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Robust adaptive signal processing methods for heterogeneous radar clutter scenarios

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Abstract

This paper addresses the problem of radar target detection in severely heterogeneous clutter environments. Specifically, we present the performance of the normalized matched filter test in a background of disturbance consisting of clutter having a covariance matrix with known structure and unknown scaling plus background white Gaussian noise. It is shown that when the clutter covariance matrix is low rank, the (LRNMF) test retains invariance with respect to the unknown scaling as well as the background noise level and has an approximately constant false alarm rate (CFAR). Performance of the test depends only upon the number of elements, the number of pulses processed in a coherent processing interval, and the rank of the clutter covariance matrix. Analytical expressions for calculating the false alarm and detection probabilities are presented. Performance of the method is shown to degrade with increasing clutter rank especially for low false alarm rates. An adaptive version of the test (LRNAMF) is developed and its performance is studied with simulated data from the KASSPER program. Results pertaining to sample support for subspace estimation, CFAR, and detection performance are presented. Target contamination of training data has a deleterious impact on the performance of the test. Therefore, a technique known as self-censoring reiterative fast maximum likelihood/adaptive power residue (SCRFML/APR) is developed to treat this problem and its performance is discussed. The SCRFML/APR method is used to estimate the unknown covariance matrix in the presence of outliers. This covariance matrix estimate can then be used in the LRNAMF or any other eigen-based adaptive processing technique.

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Keywords: STAP; Invariance; Low rank; LRNMF; LRNAMF; CFAR; SNR; P_{fa} ; P_d ; KASSPER; RLSTAP; Outlier; SCRFML/APR; SCRFML/GIP

1. Introduction

This paper addresses the problem of signal detection in interference composed of clutter (and possibly jamming), having a covariance matrix with known

structure but unknown level and background white noise. The technique developed in this paper ensures invariance with respect to the unknown level and the background noise power. The research is motivated by the problem of space-time adaptive processing (STAP) for airborne phased-array radar applications. Typically, a radar receiver front end consists of an array of J antenna elements processing N pulses in a coherent processing interval. We are interested in

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the problem of target detection given the $JN \times 1$ spatio-temporal data vector.

Previous efforts [9,10,13] derived the normalized matched filter (NMF) test for the problem of detecting a rank one signal in additive clutter modeled as a spherically invariant random process [37,53]. The NMF test is given by

$$A_{\text{NMF}} = \frac{|\mathbf{e}^H \mathbf{R}_c^{-1} \mathbf{x}|^2}{[\mathbf{e}^H \mathbf{R}_c^{-1} \mathbf{e}][\mathbf{x}^H \mathbf{R}_c^{-1} \mathbf{x}]} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{\text{NMF}}, \quad (1)$$

where \mathbf{x} is the observed data vector, \mathbf{e} is the known spatio-temporal signal steering vector, and \mathbf{R}_c is the known clutter covariance matrix. A statistic similar in spirit was also considered in [14,15] for vector subspace detection in compound-Gaussian clutter.

Related work of [21,22,41,42,44,45] considered an invariance framework for the problem of signal detection in a background of Gaussian clutter having a covariance matrix with known structure but unknown level. For a rank one signal, the test statistic derived for this problem in [21–23,41,42,44,45] reduces to the NMF test of (1). Performance of the NMF test in Gaussian [23] and non-Gaussian clutter scenarios [9,10,13,28–31] has been addressed in some detail. Recent work [43] discusses adaptive subspace detection and beamforming using oblique projections at the heart of which lies a reduced rank Wiener filter.

This paper seeks to extend previous work by including the effect of additive white Gaussian noise. Specifically, we consider the binary hypothesis testing problem given by

$$H_0: \mathbf{x} = \mathbf{d} = \mathbf{c} + \mathbf{n},$$

$$H_1: \mathbf{x} = a\mathbf{e} + \mathbf{d} = a\mathbf{e} + \mathbf{c} + \mathbf{n}, \quad (2)$$

where \mathbf{x} is the observed data vector, \mathbf{c} denotes the Gaussian clutter vector having a covariance matrix $s\mathbf{R}_c$ with known structure and unknown level s , \mathbf{n} denotes the additive white Gaussian noise vector having covariance matrix $\sigma^2\mathbf{I}$, where \mathbf{I} is the $JN \times JN$ identity matrix and σ^2 is the unknown noise power, \mathbf{e} denotes the steering vector and a is the unknown complex amplitude of the target. For the sake of compactness, \mathbf{d} is used to denote disturbance consisting of clutter plus white noise. Consequently, the disturbance covariance matrix is given by $\mathbf{R}_d = s\mathbf{R}_c + \sigma^2\mathbf{I}$. The invariance properties discussed in [21,22,41,42,44,45]

fail for the problem where the clutter power and noise variance are unknown and different from each other. This is due to the fact that invariance condition of [21,22,41,42,44,45] requires a common unknown scaling on the clutter and background white noise—a condition that is seldom satisfied in practice. A uniformly most powerful invariant (UMPI) [41] test for this problem becomes mathematically intractable in general. However, in many practical airborne radar applications \mathbf{R}_c has rank r which is much less than the spatio-temporal product $M = JN$. For example, the clutter rank in the airborne linear phased array radar problem under ideal conditions (no mutual coupling between array elements), is given by the Brennan rule [52]

$$r = J + \gamma(N - 1), \quad (3)$$

where $\gamma = 2v_p T/d$ is the slope of the clutter ridge, with v_p denoting the platform velocity, T denoting the pulse repetition interval, and d denoting the inter-element spacing. A nominal value of $\gamma = 1$, yields a clutter rank $r \approx J + (N - 1) \ll M$ especially for large J and N . This fact is advantageously used to obtain a test which offers invariance to the unknown clutter power and noise level. Additionally, the low rank approximation enables reduction of training data support compared to full dimension STAP processing. An adaptive version of the test is also developed and its performance is studied. Target contamination of training data has a deleterious impact on the performance of the test. Therefore, a technique known as self-censoring reiterative fast maximum likelihood/adaptive power residue (SCRFML/APR) is developed to treat this problem and its performance is discussed. The SCRFML/APR method is used to estimate the unknown covariance matrix in the presence of outliers. This covariance matrix estimate can then be used in the low rank normalized adaptive matched filter (LRNAMF) or any other eigen-based adaptive processing technique. The remainder of the paper is organized as follows:

In Section 2, we introduce the low-rank normalized matched filter (LRNMF). The performance of the LRNMF in terms of analytical calculation of false alarm probability (P_{fa}) and detection probability (P_d) is presented in Section 3. Section 4 introduces an adaptive version of the LRNMF known as the LRNAMF and discusses its performance with respect to CFAR, sample support for subspace estimation and detection

performance using data from the KASSPER program. A new algorithm for outlier removal in training data is developed in Section 5 and its performance analysis is carried out. Conclusions are presented in Section 6.

2. Low rank NMF test

The disturbance covariance matrix can be expressed as $\mathbf{R}_d = \mathbf{U}\mathbf{D}\mathbf{U}^H$, where \mathbf{U} is the matrix whose columns are the normalized eigenvectors of \mathbf{R}_d and \mathbf{D} is the diagonal matrix of eigenvalues of \mathbf{R}_d . When \mathbf{R}_c has rank $r \ll M$, \mathbf{R}_d can be expressed as [7]

$$\mathbf{R}_d = \sum_{i=1}^r (s\lambda_i + \sigma^2) \mathbf{u}_i \mathbf{u}_i^H + \sum_{i=r+1}^M \sigma^2 \mathbf{u}_i \mathbf{u}_i^H. \quad (4)$$

For $s\lambda_i \gg \sigma^2$, it follows from [20] that the inverse covariance matrix can be approximated as

$$\mathbf{R}_d^{-1} \approx \frac{1}{\sigma^2} (\mathbf{I} - \mathbf{P}), \quad (5)$$

where $\mathbf{P} = \sum_{i=1}^r \mathbf{u}_i \mathbf{u}_i^H$ is a rank r projection matrix formed from the eigenvectors corresponding to the dominant eigenvalues of \mathbf{R}_d . For \mathbf{R}_c with known structure, the dominant modes are readily determined and are unaffected by s .

We now use the form of \mathbf{R}_d^{-1} given by (5) to express the LRNMF test as

$$A_{lr} = \frac{|\mathbf{e}^H(\mathbf{I} - \mathbf{P})\mathbf{x}|^2}{[\mathbf{e}^H(\mathbf{I} - \mathbf{P})\mathbf{e}][\mathbf{x}^H(\mathbf{I} - \mathbf{P})\mathbf{x}]} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{lr}. \quad (6)$$

Observe that the LRNMF test is invariant to s and σ^2 . Furthermore, let $\mathbf{e}_1 = (\mathbf{I} - \mathbf{P})\mathbf{e}$ and $\mathbf{x}_1 = (\mathbf{I} - \mathbf{P})\mathbf{x}$. Thus, the LRNMF test can be expressed as

$$A_{lr} = \frac{|\mathbf{e}_1^H \mathbf{x}_1|^2}{[\mathbf{e}_1^H \mathbf{e}_1][\mathbf{x}_1^H \mathbf{x}_1]} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{lr}, \quad (7)$$

which allows for important interpretations of the test statistic as normalized matched filtering in the sub-dominant disturbance subspace or a dominant mode rejector followed by quadratic normalizations to ensure CFAR.

It is helpful to note in this context that the low rank approximation to the clairvoyant RMB beamformer

[38] given by

$$A_{LRRMB} = \frac{1}{\sigma^4} |\mathbf{e}^H(\mathbf{I} - \mathbf{P})\mathbf{x}|^2 \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{LRRMB} \quad (8)$$

and the low rank approximation to the matched filter [40] for rank one signal detection in Gaussian noise given by

$$A_{MFLR} = \frac{1}{\sigma^2} \frac{|\mathbf{e}^H(\mathbf{I} - \mathbf{P})\mathbf{x}|^2}{[\mathbf{e}^H(\mathbf{I} - \mathbf{P})\mathbf{e}]} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{MFLR} \quad (9)$$

incur an explicit dependence on σ^2 . Consequently, they do not offer CFAR with respect to σ^2 .

The work of [18–20] considered a test involving the numerator of the test statistic of (8) and its adaptive version. However, such a test incurs explicit dependence on σ^2 . Therefore, it lacks CFAR. Consequently, performance analysis in [18–20] was presented in terms of the output signal-to-noise ratio (SNR), with elegant derivations for the output SNR probability density function (PDF). In this paper, we concern ourselves with the performance of the test of (6) and its adaptive version.

3. Performance of the LRNMF Test

We now consider the performance of the test of (6). Analytical expressions are derived for the probability of false alarm and probability of detection for the LRNMF. For convenience, we work with the test of the form of (7) to carry out the analysis. Noting that a unit vector in the direction of \mathbf{e}_1 is given by $\mathbf{e}_2 = \mathbf{e}_1 / \sqrt{\mathbf{e}_1^H \mathbf{e}_1}$, $\mathbf{x}_1^H \mathbf{x}_1$ can be expressed as the sum of the squared magnitudes of projections along the subspace of \mathbf{e}_2 and the orthogonal complement space of \mathbf{e}_2 denoted by Ψ_{\perp} . Let $\mathbf{w}_i, i = 1, 2, \dots, M - r - 1$ denote an orthonormal basis set for Ψ_{\perp} and $X_0 = \mathbf{e}_2^H \mathbf{x}_1, X_i = \mathbf{w}_i^H \mathbf{x}_1, i = 1, 2, \dots, M - r - 1$. Then, $X_i, i = 0, 1, \dots, M - r - 1$ are statistically independent complex-Gaussian random variables. Let $\xi_1 = |X_0|^2 / \sigma^2, \xi_2 = (1/\sigma^2) \sum_{i=1}^{M-r-1} |X_i|^2$, and $\Phi = \xi_1 / \xi_2$. The test statistic of (7) admits a representation of the form

$$A_{lr} = \frac{\Phi}{(1 + \Phi)}. \quad (10)$$

Under $H_0, X_i, i = 1, \dots, M - r - 1$ are complex-Gaussian random variables distributed as $CN(0, \sigma^2)$.

Consequently, ξ_2 is a Chi-squared distributed random variable with $(M - r - 1)$ complex degrees-of-freedom [33]. Also under H_0 , X_0 is a complex-Gaussian random variable distributed as $CN(0, \sigma^2)$. Hence, ξ_1 is a chi-squared distributed random variable with one complex degree-of-freedom. It follows from [1] that Φ is a central-F distributed random variable, whose probability density function (PDF) is given by

$$f_\Phi(\phi) = \frac{1}{\beta(1, M - r - 1)} \frac{1}{(1 + \phi)^{M-r}}, \quad (11)$$

where

$$\beta(m, n) = \int_0^1 \varphi^{m-1} (1 - \varphi)^{n-1} d\varphi. \quad (12)$$

Using a straightforward transformation of random variables, we show that the PDF of A_{lr} under H_0 follows a beta distribution given by

$$f_{A_{lr}}(y) = (M - r - 1)(1 - y)^{M-r-2}. \quad (13)$$

The probability of false alarm is given by

$$P_{fa} = P(A_{lr} > \lambda_{lr} | H_0) = (1 - \lambda_{lr})^{M-r-1}. \quad (14)$$

Observe that the false alarm probability is independent of the nuisance parameters s and σ^2 . Instead, it depends only on M and r , which are functions of system parameters such as the number of array elements, number of pulses in a CPI and the slope of the clutter ridge. Thus, a low rank approximation of \mathbf{R}_d^{-1} results in CFAR for the LRNMF test.

We then proceed to calculate the probability of detection for the test of (6). Under H_1 , the PDF of ξ_2 remains unchanged. However under H_1 , X_0 is a complex Gaussian random variable distributed as $CN(a\sqrt{\mathbf{e}_1^H \mathbf{e}_1}, \sigma^2)$. Consequently, ξ_1 is a non-central chi-squared distributed random variable with one complex degrees-of-freedom having non-centrality parameter $A = |a|\sqrt{\mathbf{e}_1^H \mathbf{e}_1}/\sigma$. Noting that $|a|^2 \mathbf{e}_1^H \mathbf{e}_1$ is the signal energy in the sub-dominant disturbance subspace it follows that $A^2 = |a|^2 \mathbf{e}_1^H \mathbf{e}_1 / \sigma^2$ is simply the SNR arising in the sub-dominant disturbance subspace. Thus, the non-centrality parameter is related to the SNR in a straightforward manner. Hence, Φ has a non-central F distribution in this instance. Again using a straightforward transformation of random variables, it follows that A_{lr} follows a non-central

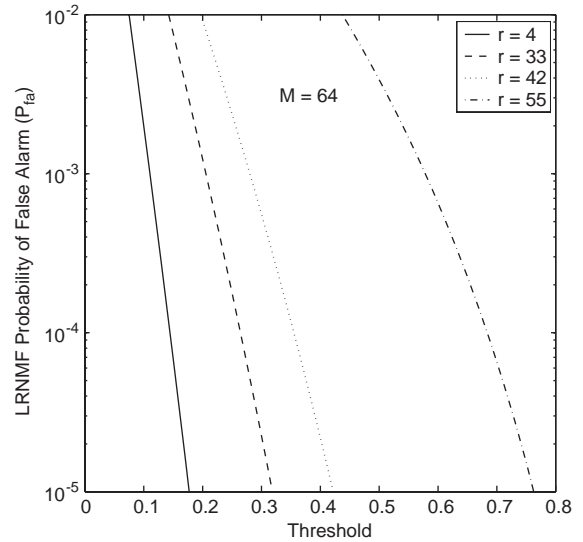


Fig. 1. P_{fa} versus threshold.

beta PDF given by

$$f_{A_{lr}}(y) = \sum_{k=0}^{\infty} \exp(-A^2) \frac{A^{2k} y^k (1 - y)^{M-r-2}}{k! \beta(M - r - 1, k + 1)}, \quad (15)$$

$$0 \leq y \leq 1.$$

The probability of detection is given by

$$P_d = P(A_{lr} > \lambda_{lr} | H_1)$$

$$= 1 - E,$$

$$E = (1 - \lambda_{lr})^{M-r-1} \sum_{k=1}^{M-r-1} \frac{\Gamma(M - r)}{\Gamma(k + 1) \Gamma(M - r - k)} F,$$

$$F = \left(\frac{\lambda_{lr}}{1 - \lambda_{lr}} \right)^k \times [1 - \text{gammainc}(A^2(1 - \lambda_{lr}), k + 1)], \quad (16)$$

where $\Gamma(\cdot)$ is the Eulero–Gamma function and

$$\text{gammainc}(\theta, M) = \frac{1}{\Gamma(M)} \int_0^\theta z^{M-1} \exp(-z) dz. \quad (17)$$

It is important to note that P_d depends on σ^2 only through A^2 (SNR) and not on nuisance parameters such as exact signal shape, signal complex amplitude or exact noise variance. Fig. 1 shows a plot of the false

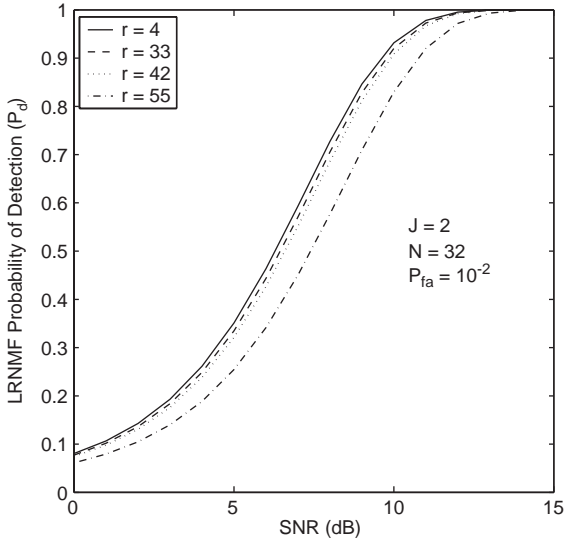


Fig. 2. P_d versus SNR.

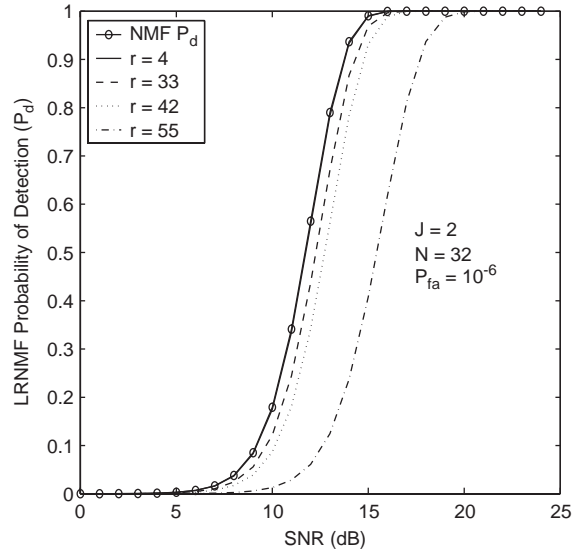


Fig. 3. P_d versus SNR.

alarm probability versus threshold for the LRNMF test with the clutter rank r as a parameter. We observe a significant increase in the threshold with increasing clutter rank for a given P_{fa} value. A plot of P_d versus SNR for the LRNMF test with r as a parameter is shown in Fig. 2. Relevant test parameters are reported in the plot. We note that a 1.32 dB loss in detection performance is encountered as the rank varies from $r=4$ to 55. Fig. 3 presents a comparison of the performance for the LRNMF with $r=4$ and 55 with the full rank NMF for the case where the disturbance consists of white noise alone with a full rank covariance matrix and no clutter. Relevant test parameters are reported in the plot. For completeness, we reproduce below the analytical expressions for the false alarm and detection probabilities of the full rank NMF [31]. Specifically,

$$P_{fa-NMF} = P(A_{NMF} > \lambda_{NMF} | H_0) = (1 - \lambda_{NMF})^{M-1} \quad (18)$$

$$P_{d-NMF} = 1 - (1 - \lambda_{NMF})^{M-1} G,$$

$$G = \sum_{k=1}^{M-1} \frac{\Gamma(M)}{\Gamma(k+1)\Gamma(M-k)} H,$$

$$H = \left(\frac{\lambda_{NMF}}{1 - \lambda_{NMF}} \right)^k$$

$$\times [1 - \text{gammainc}(A_1^2(1 - \lambda_{NMF}), k + 1)], \quad (19)$$

$$A_1^2 = \frac{|a|^2 \mathbf{e}^H \mathbf{e}}{\sigma^2},$$

where A_1^2 is the full rank NMF output SNR. The curves in Fig. 3 reveal important features of the low rank approximation to the covariance matrix. Curve 1 corresponding to $r=0$ (no clutter) upper bounds the performance of the low-rank approximation. Furthermore, for the clutter rank $r=4$ of the LRNMF test attains performance close to its upper bound, i.e., the full rank ($M \times M$ covariance matrix) NMF test performance in background white noise with unknown power level. This is due to the fact that the for $P_{fa} = 10^{-6}$, $\lambda_{NMF} = 0.1969$, while $\lambda_{lr} = 0.2088$ (a slight threshold increase). Furthermore, for $r=4$, $A^2 \approx A_1^2$ (negligible SNR loss). For instance, if $\mathbf{e} = (1/\sqrt{M})[1 \ 1 \ \dots \ 1]$ and \mathbf{P} is a rank four projection matrix, $A^2 = 0.9375A_1^2$. Hence, the LRNMF for $r=4$ attains performance close to the upper bound (indistinguishable from the upper bound performance in Fig. 3).

As the clutter rank increases, performance of the LRNMF degrades. The performance degradation (approximately 4 dB loss) with increasing rank (from $r=4$ to 55) can be accounted for due to the fact that the threshold incurs an increase with

increasing clutter rank. Furthermore, A^2 , which is a measure of the output SNR, is also decreased with increasing clutter rank. Since P_d is a monotonic function of A^2 , performance is degraded with increasing r .

This is expected since the full rank NMF test for $r = 0$ is invariant to the unknown white noise level. However, addition of clutter results in the loss of gain invariance in general. Nevertheless, imposing a low rank structure approximation of the clutter covariance matrix restores the gain invariance for small values of clutter rank. When the clutter rank follows the Brennan's rule ($r = 33$), we note that there is a slight detection loss of the LRNMF compared to the full rank NMF test with $r = 0$. However, the LRNMF test still retains the advantage of not requiring knowledge of s and σ^2 .

4. Low rank normalized adaptive matched filter

In this section, we present the performance analysis of an adaptive version of the LRNMF test of (6). The disturbance covariance matrix is seldom known in practice and thus must be estimated using representative training data. Specifically, we consider the LRNMF test of (6) with \mathbf{P} replaced by its estimate $\hat{\mathbf{P}}$ formed from a singular value decomposition (SVD) of a data matrix \mathbf{Z} whose columns \mathbf{z}_i , $i = 1, 2, \dots, K$ contain representative training data. The resulting test is called the LRNAMF. It can be readily demonstrated using arguments similar to those employed for the LRNMF test that the LRNAMF offers invariance to the unknown clutter power as well as the background noise power for large clutter-to-noise ratio (CNR), i.e., $s\lambda_i \gg \sigma^2$. In radar applications this condition is satisfied in most instances. For example, the MCARM [25] and KASSPER [2] data sets offer an average CNR of 40 dB.

Typically r is unknown in practice. Consequently, a key issue in this context is the determination of r from the training data. Several techniques for determining r are available in the literature [6,39,46,54]. The method of [6] is best suited for our analysis since it does not require explicit knowledge of σ^2 . Furthermore, the method of [6] has been successfully applied to radar data from the multichannel airborne radar measurement (MCARM) [25] and research labora-

Table 1
KASSPER data parameters

Parameter	Value
Carrier frequency (MHz)	1240
Bandwidth (MHz)	10
Number of antenna elements	11
Number of pulses	32
Pulse repetition frequency (Hz)	1984
1000 range bins (km)	35–50
91 azimuth angles (deg)	87, 89, ..., 267
128 Doppler frequencies (Hz)	–992 to 992
Clutter power (dB)	40
Platform speed (m/s)	100
Target speed (m/s)	26.8
Number of targets	226
Target Doppler frequency range (Hz)	–99.2 to 372

Table 2
RLSTAP data parameters

J	14
N	8
K	80
Number of jammers	2
Jammer angles (deg)	50, 80
Jammer powers (dB)	30, 30
Clutter power (dB)	45
Clutter power error (dB)	10
y	2.32
Signal Doppler (deg)	171
Signal spatial angle off-boresight (deg)	0
Antenna boresight angle (deg)	315

tory space-time adaptive processing (RLSTAP) [24] programs.

Data from the L-band data set of the KASSPER program [2] is used for carrying out performance analysis of the LRNAMF. The L-band data set consists of a datacube of 1000 range bins corresponding to the returns from a single coherent processing interval (CPI) from 11 channels and 32 pulses resulting in a spatio-temporal product of 352. Relevant system parameters for the L-band data sets from the KASSPER and RLSTAP programs are provided in Tables 1 and 2, respectively. Further detail is contained in [2]. Since analytical expressions for P_d and P_{fa} for the LRNAMF are mathematically intractable, we resort to performance evaluation using Monte Carlo simulation.

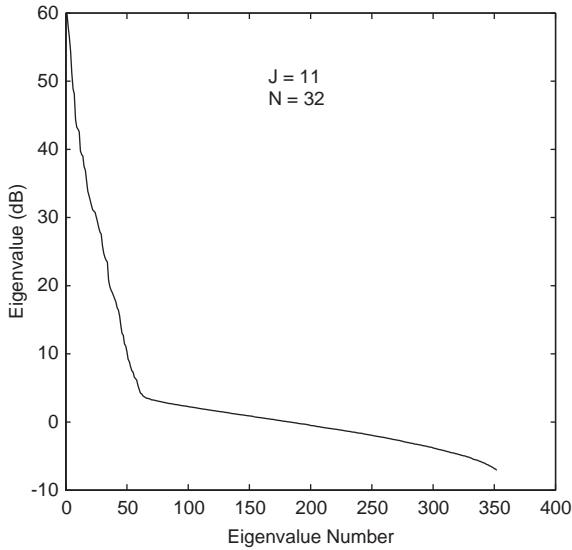


Fig. 4. Eigenspectrum of KASSPER data.

Fig. 4 shows the eigenspectrum of the KASSPER data. This plot is obtained from an SVD of the KASSPER datacube. Relevant test parameters are reported in the plot. We observe that the eigenspectrum exhibits a significant roll off (nearly 60 dB) after approximately 50 eigenvalues. The rank of the clutter subspace is determined using the procedure outlined in [6]. The procedure of [6] yields a clutter rank of 42 for this example, which is in agreement with the Brennan rule (3).

Fig. 5 shows the variation of the LRNAMF P_{fa} as a function of normalized Doppler beam position for a fixed steering angle. Relevant test parameters are reported in the plot. For ease of simulation, we present the results for the case of two channels and 32 pulses. The curve which reflects a constant false alarm probability as a function of Doppler corresponds to the LRNAMF and is obtained using (14). A large increase of the LRNAMF threshold in the vicinity of zero Doppler is observed resulting in increased false alarm probability. Also the LRNAMF threshold decreases with increasing K , the sample support used in forming $\hat{\mathbf{P}}$. Hence, P_{fa} has less variability with increasing sample support. A modest CFAR loss is incurred with respect to the normalized Doppler beam position (particularly in the vicinity of zero Doppler). Our simulations also reveal that the threshold is insensitive

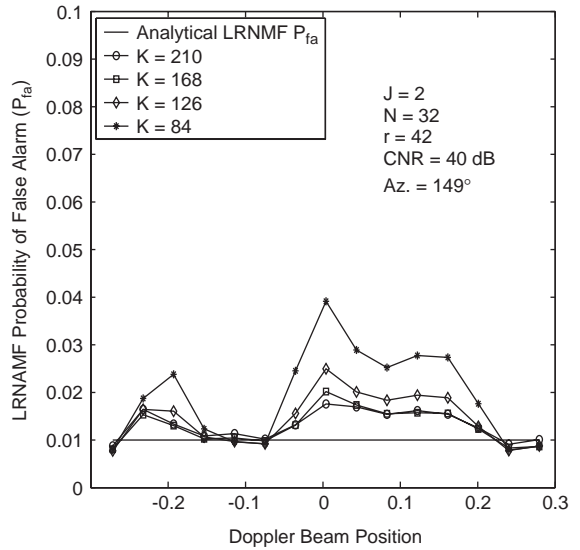


Fig. 5. P_{fa} versus normalized Doppler beam position.

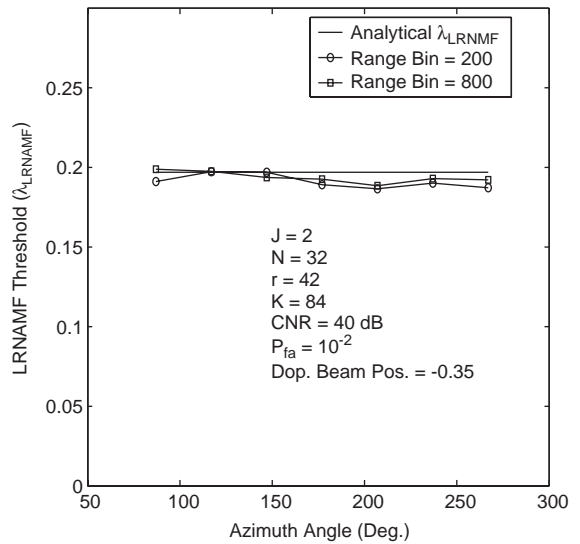
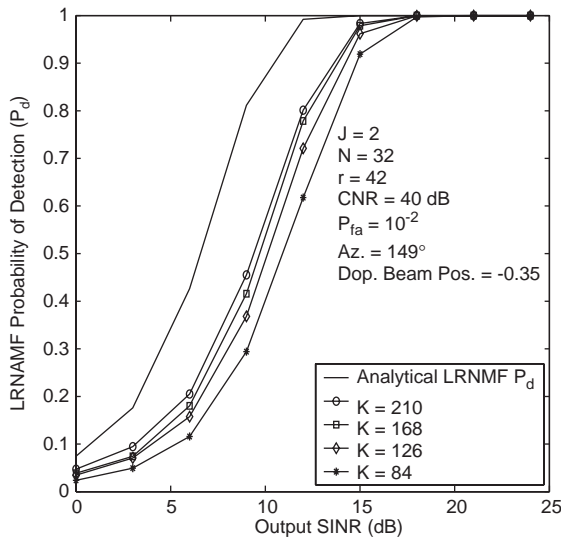
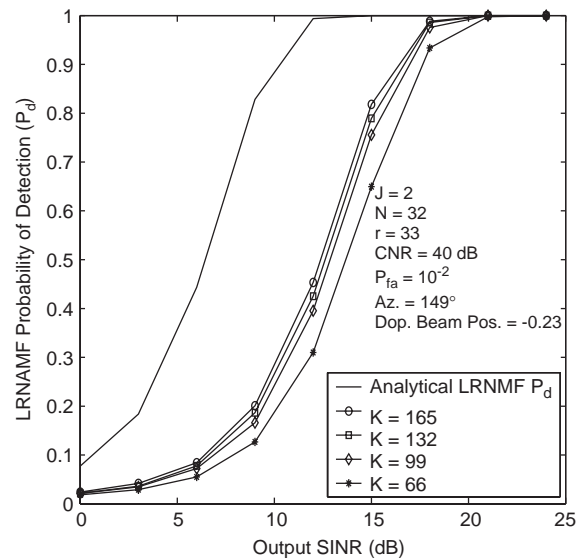


Fig. 6. Threshold versus angular beam position.

to the unknown clutter power level and noise variance as long as the CNR is high. Invariance breaks down when the CNR attains a threshold value in accordance with the results of [49,50].

Fig. 6 presents a similar result for the threshold variation as a function of angle with a fixed Doppler.

Fig. 7. P_d versus SINR.Fig. 8. P_d versus SINR: mismatched clutter subspace.

These plots correspond to range bins 200 and 800, respectively. The true covariance matrix corresponding to these two range bins [2] is used to generate data for the Monte Carlo simulations. Relevant test parameters are reported in the plots. A slight variation of the threshold as a function of angle is seen in both cases. However, this has minimal impact on the false alarm probability. Therefore, the LRNAMF offers CFAR like behavior with respect to normalized angular beam position.

Fig. 7 shows a plot of the LRNAMF P_d as a function of signal-to-interference-plus-noise ratio (SINR). Relevant test parameters are reported in the plot. The analytical curve corresponding to the LRNAMF represents the upper bound on performance of the LRNAMF. With $K = 2r$ training data samples for estimating $\hat{\mathbf{P}}$, we observe a 4 dB decrease in performance with respect to the analytical curve. The performance loss is reduced with increasing K . Specifically, for 3 dB detection (P_d) performance $K = 5r$ training data vectors are needed. The work of [38] noted that 3 dB performance in terms of SNR requires the use of $K = 2M$ target-free training data vectors. In [32] it was shown that 3 dB performance for P_d using sample matrix inversion calls for $K = 5M$ training data vectors. This is due to the fact that although P_d is a monotonic function of SNR, it is a highly nonlinear function. Con-

sequently, the training data support needed for 3 dB SNR performance is quite different from the 3 dB performance for P_d . The work of [19,20] derives an expression for the PDF of the output SINR of the principal components inverse technique and shows that the training data support for 3 dB performance is $K = 2r$. A similar result is also noted in [48] while considering the low rank problem in a maximum likelihood estimation framework and in [12] while dealing with eigenprojection methods. Fig. 7 reports for the first time the corresponding training data support for 3 dB performance in terms of P_d for low rank adaptive processing methods. This result is very similar in spirit to that reported in [32] for the sample matrix inversion technique.

In Fig. 8, we present an example where the rank of the clutter subspace is incorrectly estimated. Specifically, we consider the case where r has been underestimated and show a plot of P_d versus SINR. The predicted r from the Brennan rule is 42, whereas the estimated rank is 33. The error in estimating r is caused due to the use of an ad hoc procedure for determining r instead of the method of [6]. The specific method used in determining r is based on calculating the ratio of the sum of the squared magnitude of the dominant singular values and the sum of the squared magnitude of all the singular values arising

in the SVD and setting the ratio to 0.9. Significant performance degradation of the LRNAMF is noted for this case. A similar result (not reported here) was noted for overestimation of the clutter rank. Thus, it is extremely important to estimate the clutter subspace rank properly.

5. Self-censoring reiterative fast maximum likelihood method

Crucial to the performance of the LRNAMF is the availability of target free training data to estimate the projection matrix pertaining to the clutter subspace. In practice, strong clutter discretizes and dense target scenarios result in severe training data contamination. Consequently, this causes signal cancellation of the targets of interest. Therefore, it becomes imperative to test (for target presence) and train against the same data set. This section presents the SCRFML/APR method developed for this purpose. For this method, a single adaptive weight vector is calculated from the training data and applied back to each training data vector. There are no guard cells or the sliding window processing which is normally associated with using guard cells. In this case, the disturbance covariance matrix structure is assumed to have the form $\mathbf{R}_d = \mathbf{R}_i + \mathbf{I}$, where \mathbf{R}_i is a positive semidefinite matrix. The system noise covariance matrix is the identity matrix component of \mathbf{R}_d . Let \mathbf{Z} denote a data matrix, whose columns \mathbf{z}_i , $i = 1, 2, \dots, K$ denote K realizations of training data. The SCRFML/APR method consists of two parts: a self censoring reiterative (SCR) method for outlier removal in training data and the fast maximum likelihood (FML) method [48] for reducing training data support in covariance matrix estimation. For the sake of completeness, the FML method is briefly summarized below and the interested reader is pointed to [48] for details. Let \mathbf{Z} denote a data matrix, whose K columns contain M -tuple training data vectors.

(1) Set $[\mathbf{U}, \mathbf{S}, \mathbf{V}] = (1/\sqrt{K}) \text{svd}(\mathbf{Z})$ where $\text{svd}(\cdot)$ denotes the MATLAB operation for singular value decomposition and $\mathbf{Z} = \mathbf{U}\mathbf{S}\mathbf{V}^H$, where \mathbf{U} , and \mathbf{V} are unitary matrices, whose columns correspond to the left and right singular vectors of \mathbf{Z} and \mathbf{S} is a diagonal matrix, whose elements

- s_{nm} , $n = 1, 2, \dots, M$ denote the singular values of \mathbf{Z} .
- (2) Set $\tilde{\mathbf{D}} = \text{diag}[\min(s_{nm}^2, 1)]$, $n = 1, 2, \dots, M$.
 - (3) $\tilde{\mathbf{R}} = \mathbf{U}\tilde{\mathbf{D}}\mathbf{U}^H$.

Iterative techniques [11] employing the FML method in conjunction with the adaptive power residue (APR) and the generalized inner product (GIP) approach [3], respectively, are proposed for outlier removal and their performance is compared here. The resulting approaches are termed SCRFML/APR, and SCRFML/GIP, respectively. We now summarize the SCRFML using the APR and GIP approaches. Both methods assume that the system noise covariance matrix is known to be the $M \times M$ identity matrix \mathbf{I}_M and a steering vector \mathbf{e} is available. Certainly, the thermal noise level is unknown in applications like sun spot data time series analysis. However, for systems operating at microwave frequencies, the noise is not dominated by unknown external thermal noise but by the receiver thermal noise, which can be readily measured [47]. Furthermore, radar hardware employs only the last two significant bits to represent system noise, thereby providing an upper bound on the system noise level. Without loss of generality, this can be normalized to unity. The SCRFML/APR method consists of starting with the FML of an estimated covariance matrix, normalizing the estimate by the square root of the APR and iterating till convergence results. This method is summarized in the following steps. The APR approach is reported here for the first time.

- (1) Set $\tilde{\mathbf{R}}_0 = \text{FML}(\hat{\mathbf{R}}_0)$ where $\hat{\mathbf{R}}_0 = \mathbf{Z}\mathbf{Z}^H/K$ and $\text{FML}(\cdot)$ denotes the FML algorithm operator on the argument.
- (2) For $l = 1, 2, \dots, L$ (L =number of iterations)

$$\hat{\mathbf{R}}_{l+1} = \frac{1}{K} \frac{1}{\alpha_l} \sum_{i=1}^K \frac{\mathbf{z}_i \mathbf{z}_i^H}{|\mathbf{z}_i^H \tilde{\mathbf{R}}_l^{-1} \mathbf{e}|},$$

$$\alpha_l = \frac{1}{K} \sum_{i=1}^K \frac{1}{|\mathbf{z}_i^H \tilde{\mathbf{R}}_l^{-1} \mathbf{e}|},$$

$$\tilde{\mathbf{R}}_{l+1} = \text{FML}(\hat{\mathbf{R}}_{l+1}). \tag{20}$$

- (3) The adaptive power residue is simply $|\mathbf{z}_i^H \tilde{\mathbf{R}}_l^{-1} \mathbf{e}|^2$. No specific stopping criterion has been applied here. This is a topic for further investigation.

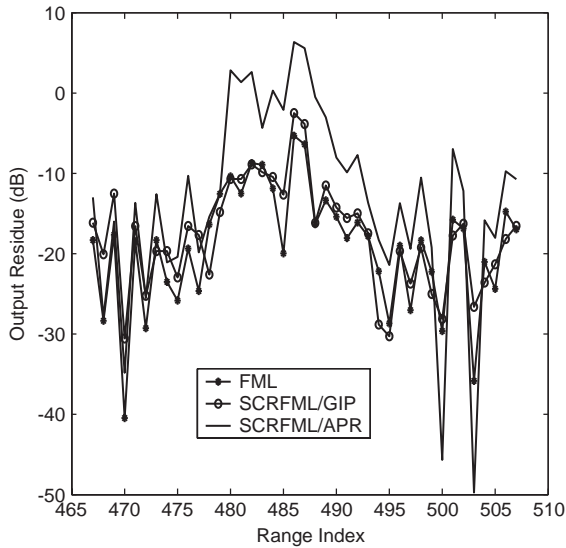


Fig. 9. APR versus range: number of iterations = 1.

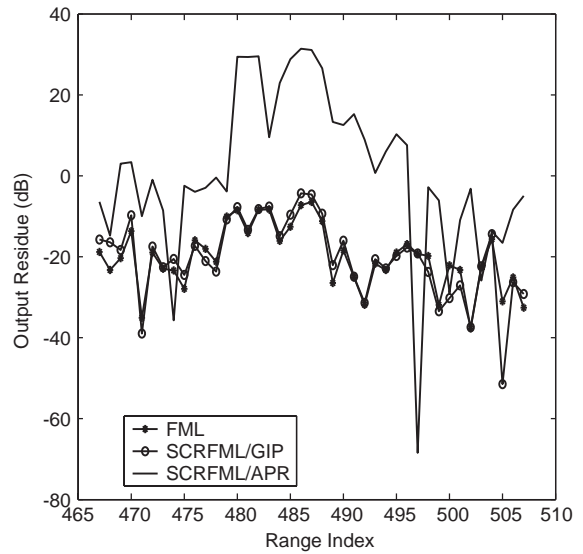


Fig. 11. APR versus range: number of iterations = 15.

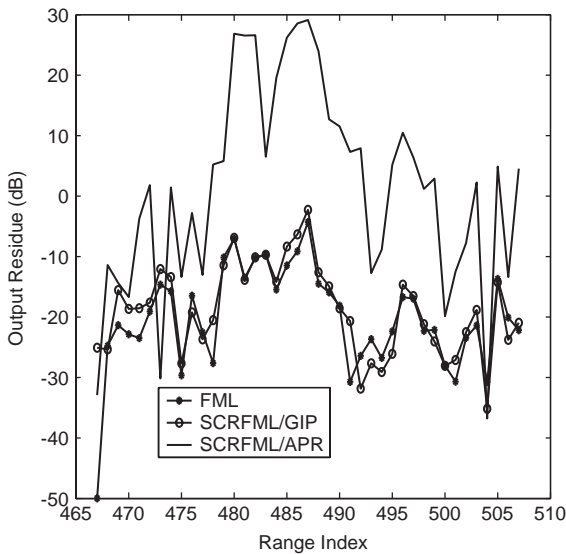


Fig. 10. APR versus range: number of iterations = 3.

Although the approach cannot be cast in a rigorous statistical formalism, an intuitive explanation of the technique can be offered in terms of the output of an adaptive RMB beamformer [38]. The rationale underlying our technique is that a peak in the RMB beamformer output occurs when training data is contaminated by

a target signal sharing the same angle-Doppler information as that of a target of interest. The RMB beamformer output energy designated as the adaptive power residue provides a basis for separating contaminated training data from target-free training data. The SCRFML/GIP is a variant of the above procedure in that the GIP, $\mathbf{z}_i^H \hat{\mathbf{R}}_i^{-1} \mathbf{z}_i$, is used in step 2 as opposed to the square root of the APR. A number of efforts [3–5,17,26,27,34–36] have considered the use of the GIP for selecting representative training data while [8,16] use the GIP normalization for covariance estimation in compound Gaussian clutter. A comparison of the FML, SCRFML/APR, and SCRFML/GIP methods is shown using data from the RLSTAP hi-fidelity clutter model [24]. Two targets of range extent 3 are centered at range bins 480 and 487. The relevant test parameters used in these examples are reported in Table 1. In order to demonstrate the generality of the SCRFML/APR technique, jamming is included in addition to clutter and white noise.

Figs. 9–11 depict the adaptive power residue as a function of range for each method. The figures correspond to number of iterations for the reiterative methods varied from one to fifteen. Observe that with as few as three iterations, the SCRFML/APR method converges to the steady-state solution. Furthermore, in all cases, the SCRFML/APR method is shown to

significantly outperform the SCRFML/GIP and FML methods; i.e., the two targets are clearly observable from the output residue (about 22 dB above the APR associated with the target free range cells). The latter two methods show poor but similar performance in all cases. The SCRFML/APR method is most useful in instances where detection and training have to be performed on a common data set. While no optimality property of the approach is claimed, the ad hoc SCRFML/APR method offers a powerful tool for removing outliers in training data.

6. Conclusions

This paper presents an analysis of the NMF test for the case of clutter plus white noise. Imposing a low rank structure on the known clutter covariance matrix enables approximate CFAR behavior yielding robustness with respect to unknown clutter scaling and unknown background noise level. Analytical expressions for the detection and false alarm probabilities are presented and illustrated with numerical examples in the form of plots of P_d versus SNR. We observe a degradation in performance as the clutter rank increases. This loss (approximately 4 dB) is quite significant at low false alarm rates.

Performance of the LRNAMF, an adaptive version of the LRNMF is studied using the KASSPER radar data. We observe a 4 dB degradation in performance due to the finite sample support used in estimating the clutter subspace. Furthermore, we note a loss of CFAR for the LRNAMF due to the threshold dependence on the Doppler beam position. An important feature of the LRNAMF is the ability to reduce the training data support for subspace estimation. Finally, we note that accurate determination of the rank of the clutter subspace significantly impacts detection performance. Critical to the performance of the LRNAMF is the ability to obtain representative training data. However, in dense target environments, significant performance penalty is incurred due to target contamination of the training data. This results in signal cancellation causing a degradation in the SNR. Consequently, the SCRFML/APR method presented here is useful for rejecting outliers in the training data and obtaining good estimates of the projection matrix. Further performance analysis using this technique with the

LRNAMF will be investigated in future. An important issue in this context is the development of a suitable stopping criterion for the SCRFML/APR method.

Additionally, finite sample support used in clutter subspace estimation causes subspace perturbation [51] and subspace swapping [49,50]. The impact of these effects on LRNAMF performance is currently under investigation. These issues will be reported on in a future publication.

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