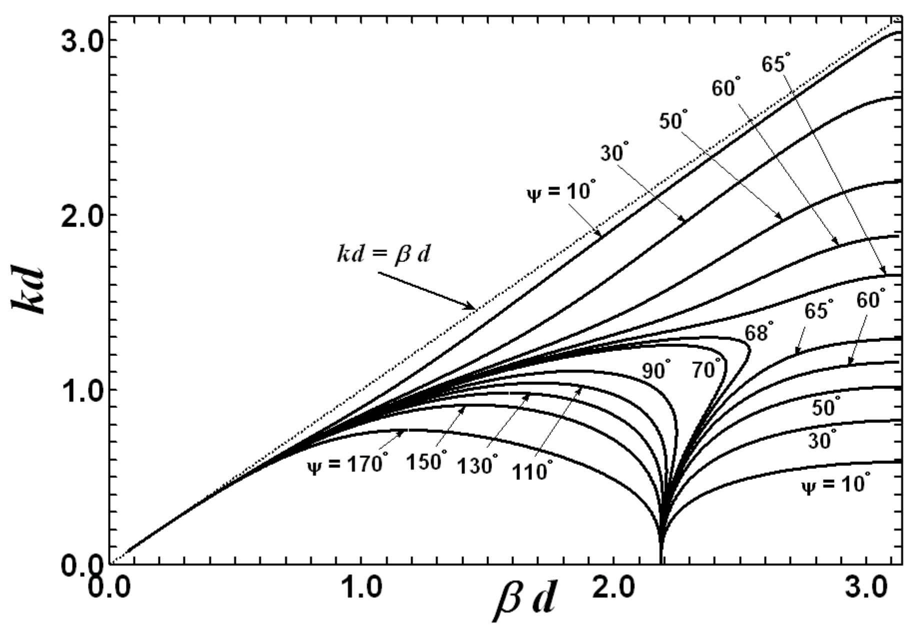

Figure 4: Family of kd–βd curves for 3D array of dipoles (perpendicular to the array plane) with constant values of the phase ψ of the scattering coefficient S.

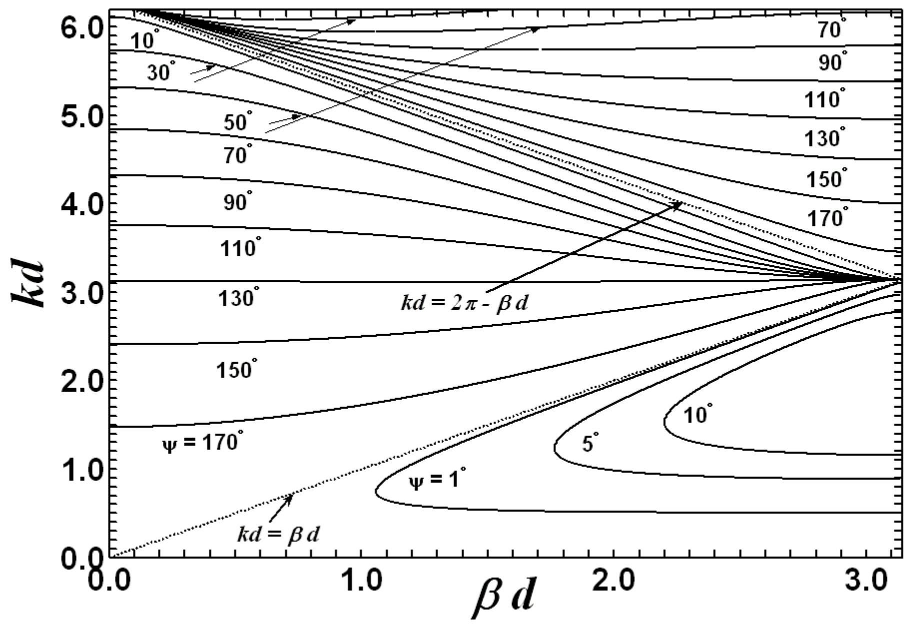

Figure 5: Family of kd–βd curves for 3D array of dipoles (normal to the propagation direction) with constant values of the phase ψ of the scattering coefficient S.

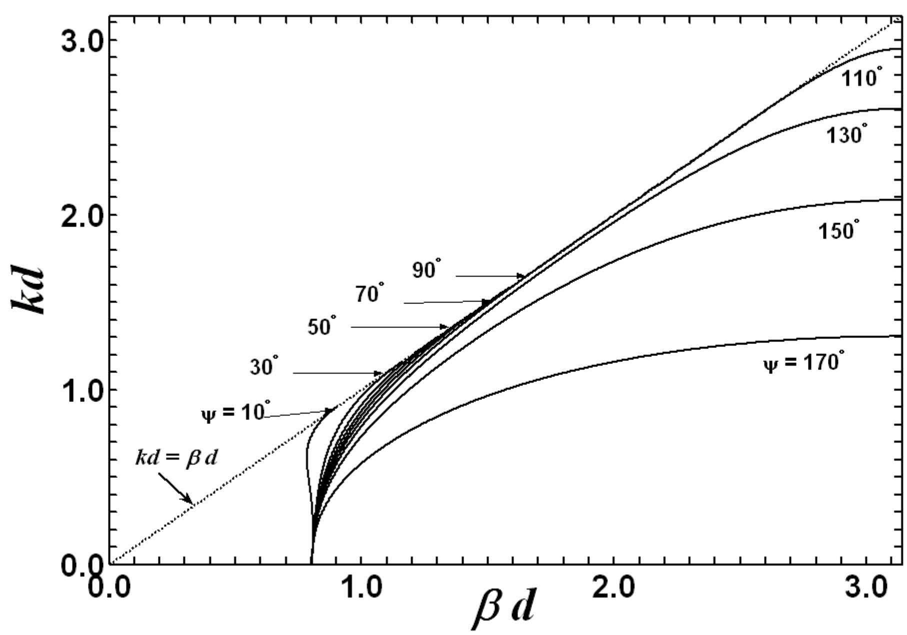

Figure 6: Family of kd–βd curves for 2D array of dipoles (parallel to the propagation direction) with constant values of the phase ψ of the scattering coefficient S.

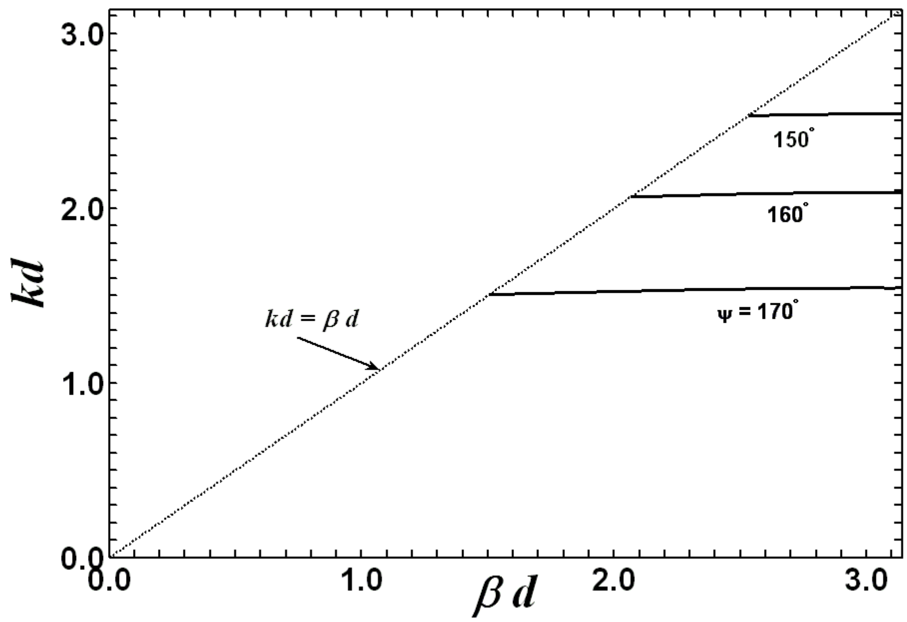

Figure 7: Family of kd–βd curves for 3D array of dipoles (parallel to the propagation direction) with constant values of the phase ψ of the scattering coefficient S.

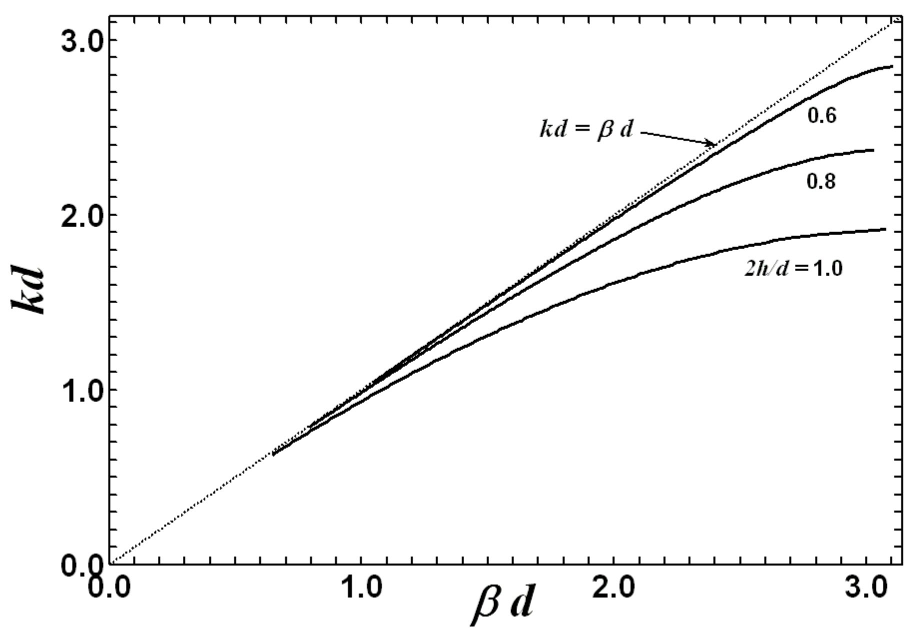

Figure 8: kd–βd curves for 2D array of short-wire electric dipoles (parallel to the array plane) with ψ obtained from NEC code.

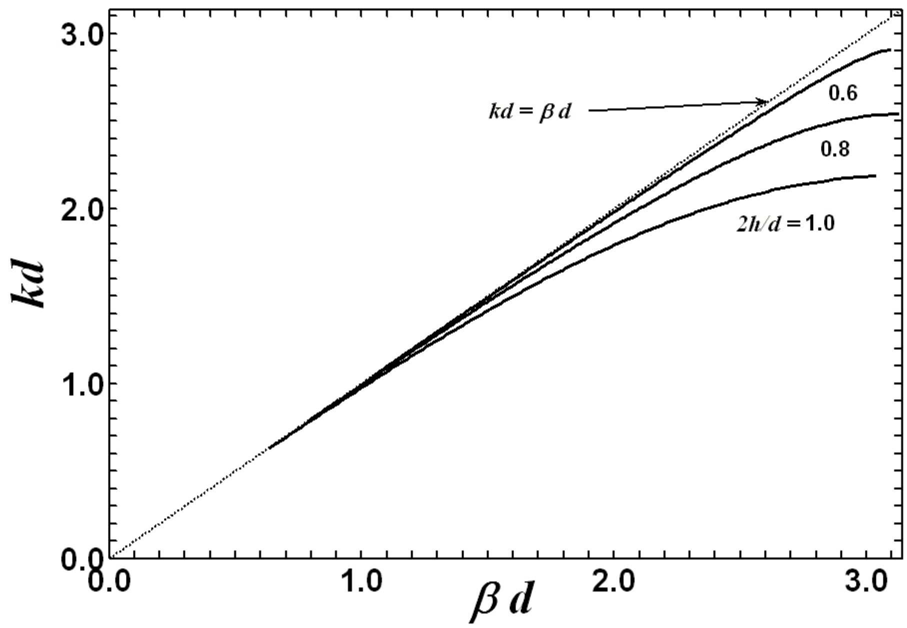

Figure 9: kd–βd curves for 2D array of short-wire electric dipoles (perpendicular to the array plane) with ψ obtained from NEC code.

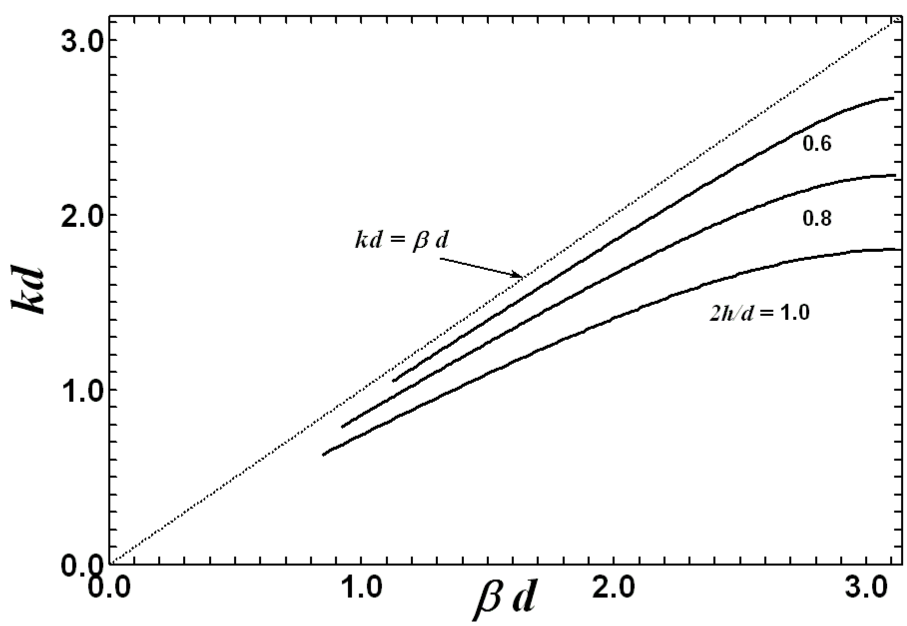

Figure 10: kd–βd curves for 3D array of short-wire electric dipoles (normal to the propagation direction) with ψ obtained from NEC code.

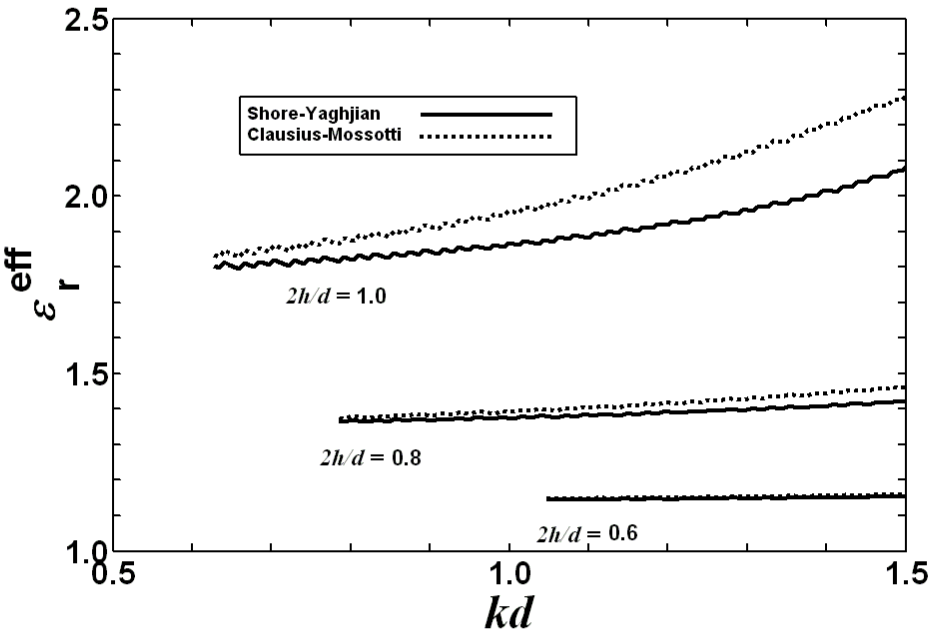

Figure 11: Effective relative permittivity for 3D array of short-wire electric dipoles (normal to the propagation direction) with ψ obtained from NEC code.

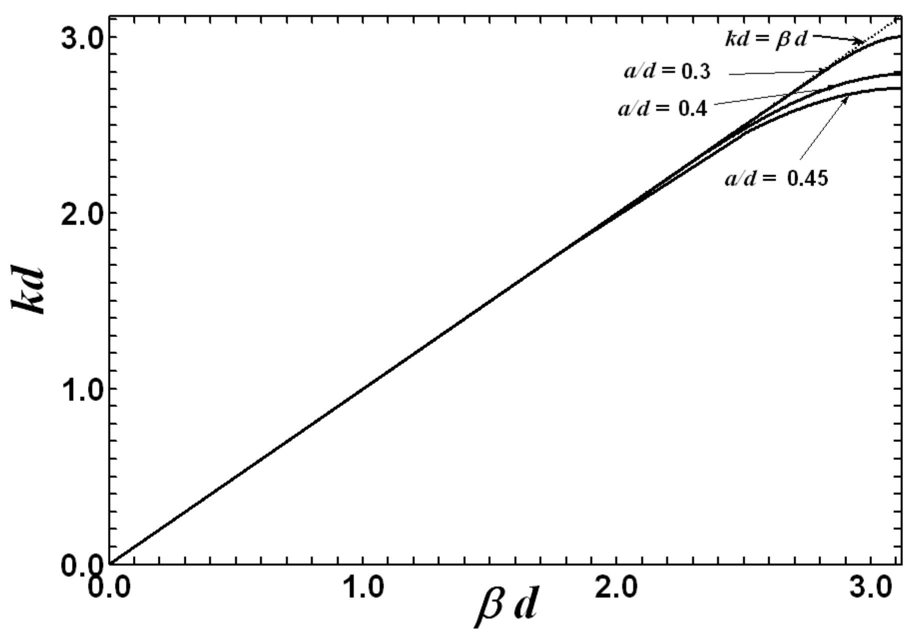

Figure 12: kd–βd curves for 1D array of PEC spheres (dipole moments normal to the propagation direction) with dipole scattering coefficients obtained from Mie solution.

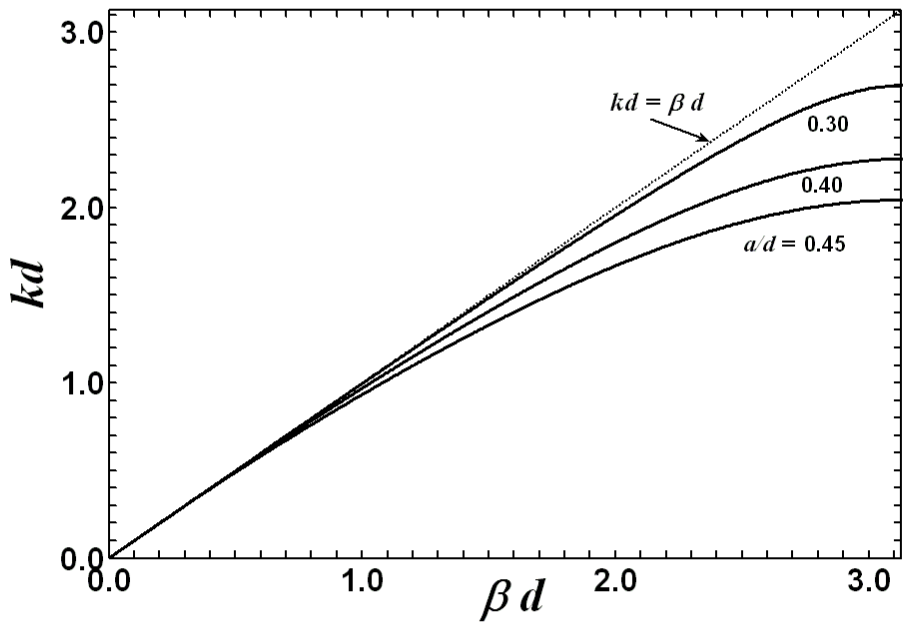

Figure 13: kd–βd curves for 2D array of PEC spheres (electric dipole moments parallel to the array plane) with dipole scattering coefficients obtained from Mie solution.

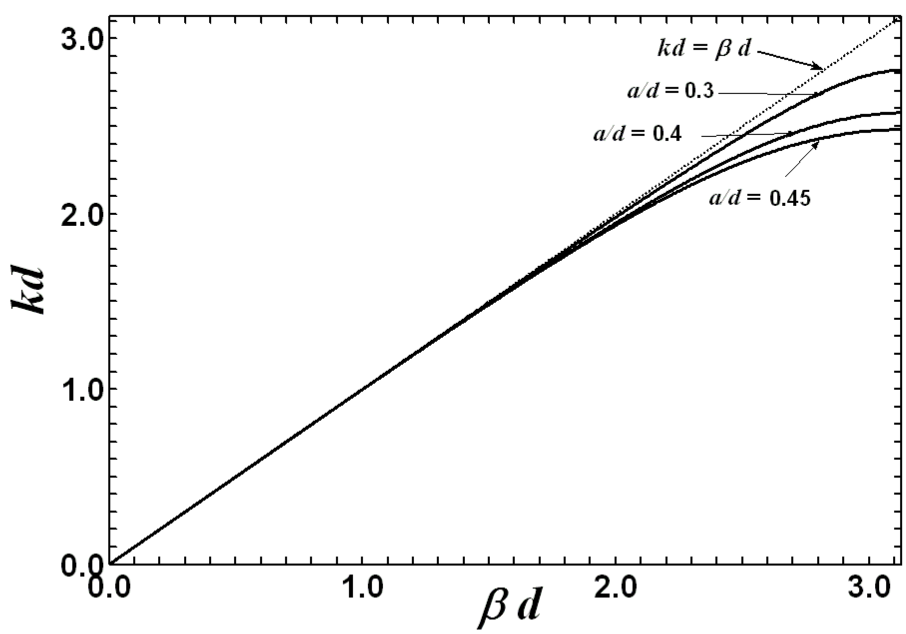

Figure 14: kd–βd curves for 2D array of PEC spheres (electric dipole moments perpendicular to the array plane) with dipole scattering coefficients obtained from Mie solution.

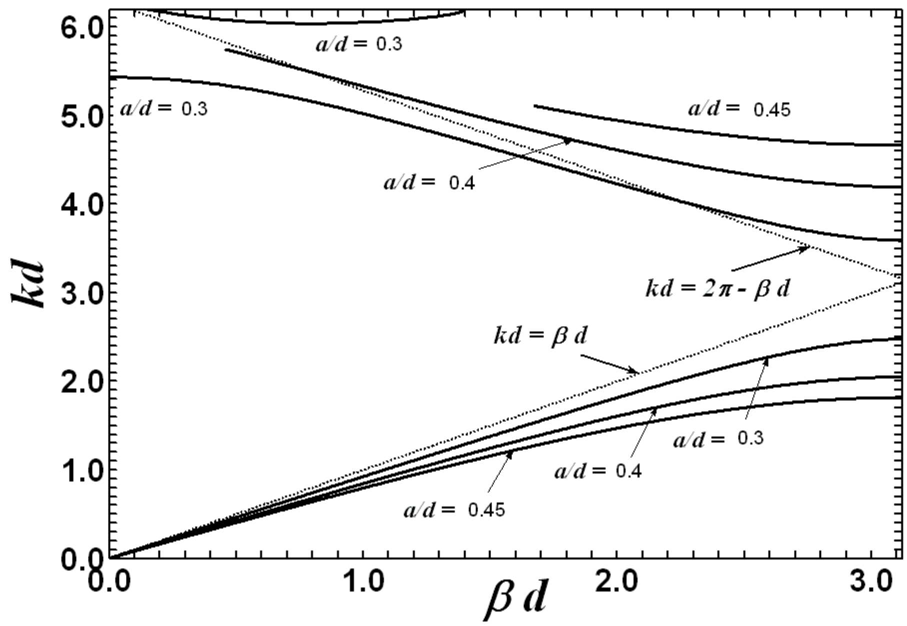

Figure 15: kd–βd curves for 3D array of PEC spheres (dipole moments normal to the propagation direction) with dipole scattering coefficients obtained from Mie solution.

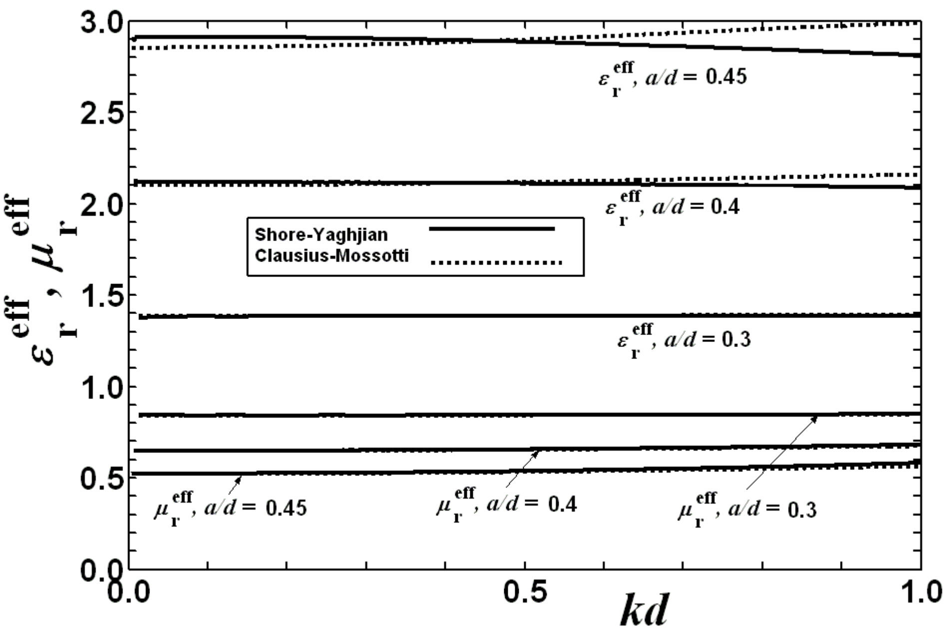

Figure 16: Effective relative permittivity and permeability for 3D array of PEC spheres (dipole moments normal to the propagation direction) with dipole scattering coefficients obtained from Mie solution.

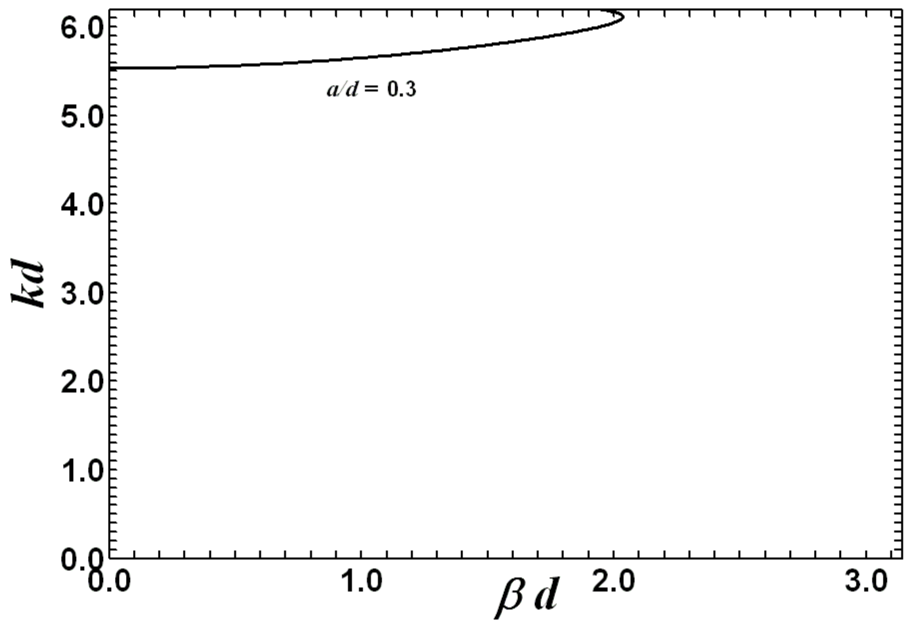

Figure 17: kd–βd curve for 3D array of PEC spheres (electric dipole moments parallel to the propagation direction) with electric dipole scattering coefficients obtained from Mie solution.

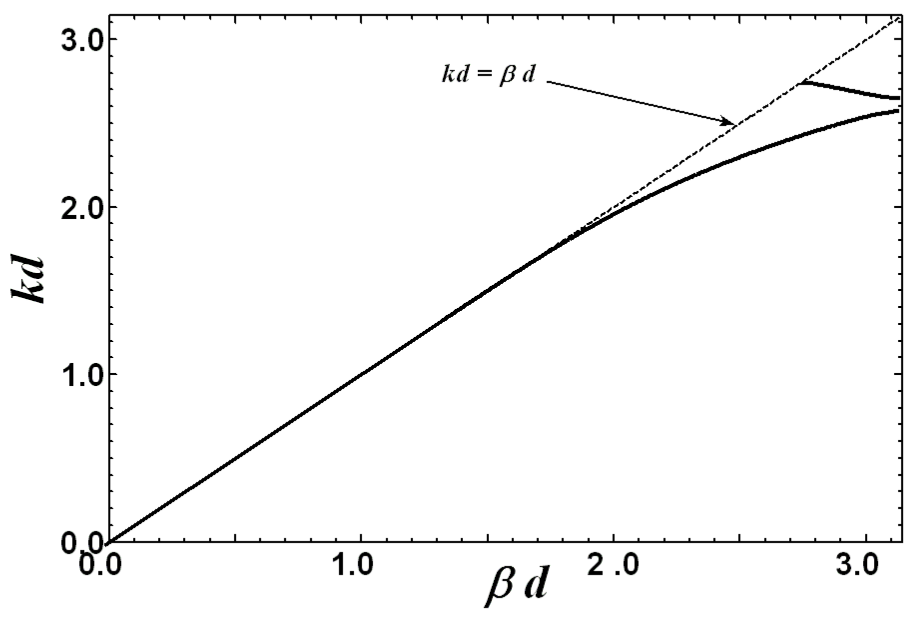

Figure 18: kd–βd diagram for 1D array of diamond spheres (dipole moments normal to the propagation direction) with $a/d = .45$ and dipole scattering coefficients obtained from Mie solution.

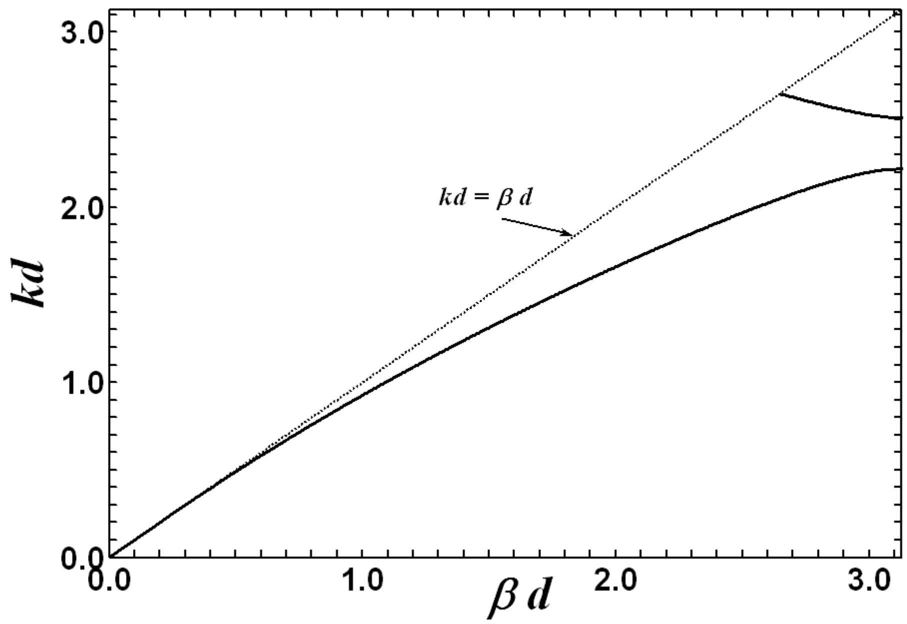

Figure 19: kd–βd diagram for 2D array of diamond spheres (electric dipole moments parallel to the array plane) with $a/d = .45$ and dipole scattering coefficients obtained from Mie solution.

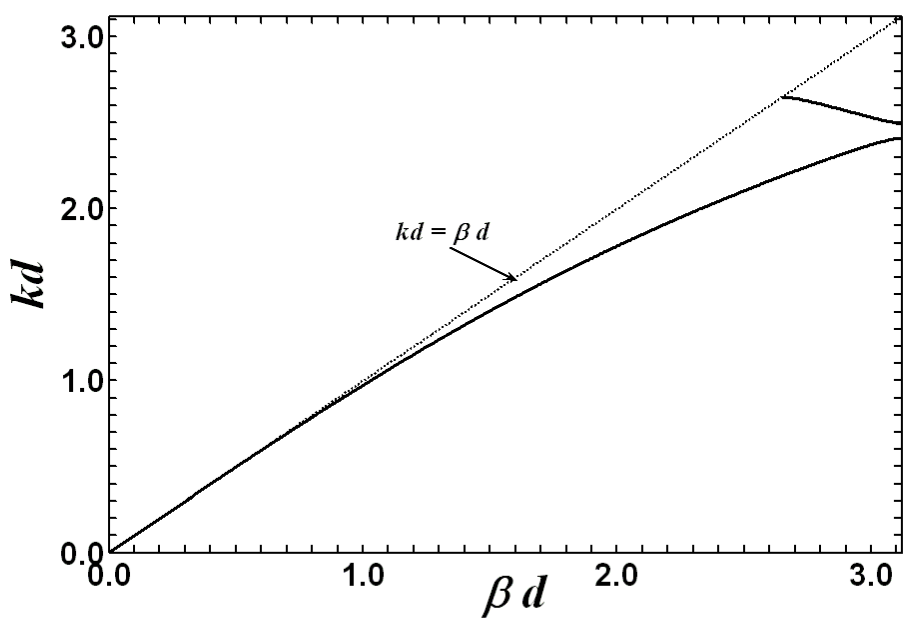

Figure 20: kd–βd diagram for 2D array of diamond spheres (electric dipole moments perpendicular to the array plane) with $a/d = .45$ and dipole scattering coefficients obtained from Mie solution.

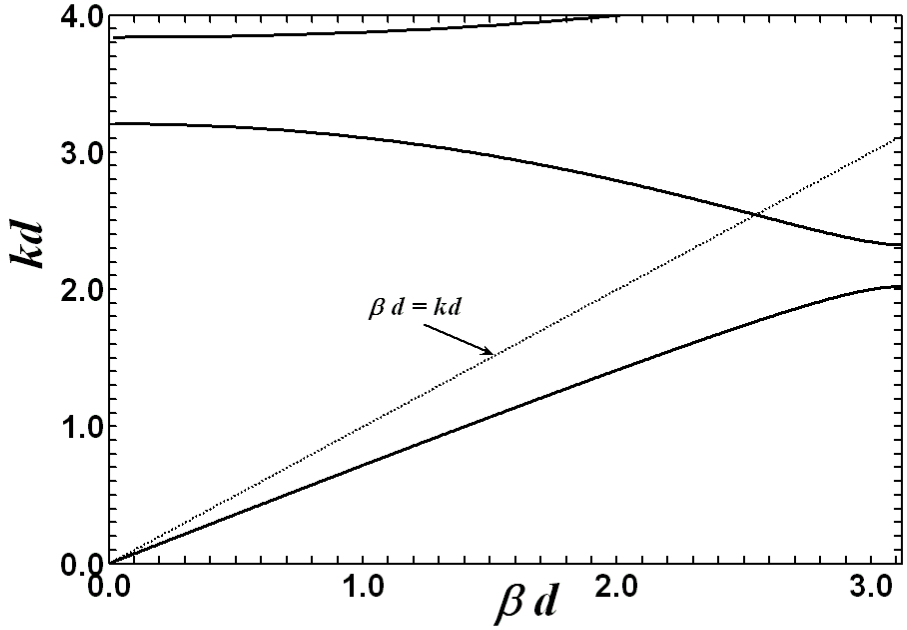

Figure 21: kd–βd diagram for 3D array of diamond spheres (dipole moments normal to the propagation direction) with $a/d = .45$ and dipole scattering coefficients obtained from Mie solution.

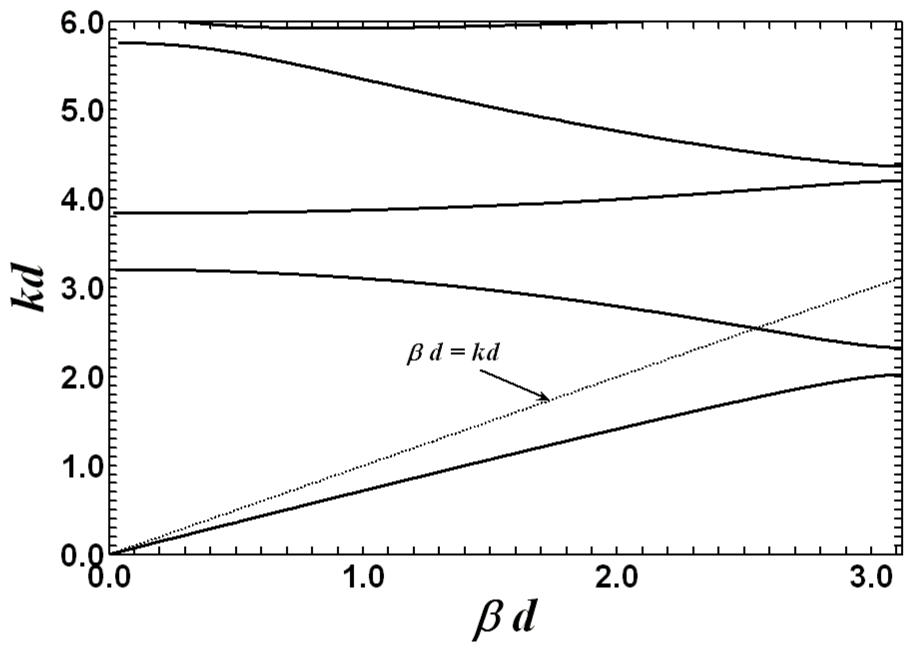

Figure 22: Extended kd–βd diagram for 3D array of diamond spheres (dipole moments normal to the propagation direction) with $a/d = .45$ and dipole scattering coefficients obtained from Mie solution.

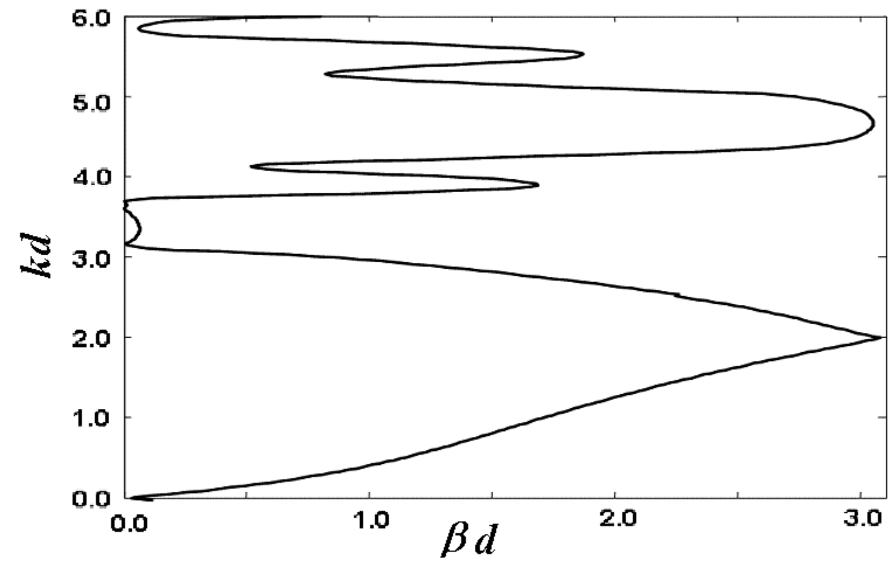

Figure 23: Extended kd–βd diagram for 3D array of diamond spheres (dipole moments normal to the propagation direction) with $a/d = .45$ and dipole scattering coefficients obtained from FDTD solution.

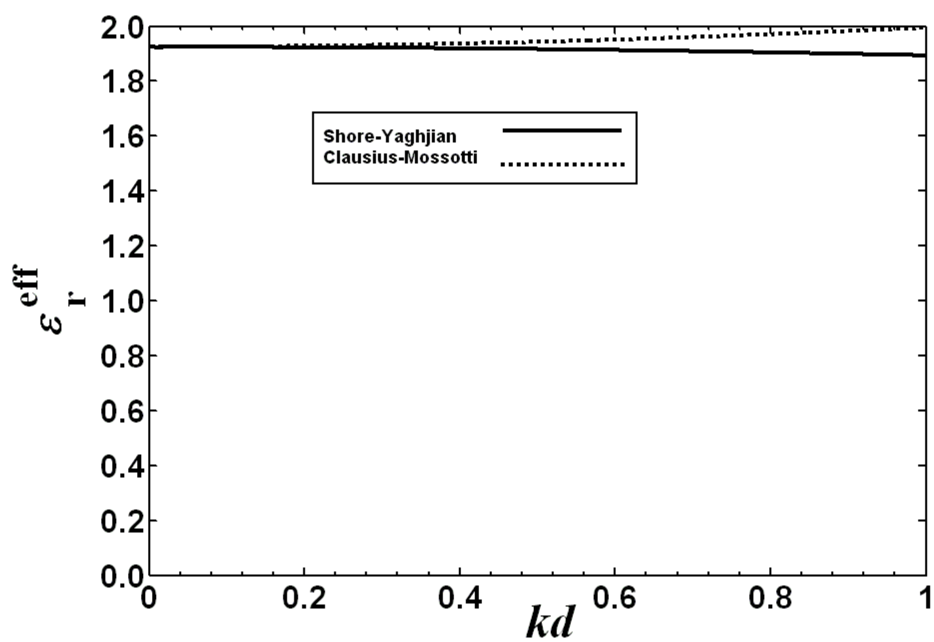

Figure 24: Effective relative permittivity for 3D array of diamond spheres (dipole moments normal to the propagation direction) with $a/d = .45$ and dipole scattering coefficients obtained from Mie solution.

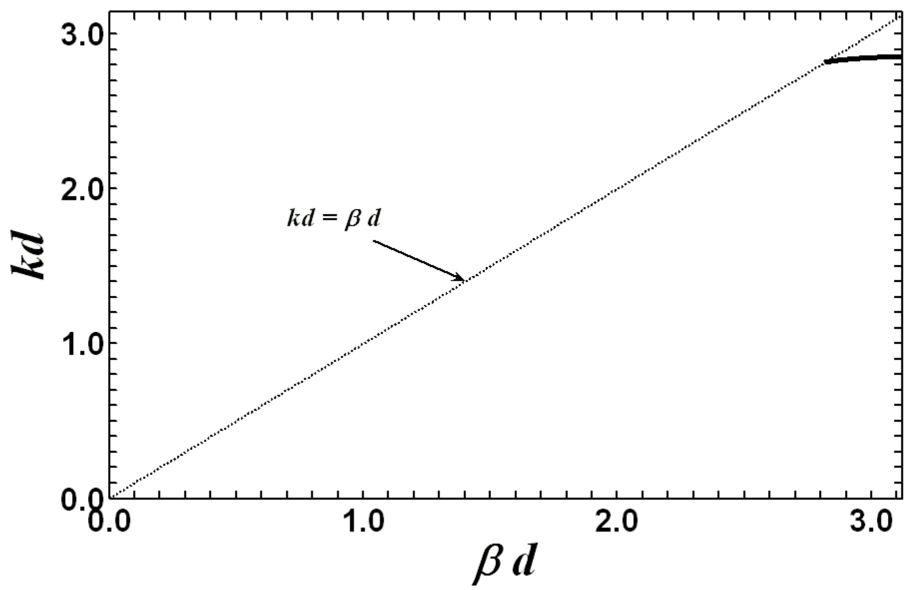

Figure 25: kd–βd diagram for 1D array of diamond spheres (magnetic dipole moments parallel to the propagation direction) with $a/d = .45$ and magnetic dipole scattering coefficients obtained from Mie solution.

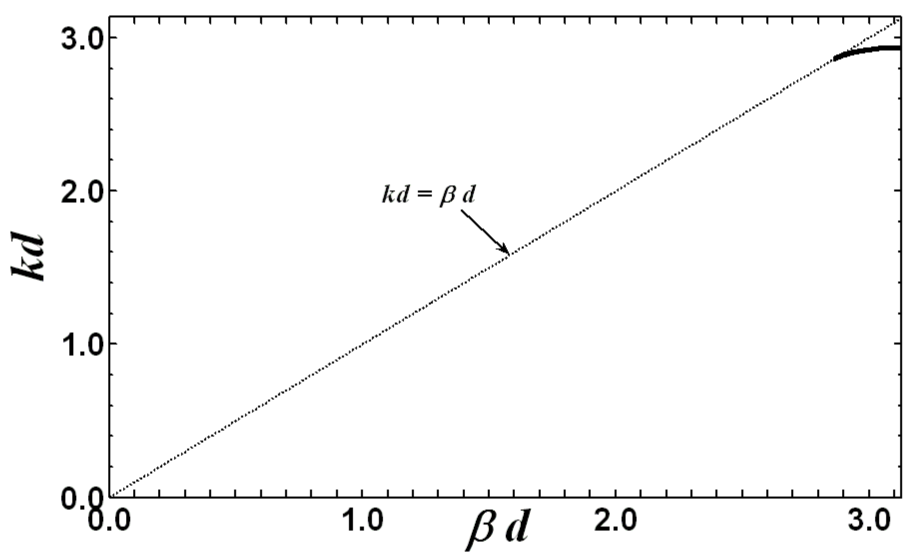

Figure 26: kd–βd diagram for 2D array of diamond spheres (magnetic dipole moments parallel to the propagation direction) with $a/d = .45$ and magnetic dipole scattering coefficients obtained from Mie solution.

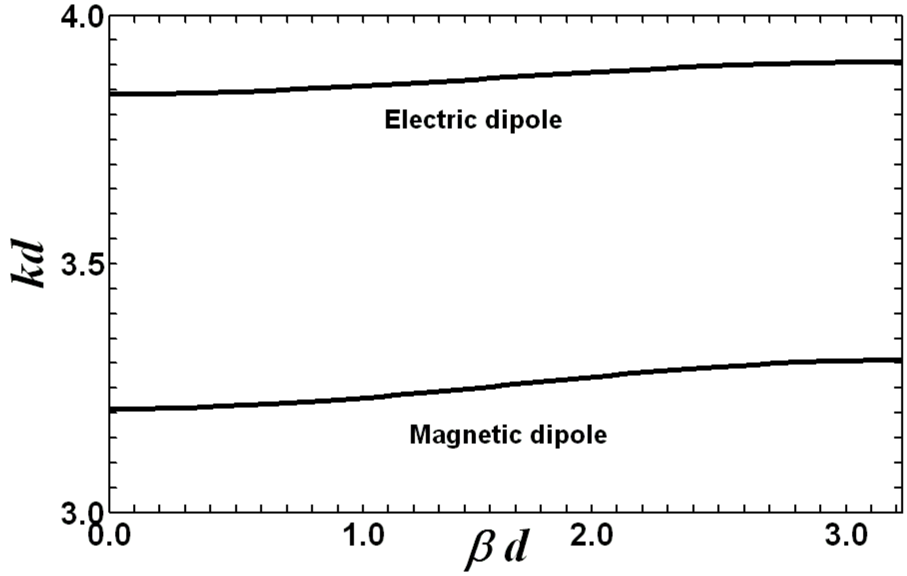

Figure 27: kd–βd diagram for 3D array of diamond spheres (magnetic dipole moments parallel to the propagation direction) with $a/d = .45$ and magnetic dipole scattering coefficients obtained from Mie solution.

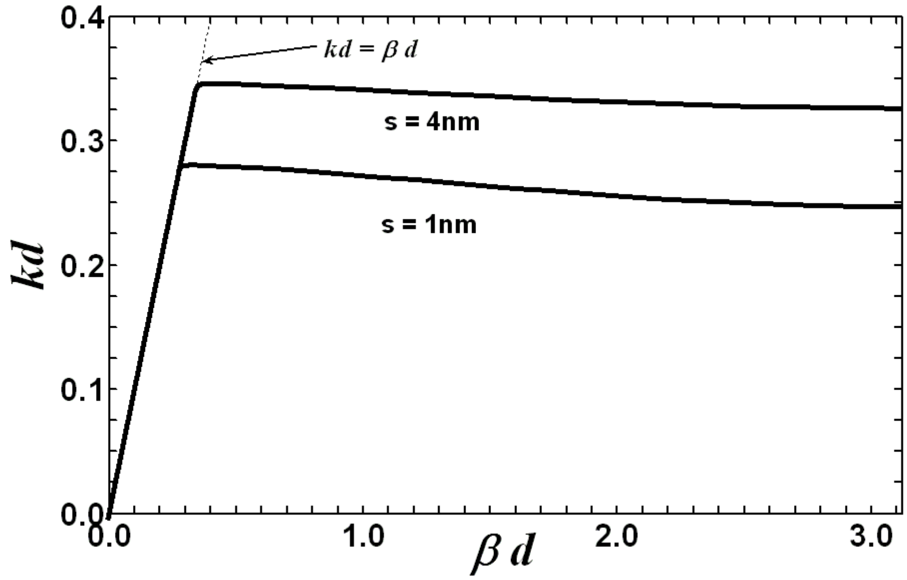

Figure 28: kd–βd curves for 1D array of glass-embedded silver nanospheres (dipole moments normal to the propagation direction) with $a = 5$ nm and dipole scattering coefficients obtained from Mie solution.

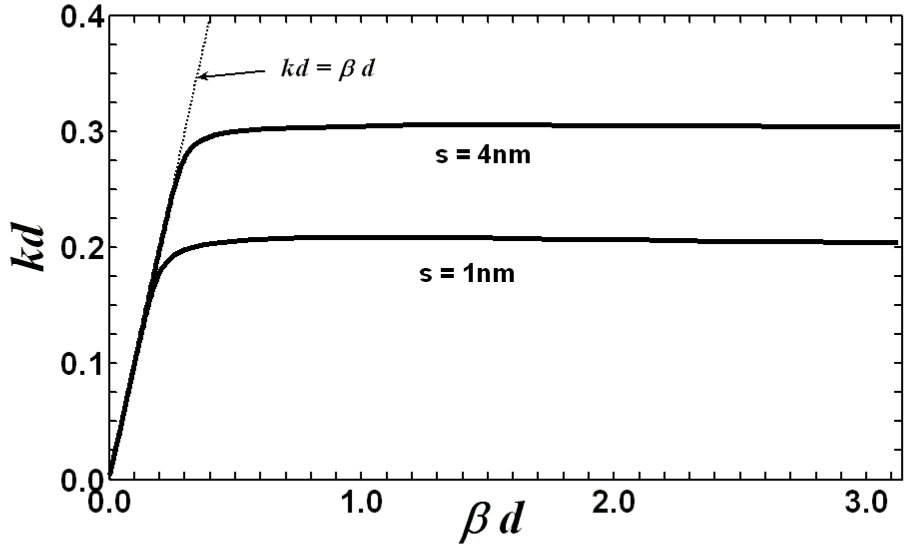

Figure 29: kd–βd curves for 2D array of glass-embedded silver nanospheres (electric dipole moments parallel to the array plane) with $a = 5$ nm and dipole scattering coefficients obtained from Mie solution.

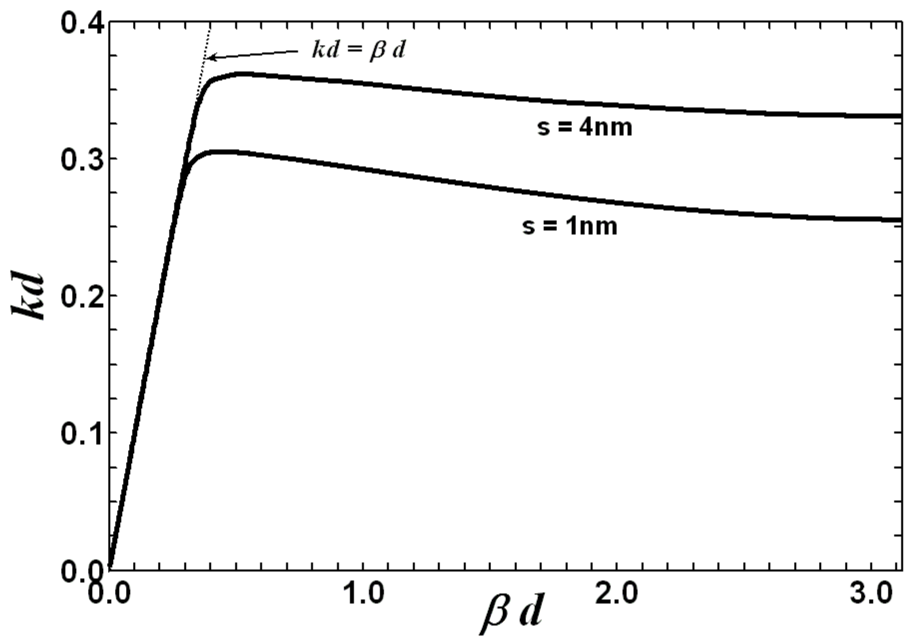

Figure 30: kd–βd curves for 2D array of glass-embedded silver nanospheres (electric dipole moments perpendicular to the array plane) with $a = 5$ nm and dipole scattering coefficients obtained from Mie solution.

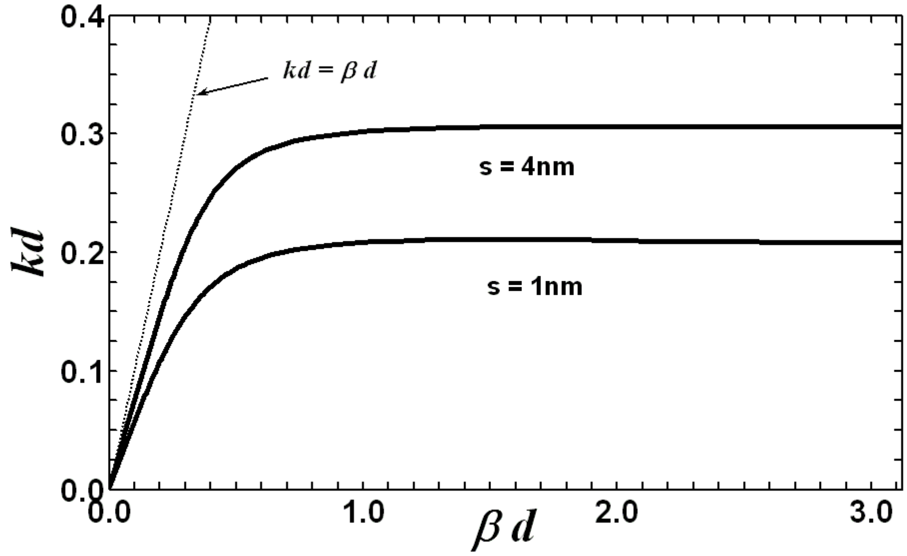

Figure 31: kd–βd curves for 3D array of glass-embedded silver nanospheres (dipole moments normal to the propagation direction) with $a = 5$ nm and dipole scattering coefficients obtained from Mie solution.

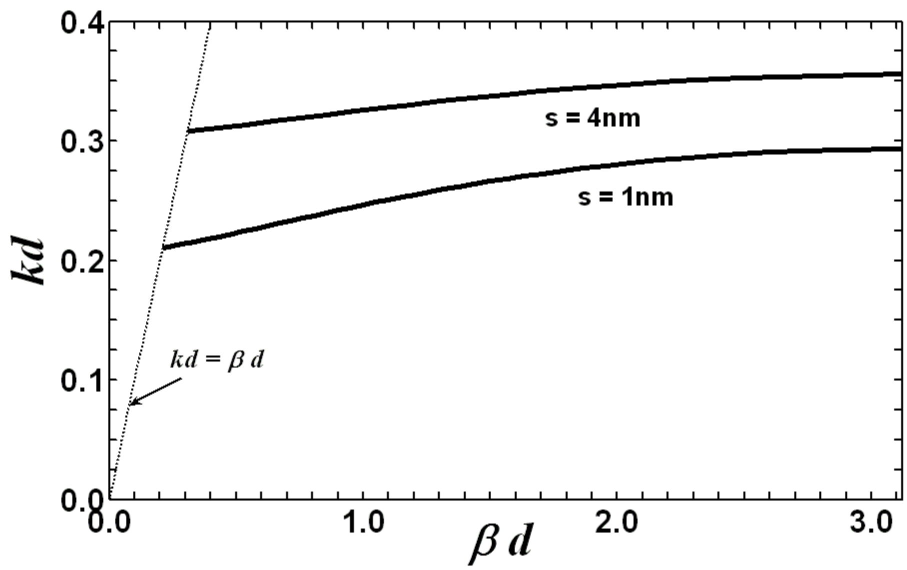

Figure 32: kd–βd curves for 1D array of glass-embedded silver nanospheres (electric dipole moments parallel to the direction of propagation) with $a = 5$ nm and electric dipole scattering coefficients obtained from Mie solution.

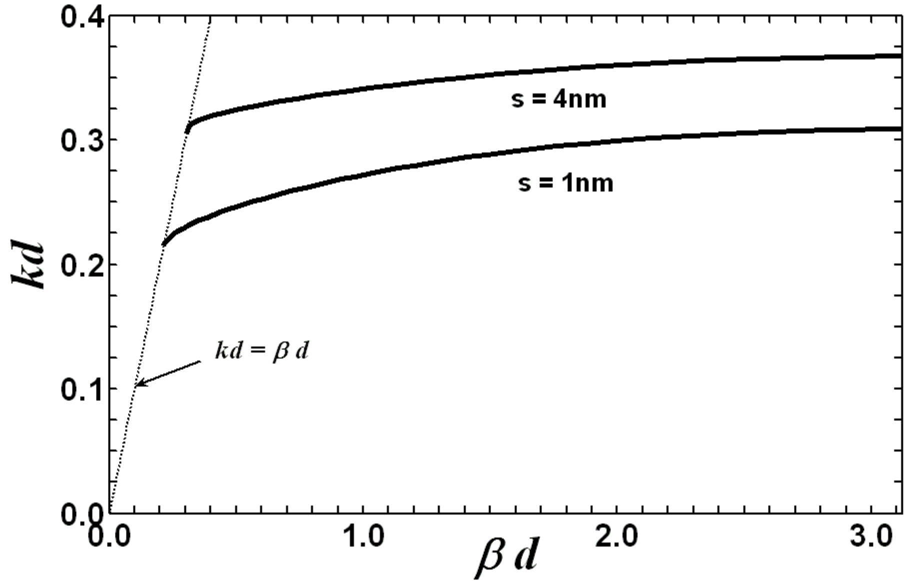

Figure 33: kd–βd curves for 2D array of glass-embedded silver nanospheres (electric dipole moments parallel to the direction of propagation) with $a = 5$ nm and electric dipole scattering coefficients obtained from Mie solution.

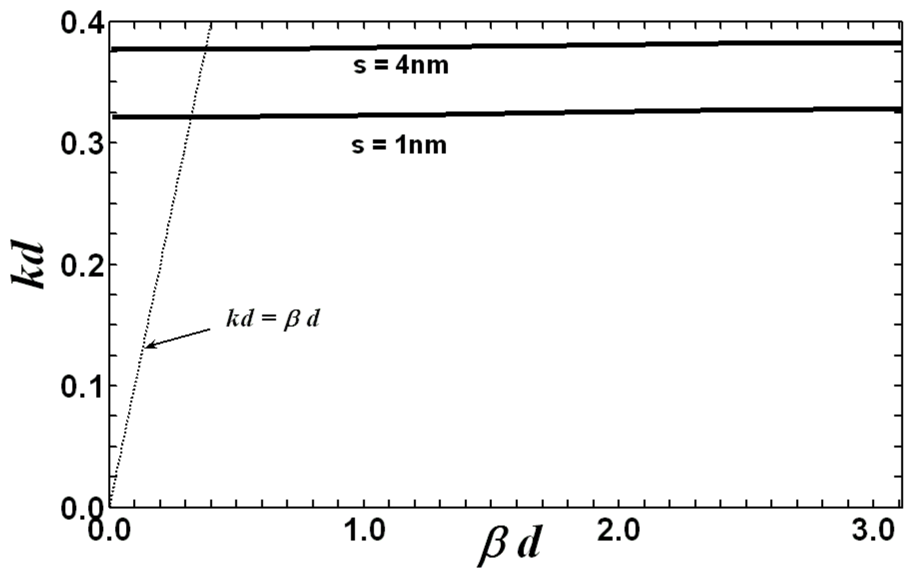

Figure 34: kd–βd curves for 3D array of glass-embedded silver nanospheres (electric dipole moments parallel to the direction of propagation) with $a = 5$ nm and electric dipole scattering coefficients obtained from Mie solution.

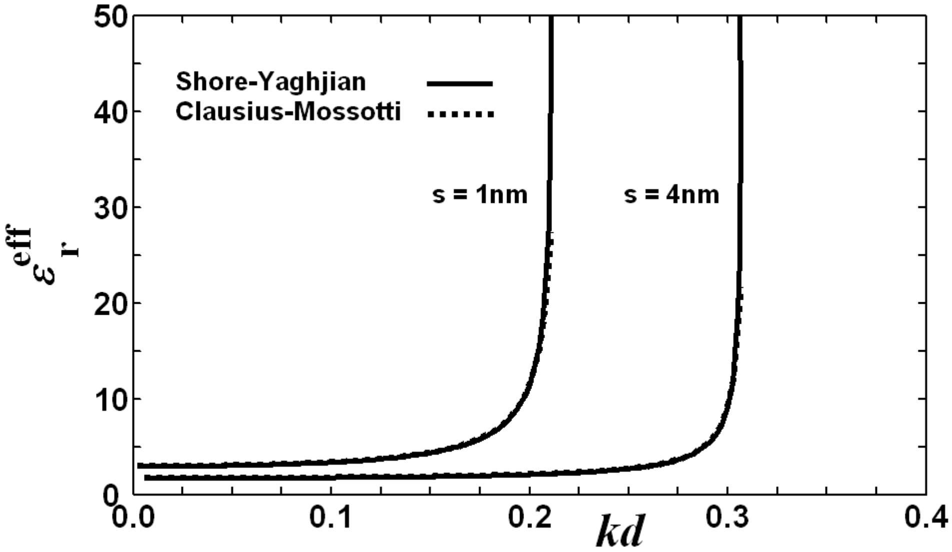

Figure 35: Effective relative permittivity for 3D array of glass-embedded silver nanospheres (dipole moments normal to the propagation direction) with $a = 5$ nm and dipole scattering coefficients obtained from Mie solution.

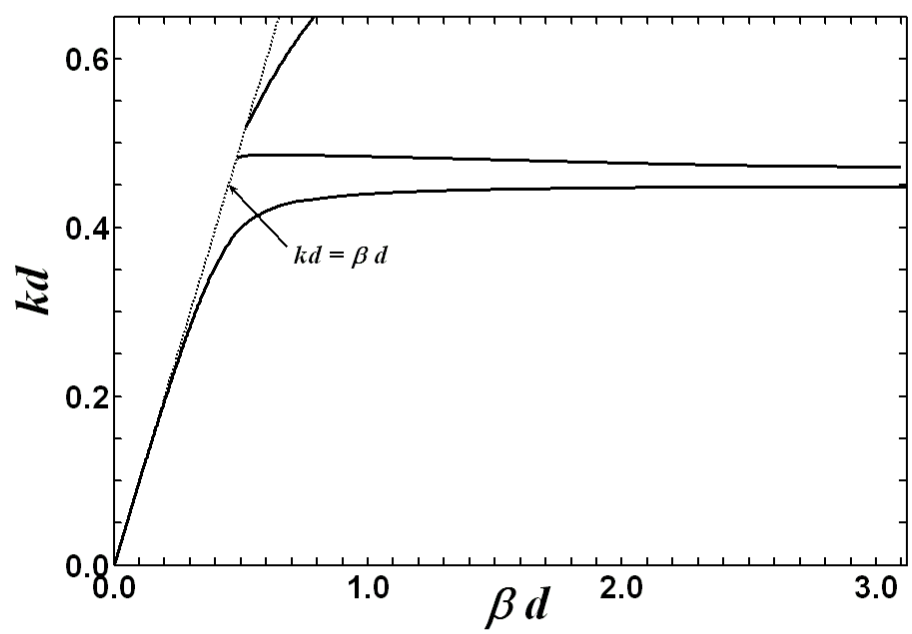

Figure 36: kd–βd diagram for 2D array of $\epsilon_r = \mu_r = 20$ magnetodielectric spheres (electric dipole moments parallel or perpendicular to the array plane) with $a/d = .45$ and dipole scattering coefficients obtained from Mie solution.

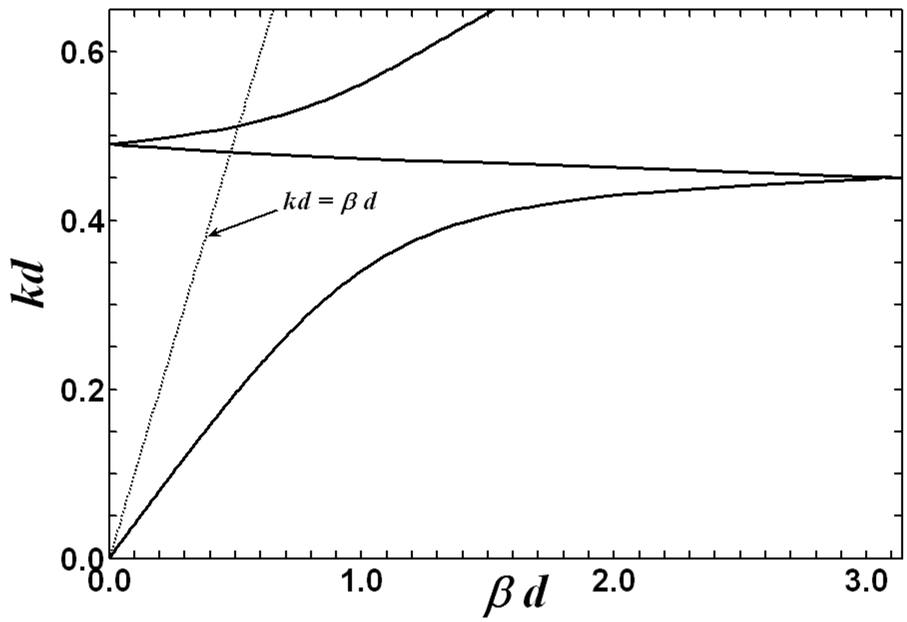

Figure 37: kd–βd diagram for 3D array of $\epsilon_{\rm r} = \mu_{\rm r} = 20$ magnetodielectric spheres (dipole moments normal to the propagation direction) with $a/d = .45$ and dipole scattering coefficients obtained from Mie solution.

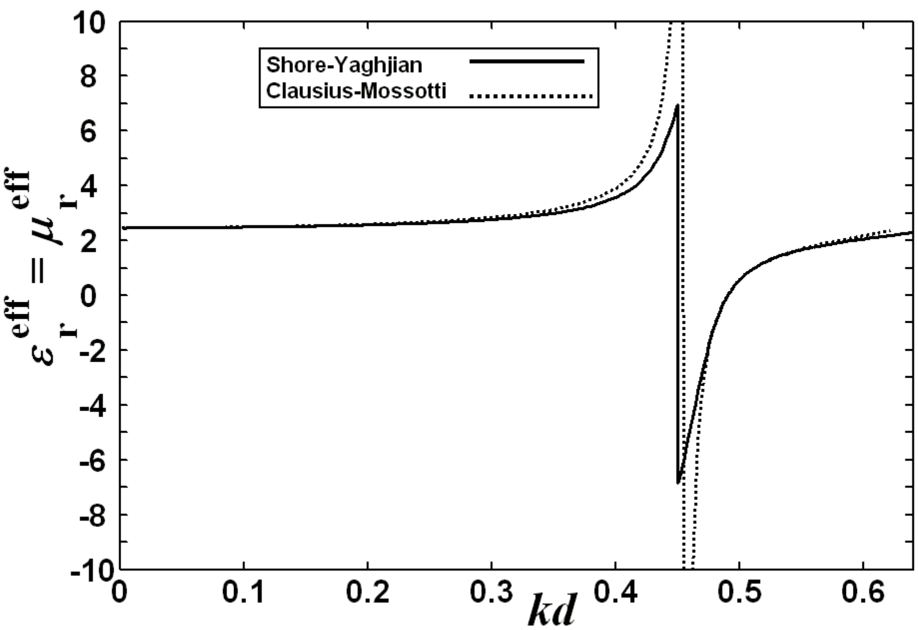

Figure 38: Effective relative permittivity and permeability for 3D array of $\epsilon_{\rm r} = \mu_{\rm r} = 20$ magnetodielectric spheres (dipole moments normal to the propagation direction) with dipole scattering coefficients obtained from Mie solution.

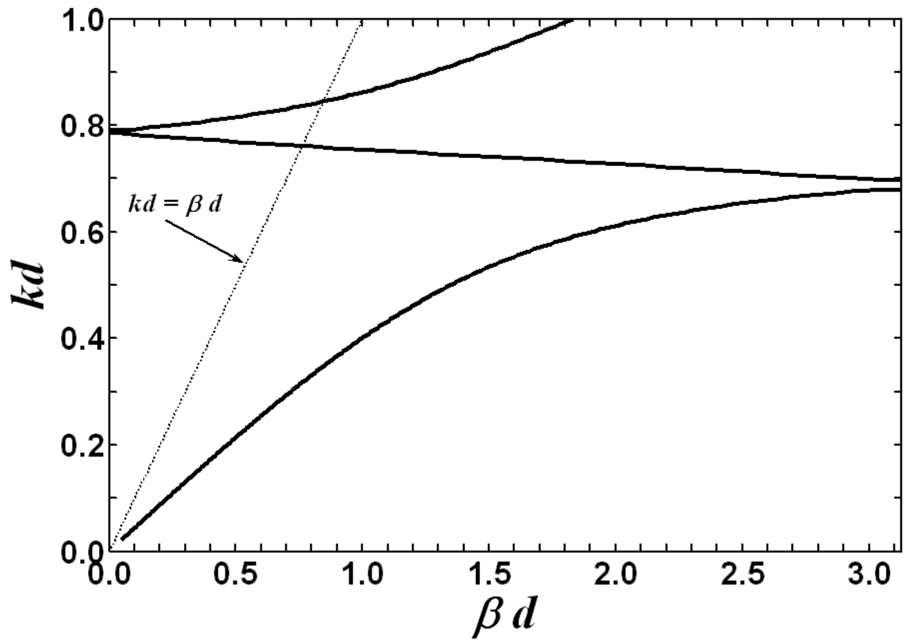

Figure 39: kd–βd diagram for 3D array of $\epsilon_{\rm r} = 13.8$, $\mu_{\rm r} = 11.0$ magnetodielectric spheres (dipole moments normal to the propagation direction) with $a/d = .45$ and dipole scattering coefficients obtained from Mie solution.

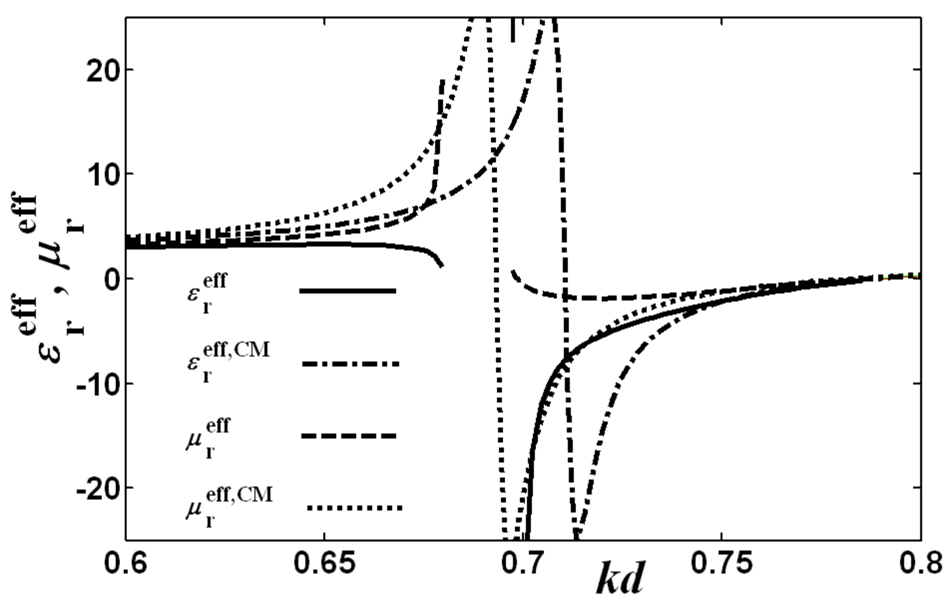

Figure 40: Effective relative permittivity and permeability for 3D array of $\epsilon_{\rm r} = 13.8$, $\mu_{\rm r} = 11.0$ magnetodielectric spheres (dipole moments normal to the propagation direction) with dipole scattering coefficients obtained from Mie solution.

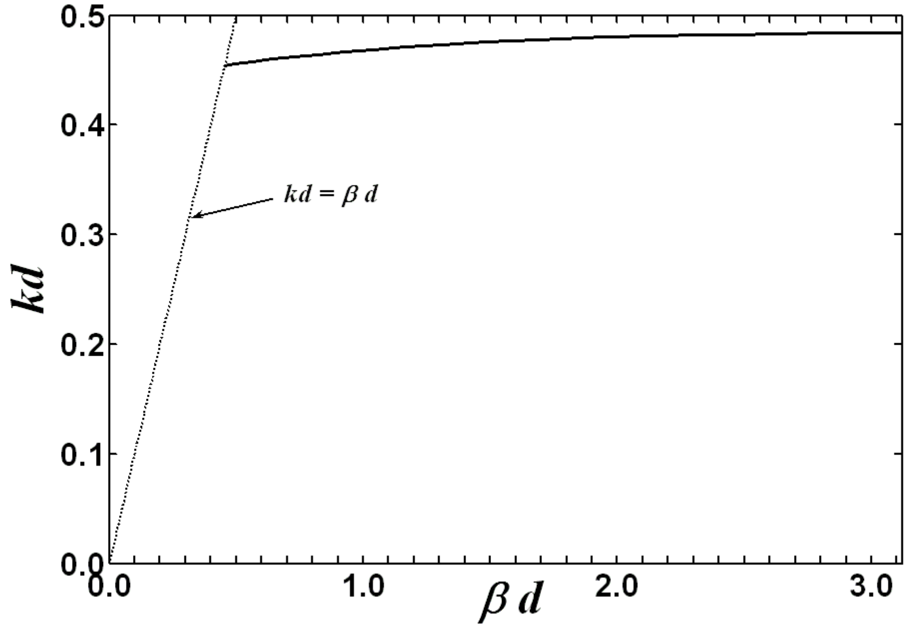

Figure 41: kd–βd diagram for 1D array of $\epsilon_r = \mu_r = 20$ magnetodielectric spheres (dipole moments parallel to the propagation direction) with $a/d = .45$ and dipole scattering coefficients obtained from Mie solution.

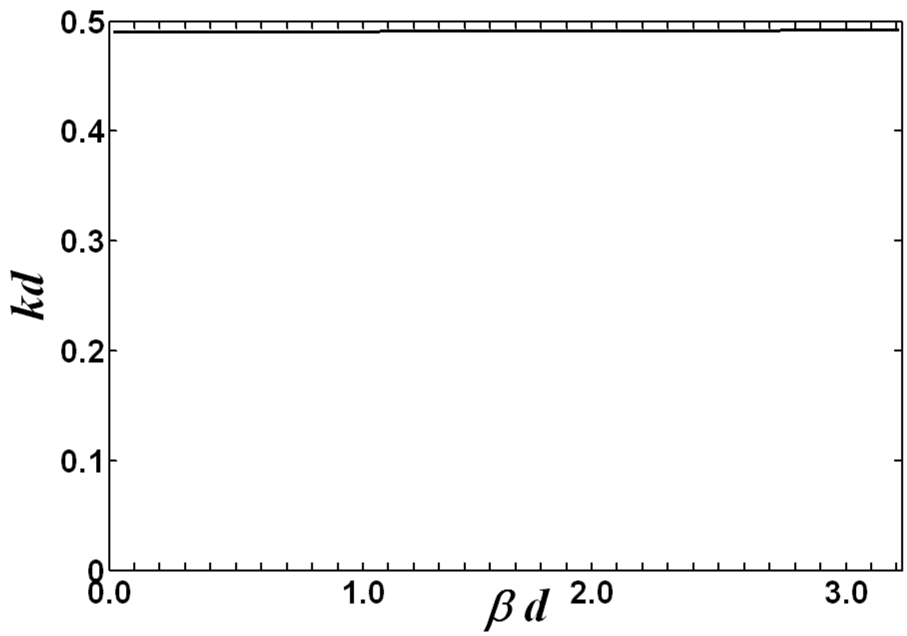

Figure 42: kd–βd diagram for 3D array of $\epsilon_r = \mu_r = 20$ magnetodielectric spheres (dipole moments parallel to the propagation direction) with $a/d = .45$ and dipole scattering coefficients obtained from Mie solution.

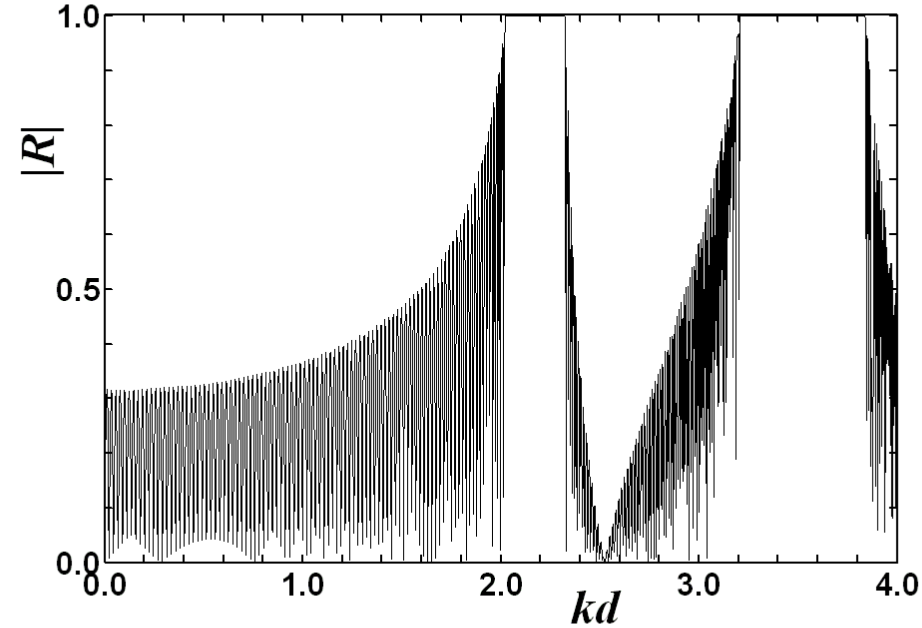

Figure 43: Reflection coefficient of a lossless partially finite 3D array of diamond spheres (dipole moments normal to the propagation direction) with $\epsilon_{\mathrm{r}} = 5.84,\ \mu_{\mathrm{r}} = 1,\ a/d = .45$, and dipole scattering coefficients obtained from Mie solution.

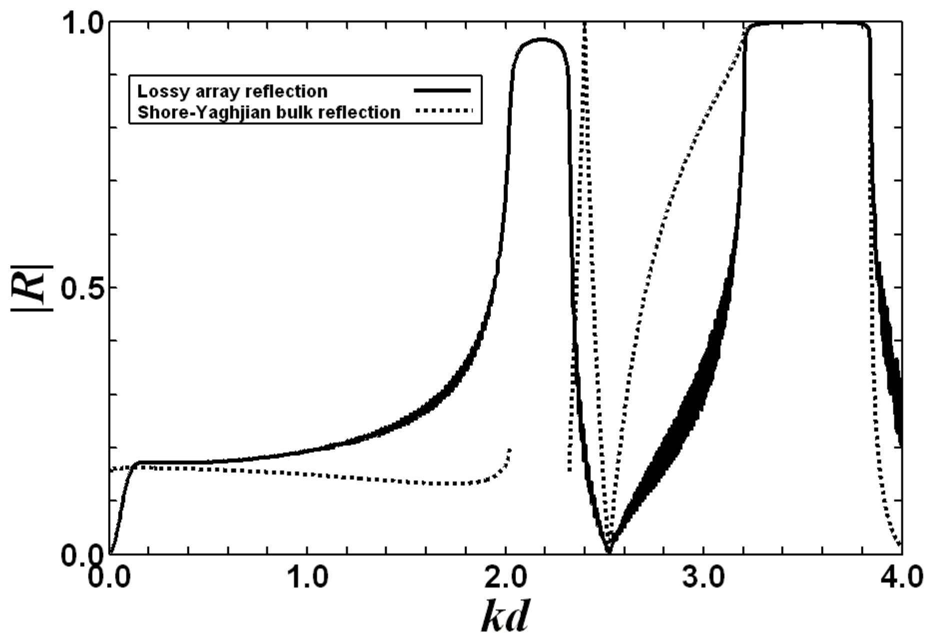

Figure 44: Reflection coefficient of a lossy partially finite 3D array of diamond spheres (dipole moments normal to the propagation direction) with $\epsilon_{\mathrm{r}} = 5.84,\ \mu_{\mathrm{r}} = 1,\ a/d = .45$, and dipole scattering coefficients obtained from Mie solution; and the Shore-Yaghjian reflection coefficient.

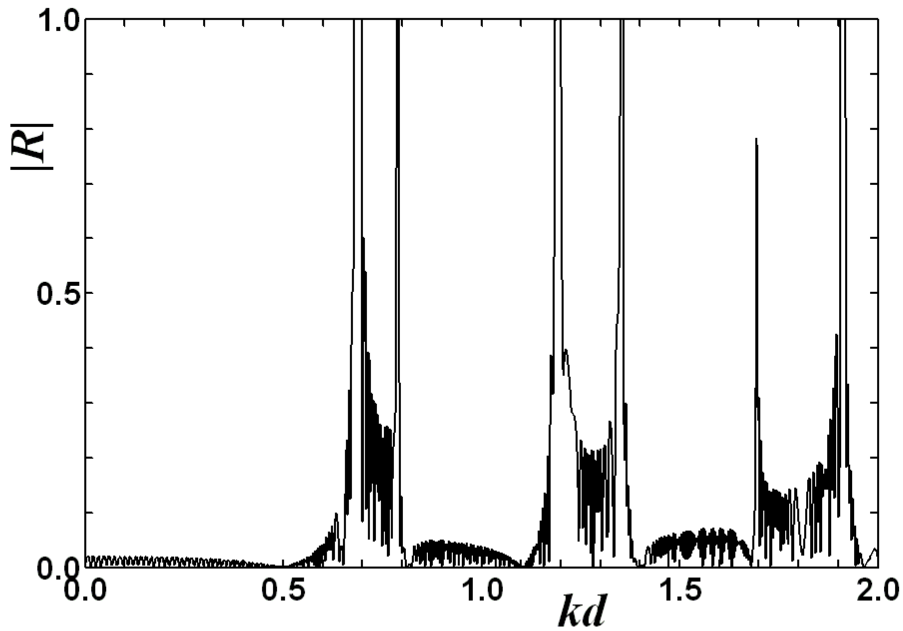

Figure 45: Reflection coefficient of a lossless partially finite 3D array of $\epsilon_{\rm r} = 13.8,\ \mu_{\rm r} = 11$ magnetodielectric spheres (dipole moments normal to the array axis) with $a/d = .45$ and dipole scattering coefficients obtained from Mie solution.

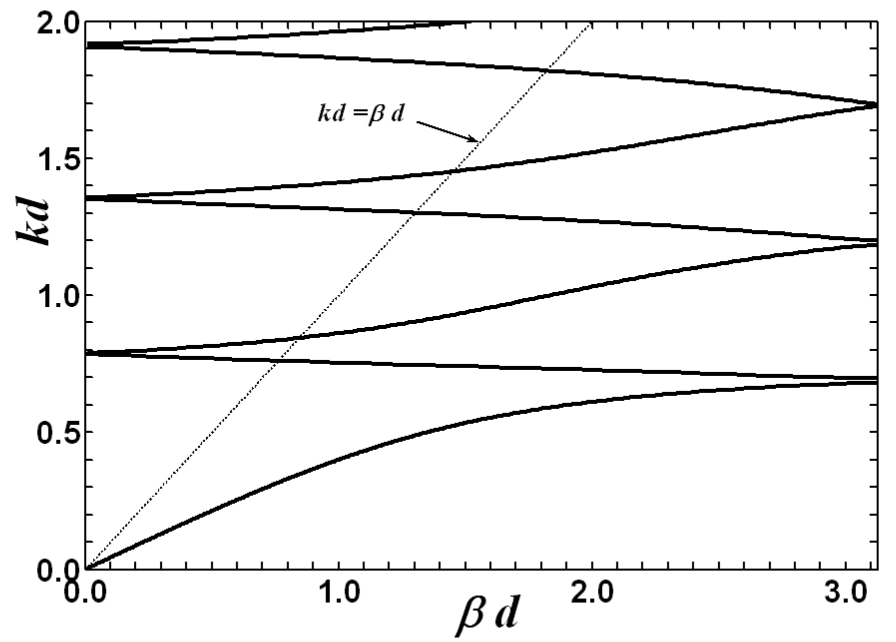

Figure 46: Extended kd–βd diagram for an infinite 3D array of $\epsilon_{\rm r} = 13.8,\ \mu_{\rm r} = 11$ magnetodielectric spheres, (dipoles normal to the array axis) with $a/d = .45$ and dipole scattering coefficients obtained from Mie solution.

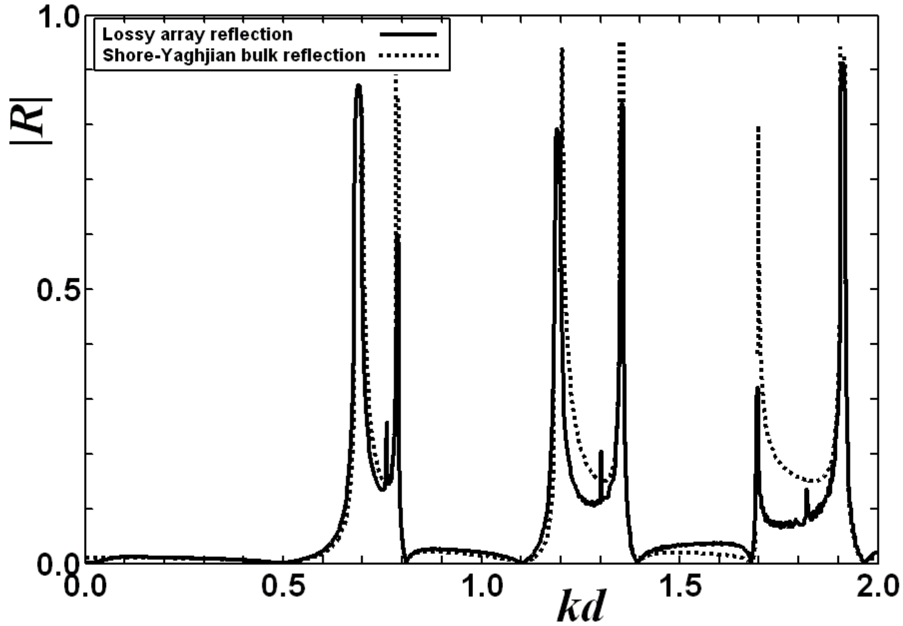

Figure 47: Reflection coefficient of a partially finite 3D array of $\epsilon_{\rm r} = 13.8,\ \mu_{\rm r} = 11$ magnetodielectric spheres, (dipoles normal to the array axis) with $a/d = .45$ and dipole scattering coefficients obtained from Mie solution; and the Shore-Yaghjian reflection coefficient.

A BIDIRECTIONALITY OF RECIPROCAL, LOSSY OR LOSSLESS, UNIFORM OR PERIODIC WAVEGUIDES

The main purpose of this Appendix is to prove that a reciprocal (lossy or lossless) waveguide (uniform or periodic) that supports a traveling wave with propagation constant β also supports a corresponding traveling wave with propagation constant $-\beta$. In other words, all reciprocal waveguides are bidirectional.

We see the need to prove this result because of some differing conclusions found in the published literature. McIsaac [29] stated without proof that "reciprocity is not a sufficient condition for bidirectionality in a lossy waveguide." Harrington and Villeneuve [30], [31] concluded without showing their proof that "for every β in the original media there exists in the transposed media a propagation constant $\hat{\beta} = -\beta$." They base their unproven conclusion on the idea that a time-harmonic traveling wave on any waveguide has a network or transmission line equivalent circuit and that this transmission line equivalent circuit is bidirectional if it is reciprocal. Although it is true that a traveling wave has a transmission line equivalent circuit at each frequency, it remains to be proven that the same transmission line equivalent circuit applies to a traveling wave propagating in the opposite direction. Pissoort and Olyslager [32] give a proof of the bidirectionality of reciprocal waveguides based on the assumption that the propagation constants and the geometry of the waveguides can be expressed as analytic functions of a parameter. Since it is uncertain that this premise of parametric analyticity is always valid, we provide an alternative proof of bidirectionality for reciprocal waveguides.

We begin by proving that for every traveling wave with a complex propagation constant β on a lossless reciprocal uniform or periodic waveguide there exists a corresponding traveling wave with complex propagation constant $-\beta^*$, where the superscript $*$ denotes the complex conjugate. It should be noted that some lossless waveguides can support traveling waves with complex propagation constants. These "complex waves" must, of course, carry zero total power [33], [34], [35].

Maxwell's homogeneous equations for the most general linear spatially nondispersive material can be written as

$$\nabla \times \mathbf{E} = \mathrm{i}\omega(\overline{\boldsymbol{\mu}} \cdot \mathbf{H} + \overline{\boldsymbol{\nu}} \cdot \mathbf{E}), \quad \nabla \times \mathbf{H} = -\mathrm{i}\omega(\overline{\boldsymbol{\epsilon}} \cdot \mathbf{E} + \overline{\boldsymbol{\tau}} \cdot \mathbf{H}) \tag{A.1}$$

where $\overline{\boldsymbol{\mu}}$ and $\overline{\boldsymbol{\epsilon}}$ are the permeability dyadic and permittivity dyadic, respectively, and $\overline{\boldsymbol{\nu}}$ and $\overline{\boldsymbol{\tau}}$ are the magnetoelectric dyadics. Consider the possibility of a second solution ($\mathbf{E}^*$, $-\mathbf{H}^*$) satisfying Maxwell's equations (A.1). Then taking the complex conjugate of Maxwell equations for this second solution gives

$$\nabla \times \mathbf{E} = \mathrm{i}\omega(\overline{\boldsymbol{\mu}}^* \cdot \mathbf{H} - \overline{\boldsymbol{\nu}}^* \cdot \mathbf{E}), \quad \nabla \times \mathbf{H} = -\mathrm{i}\omega(\overline{\boldsymbol{\epsilon}}^* \cdot \mathbf{E} - \overline{\boldsymbol{\tau}}^* \cdot \mathbf{H})\,. \tag{A.2}$$

However, for a lossless reciprocal material [36, sec. 5.1]

$$\overline{\boldsymbol{\mu}}^* = \overline{\boldsymbol{\mu}}, \quad \overline{\boldsymbol{\epsilon}}^* = \overline{\boldsymbol{\epsilon}}, \quad \overline{\boldsymbol{\nu}}^* = -\overline{\boldsymbol{\nu}}, \quad \overline{\boldsymbol{\tau}}^* = -\overline{\boldsymbol{\tau}} \tag{A.3}$$

and thus $(\mathbf{E}^*, -\mathbf{H}^*)$ is a solution if $(\mathbf{E}, \mathbf{H})$ is a solution. Consequently, for lossless uniform waveguides with traveling waves satisfying

$$\mathbf{E}(x,y,z) = \mathbf{e}(x,y)\,\mathrm{e}^{\mathrm{i}\beta z}, \quad \mathbf{H}(x,y,z) = \mathbf{h}(x,y)\,\mathrm{e}^{\mathrm{i}\beta z} \tag{A.4}$$

we also have the solution

$$\mathbf{E}^*(x,y,z) = \mathbf{e}(x,y)\,\mathrm{e}^{-\mathrm{i}\beta^* z}, \quad -\mathbf{H}^*(x,y,z) = -\mathbf{h}^*(x,y)\,\mathrm{e}^{-\mathrm{i}\beta^* z} \tag{A.5}$$

so that lossless uniform traveling waves come in pairs with propagation constants $(\beta, -\beta^*)$. Likewise, for lossless periodic waveguides with traveling waves satisfying

$$\mathbf{E}(x,y,z+d) = \mathbf{E}(x,y,z)\,\mathrm{e}^{\mathrm{i}\beta d}, \quad \mathbf{H}(x,y,z+d) = \mathbf{H}(x,y,z)\,\mathrm{e}^{\mathrm{i}\beta d} \tag{A.6}$$

where d is the spatial period, we also have the solution

$$\mathbf{E}^*(x,y,z+d) = \mathbf{E}^*(x,y,z)\,\mathrm{e}^{-\mathrm{i}\beta^* d}, \quad -\mathbf{H}^*(x,y,z+d) = -\mathbf{H}^*(x,y,z)\,\mathrm{e}^{-\mathrm{i}\beta^* d} \tag{A.7}$$

so that lossless periodic traveling waves also come in pairs with propagation constants $(\beta, -\beta^*)$.

The following proof of the bidirectionality of traveling waves on reciprocal lossy (or lossless) waveguides is not as direct as the foregoing proof that applies to only lossless waveguides. Begin by placing two linear, single-port, reciprocal antennas (labeled A1 and A2) in the fields of a waveguide as shown in Fig. 48. (Although an open periodic waveguide is shown in Fig. 48, the following proof applies to closed and uniform waveguides as well.) The two antennas are separated by a large distance L (which, for periodic waveguides, is

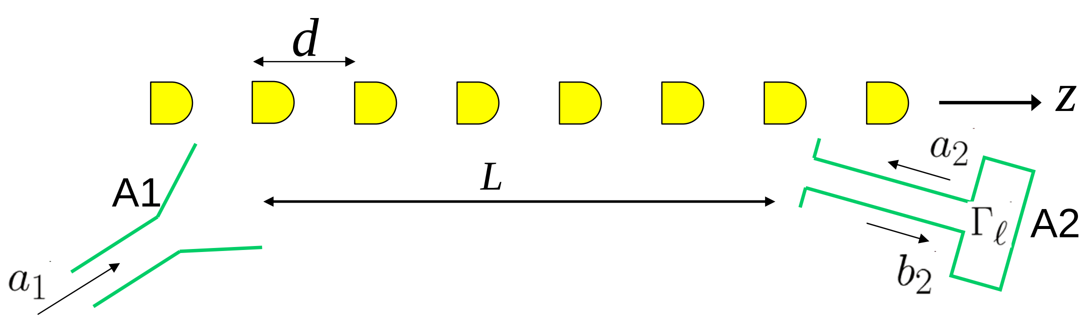

Figure 48: Reciprocal waveguide with two linear, single-port, reciprocal antennas.

assumed to increase or decrease only in increments of the periodic spacing d). Let antenna A1 be designed and positioned to excite a single waveguide mode traveling in the $+z$ direction ($\mathrm{e}^{\mathrm{i}\beta z}$) with complex propagation constant β.[14] For the sake of simplifying the scattering

[14]For total power flow in the $+z$ direction, the imaginary part of β is greater than or equal to zero, but the real part of β can be greater than zero or less than zero depending on whether the mode is a forward or backward traveling wave. For complex waves on lossless waveguides, the total power flow is zero, but the exponential decay of the wave is always in the $+z$ direction.

matrix equations below, let each of the feed lines be identical, for example, 50-ohm coaxial cables. Also, let antenna A2 be a dipole antenna that does not scatter when it is terminated in a load with reflection coefficient Γ_ℓ (a nonscattering antenna (NSA)[36, pp. 140–141]).

Feed antenna A1 with ingoing feed modal coefficient a_1 and terminate antenna A2 in its NSA load so that $a_2 = \Gamma_\ell b_2$, where a_2 and b_2 are the ingoing and outgoing feed modal coefficients of antenna A2. Then we have from the general two-port scattering-matrix equations [36, eq. (2–10)]

$$b_2 = S_{21}a_1 + S_{22}a_2 = S_{21}a_1 + S_{22}\Gamma_\ell b_2 \tag{A.8}$$

so that

$$b_2 = \frac{S_{21}a_1}{1 - \Gamma_\ell S_{22}}\,. \tag{A.9}$$

Now $S_{22} = b_{20}/a_{20}$, where b_{20} and a_{20} are values of b_2 and a_2 when $a_1 = 0$. Because of scattering from antenna A1, S_{22} depends on the antenna separation distance L. Furthermore, we can write $b_{20} = b_{200} + \Delta b_{20}$, where b_{200} is the value of b_{20} when antenna A1 is removed. If a small loss is inserted into free space so that the propagation constant of free space has an imaginary part ($k = k_r + \mathrm{i}\alpha$, $\alpha > 0$), then $\Delta b_{20} = O\left(\mathrm{e}^{-2\alpha L}\right)$ and (A.9) can be rewritten as

$$b_2 = \frac{S_{21\alpha}a_1}{1 - \Gamma_\ell S_{220}}\left[1 + O\left(\mathrm{e}^{-2\alpha L}\right)\right] \tag{A.10}$$

where $S_{220} = b_{200}/a_{20}$ is independent of the separation distance L and $S_{21\alpha}$ is the value of S_{21} with the small loss α inserted into the free-space propagation constant.

Next feed antenna A2 with ingoing feed modal coefficient a_2' and terminate A1 in a matched load so that $a_1' = 0$. Then we have from the two-port scattering-matrix equations

$$b_1' = S_{12\alpha}a_2' \tag{A.11}$$

where $S_{12\alpha}$ is the value of S_{12} with the small loss α in the free-space propagation constant. If all the material in the waveguide is reciprocal

$$S_{21\alpha} = S_{12\alpha}. \tag{A.12}$$

Now translate without rotation the antenna A2 a distance d in the $+z$ direction and repeat the above measurement procedure to obtain the equations

$$b_2^d = \frac{S_{21\alpha}^d a_1}{1 - \Gamma_\ell S_{220}}\left[1 + O\left(\mathrm{e}^{-2\alpha(L+d)}\right)\right] \tag{A.13}$$

$$b_1^{d\prime} = S_{12\alpha}^d a_2' \tag{A.14}$$

$$S_{21\alpha}^d = S_{12\alpha}^d. \tag{A.15}$$

Since antenna A2 is an NSA for the measurements leading to (A.10) and (A.13), and antenna A1 excites only a traveling wave on the waveguide

$$b_2^d = b_2\,\mathrm{e}^{\mathrm{i}\beta d}\,[1 + O(\alpha d)]. \tag{A.16}$$

Inserting b_2^d from (A.16) into (A.13) and comparing with (A.10) gives

$$S_{21\alpha}\, \mathrm{e}^{\mathrm{i}\beta d}\, [1 + O(\alpha d)] \left[1 + O\left(\mathrm{e}^{-2\alpha L}\right)\right] = S_{21\alpha}^d \left[1 + O\left(\mathrm{e}^{-2\alpha(L+d)}\right)\right]. \tag{A.17}$$

Letting $\alpha \to 0$ and $L \to \infty$ such that $\alpha L \to \infty$ reduces (A.17) to

$$\lim_{\alpha \to 0} \frac{S_{21\alpha}^d}{S_{21\alpha}} = \frac{S_{21}^d}{S_{21}} = \mathrm{e}^{\mathrm{i}\beta d} \tag{A.18}$$

or, since reciprocity demands that $S_{21} = S_{12}$ and $S_{21}^d = S_{12}^d$

$$S_{12}^d = S_{12}\, \mathrm{e}^{\mathrm{i}\beta d}. \tag{A.19}$$

Dividing (A.14) by (A.11), taking the limit as $\alpha \to 0$, using the fact that $\lim_{\alpha\to 0}[S_{12\alpha}^d/S_{12\alpha}] = [S_{12}^d/S_{12}]$, and substituting from (A.19) yields

$$b_1^{d\prime} = b_1' \mathrm{e}^{\mathrm{i}\beta d} \tag{A.20}$$

which means that the antenna A1 is receiving a traveling wave with phase varying as $\mathrm{e}^{-\mathrm{i}\beta z}$ when antenna A2 is transmitting. In other words, every traveling wave on a reciprocal, lossy or lossless, uniform or periodic waveguide is bidirectional. That is, for every traveling wave on such a waveguide with propagation constant β, there exists a corresponding traveling wave with propagation constant $-\beta$; the propagation constants come in pairs $(\beta, -\beta)$.

Since we also proved above that for lossless, reciprocal, uniform or periodic waveguides, traveling waves also come in pairs having propagation constants $(\beta, -\beta^*)$, we have the result that complex waves on lossless, reciprocal, uniform or periodic waveguides come in quadruplets with propagation constants $(\beta, -\beta, \beta^*, -\beta^*)$.

Finally, we note that although it is not true in general that nonreciprocal waveguides are bidirectional, many nonreciprocal waveguides are nevertheless bidirectional [33].

B RAPIDLY CONVERGENT EXPRESSIONS FOR SCHLÖMILCH SERIES

In this Appendix we collect the rapidly convergent expressions for Schlömilch series used in this report.

[18, eq. 8.524(1)]:

$$\sum_{n=1}^{\infty} \cos(n\beta d) J_0(nkd) = -\frac{1}{2}, \quad 0 < kd < \beta d \leq \pi \tag{B.1a}$$

[18, eq. 8.522(1)]:

$$\sum_{n=1}^{\infty} \cos(n\beta d) J_0(nkd) = -\frac{1}{2} + \frac{1}{kd\sqrt{1-\left(\dfrac{\beta d}{kd}\right)^2}}, \quad 0 < \beta d < kd \leq \pi \tag{B.1b}$$

[18, eq. 8.524(3)]:

$$\sum_{n=1}^{\infty} \cos(n\beta d) Y_0(nkd) = -\frac{1}{\pi}\left(\gamma + \ln\frac{kd}{4\pi}\right) - \frac{1}{\sqrt{(\beta d)^2 - (kd)^2}}$$

$$-\sum_{l=1}^{\infty}\left[\frac{1}{\sqrt{(2l\pi+\beta d)^2-(kd)^2}} - \frac{1}{2l\pi}\right] - \sum_{l=1}^{\infty}\left[\frac{1}{\sqrt{(2l\pi-\beta d)^2-(kd)^2}} - \frac{1}{2l\pi}\right] \tag{B.2a}$$

$$0 < kd < \beta d \leq \pi$$

[18, eq. 8.522(3)]:

$$\sum_{n=1}^{\infty} \cos(n\beta d) Y_0(nkd) = -\frac{1}{\pi}\left(\gamma + \ln\frac{kd}{4\pi}\right)$$

$$-\sum_{l=1}^{\infty}\left[\frac{1}{\sqrt{(2l\pi+\beta d)^2-(kd)^2}} - \frac{1}{2l\pi}\right] - \sum_{l=1}^{\infty}\left[\frac{1}{\sqrt{(2l\pi-\beta d)^2-(kd)^2}} - \frac{1}{2l\pi}\right] \tag{B.2b}$$

$$0 < \beta d < kd \leq \pi$$

In (B.1) and (B.2) γ, referred to as C in [18], is the Euler constant [19, Table 1.1],

$$\gamma = 0.577215665\ldots \tag{B.3}$$

and J_0 and Y_0 are the Bessel and Neumann functions of order 0, respectively. We have found that truncating the expressions on the RHS of (B.2a) and (B.2b) at $l = 10$ yields sufficient accuracy for our purposes.

[18, eq. 8.5.26]:

$$\sum_{n=1}^{\infty} \cos(n\beta d)\; K_0\left(n(d/h)\;\sqrt{(2\pi)^2-(kh)^2}\right) = \frac{1}{2}\left(\gamma + \ln\frac{1}{4\pi}\frac{d}{h}\sqrt{(2\pi)^2-(kh)^2}\right)$$

$$+\frac{\pi/2}{(d/h)\sqrt{(2\pi)^2-(kh)^2}\sqrt{1+\left(\frac{\beta d}{(d/h)\sqrt{(2\pi)^2-(kh)^2}}\right)^2}}$$
$$+\frac{\pi}{2}\left[\sum_{l=1}^{\infty}\left(\frac{1}{\sqrt{(d/h)^2[(2\pi)^2-(kh)^2]+(2l\pi-\beta d)^2}}-\frac{1}{2l\pi}\right)\right.$$
$$\left.+\sum_{l=1}^{\infty}\left(\frac{1}{\sqrt{(d/h)^2[(2\pi)^2-(kh)^2]+(2l\pi-\beta d)^2}}-\frac{1}{2l\pi}\right)\right] \tag{B.4}$$

In (B4) K_0 is the modified Bessel function of order 0.

[37]-[39]:

$$\sum_{n=1}^{\infty}\sin(n\beta d)J_1(nkd)=0,\quad 0<kd\leq\beta d\leq\pi \tag{B.5}$$

[37]-[39]:

$$\sum_{n=1}^{\infty}\sin(n\beta d)Y_1(nkd)=-\sum_{m=-\infty}^{\infty}\frac{\mathrm{sgn}(m)\,\mathrm{e}^{-q_m}}{kd\ \sinh q_m}+\lambda_1,\quad 0<kd\leq\beta d\leq\pi \tag{B.6}$$

where

$$\sinh q_m=\sqrt{\beta_m^2-1} \tag{B.7a}$$
$$\beta_m=\frac{\beta d+2m\pi}{kd} \tag{B.7b}$$
$$\lambda_1=\frac{1}{\pi}\left(\frac{2\pi}{kd}\right)B_1\left(\frac{\beta d}{2\pi}\right) \tag{B.7c}$$

and B_1 is the Bernoulli polynomial [18, eq. 9.62]

$$B_1(x)=x-\frac{1}{2}\,. \tag{B.7d}$$

[37]-[39]:

$$\sum_{n=1}^{\infty}\cos(n\beta d)J_2(nkd)=0,\quad 0<kd\leq\beta d\leq\pi \tag{B.8a}$$

$$\sum_{n=1}^{\infty}\cos(n\beta d)J_2(nkd)=-\frac{\cos\left[2\cos^{-1}\left(\frac{\beta d}{kd}\right)\right]}{kd\sqrt{1-\left(\frac{\beta d}{kd}\right)^2}},\quad 0<\beta d<kd\leq\pi \tag{B.8b}$$

[37]-[39]:

$$\sum_{n=1}^{\infty}\cos(n\beta d)Y_2(nkd)=\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{-2q_m}}{kd\ \sinh q_m}+\lambda_2,\quad 0<kd\leq\beta d\leq\pi \tag{B.9}$$

where

$$\sinh q_m = \sqrt{\beta_m^2 - 1} \tag{B.10a}$$

$$\beta_m = \frac{\beta d + 2m\pi}{kd} \tag{B.10b}$$

$$\lambda_2 = \frac{1}{2\pi}\left[B_0\left(\frac{\beta d}{2\pi}\right) - 2\left(\frac{2\pi}{kd}\right)^2 B_2\left(\frac{\beta d}{2\pi}\right)\right] \tag{B.10c}$$

and B_0 and B_2 are the Bernoulli polynomials [18, eq. 9.62]

$$B_0(x) = 1, \quad B_2(x) = x^2 - x + \frac{1}{6}\ . \tag{B.10d}$$

[18, eqs. 8.521(1), 8.522(3)]:

$$\sum_{l=1}^{\infty} J_0(lkh) = -\frac{1}{2} + \frac{1}{kh}\ , \ 0 < kh < 2\pi \tag{B.11}$$

$$\sum_{l=1}^{\infty} Y_0(lkh) = -\frac{1}{\pi}\left(\gamma + \ln\frac{kh}{4\pi}\right) - 2\sum_{l=1}^{\infty}\left[\frac{1}{\sqrt{(2\pi l)^2 - (kh)^2}} - \frac{1}{2\pi l}\right],\ 0 < kh < 2\pi \tag{B.12}$$

with γ the Euler constant given by (B.3). Truncating the series on the RHS of (B.12) at $l = 10$ gives sufficient accuracy for our purposes.

[37]-[39]:

$$\sum_{l=1}^{\infty} J_2(lkh) = \frac{1}{kh}, \quad 0 < kh < 2\pi \tag{B.13}$$

$$\sum_{l=1}^{\infty} Y_2(lkh) \approx 2\sum_{m=1}^{\infty}\frac{\mathrm{e}^{-2q_m}}{kh\ \sinh q_m} + \lambda_2 \tag{B.14}$$

where

$$\sinh q_m = \sqrt{\beta_m^2 - 1} \tag{B.15a}$$

$$\beta_m = \frac{2m\pi}{kh} \tag{B.15b}$$

$$\lambda_2 = \frac{1}{2\pi}\left[B_0(0) - 2\left(\frac{2\pi}{kh}\right)^2 B_2(0)\right] = \frac{1}{2\pi}\left[1 - 2\left(\frac{2\pi}{kh}\right)^2\frac{1}{6}\right] \tag{B.15c}$$

and B_0 and B_2 are the Bernoulli polynomials given by (B.10d).

C BESSEL FUNCTION RELATIONS

In this Appendix we assemble a number of Bessel function relations that we make frequent use of in the report.

1) Relations between modified Bessel functions and Hankel functions:

[19, eq. 9.6.4]

$$K_0(z) = \frac{\pi}{2} \mathrm{i} H_0^{(1)}(\mathrm{i}z) \tag{C.1}$$

$$K_1(z) = -\frac{\pi}{2} H_1^{(1)}(\mathrm{i}z) \tag{C.2}$$

$$K_2(z) = -\frac{\pi}{2} \mathrm{i} H_2^{(1)}(\mathrm{i}z) \tag{C.3}$$

2) Small argument forms:

[19, eq. 9.1.8]:

$$H_0^{(1)}(z) \overset{|z|<<1}{\sim} \mathrm{i}\frac{2}{\pi} \ln z \tag{C.4}$$

[19, eq. 9.6.9]:

$$K_1(z) \sim \frac{1}{2}\left(\frac{z}{2}\right)^{-1} \tag{C.5}$$

[19, eq. 9.6.9]:

$$K_2(z) \overset{|z|<<1}{\sim} \left(\frac{z}{2}\right)^{-2} \tag{C.6}$$

3) Integral representation of $J_0(z)$:

[19, eq. 9.1.21]:

$$J_0(z) = \frac{1}{\pi} \int_0^{\pi} \mathrm{e}^{\mathrm{i}z\cos\theta} \mathrm{d}\theta \tag{C.7}$$

4) Recursion relations:

[19, eqs. 9.1.27, 9.6.26]:

$$2\frac{H_1^{(1)}(z)}{z} = H_0^{(1)}(z) + H_2^{(1)}(z) \tag{C.8}$$

$$2\frac{K_1(z)}{z} = K_2(z) - K_0(z) \tag{C.9}$$

5) Differential equation satisfied by $J_{\pm\nu}(z), Y_\nu(z), H_\nu^{(1,2)}(z)$:

[19, eq. 9.1.1]:

$$z^2 \, \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + z \, \frac{\mathrm{d}w}{\mathrm{d}z} + z^2 \; w = 0 \tag{C.10}$$

D SUMMATION FORMULAS

In this Appendix we collect a number of summation formulas used frequently in the report.

[18, eqs. 1.441(1), (2)]:

$$2\sum_{m=1}^{\infty}\frac{\mathrm{e}^{ikhm}}{m} = -2\ln\left[2\sin\left(\frac{kh}{2}\right)\right] + \mathrm{i}(\pi - kh),\ 0 < kh < 2\pi. \tag{D.1}$$

The sums

$$\sum_{n=1}^{\infty}\cos(n\beta d)\,\mathrm{e}^{inkd} = \frac{1}{2}\sum_{n=1}^{\infty}\left[\mathrm{e}^{in(\beta d + kd)} + \mathrm{e}^{in(kd - \beta d)}\right] \tag{D.2}$$

and

$$\sum_{n=1}^{\infty}\sin(n\beta d)\,\mathrm{e}^{inkd} = \frac{1}{2\mathrm{i}}\sum_{n=1}^{\infty}\left[\mathrm{e}^{in(\beta d + kd)} - \mathrm{e}^{in(kd - \beta d)}\right] \tag{D.3}$$

can be evaluated in closed form by formally using the formula for the sum of an infinite geometric progression

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}, \quad z = \mathrm{e}^{\mathrm{i}(kd \pm \beta d)} \tag{D.4}$$

even though $|z| = 1$ here (rigorously, k can be thought of as having a very small imaginary part, consistent with the implicit harmonic time dependence $\mathrm{e}^{\mathrm{i}\omega t}$, which is allowed to go to zero at the end) yielding

$$\sum_{n=1}^{\infty}\cos(n\beta d)\,\mathrm{e}^{inkd} = -\frac{1}{2} + \mathrm{i}\,\frac{1}{2}\frac{\sin kd}{\cos\beta d - \cos kd}\ ,\ kd \neq \beta d. \tag{D.5}$$

and

$$\sum_{n=1}^{\infty}\sin(n\beta d)\,\mathrm{e}^{inkd} = -\frac{1}{2}\frac{\sin\beta d}{\cos\beta d - \cos kd}\ ,\ kd \neq \beta d\ . \tag{D.6}$$

[18, eqs. 1.443(3), 1.443(5)]:

$$\sum_{n=1}^{\infty}\frac{\cos na}{n^2} = \frac{\pi^2}{6} - \frac{\pi a}{2} + \frac{a^2}{4}, \quad 0 < a < 2\pi \tag{D.7a}$$

$$\sum_{n=1}^{\infty}\frac{\sin na}{n^3} = \frac{\pi^2 a}{6} - \frac{\pi a^2}{4} + \frac{a^3}{12}, \quad 0 < a < 2\pi \tag{D.7b}$$

[4, Appendix C],[40]:

$$\sum_{n=1}^{\infty} \frac{\sin na}{n^2} \equiv \mathrm{Cl}_2(a) \approx -0.1381 \sin a + 0.03212 \sin 2a - 0.9653 a \ln(a/\pi), \quad 0 < a < \pi \tag{D.8a}$$

$$\mathrm{Cl}_2(a) = -\mathrm{Cl}_2(2\pi - a), \quad \pi \le a < 2\pi \tag{D.8b}$$

$$\sum_{n=1}^{\infty} \frac{\cos na}{n^3} \equiv \mathrm{Cl}_3(a) \approx 1.3328 \; - \; 0.1424 \cos a + 0.01094 \cos 2a$$

$$- \; 0.4902 a^2 \ln(a/\pi) - 0.2417 a^2, \quad 0 < a < \pi \tag{D.8c}$$

$$\mathrm{Cl}_3(a) = \mathrm{Cl}_3(2\pi - a), \quad \pi \le a < 2\pi \; . \tag{D.8d}$$

From (D.8a)

$$\mathrm{Cl}_2(\pi) = \mathrm{Cl}_2(2\pi) = 0 \tag{D.9}$$

and from (D.8c)

$$\mathrm{Cl}_3(2\pi) = \sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3) = 1.20205 \cdots \tag{D.10}$$

where ζ is the Riemann zeta function tabulated for positive integer arguments in [19, Table 23.3].[15]

[19, Table 23.3]:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = 1.64493 \cdots \tag{D.12}$$

[18, eqs. 1.441, 1.443]:

$$\sum_{n=1}^{\infty} \frac{\cos na}{n} = \frac{1}{2} \ln \frac{1}{2(1 - \cos a)} = -\ln \left[2 \sin \left(\frac{a}{2} \right) \right], \quad 0 < a < 2\pi \tag{D.13a}$$

$$\sum_{n=1}^{\infty} \frac{\sin na}{n} = \frac{\pi - a}{2}, \quad 0 < a < 2\pi \tag{D.13b}$$

[15] The Clausen functions Cl_2 and Cl_3 are also given by the integral expressions [40]

$$\mathrm{Cl}_2(a) = -\int_0^a \ln |2 \sin \frac{t}{2}| \, \mathrm{d}t \tag{D.11a}$$

and

$$\mathrm{Cl}_3(a) = \zeta(3) - \int_0^a \mathrm{Cl}_2(t) \, \mathrm{d}t \; . \tag{D.11b}$$

$$\sum_{n=1}^{\infty} \frac{\cos na}{n^2} = \frac{\pi^2}{6} - \frac{\pi a}{2} + \frac{a^2}{4}, \quad 0 < a < 2\pi \tag{D.13c}$$

$$\sum_{n=1}^{\infty} \frac{\sin na}{n^3} = \frac{\pi^2 a}{6} - \frac{\pi a^2}{4} + \frac{a^3}{12}, \quad 0 < a < 2\pi \tag{D.13d}$$

E LIST OF kd–βd EQUATIONS

In this Appendix we list for easy reference the rapidly convergent forms of the kd–βd equations for all the 2D and 3D arrays investigated in Sections 2 to 9.

1) 2D Acoustic Monopole Arrays:

$$kh\cos\psi - \Re\sin\psi = 0 \tag{E.1}$$

$$\Re = -2\pi\sum_{n=1}^{\infty}\cos(n\beta d)Y_0(nkd) + 8\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=1}^{\infty}K_0\left(n(d/h)\,\sqrt{(2\pi m)^2-(kh)^2}\right)$$
$$-2\ln\left[2\sin\left(\frac{kh}{2}\right)\right],\ kh<2\pi \tag{E.2}$$

The Neumann function sum is evaluated using (B.2), and the modified Bessel function sum converges extremely rapidly because of the exponential decay of K_0.

2) 3D Acoustic Monopole Arrays:

$$kh\cos\psi - \Re\sin\psi = 0 \tag{E.3}$$

$$\Re = -\frac{2\pi}{kh}\,\frac{\sin kd}{\cos\beta d-\cos kd}+4\pi\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=-\infty}^{\infty}\sum_{\substack{l=-\infty\\(m,l)\neq(0,0)}}^{\infty}\frac{\mathrm{e}^{-n(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}$$
$$-2\pi\sum_{l=1}^{\infty}Y_0(lkh)+8\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\right)-2\ln\left[2\sin\left(\frac{kh}{2}\right)\right],\ kh<2\pi \tag{E.4}$$

The sum of exponentials converges very rapidly because of the negative exponential so that it is necessary to include only a few terms in the sum, for example n from 1 to 2 and m, l from -2 to 2 for sufficient accuracy. Alternately an approximation to the sum can be obtained by first performing the summation over n from 1 to ∞ in closed form using (D.4) and then including only terms in the summation over m and l from -1 to 1. This yields

$$\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=-\infty}^{\infty}\sum_{\substack{l=-\infty\\(m,l)\neq(0,0)}}^{\infty}\frac{\mathrm{e}^{-n(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}$$
$$\approx 4\left(\frac{1}{r_1}\frac{\mathrm{e}^{-(d/h)r_1}\cos\beta d-\mathrm{e}^{-2(d/h)r_1}}{1-2\cos\beta d\,\mathrm{e}^{-(d/h)r_1}+\mathrm{e}^{-2(d/h)r_1}}+\frac{1}{r_2}\frac{\mathrm{e}^{-(d/h)r_2}\cos\beta d-\mathrm{e}^{-2(d/h)r_2}}{1-2\cos\beta d\,\mathrm{e}^{-(d/h)r_2}+\mathrm{e}^{-2(d/h)r_2}}\right) \tag{E.5}$$

where $r_1=\sqrt{(2\pi)^2-(kh)^2}$, and $r_2=\sqrt{8\pi^2-(kh)^2}$. The Neumann function sum is evaluated using (B.12), and the modified Bessel function sum converges extremely rapidly because of the exponential decay of K_0.

3) 2D Electric Dipole Arrays, Dipoles Oriented Perpendicular to the Array Axis

3a) Electric Dipoles in the Array Plane:

$$\frac{2}{3}(kh)^3 \cos\psi - \Re \sin\psi = 0 \tag{E.6}$$

$$\Re = -2\pi (kh)^2 \sum_{n=1}^{\infty} \cos(n\beta d) Y_0(nkd)$$

$$- 8 \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{m=1}^{\infty} \left[(2\pi m)^2 - (kh)^2\right] K_0 \left(n(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2}\right)$$

$$+ 4(kh)\ \mathrm{Cl}_2(kh) + 4\ \mathrm{Cl}_3(kh),\ kh < 2\pi \tag{E.7}$$

The Neumann function sum is evaluated using (B.2), and the modified Bessel function sum converges extremely rapidly because of the exponential decay of K_0. The Clausen functions Cl_2 and Cl_3 are defined and approximated by (D.8).

3b) Electric Dipoles Perpendicular to the Array Plane:

$$\frac{2}{3}(kh)^3 \cos\psi - \Re \sin\psi = 0 \tag{E.8}$$

$$\Re = -\pi (kh)^2 \left[\sum_{n=1}^{\infty} \cos(n\beta d) Y_0(nkd) - \sum_{n=1}^{\infty} \cos(n\beta d) Y_2(nkd)\right]$$

$$+ 4 \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{m=1}^{\infty} \Bigg[\left[(2\pi m)^2 + (kh)^2\right] K_0 \left(n(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2}\right)$$

$$- \left[(2\pi m)^2 - (kh)^2\right] K_2 \left(n(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2}\right) \Bigg]$$

$$- 2 \left((kh)^2 \ln\left[2 \sin\left(\frac{kh}{2}\right)\right] + kh\mathrm{Cl}_2(kh) + \mathrm{Cl}_3(kh) \right),\ kh < 2\pi \tag{E.9}$$

The sum $\sum \cos(n\beta d) Y_0(nkd)$ can be evaluated using (B.2), the sum $\sum \cos(n\beta d) Y_2(nkd)$ can be evaluated very efficiently by using (B.9) and (B.10), and the K_0 and K_2 series converge extremely rapidly because of the exponential decay of K_0 and K_2.

4) 3D Electric Dipole Arrays, Dipoles Oriented Perpendicular to the Array Axis:

$$\frac{2}{3}(kh)^3 \cos\psi - \Re \sin\psi = 0 \tag{E.10}$$

$$\Re = -2\pi\ kh\ \frac{\sin kd}{\cos\beta d - \cos kd}$$

$$
-\,4\pi\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{\substack{l=-\infty\\(l,m)}}^{\infty}\;\neq\;\sum_{\substack{m=-\infty\\(0,0)}}^{\infty}\left[(2\pi m)^2-(kh)^2\right]\frac{\mathrm{e}^{-n(d/h)\,\sqrt{(2\pi)^2(l^2+m^2)-(kh)^2}}}{\sqrt{(2\pi)^2(l^2+m^2)-(kh)^2}}
$$

$$
-\,2\sum_{l=1}^{\infty}\left[\pi(kh)^2\,Y_0(lkh)+\;4\sum_{m=1}^{\infty}\left[(2\pi m)^2-(kh)^2\right]K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\,\right)\right]
$$

$$
+\,4\;kh\;\mathrm{Cl}_2(kh)+4\;\mathrm{Cl}_3(kh),\;\;kh<2\pi \tag{E.11}
$$

The sum of exponentials converges very rapidly because of the negative exponential so that it is necessary to include only a few terms in the sum, for example n from 1 to 2 and m, l from -2 to 2 for sufficient accuracy, the Neumann function sum is evaluated using (B.2), and the modified Bessel function sum converges extremely rapidly because of the exponential decay of K_0. The Clausen functions Cl_2 and Cl_3 are defined and approximated by (D.8).

5) 2D Electric Dipole Arrays, Dipoles Oriented Parallel to the Array Axis:

$$
\frac{2}{3}(kh)^3\cos\psi-\Re\sin\psi=0 \tag{E.12}
$$

$$
\Re=-\pi(kh)^2\left[\sum_{n=1}^{\infty}\cos(n\beta d)Y_0(nkd)+\sum_{n=1}^{\infty}\cos(n\beta d)Y_2(nkd)\right]
$$

$$
+\,4\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=1}^{\infty}\Bigg[\left[(2\pi m)^2+(kh)^2\right]K_0\left(n(d/h)\,\sqrt{(2\pi m)^2-(kh)^2}\right)
$$

$$
+\;\left[(2\pi m)^2-(kh)^2\right]K_2\left(n(d/h)\,\sqrt{(2\pi m)^2-(kh)^2}\right)\Bigg]
$$

$$
-\,2\left((kh)^2\ln\left[2\sin\left(\frac{kh}{2}\right)\right]+kh\mathrm{Cl}_2(kh)+\mathrm{Cl}_3(kh)\right),\;\;kh<2\pi \tag{E.13}
$$

The sum $\sum\cos(n\beta d)Y_0(nkd)$ can be evaluated using (B.2), the sum $\sum\cos(n\beta d)Y_2(nkd)$ can be evaluated very efficiently by using (B.9)-(B.10), and the modified Bessel function K_0 and K_2 series converge extremely rapidly because of the exponential decay of K_0 and K_2. The Clausen functions Cl_2 and Cl_3 are defined and approximated by (D.8).

6) 3D Electric Dipole Arrays, Dipoles Oriented Parallel to the Array Axis:

$$
\frac{2}{3}(kh)^3\cos\psi-\Re\sin\psi=0 \tag{E.14}
$$

$$
\Re=16\pi^3\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{(m^2+l^2)\;\mathrm{e}^{-n(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}
$$

$$
-\,\pi(kh)^2\left[\sum_{l=1}^{\infty}Y_0(lkh)-\sum_{l=1}^{\infty}Y_2(lkh)\right]
$$

$$+ 4\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\Bigg[\left[(2\pi m)^2+(kh)^2\right] K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\right)$$

$$-\left[(2\pi m)^2-(kh)^2\right] K_2\left(l\sqrt{(2\pi m)^2-(kh)^2}\right)\Bigg]$$

$$-2\left((kh)^2\ln\left[2\sin\left(\frac{kh}{2}\right)\right]+kh\mathrm{Cl}_2(kh)+\mathrm{Cl}_3(kh)\right),\ kh<2\pi \tag{E.15}$$

The sum

$$\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{(m^2+l^2)\,\mathrm{e}^{-n(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}} \tag{E.16}$$

converges very rapidly because of the negative exponential so that it is necessary to include only a few terms in the sum, for example, n from 1 to 2 and m, l from -2 to 2, for sufficient accuracy. Alternately an approximation to the sum can be obtained by first performing the summation over n from 1 to ∞ in closed form using (D.4) and then including only terms in the summation over m and l from -1 to 1. When this is done we obtain

$$\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\frac{(m^2+l^2)\,\mathrm{e}^{-n(d/h)\,\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}$$

$$\approx 4\left(\frac{1}{r_1}\frac{\mathrm{e}^{-(d/h)r_1}\cos\beta d-\mathrm{e}^{-2(d/h)r_1}}{1-2\cos\beta d\,\mathrm{e}^{-(d/h)r_1}+\mathrm{e}^{-2(d/h)r_1}}+\frac{2}{r_2}\frac{\mathrm{e}^{-(d/h)r_2}\cos\beta d-\mathrm{e}^{-2(d/h)r_2}}{1-2\cos\beta d\,\mathrm{e}^{-(d/h)r_2}+\mathrm{e}^{-2(d/h)r_2}}\right) \tag{E.17}$$

where $r_1=\sqrt{(2\pi)^2-(kh)^2}$, and $r_2=\sqrt{8\pi^2-(kh)^2}$. Accelerated convergence expressions for the Schlömilch series $\sum Y_0(lkh)$ and $Y_2(lkh)$ are given in (B.12) and (B.14), respectively. The modified Bessel function series

$$\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\left[(2\pi m)^2+(kh)^2\right] K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\right) \tag{E.18a}$$

and

$$\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\left[(2\pi m)^2-(kh)^2\right] K_2\left(l\sqrt{(2\pi m)^2-(kh)^2}\right) \tag{E.18b}$$

converge extremely rapidly because of the exponential decay of K_0 and K_1 so that only a few terms of the series need be included. The Clausen functions Cl_2 and Cl_3 are defined and approximated by (D.8).

7) 2D Magnetodielectric Sphere Arrays

7a) Electric Dipoles in the Array Plane:

$$\frac{(kh)^3-S_-\Sigma_1}{S_-\Sigma_2}=\frac{S_+\Sigma_2}{(kh)^3-S_+\Sigma_3} \tag{E.19}$$

S_- and S_+ are the normalized magnetodielectric sphere electric and magnetic dipole scattering coefficients, respectively,

$$S_- = -\mathrm{i}\frac{3}{2}b_1^{sc} \tag{E.20a}$$

$$S_+ = -\mathrm{i}\frac{3}{2}a_1^{sc} \tag{E.20b}$$

where b_1^{sc} and a_1^{sc} are the electric and magnetic Mie dipole scattering coefficients defined in [20],

$$\begin{aligned}\Sigma_1 &= -2\pi(kh)^2\sum_{n=1}^{\infty}\cos(n\beta d)Y_0(nkd)\\ &- 8\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=1}^{\infty}\left[(2\pi m)^2-(kh)^2\right]K_0\left(n(d/h)\sqrt{(2\pi m)^2-(kh)^2}\right)\\ &+ 4\,kh\,\mathrm{Cl}_2(kh)+4\,\mathrm{Cl}_3(kh)-\mathrm{i}\frac{2}{3}(kh)^3 \quad kh<2\pi\end{aligned} \tag{E.21}$$

$$\begin{aligned}\Sigma_2 &= 2\pi(kh)^2\sum_{n=1}^{\infty}\sin(n\beta d)Y_1(nkd)\\ &- 8(kh)\sum_{n=1}^{\infty}\sin(n\beta d)\sum_{m=1}^{\infty}\sqrt{(2\pi m)^2-(kh)^2}\,K_1\left(n(d/h)\sqrt{(2\pi m)^2-(kh)^2}\right),\ kh<2\pi\end{aligned} \tag{E.22}$$

and

$$\begin{aligned}\Sigma_3 &= -\pi(kh)^2\left[\sum_{n=1}^{\infty}\cos(n\beta d)Y_0(nkd)-\sum_{n=1}^{\infty}\cos(n\beta d)Y_2(nkd)\right]\\ &+ 4\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{m=1}^{\infty}\Bigg[\left[(2\pi m)^2+(kh)^2\right]K_0\left(n(d/h)\sqrt{(2\pi m)^2-(kh)^2}\right)\\ &\quad - \left[(2\pi m)^2-(kh)^2\right]K_2\left(n(d/h)\sqrt{(2\pi m)^2-(kh)^2}\right)\Bigg]\\ &- 2(kh)^2\ln\left[2\sin\left(\frac{kh}{2}\right)\right]-2kh\mathrm{Cl}_2(kh)-2\mathrm{Cl}_3(kh)-\mathrm{i}\frac{2}{3}(kh)^3,\ kh<2\pi\end{aligned} \tag{E.23}$$

Rapidly convergent expressions for the slowly convergent Schlömilch series $\sum\cos(n\beta d)\,Y_0(nkd)$ and $\sum\cos(n\beta d)\,Y_2(nkd)$ are given in (B.2a) and (B.9)-(B.10), respectively. All series involving the modified Bessel functions K_0, K_1, and K_2, converge very rapidly because of the exponential decay of these functions so that only a few terms of the series give sufficient accuracy. The convergence of the series $\sum\sin(n\beta d)Y_1(nkd)$ can be greatly accelerated by using (B.6)-(B.7). The Clausen functions Cl_2 and Cl_3 are defined and approximated by (D.8).

7b) Electric Dipoles Perpendicular to the Array Plane:

$$\frac{(kh)^3-S_-\Sigma_1}{S_-\Sigma_2}=\frac{S_+\Sigma_2}{(kh)^3-S_+\Sigma_3} \tag{E.24}$$

S_- and S_+ are the normalized magnetodielectric sphere electric and magnetic dipole scattering coefficients, respectively, given by (E.20).

$$\Sigma_1 = -\pi(kh)^2 \left[\sum_{n=1}^{\infty} \cos(n\beta d) Y_0(nkd) - \sum_{n=1}^{\infty} \cos(n\beta d) Y_2(nkd)\right]$$

$$+ 4 \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{m=1}^{\infty} \Bigg[\left[(2\pi m)^2 + (kh)^2\right] K_0 \left(n(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2}\right)$$

$$- \left[(2\pi m)^2 - (kh)^2\right] K_2 \left(n(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2}\right) \Bigg]$$

$$-2\left((kh)^2 \ln\left[2\sin\left(\frac{kh}{2}\right)\right] + kh\ \mathrm{Cl}_2(kh) + \mathrm{Cl}_3(kh)\right) - \mathrm{i}\frac{2}{3}(kh)^3,\ kh < 2\pi \tag{E.25}$$

$$\Sigma_2 = 2\pi(kh)^2 \sum_{n=1}^{\infty} \sin(n\beta d) Y_1(nkd)$$

$$- 8\ kh \sum_{n=1}^{\infty} \sin(n\beta d) \sum_{m=1}^{\infty} \sqrt{(2\pi m)^2 - (kh)^2}\ K_1\left(n(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2}\right),\ kh < 2\pi \tag{E.26}$$

$$\Sigma_3 = -2\pi(kh)^2 \sum_{n=1}^{\infty} \cos(n\beta d) Y_0(nkd)$$

$$- 8 \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{m=1}^{\infty} \left[(2\pi m)^2 - (kh)^2\right] K_0 \left(n(d/h)\ \sqrt{(2\pi m)^2 - (kh)^2}\right)$$

$$+ 4\ kh\ \mathrm{Cl}_2(kh) + 4\ \mathrm{Cl}_3(kh) - \mathrm{i}\frac{2}{3}(kh)^3,\ kh < 2\pi \tag{E.27}$$

The series $\sum \cos(n\beta d)\, Y_0(nkd)$ is treated in (B.2) and the series $\sum \cos(n\beta d)\, Y_2(nkd)$ is evaluated by (B.9)-(B.10). All series involving the modified Bessel functions K_0, K_1, or K_2 converge very rapidly because of the exponential decay of these functions so that only a few terms of the series give sufficient accuracy. The convergence of the series $\sum \sin(n\beta d) Y_1(nkd)$ can be greatly accelerated by using (B.6)-(B.7). The Clausen functions Cl_2 and Cl_3 are defined and approximated by (D.8).

8) 3D Magnetodielectric Sphere Arrays:

$$\frac{(kh)^3 - S_-\Sigma_1}{S_-\Sigma_2} = \frac{S_+\Sigma_2}{(kh)^3 - S_+\Sigma_1} \tag{E.28}$$

S_- and S_+ are the normalized magnetodielectric sphere electric and magnetic dipole scattering coefficients, respectively, given by (E.20).

$$\Sigma_1 = -2\pi kh\ \frac{\sin kd}{\cos\beta d - \cos kd}$$

$$- 4\pi \sum_{n=1}^{\infty} \cos(n\beta d) \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \neq \sum_{\substack{l=-\infty \\ (0,0)}}^{\infty} \left[(2\pi m)^2 - (kh)^2\right] \frac{\mathrm{e}^{-n(d/h)\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}$$

$$- 2\pi(kh)^2 \sum_{l=1}^{\infty} Y_0(lkh) - 8 \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \left[(2\pi m)^2 - (kh)^2\right] K_0\left(l\sqrt{(2\pi m)^2-(kh)^2}\right)$$

$$+ 4\, kh\, \mathrm{Cl}_2(kh) + 4\, \mathrm{Cl}_3(kh) - \mathrm{i}\frac{2}{3}(kh)^3,\ kh < 2\pi \tag{E.29}$$

and

$$\Sigma_2 = 2\pi\ kh \frac{\sin\beta d}{\cos\beta d - \cos kd}$$

$$- 4\pi\ kh \sum_{n=1}^{\infty} \sin(n\beta d) \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \neq \sum_{\substack{l=-\infty \\ (0,0)}}^{\infty} \mathrm{e}^{-n(d/h)\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}},\ kh < 2\pi \tag{E.30}$$

A rapidly convergent expression for the slowly convergent Schlömilch series $\sum Y_0(lkh)$ is given by (B.12), and all series involving negative exponentials and the modified Bessel function K_0 (which decays exponentially) converge very rapidly so that only a few terms of these series gives sufficient accuracy. Alternately, approximate closed form expressions for the summations involving negative exponentials can be obtained by first performing the summation over n from 1 to ∞ using (D.4) and then including only terms in the summations over m and l for which $|m| \leq 1$ and $|l| \leq 1$ thus yielding

$$\sum_{n=1}^{\infty} \cos(n\beta d) \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \neq \sum_{\substack{l=-\infty \\ (0,0)}}^{\infty} \left[(2\pi m)^2 - (kh)^2\right] \frac{\mathrm{e}^{-n(d/h)\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}$$

$$\approx 2\, \frac{(2\pi)^2 - 2(kh)^2}{\sqrt{(2\pi)^2-(kh)^2}} \frac{\mathrm{e}^{-(d/h)r_1} \cos\beta d - \mathrm{e}^{-2(d/h)r_1}}{1 - 2\cos\beta d\, \mathrm{e}^{-(d/h)r_1} + \mathrm{e}^{-2(d/h)r_1}}$$

$$+ 4\, \frac{(2\pi)^2 - (kh)^2}{\sqrt{8\pi^2-(kh)^2}} \frac{\mathrm{e}^{-(d/h)r_2} \cos\beta d - \mathrm{e}^{-2(d/h)r_2}}{1 - 2\cos\beta d\, \mathrm{e}^{-(d/h)r_2} + \mathrm{e}^{-2(d/h)r_2}} \tag{E.31}$$

where $r_1 = \sqrt{(2\pi)^2 - (kh)^2}$, and $r_2 = \sqrt{8\pi^2 - (kh)^2}$. The corresponding approximate closed form expression for the sum of negative exponentials in Σ_2 is

$$\sum_{n=1}^{\infty} \sin(n\beta d) \sum_{\substack{m=-\infty \\ (m,l)}}^{\infty} \neq \sum_{\substack{l=-\infty \\ (0,0)}}^{\infty} \mathrm{e}^{-n(d/h)\sqrt{(2\pi)^2(m^2+l^2)-(kh)^2}}$$

$$\approx 4\sin\beta d \left(\frac{\mathrm{e}^{-(d/h)r_1}}{1 - 2\cos\beta d\, \mathrm{e}^{-(d/h)r_1} + \mathrm{e}^{-2(d/h)r_1}} + \frac{\mathrm{e}^{-(d/h)r_2}}{1 - 2\cos\beta d\, \mathrm{e}^{-(d/h)r_2} + \mathrm{e}^{-2(d/h)r_2}} \right) \tag{E.32}$$

where $r_1 = \sqrt{(2\pi)^2 - (kh)^2}$, and $r_2 = \sqrt{8\pi^2 - (kh)^2}$. The Clausen functions Cl_2 and Cl_3 are defined and approximated by (D.8).

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