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AN UPPER BOUND FOR THE LINEARIZED MAP OF AN INVERSE PROBLEM FOR THE WAVE EQUATION [1]

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1 Introduction

A simplified model which governs many physical processes such as seismic and acoustic wave propagation is the following linear acoustic wave equation:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta - \nabla\sigma\cdot\nabla\right) u = f \;, \tag{1.1}$$

where $\sigma = \sigma(x)$ is the logarithm of the density, $c = c(x)$ is the sound speed of the medium, and $f = f(x,t)$ is the source term which introduces the energy to the problem. If σ, c and f are given along with appropriate side conditions, the forward (or direct) problem is to determine $u = u(x,t)$, the excess pressure. For appropriate choices of σ, c, and f, u is determined uniquely by standard linear hyperbolic theory of partial differential equations (*p.d.e.*). Thus the problem stated above defines a map from the coefficients to the solution of the wave equation. In this paper, we study an aspect of the *regularity* of this map, or rather its composition with the trace on a time-like hypersurface.

Throughout this work we shall restrict ourselves to the special case of constant velocity c, though we believe that the ideas in this work may be extended to cover some more general cases.

To fix ideas, write $x \in \mathbb{R}^n$ as (x', x_n), where $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. We assume that the problem is set in the whole space $\mathbb{R}^n$ and $u = 0$ in the past $(t < 0)$. Take $f(x,t) = \delta(x,t)$ as an ideal point source. Thus u is the retarded fundamental solution:

$$\Box u - \nabla\sigma\cdot\nabla u = \delta(x,t)\;,\quad (x,t)\in\mathbb{R}^n\times\mathbb{R} \tag{1.2}$$

$$u = 0\;,\quad t<0\;, \tag{1.3}$$

where $\Box$ is defined to be $\partial_t^2 - \Delta$, and Δ is the Laplacian.

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Define the *forward map* F as:

$$F : \sigma \to (\phi u) \mid_{x_n=0} , \tag{1.4}$$

where $\phi \in C_0^\infty(\mathrm{IR}^{n+1})$ is supported inside the conoid $\{t > |x|\}$ and near$\{x_n = 0\}$.

Because F is nonlinear, one wants to work with the formal linearization (or formal derivative) DF, with respect to the reference state (σ_0, u_0), defined by first order perturbation theory (Born-approximation).

$$\Box\delta u - \nabla\sigma_0 \cdot \nabla\delta u = \nabla\delta\sigma \cdot \nabla u_0 \tag{1.5}$$

$$\delta u = 0 , \; t < 0 . \tag{1.6}$$

The formal derivative $DF(\sigma_0)$ is given by

$$DF(\sigma_0)\delta\sigma = (\phi\delta u) \mid_{x_n=0} . \tag{1.7}$$

It is our main goal in this work to determine appropriate spaces of the domain and range of F for which

the formal derivative DF is bounded.

Throughout this paper, we shall always assume that

(A) $supp(\delta\sigma) \subset \{x_n > \epsilon\}$,

for $\epsilon > 0$ small. In some applications, this assumption is realistic, as the density can be measured directly, near the location of receivers (i.e. $x_n = 0$). Also the domain of dependence properties of the wave equation implies that ϕu and $\phi\delta u$ depend on σ and $\delta\sigma$ only in a bounded subset. Thus we shall throughout assume that σ and $\delta\sigma$ have fixed, bounded support.

Let $\Omega \subset \mathrm{IR}^k$ be open and bounded, $\gamma \subset T^*(\Omega)$. A constant C is said to depend on the $H^s \cap H^r_{m\ell}(\gamma)$-norm of $w \in C_0^\infty(\mathrm{IR}^k)$ if for any conic neighborhood Γ of γ there exists a ψ*.d.o.* Q of order zero with $ES(Q) \subseteq \Gamma$ and $q = 1$ on $\gamma \cap \{(x,\xi) : \; |\xi| > 1\}$ such that C can be bounded in terms of $||w||_{s,\Omega} + ||Qw||_{r,\Omega}$.

The following is the statement of our main result which will be proved in the sections which follow.

Theorem 1.1 *Assume either that*

(i) $n \geq 3$, $s > max\{3 + n/2, \; l + n - 1\}$, *and* $\sigma_0 \in H^s(\mathrm{IR}^n)$ *or*

(ii) $n = 2$, $s > max\{4, \; l + 1\}$, *and* $\sigma_0 \in H^s \cap H^{l+3/2}_{m\ell}(\{(x,\xi) : \xi_n = 0\})$.

Then under the assumption (A), the following estimate holds

$$||DF(\sigma_0)\delta\sigma||_l \leq C||\delta\sigma||_{l+\frac{n-1}{2}} \tag{1.8}$$

where the constant C depends on the H^s-norm ($n \geq 3$) or the $H^s \cap H^{l+3/2}_{m\ell}(\{(x,\xi) : \xi_n = 0\})$-norm of σ_0 ($n = 2$) but is independent of $\delta\sigma$.

The study of the forward map is motivated by the *inverse problem* which arises in reflection seismology, oil exploration, ground-penetrating radar, etc. Mathematically, the inverse problem is to determine the coefficient σ by knowing additional boundary value conditions of u. Since the inverse problem is just to invert the functional relation F, we are naturally interested in all the properties of this forward map.

To understand the problem, let us look at a simple exploration seismology experiment: Near the surface of the earth, a seismic source is fired at some point (point source). The seismic waves propagate into the earth. Since the earth's structure varies (as do its physical properties) part of the energy of the wave will be reflected back to the surface and can be measured. The inverse problem is to deduce the interior properties of the earth from the recorded data.

A simple model of this reflection seismic inverse problem in this context is: given data $F_{data}(x', t)$, find a coefficient $\sigma(x)$ so that

$$F(\sigma) = F_{data}$$

or perhaps minimizing the error $(F_{data} - F(\sigma))$ in some norm.

Numerical solution of this problem by means of Newton's method and its relatives requires a choice of Banach space structure in the space of models σ and in the space of data $F(\sigma)$ (see e.g. Kantorovich and Akilov [15]), in such a way that F is singular. The simplest regularity property of F is boundedness of DF, which is discussed in this paper. We believe that similar arguments will establish smoothness of F and allow investigation of coercive properties of DF, as is required by the theory of optimization.

The most efficient relatives of Newton's method, the quasi-Newton, conjugate gradient, and variable metric methods, all require a Hilbert space structure on the space and models. This fact accounts for our reliance on the L^2-based Sobolev spaces in this work.

When the spatial dimension is one or c and σ depend only on x_n (layered problem) there is a large literature available. For a similar problem in which the medium was assumed to be excited by an impulsive load on the surface $\{x_n = 0\}$ instead of point sources, the properties of the forward map have been studied fairly satisfactorily by Symes and others (see Symes [25] for references). It was shown by Symes that, for the constant wave speed case, the forward map defines a C^1-*diffeomorphism* between open sets in certain Hilbert spaces by applying the method of geometrical optics together with energy estimates.

When the spatial dimension $n > 1$ and c, σ depend on all space variables (nonlayered problem), very little is known in mathematics. Symes [23, 24], Sacks and Symes [21], Rakesh [18], and Sun [22] have some partial results. The difficulties are essentially due to the ill-posed nature of the timelike hyperbolic Cauchy problem and the presence of nonsmooth coefficients. For the one dimensional wave equation, both coordinate directions are spacelike, which indicates that the problem is hyperbolic with respect to both directions. Apparently, this is not the case when the spatial dimension is larger than one.

Rakesh in [18] looked at a related linearized velocity inversion problem with constant density and point sources. Assuming smooth background velocity, he obtained some results on both upper and lower bounds for the linearized forward map. The essential observation in Rakesh's work is that DF is a Fourier integral operator (see also Beylkin [7]). Unfortunately, the calculus of Fourier integral operators employed in Rakesh's work is not applicable to the nonsmooth reference velocity case since the linearized forward map is a Fourier integral operator only when the reference velocity is smooth.

In [23], Symes gave a pair of examples, based on the geometric optics construction, which show that both $DF(1)$ and $DF(1)^{-1}$ are unbounded for a slightly different problem. As the examples show, within the Sobolev scales no strengthening or weakening of topologies of the domain and range can make both DF and DF^{-1} bounded. This fact also implies a strategy of regularization: Change the topology in the domain so that DF becomes bounded, then ask for optimal regularization of DF^{-1} in the sense of best possible lower bound estimate for DF. In both examples of Symes, the unboundedness was caused by rapid oscillation of σ in the x'-direction or the tangential directions, hence the problem is actually "partially well-posed", i.e., only more smoothness of the coefficients in tangential directions (essentially grazing ray directions) will be required to cure the difficulty. This might be the main reason the anisotropic Sobolev spaces $H^{m,s}(\mathbb{R}^n)$ or Hörmander spaces, were introduced in [21] and [22].

In Theorem 4.1 of [21] Sacks and Symes showed by using the method of sideways energy estimates that for a linearized density determination problem with constant velocity and plane wave sources, DF is bounded from $H^{1,1}$ to H^1, provided the reference coefficient is in $H^{1,s}$ for some $s > n+2$. They also proved the injectivity of DF. Our techniques and results are quite different from theirs. We intend to assure the optimal regularity of the timelike trace under different hypotheses, which are in some ways weaker.

There remains an extremely important issue to be addressed, namely,

What is an appropriate space for the domain of DF ?

In 1983, Symes suggested that microlocal restrictions on the coefficients might regularize the inverse problem (see [24]). In some sense, this was confirmed by Bao and Symes [2] where we were able to prove a trace theorem for the solutions of general linear *p.d.e.* with smooth coefficients. Roughly speaking, our theorem asserts that the solution will belong to H^s along a codimension one hypersurface if it belongs to H^s in a neighborhood of the hypersurface and to H^{s+1} microlocally in those directions where the *p.d.e.* is not microlocally strictly hyperbolic. Note that we gained back the half derivative from the standard trace theorem. In a recent paper [3], we proved a similar time like trace regularity result for a second order hyperbolic equation with nonsmooth coefficients. It is obvious that the presence of nonsmooth coefficients will introduce new singularities to the solutions so that only limited initial regularity can be propagated. A crucial step in [3] was to develop an extended Beals-Reed theorem (Theorem 1 in [6]) on propagation of singularities.

The main result of this paper is a boundedness theorem for the linearized forward map $DF(\sigma_0)$ for the (sufficiently regular) nonsmooth σ_0. The main ingredients of our proof are the method of energy estimates, a microlocal regularity study of the fundamental solution, results on propagation of singularities, several trace regularity results, and a useful duality technique.

It is known that in their applications to nonlinear wave equations, most of the results based on Rauch's lemma (or the method of Fourier analysis) are limited to relatively weak singularities. This work exhibits that to some extent, strong singularities appearing in the linear wave equation (*e.g.* the fundamental solution) can also be tackled by this Fourier analysis method with the help of a duality argument and the progressing wave expansion. The relation between the coefficients and solution with strong singularities remains to be fully understood, especially when the coefficients are less regular.

The plan of this paper is as follows. In Section 2, a regularity theorem for the solution of the model problem is established. To serve this purpose, a dual problem is introduced. A crucial step is the derivation of an explicit estimate from Hömander's result on propagation of singularities. Section 3 is devoted to the proof of our main theorem. The main ingredients of our proof are: microlocal regularity analysis of the solutions for the transport equations; a microlocal version of the classical trace theorem; and regularity study of the dual problem.

Notation. Throughout this paper, the reader is assumed to be familiar with the basic calculus of *Pseudodifferential Operators* (" ψ.*d.o.* ") as stated in Taylor [26] or Nirenberg [17]. A classical ψ.*d.o.* P of order m is denoted as $P \in OPS^m$ with its symbol $p \in S^m$. $ES(P)$ stands for *the essential support* of operator P. $WF(u)$ denotes *the wave front set* of a distribution u. H^s is the standard L^2-type Sobolev space and H^s_{loc} means a local Sobolev space. $\langle\xi\rangle$ means $(1+|\xi|^2)^{1/2}$. For a nice discussion on microlocal Sobolev spaces $H^s \cap H^r_{m\ell}(x_0, \xi_0)$, we refer the reader to Beals [5], see also Rauch [19]. For simplicity, C serves as a generalized positive constant the precise value of which is not needed.

Warning. When the reference density σ_0 is smooth, most of the regularity results for the forward map in this work will follow more easily from the calculus of *Fourier Integral Operators*. For a standard text on *F. I. O.* we refer to Duistermaat [11] or Hörmander [13]. However, this technique fails with the appearance of the nonsmooth reference density, an assumption important in this work.

2 Regularity of the Fundamental Solution

Since the excess pressure u in the model equation is in fact the fundamental solution, in order to study the regularity of the forward map, the regularity of the fundamental solution must be understood. It is evident that the real obstacle here is the singular right-hand side so that none of the propagation of singularity results could be applied directly. A natural way to cure this difficulty is by employing the Hadamard theory of progressing wave expansion. We refer the reader to Courant and Hilbert [10] or Friedlander [12] for a detail study on the method of progressing wave expansions. According to Hadamard's construction, the fundamental solution may be represented as a sum of the principal part and remainder. One can then study the remainder by the Beals-Reed type propagation of singularity theorem. However, a great drawback of this idea is that additional regularity is needed to regularize the remainder term. In this section, we develop a new approach based on the method of microlocal analysis. A dual problem is introduced so that the regularity study of the fundamental solution may be transformed into regularity study of the dual problem which has a smooth right hand side. In this process, a crucial step is to derive an estimate out of Hörmander's theorem on propagation of singularities. As one might expect, with the presence of nonsmooth coefficients, the Rauch-type results and some commutator results will be demanded.

2.1 Statement of result, preliminaries

Consider a problem obtained by integrating the model problem in the time variable,

$$\begin{aligned} &(\Box - \nabla\sigma_0 \cdot \nabla) v_0 = \delta^{-\frac{n-1}{2}}(t)\delta(x) \ , \quad (x,t) \in \mathbb{R}^{n+1} \\ &v_0 = 0 \ , \quad t < 0 \ . \end{aligned} \tag{2.1}$$

The following is a regularity theorem for the fundamental solution.

Theorem 2.1 *Suppose that $1 + n/2 < s$ and $\sigma_0 \in H^s(\mathbb{R}^n)$. Then for v_0 solving the equation (2.1), $l < s - n/2$*

$$v_0 \in H^l_{loc}(U)$$

where $U = \{\mathbb{R}^n \times (0, T_1)\} \cap \{t > |x|\}$ ($T_1 > 0$). And for $\phi \in C_0^\infty(U)$, the following estimate holds

$$||\phi v_0||_l \le C \tag{2.2}$$

where the constant C depends on ϕ and $||\sigma_0||_s$.

In order to establish Theorem 2.1, we need the following results. The first was originally established by Bony [8] and was extended by Meyer [16]. See also Beals [4] for a different proof.

Proposition 2.1 *Suppose that for some $(x_0, \xi_0) \in T^*(\mathbb{R}^n)\backslash 0$, $u \in H^s \cap H^r_{m\ell}(x_0, \xi_0)$, $n/2 < s \le r \le 2s - n/2$, and $g \in C^\infty$, then*

$$g(x, u) \in H^s \cap H^r_{m\ell}(x_0, \xi_0) \ .$$

We also need a Gårding's type inequality concerning the microlocal ellipticity.

Lemma 2.1 *Assume that $Q_1 \in OPS^{m_1}$, $Q_2 \in OPS^{m_2}$, with $m_1, m_2 \in \mathbb{R}$. Furthermore assume that Q_2 is elliptic on $ES(Q_1)$. Then for any $r \in \mathbb{R}$, Ω and Ω' two open bounded sets of $\mathbb{R}^n$ with $\Omega \subset\subset \Omega'$, and $u \in C_0^\infty(\Omega)$,*

$$||Q_1 u||_{s,\Omega} \le C||Q_2 u||_{s+m_1-m_2,\Omega'} + C||u||_{r,\Omega'} \ .$$

Proof. Let Ω_1 and Ω_2 be open sets with $\Omega \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega'$. Construct a cut-off function $\phi \in C_0^\infty(\Omega')$, $\phi = 1$ on Ω_1, and $\phi = 0$ on $\Omega'\backslash\Omega_2$.

The assumption Q_2 is elliptic on $ES(Q_1)$ implies that a $\psi.d.o$ R, a parametrix of Q_2 on $ES(Q_1)$, may be found such that

$$Q_1 R Q_2 = Q_1 + K \tag{2.3}$$

with K a smoothing operator.

Having defined ϕ, we can now rewrite

$$Q_1 R Q_2 u = Q_1 R \phi Q_2 u + Q_1 R(1-\phi) Q_2 u \ .v$$

It follows that, for any r,

$$\begin{aligned}||Q_1RQ_2u||_{s,\Omega} &\leq ||Q_1R\phi Q_2u||_{s,\Omega} + ||Q_1R(1-\phi)Q_2u||_{s,\Omega} \\ &\leq C||\phi Q_2u||_{s+m_1-m_2,\mathbb{R}^n} + C||(1-\phi)Q_2u||_{r,\Omega'} \\ &\leq C||Q_2u||_{s+m_1-m_2,\Omega'} + C||u||_{r,\Omega'}\ .\end{aligned}$$

On the other hand, from (2.3), it is obvious that

$$||Q_1RQ_2u||_{s,\Omega} \geq ||Q_1u||_{s,\Omega} - ||Ku||_{s,\Omega}\ .$$

A combination of the above discussions will complete the proof. □

Finally, the proof of Theorem 2.1 demands the use of the following two lemmas. Lemma 2.2 gives an estimate based on Nirenberg's proof [17] of Hörmander's theorem which describes the propagation of regularity along bicharacteristics. With nonsmooth coefficients, only a limited amount of regularity propagates. It indicates that an estimate may be derived near any bicharacteristic, hence near the characteristic variety of operator $\Box = \partial_t^2 - \Delta$. We then proceed in Lemma 2.3 to argue that in the elliptic region of the operator $\Box$ an estimate may also be formed. With a concern about the nonsmooth σ_0, it should not be surprising that both results require a commutator argument.

Let $\Pi : T^*(\Omega_0) \to \Omega_0$ denote the projection of $T^*(\Omega_0)$ onto its base space.

Lemma 2.2 *Assume that $s > n/2$, $\sigma_0 \in H^{s+1}(\mathbb{R}^n)$. Suppose that β is a null bicharacteristic of $\Box$, $(x_0, \xi_0) \in \beta$, w is smooth in a neighborhood of x_0, and*

$$\Box w - \nabla\sigma_0 \cdot \nabla w \in L^2(\mathbb{R}^{n+1})$$

vanishes in a neighborhood of $\Pi\beta$. Then there exists a ψ.d.o. B of order zero (essentially supported near β) so that for any $\phi_1 \in C_0^\infty(\mathbb{R}^{n+1})$ and $k < s - n/2 + 2$, there is $C > 0$,

$$||\phi_1 Bw||_k \leq C||\Box w - \nabla\sigma_0 \cdot \nabla w||_0\ .$$

Here C depends on σ_0, k, B, and ϕ_1, but not on w.

Proof. Let Ω be an open bounded set containing $supp(\phi_1)$ and

$$\Box w - \nabla\sigma_0 \cdot \nabla w = f\ .$$

According to Nirenberg's construction, one can find a ψ.d.o. B_0 of order zero with

(1) b_0 supported in a small conic neighborhood of β, B_0 elliptic near β,

(2) $\Pi supp(b_0) \cap supp(f) = \emptyset$, and

(3) $[\Box, B_0] \in OPS^0$.

Since w solves (2.1), the method of energy estimates yields

$$||w||_1 \leq C||f||_0 ,$$

where C is a constant depending on $||\nabla\sigma_0||_{\tilde{s}}$ for $\tilde{s} > n/2$.

Observe that from (2.1),

$$\Box_1' B_0 w = [\Box, B_0]w + [B_0, \nabla\sigma_0 \cdot \nabla]w + B_0 f .$$

Since $\Pi supp(b_0) \cap supp(f) = \emptyset$, we have

$$B_0 f = 0 .$$

Now energy estimates give

$$||B_0 w||_{2,\Omega} \leq C(||[\Box, B_0]w||_{1,\Omega} + ||[B_0, \nabla\sigma_0 \cdot \nabla]w||_{1,\Omega}) . \tag{2.4}$$

Since $[\Box, B_0]$ is of order 0,

$$||[\Box, B_0]w||_{1,\Omega} \leq C||w||_1 \leq C||f||_0 .$$

The third term in (2.4) may be estimated by applying the generalized commutator lemma, Lemma 2.4 in [3] and the corresponding estimate. In fact, let $1 + n/2 < s_0$, we then have

$$||[B_0, \nabla\sigma_0 \cdot \nabla w]||_{1,\Omega} \leq C||w||_1 \leq C||f||_0 ,$$

where C depends on $||\nabla\sigma_0||_{s_0}$.

Thus

$$||B_0 w||_{2,\Omega} \leq C_0||f||_0 , \tag{2.5}$$

with C_0 depending on $||\nabla\sigma_0||_{s_0}$.

Applying Nirenberg's construction once again, we can find a $\psi.d.o.$ B_1 of order zero such that $ES(B_1) \subset ES(B_0)$(strictly), B_1 also has properties (1) and (2) above; moreover $[\Box, B_1] \in OPS^{-1}$ and B_0 is elliptic near $ES(B_1)$. From (2.1) and $B_1 f = 0$,

$$\Box_1' B_1 w = [\Box, B_1]w + [B_1, \nabla\sigma_0 \cdot \nabla]w .$$

If we write down the energy estimates, after a simple $\psi.d.o.$ cut-off on B_1, we will find

$$||B_1 w||_{3,\Omega}^2 \leq C||w||_1^2 + C||A_1[B_1, \nabla\sigma_0 \cdot \nabla]w||_{2,\Omega}||B_1 w||_{3,\Omega} ,$$

where $A_1 \in OPS^0$, $ES(B_1) \subset ES(A_1) \subset ES(B_0)$, B_0 is elliptic on $ES(A_1)$, and $a_1 = 1$ on $ES(B_1) \cap \{(x,\xi), |\xi| \geq 1\}$.

Now since $w \in H^1 \cap H^2_{m\ell}(ES(B_0))$, Lemma 2.4 in [3] again implies that $[B_1, \nabla\sigma_0 \cdot \nabla]w \in H^1 \cap H^2_{m\ell}(ES(A_1))$ and

$$||A_1[B_1, \nabla\sigma_0 \cdot \nabla]w||_{2,\Omega} \leq C(||w||_1 + ||A_1 w||_{2,\Omega}) .$$

Here C depends on $||\nabla\sigma_0||_{s_1}$ for $2 + n/2 \leq s_1$.

Because of our construction, B_0 is elliptic on $ES(A_1)$; therefore Gårding's type inequality Lemma 2.1 leads to, for any real r and $\Omega \subset\subset \Omega_1$

$$||A_1 w||_{2,\Omega} \leq C||B_0 w||_{2,\Omega_1} + C||w||_r \leq C||f||_0$$

by (2.5).

Therefore we have shown that

$$||B_1 w||_{3,\Omega} \leq C_1||f||_0 ,$$

where C_1 depends on $||\nabla\sigma_0||_{s_1}$.

We can continue this process by constructing a sequence of $\psi.d.o.$ B_i and A_i ($i = 1, \cdots, k-2$), such that

- B_i has properties (1), (2), and $[\Box, B_i] \in OPS^{-i}$,
- $ES(B_{i-1}) \subset ES(A_{i-1}) \subset ES(B_i)$, and
- B_i is elliptic on $ES(A_{i-1})$, $a_{i-1} = 1$ on $ES(B_{i-1}) \cap \{(x,\xi), |\xi| \geq 1\}$,
- Also

  $$||B_i w||_{i+2,\Omega} \leq C_i||f||_0 ,$$

  where C_i depends on $||\nabla\sigma_0||_{s_i}$ for $i + n/2 < s_i$.

Eventually we conclude by choosing $B = B_{k-2}$ so that, for $k - 2 + n/2 < s$,

$$||Bw||_{k,\Omega} \leq C||f||_0$$

with C depending on $||\nabla\sigma_0||_s$. □

The proof of Theorem 2.1 requires a slightly different form of Lemma 2.2.

Corollary 2.1 *Assume that $s > n/2$, $\sigma_0 \in H^{s+1}(\mathbb{R}^n)$. Suppose that γ is a set of null bicharacteristics of $\Box$, $(x_0,\xi_0) \in \gamma$, w is smooth in a neighborhood of x_0, and*

$$\Box w - \nabla\sigma_0 \cdot \nabla w \in L^2(\mathbb{R}^{n+1})$$

vanishes in a neighborhood of $\Pi\gamma$. Then there exists a $\psi.d.o.$ Q of order zero (essentially supported near γ) so that for any $\phi_1 \in C_0^\infty(\mathbb{R}^{n+1})$ and $k < s - n/2 + 2$, there is $C > 0$,

$$||\phi_1 Qw||_k \leq C||\Box w - \nabla\sigma_0 \cdot \nabla w||_0 .$$

Here C depends on σ_0, k, Q, and ϕ_1, but not on w.

Proof. For every null bicharacteristic of the set γ, Lemma 2.2 indicates that a $\psi.d.o.$ B of order zero may be found so that

$$||\phi_1 Bw||_k \leq C||\Box w - \nabla\sigma_0 \cdot \nabla w||_0 .$$

Now Q may be constructed as $Q = \sum B$. Moreover, the local compactness of the unit sphere ensures that the summation is finite. □

Lemma 2.3 *Assume that $s > n/2$, $\sigma_0 \in H^{s+1}(\mathbb{R}^n)$. Suppose that P is a ψ.d.o. of order zero such that a conic neighborhood of its essential support is contained in the elliptic region of the wave operator $\Box$. Assume also that*

$$\Box w - \nabla\sigma_0 \cdot \nabla w \in L^2(\mathbb{R}^{n+1})$$

vanishes in a neighborhood of Πp. Then for any $\phi_1 \in C_0^\infty(\mathbb{R}^{n+1})$ and $k < s - n/2 + 2$, there is $C > 0$ so that

$$||\phi_1 P w||_k \le C||\Box w - \nabla\sigma_0 \cdot \nabla w||_0 \, .$$

Here the constant C depends on σ_0, k, P, and ϕ_1, but not on w.

Proof. The proof is based on the same type of bootstrap arguments as in the proof of last lemma.

Let

$$\Box w - \nabla\sigma_0 \cdot \nabla w = f \, . \tag{2.6}$$

Assume that $\Omega \supset supp(\phi_1)$ be a bounded open set. From the support assumption on p, we see that $Pf = 0$. Hence, by applying P to both sides of (2.6), we find

$$\Box P w = [\Box, P]w + [P, \nabla\sigma_0 \cdot \nabla]w + \nabla\sigma_0 \cdot \nabla P w \, . \tag{2.7}$$

Now since $\Box$ is elliptic in a small conic neighborhood of $ES(P)$, there exists a ψ*.d.o.* P_0 of order zero, such that $ES(P) \subset ES(P_0)$, P_0 is elliptic near $ES(P)$, and $\Box$ is elliptic in a small conic neighborhood of $ES(P_0)$. From the ellipticity of $P_0\Box$ on $ES(P)$, Lemma 2.1 gives, for any real number r and $\Omega \subset\subset \Omega'$,

$$||Pw||_{k,\Omega} \le C||P_0 \Box P w||_{k-2,\Omega'} + C||w||_r \, ,$$

or from (2.7)

$$||Pw||_{k,\Omega} \le C(||P_0[\Box, P]w||_{k-2,\Omega'} + ||P_0[P, \nabla\sigma_0 \cdot \nabla]w||_{k-2,\Omega'} + ||P_0 \nabla\sigma_0 \cdot \nabla P w||_{k-2,\Omega'}) \, .$$

Therefore an application of Lemma 2.4 and the generalized Rauch's lemma in [3] yields

$$\begin{aligned} ||Pw||_{k,\Omega} &\le C_1||P_0 w||_{k-1,\Omega'} + C_2(||w||_1 + ||P_0 w||_{k-2,\Omega'}) \\ &\quad + C_3(||w||_1 + ||P_0 w||_{k-1,\Omega'}) \\ &\le C||f||_0 + C||P_0 w||_{k-1,\Omega'} \, . \end{aligned}$$

Here constants C_2 and C_3 depend on $||\nabla\sigma_0||_s$ for $k - 2 + n/2 < s$.

Thus the bootstrap arguments on P_0 will accomplish the proof. $\Box$

2.2 Proof of Theorem 2.1

We study the regularity of v_0 through its dual problem. To simplify the arguments on the dual problem, we make use of the symmetric form of (2.1) by introducing $\rho(x) = e^{-\sigma_0}$. Then (2.1) becomes

$$\begin{aligned} \Box_1 v_0 = [\frac{1}{\rho}\partial_t^2 - \nabla\cdot(\frac{1}{\rho}\nabla)]v_0 = \frac{1}{\rho}\delta^{-\frac{n-1}{2}}(t)\delta(x) \\ v_0 = 0 \quad t < 0\ . \end{aligned} \tag{2.8}$$

Now let us look at a dual problem to (2.8),

$$\begin{aligned} \Box_1 w = [\frac{1}{\rho}\partial_t^2 - \nabla\cdot(\frac{1}{\rho}\nabla)]w = \Psi \\ w = 0 \quad t >> T_1\ , \end{aligned} \tag{2.9}$$

where $\Psi \in C_0^\infty(\Omega)$ with $\Omega = \{\mathbb{R}^n \times (0, T_1)\} \cap \{t > |x| + \epsilon_0\}$, for $\epsilon_0 > 0$ small. Note that, this equation may be reformulated as

$$\begin{aligned} \Box_1' w = \Box w - \nabla\sigma_0\cdot\nabla w = e^{-\sigma_0}\Psi \\ w = 0 \quad t >> T_1\ . \end{aligned} \tag{2.10}$$

Thus if we can show that for any $\Psi \in C_0^\infty(\Omega)$

$$|(\partial_t^l v_0, \Psi)| \le C||\Psi||_0\ , \tag{2.11}$$

then it can be concluded that

$$||\partial_t^l v_0||_{0,\Omega} \le C\ . \tag{2.12}$$

From (2.8), integration by parts leads

$$\begin{aligned} |(\partial_t^l v_0, \Psi)| &= |(\Box_1 \partial_t^l v_0, w)| \\ &= |(\frac{1}{\rho}\delta(t)\delta(x), \partial_t^{l-(n-1)/2} w)| \\ &\le C|(\partial_t^{l-(n-1)/2} w)(0,0)|\ . \end{aligned}$$

The trace theorem (see for example [26]) yields that

$$|(\partial_t^l v_0, \Psi)| \le C||\phi_1 w||_{l+1} \tag{2.13}$$

with $\phi_1 \in C_0^\infty(\Omega_1)$, Ω_1 a small neighborhood of the origin and $\Omega_1 \cap supp(\Psi) = \emptyset$.

Construct two $\psi.d.o.$ $Q_1, Q_2 \in OPS^0(\mathbb{R}^{n+1})$, such that

- $Q_1 + Q_2 = R$; where R is an elliptic $\psi.d.o.$ of order zero in Ω_1;
- $\Pi supp(q_i) \cap supp(\Psi) = \emptyset$, for $i = 1, 2$;
- $ES(Q_2)$ is a small conic neighborhood of set of null bicharacteristics of the wave operator $\Box$ passing over Ω_1;

- Q_1 is microlocally smoothing on the null bicharacteristics passing over Ω_1.

Therefore, with (2.13), we have

$$|(\partial_t^l v_0, \Psi)| \leq C||Q_1\phi_1 w||_{l+1,\Omega_1} + C||Q_2\phi_2 w||_{l+1,\Omega_1} \tag{2.14}$$

here the expression makes sense because the domain of dependence for w and the pseudo-local properties of Q_1 and Q_2.

Now, we can apply Corollary 2.1 to obtain that

$$||Q_2 w||_{l+1,\Omega_1} \leq C||\Psi||_0 \,. \tag{2.15}$$

Lemma 2.3 yields

$$||Q_2 w||_{l+1,\Omega_1} \leq C||\Psi||_0 \tag{2.16}$$

where the constants here depend on $||\sigma_0||_s$.

Therefore, we have shown

$$|(\partial_t^l v_0, \Psi)| \leq C||\phi_1 w||_{l+1} \leq C||\Psi|| \,. \tag{2.17}$$

To complete the proof, we have one more step to go, that is, to estimate all the x-derivatives as well as the corresponding mixed derivatives of v_0 up to order of l. This may be done by introducing another dual problem. Here, we demonstrate this method by looking at $\partial_{x_i}^l v_0$ $(i = 1, \cdots, n)$. The rest terms can be estimated by the same fashion. It suffices to show that for any $\Psi \in C_0^\infty(\Omega)$

$$|(\partial_{x_i}^l v_0, \Psi)| \leq C||\Psi||_0 \tag{2.18}$$

for $i = 1, \cdots, n$.

Let w solve the dual problem (2.9), w_1 solve the following problem

$$\begin{aligned} \Box_1 w_1 = [\frac{1}{\rho}\partial_t^2 - \nabla\cdot(\frac{1}{\rho}\nabla)]w_1 = [\partial_{x_i}^l, \Box_1]w \\ w = 0 \quad t >> T_1 \,. \end{aligned} \tag{2.19}$$

We have

$$\begin{aligned} |(\partial_{x_i}^l v_0, \Psi)| &= |(v_0, \partial_{x_i}^l \Box_1 w)| \\ &\leq |(\Box_1 v_0, \partial_{x_i}^l w)| + |(\Box_1 v_0, w_1)| \\ &\leq C|(\partial_{x_i}^l \partial_t^{-(n-1)/2} w)(0,0)| + C|(\partial_t^{-(n-1)/2} w_1)(0,0)| \,. \end{aligned}$$

Again, the trace theorem gives

$$\begin{aligned} |(\partial_{x_i}^l v_0, \Psi)| &\leq C||\phi_1 w||_{l+1} + C||\phi_1 w_1||_1 \\ &\leq C||\phi_1 w||_{l+1} \end{aligned}$$

where in order to obtain the second inequality, we have applied the standard energy estimates and a commutator argument. From (2.17), the estimate (2.18) can then be proved. $\square$

3 Proof of the Main Theorem

Our goal in this section is to determine the appropriate hypotheses under which $DF(\sigma_0)$, the linearization of F about a reference state σ_0, is bounded. Our proof is based on the microlocal regularity analysis of the transport equations and regularity study of the dual problem.

Recall the linearized problem corresponding to the reference state (u_0, σ_0), for $(t, x) \in \mathbb{R}^{n+1}$, $x = (x', x_n)$,

$$\begin{aligned} (\Box - \nabla\sigma_0 \cdot \nabla)\delta u &= \nabla\delta\sigma \cdot \nabla u_0 \\ \delta u = 0 \ , &\quad t < 0 \ , \end{aligned} \tag{3.1}$$

where u_0 is the solution of the model problem corresponding to the reference density σ_0. The linearized forward map can be defined as

$$DF(\sigma_0)\delta\sigma = (\phi\delta u)\mid_{x_n=0} \ , \tag{3.2}$$

where $\phi(x,t) \in C_0^\infty(\mathbb{R}^{n+1})$ is supported inside the conoid $\{t > |x|\}$, and near $\{x_n = 0\}$.

Once again we consider a related problem,

$$\begin{aligned} (\Box - \nabla\sigma_0 \cdot \nabla)\ v &= \nabla\delta\sigma \cdot \nabla v_0 \\ v = 0 \ , &\quad t < 0 \ , \end{aligned} \tag{3.3}$$

where $\delta u = \partial_t^{\frac{n-1}{2}} v$ and v_0 solves

$$\begin{aligned} (\Box - \nabla\sigma_0 \cdot \nabla)v_0 &= \delta^{-\frac{n-1}{2}}(t)\delta(x) \\ v_0 = 0 \ , &\quad t < 0 \ . \end{aligned} \tag{3.4}$$

Observe that for $l \in \mathbb{R}$,

$$\begin{aligned} ||DF(\sigma_0)\delta\sigma||_l &= ||\ (\phi\delta u)\mid_{x_n=0}||_l \\ &\leq C||(\phi v)\mid_{x_n=0}||_{l_1}, \end{aligned} \tag{3.5}$$

where l_1 denotes $l + (n-1)/2$. Thus the real challenge here is to get an appropriate trace regularity estimate for v on a time-like hypersurface $\{x_n = 0\}$.

Before getting into the details of the proof, let us first make the following general remarks on this theorem:

The estimate (1.8) has a similar form to a Rakesh's theorem (Theorem 2.5 in [18]). Actually, we conjecture that a formal extension of our proof here could lead to an elementary proof of his theorem. On the contrary, the principal tool of Rakesh's proof, calculus of Fourier integral operators, is not available when the reference density is nonsmooth.

Our approach enjoys the beauty of the method of energy estimates, that is, it possesses useful information on various parameters involved in the estimates.

3.1 Some useful results

Hadamard's construction leads to the progressing wave expansion for v_0,

$$v_0 = \sum_{k=0}^{s} b_k S_k(t - \tau(x)) + R_{v_0}(x,t) \tag{3.6}$$

where $\tau(x) = |x|$, S_0 is the Heaviside function, $S_k' = S_{k-1}$ $(k \geq 1)$, and R_{v_0} vanishes at $t = \tau(x)$. Moreover $\{b_k\}$ solve the transport equations, for $k = 1, \cdots, s$,

$$\begin{aligned} 2\nabla\tau \cdot \nabla b_0 + (\Delta\tau + \nabla\tau \cdot \nabla\sigma_0)b_0 &= 0 &(3.7)\\ 2\nabla\tau \cdot \nabla b_k + (\Delta\tau + \nabla\tau \cdot \nabla\sigma_0)b_k &= \Delta b_{k-1} + \nabla\sigma_0 \cdot \nabla b_{k-1} . &(3.8)\end{aligned}$$

From (3.3), Hadamard's construction also gives the progressing wave expansion of v,

$$v = \sum_{k=0}^{s} a_k S_k(t - \tau(x)) + R_v(x,t) , \tag{3.9}$$

where $\tau(x) = |x|$, S_0 is the Heaviside function, $S_k' = S_{k-1}$, R_v vanishes at $t = \tau(x)$, and $\{a_k\}$ solve the transport equations, for $k = 0, \cdots, s-1$,

$$2\nabla\tau \cdot \nabla a_0 + (\Delta\tau + \nabla\tau \cdot \nabla\sigma_0)a_0 = -b_0\nabla\tau \cdot \nabla\delta\sigma \tag{3.10}$$

$$2\nabla\tau \cdot \nabla a_{k+1} + (\Delta\tau + \nabla\tau \cdot \nabla\sigma_0)a_{k+1} = \Delta a_k + \nabla\sigma_0 \cdot \nabla a_k + \nabla\delta\sigma(\nabla b_k - b_{k+1}\nabla\tau) . \tag{3.11}$$

In fact, the equations (3.10) and (3.11) are the first order perturbation of the equations (3.7) and (3.8), respectively.

For convenience, we introduce a function $q = \sigma_0/2 + q_0$ with $\nabla\tau \cdot \nabla q_0 = \Delta\tau/2$. Thus away from the origin, q is nothing more than a smooth perturbation of $\sigma_0/2$. Then the transport equations (3.7), (3.8) may be transformed to equations

$$\begin{aligned} \nabla\tau \cdot \nabla b_0 e^q &= 0 &(3.12)\\ \nabla\tau \cdot \nabla b_k e^q &= (\Delta b_{k-1}/2 + \nabla\sigma_0 \cdot \nabla b_{k-1}/2)e^q , &(3.13)\end{aligned}$$

for $k = 1, \cdots, l_1$.

Similarly, equations (3.10) and (3.11) may be transformed to equations

$$\begin{aligned} \nabla\tau \cdot \nabla a_0 e^q &= \nabla\tau \cdot \nabla(\delta\sigma b_0 e^q) &(3.14)\\ \nabla\tau \cdot \nabla a_k e^q &= (\Delta a_{k-1}/2 + \nabla\sigma_0 \cdot \nabla a_{k-1}/2)e^q + \nabla\delta\sigma(\nabla b_k - b_{k+1}\nabla\tau)e^q/2 , &(3.15)\end{aligned}$$

for $k = 1, \cdots, l_1$, where to obtain the first equation, we have used the equation $\nabla\tau \cdot \nabla b_0 e^q = 0$.

Observation. The right hand side of the equation (3.13) may be rewritten in terms of $b_{k-1}e^q$ as

$$\Delta(b_{k-1}e^q)/2 - |\nabla q|^2 b_{k-1}e^q/2 - b_{k-1}e^q\Delta q/2 - \nabla(b_{k-1}e^q) \cdot \nabla q_0 + b_{k-1}e^q\nabla q \cdot \nabla q_0 . \tag{3.16}$$

Similar observation may be made for the right hand side of (3.15).

Observe that all the transport equations have the same principal part $\nabla\tau \cdot \nabla$ which is a smooth vector field. Therefore, in order to understand the regularity of the solutions to (3.12)-(3.15) it is essential to study the properties of this smooth vector field. The following is a microlocal regularity result for the solutions of transport equations. It indicates that a refined regularity result can be achieved microlocally away from the characteristic variety of the wave operator, in contrast to a direct application of the energy estimates. This result will be used frequently throughout the proof of Theorem 1.1.

Lemma 3.1 *Let V be a smooth vector field. Assume that u is a smooth function, $Q \in OPS^0$ supported away from $Char(V)$, and $\phi \in C_0^\infty(\Omega)$ where Ω is a bounded open set. Then there exist $Q' \in OPS^0$ also supported away from $Char(V)$ and ϕ', $\bar{\phi} \in C_0^\infty(\Omega')$ with $\Omega \subset\subset \Omega'$, such that for any $r \in \mathbb{R}$,*

$$||Q\phi u||_{s,\Omega} \leq C||Q'\phi' Vu||_{s-1,\Omega'} + C||\bar{\phi}u||_{r,\Omega'} \, .$$

Proof. Let Q_1 be another $\psi.d.o.$ of order zero, such that Q_1 is elliptic on $ES(Q)$ and $ES(Q_1)$ is away from $Char(V)$. It follows from Lemma 2.1 that there exists $\Omega_1 \supset\supset \Omega$ an open bounded set such that

$$\begin{aligned} ||Q\phi u||_{s,\Omega} &\leq C||Q_1 V\phi u||_{s,\Omega_1} + C||\phi u||_{r,\Omega_1} && (3.17)\\ &\leq C||Q_1[V,\phi]u||_{s-1,\Omega_1} + C||Q_1\phi Vu||_{s-1,\Omega_1} + C||\phi u||_{r,\Omega_1} \, . && (3.18)\end{aligned}$$

Now choose $\phi_1 \in C_0^\infty(\Omega_1)$ and $\phi_1 = 1$ on $supp(\phi)$, then

$$Q_1[V,\phi]u = Q_1[V,\phi]\phi_1 u = \tilde{Q}_1\phi_1 u$$

where $\tilde{Q}_1 = Q_1[V,\phi]$ is a $\psi.d.o.$ of order zero and $ES(\tilde{Q}_1)$ is away from $Char(V)$.

Thus

$$||Q\phi u||_{s,\Omega} \leq C||\tilde{Q}_1\phi_1 u||_{s-1,\Omega_1} + C||Q_1\phi Vu||_{s-1,\Omega_1} + C||\phi u||_{r,\Omega_1} \, .$$

We can then repeat the above process to get a similar estimate on $||\tilde{Q}_1\phi_1 u||_{s-1,\Omega_1}$. In fact, one can construct: $Q_2 \in OPS^0$, Q_2 is elliptic on $ES(\tilde{Q}_1)$ and $ES(Q_2)$ is away from $Char(V)$; $\phi_2 \in C_0^\infty(\Omega_2)$, $\phi_2 = 1$ on $supp(\phi)$, $\Omega_1 \subset\subset \Omega_2$; $\tilde{Q}_2 = Q_2[V,\phi_1]$.

Then

$$||\tilde{Q}_1\phi_1 u||_{s-1,\Omega_1} \leq C||\tilde{Q}_2\phi_2 u||_{s-2,\Omega_2} + C||Q_2\phi_1 Vu||_{s-2,\Omega_2} + C||\phi_1 u||_{r,\Omega_2} \, .$$

Following the same idea, one may keep going to get similar estimates: (the i-th step) Construct: $Q_i \in OPS^0$, Q_i is elliptic on $ES(\tilde{Q}_{i-1})$ and $ES(Q_i)$ is away from $Char(V)$; $\phi_i \in C_0^\infty(\Omega_i)$, $\phi_i = 1$ on $supp(\phi_{i-1})$, $\Omega_{i-1} \subset\subset \Omega_i$; $\tilde{Q}_i = Q_i[V,\phi_i]$, $0 \leq i \leq k-1$. (k is chosen to satisfy $s - k \leq r$).

Then

$$||\tilde{Q}_i\phi_i u||_{s-i,\Omega_i} \leq C||\tilde{Q}_{i+1}\phi_{i+1} u||_{s-i-1,\Omega_{i+1}} + C||Q_{i+1}\phi_i Vu||_{s-i-1,\Omega_{i+1}} + C||\phi_i u||_{r,\Omega_{i+1}} \, .$$

Combining all of the estimates above, we have

$$||Q\phi u||_{s,\Omega} \leq C||\tilde{Q}_k\phi_k u||_{s-k,\Omega_k} + C\sum_{i=0}^{k-1}||Q_{i+1}\phi_i Vu||_{s-i-1,\Omega_{i+1}2} + C\sum_{i=0}^{k-1}||\phi_i u||_{r,\Omega_{i+1}} .$$

In order to get the desired estimate, some further simplifications are needed. Clearly, for $j = 0, \cdots, k-2$,

$$||\phi_j u||_{r,\Omega_{j+1}} \leq C||\phi_{j+1}u||_{r,\Omega_{j+2}} ,$$

hence

$$\sum_{i=0}^{k-1} C||\phi_i u||_{r,\Omega_{i+1}} \leq C||\phi_{k-1}u||_{r,\Omega_k} .$$

Since $s-k \leq r$, $||\tilde{Q}_k\phi_k u||_{s-k,\Omega_k} \leq C||\phi_k u||_{r,\Omega_k}$, therefore,

$$||Q\phi u||_{s,\Omega} \leq C||\phi_k u||_{r,\Omega_k} + \sum_{i=0}^{k-1}||Q_{i+1}\phi_i Vu||_{s-i-1,\Omega_{i+1}} .$$

Construct a $\psi.d.o.$ of order zero, Q', with the following properties:

Q' is elliptic on $\cup_{i=0}^{k-1} ES(Q_{i+1}\phi_i)$; $ES(Q')$ is away from $Char(V)$. Lemma 2.1 then yields that

$$\begin{aligned} ||Q_{i+1}\phi_i Vu||_{s-1,\Omega_k} &= ||Q_{i+1}\phi_i\phi_k Vu||_{s-1,\Omega_k} && (3.19)\\ &\leq C||Q'\phi_k Vu||_{s-1,\Omega'} + C||\phi_k Vu||_{r-1,\Omega'} && (3.20)\end{aligned}$$

where $\Omega_k \subset\subset \Omega'$, $i = 0, 1, \cdots, k-1$. Therefore, if we construct $\bar{\phi} \in C_0^\infty(\Omega')$ in such a way that $ar\phi > 0$ on $supp(\phi_k)$, then

$$||\phi_k Vu||_{r-1,\Omega'} \leq C||\bar{\phi}u||_{r,\Omega'} .$$

Let ϕ' be ϕ_k, we finally obtain

$$||Q\phi u||_{s,\Omega} \leq C||Q'\phi' Vu||_{s-1,\Omega'} + C||\bar{\phi}u||_{r,\Omega'}$$

which completes our proof. □

We also need the following standard result for hyperbolic *p.d.e*, as well as the estimates involving in its proof. See, for example, Chazarain-Piriou [9] for the idea of the proof. The following is the version stated in Beals [5].

Lemma 3.2 *(Linear Energy Inequality) Let $p(x, D)$ be a partial differential operator of order m on $\mathbb{R}^{n+1}$, strictly hyperbolic with respect to the plane $\{x_{n+1} = 0\}$, and let u satisfy $p(x, D)u = f(x)$. If $f \in H_{loc}^{s-m+1}(\mathbb{R}^{n+1})$ and $u \in H_{loc}^s(x : |x_{n+1}| \leq \epsilon)$ for some $\epsilon > 0$, then $u \in H_{loc}^s(\mathbb{R}^{n+1})$.*

Next, we present a trace regularity result.

Lemma 3.3 *Assume that $s > 3 + n/2$, $1 \leq l_1 \leq s$, and v solves problem (3.3) then there is a $\phi_0 \in C_0^\infty$ supported near $supp(\phi)$ such that the following estimate holds,*

$$||(\phi v)|_{x_n=0}||_{l_1} \leq C||\phi_0 v||_{l_1} , \tag{3.21}$$

where C is a constant depending on the $H^s \cap H^{l_1+1}_{m\ell}(K)$-norm of σ_0, but is independent of $\delta\sigma$.

Proof. This lemma is a direct application of Theorem 3.1 in [3] by taking into account of the fact that ϕ and $\delta\sigma$ have disjoint supports. □

3.2 Microlocal version of trace theorem

The classical trace theorem in Sobolev spaces characterizes the regularity of a distribution restricted to a hypersurface. Dealing with inverse problems, one always has to face a difficult but crucial question: When does the restriction operator commute with another operator of interest? The result in this subsection indicates that a simple microlocal trace theorem, which not only works on the space restriction but also on the phase restriction (*i.e.* a trace theorem on cotangent bundles), may lead to a way to cure this difficulty.

Let K be a conic set in $\mathbb{R}^n$, $i : x \in \mathbb{R}^n \to (x, 0) \in \mathbb{R}^{n+1}$. Define a semi-norm: for γ a conic set of $\mathbb{R}^k$ and $u \in C_0^\infty(\mathbb{R}^k)$,

$$|u|_{\gamma,s} = \left(\int_{\xi\in\gamma} d\xi |\hat{u}(\xi)|^2 \langle\xi\rangle^{2s}\right)^{1/2} .$$

Then, a proof of the classical trace theorem (see *e.g.* in Taylor [26], pages 20-21) implies the following inequality.

Proposition 3.1 *For $s > 1/2$, $u \in C_0^\infty(\mathbb{R}^{n+1})$,*

$$|i^* u|_{K,s-1/2} \leq C|u|_{K\times\mathbb{R},s} .$$

Thus the map i^* may be extended to be a bounded map from $H^s_{m\ell}(x \times \mathbb{R}, K \times \mathbb{R})$ to $H^{s-1/2}_{m\ell}(x, K)$, provided $s > 1/2$.

Let Π_2 be the projection map to the frequency space (or the second factor). We may reformulate this result in terms of $\psi.d.o.$.

Proposition 3.2 *If P_1 is a $\psi.d.o.$ of order zero in $\mathbb{R}^n$, with $\Pi_2 ES(P_1) \subset K$, then there exists a $\psi.d.o.$ P_2 of order zero in $\mathbb{R}^{n+1}$, and $\Pi_2 ES(P_2) \subset K \times \mathbb{R}$, such that for $s > 1/2$, $u \in C_0^\infty(\Omega)$ with Ω an open bounded subset of $\mathbb{R}^{n+1}$,*

$$||P_1 i^* u||_{s-1/2,\Omega_0} \leq C||P_2 u||_{s,\Omega} ,$$

where i^* again denotes a restriction operator to a codimension one hypersurface and $\Omega_0 = i^*\Omega$.

The above results together with our Gårding's type result Lemma 2.1 yield a microlocal version of trace theorem.

Lemma 3.4 *Assume that E is an elliptic operator of order m in $\mathbb{R}^{n+1} \times K \times \mathbb{R}$, $P \in OPS^0(\mathbb{R}^n)$ and $\Pi_2 ES(P) \subset K$. Then for $s > 1/2$, $u \in C_0^\infty(\Omega)$ where Ω and Ω' are open bounded subsets of $\mathbb{R}^{n+1}$ with $\Omega \subset\subset \Omega'$, and $\Omega_0 = i^*\Omega$,*

$$||Pi^*u||_{s-1/2,\Omega_0} \leq C||Eu||_{s-m,\Omega'} + C||u||_{r,\Omega'}$$

for any $r \in \mathbb{R}$.

A result on propagation of singularities, see Proposition 1.3.3 in Duistermaat [11] or Theorem 8.2.13 in Hörmander [14], demonstrates the relation between the wavefront of the restriction of a distribution and the wavefront set of its own. However the result does not directly lead to any explicit bound. Here, in this direction, we present a result together with an estimate. The proof follows directly from Lemma 2.2 and Lemma 2.3.

Lemma 3.5 *Assume that $s > n/2$, $\sigma_0 \in H^{s+1}(\mathbb{R}^n)$. Suppose that w is smooth near $\{t = 0\}$ and $\Box w - \nabla\sigma_0 \cdot \nabla w \in L^2(\mathbb{R}^{n+1})$. Then there exists an elliptic ψ.d.o. $\tilde{B}$ of order zero, such that $ES(\tilde{B})$ is contained in Cy, a "cylindrical" conic neighborhood of*

$$\{(x, t, \xi, \omega) \in T^*\mathbb{R}^{n+1}\backslash 0,\ t^2 - |x|^2 = 0, \omega = \nabla\tau \cdot \xi\}$$

along ω direction, and the symbol of $\tilde{B}$, $\tilde{b}$ satisfies

$$\Pi supp(\tilde{b}) \cap supp(\Box w - \nabla\sigma_0 \cdot \nabla w) = \emptyset\ .$$

Then, for any $\phi_1 \in C_0^\infty(\mathbb{R}^{n+1})$ and $k < s - n/2 + 2$,

$$||\phi_1 \tilde{B} w||_k \leq C||\Box w - \nabla\sigma_0 \cdot \nabla w||_0\ , \tag{3.22}$$

where the constant C depends on σ_0, k, $\tilde{B}$, and ϕ_1, but not on w.

3.3 Proof of Theorem 1.1

Recall the assumption made in Section 1,

(A) $supp(\delta\sigma) \subset \{x_n > \epsilon\}$,

for $\epsilon > 0$ small.

For simplicity, we shall also assume that l_1 is an integer. Without any further difficulty, the proof may be extended formally to cover the general case.

Let $\phi \in C_0^\infty$ be supported inside the characteristic surface and the set $\{x_n < \epsilon/2\}$. Multiplying ϕ to both sides of equation (3.3), we have

$$\begin{aligned} \Box\phi v &= \phi\nabla\sigma_0 \cdot \nabla v + [\Box, \phi]v \\ v &= 0\ ,\ \ t < 0\ . \end{aligned} \tag{3.23}$$

Here we have used the fact that according to the assumption (A), ϕ and $\delta\sigma$ have disjoint supports, so that $\phi\nabla\delta\sigma\cdot\nabla v_0 = 0$.

Having Lemma 3.3, the estimate of $||(\phi v)|_{x_n=0}||_{l_1}$ may be reduced to estimating $||\phi_0 v||_{l_1}$. For the sake of simplicity, we shall only discuss the estimates of the t-derivatives of v. The rest may be estimated as in the proof of Theorem 2.1.

The rest of this section is devoted to the estimate of $||\phi_0 v||_{l_1}$.

We study the regularity of v through its dual problem. Once again, we look at the symmetric form, for $\rho(x) = e^{-\sigma_0}$.

$$\begin{aligned} \Box_1 v = [\frac{1}{\rho}\partial_t^2 - \nabla\cdot(\frac{1}{\rho}\nabla)]v_0 = \frac{1}{\rho}\nabla\delta\sigma\cdot\nabla v_0 \\ v = 0 \quad t < 0\ . \end{aligned} \tag{3.24}$$

A dual problem to (3.24),

$$\begin{aligned} \Box_1' w = [\frac{1}{\rho}\partial_t^2 - \nabla\cdot(\frac{1}{\rho}\nabla)]w = \Psi \\ w = 0 \quad t >> T_1\ , \end{aligned} \tag{3.25}$$

where $\Psi \in C_0^\infty(\Omega)$ with $\Omega = \{\mathbb{R}^n \times (0, T_1)\} \cap \{t > |x| + \epsilon\}$, for a small $\epsilon > 0$.

Thus if we can show that for any $\Psi \in C_0^\infty(\Omega)$

$$|(\partial_t^{l_1} v_2, \Psi)| \le C||\delta\sigma||_{l_1}\ ||\Psi||_0\ , \tag{3.26}$$

then it can be concluded that

$$||\partial_t^{l_1} v_2||_{0,\Omega} \le C||\delta\sigma||_{l_1}\ . \tag{3.27}$$

Green's identity and integration by parts lead to

$$\begin{aligned} &(\partial_t^{l_1} v, \Psi) = (\partial_t^{l_1} v, \Box_1' w) \\ &= (\Box_1\partial_t^{l_1} v, w) - 2\int_{t=\tau(x)} \frac{1}{\rho}[\partial_t^{l_1} v\frac{\partial}{\partial n}w - w\frac{\partial}{\partial n}\partial_t^{l_1} v]ds \end{aligned} \tag{3.28}$$

Let us look at the first term. From equation (3.3),

$$(\Box_1\partial_t^{l_1} v, w) = (\nabla\delta\sigma\cdot\nabla\partial_t^{l_1} v_0, w)\ .$$

Thus

$$|(\Box_1\partial_t^{l_1} v, w)| \le C||\delta\sigma||\ ||\partial_t^{l_1} v_0||_{0,\Omega}\ ||w||_{1,\Omega}\ .$$

Applying Theorem 2.1 and energy estimates to the right hand side, we have

$$|(\Box_1\partial_t^{l_1} v, w)| \le C||\delta\sigma||\ ||\Psi|| \tag{3.29}$$

with the constant C depending on $||\sigma_0||_s$, for $s > max\{1 + n/2, l_1 + n/2\}$.

Using Hardamard's construction, one may get:
$(\partial_t^{l_1} v)|_{t=\tau(x)} = a_{l_1}$, $(\partial_t^{l_1+1} v)|_{t=\tau(x)} = a_{l_1+1}$, $(\partial_x\partial_t^{l_1} v)|_{t=\tau(x)} = \partial_x a_{l_1} + a_{l_1+1}$.

Denote

$$I_1 = \int_{t=\tau(x)} \frac{1}{\rho}\partial_t^{l_1} v \frac{\partial}{\partial n} w ds \ , \quad I_2 = \int_{t=\tau(x)} \frac{1}{\rho}\frac{\partial}{\partial n}(\partial_t^{l_1} v) w ds \ .$$

Observation. The integrands are all compactly supported even though neither v nor w is compactly supported.

Since the above integrals have similar forms, we will only give a detail estimate of I_1 by making the following remark on the estimate of I_2:

$$I_2 = -2\int_{t=\tau(x)} \frac{1}{\rho}\nabla\tau\cdot\nabla(\partial_t^{l_1} v|_{t=\tau}) w ds + \int_{t=\tau(x)} \frac{1}{\rho}\partial_t^{l_1+1} v w ds \ .$$

Now, by integration by parts, the first term of the right hand side can be studied just like I_1; while the second term may also be handled in a similar fashion as for I_1.

3.4 The estimate of I_1

A function $\phi(x) \in C_0^\infty(\Omega)$, Ω an open bounded set in $\mathbb{R}^n$ and $\Omega \supset supp\{\partial_t^l v|_{t=\tau}\frac{\partial}{\partial n} w|_{t=\tau}\}$, may be found because the integrand of I_1 is compactly supported, so that

$$I_1 = \int_{t=\tau(x)} \phi\frac{1}{\rho}\partial_t^l v \frac{\partial}{\partial n} w ds \ .$$

Now construct two $\psi.d.o.$ Q_1, $Q_2 \in OPS^0(\mathbb{R}^n)$

- $Q_1 + Q_2 = I$;
- $ES(Q_2)$ is a small conic neighborhood of $\{\nabla\tau\cdot\xi = 0\}$;
- Q_2's symbol $q_2 = 1$ near $\{\nabla\tau\cdot\xi = 0\}\cap\{(x,\xi),\ |\xi|\geq 1\}$.

Recall the definitions of Q_1 and Q_2. We can again rewrite I_1 as, for a $\phi_1(x) \in C_0^\infty(\Omega')$ with $\Omega \subset \Omega'$,

$$\begin{aligned} I_1 &= \int_{t=\tau} Q_1\phi\frac{1}{\rho}\partial_t^{l_1} v Q_1\phi_1\frac{\partial}{\partial n} w ds + \int_{t=\tau} Q_1\phi\frac{1}{\rho}\partial_t^{l_1} v Q_2\phi_1\frac{\partial}{\partial n} w ds + \int_{t=\tau} Q_2\phi\frac{1}{\rho}\partial_t^{l_1} v \phi_1\frac{\partial}{\partial n} w ds \\ &= I_{1,a} + I_{1,b} + I_{1,c} \ . \end{aligned}$$

Now, we estimate $I_{1,a}$, $I_{1,b}$, and $I_{1,c}$ separately.

(a) The estimate of $I_{1,a}$

Clearly, in order to estimate $I_{1,a}$, it suffices to estimate $||Q_1\phi\frac{1}{\rho}a_{l_1}||$ and $||Q_1\phi_1\frac{\partial}{\partial n}w|_{t=\tau}||$. These terms may be handled by the following propositions.

Proposition 3.3 *The following estimate holds*

$$||Q_1\phi b_k e^q||_{0,\Omega} \leq C_k$$

where $\Omega \subset\subset \hat{\Omega}$ and C_k depends on $||\sigma_0||_k$.

Proof. The proof is based on an application of Lemma 3.2. We also have to use variants of Rauch's Lemma, Schauder's Lemma, together with Proposition 2.1.

In fact, Lemma 2.2 leads to, for any r,

$$||Q_1\phi b_k e^q||_{0,\Omega} \leq C||Q_1'\phi_1'\nabla\tau\cdot\nabla b_k e^q||_{-1,\Omega_1} + C||\bar{\phi}_1 b_k e^q||_{r,\Omega_1}$$

where $\Omega \subset\subset \Omega_1$, ϕ_1', $\bar{\phi}_1 \in C_0^\infty(\Omega_1)$, and Q' supported away from $Char(\nabla\tau\cdot\nabla)$. From (3.13), it is easy to represent the right hand side in terms of $b_{k-1}e^q$. Therefore, Rauch's Lemma and Proposition 2.1 yield

$$||Q_1\phi b_k e^q||_{0,\Omega} \leq C||Q_1'\phi_2 b_{k-1}e^q||_{1,\Omega_1} + C||\bar{\phi}_1 b_k||_{r,\Omega_1}$$

where $s > 1 + n/2$ and the constant C depends on $||\sigma_0||_s$.

One can then continue this process by applying Lemma 2.2 to estimate $||Q_1'\phi_2 b_{k-1}e^q||_{1,\Omega_1}$. After l similar steps, we can write the following estimate, for any r,

$$||Q_1\phi b_k e^q||_{0,\Omega} \leq C||Q_l'\phi_l b_0 e^q||_{k,\Omega_l} + \sum_{i=1}^{k} C||\bar{\phi}_i b_i||_{r,\Omega_i} \tag{3.30}$$

where C depends on $||\sigma_0||_s$ for $s > 1 + n/2$, $\Omega \subset\subset \Omega_1 \subset\subset \cdots \subset\subset \Omega_k$, and $\bar{\phi}_i \in C_0^\infty(\Omega_i)$.

We next discuss why the smoothing terms $||\bar{\phi}_i b_i||_{r,\Omega_i}$ may be negligible.

Note that the transport equations are hyperbolic along the λ direction ($\lambda = |x|$). Actually, introducing polar coordinates, we then get $\nabla\tau\cdot\nabla = \frac{\partial}{\partial\lambda}$ ($\lambda = |x|$). Thus, λ may be treated as the "time" variable for a standard hyperbolic problem. The right hand sides of transport equations may be expressed under the polar coordinates as

$$\frac{db_k e^q}{d\lambda} \tag{3.31}$$

Because of the main assumption, $\sigma(x)$ is constant near $|x| = 0$, that is, v_0 is the fundamental solution for the same equation with constant coefficients. The Hadamard construction yields that b_k ($k = 0, \cdots, l_1$) are smooth function near $|x| = 0$ (or $\lambda = 0$), see [20] or [1] for more discussions. Lemma 3.2 then becomes applicable.

Eventually, there exists a constant C_k depending on $||\sigma_0||_k$, so that

$$||Q_1\phi b_k e^q|| \leq C_k \tag{3.32}$$

□

Proposition 3.4 *The following estimate holds*

$$||Q_1\phi a_{l_1} e^q||_{0,\Omega} \leq C||\delta\sigma||_{l_1}$$

where C depends on $||\sigma_0||_s$ for $s > \{1 + n/2, l_1\}$.

Remark. With a few necessary changes, the proof of Proposition 3.3will still work. Observe that the only major difference to the analysis before is that each one of the transport equations now possesses an extra term, which may be viewed as a "source" term. Certainly, these extra terms are responsible for the $\delta\sigma$ term in the estimate of our main theorem.

We may also rewrite the right hand side of (3.15) in terms of $a_{k-1}e^{q}$ plus a source term containing $\delta\sigma$. The desired estimate then follows by applying Lemma 3.4, together with algebraic estimates, various forms of Schauder's Lemma, Rauch's Lemma Proposition 2.1, and commutator lemmas.

Note that the transport equations of a_k may be viewed as the first order perturbation of the corresponding equations of b_k. Hence near $|x| = 0$, by the main assumption (A) $\delta\sigma = 0$, $a_k = 0$ for $k = 1, \cdots, l_1$.

Next, we want to get an estimate of $||Q_1\phi_1\frac{\partial}{\partial n}w|_{t=\tau}||_{0,\Omega'}$.

Proposition 3.5 *For Q_1, w, Ψ, and ϕ_1 as previously defined, we have*

$$||Q_1\phi_1\frac{\partial}{\partial n}w|_{t=\tau}||_{0,\Omega'} \leq C||\Psi||$$

where the constant C depends on $||\sigma_0||_s$, for $s > 1 + n/2$.

Proof. Since w solves the dual problem (2.9), which has smooth right hand side, we need a characteristic trace regularity result for w. Note that $ES(Q_1)$ is away from the characteristic variety of the wave equation. In fact, an application of a characteristic trace regularity result, Corollary 2 in [2], yields that there exists a $\tilde{\phi} \in C_0^\infty(\mathbb{R}^{n+1})$, such that

$$||Q_1\phi_1\frac{\partial}{\partial n}w|_{t=\tau}||_{0,\Omega'} \leq C||\tilde{\phi}w||_1 \leq C||\Psi||$$

the second inequality comes from the standard energy estimates. □

(b). The estimate of $I_{1,b}$

Because of Proposition 3.4, in order to bound $I_{1,b}$, it suffices to estimate $||Q_2\phi_1\frac{\partial}{\partial n}w|_{t=\tau}||_{0,\Omega'}$. This requires the microlocal trace regularity result Lemma 3.4 and the regularity analysis of the dual problem Lemma 3.5. The estimate can be easily derived by applying Lemma 3.4 and Lemma 3.5.

Proposition 3.6 *For Q_2, w, and ϕ_1 defined above. The following holds*

$$||Q_2\phi_1\frac{\partial}{\partial n}w|_{t=\tau}||_{0,\Omega'} \leq C||\Psi||$$

with the constant C depending on $||\sigma_0||_s$ for $s > 1 + n/2$

(c). The estimate of $I_{1,c}$

We want to estimate

$$\int_{t=\tau(x)} Q_2\phi\frac{1}{\rho}a_{l_1}\ \phi_1\frac{\partial}{\partial n}wds\ .$$

Obviously, this term has to be handled very differently. In fact, the transport operator (the operator appears in the transport equations) is elliptic on $ES(Q_1)$, by Lemma 3.1 the L^2 norm of $Q_1\phi a_{l_1}$ may be bounded by the H^{l_1} norm of $Q_1\phi a_0$. However, there is no ellipticity on $ES(Q_2)$, and the only direct estimate one may expect is by applying Lemma 3.2, which implies that essentially the L^2 norm of $Q_2\phi a_{l_1}$ has to be bounded by the H^{2l_1} norm of ϕa_0.

To cure this difficulty, we introduce a "bootstrap" argument by taking advantage of the transport equations and the fact that $Q_2\phi_1\frac{\partial}{\partial n}w|_{t=\tau}$ is a much smoother term (it would be smooth if σ_0 were smooth). Again, we start with the corresponding estimate for b_{l_1}.

Proposition 3.7 *The estimate*

$$|\int_{t=\tau(x)} Q_2\phi\frac{1}{\rho}b_{l_1}\ \phi_1\frac{\partial}{\partial n}wds| \leq C||\Psi||$$

holds with the constant C depending on $||\sigma_0||_s$, for $s > l_1 + (n-3)/2$.

Proof. Without loss of generalities, Q_2 may be assumed to be self-adjoint. Thus

$$\int_{t=\tau(x)} Q_2\phi\frac{1}{\rho}b_{l_1}\phi_1\frac{\partial}{\partial n}wds = \int_{t=\tau(x)}\frac{1}{\rho}b_{l_1}\phi Q_2\phi_1\frac{\partial}{\partial n}wds\ . \tag{3.33}$$

Since $Q_2\phi_1\frac{\partial}{\partial n}w$ is much smoother than the other part, one wants to apply transport equations to reduce b_{l_1} to the one requires the least regularity.

Consider a problem

$$(\nabla\tau\cdot\nabla)^*W_1 = \phi e^{\sigma_0-q}Q_2\phi_1(\frac{\partial}{\partial n}w)|_{t=\tau(x)} \tag{3.34}$$

$$W_1 = 0 \quad \text{near } |x| = 0\ . \tag{3.35}$$

Then (3.33) becomes

$$\int_{t=\tau(x)} Q_2\phi\frac{1}{\rho}b_{l_1}\phi_1\frac{\partial}{\partial n}wds = \int_{t=\tau(x)}\nabla\tau\cdot\nabla b_{l_1}e^qW_1ds\ .$$

From the transport equation (3.16), we have

$$\begin{aligned}\int_{t=\tau(x)}\nabla\tau\cdot\nabla b_{l_1}e^qW_1ds &= \int_{t=\tau(x)}(\Delta(b_{l_1-1}e^q)/2 - |\nabla q|^2b_{l_1-1}e^q/2 - \Delta qb_{l_1-1}e^q/2\\ &\quad -\nabla(b_{l_1-1}e^q)\nabla q_0 + b_{l_1-1}e^q\nabla q\nabla q_0)W_1ds\\ &= \int_{t=\tau(x)} b_{l_1-1}e^qP_1W_1ds\end{aligned}$$

where $P = \Delta - |\nabla q|^2 - \Delta q + \nabla\cdot\nabla\sigma_0 + \nabla q\cdot\nabla q_0$.

One can then introduce another problem

$$(\nabla\tau\cdot\nabla)^*W_2 = \phi PW_1 \tag{3.36}$$

$$W_2 = 0 \quad \text{near } |x| = 0\ . \tag{3.37}$$

Hence

$$\int_{t=\tau(x)} Q_2\phi\frac{1}{\rho}b_{l_1}\phi_1\frac{\partial}{\partial n}wds = \int_{t=\tau(x)} \nabla\tau\cdot\nabla b_{l_1-1}e^q PW_2 ds\ .$$

We can continue this procedure. Indeed, it follows, after $[l_1/2]$ steps where $[l_1/2] = l_1/2$ if l_1 is even and is $(l_1-1)/2$ if l_1 is odd, that

$$\int_{t=\tau(x)} Q_2\phi\frac{1}{\rho}b_{l_1}\phi_1\frac{\partial}{\partial n}wds = \int_{t=\tau(x)} \nabla\tau\cdot\nabla b_{[l_1/2]}e^q PW_{[l_1/2]+1}ds$$

where W_i solves

$$\begin{aligned} (\nabla\tau\cdot\nabla)^* W_i &= \phi PW_{i-1} && (3.38)\\ W_i &= 0 \quad \text{near } |x| = 0 && (3.39)\end{aligned}$$

for $i = 1,\cdots,[l_1/2]+1$.

Therefore, by the Cauchy-Schwatz inequality, Proposition 2.1, and Rauch's Lemma

$$|\int_{t=\tau(x)} Q_2\phi\frac{1}{\rho}b_{l_1}\phi_1\frac{\partial}{\partial n}wds| \le C||\psi b_{[l_1/2]}e^q||_{l_1-2[l_1/2]}\ ||PW_{[l_1/2]+1}||_{2[l_1/2]-l_1} \tag{3.40}$$

with C depending on $||\sigma_0||_s$, for some $s > n/2$. Since $b_i e^q$ solves the i-th transport equation, Lemma 3.2 then implies that

$$||\psi b_{[l_1/2]}e^q||_{l_1-2[l_1/2]} \le C||\phi\sigma_0||_{l_1}\ . \tag{3.41}$$

Next, applying Lemma 3.2, after some simple calculations, we can obtain

$$||PW_{[l_1/2]+1}||_{2[l_1/2]-l_1} \le C||\phi e^{\sigma_0-q}Q_2\phi_1(\frac{\partial}{\partial n}w)|_{t=\tau(x)}||_{l_1-1}\ .$$

Now Lemma 3.4 and Lemma 3.5 may be applied to get

$$||PW_{[l_1/2]+1}||_{2[l_1/2]-l_1} \le C||\Psi||$$

where the constant C depends on $||\sigma_0||_s$ with $s > l_1 - 1/2 + n/2$. □

Once again, a variant of the proof of Proposition 3.6, with no additional technical difficulty, will lead to the following result.

Proposition 3.8 *The estimate*

$$|\int_{t=\tau(x)} Q_2\phi\frac{1}{\rho}a_{l_1}\ \phi_1\frac{\partial}{\partial n}wds| \le C||\delta\sigma||_{l_1}||\Psi||$$

holds with the constant C depending on $||\sigma_0||_s$, for $s > l_1 + (n-1)/2$.

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