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Assessing Resource Requirements for Maritime Domain Awareness and Protection (Security)

by

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September 2006

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Prepared for: Wayne E. Meyer Institute for Systems Engineering, 777 Dyer Road, Bldg. 233, Naval Postgraduate School, Monterey, CA 93943-5194 and the Center for Defense Technology and Education for the Military Services Initiative (CDTEMS)

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13. ABSTRACT (maximum 200 words)

A maritime domain or region contains a number w of nonhostile W (White) vessels of interest. Hostile R (Red) vessels enter the domain. The Rs are traveling through the domain toward targets. Overhead, friendly (Blue) sensors ($\mathcal{S}$) patrol the domain and classify (perhaps incorrectly) detected vessels of interest as R or W. The misclassification of a W as an R is a false positive. An overhead sensor follows (or tracks) any vessel it classifies as R until it is relieved by another platform, perhaps a destroyer pair (DD). The overhead sensor is here assumed unable to detect and classify additional vessels while it is following a suspicious vessel; this may well be a somewhat pessimistic assumption, very possibly "richer possibilities" based on additional assets (such as unmanned aerial vehicles (UAVs)) are available, but loss of track may occur as well as misclassification.

Deterministic and stochastic models are formulated and studied to evaluate the probability that Rs are successfully neutralized before reaching their destination. The model results quantify the effect of the resources and time needed to prosecute misclassified Ws (false positives) on the probability of successfully neutralizing R.

The results indicate that the probability of neutralizing an R vessel is very sensitive to the false positive rate. Technologies, processes, and procedures that can decrease the false positive rate will increase the effectiveness of the Maritime Intercept Operation (MIO).

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ABSTRACT

A maritime domain or region contains a number w of nonhostile W (White) vessels of interest. Hostile R (Red) vessels enter the domain. The Rs are traveling through the domain toward targets. Overhead, friendly (Blue) sensors ($\mathcal{S}$) patrol the domain and classify (perhaps incorrectly) detected vessels of interest as R or W. The misclassification of a W as an R is a false positive. An overhead sensor follows (or tracks) any vessel it classifies as R until it is relieved by another platform, perhaps a destroyer pair (DD). The overhead sensor is here assumed unable to detect and classify additional vessels while it is following a suspicious vessel; this may well be a somewhat pessimistic assumption, very possibly "richer possibilities" based on additional assets (such as unmanned aerial vehicles (UAVs)) are available, but loss of track may occur as well as misclassification.

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0. Background

The general term Maritime Domain Awareness (MDA) includes a broad range of initiatives to enhance both the security of ports and approaches to the United States and the force protection of U.S. and/or allied maritime assets (e.g., Japan, Singapore, Bahrain, etc.) in ports and choke points throughout the world. An essential requirement is to furnish adequate Blue surveillance force size and composition, and resource-assignment-effective CONOPS to maintain useful knowledge of hostile (Red) elements.

We consider the following scenario. Hostile Red vessels (Rs) enter a domain or region. The Rs are transiting through the region toward targets. There are Blue overhead sensors that patrol the domain. Blue wishes to neutralize the Rs before they reach their targets. In addition to the Rs, there are neutral White (W) vessels of interest in the region that can be misclassified as Rs. A sensor system classifies vessels of interest as W or R. If the sensor misclassifies a W as an R, a false positive occurs. The sensor follows (tracks) any vessel classified as an R until it is relieved by another platform. During this following time, the sensor is unable to classify any further vessels. The probability of neutralizing an R depends on the area of the domain being patrolled, the number of sensors, the velocity of the sensors, the time needed to classify a vessel of interest, the ability to correctly classify vessels of interest, the time until a sensor following a suspicious vessel is relieved, and the false positive rate. The false positive rate is a function of the probability of correct classification, which may well be environmentally dependent (see Frederickson and Davidson (2003)) and the number of vessels of interest that are Ws.

We present five models to study Blue force size requirements in this scenario. The results of the models highlight the importance of minimizing the false positive rate; a false positive occurs when a friendly (White) vessel is misclassified as hostile (Red) and increases as the number of friendly (White) vessels that are subject to surveillance and possible classification as hostile (Red) increases.

Section 1 presents a deterministic model, a so-called “fluid approximation” model, consisting of a system of differential equations. Models 2, 3, 4, and 5 are stochastic models. All of these provide insights, but are also useful to help validate and verify more complex simulations. Sections 2 and 3 present Markovian stochastic models. Section 4 presents examples comparing the results of the Deterministic and Markovian models. The sensitivity of the ability to neutralize Rs to model parameters, such as the size of the domain, the speed of the sensor, the ability to correctly classify the vessel type, and the rate of false positives, is also discussed in Section 4. Section 5 presents a non-Markovian model and approximations for the probability that R is neutralized. The usefulness of the approximations is studied by comparing their results to those obtained from more detailed Monte Carlo simulations. The approximations give very reasonable results. Numerical results using the approximations are presented to illustrate the evaluation of the number of overhead sensors that are needed to patrol a domain to achieve a probability of neutralizing an R larger than a specified value.

One technology that is being proposed to reduce the false positive rate is to use the Automatic Identification System (AIS) to classify and track vessels in the maritime domain. In Section 6, a stochastic model is presented for the ability to detect and classify vessels in the maritime domain using AIS; in particular, the model is used to obtain an expression for the probability that a vessel with AIS will not be detected by an overhead surveillance system that is periodically over the maritime domain; the vessel will not be detected if all of its AIS messages are blocked during one pass of the overhead surveillance system.

1. Model 1: A First Deterministic "Fluid" Model

1.1. Formulation

An Area of Interest (AoI), the Domain, $\mathcal{D}$ for short, is entered via an upper boundary, $\bar{\mathcal{D}}$, by both Ws and Rs. The Rs aim to transit to the lower boundary $\underline{\mathcal{D}}$, cross it, and enter Blue Homeland ($\underline{\mathcal{D}}(H)$) in order to damage/destroy infrastructure and population. The Ws transit within and through the Domain, $\mathcal{D}$, with no hostile intent. They may be ferries, fishing, and other pleasure boats, or innocuous cargo-carrying vessels.

To protect $\underline{\mathcal{D}}(H)$, $\mathcal{D}$ is under surveillance by a force of $\bar{S}$ surveillance platforms (denoted by S) such as helicopters (helos), UAVs, and fixed wing (FW) aircraft (e.g., P3s); this is the Overhead Force (OH); their purpose is to detect and (correctly) identify Reds/Hostiles in transit from $\bar{\mathcal{D}}$ to $\underline{\mathcal{D}}$ and prevent them from penetrating $\underline{\mathcal{D}}$ to reach $\underline{\mathcal{D}}(H)$. Note that, in the present simplified model, when the number of Ws (respectively Rs) equal w (respectively r for Rs), the Ss act independently, coming at random with rate $\delta(w+r)$, on potential targets for escort off-$\mathcal{D}$ by armed and lethal Blue (B) vessels (e.g., destroyers (DDs)), but may do so erroneously: a W may, for instance, be misidentified as an R and followed by a previously free S until DD escorts appear to take it to a segregated Pound/Quarantine (PQ) area. If the followed vessel is indeed an R, then the flow of Rs through $\mathcal{D}$ is lessened, to the advantage of B; but, if it turns out to actually be W, then the $\bar{S}$ force is temporarily reduced, to the clear disadvantage of Blue.

Assume that the mean time that an S requires to service/follow-to-escort a detected and classified-as-R vessel (could actually be W or R) is $1/\phi$. Note: We do not here consider that the escort fleet (DD numbers) may be limited.

We now propose a simple deterministic/"fluid approximation" model to describe the state of the system, i.e., the numbers of free and followed (misidentified) Ws and Rs in $\mathcal{D}$ at time *t* after some initial instant. We assume that once a W is classified as an R it is followed (tracked) and removed from further consideration; a sensor that is following a vessel classified as an R, while awaiting the surface escort vessel (e.g., DDs), is unable to search for Rs.

Objective (Principle). Characterize/model to reveal the rate of leakage of hostile Rs through $\mathcal{D}$ as it depends on the magnitude of $\bar{S}$ and various operational parameters. Let leakage be defined as

- $L\left(\bar{S};\mathcal{D},\text{ parameters and CONOPS}\right) = \text{rate at which Rs cross the Homeland Boundary } \underline{\mathcal{D}}$

The mean time a free R spends in $\mathcal{D}$ is $1/\mu_r$; the mean time a followed R (number is $R_f(t)$ at *t*) spends in $\mathcal{D}$ is $1/\mu_r^*$; in more detailed models, the R may evade or dash to a boundary if detected, but such a move may well be to his disadvantage. In the simplest case, the leakage rate at time *t* is

- $L(t) = \mu_r R(t) + \mu_r^* R_f(t)$, or $L(\infty) = \mu_r \lim_{t\to\infty} R(t) + \mu_r^* \lim_{t\to\infty} R_f(t) = \mu_r r + \mu_r^* r_f$.

This assumes that there is a finite long-run steady state value of both $R(t)$ and $R_f(t)$, which follows if the (leakage) rates, μ_r and μ_r^* are both positive.

1.2. Initial Fluid Model

Let

$W(t)$ =number of free, undetected and unfollowed Ws in $\mathcal{D}$ at time *t*;

$W_f(t)$ =same, but now detected and followed until it is induced/directed to meet an escort (e.g., DD-pair). Each S so involved—one per W_f, is unprofitably employed not as an active searcher capable of detecting a "true R";

$R(t)$ =number of lethal Rs in $\mathcal{D}$ at time *t* that are undetected and unfollowed;

$R_f(t)$ =number of potentially lethal Rs (to Blue (B) Homeland) that have been detected and *are* being followed, in this case profitably.

Note: The number of Ss engaged in following, and therefore not free to search/carry out detection and (mis) classification, is $\bar{S}(t) = W_f(t) + R_f(t)$, provided (as assumed) there is one S assigned to each following operation. There is no swarming to verify detection/classification, although this is an emerging alternative.

1.2.1. Example State Transitions in the First Fluid Model

For $h > 0$ "small" we write (for example)

$$W(t+h) = W(t) + \underbrace{\lambda_w(t)h}_{\substack{\text{W arrivals in } \mathcal{D} \\ \text{during (t,t+h]}}} - \underbrace{\mu_w W(t)h}_{\substack{\text{W departures} \\ \text{from } \mathcal{D} \\ \text{during (t,t+h]}}}$$

$$-\underbrace{\left[\bar{S} - W_f(t) - R_f(t)\right]^+}_{\substack{\text{free-to-search} \\ \text{OH force} \\ \text{elements}}} \underbrace{\delta\left(W(t)+R(t)\right)}_{\substack{\text{detection} \\ \text{rate} \\ \text{when} \\ W(t)+R(t) \\ \text{vessels} \\ \text{in } \mathcal{D}}} \underbrace{\frac{W(t)}{W(t)+R(t)}}_{\substack{\text{Prob.} \\ \text{detect} \\ \text{Free W}}} c_{wr} h + o(h), \qquad (1.2.1)$$

where c_{wr} is the probability that a W is misclassified as an R.

Note that $\delta(\cdot)$ is the overall search rate; rate of contact between Free Ss and Free Ws; the rate of detection depends on the size of $\mathcal{D}$ and the number of Ws and Rs in $\mathcal{D}$; the expression above, with the term $[\ ]^+$ expresses explicitly the fact that $W_f(t)+R_f(t)\le\bar{S}$; the probability c_{wr} is that of mistaking a W for an R and thus initiating a counterproductive "follow."

Manipulation of (1.2.1) (subtraction of $W(t)$, division by h, letting $h\to 0$) yields the differential equation

$$\frac{dW(t)}{dt}=\lambda_w(t)-\mu_w W(t)-\left[\bar{S}-W_f(t)-R_f(t)\right]^+\delta\big(W(t)+R(t)\big)c_{wr}\frac{W(t)}{W(t)+R(t)}.$$

(1.2.2)

An analogous argument leads to

$$\frac{dW_f(t)}{dt}=\underbrace{\left[\bar{S}-W_f(t)-R_f(t)\right]^+\delta\big(W(t)+R(t)\big)c_{wr}\frac{W(t)}{W(t)+R(t)}}_{\text{follow of Ws as (mistakenly) Rs initiated}}-\underbrace{W_f(t)\phi_w}_{\text{follow ends with meeting DD escort}}-\underbrace{\mu_w^* W_f(t)}_{\text{follow ends by followed vessel leaving domain}}. \qquad (1.2.3)$$

Likewise and analogously for Rs:

$$\frac{dR(t)}{dt}=\lambda_r(t)-\mu_r R(t)-\left[\bar{S}-W_f(t)-R_f(t)\right]^+\delta\big(W(t)+R(t)\big)c_{rr}\frac{R(t)}{W(t)+R(t)}$$

(1.2.4)

and

$$\frac{dR_f(t)}{dt} = \underbrace{\left[\bar{S} - W_f(t) - R_f(t)\right]^+ \delta\big(W(t)+R(t)\big)c_{rr}\frac{R(t)}{W(t)+R(t)}}_{\substack{\text{follow of R legitimately}\\ \text{and correctly}\\ \text{by S}}} - \underbrace{R_f(t)\phi_r}_{\substack{\text{follow ends with}\\ \text{meeting DD}\\ \text{escort}}} - \underbrace{\mu_r^* R_f(t)}_{\substack{\text{follow ends by}\\ \text{followed vessel}\\ \text{leaving}\\ \text{domain}}} . \qquad (1.2.5)$$

1.3. Steady-State or Long-Run ($t \to \infty$) Equations for the Simple Fluid Model

Assume $\lambda_w(t) = \lambda_w,\ \lambda_r(t) = \lambda_r$. Then $W(t) \to W(\infty) = w, R(t) \to R(\infty) = r, etc.$; the values are obtained by setting $\frac{dR}{dt} = 0$, etc. in (1.2.2)-(1.2.5).

$$0 = \lambda_w - \mu_w w - \left[\bar{S} - w_f - r_f\right]^+ \delta(w+r)c_{wr}\frac{w}{w+r} \qquad (1.3.1)$$

$$0 = \underbrace{\left[\bar{S} - w_f - r_f\right]^+ \delta(w+r)c_{wr}\frac{w}{w+r}}_{\text{follow of W as R}} - \underbrace{w_f\phi_w}_{\substack{\text{follow}\\ \text{ends}\\ \text{with}\\ \text{meeting}\\ \text{DD}\\ \text{escort}}} - \underbrace{\mu_w^* w_f}_{\substack{\text{follow ends by}\\ \text{followed vessel}\\ \text{leaving}\\ \text{domain}}} \qquad (1.3.2)$$

$$0 = \lambda_r - \mu_r r - \left[\bar{S} - w_f - r_f\right]^+ \delta(w+r)c_{rr}\frac{r}{w+r} \qquad (1.3.3)$$

$$0 = \underbrace{\left[\bar{S} - w_f - r_f\right]^+ \delta(w+r)c_{rr}\frac{r}{w+r}}_{\text{follow of R as R}} - \underbrace{r_f\phi_r}_{\substack{\text{follow}\\ \text{ends}\\ \text{with}\\ \text{meeting}\\ \text{DD}\\ \text{escort}}} - \underbrace{\mu_r^* r_f}_{\substack{\text{follow ends by}\\ \text{followed vessel}\\ \text{leaving}\\ \text{domain}}} \qquad (1.3.4)$$

A Special Case

Let $\rho = \dfrac{w_f + r_f}{\bar{S}}$. Assume $\delta(w+r) = \delta_0 \times [w+r]$, where δ_0 is a constant; $\phi_w = \phi_r$; $\mu_w^* = \mu_r^*$ and $w_f + r_f < \bar{S}$. Then

$$w = \frac{\lambda_w}{\mu_w + \bar{S}[1-\rho]\delta_0 c_{wr}};$$

(1.3.5)

$$w_f = \frac{\bar{S}(1-\rho)\delta_0 c_{wr}}{\phi_w + \mu_w^*} w = \left[\frac{\bar{S}(1-\rho)\delta_0 c_{wr}}{\phi_w + \mu_w^*}\right]\left[\frac{\lambda_w}{\mu_w + \bar{S}(1-\rho)\delta_0 c_{wr}}\right]; \qquad (1.3.6)$$

$$r = \frac{\lambda_r}{\mu_r + \bar{S}[1-\rho]\delta_0 c_{rr}}; \qquad (1.3.7)$$

$$r_f = \frac{\bar{S}(1-\rho)\delta_0 c_{rr}}{\phi_r + \mu_r^*} r = \left[\frac{\bar{S}(1-\rho)\delta_0 c_{rr}}{\phi_r + \mu_r^*}\right]\left[\frac{\lambda_r}{\mu_r + \bar{S}(1-\rho)\delta_0 c_{rr}}\right]. \qquad (1.3.8)$$

Since

$$\rho = \frac{w_f + r_f}{\bar{S}}$$
$$= \frac{1}{\bar{S}}\left[\left[\frac{\bar{S}(1-\rho)\delta_0 c_{wr}}{\phi_w + \mu_w^*}\right]\left[\frac{\lambda_w}{\mu_w + \bar{S}(1-\rho)\delta_0 c_{wr}}\right] + \left[\frac{\bar{S}(1-\rho)\delta_0 c_{rr}}{\phi_r + \mu_r^*}\right]\left[\frac{\lambda_r}{\mu_r + \bar{S}(1-\rho)\delta_0 c_{rr}}\right]\right]. \qquad (1.3.9)$$

Note that the RHS of the equation (1.3.9) decreases as ρ increases. Thus, there is one solution. For given $\bar{S}$ and parameters one can solve for ρ; ρ satisfies

$$\frac{\rho}{1-\rho}$$
$$= \left[\left[\frac{\delta_0 c_{wr}}{\phi_w + \mu_w^*}\right]\left[\frac{\lambda_w}{\mu_w + \bar{S}(1-\rho)\delta_0 c_{wr}}\right] + \left[\frac{\delta_0 c_{rr}}{\phi_r + \mu_r^*}\right]\left[\frac{\lambda_r}{\mu_r + \bar{S}(1-\rho)\delta_0 c_{rr}}\right]\right]. \qquad (1.3.10)$$

If $\delta_0 = 0$, then the solution is $\rho = 0$; thus $w = \dfrac{\lambda_w}{\mu_w}$ $r = \dfrac{\lambda_r}{\mu_r}$.

1.4. Linearization of Dynamic Equations ((1.2.2)-(1.2.5)) via a Steady-State Term Replacement

The analytical intractability and computational awkwardness in solving the time-dependent equations (1.2.2)-(1.2.5) owes to the term $\left[\overline{S}-W_f(t)-R(t)\right]^{+}$ that appears in each. Assume $\delta(w+r)=\delta_0\times(w+r)$. We now investigate the effect of substituting the corresponding constant steady-state term for it, and thus solving the following linear equations:

$$\frac{d\tilde{W}(t)}{dt}=\lambda_w-\left(\mu_w+\overline{S}(1-\rho)\delta_0 c_{wr}\right)\tilde{W}(t) \tag{1.4.1}$$

$$\frac{d\tilde{W}_f(t)}{dt}=\overline{S}(1-\rho)\delta_0 c_{wr}\tilde{W}(t)-\left(\phi_w+\mu_w^*\right)\tilde{W}_f(t) \tag{1.4.2}$$

$$\frac{d\tilde{R}(t)}{dt}=\lambda_r-\left(\mu_r+\overline{S}(1-\rho)\delta_0 c_{rr}\right)\tilde{R}(t) \tag{1.4.3}$$

$$\frac{d\tilde{R}_f(t)}{dt}=\overline{S}(1-\rho)\delta_0 c_{rr}\tilde{R}(t)-\left(\phi_r+\mu_r^*\right)\tilde{R}_f(t)\,. \tag{1.4.4}$$

The solutions to these are elementary. To simplify writing put

$$\begin{aligned}
A&=\mu_w+\overline{S}(1-\rho)\delta_0 c_{wr}\\
B&=\overline{S}(1-\rho)\delta_0 c_{wr}\\
C&=\phi_w+\mu_w^*\\
D&=\mu_r+\overline{S}(1-\rho)\delta_0 c_{rr}\\
E&=\overline{S}(1-\rho)\delta_0 c_{rr}\\
F&=\phi_r+\mu_r^*
\end{aligned}\;. \tag{1.4.5}$$

Start by solving (1.4.1)

$$\tilde{W}(t) = \tilde{W}(0)e^{-At} + \frac{\lambda_w}{A}\left(1 - e^{-At}\right), \tag{1.4.6}$$

then substitute into (1.4.2):

$$\tilde{W}_f(t) = \tilde{W}_f(0)e^{-Ct} + B\left[\left(\tilde{W}(0) - \frac{\lambda_w}{A}\right)\left(\frac{e^{-At} - e^{-Ct}}{C - A}\right) + \frac{\lambda_w}{A}\frac{1 - e^{-Ct}}{C}\right]. \tag{1.4.7}$$

Simply changing parameter values solves (1.4.3) and (1.4.4):

$$\tilde{R}(t) = \tilde{R}(0)e^{-Ft} + \frac{\lambda_r}{D}\left(1 - e^{-Dt}\right) \tag{1.4.8}$$

$$\tilde{R}_f(t) = \tilde{R}_f(0)e^{-Ft} + E\left[\left(\tilde{R}(0) - \frac{\lambda_r}{D}\right)\left(\frac{e^{-Dt} - e^{-Ft}}{F - D}\right) + \frac{\lambda_r}{D}\left[\frac{1 - e^{-Ft}}{F}\right]\right]. \tag{1.4.9}$$

These are all simple solutions involving exponentials and constants. They perform correctly at $t = 0$ and $t = \infty$ (compare to results (1.3.5)-(1.3.8)).

These can be compared to simulation results.

A fundamental Measure of Effectiveness (actually Defectiveness) is the rate of leakage of Rs through $\underline{\mathcal{D}}$. Note that both $R(t)$ and $R_f(t)$ can reach $\underline{\mathcal{D}}$ before arrival of DD escorts, so the total leakage over time *t* is given by

$$L(t) = \int_0^t R(s)\mu_r ds + \int_0^t R_f(s)\mu_r^* ds \tag{1.4.10}$$

and approximately by

$$L(t) = \int_0^t \tilde{R}(s)\mu_r ds + \int_0^t \tilde{R}_f(s)\mu_r^* ds. \tag{1.4.11}$$

In steady-state or long-run, the average rate of leakage is

$$L(\infty) = \lim_{t \to \infty} \frac{L(t)}{t} = r\mu_r + r_f \mu_r^* \tag{1.4.12}$$

1.5. Generalization to More Awareness States

Here are some directions that generalizations and approximations with somewhat more reality can take.

1.5.1. Time-Consuming Classification

Suppose it is acknowledged that when a free-searching Blue surveillance agent, an $\mathcal{S}$, detects a potential threat it may spend a nonnegligible time classifying it; let that time have mean τ or rate $1/\tau$, and the resulting classification accuracy depends on that (mean) time: i.e.,

(a) $c_{wr}(\tau)$ decreases with τ;

(b) $c_{rr}(\tau)$ increases with τ.

The specific form of the dependence can be established by field experiment, or at least $c_{wr}(\underline{\alpha}_w)$ and $c_{rr}(\underline{\alpha}_r)$ can be made plausibly parametric, e.g., the logistic or Weibull with parameters (vectors) $\underline{\alpha}_w$ and $\underline{\alpha}_r$, respectively.

One possibility to account for the influence of a nonzero classification time is to add the means of the detection times and the classification times; then invert to obtain the effective rate of detection and classification:

$$\delta(w,\tau) = \left(\frac{1}{\delta(w)} + \tau\right)^{-1}. \tag{1.5.1}$$

A more physical version of the above is given in Section (4.1). In the fluid approximation proposed, with dynamic equations (1.3.1)-(1.3.4) and steady-state equations (1.3.5)-(1.3.9) replace δ_0 by $\delta_0(\tau)=1/\left(\frac{1}{\delta_0}+\tau\right)$ and

$$\begin{array}{l} c_{wr} \text{ by } c_{wr}(\tau;\underline{\alpha}_w) \\ c_{rr} \text{ by } c_{rr}(\tau;\underline{\alpha}_r) \end{array}. \tag{1.5.2}$$

Apparently, τ is a decision variable available for adjustment by S (Blue Surveillance) to minimize the rate at which Rs cross $\underline{\mathcal{D}}$ into $\underline{\mathcal{D}}(H)$. See below.

1.5.2. A Parametric Model for the Probability of Correct Classification as a Function of Classification Time

Assume that the probability of correct classification of a vessel of type *j* is a function of the time spent classifying the vessel. One functional form for the conditional probability that a vessel of type *j* is correctly classified as type *j* when τ time units are spent classifying it is $c_{jj}(\tau)=0.5+0.5\frac{(\beta\tau)^{\alpha}}{1+(\beta\tau)^{\alpha}}$; the parameter α, a classification time τ_0, and a target value c_{jj}^{0} are specified; β is chosen so that $c_{jj}(\tau_0)=c_{jj}^{0}$. More rationally, use τ to optimize an MOE/MOP: e.g., the probability that the Red leaks through $\mathcal{D}$ is minimized. See Section 4.1, especially Figures 4.5a, 4.5b, 4.6a, 4.6b, and 4.5ad, 4.5bd, 4.6ad, and 4.6bd, where it is shown that the qualitative effect carries over between stochastic and deterministic/fluid models.

1.6. A Partially-Spatial Detection (“Strip Search”) Model

Suppose domain $\mathcal{D}$ is divided spatially into $i\in\{1,2,3,...I\}$ *strips* (*subdomains*) that roughly parallel the shoreline. Let $\mathcal{D}_i$ be the ith such strip, with $\mathcal{D}_1$ the furthest toward open sea, and $\mathcal{D}_I$ the closest to (one border being) the shoreline/beach. Any vessel, say a hostile Red, that enters $\mathcal{D}$ must first enter $\mathcal{D}_1$, proceed from that to $\mathcal{D}_2$, to $\mathcal{D}_3$, etc., and finally to $\mathcal{D}_I$. For simplicity, each subdomain $\mathcal{D}_i$ $(i=1,\ldots,I)$ can be rectangular; they cover $\mathcal{D}$ without overlap.

The state transition equations must be augmented as follows: Introduce $\{W_i(t), W_{f,i}(t), R_i(t), R_{f,i}(t); i=1,2,\ldots I\}$ vector(s) of numbers of W and R units occupying each subdomain $\mathcal{D}_i$ $i=1,\ldots,I$. Then differential equations can be written that express transitions into and out of $\mathcal{D}_i$, following arguments similar to those leading to (1.2.2)-(1.2.5). Here are some important special cases. The parameters, as in Section 1.2, are now stage dependent, including the number of sensors/level of surveillance associated with $\mathcal{D}_i$; the latter can change in response to need in a particular geographical stage.

<u>Case 1</u>. Ws enter at $\bar{\mathcal{D}}_1$ migrate to $\mathcal{D}_2, \ldots$ *or* leave $\mathcal{D}_1$ for outside, all if uninterrupted by S and followed; Rs enter at $\bar{\mathcal{D}}_1$, migrate to $\mathcal{D}_2$, …eventually leak through $\underline{\mathcal{D}}_I$ if not intercepted and followed by S to DDs. Innocuous Ws "leak" downwards and sideways, whereas hostile Rs leak purposefully toward $\underline{\mathcal{D}}_I(H)$, the sea-land Blue Homeland barrier. The dynamic equations akin to (1.2.2)-(1.2.5) are now these:

$$\begin{aligned}\frac{dW_1(t)}{dt} &= \lambda_w(t) - \mu_{w,1}(t)W_1(t) \\ &\quad - \left[\bar{S}_1 - W_{f1}(t) - R_{f1}(t)\right]^+ \delta_1\left(W_1(t)+R_1(t)\right)c_{wr,1}\frac{W_1(t)}{W_1(t)+R_1(t)}\end{aligned} \tag{1.6.1}$$

$$\begin{aligned}\frac{dW_{f,1}(t)}{dt} &= \left[\bar{S}_1 - W_{f,1}(t) - R_{f,1}(t)\right]^+ \delta_1\left(W_1(t)+R_1(t)\right)c_{wr,1}\frac{W_1(t)}{W_1(t)+R_1(t)} \\ &\quad - W_{f,1}(t)\phi_{w,1} - \mu^*_{w,1}W_{f,1}(t)\end{aligned} \tag{1.6.2}$$

$$\begin{aligned}\frac{dR_1(t)}{dt} &= \lambda_r(t) - \mu_{r,1}(t)R_1(t) \\ &\quad -\left[\bar{S}_1 - W_{f,1}(t) - R_{f,1}(t)\right]^+ \delta_1\left(W_1(t)+R_1(t)\right)c_{rr,1}\frac{R_1(t)}{W_1(t)+R_1(t)}\end{aligned} \tag{1.6.3}$$

$$\begin{aligned}\frac{dR_{f,1}(t)}{dt} &= \left[\bar{S}_1 - W_{f,1}(t) - R_{f,1}(t)\right]^+ \delta_1\left(W_1(t)+R_1(t)\right)c_{rr,1}\frac{R_1(t)}{W_1(t)+R_1(t)}. \\ &\quad -R_{f,1}(t)\phi_{r,1} - \mu^*_{r,1}R_{f,1}(t)\end{aligned} \tag{1.6.4}$$

Next, for interstages, $i = 2,3,\ldots,I;$ simply replace $\lambda_w(t)$ by the arrival rate of Ws from the previous stage, where $0 < \omega_{\square,i} \le 1$ is the fraction of platforms of type □ that "leak" into strip i from strip i-1 (closer to entry strip 1). We do not consider entries at the sides of $\mathcal{D}$, although this could be represented.

$$\begin{aligned}\frac{dW_i(t)}{dt} &= \omega_{w,i}\mu_{w,i-1}W_{i-1}(t) - \mu_{w,i}(t)W_i(t) \\ &\quad -\left[\bar{S}_i - W_{f,i}(t) - R_{f,i}(t)\right]^+ \delta_i\left(W_i(t)+R_i(t)\right)c_{wr,i}\frac{W_i(t)}{W_i(t)+R_i(t)}\end{aligned} \tag{1.6.5}$$

$$\begin{aligned}\frac{dW_{f,i}(t)}{dt} &= \omega_{w,i}\mu^*_{w,i-1}(t)W_{f,(i-1)}(t) \\ &\quad +\left[\bar{S}_i - W_{f,i}(t) - R_{f,i}(t)\right]^+ \delta_i\left(W_i(t)+R_i(t)\right)c_{wr,i}\frac{W_i(t)}{W_i(t)+R_i(t)} \\ &\quad -W_{f,i}(t)\phi_{w,i} - \mu^*_{w,i}W_{f,i}(t)\end{aligned} \tag{1.6.6}$$

$$\begin{aligned}\frac{dR_i(t)}{dt} &= \omega_{r,i}\mu_{r,i-1}R_{i-1}(t) - \mu_{r,i}(t)R_i(t) \\ &\quad -\left[\bar{S}_i - W_{f,i}(t) - R_{f,i}(t)\right]^+ \delta_i\left(W_i(t)+R_i(t)\right)c_{rr,i}\frac{R_i(t)}{W_i(t)+R_i(t)}\end{aligned} \tag{1.6.7}$$

$$\begin{aligned}\frac{dR_{f,i}(t)}{dt} &= \omega_{r,i}\mu^*_{r,i-1}(t)R_{f,(i-1)}(t) \\ &\quad +\left[\bar{S}_i - W_{f,i}(t) - R_{f,i}(t)\right]^+ \delta_i\left(W_i(t)+R_i(t)\right)c_{rr,i}\frac{R_i(t)}{W_i(t)+R_i(t)} \\ &\quad -R_{f,i}(t)\phi_{r,i} - \mu^*_{r,i}R_{f,i}(t)\end{aligned} \tag{1.6.8}$$

Finally, at the strip nearest the beach/Homeland border, evaluate (1.6.7) at $i = I$ and interpret the term $\mu_{rI} R_I(t)$ as the instantaneous rate at which hostile Reds that are not being followed enter the homeland (i.e., "leak").

Case 2. Same as above, but each subdomain $\mathcal{D}_i$ always contains the same number of Ws, i.e. $w_i = W_i(t)$. If a W is erroneously identified as R and escorted to Quarantine, it is examined and released after a delay. Assume that replacement by another W occurs immediately.

1.6.1. Steady-State or Long-Run Equations for the Strip Search Model

Again, as in Section 1.3, assume that all rate parameters are constant. We obtain a collection of 4I non-linear equations. If $W_1(t) \to w_1, ..., R_{f,1}(t) \to r_{f,1}$, and $W_i(t) \to w_i, ..., R_{f,i}(t) \to r_{f,i}$, then if as before $\delta_i\left(W_i(t) + R_i(t)\right) = \delta_i \times \left(w_i + r_i\right)$, δ_i a constant, we find

$$\begin{aligned}
0 &= \lambda_w - \mu_{w,1} w_1 - \left[\bar{S}_1 - w_{f,1} - r_{f,1}\right]^+ \delta_1 w_1 c_{wr,1} \\
0 &= \left[\bar{S}_1 - w_{f,1} - r_{f,1}\right]^+ \delta_1 w_1 c_{wr,1} - w_{f,1}\left(\phi_{w,1} + \mu^*_{w,1}\right) \\
0 &= \lambda_r - \mu_{r,1} r_1 - \left[\bar{S}_1 - w_{f,1} - r_{f,1}\right]^+ \delta_1 r_1 c_{rr,1} \\
0 &= \left[\bar{S}_1 - w_{f,1} - r_{f,1}\right]^+ \delta_1 r_1 c_{rr,1} - r_{f,1}\left(\phi_{r,1} + \mu^*_{r,1}\right)
\end{aligned}. \qquad (1.6.9a)$$

For subsequent slices we get for i=2,3,…,I,

$$\begin{aligned}
0 &= \omega_{w,i} \mu_{w,i-1} w_{i-1} - \mu_{w,i} w_i - \left[\bar{S}_i - w_{f,i} - r_{f,i}\right]^+ \delta_i w_i c_{wr,i} \\
0 &= \omega_{w,i} \mu^*_{w,i-1} w_{f,i-1} + \left[\bar{S}_i - w_{f,i} - r_{f,i}\right]^+ \delta_i w_i c_{wr,i} - w_{f,i}\left(\phi_{w,i} + \mu^*_{w,i}\right) \\
0 &= \omega_{r,i} \mu_{r,i-1} r_{i-1} - \mu_{r,i} r_i - \left[\bar{S}_i - w_{f,i} - r_{f,i}\right]^+ \delta_i r_i c_{rr,i} \\
0 &= \omega_{r,i} \mu^*_{r,i-1} r_{f,i-1} + \left[\bar{S}_i - w_{f,i} - r_{f,i}\right]^+ \delta_i r_i c_{rr,i} - r_{f,i}\left(\phi_{r,i} + \mu^*_{r,i}\right)
\end{aligned}. \qquad (1.6.9b)$$

Letting $\rho_1 = \frac{w_{f,1} + r_{f,1}}{\bar{S}_1}$

$$\lambda_w = \mu_{w,1} w_1 + (1-\rho_1)\bar{S}_1 \delta_1 w_1 c_{wr,1}$$

or (1.6.7a)

$$w_1 = \frac{\lambda_w}{\mu_{w,1} + (1-\rho_1)\bar{S}_1 \delta_1 c_{wr,1}}$$

$$\begin{aligned} w_{f,1} &= \frac{\bar{S}_1 (1-\rho_1) \delta_1 c_{wr,1}}{\phi_{w,1} + \mu^*_{w,1}} w_1 \\ &= \left[\frac{\bar{S}_1 (1-\rho_1) \delta_1 c_{wr,1}}{\phi_{w,1} + \mu^*_{w,1}} \right] \left[\frac{\lambda_w}{\mu_{w,1} + (1-\rho_1)\bar{S}_1 \delta_1 c_{wr,1}} \right] \end{aligned} \tag{1.6.7b}$$

$$r_1 = \frac{\lambda_r}{\mu_{r,1} + (1-\rho_1)\bar{S}_1 \delta_1 c_{rr,1}} \tag{1.6.7c}$$

$$\begin{aligned} r_{f,1} &= \frac{\bar{S}_1 (1-\rho_1) \delta_1 c_{rr,1}}{\phi_{r,1} + \mu^*_{r,1}} r_1 \\ &= \left[\frac{\bar{S}_1 (1-\rho_1) \delta_1 c_{rr,1}}{\phi_{r,1} + \mu^*_{r,1}} \right] \left[\frac{\lambda_r}{\mu_{r,1} + (1-\rho_1)\bar{S}_1 \delta_1 c_{rr,1}} \right], \end{aligned} \tag{1.6.7d}$$

where ρ_1 satisfies the equation

$$\begin{aligned} \rho_1 &= \frac{w_{f,1} + r_{f,1}}{\bar{S}_1} \\ &= \frac{1}{\bar{S}_1} \left[\left[\frac{\bar{S}_1 (1-\rho_1) \delta_1 c_{wr,1}}{\phi_{w,1} + \mu^*_{w,1}} \right] \left[\frac{\lambda_w}{\mu_w + \bar{S}_1 (1-\rho_1) \delta_1 c_{wr,1}} \right] \right]. \\ &+ \frac{1}{\bar{S}_1} \left[\frac{\bar{S}_1 (1-\rho_1) \delta_1 c_{rr,11}}{\phi_{r,1} + \mu^*_{r,1}} \right] \left[\frac{\lambda_r}{\mu_{r,1} + \bar{S}_1 (1-\rho_1) \delta_1 c_{rr,1}} \right] \end{aligned} \tag{1.6.7e}$$

For i=2,3,…,I, letting

$$\rho_i = \frac{w_{f,i} + r_{f,i}}{\bar{S}_i}$$

$$w_i = \frac{\omega_{w,i}\mu_{w,i-1}w_{i-1}}{\mu_{w,i} + (1-\rho_i)\bar{S}_i\delta_i c_{wr,i}} \tag{1.6.8a}$$

$$\begin{aligned} w_{f,i} &= \left[\frac{\omega_{w,i}\mu^*_{w,i-1}w_{f,i-1}}{\phi_{w,i}+\mu^*_{w,i}}\right] + \left[\frac{\bar{S}_i(1-\rho_i)\delta_i c_{wr,i}}{\phi_{w,i}+\mu^*_{w,i}} w_i\right] \\ &= \left[\frac{\omega_{w,i}\mu^*_{w,i-1}w_{f,i-1}}{\phi_{w,i}+\mu^*_{w,i}}\right] + \left[\frac{\bar{S}_i(1-\rho_i)\delta_i c_{wr,i}}{\phi_{w,i}+\mu^*_{w,i}}\right]\left[\frac{\omega_{w,i}\mu_{w,i-1}w_{i-1}}{\mu_{w,i}+(1-\rho_i)\bar{S}_i\delta_i c_{wr,i}}\right] \end{aligned} \tag{1.6.8b}$$

$$r_i = \frac{\omega_{r,i}\mu_{r,i-1}r_{i-1}}{\mu_{r,i} + (1-\rho_i)\bar{S}_i\delta_i c_{rr,i}} \tag{1.6.8c}$$

$$\begin{aligned} r_{f,i} &= \left[\frac{\omega_{r,i}\mu^*_{r,i-1}r_{f,i-1}}{\phi_{r,i}+\mu^*_{r,i}}\right] + \left[\frac{\bar{S}_i(1-\rho_i)\delta_i c_{rr,i}}{\phi_{r,i}+\mu^*_{r,i}} r_i\right] \\ &= \left[\frac{\omega_{r,i}\mu^*_{r,i-1}r_{f,i-1}}{\phi_{r,i}+\mu^*_{r,i}}\right] + \left[\frac{\bar{S}_i(1-\rho_i)\delta_i c_{rr,i}}{\phi_{r,i}+\mu^*_{r,i}}\right]\left[\frac{\omega_{r,i}\mu_{r,i-1}w_{i-1}}{\mu_{r,i}+(1-\rho_i)\bar{S}_i\delta_i c_{rr,i}}\right], \end{aligned} \tag{1.6.8d}$$

where ρ_i satisfies the equation

$$\begin{aligned} \rho_i &= \frac{w_{f,i}+r_{f,i}}{\bar{S}} \\ &= \frac{1}{\bar{S}_i}\left[\left[\frac{\omega_{w,i}\mu^*_{w,i-1}w_{f,i-1}}{\phi_{w,i}+\mu^*_{w,i}}\right] + \left[\frac{\bar{S}_i(1-\rho_i)\delta_i c_{wr,i}}{\phi_{w,i}+\mu^*_{w,i}}\right]\left[\frac{\omega_{w,i}\mu_{w,i-1}w_{i-1}}{\mu_{w,i}+(1-\rho_i)\bar{S}_i\delta_i c_{wr,i}}\right]\right]. \\ &+ \frac{1}{\bar{S}}\left[\left[\frac{\omega_{r,i}\mu^*_{r,i-1}r_{f,i-1}}{\phi_{r,i}+\mu^*_{r,i}}\right] + \left[\frac{\bar{S}_i(1-\rho_i)\delta_i c_{rr,i}}{\phi_{r,i}+\mu^*_{r,i}}\right]\left[\frac{\omega_{r,i}\mu_{r,i-1}r_{i-1}}{\mu_{r,i}+(1-\rho_i)\bar{S}_i\delta_i c_{rr,i}}\right]\right] \end{aligned} \tag{1.6.8e}$$

2. Model 2: A Stochastic Model with One Red Entering the Domain at Time 0

2.1. Simple MDA Scenarios

Start with the number of white vessels (Ws) at time 0, $W(0)=w$ in a domain $\mathcal{D}$, and allow either new W arrivals if a W is wrongly removed, or allow replacements (there are several arrivals); there are a constant number of Ws in the domain. At $t=0$ a single hostile/lethal R enters the domain. When in free/search the (single) overhead (OH) sensor

($\mathcal{S}$) encounters any platform W or R at time t with probability $\delta(W(t)+1)dt$ if the R has not been identified earlier (time <t), where $W(t)$ is the number of White vessels at time t. If the W is misidentified as an R, the $\mathcal{S}$ follows (tracks) it for a random time until a DD (pair) approaches and takes over from the $\mathcal{S}$, and the $\mathcal{S}$ returns to searching $\mathcal{D}$.

We will concentrate on the set of cases that become progressively more mathematically involved (but also more operationally realistic and significant).

2.2. The Number of Ws in $\mathcal{D}$ is Constant (=w)

Suppose $\mathcal{S}$ searches $\mathcal{D}$ at random at rate $\delta(w)$ (or $\delta(\tau;w)$ (see Section 1.5, where τ is the time $\mathcal{S}$ spends classifying a vessel) and encounters a W (before an R) with probability $w/(w+1)$. If it encounters "something" and it is a W, it classifies it correctly with probability $c_{ww}(\tau)$, and releases it and continues searching after the classification time τ.

Note that another case is one in which such a discovery might be followed by tagging the particular platform correctly as a W, in which case it is ignored on future encounters; the Automatic Identification System (AIS) is a possible way of tagging/identifying a (subset of) Ws, but is not infallible, being subject to interference, system failure, and conceivably jamming or deception. But if it is the R that is encountered and misclassified as a W, then the R may be ignored in future searches, and should make it to the lower boundary $\underline{\mathcal{D}}$ of $\mathcal{D}$ and be able to invade the homeland $\underline{\mathcal{D}}(H)$. We analyze this kind of situation later.

2.2.1. Backward Equation for Time of Capture and Probability of Capture of Single Hostile R for Quarantine Before Homeland $\underline{\mathcal{D}}(H)$ is Entered

Let a single R enter $\mathcal{D}$ at the upper boundary $\bar{\mathcal{D}}$ at t=0. There are $W(0)=w$ Ws in $\mathcal{D}$ initially, and this number remains constant, at w, perhaps by replacement if any W misidentified as R and followed and escorted to quarantine (Q). Let $\boldsymbol{T}_{rq}(w)$ denote the time an R survives being taken to quarantine.

Assume initially that $\boldsymbol{X}_i$ (i=1,2,…) are successive times between detection of platforms; they are independent identically distributed (iid) random variables (rvs).

2.2.2. Benchmark Example

Let the time between detections be $\boldsymbol{X}_i$. Then a conventional example is that time to detection is exponential with rate proportional to the number of targets present:

$$P\left\{\boldsymbol{X}_i(w)>t\right\}=\exp\left(-\delta(w+1)t\right). \tag{2.2.1}$$

This need not be assumed, but is convenient. In fact, any sequence of independent identically distributed random variables (iid rvs) is possible, and analytically tractable in the present situation. Let $\boldsymbol{D}_i$ be the ith follow (or tracking) period for any platform selected to follow (that has been classified correctly or incorrectly, as R). At the end of follow periods, the platform is released to the Diverters (e.g., destroyers (DDs)).

Note: We only study the stages involved in Detection and Follow to the DDs at present. We assume that a correct classification is made (almost) as soon as the diverter (D) reaches the platform.

2.2.3. Backward Equation Logic

The R is allowed to move into $\mathcal{D}$, and the overhead sensor $\mathcal{S}$ is in search until a platform/(potential quarry) is detected; the platform is then classified for time τ; it is classified as a W with probability $c_{\square w}(\tau)$, and as an R with probability $c_{\square r}(\tau)$.

Consider the time until first detection, and the subsequent possibilities:

(a) If the R is detected first, and correctly classified as an R, it is followed to a signaled D; this requires a relatively long random time $\boldsymbol{D}$ (a strong assumption that can be relaxed). Assume also that there is always a D available upon request. In this case, the R is detected first and is correctly followed to D in time $\boldsymbol{X}+\boldsymbol{D}$ with probability $\frac{1}{w+1}c_{rr}(\tau)$.

(b) If the R is detected first and misidentified as a W it is released. The time this event takes place is at $t=\boldsymbol{X}$, and its probability is $\frac{1}{w+1}c_{rw}(\tau)$. The process restarts.

(c) If a W is detected first and correctly identified as a W, then it is released and $\mathcal{S}$ immediately begins to search again. In this case, the first time length is $\boldsymbol{X}$ with probability $\frac{w}{w+1}c_{ww}(\tau)$. The search for the R begins again from scratch.

(d) If a W is detected first and misclassified as an R, then a D is summoned with probability $c_{wr}(\tau)$. In this case, the time taken by a complete first step/event is

$\boldsymbol{X}+\boldsymbol{D}$ with probability $\frac{w}{w+1}c_{wr}$; at the end of this step, the $\mathcal{S}$ begins to search again from scratch.

Note that there are natural exceptions to the behavior described, such as pointed out in (a). We address these later.

Then for this model

(a') $\boldsymbol{T}_{rq}(w)=\boldsymbol{X}+\boldsymbol{D}$ with probability $\frac{1}{w+1}c_{rr}(\tau)$

(b') $\boldsymbol{T}_{rq}(w)=\boldsymbol{X}'+\boldsymbol{T}_{rq}^{'}(w)$ with probability $\frac{1}{w+1}c_{rw}(\tau)$

(c') $\boldsymbol{T}_{rq}(w)=\boldsymbol{X}''+\boldsymbol{T}_{rq}^{''}(w)$ with probability $\frac{w}{w+1}c_{ww}(\tau)$

(d') $\boldsymbol{T}_{rq}(w)=\boldsymbol{X}'''+\boldsymbol{D}'+\boldsymbol{T}_{rq}^{'''}(w)$ with probability $\frac{w}{w+1}c_{wr}(\tau)$

the various different apostrophes indicate that the rvs are iid replicas of the basic $\boldsymbol{X},\boldsymbol{D}$ components, and those on $\boldsymbol{T}_{rq}$ on the right side indicate that the search process starts over "from scratch."

Now compute the Laplace-Stieltjes transform of $T_{rq}(w)$: at least for transform variable $s\geq 0$ and using independence where needed, conditioning as needed gives

$$
\begin{aligned}
&E\left[e^{-s\boldsymbol{T}_{rq}(w)}\right]=E\left[e^{-s(\boldsymbol{X}+\boldsymbol{D})}\right]\frac{1}{w+1}c_{rr}(\tau)\\
&+E\left[e^{-s\boldsymbol{X}}\right]E\left[e^{-s\boldsymbol{T}_{rq}(w)}\right]\left[\frac{1}{w+1}c_{rw}(\tau)+\frac{w}{w+1}c_{ww}(\tau)\right]\\
&+E\left[e^{-s(\boldsymbol{X}+\boldsymbol{D})}\right]E\left[e^{-s\boldsymbol{T}_{rq}(w)}\right]\frac{w}{w+1}c_{wr}(\tau)\\
&=\frac{E\left[e^{-s(\boldsymbol{X}+\boldsymbol{D})}\right]\frac{1}{w+1}c_{rr}(\tau)}{1-E\left[e^{-s\boldsymbol{X}}\right]\left[\frac{1}{w+1}c_{rw}(\tau)+\frac{w}{w+1}c_{ww}(\tau)+E\left[e^{-s\boldsymbol{D}}\right]\frac{w}{w+1}c_{wr}(\tau)\right]}
\end{aligned} \quad . \qquad (2.2.2)
$$

Let $p_q(t;w)$ be the probability that R is captured by D in time *t* given there are *w* Ws in the domain. If $s \to 0$, $E\left[e^{-s\boldsymbol{T}_{rq}(w)}\right] \to p_q(\infty;w)$ the probability that, if the domain $\mathcal{D}$ is of infinite width, or requires an infinite traversal time, the R is eventually captured by D and escorted to Q; then we see from (2.2.2) that $p_q(\infty;w)=1$.

Special Tractable Case: Exponential Red Transit Time

Let the probability distribution of the time for R to reach the lower boundary $\underline{\mathcal{D}}$ starting at any point in the domain $\mathcal{D}$ if it has not yet been linked to D be exp(μ_r), i.e., exponential with mean $1/\mu_r$. Then it can be seen that, conditional on $\boldsymbol{T}_{qr}(w)$, the probability of R capture before leakage is just $E\left[e^{-\mu_r \boldsymbol{T}_{rq}(w)}\right]$; the result of setting $s=\mu_r$ in (2.2.2). Notice that this probability is 1 if $\mu_r=0$, so the mean transit time through $\mathcal{D}$ of R is very large: $1/\mu_r=\infty$. Otherwise if R’s mean transit time $1/\mu_r=0$ or $\mu_r \to \infty$, then there is no time to detect and follow such a fast-moving target.

Differentiate $E\left[e^{-s\boldsymbol{T}_{rq}(w)}\right]$ with respect to *s* and change sign to obtain $E\left[\boldsymbol{T}_{rq}(w)e^{-s\boldsymbol{T}_{rq}(w)}\right]$. If $s=\mu_r$; this is the mean time of capture by D in the event that R has not yet reached $\mathcal{D}$. The conditional expectation/mean time to capture R for quarantine is

$$E\left[\boldsymbol{T}_{rq}(w) \mid \boldsymbol{T}_{rq}(w) < \infty\right] = \frac{E\left[\boldsymbol{T}_{rq}(w)\exp\left\{-\mu_r \boldsymbol{T}_{rq}(w)\right\}\right]}{E\left[\exp\left\{-\mu_r \boldsymbol{T}_{rq}(w)\right\}\right]}, \tag{2.2.3}$$

which can be evaluated by evaluating the Laplace transform (2.2.2) and its derivative at $s = \mu_r$.

Appendix A presents results for a special case in which $\boldsymbol{X}$ and $\boldsymbol{D}$ are independent and exponentially distributed.

2.3. A Strip-Search Approximation for Gamma (Erlang) Red Transit Times

It has been initially assumed that a Red's unopposed transit time of $\mathcal{D}$ is exponentially distributed with mean μ_r. This assumption can conveniently be made more physically plausible by dividing $\mathcal{D}$ into parallel strips, as in (1.6), and assuming that the times to pass through consecutive strips are independently exponential; if there are $I \geq 1$ strips, say $I = 12$, then the mean transit time is I/μ_r, with variance I/μ_r^2 and coefficient of variation $1/I$, the square root, $1/\sqrt{I}$, giving an assessment of the variability of transit time, expressed as a fraction of its mean. Approximations for the probability that R is detected, correctly classified, and escorted in this case are presented in Section 5.

2.4. W Reduction and Identification

A hostile R wishing to cross a littoral $\mathcal{D}$ is searched-for by an OH facility (e.g., a P3 a/c), $\mathcal{S}$, but that search is inhibited by the presence of many "false targets," the Ws. One technology that the Blue $\mathcal{S}$, and other Blue assets can exploit to deal with the many Ws or single (or multiple) Rs situation is the Automatic Identification System (AIS). This system places an RF transponder on friendly Ws (or a subset of all). The transponder periodically transmits the identity, location, and properties such as course and speed, to the W platforms in line of sight, and there exist plans to transmit this information to a Far Overhead Satellite, or the equivalent, from which it is sent to a central database. The AIS is not perfect, (e.g., two vessels within a short distance of each other can block each other's signals; there can also be system failures and environmental miseffects). However, AIS-equipped Ws (a subset of all) ease the task of $\mathcal{S}$. A model for assessing vessel detection/classification by an overhead AIS receiver appears in Section 6.

3. Model 3: A Level-Dependent Quasi-Birth and -Death Model: A Markovian Model in Which More than One R Travels Through the Region

3.1. Formulation: MDA Situation and Model

In this section, more than one Red vessel can be in the Maritime Domain at a time. Assume Red vessels arrive at a Maritime Domain $\mathcal{D}$ according to a Poisson process with rate λ_r. Unless intercepted first, Red (R) spends an exponential time in the domain with mean $1/\mu$. There is one overhead surveillance sensor $\mathcal{S}$. The time until the sensor finds and correctly classifies each R in the domain is an exponential random variable with mean $1/\delta_0$ independent of everything; successive detections are independent. We assume the number of White (W) vessels in $\mathcal{D}$ is constant; if one moves from $\mathcal{D}$ another quickly replaces it. Assume the $\mathcal{S}$ will become busy following a (misclassified) W *after* an exponential time with mean $1/\delta_w$; all times are assumed independent. The $\mathcal{S}$ follows (tracks) any vessels classified as R for an exponential time with mean $1/\phi$. The vessel classified as R is escorted by destroyer pair (DD) after the $\mathcal{S}$ following time.

Let $\boldsymbol{R}(t)=(n,f)$ if there are n unescorted R vessels in the domain and the OH is busy following (tracking) a misclassified W at time t. Let $\boldsymbol{R}(t)=(n,s)$ if there are n unescorted R vessels in the domain and OH is searching for Rs at time t. Let $\boldsymbol{R}(t)=(n,c)$ if there are n unescorted R vessels in the domain and the $\mathcal{S}$ is following a R at time t.

3.2. States and Their Transitions

$P\{\boldsymbol{R}(t+h)=(n+1,f)\,|\,\boldsymbol{R}(t)=(n,f)\}=\lambda_r h+o(h)$: New R arrival; B following W
$P\{\boldsymbol{R}(t+h)=(n-1,f)\,|\,\boldsymbol{R}(t)=(n,f)\}=n\mu h+o(h)$: An R leaves $\mathcal{D}$; B following W
$P\{\boldsymbol{R}(t+h)=(n,s)\,|\,\boldsymbol{R}(t)=(n,f)\}=\phi h+o(h)$: R unchanged; B stops following W ("free")
$P\{\boldsymbol{R}(t+h)=(n+1,s)\,|\,\boldsymbol{R}(t)=(n,s)\}=\lambda_r h+o(h)$: New R arrival; B "free" searching
$P\{\boldsymbol{R}(t+h)=(n-1,s)\,|\,\boldsymbol{R}(t)=(n,s)\}=n\mu h+o(h)$: An R leaves $\mathcal{D}$; B "free" searching
$P\{\boldsymbol{R}(t+h)=(n,f)\,|\,\boldsymbol{R}(t)=(n,s)\}=\delta_w h+o(h)$: A W is misclassified as R; B starts to track (follow)
$P\{\boldsymbol{R}(t+h)=(n-1,c)\,|\,\boldsymbol{R}(t)=(n,s)\}=n\delta_0 h+o(h)$: An R is classified as R; B starts to follow
$P\{\boldsymbol{R}(t+h)=(n,s)\,|\,\boldsymbol{R}(t)=(n,c)\}=\phi h+o(h)$: Rs unchanged; B stops following R
$P\{\boldsymbol{R}(t+h)=(n-1,c)\,|\,\boldsymbol{R}(t)=(n,c)\}=n\mu h+o(h)$: An unescorted R leaves $\mathcal{D}$; B following an R
$P\{\boldsymbol{R}(t+h)=(n+1,c)\,|\,\boldsymbol{R}(t)=(n,c)\}=\lambda_r h+o(h)$: New R arrival; B following an R

The process $\{\boldsymbol{R}(t);t\geq 0\}$ is an example of a level-dependent quasi-birth and -death process; *cf.* (Bean et al. (2000), Bright and Taylor (1995), and Gaver et al.(1984)

3.3. Limiting Distributions

Let $\pi(n,i)=\lim_{t\to\infty}P\{\boldsymbol{R}(t)=(n,i)\}$.

The balance equations are

$$\begin{aligned}
&\left[\lambda_r+\delta_w\right]\pi(0,s)=\mu\pi(1,s)+\phi\pi(0,f)+\phi\pi(0,c)\\
&\left[\lambda_r+\phi\right]\pi(0,f)=\mu\pi(1,f)+\delta_w\pi(0,s)\\
&\left[\lambda_r+\phi\right]\pi(0,c)=\mu\pi(1,c)+\delta_0\pi(1,s)
\end{aligned} \quad . \tag{3.3.1}$$

For n>0

$$\begin{aligned}
&\left[\lambda_r+\delta_w+n\left[\mu+\delta_0\right]\right]\pi(n,s)=(n+1)\mu\pi(n+1,s)+\phi\pi(n,f)+\phi\pi(n,c)+\lambda_r\pi(n-1,s)\\
&\left[\lambda_r+\phi+n\mu\right]\pi(n,f)=(n+1)\mu\pi(n+1,f)+\delta_w\pi(n,s)+\lambda_r\pi(n-1,f)\\
&\left[\lambda_r+\phi+n\mu\right]\pi(n,c)=(n+1)\mu\pi(n+1,c)+\delta_0\pi(n+1,s)+\lambda_r\pi(n-1,c)
\end{aligned} \quad . \tag{3.3.2}$$

A recursive procedure to compute the limiting distribution for the model appears in Appendix B.

The long run fraction of Rs that exit the area without being escorted by DDs (probability of R leakage) is

$$P\{Leak\}=\frac{\mu}{\lambda_r}\sum_{n=0}^{\infty}n\left[\pi(n,c)+\pi(n,s)+\pi(n,f)\right]. \tag{3.3.3}$$

The long run fraction of time the OH follows a misclassified W is

$$\pi(\text{follow misclassfied W})=\sum_{n=0}^{\infty}\pi(n,f). \tag{3.3.4}$$

4. Numerical Examples Comparing the Results Using the Deterministic and the Markovian Stochastic Models

Consider a rectangular domain, $\mathcal{D}$, with width in the x-direction $M_x = 100$ NM and length in the y-direction M_y NM for various M_y. Assume the OH sensor footprint is a square with sides $f = 25$ NM. Assume vessels travel through the domain in the y-direction. Assume the parameters in Table 4.1.

Velocity of White vessel	v_w	15 kts
Velocity of Red vessel	v_r	15 kts
Velocity of OH sensor	v_s	250 kts
Velocity of DD (pair)	v_d	30 kts.
Arrival rate of Rs	λ_r	variable
Arrival rate of Ws	λ_w	variable
y-direction length of rectangle	M_y	variable
x-direction length of rectangle	M_x	200 NM
Side of square of OH sensor footprint	f	25 NM
Mean time to classify detected vessel	τ	2/60 hrs
Number of OH sensors	$\overline{S}$	1
Prob. correctly classifying an R	c_{rr}	variable
Prob. correctly classifying a W	c_{ww}	variable
Prob. a vessel in the footprint of the OH sensor is detected	p_d	1

Table 4.1 Parameter Values for Examples of Section 4.2

4.1. A Model for the Detection Rate of Vessels

The total time for the $\mathcal{S}$ sensor to cover a one footprint square is f / v_s. The mean time for the $\mathcal{S}$ to cover the domain is $\left[M_x \times M_y / f^2\right]\frac{f}{v_s}$. The mean time a W is in the domain is $\frac{1}{\mu_w} = M_y / v_w$; the mean time a Red vessel is in the domain is $\frac{1}{\mu_r} = M_y / v_r$. If no vessels are ever classified as Red, then the limiting mean number of Ws (respectively

Rs) in the domain is λ_w / μ_w (respectively λ_r / μ_r). We assume the constant detection rate per vessel is

$$\delta(\tau) = p_d \left[\underbrace{\frac{M_x M_y}{f v_s}}_{\substack{\text{mean time}\\ \text{for OH}\\ \text{to cover}\\ \text{domain}}} + \underbrace{\tau \left[\frac{\lambda_w}{\mu_w} + \frac{\lambda_r}{\mu_r} \right]}_{\substack{\text{mean time}\\ \text{to class.}\\ \text{vessels}\\ \text{in}\\ \text{domain}}} \right]^{-1}, \tag{4.1.1}$$

which decreases as τ increases, as it should. The mean time until a searching $\mathcal{S}$ detects and misclassifies a W is $1/\delta_w$, where $\delta_w = \dfrac{\lambda_w}{\mu_w}\delta(\tau)c_{wr}$. The mean time until a searching $\mathcal{S}$ detects and correctly classifies an R is $1/\delta_0$, where $\delta_0 = \delta(\tau)c_{rr}$. The time the $\mathcal{S}$ follows a vessel classified as R has an exponential distribution with mean $1/\phi = \dfrac{M_y/2}{v_d}$.

4.2. Numerical Examples

Figures 4.1 and 4.2 display the long run fraction of Rs that leak through defenses, (3.3), as a function of the arrival rate of Rs for a fixed number of Ws equal to 0, 100, and 1,000. The velocity of the DD pair that relieves the following $\mathcal{S}$ is 30 kts. In Figure 4.1, the probability of correct classification $c_{ww} = c_{rr} = 0.99$; in Figure 4.2 it is 0.90.

Multi-Dimensional Markov Chain Model (Section 3)
Fraction of Rs that Leak Through Defenses
Rectangular Domain: x-direction=100 NM; y-direction=200 NM
cww=0.99; crr=0.99

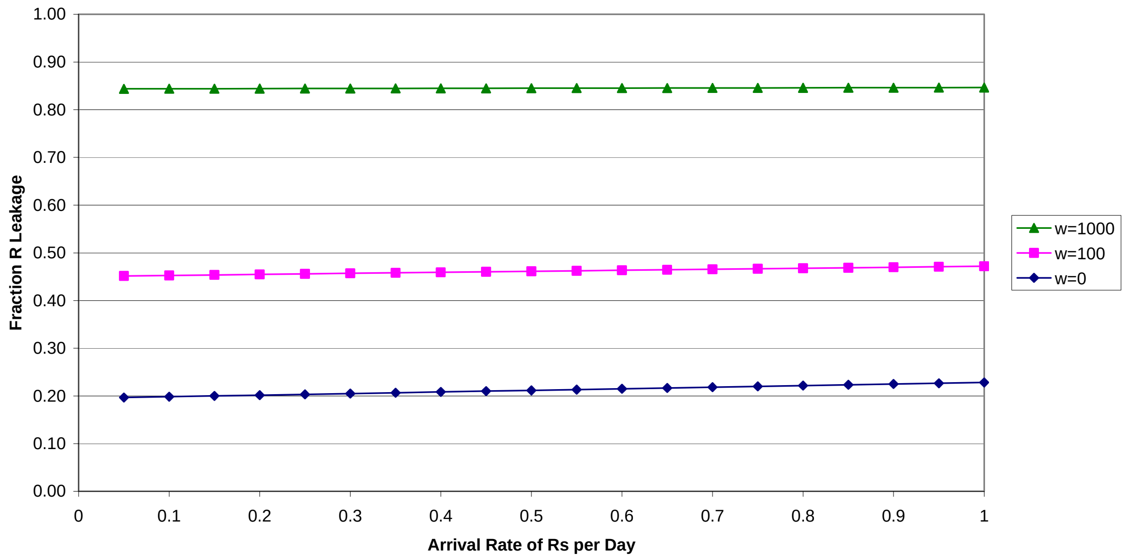


Figure 4.1

Multi-Dimensional Markov Model (Section 3)
Fraction of Rs that Leak Through Defenses
Rectangular Domain: x-direction=100 NM; y-direction=200 NM
cww=0.90; crr=0.90

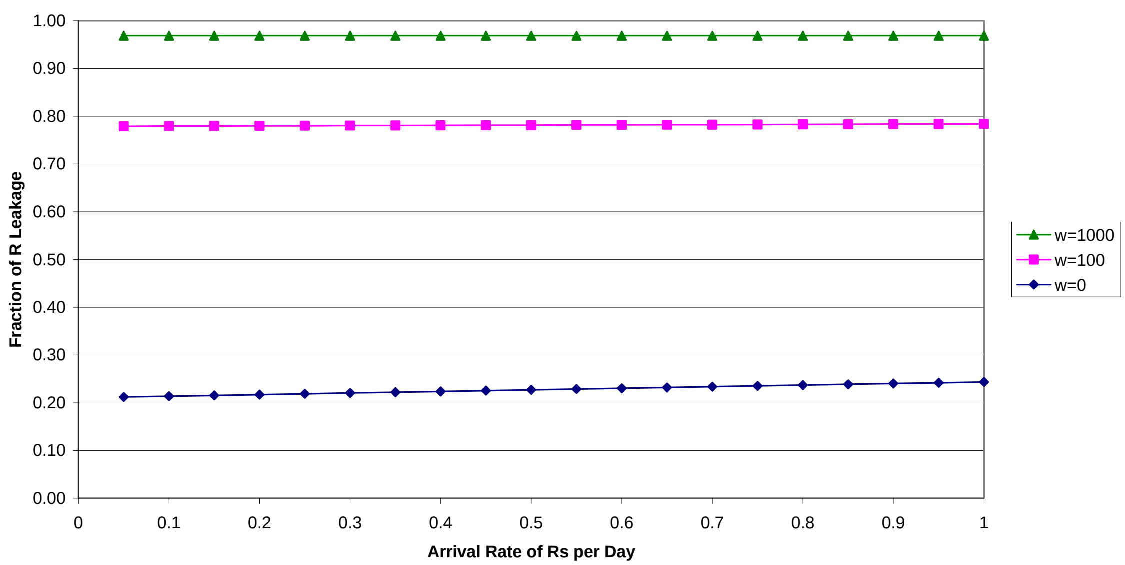


Figure 4.2

A deterministic model for this scenario was introduced in Section 1. The model is a system of differential equations. Figures 4.1d and 4.2d display the normalized R leakage rate (the long run average leakage rate divided by the R arrival rate) from that model.

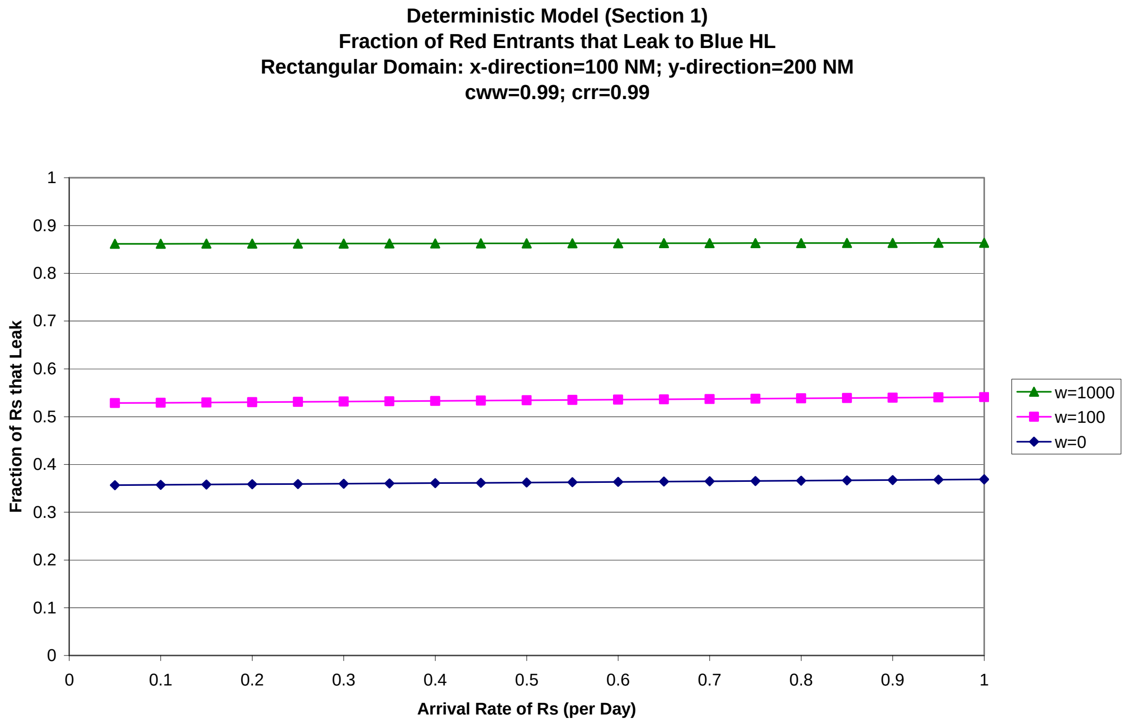


Figure 4.1d

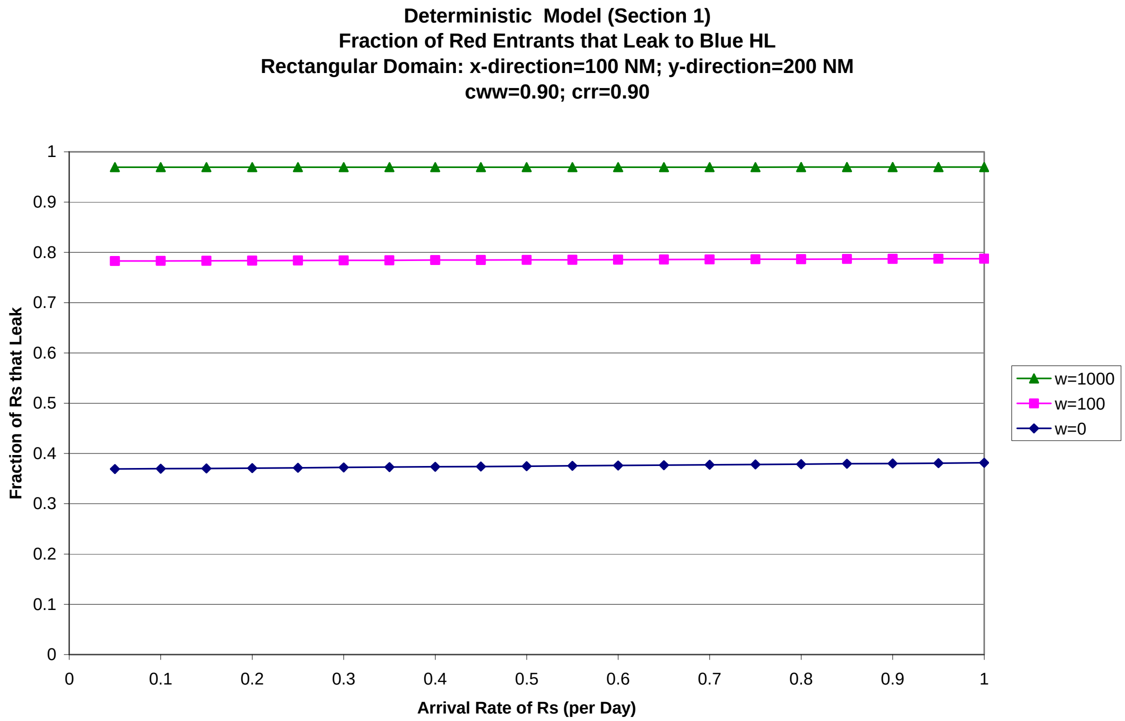


Figure 4.2d

Discussion: As the arrival rate increases the fraction of Rs leaking through the defenses increases. Comparison of Figures 4.1 and 4.2 suggests that the ability to correctly classify vessels becomes more important as the number of Ws increases. Comparison of Figures 4.1 and 4.1d and Figures 4.2 and 4.2d suggest that the two models provide the same qualitative results. The deterministic model is more pessimistic than the stochastic model. For the parameter values considered, the stochastic model represents the possibility of having 0 Rs in the domain when the arrival rate of Rs is positive; during this time there can be no leakage. For the parameters considered, the deterministic model is unable to represent this possibility and so there is always R leakage; the pessimistic behavior is the result. The number of Ws in the domain greatly influences the leakage rate.

Figure 4.3 displays the long run fraction of Rs that leak through defenses as the probability of correct classification increases. The velocity of the relief platform for the following OH is 30 kts.

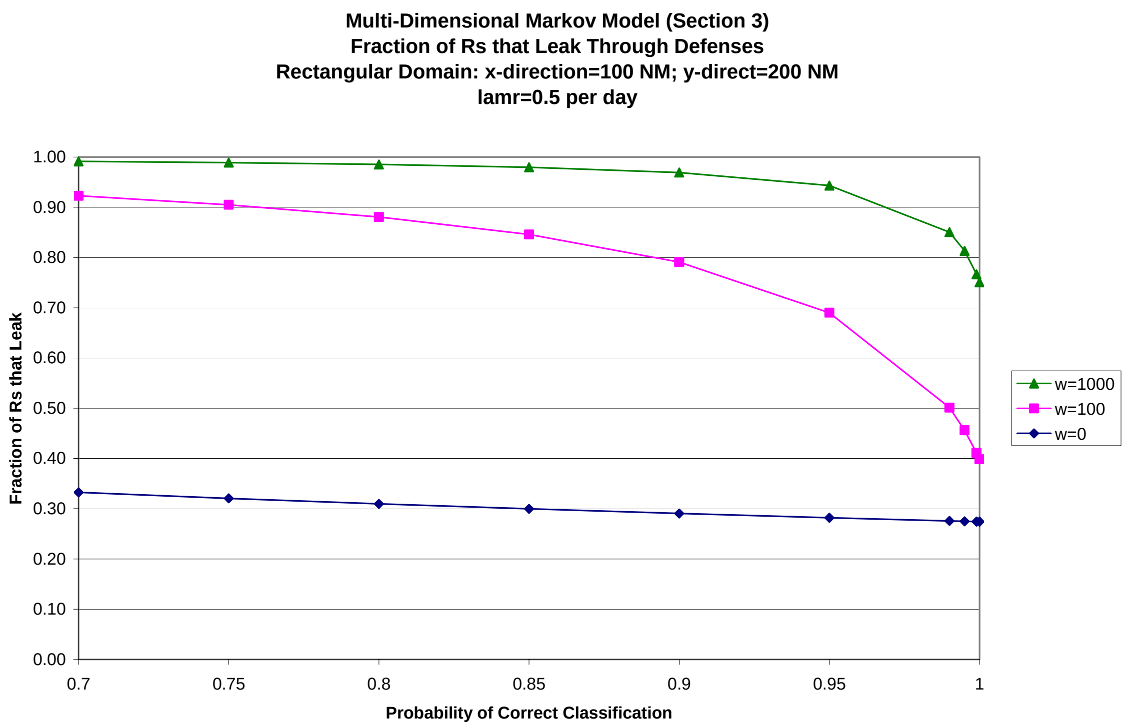


Figure 4.3

Figure 4.3d presents the normalized leakage rate resulting from the deterministic model of Section 1.

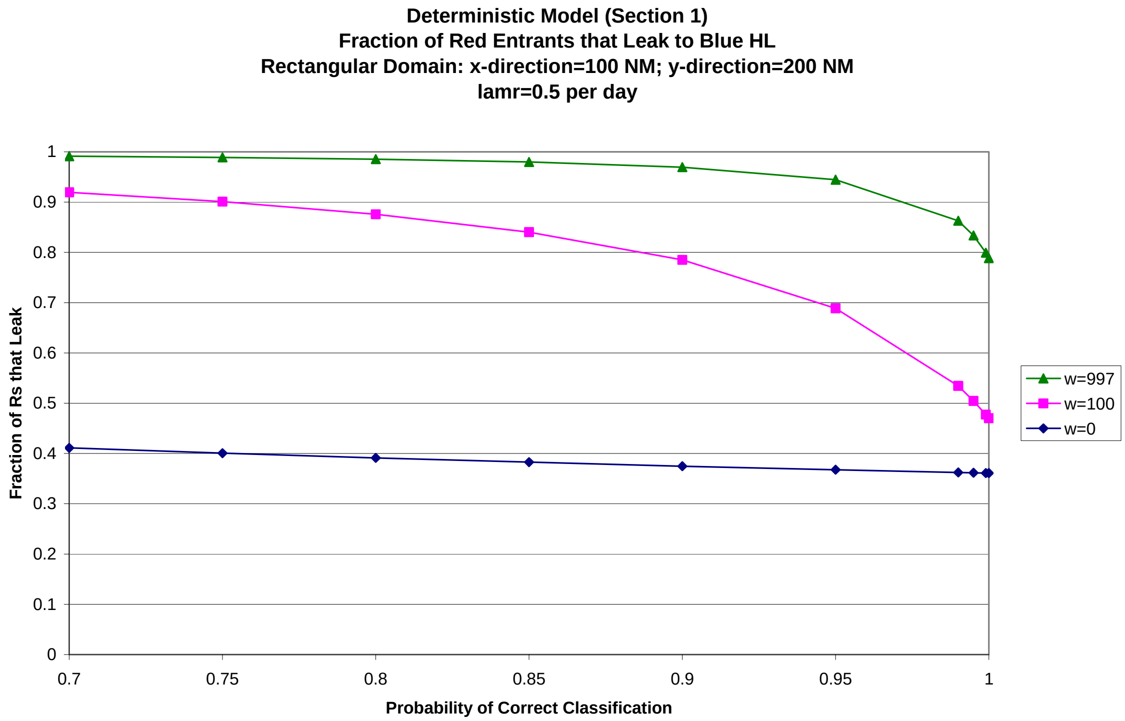


Figure 4.3d

In Section 2, a stochastic model is introduced in which there are a constant number of Ws in the domain and one R enters the domain at time 0. R leaks through the defenses if it transits through the domain before it is detected, correctly classified, and a platform arrives to escort it. This model differs from that model of Section 3.2 in that there is only one R in the domain. Figure 4.3s displays the probability that one Red leaks through the defenses as a function of the probability that a vessel is correctly classified.

One Red Enters the Region at Time 0 (Section 2)
Probability the One Red Leaks Through Defenses
Rectangular Domain: x-direction=100 NM; y-direction=200 NM

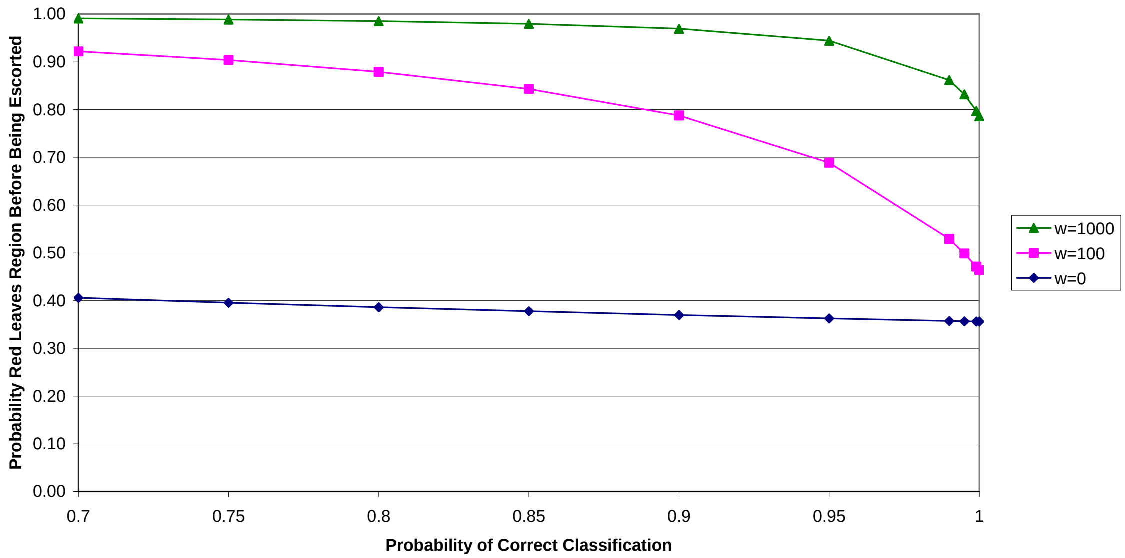


Figure 4.3s

Discussion: If there are 0 Ws, then increasing the probability of correct classification does not have much impact. Recall that each classification takes a mean time of 2 minutes. Thus, even with perfect classification, the presence of Ws increases the fraction of Rs that leak through the defenses. The decreased number of Ws from 1,000 to 100 results in a larger increase in the probability that Red can be neutralized, rather than increasing the probability of correct classification to 0.95. Decreasing the number of W to 0 results in a larger probability of correct classification than increasing the probability of correct classification to 1 when the number of Ws=100. Comparison of Figures 4.3, 4.3d, and 4.3s indicates that the three models give very similar quantitative results. Once again, the deterministic model is more pessimistic than the model of Section 3.2.

Figure 4.4 displays the long run fraction of Rs that leak through the defenses as a function of the velocity of the platform that relieves a following $\mathcal{S}$.

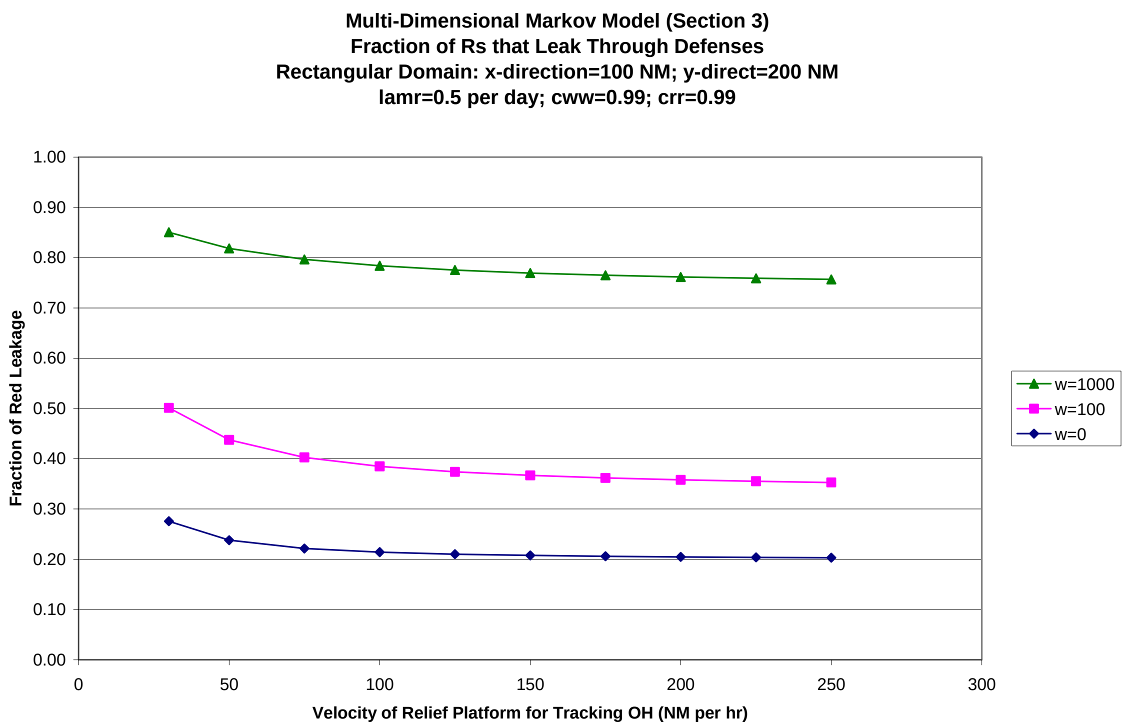


Figure 4.4

Figure 4.4d displays the normalized R leakage rates resulting from the deterministic model of Section 1. Figure 4.4s displays the probability the one R leaks through the defenses for the stochastic model, in which one R enters the domain at time 0 and there are a constant number of Ws in the domain from Section 2.

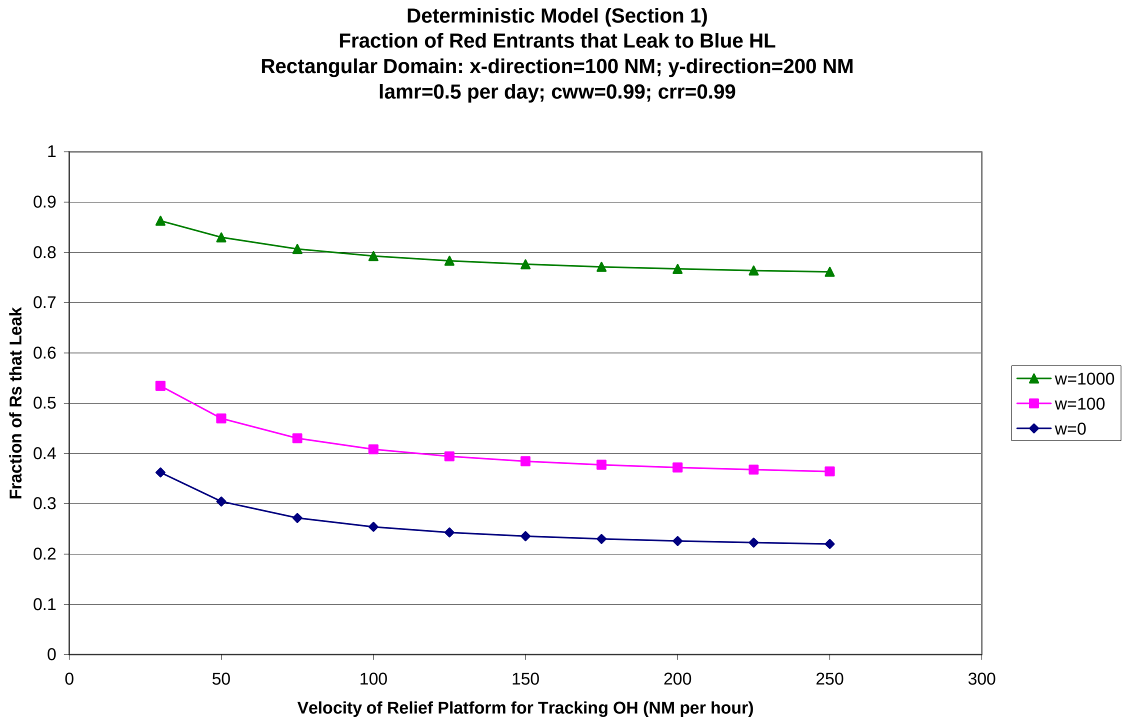


Figure 4.4d

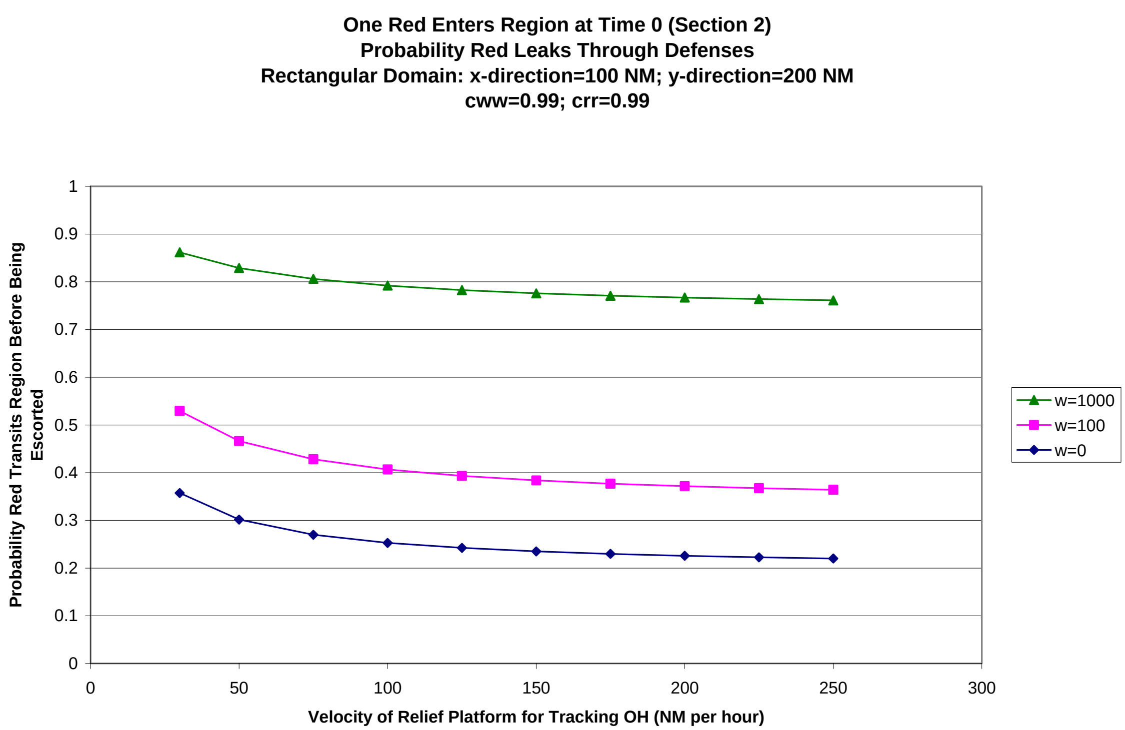


Figure 4.4s

Discussion: The baseline relief platform for a following $\mathcal{S}$ is a pair of DDs. The assumed velocity of the DD is 30 kts. The use of another air platform to relieve the $\mathcal{S}$ so that the $\mathcal{S}$ can continue to search will decrease the fraction of Rs that leak. The effect is greater the larger the number of Ws. Comparison of Figures 4.4, 4.4d, and 4.4s suggests that the three models result in very similar qualitative behavior. The deterministic model has larger leakage rates than the model of Section 3.2. Thus, while the qualitative results are similar, the quantitative results are different. Decreasing the number of Ws (the false positive rate) provides more operational benefit than decreasing the time until a sensor following a suspicious vessel is relieved.

4.3. The Probability a Vessel is Classified Correctly is a Function of the Time, τ, that the Sensor $\mathcal{S}$ Spends Classifying the Vessel

Assume that the probability of correct classification of a vessel of type j is a function of the time spent classifying the vessel. We use the logistic functional form of Section 1.5.2 and the detection rate (4.1.1).

Table 4.2 displays parameter values used in the following numerical examples. The arrival rate of W is chosen so that λ_w / μ_w is equal to the specified number of Ws in the domain.

Number of Ws	w	variable
Velocity of White vessel	v_w	15 kts
Velocity of Red vessel	v_r	15 kts
Velocity of OH sensor	v_s	250 kts
Velocity of DD (pair)	v_d	30 kts
Arrival rate of Rs	λ_r	1 per day
Arrival rate of Ws	λ_w	variable
y-direction length of rectangle	M_y	200 NM
x-direction length of rectangle	M_x	100 NM
Side of square of OH sensor footprint	f	25 NM
Mean time to classify detected vessel	τ	variable
Number of OH sensors	$\bar{S}$	1
Prob. correctly classifying an R	c_{rr}	variable
Prob. correctly classifying a W	c_{ww}	variable
Prob. a vessel in the footprint of the OH sensor is detected	p_d	1

Table 4.2 Parameter Values for Examples in Section 4.3

Figure 4.5 displays the probabilities of correct classification as a function of time for the various values of α and the specified probability of correct classification at time 5/60 hours. Figures 4.5a (respectively 4.5b) display the long run fraction of Rs that leak for $\alpha = 2$ and a specified probability of correctly classifying a W equal to 0.99 (respectively 0.999) for the model of Section 3. Figures 4.6a (respectively 4.6b) display the long run fraction of Rs that leak for $\alpha = 3$ and a specified probability of correctly classifying a W equal to 0.99 (respectively 0.999) for the model of Section 3. All of the figures have a specified probability of correctly classifying an R equal to 0.7.

Probability of Correct Classification as a Function of Time Spent

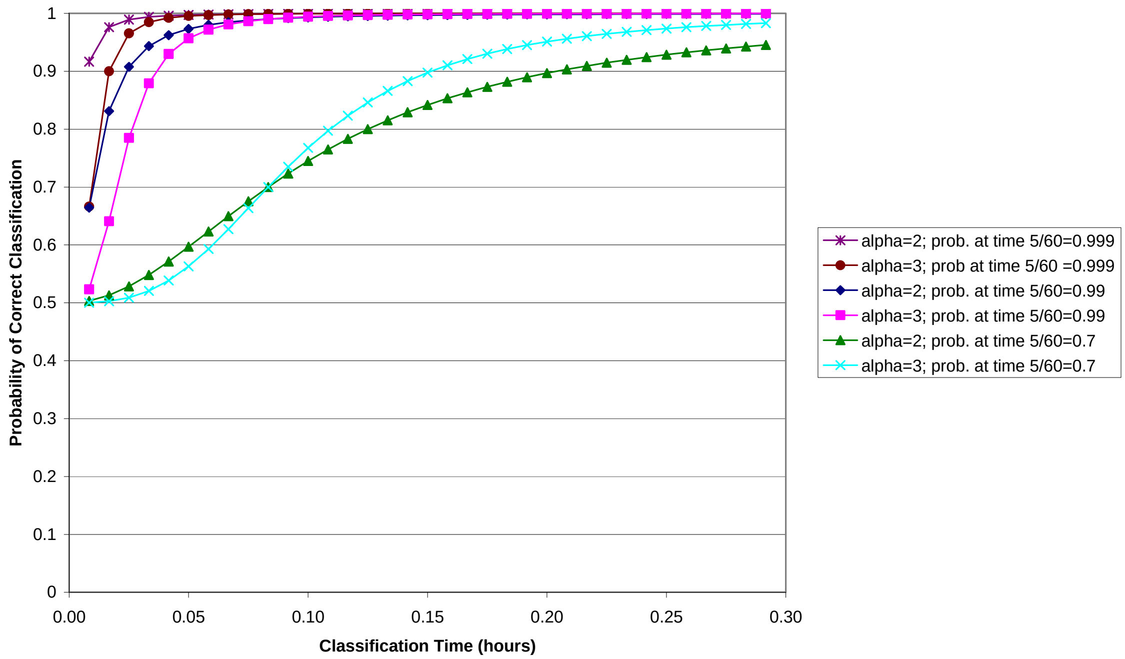


Figure 4.5

Multi-Dimensional Markov Model (Section 3)
Long Run Average Fraction of Rs Leaking Through Defenses
alpha=2; specified cww=0.99 & specified crr=0.7 at class. time 5/60 hours

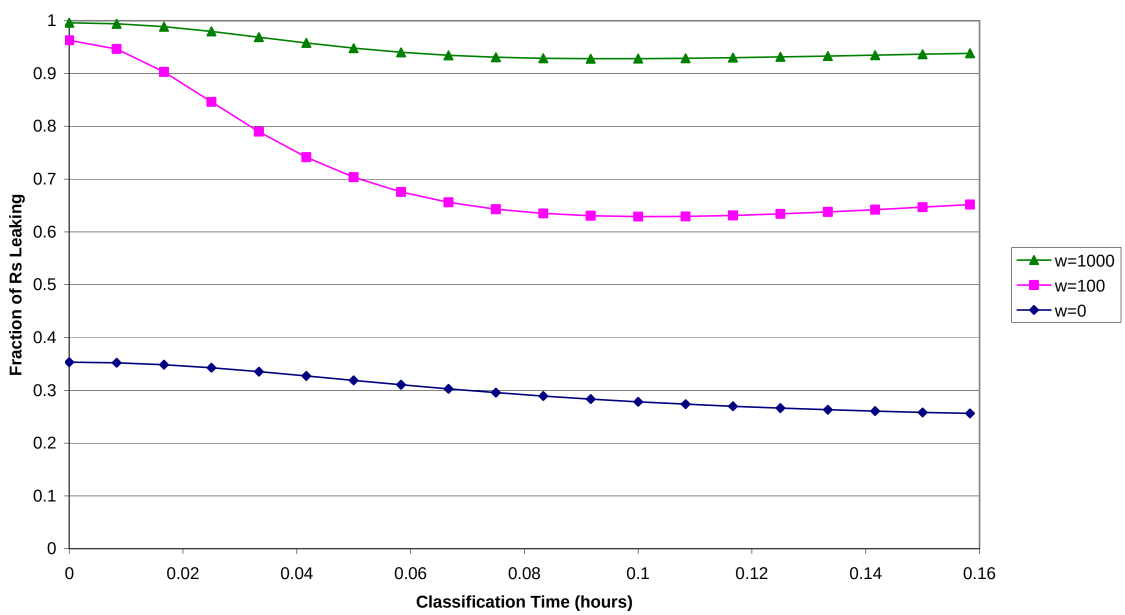


Figure 4.5a

Multi-Dimensional Markov Model (Section 3)
Long Run Average Fraction of Rs Leaking Through Defenses
alpha=2; specified cww=0.999 & specified crr=0.7 at class. time 5/60 hours

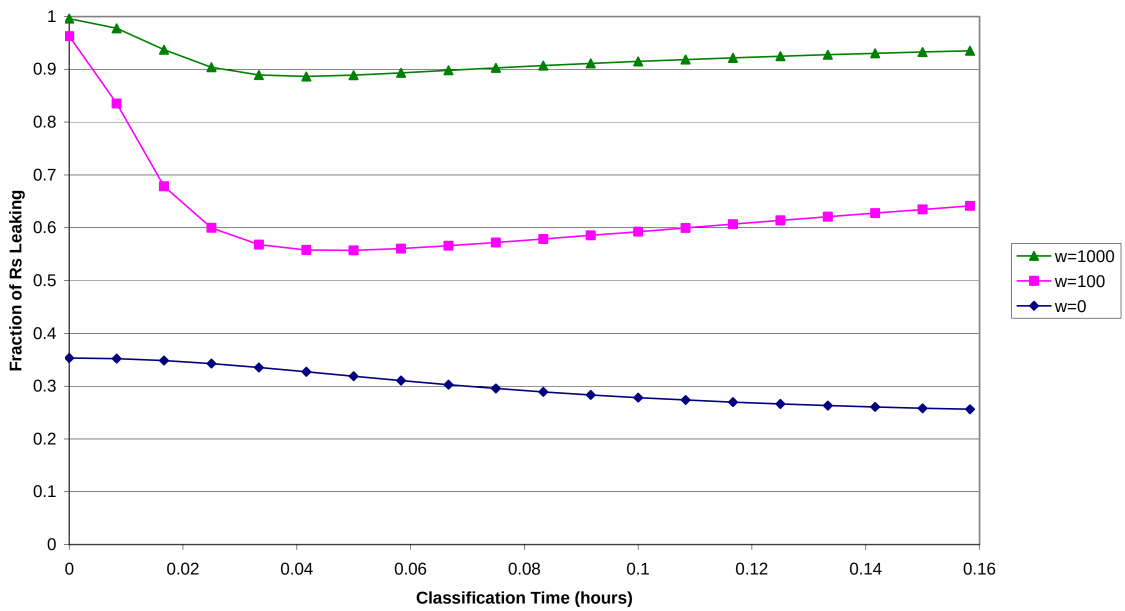


Figure 4.5b

Multi-Dimensional Markov Model (Section 3)
Long Run Average Fraction of Rs Leaking Through Defenses
alpha=3; specified cww=0.99 & specified crr=0.7 at class. time 5/60 hours

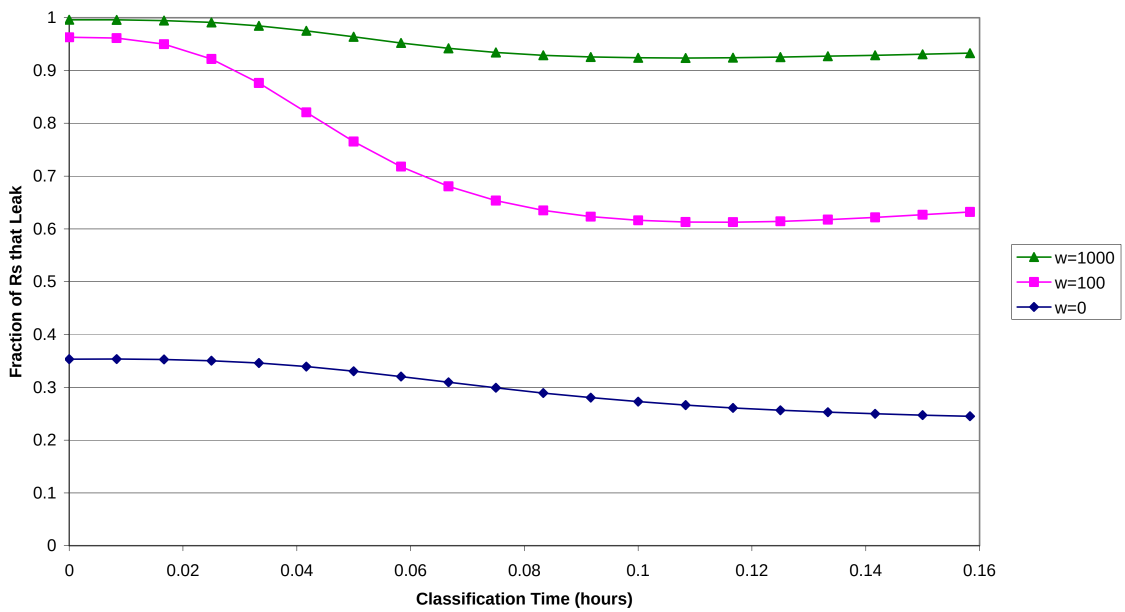
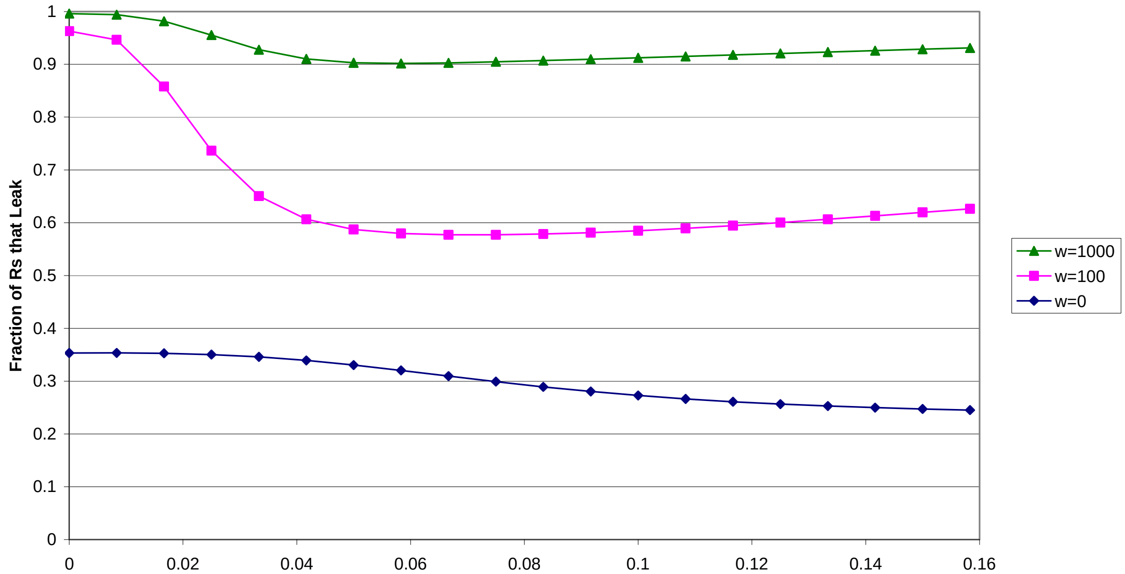


Figure 4.6a

Multi-Dimensional Markov Model (Section 3)
Long Run Average Fraction of Rs Leaking Through Defenses
alpha=3; specified cww=0.999 & specified crr=0.7 at class. time 5/60 hours



Classification Time (hours)
Figure 4.6b

Figures 4.5ad, 4.5bd, 4.6ad, and 4.6bd display the long fraction of Rs that leak resulting from the deterministic model of Section 1.

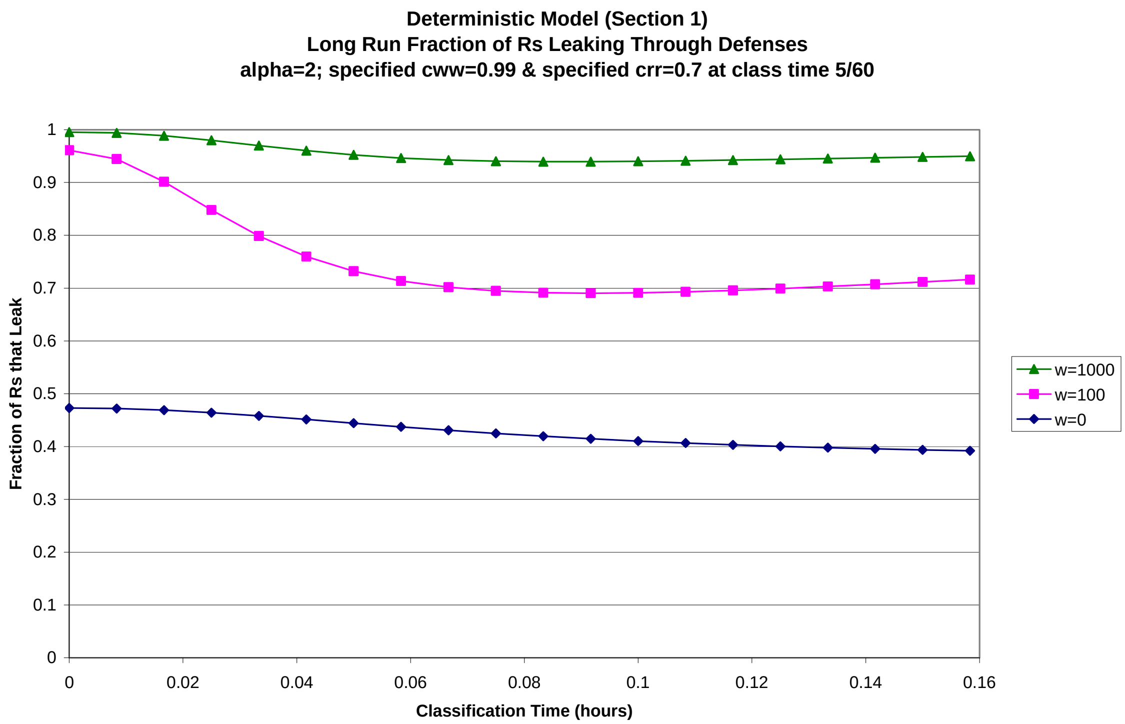


Figure 4.5ad

Deterministic Model (Section 1)
Long Run Fraction of Rs Leaking Through Defenses
alpha=2; specified cww=0.999 & specified crr=0.7 at class time 5/60

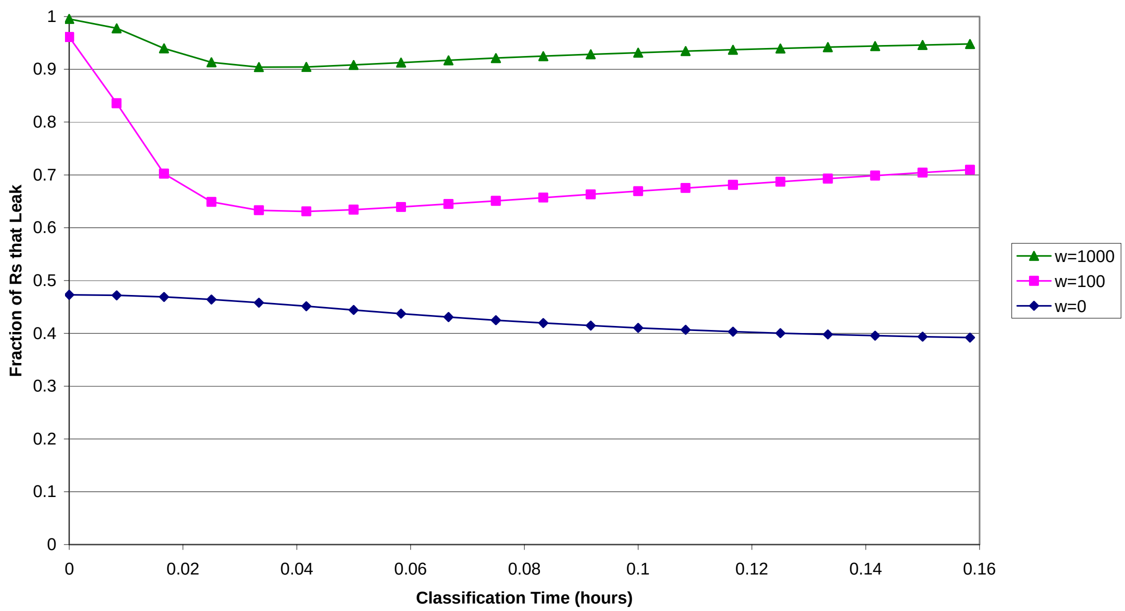


Figure 4.5bd

Deterministic Model (Section 1)
Long Run Fraction of Rs Leaking Through Defenses
alpha=3; specified cww=0.99 & specified crr=0.7 at class time 5/60

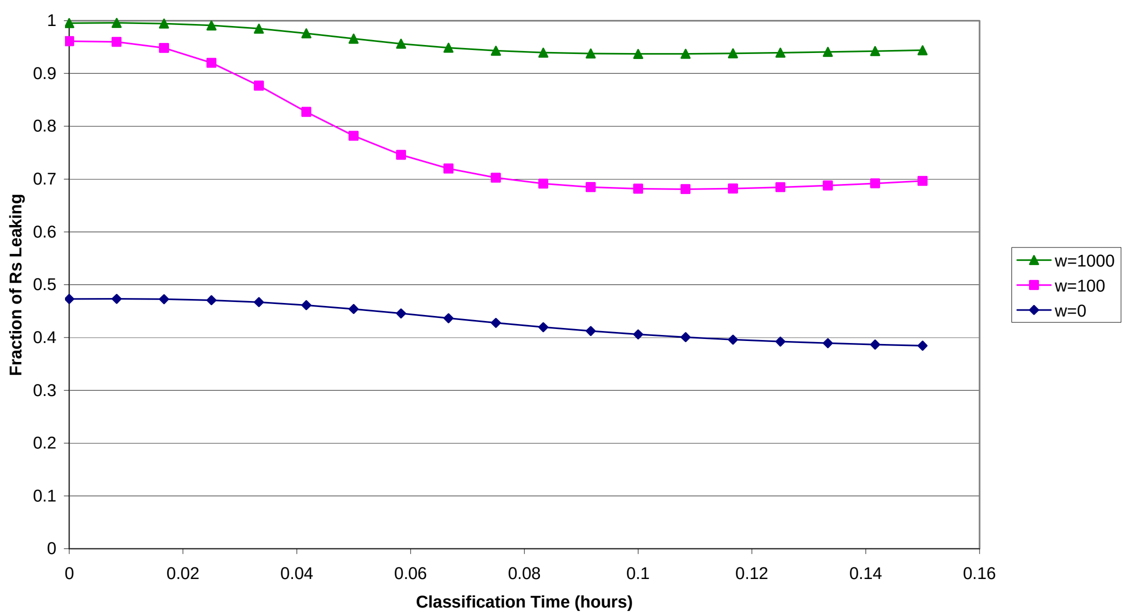


Figure 4.6ad

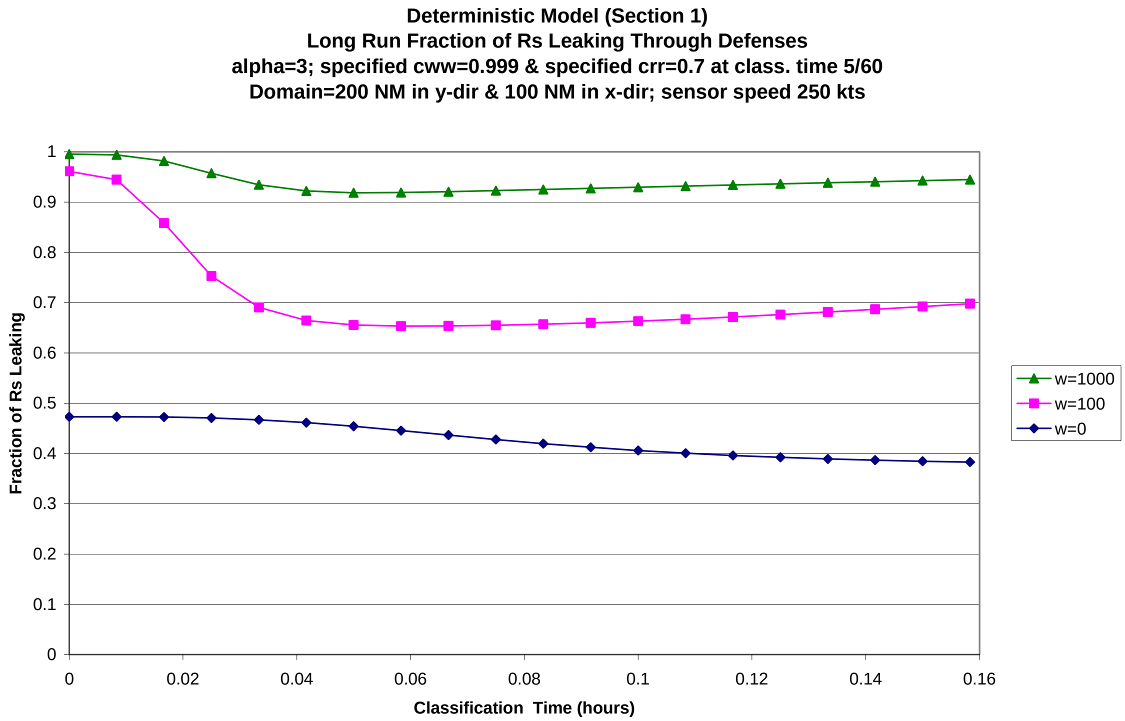


Figure 4.6bd

Discussion: For the cases of the number of Ws, *w*, explored, the minimizing classification time increases as the specified correct classification probability at 5/60 hours decreases. The minimizing classification time increases as α increases. The minimizing classification time increases as the number of Ws decreases. A comparison of the results from the stochastic model and the deterministic model suggests that both models give similar qualitative results. The resulting R leakage rate is larger for the deterministic model than that for the stochastic model. The classification time that minimizes the leakage rate for the deterministic model tends to be smaller than that for the stochastic model. Once again, the decreasing the false positive rate (number of Ws) results in larger probabilities of neutralizing Rs than increasing the probability of correct classification.

5. Non-Markovian Models

A Maritime domain or region contains a number *w* of nonhostile W vessels. At time 0, a hostile R vessel enters the domain. An overhead friendly (Blue) sensor ($\mathcal{S}$) patrols the domain and classifies (perhaps incorrectly) detected vessels as R or W. The $\mathcal{S}$ follows (or tracks) any vessel it classifies as R until it is relieved by a DD. The $\mathcal{S}$ is here assumed unable to detect and classify additional vessels while it is following a vessel. We assume an unescorted R will remain in the domain for a positive random time having a distribution function *F*; the sensor's following times of vessels classified as R are independent and identically distributed nonnegative random variables; and the distribution of the time until the patrolling sensor detects a vessel is exponential.

In this section, we present two approximations to the probability that R is detected and correctly classified before leaving the domain and the probability that R is detected, correctly classified, and escorted before leaving the domain. The first approximation uses a terminating renewal process argument. The second approximation uses an alternating renewal process approximation. The usefulness of the approximations is assessed by comparing the results of the approximations to more detailed simulation models. The approximations appear to be good and are used to study the sensitivity of the probability of neutralizing the R on model parameters.

We assume the domain is a rectangle. The parameters of the model appear in Table 5.1.

Velocity of Red vessel	v_r	15 kts
Velocity of OH sensor $\mathcal{S}$	v_s	250 kts
Velocity of DD (pair)	v_d	30 kts
Number of White vessels	w	variable
y-direction length of rectangle	M_y	200 NM
x-direction length of rectangle	M_x	200 NM
Side of square of OH sensor footprint	f	25 NM
Mean time to classify detected vessel	τ	2/60 hrs
Prob. correctly classifying an R	c_{rr}	variable
Prob. correctly classifying a W	c_{ww}	variable
Prob. a vessel in the footprint of the OH sensor is detected	p_d	1

Table 5.1 Parameters for Section 5

5.1. A Terminating Renewal Process Approximation

The detection rate of vessels in the entire domain is assumed to be

$$\delta_0(\tau) = p_d \left[\underbrace{\frac{M_x M_y}{f v_s}}_{\substack{\text{mean time}\\ \text{for OH}\\ \text{to cover}\\ \text{region}}} + \underbrace{\tau[w+1]}_{\substack{\text{mean time}\\ \text{to class.}\\ \text{vessels}\\ \text{in}\\ \text{region}}} \right]^{-1} . \qquad (5.1.1)$$

We assume the times between detections are independent and identically distributed, having an exponential distribution with mean $1/\delta_0(\tau)$. Let T_f be the first time a vessel is classified as R; the classified vessel could be White (false positive) or Red.

$$\begin{aligned} P\{T_f > t\} &= \sum_{n=0}^{\infty} e^{-\delta_0(\tau)} \frac{\delta_0(\tau)^n}{n!} \left[\frac{w}{w+1} c_{ww} + \frac{1}{w+1} c_{rw} \right]^n \\ &= \exp\left\{ -\delta_0(\tau) \left[1 - \left[\frac{w}{w+1} c_{ww} + \frac{1}{w+1} c_{rw} \right] \right] t \right\} \\ &= \exp\left\{ -\delta_0(\tau) \left[\frac{w}{w+1} c_{wr} + \frac{1}{w+1} c_{rr} \right] t \right\} \end{aligned} \qquad (5.1.2)$$

Let $\boldsymbol{T}_r$ be the time until R is correctly classified as R and $\boldsymbol{T}_w$ be the first time a W is classified as an R.

$$P\{\boldsymbol{T}_w < \boldsymbol{T}_r\} = \frac{\frac{w}{w+1}c_{wr}}{\frac{w}{w+1}c_{wr} + \frac{1}{w+1}c_{rr}} \equiv p \qquad (5.1.3)$$

In what follows, we assume $p > 0$.

When a vessel is classified as R, the sensor follows (tracks) the vessel until relieved by escort vessels (a pair of DDs). Assume the following time, $\boldsymbol{D}$, has a distribution with mean $1/\phi$ and Laplace transform $E\left[e^{-s\boldsymbol{D}}\right] = \zeta_{\boldsymbol{D}}(s)$, where the mean time the overhead sensor follows a vessel classified as R is

$$1/\phi = \left[\frac{M_y}{2}\right]\frac{1}{v_r + v_d}. \qquad (5.1.4)$$

5.1.1. An Approximation to the Time Until R is Detected and Correctly Classified

The probability that R is detected and correctly classified before time t satisfies the equation

$$P\{\boldsymbol{T}_r \le t\} = (1-p)P\{\boldsymbol{T}_f \le t\} + \int_0^t P\{\boldsymbol{T}_r \le t-s\}\, pP\{\boldsymbol{T}_f + \boldsymbol{D} \in ds\}. \qquad (5.1.5)$$

Thus, the probability that R is not detected and correctly classified before time t satisfies the equation

$$\begin{aligned} P\{\boldsymbol{T}_r > t\} &= 1-(1-p)P\{\boldsymbol{T}_f \le t\} - \int_0^t P\{\boldsymbol{T}_r \le t-s\}\, pP\{\boldsymbol{T}_f + \boldsymbol{D} \in ds\} \\ &= pP\{\boldsymbol{T}_f \le t\} + P\{\boldsymbol{T}_f > t\} - pP\{\boldsymbol{T}_f + \boldsymbol{D} \le t\} + \int_0^t P\{\boldsymbol{T}_r > t-s\}\, pP\{\boldsymbol{T}_f + \boldsymbol{D} \in ds\} \end{aligned}. \qquad (5.1.6)$$

Following Feller (1966) p. 322, assume there is a $\kappa_0 > 0$ that satisfies the equation

$$pE\left[e^{\kappa_0\left(\boldsymbol{T}_f+\boldsymbol{D}\right)}\right]=1. \tag{5.1.7}$$

Let

$$\mu^{\#}\left(\kappa_0\right)=pE\left[\left(\boldsymbol{T}_f+\boldsymbol{D}\right)e^{\kappa_0\left(\boldsymbol{T}_f+\boldsymbol{D}\right)}\right]. \tag{5.1.8}$$

$$\begin{aligned}
&e^{\kappa_0 t}P\left\{\boldsymbol{T}_r>t\right\}\\
&=e^{\kappa_0 t}\left[pP\left\{\boldsymbol{T}_f\le t\right\}+P\left\{\boldsymbol{T}_f>t\right\}-pP\left\{\boldsymbol{T}_f+\boldsymbol{D}\le t\right\}\right]\\
&+\int_0^t e^{\kappa_0(t-s)}P\left\{\boldsymbol{T}_r>t-s\right\}e^{\kappa_0 s}pP\left\{\boldsymbol{T}_f+\boldsymbol{D}\in ds\right\}.
\end{aligned} \tag{5.1.9}$$

The key renewal theorem implies

$$\lim_{t\to\infty}e^{\kappa_0 t}P\left\{T_r>t\right\}=\frac{c_R\left(\kappa_0\right)}{\mu^{\#}\left(\kappa_0\right)}, \tag{5.1.10}$$

where using (5.1.2) and (5.1.7)

$$\begin{aligned}
c_R\left(\kappa_0\right)&=\left[\frac{\delta_0(\tau)\left[\frac{w}{w+1}c_{wr}+\frac{1}{w+1}c_{rr}\right]}{\delta_0(\tau)\left[\frac{w}{w+1}c_{wr}+\frac{1}{w+1}c_{rr}\right]-\kappa_0}\right]\frac{1}{\kappa_0}\left[(1-p)+p\zeta_{\boldsymbol{D}}\left(-\kappa_0\right)\right]-\frac{1}{\kappa_0}\\
&=(1-p)\frac{1}{\kappa_0}\left[\frac{\delta_0(\tau)\left[\frac{w}{w+1}c_{wr}+\frac{1}{w+1}c_{rr}\right]}{\delta_0(\tau)\left[\frac{w}{w+1}c_{wr}+\frac{1}{w+1}c_{rr}\right]-\kappa_0}\right]
\end{aligned}. \tag{5.1.11}$$

Thus,

$$P\left\{\boldsymbol{T}_r>t\right\}\sim\frac{c_R\left(\kappa_0\right)}{\mu^{\#}\left(\kappa_0\right)}e^{-\kappa_0 t}. \tag{5.1.12}$$

Assume an unescorted R is in the domain for a time having a distribution F with Laplace transform $\zeta_{\boldsymbol{R}}(s)$. An approximation to the probability that R is not detected and correctly classified while in the domain is

$$\frac{c_R(\kappa_0)}{\mu^{\#}(\kappa_0)}\zeta_{\boldsymbol{R}}(\kappa_0). \tag{5.1.13}$$

If F is gamma with scale ϖ and shape parameter β, then an approximation to the probability R is not detected and correctly classified while in the domain is

$$\frac{c_R(\kappa_0)}{\mu^{\#}(\kappa_0)}\left[\frac{\varpi}{\varpi+\kappa_0}\right]^{\beta}. \tag{5.1.14}$$

5.1.2. An Approximation to the Probability R is Detected, Correctly Classified, and Escorted Before It Leaves the Domain

Let $\boldsymbol{T}_e$ be the time until R is escorted.

$$P\{\boldsymbol{T}_e \le t\} = (1-p)P\left\{\boldsymbol{T}_f + \boldsymbol{D} \le t\right\} + \int_0^t P\{\boldsymbol{T}_e \le t-s\}\, pP\left\{\boldsymbol{T}_f + \boldsymbol{D} \in ds\right\}. \tag{5.1.15}$$

Hence,

$$\begin{aligned} P\{\boldsymbol{T}_e > t\} &= 1-(1-p)P\left\{\boldsymbol{T}_f + \boldsymbol{D} \le t\right\} - \int_0^t P\{\boldsymbol{T}_e \le t-s\}\, pP\left\{\boldsymbol{T}_f + \boldsymbol{D} \in ds\right\} \\ &= P\left\{\boldsymbol{T}_f + \boldsymbol{D} > t\right\} + \int_0^t \left[1-P\{\boldsymbol{T}_e \le t-s\}\right] pP\left\{\boldsymbol{T}_f + \boldsymbol{D} \in ds\right\} \\ &= P\left\{\boldsymbol{T}_f + \boldsymbol{D} > t\right\} + \int_0^t P\{\boldsymbol{T}_e > t-s\}\, pP\left\{\boldsymbol{T}_f + \boldsymbol{D} \in ds\right\}. \end{aligned} \tag{5.1.16}$$

Assume (5.1.7) holds. The key renewal theorem implies that

$$P\{\boldsymbol{T}_e > t\} \sim \frac{c_E(\kappa_0)}{\mu^{\#}(\kappa_0)} e^{-\kappa_0 t}, \tag{5.1.17}$$

where $\mu^{\#}(\kappa_0)$ is given by (5.1.8) and

$$c_E(\kappa_0) = \frac{1}{\kappa_0}\frac{1-p}{p}. \tag{5.1.18}$$

Assume an unescorted and unfollowed R is in the domain for a random time having a distribution F with Laplace transform $\zeta_{\boldsymbol{R}}(s)$. An approximation to the probability that R is not detected and correctly classified while in the domain is

$$\frac{c_E(\kappa_0)}{\mu^{\#}(\kappa_0)}\zeta_{\boldsymbol{R}}(\kappa_0). \tag{5.1.19}$$

If F is a gamma distribution with scale ϖ and shape parameter β, then an approximation to the probability that R is not detected and correctly classified while in the domain is

$$\frac{c_E(\kappa_0)}{\mu^{\#}(\kappa_0)}\left[\frac{\varpi}{\varpi+\kappa_0}\right]^{\beta}. \tag{5.1.20}$$

5.2. An Alternating Renewal Process Approximation

Assume only Whites are in the domain. Let

$$\delta = p_d\left[\underbrace{\frac{M_x M_y}{f v_s}}_{\substack{\text{mean time}\\ \text{for OH}\\ \text{to cover}\\ \text{region}}}\right]^{-1}. \tag{5.2.1}$$

The sensor alternates between looking for suspicious vessels and classifying/following vessels. The expected time that the sensor is busy when it detects a vessel (busy period) is

$$E[\boldsymbol{B}] = \tau + E[\boldsymbol{D}]c_{wr}. \tag{5.2.2}$$

The expected time between busy periods is $1/\delta w$, where w is the number of White vessels.

The long run unavailability of the sensor is

$$\bar{\alpha} = \frac{E[\boldsymbol{B}]}{\frac{1}{\delta w} + E[\boldsymbol{B}]}. \tag{5.2.3}$$

The long run availability of the sensor is

$$\alpha = \frac{1/\delta w}{\frac{1}{\delta w} + E[\boldsymbol{B}]} = \frac{1}{1 + \delta w E[\boldsymbol{B}]}. \tag{5.2.4}$$

Assume R enters the domain $\mathcal{D}$ when the system is in steady state. Let

$$\delta^* = \delta\alpha. \tag{5.2.5}$$

Approximate the distribution of the time until R is detected and correctly classified with an exponential distribution with mean $\frac{1}{\delta^* c_{rr}(\tau)}$.

5.2.1. An Approximation to the Probability that R is Not Detected and Correctly Classified Before It Travels Through the Domain

Assume an unescorted and unfollowed R is in the domain for a random time having a distribution F with Laplace transform $\zeta_{\boldsymbol{R}}(s)$. An approximation to the probability that R is not detected and correctly classified while in the domain is

$$\zeta_{\boldsymbol{R}}\left(\delta^* c_{rr}(\tau)\right). \tag{5.2.6}$$

Suppose the time R travels through the domain has a gamma distribution with shape parameter β and mean m; the scale is $\varpi = \frac{\beta}{m}$. An approximation to the probability that R is not detected and correctly classified while traveling through the domain is

$$\left[\frac{\varpi}{\varpi + \delta^* c_{rr}(\tau)}\right]^{\beta}. \tag{5.2.7}$$

5.2.2. An Approximation to the Probability that Red is Not Detected, Correctly Classified and Escorted Before Traveling Through the Domain

To obtain an approximation to the probability that R is not escorted before it leaves the domain, assume the successive times the sensor follows a vessel classified as R are independent, having an exponential distribution with mean (5.1.4). The probability that R is not escorted before time *t* is approximately

$$\begin{aligned} P\{\boldsymbol{T}_e > t\} &\approx e^{-\delta^* c_{rr} t} + \int_0^t \delta^* c_{rr} e^{-\delta^* c_{rr} s} e^{-\phi(t-s)} ds \\ &= e^{-\delta^* c_{rr} t} + \frac{\delta^* c_{rr}}{\delta^* c_{rr} - \phi}\left[e^{-\phi t} - e^{-\delta^* c_{rr} t}\right]. \end{aligned} \tag{5.2.8}$$

Assume an unescorted and unfollowed R is in the domain for a random time having a distribution F with Laplace transform $\zeta_{\boldsymbol{R}}(s)$. An approximation to the probability that R is not detected, correctly classified, and escorted while in the domain is

$$\zeta_R\left(\delta^* c_{rr}\right) + \frac{\delta^* c_{rr}}{\delta^* c_{rr} - \phi}\left[\zeta_R(\phi) - \zeta_R\left(\delta^* c_{rr}\right)\right]. \tag{5.2.9}$$

Suppose the time R travels through the domain has a gamma distribution with shape parameter β and mean m; the scale is $\varpi = \frac{\beta}{m}$. An approximation to the probability that R is not escorted before it leaves the domain is

$$\left(\frac{\varpi}{\varpi + \delta^* c_{rr}}\right)^{\beta} + \frac{\delta^* c_{rr}}{\delta^* c_{rr} - \phi}\left[\left(\frac{\varpi}{\varpi + \phi}\right)^{\beta} - \left(\frac{\varpi}{\varpi + \delta^* c_{rr}}\right)^{\beta}\right]. \qquad (5.2.10)$$

5.3. Comparison of the Renewal Process Approximations to Results from More Detailed Simulations

5.3.1. Simulation 1

Sato (2005) presents the results of a simulation. In the simulation, the sensor following times and the time an unescorted R travels through the domain are independent random variables having gamma distributions with the same shape parameter. The mean of the sensor following time is given by (5.1.4). The mean of R's unescorted travel time through the domain is M_y / v_r. Table 5.2 displays the parameters of the simulation and the simulation results; f_C is the fraction of replications R is detected and correctly classified before it leaves the domain; f_E is the fraction of replications R is detected, correctly classified, and escorted before it leaves the domain. Each case has 10,000 replications.

Parameters for and Results from a Simulation with 10,000 Replications

M_x (NM)	w	v_r (kts)	τ (minutes)	c_{ww}	c_{rr}	Shape Parameter	f_C	f_E
81	22	11.3	5	0.96	0.96	50	0.9	0.87
81	13	11.3	4.8	0.94	0.86	25	0.9	0.86
144	34	10	5.8	0.99	0.85	42	0.82	0.78
50	9	22.5	7	0.91	0.89	36	0.82	0.74
50	34	22.5	4.8	0.99	0.85	11	0.77	0.68
159	6	15	4.3	0.92	0.88	19	0.74	0.67
97	31	16.3	8	0.96	0.83	39	0.64	0.57
144	41	10	6.8	0.91	0.91	8	0.63	0.56
66	13	27.5	6.3	0.9	0.86	33	0.63	0.54
128	19	26.3	4.3	0.98	0.98	50	0.61	0.53
284	22	12.5	5.3	0.9	0.96	39	0.56	0.5
97	50	16.3	7.5	0.94	0.99	33	0.55	0.49
238	6	18.8	8	0.97	0.88	47	0.55	0.48
175	25	20	6	0.95	0.9	28	0.52	0.45
175	25	20	6	0.95	0.9	28	0.52	0.44
222	31	13.8	7.8	0.93	0.83	5	0.49	0.42
113	44	21.3	4	0.93	0.93	8	0.48	0.41
300	41	17.5	5	0.99	0.91	19	0.46	0.4
300	16	17.5	7.3	0.91	0.95	44	0.46	0.4
253	19	23.8	4	0.94	0.98	16	0.45	0.36
206	9	30	5.3	0.99	0.89	47	0.44	0.36
191	44	25	7.8	0.98	0.93	36	0.42	0.34
128	50	26.3	5.5	0.93	0.99	30	0.4	0.32
206	16	30	6.3	0.91	0.95	13	0.39	0.31
238	47	18.8	4.5	0.93	0.8	42	0.35	0.29
191	47	25	5.5	0.92	0.8	44	0.31	0.25
269	38	28.8	7.3	0.96	0.94	30	0.3	0.24
269	28	28.8	7	0.94	0.84	5	0.29	0.23

Table 5.2 Parameters for and Results of the Simulation of Section 5.1

The other parameter values for the simulation are displayed in Table 5.3.

Other Parameter Values

Velocity of Red vessel	v_r	variable
Velocity of OH sensor $\mathcal{S}$	v_s	300 kts
Velocity of DD (pair)	v_d	30 kts
Number of White vessels	w	variable
y-direction length of rectangle	M_y	300 NM
x-direction length of rectangle	M_x	Variable
Side of square of OH sensor footprint	f	15 NM
Mean time to classify detected vessel	τ	variable
Number of OH sensors	$\bar{S}$	1
Prob. correctly classifying an R	c_{rr}	variable
Prob. correctly classifying a W	c_{ww}	variable
Prob. a vessel in the footprint of the OH sensor is detected	p_d	1

Table 5.3 Parameters for Simulation Environment of Section 5.1

Figure 5.1 displays the results of the two approximations for the probability R is detected and correctly classified before it leaves the domain; (5.1.14) and (5.2.7). The two approximations assume the sensor following times have an exponential distribution with the same mean as the simulation. Apparently the two approximations give very similar results.

Approximations to the Probability R is Detected and Correctly Classified Before Leaving the Domain

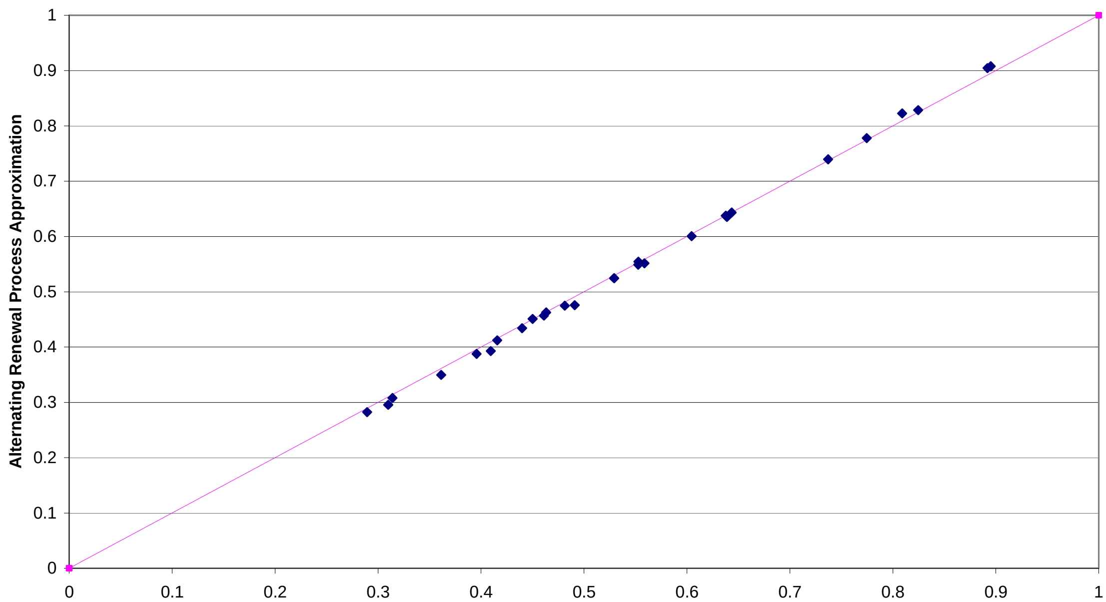


Figure 5.1

Figure 5.2 displays the results of the two approximations for the probability that R is detected, correctly classified, and escorted before it leaves the domain; (5.1.20) and (5.2.10). Once again, the two approximations are very similar. The alternating renewal process approximation (5.2.10) is easier to compute, and so is preferred.

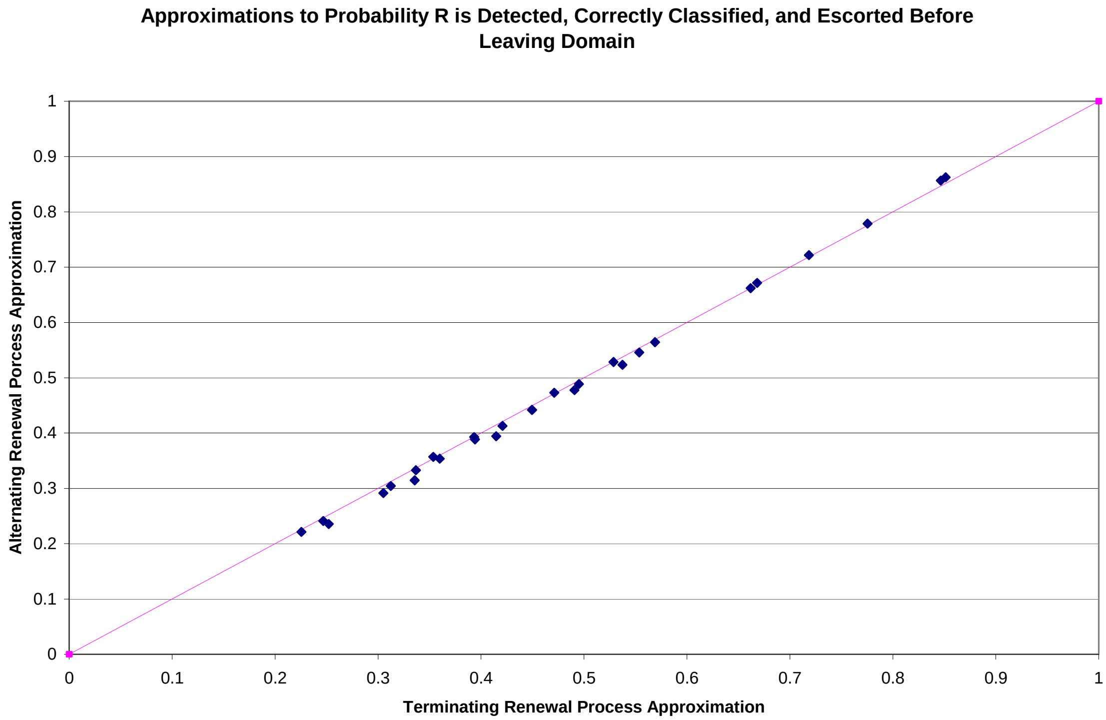


Figure 5.2

Figure 5.3 displays the fraction of simulation replications resulting in R being detected and correctly classified before leaving the domain and the alternating renewal process approximation (5.2.7). The expression (5.2.7) agrees very well with the simulation results.

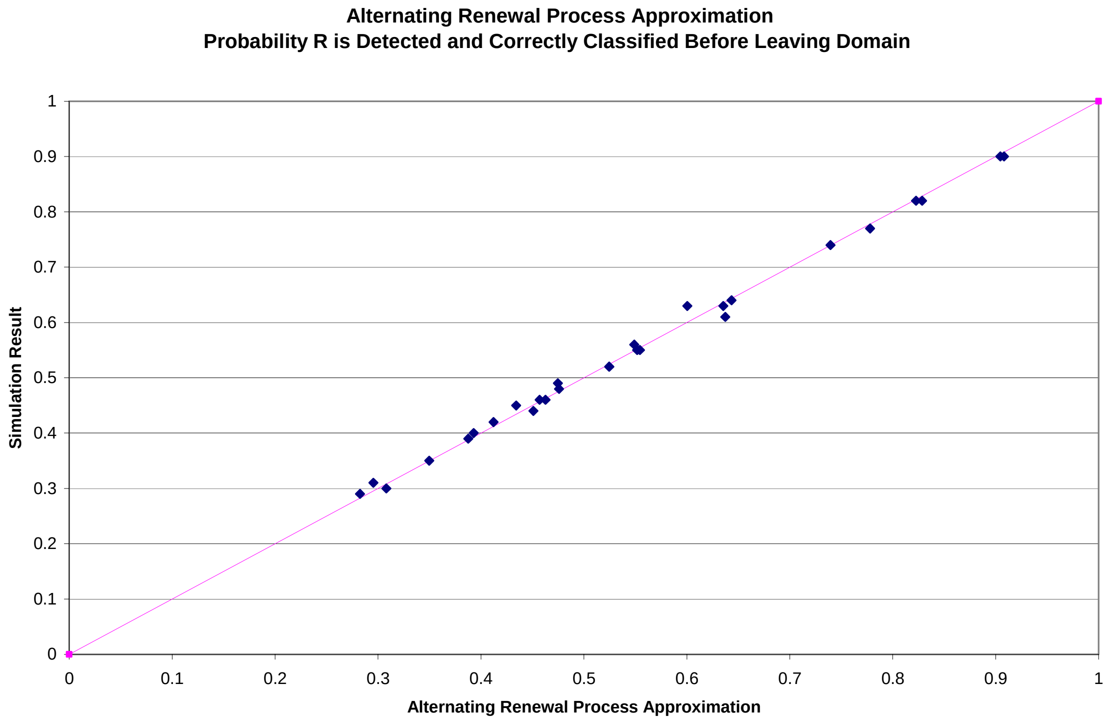


Figure 5.3

Figure 5.4 displays the fraction of simulation replications resulting in R being detected, correctly classified, and escorted before leaving the region and the alternating renewal process approximation (5.2.10). The expression (5.2.10) agrees very well with the simulation results. Note that approximations (5.2.7) and (5.2.10) assume that the sensor following times have an exponential distribution; in the simulation, the following times have a gamma distribution having the same mean. Apparently, the probabilities are relatively insensitive to the shape of the distribution of the following times. The figures corresponding to Figures 5.3 and 5.4 for the terminating renewal process approximation are similar.

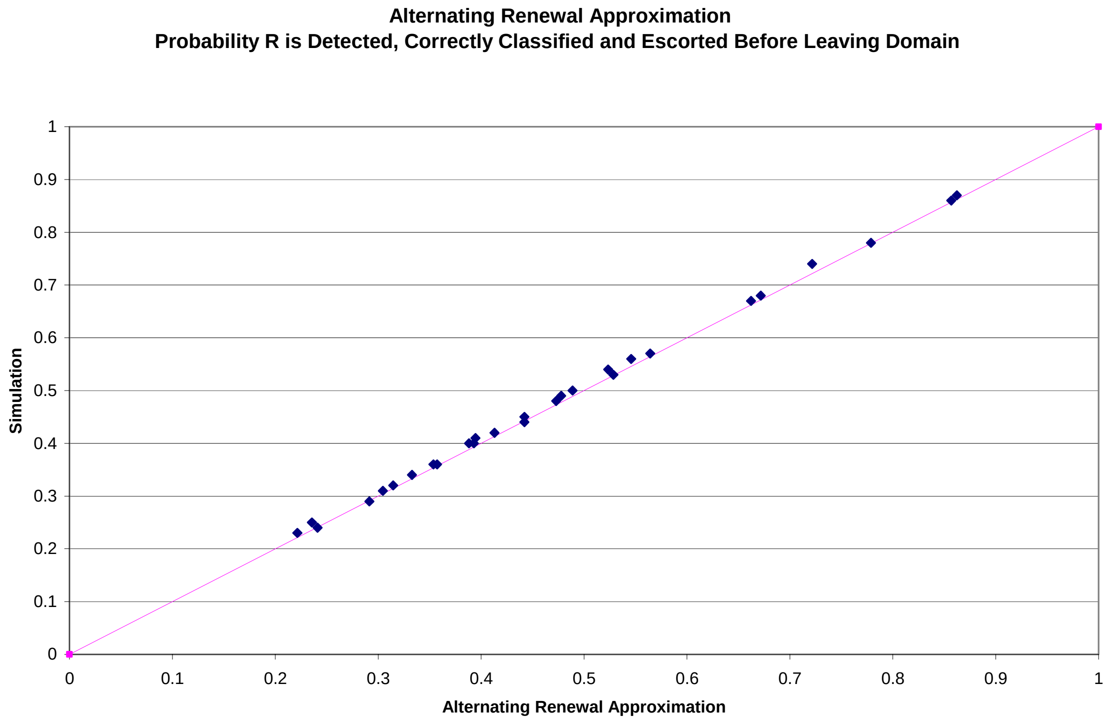


Figure 5.4

5.3.2. Simulation 2

Sato (2005) presents results from a simulation experiment. The simulation has the following assumptions. There are a constant number of White vessels in the domain, *w*. One Red vessel enters the domain at time 0. If unescorted, the time R is in the domain has a gamma distribution. The times until vessel detection have an exponential distribution. The sensor's following times of vessels classified as R are independent and identically distributed, having a gamma distribution. Each time a vessel is detected and classified as W, the sensor remembers that information for an independent random time having a gamma distribution; during this time, the vessel is not subject to detection. Tables 5.4a-b display the parameter values for the simulation. Each case has 10,000 replications.

x-dir (NM)	# Whites	Velocity of R	Time to classify detected vessel (min)	c_{ww}	c_{rr}	Shape param. R Transit Time	Shape param. sensor following times	Mean time to loss of vessel classify-cation infor-mation (hrs)	Shape param. of time to loss of vessel classifi-cation infor-mation
400	9	21.6	4.8	0.98	0.85	26	50	2.8	36
381	100	16.9	5.5	0.89	0.68	19	46	3	25
375	44	28.6	4.6	0.81	0.84	6	19	2.9	49
313	88	30	5.6	0.99	0.66	8	25	3.3	9
388	3	22	4.9	0.94	0.89	30	8	1.6	11
394	94	19.7	5.1	0.89	0.69	44	6	0.6	29
338	47	29.5	5	0.8	0.86	46	44	1.5	5
306	69	29.1	5.4	0.98	0.7	50	29	0.9	47
331	25	18.3	6.1	0.94	0.73	13	32	0	32
350	66	19.2	6.8	0.84	0.81	22	43	0.4	13
344	22	26.3	7.9	0.87	0.63	12	22	0.5	33
356	72	24.8	7.8	0.94	0.99	23	15	1.9	12
319	16	17.8	6.3	0.92	0.65	40	20	3.9	18
369	59	20.6	7.5	0.83	0.83	37	16	3.8	40
325	19	27.7	7.6	0.88	0.6	39	37	2.6	20
363	63	23.9	8	0.96	0.96	35	42	2.3	39
300	50	22.5	6	0.9	0.8	28	28	2	28
200	91	23.4	7.3	0.83	0.75	29	5	1.3	19
225	56	16.4	7.4	0.99	0.76	49	36	1.1	6
288	13	15	6.4	0.81	0.94	47	30	0.8	46
213	97	23	7.1	0.86	0.71	25	47	2.4	44
206	6	25.3	6.9	0.91	0.91	11	49	3.4	26
294	31	15.9	6.6	0.82	0.9	5	26	3.1	8
269	75	26.7	5.9	0.86	0.88	42	23	4	23
250	34	25.8	5.3	0.96	0.79	33	12	3.6	42
256	78	18.8	4.1	0.93	0.98	43	33	3.5	22
244	28	20.2	4.3	0.85	0.61	32	40	2.1	43
281	84	27.2	5.8	0.88	0.95	15	35	0.1	37
231	41	24.4	4.5	0.97	0.78	18	39	0.3	15
238	38	21.1	4	0.84	0.64	20	13	1.8	16

Table 5.4a Parameters for the Simulation of Section 5.3.2

x-dir (NM)	# Whites	Velocity of R	Time to classify detected vessel (min)	c_{ww}	c_{rr}	Shape param. R Transit Time	Shape param. sensor following times	Mean time to loss of vessel classification information (hrs)	Shape param. of time to loss of vessel classification information
250	2	17.2	4.4	0.99	0.96	26	50	2.8	36
245	20	15.6	4.8	0.97	0.92	19	46	3	25
244	9	19.5	4.3	0.95	0.96	6	19	2.9	49
247	1	17.3	4.4	0.98	0.97	30	8	1.6	11
248	19	16.6	4.6	0.97	0.92	44	6	0.6	29
234	9	19.8	4.5	0.95	0.97	46	44	1.5	5
233	5	16.1	5.1	0.99	0.93	13	32	0	32
238	13	16.4	5.4	0.96	0.95	22	43	0.4	13
236	4	18.8	5.9	0.97	0.91	12	22	0.5	33
230	3	15.9	5.1	0.98	0.91	40	20	3.9	18
242	12	16.9	5.8	0.96	0.96	37	16	3.8	40
231	4	19.2	5.8	0.97	0.9	39	37	2.6	20
241	13	18	6	0.99	0.99	35	42	2.3	39
225	10	17.5	5	0.98	0.95	28	28	2	28
200	18	17.8	5.6	0.96	0.94	29	5	1.3	19
222	3	15	5.2	0.95	0.98	47	30	0.8	46
203	19	17.7	5.6	0.97	0.93	25	47	2.4	44
202	1	18.4	5.4	0.98	0.98	11	49	3.4	26
223	6	15.3	5.3	0.95	0.98	5	26	3.1	8
217	15	18.9	4.9	0.96	0.97	42	23	4	23
213	7	18.6	4.6	0.99	0.95	33	12	3.6	42
214	16	16.3	4.1	0.98	0.99	43	33	3.5	22
211	6	16.7	4.1	0.96	0.9	32	40	2.1	43
220	17	19.1	4.9	0.97	0.99	15	35	0.1	37
208	8	18.1	4.3	0.99	0.94	18	39	0.3	15
209	8	17	4	0.96	0.91	20	13	1.8	16

Table 5.4b Parameters for the Simulation of Section 5.3.2

The models for the terminating renewal process approximation and the alternating renewal process approximation assume exponential times to vessel detection, and the unescorted R transit time through the domain has a gamma distribution; these are the same assumptions as the simulation. However, unlike the simulation model, the renewal models assume the sensor's following times have an exponential distribution and there is no memory of previous classifications.

Figure 5.5 displays the results of both approximations for the probability that R is detected and correctly classified before it leaves the domain. The two approximations give similar results. The terminating renewal process approximation tends to result in somewhat higher probabilities that R is detected and correctly classified before leaving the domain, than the alternating renewal process approximation. Figure 5.6 displays the terminating renewal process approximation and the simulation results; the approximation tends to be systematically a little larger than the simulation results. Figure 5.7 displays the alternating renewal process approximation versus the simulation results; there is good agreement. The alternating renewal process approximation agrees better with the simulation results and is numerically more stable than the terminating renewal process approximation. The agreement is satisfying since the simulation model is more complex than the alternating renewal process model.

Terminating Renewal Process Approximation and Alternating Renewal Porcess Approximation
Probability R is Detected and Correctly Classified Before Leaving the Domain

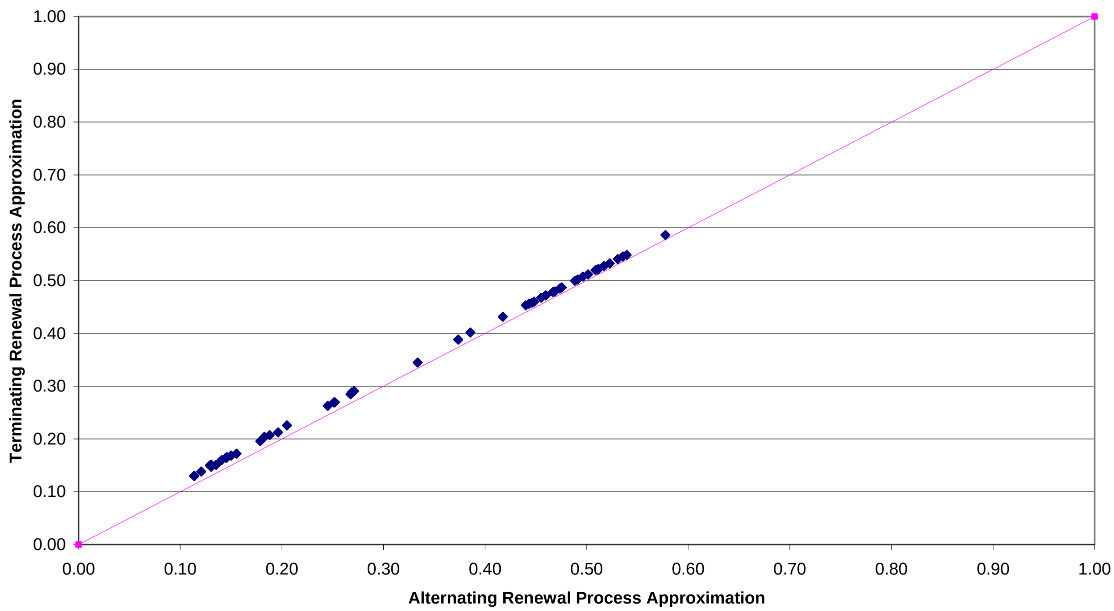


Figure 5.5

Simulation Results and Terminating Renewal Process Approximation
Probability R is Detected and Correctly Classified Before Leaving the Domain

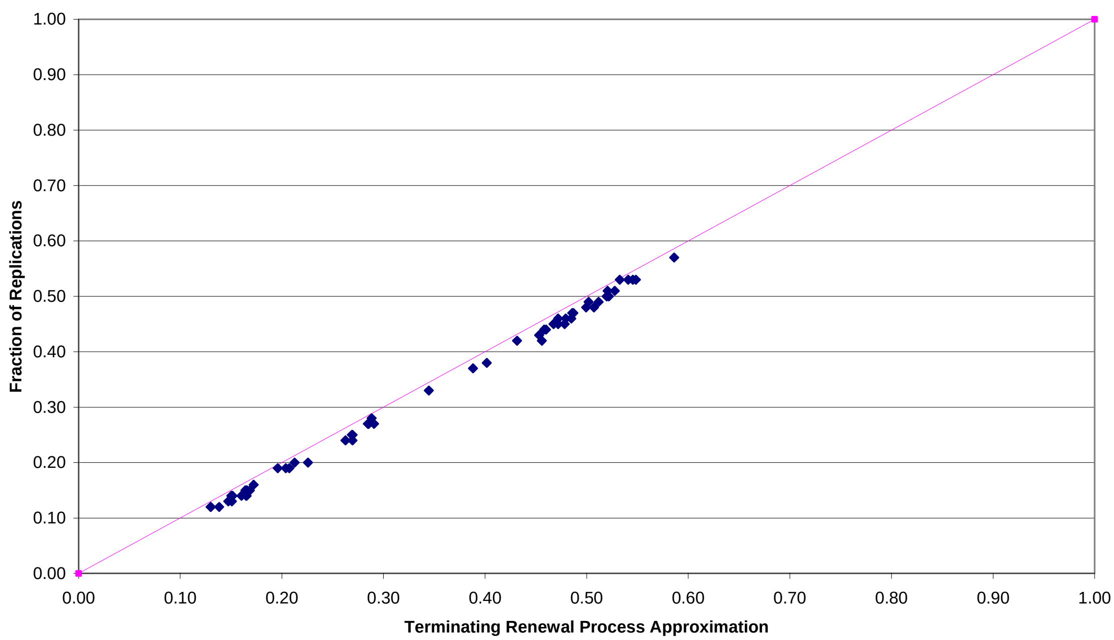


Figure 5.6

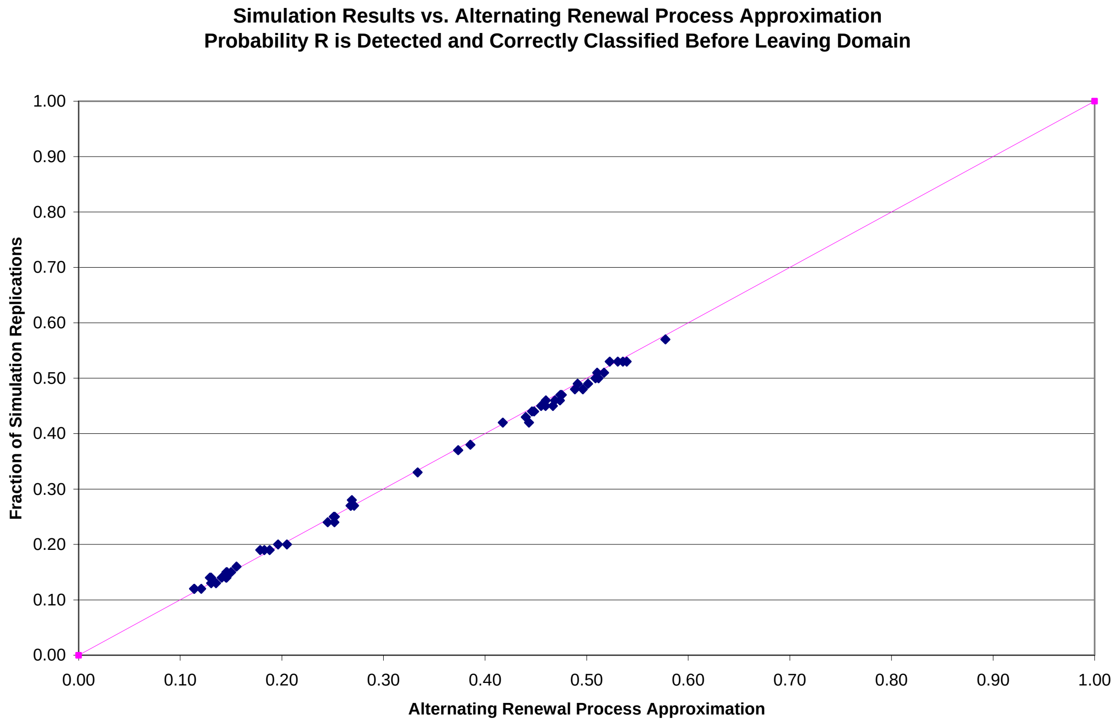


Figure 5.7

5.4. A Spatial Simulation

Results from a more detailed simulation that includes representation of the spatial movement of the overhead sensor and R are used to explore the robustness of the renewal process approximations. A description of the simulation is as follows.

The Search Domain

The domain is rectangular: M_x NM along the x-axis and M_y NM along the y-axis. The footprint of the sensor is a square with sides *f* NM. The region is tiled with squares having sides *f* NM; we assume both M_x and M_y are multiples of *f*. Label the upper row, row number 1 and column 1 is on the leftmost side. The upper left grid square is labeled $(1,1)$; the rightmost grid square in row 1 is labeled $(1, G_x)$; $G_x = M_x / f$. The grid square in the lower right hand corner row is labeled $\left(G_y, G_x\right)$; $G_y = M_y / f$. We assume there are a fixed number of neutral (White (W)) vessels in the domain.

Initializing the Simulation

1. One Red (R) vessel enters row 1 at time 0. The column it enters is chosen at random; each column is equally likely to be chosen. The R travels down the column at a constant velocity v_r. The grid square occupied by R is computed at each time the overhead sensor enters a new grid square.
2. The initial position of the overhead sensor $\mathcal{S}$ is chosen randomly from the rectangle; each position is equally likely to be chosen. The grid square of the initial position is determined; call it (y, x). Two independent trials are performed to determine the direction of $\mathcal{S}$. The x-direction is right or left; the y-direction is up or down. Each x-direction (respectively y-direction) has a probability of 0.5 of being chosen. We assume a raster scan sensor path. Assume the midpoint of the initial grid square that the sensor is in is (y,x). The path of the sensor is as follows: if the sensor's x-direction is right (respectively left) the sensor travels along the row of that square to that grid square with the largest, (say (y, G_x)) (respectively smallest, $(y,1)$) *x*-value. If the sensor's *y*-direction is up, after reaching the boundary, $\mathcal{S}$ will next travel to square $(y-1, G_x)$, (respectively $(y-1,1)$); it will next travel to $(y-1, G_x-1)$ (respectively $(y-1,2)$); when $\mathcal{S}$ enters the last grid square along row 1, say $(1, G_x)$, it then travels to grid square

$(2, G_x)$ and continues to $(2, G_x - 1)$ and so on. If the *y*-direction is down and $\mathcal{S}$ is in grid square (y, G_x), then it will next travel to square $(y+1, G_x)$. The velocity of $\mathcal{S}$ is v_s. The time spent in each grid square is f / v_s.

Detection of R

Each time $\mathcal{S}$ and R are in the same grid square there is a probability $p_d c_{rr}$ that R will be detected and correctly classified as R, where p_d is the probability that R is detected and c_{rr} is the probability that a detected R is correctly classified as R.

Number of White (Neutral) Vessels

Let *w* be the mean total number of White vessels in the region. The mean number of White vessels in each square, m_W, is the mean total number divided by the number of grid squares. Each time the sensor enters a square, a Poisson random variable having mean m_W is drawn; the resulting value is the number of White vessels in that square when it is searched. Each vessel in the square takes a time τ to be investigated. A White vessel is misclassified as an R with probability $p_d c_{wr} = p_d (1 - c_{ww})$. If a White vessel is misclassified as an R in grid square (*y*,*x*), $\mathcal{S}$ follows it for a time $(G_y - y)/(v_d + v)$, where v_d is the velocity of the DDs, $v = v_r = v_w$ is the velocity of all the vessels, and G_y is the number of grid squares in the y-direction. After this following time is completed, $\mathcal{S}$ proceeds instantaneously to the next square in its original search pattern. The total time spent in the grid square is the sum of the travel time through the square, plus the time spent investigating all vessels in the square plus any following time if applicable. If R is detected and correctly classified, it will be escorted.

5.4.1. Comparing Results from the Spatial Simulation to Those of the Renewal Process Approximations

The parameter values appear in Table 5.5. Table 5.6 displays the results of the spatial simulation and two approximations (5.1.14) and (5.2.7). In both approximations, the sensor's following time of a vessel classified as Red is exponential with mean $M_y / 2(v_d + v_r)$; the exponential distribution is an approximation to the following time distribution in the simulation. In the simulation, the time the R spends in the domain is constant and equal to M_y / v_r. In the approximations, the time R spends in the domain is assumed to have a gamma distribution with mean M_y / v_r and shape parameter equal to the number of grid squares in the y-direction, G_y.

Velocity of Red vessel	v_r	15 kts
Velocity of OH sensor $\mathcal{S}$	v_s	250 kts
Velocity of DD (pair)	v_d	30 kts
Number of neutral vessels	W	variable
y-direction length of rectangle	M_y	200 NM
x-direction length of rectangle	M_x	200 NM
Side of square of OH sensor footprint	f	25 NM
Mean time to classify detected vessel	τ	2/60 hrs
Number of OH sensors	$\bar{S}$	1
Prob. correctly classifying an R	c_{rr}	variable
Prob. correctly classifying a W	c_{ww}	variable
Prob. a vessel in the footprint of the OH sensor is detected	p_d	1

Table 5.5 Parameter Values for Simulation Results Displayed in Table 5.6

# Ws	c_{rr}	c_{ww}	**Spatial Simulation (1,000 Replications) Fraction of Replications Red is not Correctly Class. (std error)**	**Terminating Renewal Process Approx. Prob. R is not Correctly Class.**	**Alternating Renewal Process Approx. Prob. R is not Correctly Class.**
200	0.90	0.90	0.82 (0.01)	0.80	0.81
200	0.90	0.99	0.54 (0.02)	0.51	0.52
200	0.99	0.99	0.49 (0.02)	0.48	0.49
200	0.999	0.999	0.43 (0.01)	0.39	0.39
100	0.90	0.90	0.72 (0.01)	0.67	0.69
100	0.99	0.99	0.39 (0.01)	0.35	0.36
100	0.999	0.999	0.33 (0.01)	0.29	0.29
50	0.90	0.90	0.61 (0.02)	0.53	0.55
50	0.90	0.99	0.34 (0.02)	0.30	0.30
50	0.99	0.99	0.30 (0.01)	0.27	0.27
50	0.999	0.999	0.22 (0.01)	0.23	0.23

Table 5.6 Simulation and Approximation Results

Table 5.8 displays simulation and approximation results for cases with parameters displayed in Table 5.7.

Velocity of Red vessel	v_r	15 kts
Velocity of OH sensor $\mathcal{S}$	v_s	200 kts
Velocity of DD (pair)	v_d	30 kts
Number of Ws	W	variable
y-direction length of rectangle	M_y	200 NM
x-direction length of rectangle	M_x	400 NM
Side of square of OH sensor footprint	f	25 NM
Mean time to classify detected vessel	τ	2/60 hrs
Number of OH sensors	$\overline{S}$	1
Prob. correctly classifying an R	c_{rr}	variable
Prob. correctly classifying a W	c_{ww}	variable
Prob. a vessel in the footprint of the OH sensor is detected	p_d	1

Table 5.7 Parameter values for Simulation Results Displayed in Table 5.8

# Ws	c_{rr}	c_{ww}	Spatial Simulation (1,000 replications) Fraction of Replications Red is not Correctly Class. (std error)	Terminating Renewal Process Approx. Prob. R is not Correctly Class.	Alternating Renewal Process Approx. Prob. R is not Correctly Class.
100	0.7	0.99	0.65 (0.02)	0.65	0.66
100	0.7	0.999	0.58 (0.02)	0.63	0.63
50	0.7	0.99	0.62 (0.02)	0.62	0.62
50	0.7	0.999	0.58 (0.02)	0.60	0.60
100	0.8	0.99	0.58 (0.02)	0.62	0.62
100	0.8	0.999	0.56 (0.02)	0.59	0.59
50	0.8	0.99	0.54 (0.02)	0.58	0.58
50	0.8	0.999	0.56 (0.02)	0.56	0.56
100	0.9	0.9	0.75 (0.01)	0.74	0.75
50	0.9	0.9	0.65 (0.01)	0.65	0.67

Table 5.8 Simulation and Approximation Results

Figure 5.8 displays the simulation results of Tables 5.6 and 5.8 and the alternating renewal process approximation (5.2.7). The approximation gives very useful results.

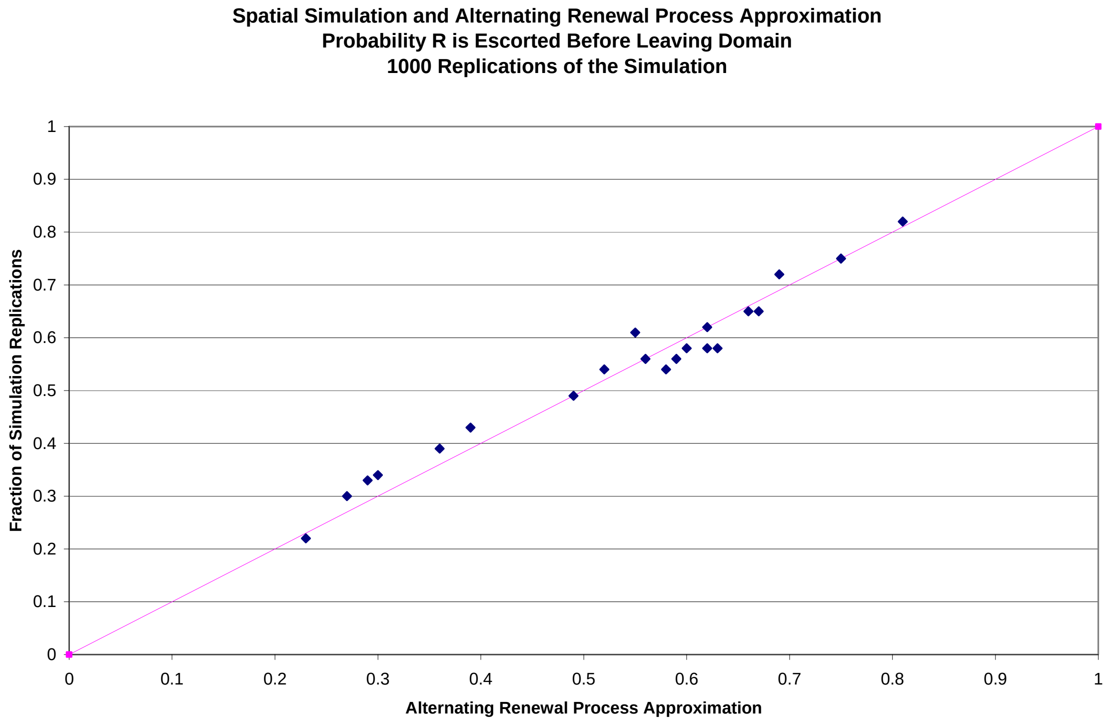


Figure 5.8

5.5 Numerical Examples Using the Alternating Renewal Process Approximation

The parameter values are those of Table 5.5. Figure 5.9 displays the alternating renewal process approximate probability that R is detected and correctly classified for various values of the probability of correct classification and number of Ws in the domain.

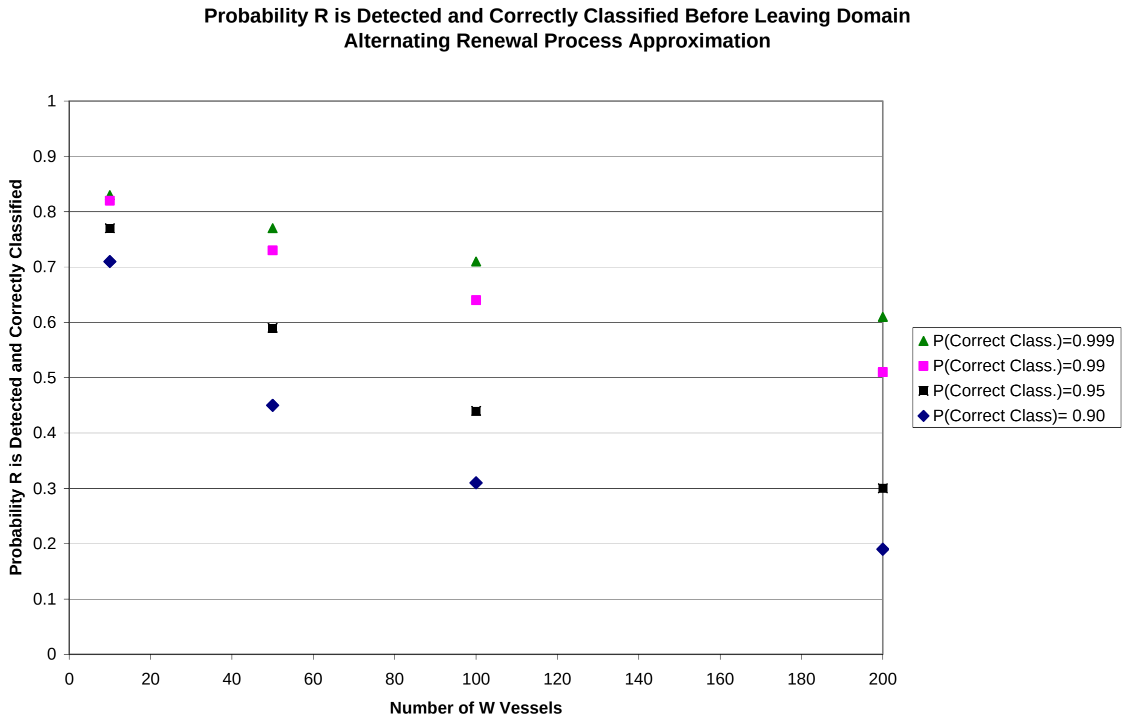


Figure 5.9

Discussion: The probability of detecting and correctly classifying R before it leaves the domain increases as the probability of correctly classifying a vessel increases; the increase is larger, the more Ws there are in the domain. Being able to decrease the number of Ws from 200 to 100 or from 100 to 50 results in a larger increase in the probability of detecting and correctly classifying R before it leaves the region than increasing the probability of correctly classifying detected vessel from 0.90 to 0.95.

Suppose that if s overhead sensors (P3s) are used to patrol a rectangular domain of 200 NM by 200 NM, then each P3 patrols a domain of length 200/s NM in the x-direction and 200 NM in the y-direction. Other CONOPS are possible. If there are w Ws in the large domain, then we assume there are $\frac{x}{200}w$ Ws in a domain of size x NM in the x-direction by 200 NM in the y-direction. Assume 1 R enters the domain at time 0 and travels straight down the domain in the y-direction. The time the unescorted R spends in the domain has a gamma distribution with mean $200/v_r$ and shape parameter $200/f$. Thus, the R will be in the domain patrolled by one sensor. All of the sensors can misclassify Ws as R. We assume there are a sufficient number of escort vessels to escort all the vessels classified as R. Table 5.9 displays the parameter values. Figure 5.10 displays the alternating renewal process approximation for the probability that R is detected and correctly classified as a function of the *x*-distance of the rectangular domain patrolled by one sensor and the number of Ws in the total domain of size 200 NM by 200 NM, *w*.

Velocity of Red vessel	v_r	15 kts
Velocity of OH sensor $\mathcal{S}$	v_s	250 kts
Velocity of DD (pair)	v_d	30 kts
Number of neutral vessels	w	variable
y-direction length of rectangle	M_y	200 NM
x-direction length of rectangle	M_x	variable
Side of square of OH sensor footprint	f	10 NM
Mean time to classify detected vessel	τ	2/60 hrs
Number of OH sensors	$\overline{S}$	1
Prob. correctly classifying an R	c_{rr}	0.99
Prob. correctly classifying a W	c_{ww}	0.99
Prob. a vessel in the footprint of the OH sensor is detected	p_d	1

Table 5.9 Parameter Values for Figure 5.10

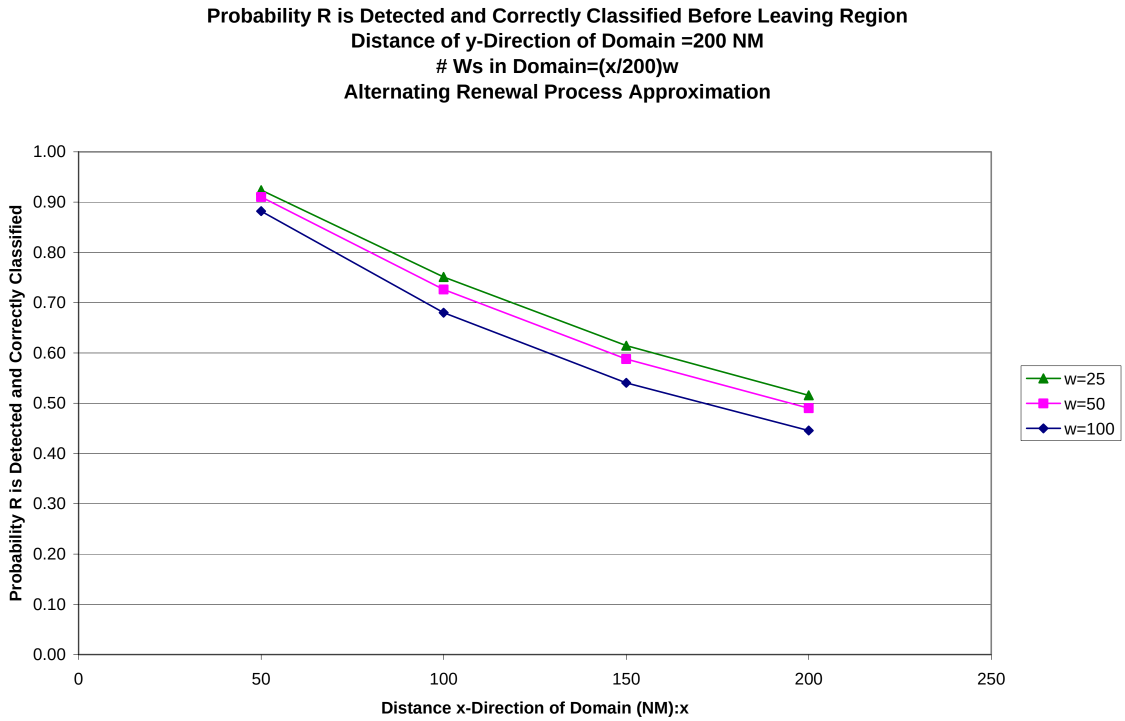


Figure 5.10

Discussion: In order to have an approximate probability of detecting and correctly classifying R before it leaves the region of about 0.9, four overhead sensors are needed. Each sensor patrols a rectangular domain 50 NM by 200 NM. For the parameter values considered, the approximate probability of detecting and correctly classifying R before it leaves the region is more sensitive to the size of the domain the sensor patrols than the number of Ws in the domain; however, the number of Ws considered are 25, 50, and 100. Decreasing the size of the domain that a single sensor patrols also decreases the number of Ws in the domain and hence the false positive rate.

6. Assessing the Performance of the Automatic Identification System (AIS) for Vessel Tracking

The Automatic Identification System (AIS) is a communication system using two frequencies in the VHS maritime band by vessels to periodically broadcast information about position, identification, etc. The rate at which a vessel transmits its information depends on the speed of the vessel and how it is maneuvering. Each vessel broadcasts its information to vessels within its line of sight. At most, 4,500 messages can be transmitted per minute when each message takes one time slot. A self-organizing time division multiple access (SOTDMA) protocol is used to minimize the chance that two messages from different vessels will use the same time slot. The international maritime organization and governments are requiring vessels of certain sizes to use the AIS system. It is hoped that AIS will decrease the number of vessel collisions and increase maritime safety and security. There is also interest in using the AIS messages to track vessels. There are land AIS receivers and there is a proposal to put AIS receivers on satellites. A receiver on a satellite may have a field of view that extends over a domain in which vessels do not have line of sight with each other. In this case vessels may be using the same time slots to transmit messages; in which case, all messages using the same time slot will not be received by a satellite receiver.

6.1. Formulation of a Continuous Time Markov Chain Model

Each vessel is in one of J subdomains. Assume a vessel uses a time slot for an independent random time having an exponential distribution with mean $1/\eta$; it then chooses a new time slot that is not being used by another vessel in its own subdomain, assuming one exists. Each vessel must use s different time slots each minute. Let c be the total number of time slots available. Let b_j be the number of vessels in subdomain j, $j = 1, \ldots, J$; here $c > b_j s$. Let $\bar{b} = \sum_{j=1}^{J} b_j$ be the total number of vessels in all the subdomains.

Consider a particular time slot. Let $\boldsymbol{X}_j(t)$ be equal to 1 if the time slot is being used at time *t* by a vessel in subdomain *j* and 0 otherwise.

$$P\left\{\boldsymbol{X}_j(t+h)=1 \mid \boldsymbol{X}_j(t)=0\right\}=\underbrace{\eta b_j s}_{\substack{\text{rate at which vessels in region j change time slots;}\\ \text{each vessel uses s time slots}}} \times \underbrace{\frac{1}{c-sb_j}}_{\text{prob. choose a particular idle time slot}} h+o(h) \tag{6.1.1}$$

$$P\left\{\boldsymbol{X}_j(t+h)=0 \mid \boldsymbol{X}_j(t)=1\right\}=\eta h+o(h)$$

Thus, the long run proportion of time one time slot is busy with a vessel from subdomain *j* is $\pi_j = b_j s/c$. The long run proportion of time one particular time slot is not being used by any vessel in any sub-domain is $\prod_{j=1}^{J}\left[1-\frac{b_j s}{c}\right]$.

Let $\underline{\boldsymbol{X}}(t)=\left[\boldsymbol{X}_1(t),\ldots,\boldsymbol{X}_J(t)\right]$ a vector indicating those subdomains that have a vessel using the particular time slot. Let 1_j be a J-dimensional row vector that has a 1 in the j$^{\text{th}}$ column and zeros elsewhere. Since the subdomains are independent

$$\pi(\underline{x})=\lim_{t\to\infty} P\left\{\underline{\boldsymbol{X}}(t)=\underline{x}\right\}=\prod_{j=1}^{J}\left[\frac{b_j a}{c}\right]^{x_j}\left[1-\left[\frac{b_j a}{c}\right]\right]^{1-x_j}. \tag{6.1.2}$$

The long run proportion of time a time slot has exactly one vessel using it is

$$\sum_{j=1}^{J}\pi\left(\underline{1}_j\right)=\sum_{j=1}^{J}\frac{b_j a}{c}\prod_{k\neq j}\left[1-\frac{b_k a}{c}\right]. \tag{6.1.3}$$

Consider a vessel in subdomain *d*. It uses *s* time slots during a minute. Let the probability that all messages from the vessel are blocked during a minute be

$$\gamma_d(s) = \left[1 - \underbrace{\prod_{j \neq d} \left[1 - \left[\frac{b_j s}{c} \right] \right]}_{\substack{prob \\ \text{no other} \\ \text{subdomain} \\ \text{using} \\ \text{time} \\ \text{slot}}} \right]^s . \qquad (6.1.4)$$

6.2 Numerical Example

There are 2,250 time slots on each of two frequencies. There are c=4,500 time slots in total. Assume each vessel transmits at s=6 times per minute. Each message requires one time slot. Suppose there are J=20 subdomains and each subdomain contains the same number of vessels. A satellite will be able to receive a message from a vessel in its field of view if a vessel transmits the message in a time slot that is not blocked by vessels in other subdomains during the observation time, T_{obs}, of the satellite. We say that a vessel is observable if at least one time slot that the vessel uses during an accessible observation time is not blocked. Let the probability that a vessel in subdomain d is detected be

$$p_d(s, T_{obs}) = 1 - \gamma_d(s)^{T_{obs}} . \qquad (6.2.1)$$

If all subdomains contain the same number of vessels, then (6.2.1) is equal to expression (3.2) in Ericksen et al. (2004). Ericksen et al. (2004) report that this simple formula agrees well with results from a more detailed simulation of the probability that a vessel will be detected.

Figure 6.1 displays the probability of vessel detection as a function of the total number of vessels in all the subdomains. The numbers of vessels in each subdomain are assumed to be equal. The observation time, $T_{obs} = 15$ minutes.

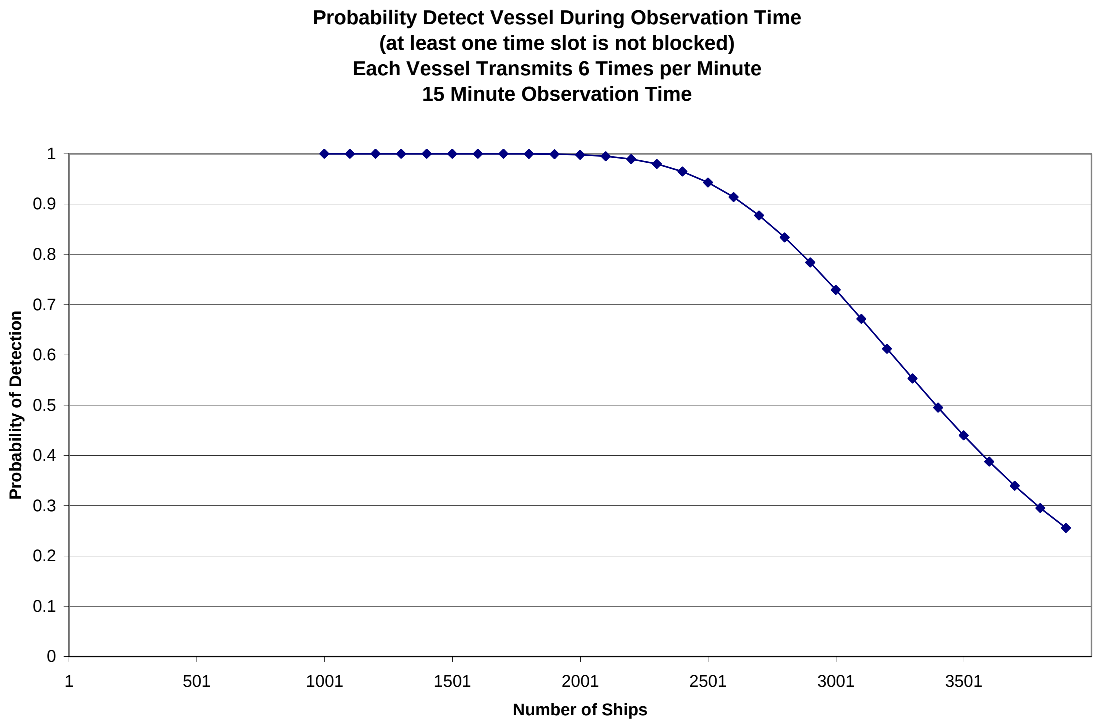


Figure 6.1

7. Conclusions

We have presented models and results for a maritime domain awareness scenario. In the scenario, there are neutral vessels, Whites (W), and hostile vessels, Reds (R), traveling within a domain. A patrolling overhead sensor detects vessels in the domain and classifies them as W or R. The overhead sensor follows each vessel that is classified (perhaps incorrectly) as R until relieved by escorting vessels; during this time it is unavailable to detect additional vessels. The ability to detect, correctly classify, and escort Rs is influenced by the size of the domain, the number of Ws that are in the domain, and the probability of correctly classifying detected vessels. The introduction of technology such as the Automatic Information System (AIS) to track vessels should decrease the number of Ws in the domain and thus increase the ability to neutralize Rs.

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Appendix A: An Exponential Model for the Probability that a Red Vessel will Leak Through a Maritime Domain

A.1. The Probability of Detecting, Classifying, and Escorting a Red Vessel

Assume there are always *w* White vessels (Ws) in the domain and there are no Red (R) vessels. The time until an overhead sensor (OH) detects a white vessel has an exponential distribution with mean $1/\delta(w)$. Each time the OH detects a W it classifies it as a W with probability c_{ww}. With probability c_{wr} it misclassifies W as an R. All vessels that are classified as R are tracked (followed) for a time having an exponential distribution with mean $1/\phi$. During the following time no further vessels are detected. The time until detection of a W that is classified as R has an exponential distribution with mean $1/\left[\delta(w)c_{wr}\right]$.

Assume a Red vessel (R) enters the domain at time 0. The R will remain in the domain for a time having an exponential distribution having mean $1/\mu$. The time until the R is detected has an exponential distribution with mean $(w+1)/\delta(w+1)$ if the OH is searching and there are w Ws in the domain. When the R is detected, it is correctly classified as R with probability c_{rr}. The time until the Red is detected and correctly classified has an exponential distribution with mean $(w+1)/\left[\delta(w+1)c_{rr}\right]$.

We assume all exponential times are independent. Let $p(w)$ be the probability that the Red is detected, correctly classified, and escorted before it leaves the domain given there are *w* Ws in the domain.

$$
\begin{aligned}
p(w) = {} & \underbrace{\left[\frac{\delta(w+1)\frac{1}{w+1}c_{rr}}{\delta(w+1)\frac{1}{w+1}c_{rr}+\delta(w+1)\frac{w}{w+1}c_{wr}+\mu}\right]}_{\substack{\text{Prob. Red}\\ \text{is detected}\\ \text{and correctly}\\ \text{class. before}\\ \text{Red leaves}\\ \text{region or}\\ \text{a W is misclass.}}} \underbrace{\frac{\phi}{\phi+\mu}}_{\substack{\text{Prob.}\\ \text{DD}\\ \text{arrives}\\ \text{before}\\ \text{R}\\ \text{leaves}\\ \text{region}}} \\
& + \underbrace{\left[\frac{\delta(w+1)\frac{w}{w+1}c_{wr}}{\delta(w+1)\frac{1}{w+1}c_{rr}+\delta(w+1)\frac{w}{w+1}c_{wr}+\mu}\right]}_{\substack{\text{Prob.}\\ \text{W is misclass.}\\ \text{before Red}\\ \text{is detected}\\ \text{and correctly}\\ \text{class. or Red}\\ \text{leaves}\\ \text{region}}} \frac{\phi}{\phi+\mu}\, p(w)
\end{aligned}
\tag{A.1.1}
$$

Solving

$$
\begin{aligned}
p(w) &= \frac{\left[\dfrac{\delta(w+1)\frac{1}{w+1}c_{rr}}{\delta(w+1)\frac{1}{w+1}c_{rr}+\delta(w+1)\frac{w}{w+1}c_{wr}+\mu}\right]\dfrac{\phi}{\phi+\mu}}{1-\left[\dfrac{\delta(w+1)\frac{w}{w+1}c_{wr}}{\delta(w+1)\frac{1}{w+1}c_{rr}+\delta(w+1)\frac{w}{w+1}c_{wr}+\mu}\right]\dfrac{\phi}{\phi+\mu}}. \\
&= \frac{\delta(w+1)\frac{1}{w+1}c_{rr}\,\frac{\phi}{\phi+\mu}}{\delta(w+1)\frac{1}{w+1}c_{rr}+\delta(w+1)\frac{w}{w+1}c_{wr}\,\frac{\mu}{\phi+\mu}+\mu}
\end{aligned}
\tag{A.1.2}
$$

Let $p_0(w)$ be the probability the Red is detected and correctly classified before it leaves the domain.

$$p_0(w) = \underbrace{\left[\frac{\delta(w+1)\frac{1}{w+1}c_{rr}}{\delta(w+1)\frac{1}{w+1}c_{rr} + \delta(w+1)\frac{w}{w+1}c_{wr} + \mu} \right]}_{\substack{\text{Prob. Red}\\ \text{is detected}\\ \text{and correctly}\\ \text{class. before}\\ \text{Red leaves}\\ \text{region or}\\ \text{a W is misclass.}}}$$

$$+ \underbrace{\left[\frac{\delta(w+1)\frac{w}{w+1}c_{wr}}{\delta(w+1)\frac{1}{w+1}c_{rr} + \delta(w+1)\frac{w}{w+1}c_{wr} + \mu} \right]}_{\substack{\text{Prob.}\\ \text{W is misclass.}\\ \text{before Red}\\ \text{is detected}\\ \text{and correctly}\\ \text{class. or Red}\\ \text{leaves}\\ \text{region}}} \frac{\phi}{\phi+\mu} p_0(w) \tag{A.1.3}$$

Solving

$$p_0(w) = \frac{\left[\dfrac{\delta(w+1)\frac{1}{w+1}c_{rr}}{\delta(w+1)\frac{1}{w+1}c_{rr} + \delta(w+1)\frac{w}{w+1}c_{wr} + \mu} \right]}{1 - \left[\dfrac{\delta(w+1)\frac{w}{w+1}c_{wr}}{\delta(w+1)\frac{1}{w+1}c_{rr} + \delta(w+1)\frac{w}{w+1}c_{wr} + \mu} \right] \dfrac{\phi}{\phi+\mu}} . \tag{A.1.4}$$

$$= \frac{\delta(w+1)\frac{1}{w+1}c_{rr}}{\delta(w+1)\frac{1}{w+1}c_{rr} + \delta(w+1)\frac{w}{w+1}c_{wr}\frac{\mu}{\phi+\mu} + \mu}$$

A.2. Misclassified White Vessels are Removed from the Population

Assume that each time a White vessel is classified as R, it is removed from the population. The above equations become systems of equations. For example, the conditional probability that Red is escorted before it leaves the domain given there are w Whites in the domain satisfies the equation

$$
\begin{aligned}
p(w) = &\left[\frac{\delta(w+1)\frac{1}{w+1}c_{rr}}{\delta(w+1)\frac{1}{w+1}c_{rr} + \delta(w+1)\frac{w}{w+1}c_{wr} + \mu} \right] \frac{\phi}{\phi+\mu} \\
&+ \left[\frac{\delta(w+1)\frac{w}{w+1}c_{wr}}{\delta(w+1)\frac{1}{w+1}c_{rr} + \delta(w+1)\frac{w}{w+1}c_{wr} + \mu} \right] \frac{\phi}{\phi+\mu} p(w-1)
\end{aligned} \quad . \tag{A.2.1}
$$

The conditional probability that Red is detected and correctly classified before it passes through the domain given there are *w* Ws in the domain is

$$
\begin{aligned}
p_0(w) = &\left[\frac{\delta(w+1)\frac{1}{w+1}c_{rr}}{\delta(w+1)\frac{1}{w+1}c_{rr} + \delta(w+1)\frac{w}{w+1}c_{wr} + \mu} \right] \\
&+ \left[\frac{\delta(w+1)\frac{w}{w+1}c_{wr}}{\delta(w+1)\frac{1}{w+1}c_{rr} + \delta(w+1)\frac{w}{w+1}c_{wr} + \mu} \right] \frac{\phi}{\phi+\mu} p_0(w-1)
\end{aligned} \quad . \tag{A.2.2}
$$

Appendix B: A Recursive Procedure to Find Limiting Probabilities for a Truncated Finite State Space Quasi-Birth and -Death Model

Assume Reds that arrive when there are N Rs in the domain are lost and do not enter the domain. The state space is $(n,s),(n,f),(n,c)$ for $n=0,...,N;$ see Section (3.1).

The balance equations for states (N,s), (N,f), and (N,c) are:

$$\left[N\left[\mu+\delta_0\right]+\delta_w\right]\pi(N,s)=\lambda_r\pi(N-1,s)+\phi\pi(N,f)+\phi\pi(N,c)$$

$$\left[N\mu+\phi\right]\pi(N,f)=\lambda_r\pi(N-1,f)+\delta_w\pi(N,s)$$

$$\left[N\mu+\phi\right]\pi(N,c)=\lambda_r\pi(N-1,c).$$

An outline of the recursive procedure to find the limiting distribution is as follows; see also Gaver *et al* (1984). For each number of Rs n>0.

1. Use the balance equation for (n,f) to express

   $$\pi(n,f)=B_T(n,s)\pi(n,s)+B_T(n,f)\pi(n-1,f)+B_T(n,c)\pi(n,c)$$

2. Use the balance equation for (n,c) to express

   $$\pi(n,c)=C_T(n,s)\pi(n,s)+C_T(n,f)\pi(n-1,f)+C_T(n,c)\pi(n-1,c)$$

3. Use the balance equation for (n,s) to express

   $$\pi(n,s)=A(n,s)\pi(n-1,s)+A(n,f)\pi(n-1,f)+A(n,c)\pi(n-1,c)$$

4. Use 2 and 3 to express

   $$\pi(n,c)=C(n,s)\pi(n-1,s)+C(n,f)\pi(n-1,f)+C(n,c)\pi(n-1,c)$$

5. Use 1, 3, and 4 to express

   $$\pi(n,f)=B(n,s)\pi(n-1,s)+B(n,f)\pi(n-1,f)+B(n,c)\pi(n-1,c)$$

6. Use the balance equation for $(0, f)$ to express

$$\pi(0, f) = B(0,s)\pi(0,s) + B(0,c)\pi(0,c)$$

7. Use the balance equation for $(0,c)$ to express

$$\pi(0,c) = C(0,s)\pi(0,s)$$

8. Find $\pi(0,s)$ so that the sum of the limiting probabilities equals 1.

   Detailed Calculations appear below.

Balance equation for (*N*,*f*):

$$\pi(N, f) = \frac{\lambda_r}{N\mu + \phi}\pi(N-1, f) + \frac{\delta_w}{N\mu + \phi}\pi(N,s)$$

Let $B_T(N, f) = \frac{\lambda_r}{D(N, f)}; B_T(N,s) = \frac{\delta_w}{D(N, f)}; D(N, f) = N\mu + \phi$

$$\pi(N, f) = B_T(N, f)\pi(N-1, f) + B_T(N,s)\pi(N,s).$$

Balance equation for (*N*,*c*):

$$\pi(N,c) = C_T(N,c)\pi(N-1,c),$$

where

$$C_T(N,c) = \frac{\lambda_r}{N\mu + \phi}.$$

Balance equation for (*N*,*s*):

$$\begin{aligned}
&\left[N[\mu + \delta_0] + \delta_w\right]\pi(N,s) = \lambda_r\pi(N-1,s) + \phi\left[\frac{\lambda_r}{N\mu + \phi}\pi(N-1, f) + \frac{\delta_w}{N\mu + \phi}\pi(N,s)\right] \\
&+\phi C_T(N,c)\pi(N-1,c) \\
&= \lambda_r\pi(N-1,s) + \phi\left[B_T(N, f)\pi(N-1, f) + B_T(N,s)\pi(N,s)\right] + \phi C_T(N,c)\pi(N-1,c)
\end{aligned}.$$

Let

$$D(N,s) = \left[N[\mu + \delta_0] + \delta_w - \phi B_T(N,s)\right]$$

$$\pi(N,s)=\frac{\lambda_r}{D(N,s)}\pi(N-1,s)+\phi\frac{B_T(N,f)}{D(N,s)}\pi(N-1,f)+\phi\frac{C_T(N,c)}{D(N,s)}\pi(N-1,c)$$
$$\equiv A(N,s)\pi(N-1,s)+A(N,f)\pi(N-1,f)+A(N,c)\pi(N-1,c)$$

$$\pi(N,f)=B_T(N,f)\pi(N-1,f)+B_T(N,s)\pi(N,s)$$
$$=B_T(N,f)\pi(N-1,f)+B_T(N,s)\left[A(N,s)\pi(N-1,s)+A(N,f)\pi(N-1,f)+A(N,c)\pi(N-1,c)\right],$$
$$\equiv B(N,s)\pi(N-1,s)+B(N,f)\pi(N-1,f)+B(N,c)\pi(N-1,c)$$

where

$$B(N,s)=B_T(N,s)A(N,s);B(N,f)=B_T(N,f)+B_T(N,s)A(N,f);$$
$$B(N,c)=B_T(N,s)A(N,c)$$
.

For 1<j<N

$$\pi(j,f)$$
$$\equiv B(j,s)\pi(j-1,s)+B(j,f)\pi(j-1,f)+B(j,c)\pi(j-1,c),$$
where
$$B(j,s)=B_T(j,s)A(j,s)+B_T(j,c)C(j,s)$$
$$B(j,f)=B_T(j,f)+B_T(j,s)A(j,f)+B_T(j,c)C(j,f)$$
$$B(j,c)=B_T(j,s)A(j,c)+B_T(j,c)C(j,c).$$

$$\pi(j,c)$$
$$\equiv C(j,s)\pi(j-1,s)+C(j,f)\pi(j-1,f)+C(j,c)\pi(j-1,c),$$
where
$$C(j,s)=C_T(j,s)A(j,s)$$
$$C(j,f)=C_T(j,s)A(j,f)+C_T(j,f)$$
$$C(j,c)=C_T(j,s)A(j,c)+C_T(j,c).$$

$$\pi(j,s)$$
$$\equiv A(j,s)\pi(j-1,s)+A(j,f)\pi(j-1,f)+A(j,c)\pi(j-1,f),$$
where
$$A(j,s)=\frac{\lambda_r}{D(j,s)}$$
$$A(j,f)=\frac{\left[\phi+(j+1)\mu A(j+1,f)\right]\left[B_T(j,f)+B_T(j,c)C_T(j,f)\right]}{D(j,s)}$$
$$+\frac{\left[\phi+(j+1)\mu A(j+1,c)\right]C_T(j,f)}{D(j,s)}$$
$$A(j,c)=\frac{\left[\phi+(j+1)\mu A(j+1,f)\right]B_T(j,c)C_T(j,c)}{D(j,s)}$$
$$+\frac{\left[\phi+(j+1)\mu A(j+1,c)\right]C_T(j,c)}{D(j,s)}$$
$$D(j,s)=\left[j\left[\mu+\delta_0\right]+\delta_w+\lambda_r\right]-\left[\phi+(j+1)\mu A(j+1,f)\right]\left[B_T(j,s)+B_T(j,c)C_T(j,s)\right]$$
$$-\left[\phi+(j+1)\mu A(j+1,c)\right]C_T(j,s)-(j+1)\mu A(j+1,s).$$

$$D(j,f)=\left[j\mu+\phi+\lambda_r-(j+1)\mu B(j+1,f)\right]$$
$$\pi(j,f)=\frac{\lambda_r}{D(j,f)}\pi(j-1,f)+\left[\frac{\delta_w}{D(j,f)}+\frac{(j+1)\mu B(j+1,s)}{D(j,f)}\right]\pi(j,s)$$
$$+\frac{(j+1)\mu B(j+1,c)}{D(j,f)}\pi(j,c)$$
$$\equiv B_T(j,f)\pi(j-1,f)+B_T(j,s)\pi(j,s)+B_T(j,c)\pi(j,c)$$

$$D(j,c)=\left[j\mu+\phi+\lambda_r-(j+1)\delta_0 A(j+1,c)\right]$$
$$-(j+1)\delta_0 A(j+1,f)B_T(j,c)-(j+1)\mu C_T(j+1,c)$$
$$\pi(j,c)=\frac{\lambda_r}{D(j,c)}\pi(j-1,c)$$
$$+\left[\frac{(j+1)\delta_0\left[A(j+1,s)+A(j+1,f)B_T(j,s)\right]+(j+1)\mu\left[C(j+1,s)+C(j+1,f)B_T(j,s)\right]}{D(j,c)}\right]\pi(j,s)$$
$$+\frac{(j+1)\delta_0 A(j+1,f)B_T(j,f)+(j+1)\mu C(j+1,f)B_T(j,f)}{D(j,c)}\pi(j,f)$$
$$\equiv C_T(j,f)\pi(j-1,f)+C_T(j,s)\pi(j,s)+C_T(j,c)\pi(j-1,c)$$

Finally, the balance equation for *j*=1 is

$$\begin{aligned}&\left[\mu+\lambda_r+\phi\right]\pi(1,f)=\lambda_r\pi(0,f)+\delta_w\pi(1,s)+2\mu\pi(2,f)\\&=\lambda\pi(0,f)+\delta_w\pi(1,s)+2\mu\left[B(2,s)\pi(1,s)+B(2,f)\pi(1,f)+B(2,c)\pi(1,c)\right].\end{aligned}$$

Let

$$\begin{aligned}&D(1,f)=\left[\mu+\lambda_r+\phi-2\mu B(2,f)\right]\\&B_T(1,f)=\frac{\lambda_r}{D(1,f)};B_T(1,s)=\frac{\delta_w+2\mu B(2,s)}{D(1,f)}\\&B_T(1,c)=\frac{2\mu B(2,c)}{D(1,f)}\end{aligned}$$

$$\pi(1,f)=B_T(1,s)\pi(1,s)+B_T(1,f)\pi(0,f)+B_T(1,c)\pi(1,c).$$

The balance equation for $\pi(1,c)$ is

$$\left[\lambda_r+\mu+\phi\right]\pi(1,c)=2\mu\pi(2,c)+\lambda_r\pi(0,c)+2\delta_0\pi(2,s)$$

$$\begin{aligned}&\left[\lambda_r+\mu+\phi\right]\pi(1,c)=2\mu\left[C(2,s)\pi(1,s)+C(2,f)\pi(1,f)+C(2,c)\pi(1,c)\right]+\lambda_r\pi(0,c)\\&+2\delta_0\left[A(2,s)\pi(1,s)+A(2,f)\pi(1,f)+A(2,c)\pi(1,c)\right]\end{aligned}$$

$$\begin{aligned}&\left[\lambda_r+\mu+\phi\right]\pi(1,c)=2\mu\left[C(2,s)\pi(1,s)+C(2,c)\pi(1,c)\right]+\lambda_r\pi(0,c)\\&+2\delta_0\left[A(2,s)\pi(1,s)+A(2,f)\pi(1,f)+A(2,c)\pi(1,c)\right]\\&+2\mu C(2,f)\left[B_T(1,s)\pi(1,s)+B_T(1,f)\pi(0,f)+B_T(1,c)\pi(1,c)\right]\\&=2\mu\left[C(2,s)\pi(1,s)+C(2,c)\pi(1,c)\right]+\lambda_r\pi(0,c)\\&+2\delta_0\left[A(2,s)\pi(1,s)+A(2,c)\pi(1,c)\right]\\&+2\delta_0A(2,f)\left[B_T(1,s)\pi(1,s)+B_T(1,f)\pi(0,f)+B_T(1,c)\pi(1,c)\right]\\&+2\mu C(2,f)\left[B_T(1,s)\pi(1,s)+B_T(1,f)\pi(0,f)+B_T(1,c)\pi(1,c)\right]\end{aligned}.$$

Let

$$\begin{aligned}&D(1,c)=\left[\lambda_r+\mu+\phi\right]-2\mu C(2,c)-2\delta_0A(2,c)-2\mu C(2,f)B_T(1,c)\\&-2\delta_0A(2,f)B_T(1,c)\end{aligned}$$

$$\pi(1,c)=C_T(1,s)\pi(1,s)+C_T(1,f)\pi(0,f)+C_T(1,c)\pi(0,c),$$

where

$$C_T(1,s) = \frac{2\mu C(2,s) + 2\delta_0 A(2,s) + 2\delta_0 A(2,f) B_T(1,s) + 2\mu C(2,f) B_T(1,s)}{D(1,c)}$$
$$C_T(1,f) = \frac{2\delta_0 A(2,f) B_T(1,f) + 2\mu C(2,f) B_T(1,f)}{D(1,c)} .$$
$$C_T(1,c) = \frac{\lambda_r}{D(1,c)}$$

The balance equation for $\pi(1,s)$

$$\begin{aligned}
&[\mu + \delta_0 + \lambda_r + \delta_w]\pi(1,s) = \lambda_r \pi(0,s) + \phi\pi(1,f) + \phi\pi(1,c) + 2\mu\pi(2,s) \\
&= \lambda\pi(0,s) + \phi\left[B_T(1,s)\pi(1,s) + B_T(1,f)\pi(0,f)\right] \\
&+\phi\left[1 + B_T(1,c)\right]\left[C_T(1,s)\pi(1,s) + C_T(1,f)\pi(0,f) + C_T(1,s)\pi(0,c)\right] \\
&+2\mu\left[A(2,s)\pi(1,s)\right] \\
&+2\mu A(2,f)\left[B_T(1,s)\pi(1,s) + B_T(1,f)\pi(0,f)\right] \\
&+2\mu A(2,f) B_T(1,c)\left[C_T(1,s)\pi(1,s) + C_T(1,f)\pi(0,f) + C_T(1,c)\pi(0,c)\right] \\
&+2\mu A(2,c)\left[C_T(1,s)\pi(1,s) + C_T(1,f)\pi(0,f) + C_T(1,c)\pi(0,c)\right]
\end{aligned} .$$

Let

$$\begin{aligned}
&D(1,s) = \mu + \delta_0 + \lambda_r + \delta_w - \phi B_T(1,s) - \phi\left[1 + B_T(1,c)\right] C_T(1,s) \\
&-2\mu A(2,s) - 2\mu A(2,f) B_T(1,s) \\
&-2\mu A(2,f) B_T(1,c) C_T(1,s) - 2\mu A(2,c) C_T(1,s)
\end{aligned} .$$

Then

$$\pi(1,s) = A(1,s)\pi(0,s) + A(1,f)\pi(0,f) + A(1,c)\pi(0,c),$$

where

$$A(1,s)=\frac{\lambda_r}{D(1,s)}$$

$$A(1,f)=\frac{\phi B_T(1,f)+\phi\left[1+B_T(1,c)\right]C_T(1,f)+2\mu A(2,f)B_T(1,f)}{D(1,s)}$$
$$+\frac{2\mu A(2,f)B_T(1,c)C_T(1,f)+2\mu A(2,c)C_T(1,f)}{D(1,s)} .$$

$$A(1,c)=\frac{\phi\left[1+B_T(1,c)\right]C_T(1,c)+2\mu A(2,f)B_T(1,c)C_T(1,c)+2\mu A(2,c)C_T(1,c)}{D(1,s)}$$

$$\pi(1,c)=C_T(1,s)\pi(1,s)+C_T(1,f)\pi(0,f)+C_T(1,c)\pi(0,c)$$
$$=C_T(1,s)\left[A(1,s)\pi(0,s)+A(1,f)\pi(0,f)+A(1,c)\pi(0,c)\right] .$$
$$+C_T(1,f)\pi(0,f)+C_T(1,c)\pi(0,c)$$

$$\pi(1,c)=C(1,s)\pi(0,s)+C(1,f)\pi(0,f)+C(1,c)\pi(0,c),$$

where

$$C(1,s)=C_T(1,s)A(1,s)$$
$$C(1,f)=C_T(1,s)A(1,f)+C_T(1,f)$$
$$C(1,c)=C_T(1,s)A(1,c)+C_T(1,c).$$

$$\pi(1,f)=B_T(1,s)\pi(1,s)+B_T(1,f)\pi(0,f)+B_T(1,c)\pi(1,c)$$
$$=B_T(1,s)\left[A(1,s)\pi(0,s)+A(1,f)\pi(0,f)+A(1,c)\pi(0,c)\right]$$
$$+B_T(1,f)\pi(0,f)+B_T(1,c)\left[C(1,s)\pi(0,s)+C(1,f)\pi(0,f)+C(1,c)\pi(0,c)\right]$$
$$\equiv B(1,s)\pi(0,s)+B(1,f)\pi(0,f)+B(1,c)\pi(0,c),$$

where

$$B(1,s)=B_T(1,s)A(1,s)+B_T(1,c)C(1,s)$$
$$B(1,f)=B_T(1,s)A(1,f)+B_T(1,f)+B_T(1,c)C(1,f)$$
$$B(1,c)=B_T(1,s)A(1,c)+B_T(1,c)C(1,c).$$

The balance equation for $\pi(0,f)$

$$\left[\lambda_r+\phi\right]\pi(0,f)=\delta_w\pi(0,s)+\mu\pi(1,f)$$
$$=\delta_w\pi(0,s)+\mu\left[B(1,s)\pi(0,s)+B(1,f)\pi(0,f)+B(1,c)\pi(0,c)\right] .$$

Let

$$D(0,f) = \lambda + \phi - \mu B(1,f)$$

$$\begin{aligned}\pi(0,f) &= \frac{\delta_w + \mu B(1,s)}{D(0,f)}\pi(0,s) + \frac{\mu B(1,c)}{D(0,f)}\pi(0,c) \\ &\equiv B(0,s)\pi(0,s) + B(0,c)\pi(0,c)\end{aligned}.$$

The balance equation for $\pi(0,c)$

$$\begin{aligned}(\lambda_r + \phi)\pi(0,c) &= \mu\pi(1,c) + \delta_0\pi(1,c) \\ &= \mu\left[C(1,s)\pi(0,s) + C(1,f)\pi(0,f) + C(1,c)\pi(0,c)\right] \\ &\quad + \delta_0\left[A(1,s)\pi(0,s) + A(1,f)\pi(0,f) + A(1,c)\pi(0,c)\right] \\ &= \mu C(1,f)\left[B(0,s)\pi(0,s) + B(0,c)\pi(0,c)\right] + \mu\left[C(1,s)\pi(0,s) + C(1,c)\pi(0,c)\right] \\ &\quad + \delta_0 A(1,f)\left[B(0,s)\pi(0,s) + B(0,c)\pi(0,c)\right] + \delta_0\left[A(1,s)\pi(0,s) + A(1,c)\pi(0,c)\right]\end{aligned}.$$

Let

$$D(0,c) = \lambda_r + \phi - \mu\left[C(1,c) + C(1,f)B(0,c)\right] - \delta_0\left[A(1,c) + A(1,f)B(0,c)\right].$$

Then

$$\pi(0,c) = C(0,s)\pi(0,s),$$

where

$$C(0,s) = \frac{\mu\left[C(1,s) + C(1,f)B(0,s)\right] + \delta_0\left[A(1,s) + A(1,f)B(0,s)\right]}{D(0,c)}.$$

$\pi(0,s)$ is found using the fact that the limiting probabilities sum to 1.

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