An Iterative Substructuring Method for Coupled Fluid-Solid Acoustic Problems[1]

Jan Mandel

Department of Mathematics and Center for Computational Mathematics, University of Colorado at Denver, Denver, CO 80217-3364, U.S.A., and Department of Aerospace Engineering Science and Center for Aerospace Structures, University of Colorado at Boulder, Boulder, CO 80309-0429, U.S.A

November 13, 2000

A fast parallel iterative method is proposed for the solution of linear systems arizing from Finite Element discretization of the time harmonic acoustics of coupled fluid-solid systems in fluid pressure and solid displacement formulation. The method generalizes the FETI-H method for the Helmholtz equation to elastic scattering. The fluid and the solid domains are decomposed into non-overlapping subdomains. Continuity of the solution enforced by Lagrange multipliers. The system is augmented by duplicating the degrees of freedom on the wet interface. The original degrees of freedom are then eliminated and the resulting system is solved by the GCR method preconditioned by a subspace correction. In each iteration, the method requires the solution of one independent acoustic problem per subdomain, and the solution of a coarse problem with several degrees of freedom per subdomain. Computational results show that the method is scalable with the problem size, frequency, and the number of subdomains. The number of iterations was mostly about same as the number of iterations of the FETI-H method for the related Helmholtz problem with Neumann boundary condition instead of elastic scatterer. Convergence is explained from the spectrum of the iteration operator.

Key Words: Lagrange multipliers, domain decomposition, iterative substructuring, elastic scattering, Helmholtz equation, coupled fluid-solid acoustics, FETI

1. INTRODUCTION

The method of Finite Element Tearing and Interconnecting (FETI) was proposed by Farhat and Roux [11] for solving linearized elasticity problems. The FETI

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE	2. REPORT TYPE	3. DATES COVERED
13 NOV 2000		**00-00-2000 to 00-00-2000**

4. TITLE AND SUBTITLE	5a. CONTRACT NUMBER
An Iterative Substructuring Method for Coupled Fluid-Solid Acoustic Problems	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **University of Colorado at Denver,Center for Computational Mathematics,PO Box 173364,Denver,CO,80217-3364**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT
see report

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**		**14**	

method consists of decomposing the domain into non-overlapping subdomains. Lagrange multipliers are used to enforce that the values of the degrees of freedom coincide on the interfaces between the subdomains interfaces. All original degrees of freedom are then eliminated, leaving a dual system for the Lagrange multipliers, which is solved by preconditioned conjugate gradients. For details, further developments, and theoretical analysis of the FETI method for positive definite problems, see [1, 4, 5, 9, 10, 13, 14, 16, 17, 18, 20, 21] and references therein.

Variants of the FETI method for the Helmholtz equation of scattering, were proposed by De La Bourdonnaye et al. [3], Farhat et al. [6, 7, 8], and further developed by Tezaur et al. [22]. In this paper, we present an extension of the method of [7, 8], called FETI-H, to the case of an elastic scatterer.

The paper is organized as follows. In Section 2, we state the boundary value problem of coupled fluid-solid acoustics. Section 3 reviews the variational formulation and Finite Element discretization. The discrete equations are put in a substructured form in Section 4, which leads to the iterative method, presented in Section 5. Section 6 contains computational results. An explanation of some of the convergence properties of the method from the spectrum of the iteration operator is given in Section 7. Section 8 is the conclusion.

2. THE SCATTERING PROBLEM

We consider an acoustic scattering problem with an elastic scatterer completely immersed in a fluid. Let Ω and Ω_e be bounded domains in $\Re^n$, $= 2, 3$, $\overline{\Omega}_e \subset \Omega$, and let $\Omega_f = \Omega \setminus \Omega_e$. Let ν denote the exterior normal. Let $\partial\Omega$ be decomposed disjoint subsets, $\partial\Omega = \Gamma_d \cup \Gamma_n \cup \Gamma_a$. The domain Ω_f is filled with a fluid. The acoustic pressure at time t is assumed to be of the form $\mathrm{Re}\, pe^{i\omega t}$, where p is complex amplitude independent of t. The amplitude p is governed by the Helmholtz equation

$$\Delta p + k^2 p = 0 \quad \text{in} \quad \Omega_f, \tag{1}$$

with the boundary conditions

$$p = p_0 \text{ on } \Gamma_d, \quad \frac{\partial p}{\partial \nu} = 0 \text{ on } \Gamma_n, \quad \frac{\partial p}{\partial \nu} + ikp = 0 \text{ on } \Gamma_a, \tag{2}$$

where $k = \omega/c_f$ is the wave number and c_f is the speed of sound in the fluid. The boundary conditions (2) model excitation, sound hard boundary, and outgoing boundary, respectively. The amplitude of the displacement u of the elastic body occupying the domain Ω_e satisfies the elastodynamic equation

$$\nabla \cdot \tau + \omega^2 \rho_e u = 0 \quad \text{in} \quad \Omega_e, \tag{3}$$

where τ is the stress tensor and ρ_e is the density of the solid. For simplicity, we consider an isotropic homogeneous material with

$$\tau = \lambda I(\nabla \cdot u) + 2\mu e(u), \quad e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{4}$$

where λ and μ are the Lamé coefficients of the solid.

Let $\Gamma = \partial\Omega_e$ be the wet interface. On Γ, the fluid pressure and the solid displacement satisfy

$$\nu \cdot u = \frac{1}{\rho_f \omega^2} \frac{\partial p}{\partial \nu}, \quad \nu \cdot \tau \cdot \nu = -p, \quad \nu \times \tau \cdot \nu = 0, \tag{5}$$

where ρ_f is the fluid density. The interface conditions (5) model the continuity of normal displacement, the balance of normal forces, and zero tangential tension, respectively, cf., e.g., [23].

3. VARIATIONAL FORM AND DISCRETIZATION

We use the following standard variational form and discretization by conforming elements, cf., e.g., [19]. Define the spaces $V_f = \{q \in H^1(\Omega_f) \mid q = 0 \text{ on } \Gamma_d\}$, $V_e = (H^1(\Omega_e))^n$, where H^1 is the Sobolev space of generalized functions with square integrable generalized first derivatives. Assuming that p_0 on Γ_d is extended to a function in $H^1(\Omega_f)$, multiplying equation (1) by a test function $q \in V_f$, equation (3) by a test function $u \in V_e$, and integrating by parts, we obtain the following variational form of (1) – (5): Find p such that $p - p_0 \in V_f$, and $u \in V_e$ such that for all $q \in V_f$ and all $v \in V_e$,

$$\begin{aligned} -\int_{\Omega_f} \nabla p \nabla q + k^2 \int_{\Omega_f} pq - ik \int_{\Gamma_a} pq - \omega^2 \int_{\Gamma} \rho_f (\nu \cdot u) q = 0, \\ -\int_{\Omega_e} \big(\lambda (\nabla \cdot u)(\nabla \cdot v) + 2\mu e(u) : e(v)\big) + \omega^2 \int_{\Omega_e} \rho_e u \cdot v - \int_{\Gamma} p(\nu \cdot v) = 0. \end{aligned}$$

Proceeding as usual in the Finite Element method [2], we replace V_f and V_e with conforming finite element spaces and obtain the algebraic system

$$\begin{bmatrix} -\mathbf{K}_f + k^2 \mathbf{M}_f - ik\mathbf{G}_f & -\rho_f \omega^2 \mathbf{T} \\ -\mathbf{T}^* & -\mathbf{K}_e + \omega^2 \mathbf{M}_e \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{r} \\ 0 \end{bmatrix}. \tag{6}$$

In the coupled system (6), $\mathbf{p}$ and $\mathbf{u}$ are the vectors of the (values of) degrees of freedom of p and u, i.e., p and u are the finite element interpolations of $\mathbf{p}$ and $\mathbf{u}$, respectively. The matrix blocks in (6) are defined by

$$\begin{aligned} \mathbf{p}^* \mathbf{K}_f \mathbf{q} &= \int_{\Omega_f} \nabla p \cdot \nabla q, \qquad \mathbf{p}^* \mathbf{M}_f \mathbf{q} = \int_{\Omega_f} pq, \\ \mathbf{p}^* \mathbf{G}_f \mathbf{q} &= \int_{\Gamma_a} pq, \qquad \mathbf{u}^* \mathbf{K}_e \mathbf{v} = \int_{\Omega_e} \big(\lambda (\nabla \cdot u)(\nabla \cdot v) + 2\mu e(u) : e(v)\big), \\ \mathbf{u}^* \mathbf{M}_e \mathbf{v} &= \int_{\Omega_e} \rho_e (u \cdot v), \qquad \mathbf{p}^* \mathbf{T} \mathbf{v} = \int_{\Gamma} p(\nu \cdot v). \end{aligned}$$

4. SUBSTRUCTURED DISCRETE EQUATIONS

The fluid and solid domains are decomposed into nonoverlapping subdomains that consist of union of elements,

$$\overline{\Omega}_f = \bigcup_{s=1}^{N_f} \overline{\Omega}_e^s, \qquad \overline{\Omega}_e = \bigcup_{s=1}^{N_e} \overline{\Omega}_e^s. \tag{7}$$

The vectors of degrees of freedom corresponding to Ω_f^s and Ω_e^s are denoted by $\mathbf{p}^s$ and $\mathbf{u}^s$, respectively.

The subdomain matrices are defined by subassembly,

$$\mathbf{p}^{s*}\mathbf{K}_f^s\mathbf{q} = \int_{\Omega_f^s} \nabla p \cdot \nabla q, \qquad \mathbf{p}^{s*}\mathbf{M}_f^s\mathbf{q}^s = \int_{\Omega_f^s} pq,$$

$$\mathbf{p}^{s*}\mathbf{G}_f^s\mathbf{q}^s = \int_{\partial\Omega_f^s\cap\Gamma_a} pq, \qquad \mathbf{u}^{s*}\mathbf{K}_e^s\mathbf{v}^s = \int_{\Omega_e^s} \lambda(\nabla\cdot u)(\nabla\cdot v) + 2\mu e(u) : e(v),$$

$$\mathbf{u}^{s*}\mathbf{M}_e^s\mathbf{v} = \int_{\Omega_e^s} \rho_e(u\cdot v), \qquad \mathbf{p}^{r*}\mathbf{T}^{rs}\mathbf{v}^s = \int_{\partial\Omega_f^r\cap\partial\Omega_e^s} p(\nu\cdot v).$$

We will use vectors consisting of all subdomain degrees of freedom,

$$\hat{\mathbf{p}} = \begin{bmatrix} \mathbf{p}^1 \\ \vdots \\ \mathbf{p}^{N_f} \end{bmatrix}, \qquad \hat{\mathbf{u}} = \begin{bmatrix} \mathbf{u}^1 \\ \vdots \\ \mathbf{u}^{N_e} \end{bmatrix},$$

and the corresponding partitioned matrices,

$$\hat{\mathbf{K}}_f = \operatorname{diag}(\mathbf{K}_f^s) = \begin{bmatrix} \mathbf{K}_f^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{K}_f^{N_f} \end{bmatrix}, \qquad \hat{\mathbf{K}}_e = \operatorname{diag}(\mathbf{K}_e^s) = \begin{bmatrix} \mathbf{K}_e^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{K}_e^{N_e} \end{bmatrix}.$$

The matrices $\hat{\mathbf{M}}_f$, $\hat{\mathbf{G}}_f$, and $\hat{\mathbf{M}}_e$ defined similarly, and

$$\hat{\mathbf{T}} = (\mathbf{T}^{rs})_{rs} = \begin{bmatrix} \mathbf{T}^{11} & \dots & \mathbf{T}^{1,N_e} \\ \vdots & \ddots & \vdots \\ \mathbf{T}^{N_f,1} & \dots & \mathbf{T}^{N_f,N_e} \end{bmatrix}.$$

Let $\mathbf{N}_f$ and $\mathbf{N}_e$ be the matrices with $0, 1$ entries of the global to local maps corresponding to the decompositions of Ω_f and Ω_e, respectively, cf., (7), so that

$$\mathbf{K}_f = \mathbf{N}_f^*\hat{\mathbf{K}}_f\mathbf{N}_f, \qquad \mathbf{K}_e = \mathbf{N}_e^*\hat{\mathbf{K}}_e\mathbf{N}_e.$$

Let $\mathbf{B}_f = (\mathbf{B}_f^1, \dots, \mathbf{B}_f^{N_f})$ and $\mathbf{B}_e = (\mathbf{B}_e^1, \dots, \mathbf{B}_e^{N_e})$ be matrices with entries $0, \pm 1$ such that the conditions $\mathbf{B}_f\hat{\mathbf{p}} = 0$ and $\mathbf{B}_e\hat{\mathbf{u}} = 0$ express the constraint that the values of the same degrees of freedom on two different subdomains coincide, that

is,

$$\mathbf{B}_f\hat{\mathbf{p}} = 0 \iff \hat{\mathbf{p}} = \mathbf{N}_f\mathbf{p} \quad \text{for some} \quad \mathbf{p} \tag{8}$$

$$\mathbf{B}_e\hat{\mathbf{u}} = 0 \iff \hat{\mathbf{u}} = \mathbf{N}_e\mathbf{u} \quad \text{for some} \quad \mathbf{u}. \tag{9}$$

Multiplying the second equation in (6) by $\omega^2\rho_f$ to symmetrize the off-diagonal block and introducing Lagrange multipliers λ_f and λ_e for the constraints $\mathbf{B}_f\mathbf{p} = 0$ and $\mathbf{B}_e\mathbf{u} = 0$, we get the system of linear equations in block form,

$$\begin{bmatrix} -\hat{\mathbf{K}}_f + k^2\hat{\mathbf{M}}_f - ik\hat{\mathbf{G}} & -\omega^2\rho_f\hat{\mathbf{T}} & \mathbf{B}_f^* & 0 \\ -\omega^2\rho_f\hat{\mathbf{T}}^* & \omega^2\rho_f(-\hat{\mathbf{K}}_e + \omega^2\hat{\mathbf{M}}_e) & 0 & \mathbf{B}_e^* \\ \mathbf{B}_f & 0 & 0 & 0 \\ 0 & \mathbf{B}_e & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{p}} \\ \hat{\mathbf{u}} \\ \lambda_f \\ \lambda_e \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{r}} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{10}$$

where $\mathbf{N}^*\hat{\mathbf{r}} = \mathbf{r}$. The system (10) is equivalent to (6) in the following sense.

LEMMA 4.1. *If $(\hat{\mathbf{p}}, \hat{\mathbf{u}}, \lambda_f, \lambda_e)$ is a solution of (10), then $\hat{\mathbf{p}} = \mathbf{N}_f\mathbf{p}$ and $\hat{\mathbf{u}} = \mathbf{N}_e\mathbf{u}$, where $(\mathbf{p}, \mathbf{u})$ solves (6). Conversely, if $(\mathbf{p}, \mathbf{u})$ is a solution of (6), then there exist λ_f and λ_e such that $(\hat{\mathbf{p}}, \hat{\mathbf{u}}, \lambda_f, \lambda_e)$ is a solution of (10).*

Proof. To simplify notation, write (6) as $\mathbf{N}^*\hat{\mathbf{S}}\mathbf{N}\mathbf{x} = \mathbf{N}^*\hat{\mathbf{b}}$ and (10) as $\hat{\mathbf{S}}\hat{\mathbf{x}} + \mathbf{B}^*\lambda = \hat{\mathbf{b}}$, $\mathbf{B}\hat{\mathbf{x}} = 0$. Then (8), (9) become $\operatorname{Null}\mathbf{B} = \operatorname{Range}\mathbf{N}$.

Suppose that $\hat{\mathbf{S}}\hat{\mathbf{x}} + \mathbf{B}^*\lambda = \hat{\mathbf{b}}$ and $\mathbf{B}\hat{\mathbf{x}} = 0$. From the second equation and $\operatorname{Null}\mathbf{B} = \operatorname{Range}\mathbf{N}$, it follows that there is $\mathbf{x}$ such that $\hat{\mathbf{x}} = \mathbf{N}\mathbf{x}$. Multiplying the first equation by $\mathbf{N}^*$ and using the fact that $\mathbf{N}^*\mathbf{B}^* = (\mathbf{B}\mathbf{N})^* = 0$, we obtain $\mathbf{N}^*\hat{\mathbf{S}}\mathbf{N}\mathbf{x} = \mathbf{N}^*\hat{\mathbf{b}}$.

Conversely, suppose that $\mathbf{N}^*\hat{\mathbf{S}}\mathbf{N}\mathbf{x} = \mathbf{N}^*\hat{\mathbf{b}}$. Let $\hat{\mathbf{x}} = \mathbf{N}\mathbf{x}$. Then $\mathbf{B}\hat{\mathbf{x}} = 0$, and

$$\hat{\mathbf{S}}\hat{\mathbf{x}} - \hat{\mathbf{b}} \in \operatorname{Null}\mathbf{N}^* = (\operatorname{Range}\mathbf{N})^\perp = (\operatorname{Null}\mathbf{B})^\perp = \operatorname{Range}\mathbf{B}^*,$$

hence $\hat{\mathbf{S}}\mathbf{x} - \hat{\mathbf{b}} = \mathbf{B}^*\lambda$ for some λ. ∎

Note that symmetry of the system or the existence of a Lagrangean functional are not needed.

We will want to eventually eliminate the variables $\hat{\mathbf{p}}$ and $\hat{\mathbf{u}}$. Because the matrices $-\hat{\mathbf{K}}_f + k^2\hat{\mathbf{M}}_f$ and $-\hat{\mathbf{K}}_e + \omega^2\hat{\mathbf{M}}_e$ are typically close to singular due to near-resonance, we replace them by *regularized matrices*

$$\begin{aligned} \hat{\mathbf{A}}_f &= -\hat{\mathbf{K}}_f + k^2\hat{\mathbf{M}}_f + ik\hat{\mathbf{G}}_f + \hat{\mathbf{R}}_f, \\ \hat{\mathbf{A}}_e &= -\hat{\mathbf{K}}_e + \omega^2\hat{\mathbf{M}}_e + \hat{\mathbf{R}}_e, \end{aligned}$$

where the regularization matrices are given by

$$\hat{\mathbf{R}}_f = \operatorname{diag}(\mathbf{R}_f^s), \qquad \mathbf{p}^{s*}\mathbf{R}_f^s\mathbf{q}^s = ik\sum_{\substack{t=1 \\ t\neq s}}^{N_f}\sigma_f^{st}\int\limits_{\partial\Omega_f^s\cap\partial\Omega_f^t} pq,$$

$$\hat{\mathbf{R}}_e = \operatorname{diag}(\mathbf{R}_e^s), \qquad \mathbf{u}^{s*}\mathbf{R}_e^s\mathbf{v}^s = i\omega\rho_e\sum_{\substack{t=1 \\ t\neq s}}^{N_e}\sigma_e^{st}\int\limits_{\partial\Omega_e^s\cap\partial\Omega_e^t} (n\cdot u)(n\cdot v),$$

where $\sigma_f^{st} \in \{0, \pm 1\}$, $\sigma_f^{st} = -\sigma_f^{ts}$, and $\sigma_e^{st} = \{0, \pm 1\}$, $\sigma_e^{st} = -\sigma_e^{ts}$.

It is shown in [6] that if for a given s, all $\sigma_f^{st} \geq 0$ or all $\sigma_f^{st} \leq 0$ with some $\sigma_f^{st} \neq 0$, then $\hat{\mathbf{A}}_f^s$ is invertible. The case of solid subdomains is similar. For details on strategies for choosing σ_f^{st} to guarantee this, see [6]. In our computations, we simply choose $\sigma_f^{st} = +1$ if $s > t$, $\sigma_f^{st} = -1$ if $s < t$, and similarly for σ_e^{st}.

Because

$$\mathbf{N}_f^* \mathbf{R}_f \mathbf{N}_f = 0, \qquad \mathbf{N}_e^* \mathbf{R}_e \mathbf{N}_e = 0,$$

the system (10) is equivalent to

$$\begin{bmatrix} \hat{\mathbf{A}}_f & -\omega^2 \rho_f \hat{\mathbf{T}} & \mathbf{B}_f^* & 0 \\ -\omega^2 \rho_f \hat{\mathbf{T}}^* & \omega^2 \rho_f \hat{\mathbf{A}}_e & 0 & \mathbf{B}_e^* \\ \mathbf{B}_f & 0 & 0 & 0 \\ 0 & \mathbf{B}_e & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{p}} \\ \hat{\mathbf{u}} \\ \lambda_f \\ \lambda_e \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{r}} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{11}$$

Since the value of $\hat{\mathbf{T}}\hat{\mathbf{u}}$ depends on the values of $\hat{\mathbf{u}}$ on Γ only, we have

$$\hat{\mathbf{T}}\hat{\mathbf{u}} = \hat{\mathbf{T}}\mathbf{J}_e \hat{\mathbf{u}}_\Gamma, \qquad \hat{\mathbf{u}}_\Gamma = \mathbf{J}_e^* \hat{\mathbf{u}},$$

where $\hat{\mathbf{J}}_e$ is the matrix of the operator of embedding a subvector that corresponds to degrees of freedom on Γ into $\hat{\mathbf{u}}$ by adding zero entries. Similarly,

$$\hat{\mathbf{T}}^* \hat{\mathbf{p}} = \hat{\mathbf{T}}^* \mathbf{J}_f \hat{\mathbf{p}}_\Gamma, \qquad \hat{\mathbf{p}}_\Gamma = \mathbf{J}_f^* \hat{\mathbf{p}}.$$

Therefore, we obtain the augmented system equivalent to (11),

$$\begin{bmatrix} \hat{\mathbf{A}}_f & 0 & \mathbf{B}_f^* & 0 & 0 & -\omega^2 \rho_f \hat{\mathbf{T}} \mathbf{J}_e \\ 0 & \omega^2 \rho_f \hat{\mathbf{A}}_e & 0 & \mathbf{B}_e^* & -\omega^2 \rho_f \hat{\mathbf{T}}^* \mathbf{J}_f & 0 \\ \mathbf{B}_f & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{B}_e & 0 & 0 & 0 & 0 \\ \mathbf{J}_f^* & 0 & 0 & 0 & -\mathbf{I} & 0 \\ 0 & \mathbf{J}_e^* & 0 & 0 & 0 & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{p}} \\ \hat{\mathbf{u}} \\ \lambda_f \\ \lambda_e \\ \hat{\mathbf{p}}_\Gamma \\ \hat{\mathbf{u}}_\Gamma \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{r}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

Because the variables in a coupled system typically have vastly different scales, we use symmetric diagonal scaling to get the scaled system

$$\begin{bmatrix} \tilde{\mathbf{A}}_f & 0 & \tilde{\mathbf{B}}_f^* & 0 & 0 & -\tilde{\mathbf{T}} \mathbf{J}_e \\ 0 & \tilde{\mathbf{A}}_e & 0 & \tilde{\mathbf{B}}_e^* & -\tilde{\mathbf{T}}^* \mathbf{J}_f & 0 \\ \tilde{\mathbf{B}}_f & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{\mathbf{B}}_e & 0 & 0 & 0 & 0 \\ \mathbf{J}_f^* & 0 & 0 & 0 & -\mathbf{I} & 0 \\ 0 & \mathbf{J}_e^* & 0 & 0 & 0 & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{p}} \\ \tilde{\mathbf{u}} \\ \tilde{\lambda}_f \\ \tilde{\lambda}_e \\ \tilde{\mathbf{p}}_\Gamma \\ \tilde{\mathbf{u}}_\Gamma \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{r}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{13}$$

where the matrices and the vectors scale as

$$\tilde{\mathbf{A}}_f = \mathbf{D}_f \hat{\mathbf{A}}_f \mathbf{D}_f, \quad \tilde{\mathbf{A}}_e = \omega^2 \rho_f \mathbf{D}_e \hat{\mathbf{A}}_e \mathbf{D}_e, \quad \tilde{\mathbf{T}} = \omega^2 \rho_f \mathbf{D}_f \hat{\mathbf{T}} \mathbf{D}_e, \tag{14}$$

$$\tilde{\mathbf{B}}_f = \mathbf{E}_f \mathbf{B}_f \mathbf{D}_f, \quad \tilde{\mathbf{B}}_e = \mathbf{E}_e \mathbf{B}_e \mathbf{D}_e, \quad \tilde{\mathbf{r}} = \mathbf{D}_f \hat{\mathbf{r}}, \tag{15}$$

$$\hat{\mathbf{p}} = \mathbf{D}_f \tilde{\mathbf{p}}, \quad \hat{\mathbf{u}} = \mathbf{D}_e \tilde{\mathbf{u}}, \quad \lambda_f = \mathbf{D}_f \tilde{\lambda}_f, \quad \lambda_e = \mathbf{D}_e \tilde{\lambda}_e. \tag{16}$$

The scaling matrices $\mathbf{D}_f$, $\mathbf{D}_e$, $\mathbf{E}_f$, and $\mathbf{E}_e$, are diagonal. We have chosen scaling matrices with positive diagonal entries such that the absolute values of the diagonal entries of $\tilde{\mathbf{A}}_f$ and $\tilde{\mathbf{A}}_e$ are one and the ℓ^2 norms of the columns of $\tilde{\mathbf{B}}_e$ and $\tilde{\mathbf{B}}_f$ are one. Other, perhaps better, scaling choices are of course possible.

The proposed method consists of eliminating the variables $\hat{\mathbf{p}}$ and $\hat{\mathbf{u}}$ from the augmented system (13) and solving the resulting reduced system by Generalized Conjugate Residuals, with preconditioning by projection on a coarse space with few degrees of freedom per subdomain.

Computing $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{u}}$ from the first two equations in (13) gives

$$\tilde{\mathbf{p}} = \tilde{\mathbf{A}}_f^{-1}(\tilde{\mathbf{r}} - \tilde{\mathbf{B}}_f^*\tilde{\lambda}_f + \tilde{\mathbf{T}}\mathbf{J}_e\tilde{\mathbf{u}}_\Gamma) \tag{17}$$

$$\tilde{\mathbf{u}} = \tilde{\mathbf{A}}_e^{-1}(-\tilde{\mathbf{B}}_e^*\tilde{\lambda}_e + \tilde{\mathbf{T}}^*\mathbf{J}_f\tilde{\mathbf{p}}_\Gamma) \tag{18}$$

Substituting $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{u}}$ from (17), (18) into the rest of the equations in (13), we obtain the reduced system

$$\mathbf{F}\mathbf{x} = \mathbf{b}, \tag{19}$$

where

$$\mathbf{F} = \begin{bmatrix} \tilde{\mathbf{B}}_f\tilde{\mathbf{A}}_f^{-1}\tilde{\mathbf{B}}_f^* & 0 & 0 & -\tilde{\mathbf{B}}_f\tilde{\mathbf{A}}_f^{-1}\tilde{\mathbf{T}}\mathbf{J}_e \\ 0 & \tilde{\mathbf{B}}_e\tilde{\mathbf{A}}_e^{-1}\tilde{\mathbf{B}}_e & -\tilde{\mathbf{B}}_e\tilde{\mathbf{A}}_e^{-1}\tilde{\mathbf{T}}^*\mathbf{J}_f & \\ -\mathbf{J}_f^*\tilde{\mathbf{A}}_f^{-1}\tilde{\mathbf{B}}_f^* & 0 & -\mathbf{I} & \mathbf{J}_f^*\tilde{\mathbf{A}}_f^{-1}\tilde{\mathbf{T}}\mathbf{J}_e \\ 0 & -\mathbf{J}_e\tilde{\mathbf{A}}_e^{-1}\tilde{\mathbf{B}}_e & \mathbf{J}_e\tilde{\mathbf{A}}_e^{-1}\tilde{\mathbf{T}}^*\mathbf{J}_f & -I \end{bmatrix}, \tag{20}$$

and

$$\mathbf{x} = \begin{bmatrix} \lambda_f \\ \lambda_e \\ \tilde{\mathbf{p}}_\Gamma \\ \tilde{\mathbf{u}}_\Gamma \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} \tilde{\mathbf{B}}_f\tilde{\mathbf{A}}_f^{-1}\tilde{\mathbf{r}} \\ 0 \\ -\mathbf{J}_f^*\tilde{\mathbf{A}}_f^{-1}\tilde{\mathbf{r}} \\ 0 \end{bmatrix}.$$

In equation (20), the first diagonal block $\tilde{\mathbf{B}}_f\tilde{\mathbf{A}}_f^{-1}\tilde{\mathbf{B}}_f^*$ is exactly same as in the FETI-H method for the Helmholtz equation. The second diagonal block $\tilde{\mathbf{B}}_f\tilde{\mathbf{A}}_f^{-1}\tilde{\mathbf{B}}_f^*$ is the analogue of FETI-H for the elastodynamic problem. FETI-H is known to converge fast, so one can expect that on the subspace defined by the coarse correction (cf., Section 5 below), these two diagonal blocks will be well conditioned. The off-diagonal blocks all contain the inverse of $\mathbf{A}_f$, resp. $\mathbf{A}_e$, which are discretizations of differential operators. Hence, the off-diagonal blocks are discretizations of compact operators, and the structure of the equation (19) resembles discretization of a Fredholm integral equation of the second kind, which is well conditioned because its eigenvalues cluster about a point different from zero. Such clustering is indeed observed computationally, cf., Section 7 below.

Evaluating the matrix vector product $\mathbf{F}\mathbf{x}$ requires the solution of one independent problem per subdomain, because

$$\mathbf{F}\begin{bmatrix} \lambda_f \\ \lambda_e \\ \tilde{\mathbf{p}}_\Gamma \\ \tilde{\mathbf{u}}_\Gamma \end{bmatrix} = \begin{bmatrix} -\tilde{\mathbf{B}}_f\tilde{\mathbf{q}} \\ -\tilde{\mathbf{B}}_e\tilde{\mathbf{v}} \\ \mathbf{J}_f^*\tilde{\mathbf{q}} - \tilde{\mathbf{p}}_\Gamma \\ \mathbf{J}_e^*\tilde{\mathbf{v}} - \tilde{\mathbf{u}}_\Gamma \end{bmatrix}, \quad \text{where} \quad \begin{cases} \tilde{\mathbf{q}} = \tilde{\mathbf{A}}_f^{-1}(-\tilde{\mathbf{B}}_f^*\tilde{\lambda}_f + \tilde{\mathbf{T}}\mathbf{J}_e\tilde{\mathbf{u}}_\Gamma), \\ \tilde{\mathbf{v}} = \tilde{\mathbf{A}}_e^{-1}(-\tilde{\mathbf{B}}_e^*\tilde{\lambda}_e + \tilde{\mathbf{T}}^*\mathbf{J}_f\tilde{\mathbf{p}}_\Gamma). \end{cases}$$

5. ITERATIVE SOLUTION

The present method consists of solving the linear system (19) by the Generalized Conjugate Residual (GCR) method [12] preconditioned by a subspace correction. GCR is mathematically equivalent to GMRES and easier to program, but less numerically stable. Because the number of iterations was small, GCR was sufficient for our purposes.

Let $\mathbf{Q}$ be a matrix with the same number of rows as $\mathbf{F}$ and linearly independent columns. The columns of $\mathbf{Q}$ form the basis of the *coarse space*. We wish to enforce the condition that the residual is orthogonal to the coarse space,

$$\mathbf{Q}^*(\mathbf{F}\mathbf{x} - \mathbf{b}) = 0, \tag{21}$$

by adding a correction from the coarse space in each iteration. Define the correction operator $\mathcal{C}$ by

$$\mathcal{C}(\mathbf{v}, \mathbf{b}) = \mathbf{v} + \mathbf{Q}\mathbf{w}, \qquad \mathbf{Q}^*(\mathbf{F}(\mathbf{v} + \mathbf{Q}\mathbf{w}) - \mathbf{b}) = 0.$$

We first make make sure that the initial approximation satisfies (21): If $\mathbf{v}$ is given, the initial approximation is $\mathbf{x}^{(0)} = \mathcal{C}(\mathbf{v}, \mathbf{b})$. The system (19) is then solved by the GCR method with left preconditioning by the action of the projection $\mathbf{P}$, given by

$$\mathbf{P}\mathbf{v} = \mathcal{C}(\mathbf{v}, 0) = (\mathbf{I} - \mathbf{Q}(\mathbf{Q}^*\mathbf{F}\mathbf{Q})^{-1}\mathbf{Q}^*\mathbf{F})\mathbf{v}.$$

Equivalently, the GCR method is applied to the preconditioned system

$$\mathbf{P}\mathbf{F}\mathbf{x} = \mathbf{P}\mathbf{b}. \tag{22}$$

In step m, the GCR method computes $\mathbf{x}^{(m)}$ by adding to $\mathbf{x}^{(m-1)}$ a linear combination of preconditioned residuals $\mathbf{P}(\mathbf{F}\mathbf{x}^{(k)} - \mathbf{b})$, $k = 0, \ldots, m-1$, so that the ℓ^2 norm of the residual $\|\mathbf{P}(\mathbf{F}\mathbf{x}^{(m)} - \mathbf{b})\|$ is minimal. Because $\mathbf{Q}^*\mathbf{F}\mathbf{P} = 0$, it follows that the iterates run in a subspace,

$$\mathbf{Q}^*\mathbf{F}(\mathbf{x}^{(m)} - \mathbf{x}^{(0)}) = 0,$$

hence, from the selection of $\mathbf{x}^{(0)}$,

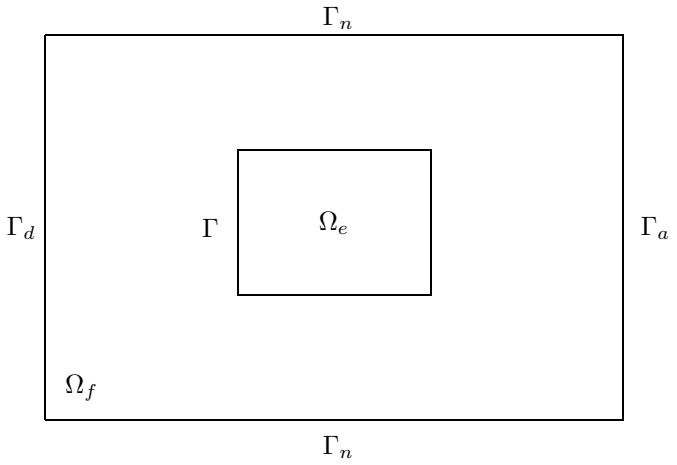
$$\mathbf{Q}^*(\mathbf{F}\mathbf{x}^{(m)} - \mathbf{b}) = 0.$$

We choose the matrix $\mathbf{Q}$ of the form

$$\mathbf{Q} = \begin{bmatrix} \mathbf{D}_f\mathbf{B}_f\mathrm{diag}(\mathbf{Y}_f^s)_s & 0 & 0 & 0 \\ 0 & \mathbf{D}_e\mathbf{B}_e\mathrm{diag}(\mathbf{Y}_e^s)_s & 0 & 0 \\ 0 & 0 & \mathbf{D}_f\mathbf{J}_f^*\mathrm{diag}(\mathbf{Z}_f^s)_s & 0 \\ 0 & 0 & 0 & \mathbf{D}_e\mathbf{J}_e^*\mathrm{diag}(\mathbf{Z}_e^s)_s \end{bmatrix}.$$

For a fluid subdomain Ω_f^s, we choose $\mathbf{Y}_f^s$ as the matrix of columns that are discrete representations of plane waves in a small number of equally distributed directions, or discrete representation of the constant function. For a solid subdomain Ω_e^s, the

FIG. 1. Model 2D Problem



columns of $\mathbf{Y}_e^s$ are discrete representations of plane pressure and shear waves, or of rigid body motions. The matrices $\mathbf{Z}_f^s$ and $\mathbf{Z}_e^s$ are chosen in the same way as $\mathbf{Y}_f^s$ and $\mathbf{Y}_e^s$, with possibly different selection of the number of directions and selection of constant or rigid body modes. Some of the matrices $\mathbf{Y}_f^s$, $\mathbf{Y}_e^s$, $\mathbf{Z}_e^s$, or $\mathbf{Y}_e^s$ may be void.

6. COMPUTATIONAL RESULTS

We consider a model 2D problem with a scatterer in the center a waveguide, cf., Fig. 1. The fluid domain Ω_f is a square with side $1\,m$, filled with water with density $\rho_f = 1000\,kg\,m^{-3}$ and speed of sound $c_f = 1500\,m\,s^{-1}$. The scatterer is a square in the center o the fluid domain, with side $0.2\,m$, and consisting of aluminum with dnsity $\rho_e = 2700\,kg\,m^{-3}$ and Lamé elasticity coefficients $\lambda = 5.5263.10^{10}\,N\,m^{-2}$, $\mu = 2.595.10^{10}\,N\,m^{-2}$.

For numerical stability, the rows of the matrices $\mathbf{B}_e$, $\mathbf{B}_f$, and the columns of $\mathbf{Q}$ were orthogonalized. The number of reduced variables reported in the tables is the number of the Lagrange multipliers to enforce the linearly independent constraints. The number of coarse degrees of freedom is the size of the basis of the columns of the matrix $\mathbf{Q}$.

The fluid domain and the solid domain are discretized by bilinear square elements on a uniform mesh with meshsize h. Both domains are divided independently into m by n subdomains by dividing their horizontal sides into m intervals of the same length and the vertical sides into n intervals of the same length. The iterations were stopped when the relative residual in the ℓ^2 norm of the reduced and scaled system (19) was less than 10^{-6}.

The numerical results for the model problem are summarized in Tables 1 to 3. The lines in the tables are organized in groups. The first line in the group is the result for the FETI-H method with Neumann condition instead of the elastic scatterer. The following lines are for the elastic scatterer, and they show the number of iterations for increasing the number of coarse space functions, that is, the columns of $\mathbf{Q}$. In all cases, the constant function is included in the coarse space for the fluid and the rigid body modes are included in the coarse space for the solid, both for

TABLE 1

Decreasing mesh size and wavelength, 1 solid subdomain

Problem description				Coarse directions				Number of degrees of freedom			
		Subdomains		Multipliers		Wet interf.					
h	k	Fluid	Solid	Fl.	Sol.	Fl.	Sol.	Orig.	Red.	Coarse	Iter.
1/40	10	4x4	rigid	4				1632	240	63	11
1/40	10	4x4	1x1	4	0	0	0	1794	312	63	18
1/40	10	4x4	1x1	4	0	4	0	1794	312	86	13
1/80	20	4x4	rigid	4				6336	464	63	20
1/80	20	4x4	1x1	4	0	0	0	6914	600	63	24
1/80	20	4x4	1x1	4	0	4	0	6914	600	86	20
1/160	40	4x4	rigid	4				24960	912	63	52
1/160	40	4x4	1x1	4	0	0	0	27138	1176	63	64
1/160	40	4x4	1x1	4	0	4	0	27138	1176	86	55
1/160	40	4x4	rigid	8				24960	912	102	20
1/160	40	4x4	1x1	8	0	4	0	27138	1176	125	21
1/160	40	4x4	1x1	8	4	4	4	27138	1176	126	22

TABLE 2

Increasing number of solid subdomains

Problem description				Coarse directions				Number of degrees of freedom			
		Subdomains		Multipliers		Wet interf.					
h	k	Fluid	Solid	Fl.	Sol.	Fl.	Sol.	Orig.	Red.	Coarse	Iter.
1/250	20	4x4	rigid	4				60600	1416	63	32
1/250	20	4x4	1x1	4	0	0	0	65802	1824	63	37
1/250	20	4x4	1x1	4	0	4	0	65802	1824	86	32
1/250	20	4x4	rigid	4				60600	1416	63	32
1/250	20	4x4	2x2	4	0	0	0	65802	2034	72	48
1/250	20	4x4	2x2	4	0	4	0	65802	2034	100	42
1/250	20	4x4	rigid	4				60600	1416	63	32
1/250	20	4x4	3x3	4	0	0	0	65802	1824	63	37
1/250	20	4x4	3x3	4	0	4	0	65802	1824	86	32
1/250	20	4x4	rigid	8				60600	1416	102	26
1/250	20	4x4	1x1	8	0	4	0	65802	1824	125	27
1/250	20	4x4	1x1	8	4	4	4	65802	1824	126	27
1/250	20	4x4	rigid	8				60600	1416	102	26
1/250	20	4x4	2x2	8	0	4	0	65802	2034	139	39
1/250	20	4x4	2x2	8	4	4	4	65802	2034	148	36
1/250	20	4x4	rigid	8				60600	1416	102	26
1/250	20	4x4	3x3	8	0	4	0	65802	1824	125	27
1/250	20	4x4	3x3	8	4	4	4	65802	1824	126	27

TABLE 3
Increasing number of fluid and solid subdomains

Problem description				Coarse directions				Number of degrees of freedom			
		Subdomains		Multipliers		Wet interf.					
h	k	Fluid	Solid	Fl.	Sol.	Fl.	Sol.	Orig.	Red.	Coarse	Iter.
1/315	32	3x3	rigid	4				96012	1268	32	45
1/315	32	3x3	1x1	4	0	0	0	104204	1776	32	56
1/315	32	3x3	1x1	4	0	4	0	104204	1776	40	51
1/315	32	5x5	rigid	4				96012	2292	96	45
1/315	32	5x5	3x3	4	0	0	0	104204	2804	96	57
1/315	32	5x5	3x3	4	0	4	0	104204	2804	111	46
1/315	32	3x3	rigid	8				96012	1268	52	27
1/315	32	3x3	1x1	8	0	4	0	104204	1776	60	34
1/315	32	3x3	1x1	8	4	4	4	104204	1776	61	34
1/315	32	5x5	rigid	8				96012	2292	156	28
1/315	32	5x5	3x3	8	0	4	0	104204	2804	171	31
1/315	32	5x5	3x3	8	4	4	4	104204	2804	172	31

the multipliers and for the wet interface. The prototype implementation was done in MATLAB, hence we do not report timings.

Table 1 shows that the method is scalable when the frequency is increased and the mesh is correspondingly refined. Since there is only one solid subdomain, the effect of increasing the coarse space for the solid and for the wet interface was minor. The results in Table 2 show that the number of iterations does not increase when the scattered is decomposed into increasing number of subdomains. Finally, Table 3 shows the number of iterations does not grow when a fixed problem is decomposed into a growing number of both fluid and solid subdomains.

7. SPECTRUM OF THE ITERATION OPERATOR

Convergence of the method can be explained by an analysis of the spectrum of the iteration operator. From well-known properties of Krylov space methods [12], it follows that in m iterations, the residual is reduced by the factor of at least $Cr(m)$, where

$$r(m) = \min_{\substack{\deg(p)=m \\ p(0)=1}} \max_{\substack{\lambda\in\sigma(\mathbf{PF}) \\ \lambda\neq 0}} |p(\lambda)|.$$

Here, the minimum is taken over complex polynomials and the maximum over eigenvalues of $\mathbf{PF}$.

Figures 2 and 3 show one representative case of the spectrum of the iteration operator $\mathbf{PF}$ from (22) for Neumann boundary condition on the scatterer, and for the elastic scatterer, respectively. The problem setting was $h = 1/40$, $k = 10$, 4×4 fluid subdomains and, for Figure 3, 2×2 solid subdomains. The coarse space had 4 directions for fluid waves, both for the multipliers and for the wet interface.

The figures show that in both cases, the spectrum is clustered around a point other than the origin. For the coupled problem, there are few more eigenvalues near

FIG. 2. Spectrum of the FETI-H iteration operator

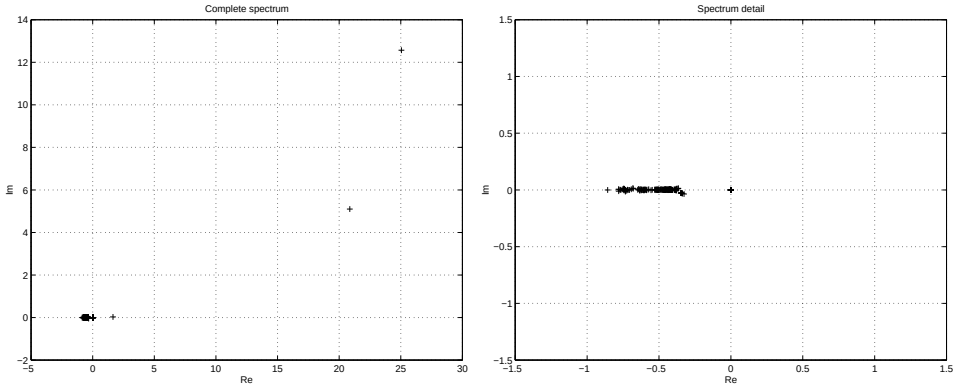
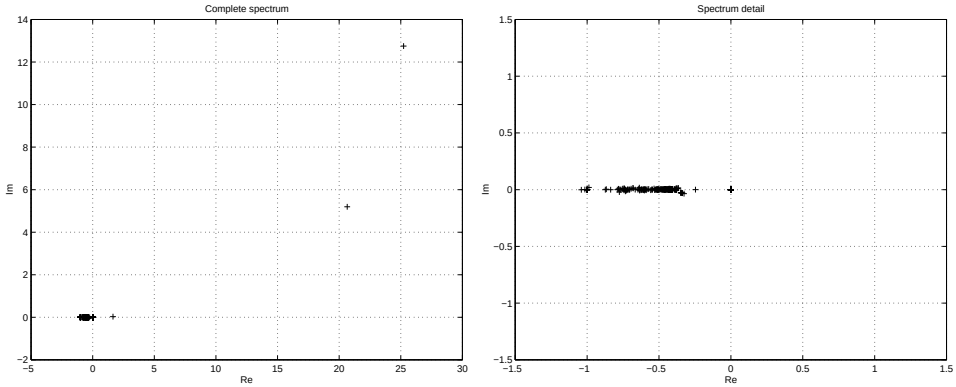


FIG. 3. Spectrum of the iteration operator for coupled problem



the origin and some clustered around -1. Because the eigenvalues are clustered around a point different from zero, a polynomial with a low value of $r(m)$ exists, and convergence is fast. The effect of adding the elastic scatterer shows in few extra eigenvalues close to the cluster and a minor change in other eigenvalues.

The change in the spectrum can be heuristically explained as follows, cf., also the remarks at the end of Section 5. The last two by two diagonal block (with $-\mathbf{I}$ on the diagonal) in (20) is a discretization of a Fredholm integral equation of the second kind, with eigenvalues clustered around -1. The influence of the other off-diagonal blocks is relatively weak. The FETI-H method already has eigenvalues around -1, so the overall effect of adding the elastic scatterer on the spectrum of the iteration operator is small.

8. CONCLUSION

We have presented a new fast substructuring method for coupled fluid-solid acoustic problems. The computational results indicate that the method is scalable with respect to mesh size, frequency, and the number of subdomains. The growth of the number of iterations can be controlled by increasing the size of the coarse space. Numerical calculation of the spectrum of the iteration operator suggests that the fast convergence is due to clustering of the spectrum. In most cases, the resulting number of iterations is same or slightly larger than for the FETI-H method

for the same problem with the Neumann boundary condition instead of an elastic scatterer.

Performance study of the method for realistic problems, 3D computations, parallel implementation, and theoretical analysis will be reported elsewhere.

ACKNOWLEDGMENT

This research was supported in part by the Office of Naval Research under grant N-00014-95-1-0663, and the National Science foundation under grants ECS-9725504 and DMS-007428. The author would like to thank Charbel Farhat, Rabia Djellouli, and other coworkers at the Center for Aerospace Structures at the University of Colorado at Boulder for useful discussions, stimulating remarks, and their patience during several presentations of early versions of this work. The prototype MATLAB code used parts of a code originally written by Mirela Popa and the author [15].

REFERENCES

1. S. C. Brenner, *An additive Schwarz preconditioner for the FETI method*. Submitted.
2. P. G. Ciarlet, *Basic error estimates for elliptic problems*, in Handbook of Numerical Analysis, P. Ciarlet and J. L. Lions, eds., vol. II, North-Holland, Amsterdam, 1989, pp. 17–352.
3. A. De La Bourdonnaye, C. Farhat, A. Macedo, F. Magoulès, and F.-X. Roux, *A non-overlapping domain decomposition method for the exterior Helmholtz problem*, Contemporary Mathematics, 218 (1998), pp. 42–66. Proceedings of the Tenth Conference on Domain Decomposition, J. Mandel, C. Farhat, and X.-C. Cai, eds.
4. C. Farhat, P. S. Chen, and J. Mandel, *Scalable Lagrange multiplier based domain decomposition method for time-dependent problems*, Int. J. Numer. Meth. Engrg., 38 (1995), pp. 3831–3853.
5. C. Farhat, P.-S. Chen, J. Mandel, and F.-X. Roux, *The two-level FETI method - Part II: Extension to shell problems, parallel implementation and performance results*, Comp. Meth. Appl. Mech. Engrg., 155 (1998), pp. 153–180.
6. C. Farhat, A. Macedo, and M. Lesoinne, *A two-level domain decomposition method for the iterative solution of high frequency exterior Helmholtz problem*, Numerische Mathematik, 85 (2000), pp. 283–308.
7. C. Farhat, A. Macedo, M. Lesoinne, F.-X. Roux, F. Magoulès, and A. De La Bourdonnaye, *Two-level domain decomposition methods with Lagrange multipliers for the fast solution of acoustic scattering problems*, Comput. Methds. Appl. Mech. Engrg., 184 (2000), pp. 213–239.
8. C. Farhat, A. Macedo, and R. Tezaur, *FETI-H: a scalable domain decomposition method for high frequency exterior Helmholtz problems*, in Eleventh International Conference on Domain Decomposition Method, C.-H. Lai, P. Bjørstad, M. Cross, and O. Widlund, eds., DDM.ORG, 1999, pp. 231–241.
9. C. Farhat and J. Mandel, *Scalable substructuring by Lagrange multipliers in theory and practice*, in Proceedings of the 9th International Conference on Domain Decomposition, Bergen, Norway, June 1996, DDM.ORG, 1998, pp. 20–30.
10. ———, *The two-level FETI method for static and dynamic plate problems - Part I: An optimal iterative solver for biharmonic systems*, Comp. Meth. Appl. Mech. Engrg., 155 (1998), pp. 129–152.
11. C. Farhat and F.-X. Roux, *An unconventional domain decomposition method for an efficient parallel solution of large-scale finite element systems*, SIAM J. Sci. Stat. Comput., 13 (1992), pp. 379–396.
12. A. Greenbaum, *Iterative Methods for Solving Linear Systems*, SIAM, Philadelphia, 1997.
13. A. Klawonn and O. B. Widlund, *A domain decomposition method with Lagrange multipliers and inexact solvers for linear elasticity*, SIAM J. Sci. Comput., 22 (2000), pp. 1199–1219.
14. ———, *FETI and Neumann-Neumann iterative substructuring methods: Connections and new results*, Comm. Pure Appl. Math., 54 (2001), pp. 57–90.
15. J. Mandel and M. Popa, *A multigrid method for elastic scattering*, UCD/CCM Report 109, Center for Computational Mathematics, University of Colorado at Denver, September 1997. `http://www-math.cudenver.edu/ccmreports/rep109.ps.gz`.

16. J. Mandel and R. Tezaur, *On the convergence of a dual-primal substructuring method*. Numerische Mathematik, in print.

17. ———, *Convergence of a substructuring method with Lagrange multipliers*, Numerische Mathematik, 73 (1996), pp. 473–487.

18. J. Mandel, R. Tezaur, and C. Farhat, *A scalable substructuring method by Lagrange multipliers for plate bending problems*, SIAM J. Numer. Anal., 36 (1999), pp. 1370–1391.

19. H. J.-P. Morand and R. Ohayon, *Fluid Structure Interaction*, John Wiley and Sons, Chichester, 1995.

20. K. C. Park, M. R. Justino, Jr., and C. A. Felippa, *An algebraically partitioned FETI method for parallel structural analysis: Algorithm description*, Int. J. Numer. Meth. Engrg., 40 (1997), pp. 2717–2737.

21. D. J. Rixen, C. Farhat, R. Tezaur, and J. Mandel, *Theoretical comparison of the FETI and algebraically partitioned FETI methods, and performance comparisons with a direct sparse solver*, Int. J. Numer. Meth. Engrg., 46 (1999), pp. 501–534.

22. R. Tezaur, A. Macedo, and C. Farhat, *Iterative solution of large-scale acoustic scattering problems with multiple right hand sides by a domain decomposition method with Lagrange multipliers*. Int. J. Numer. Meth. Engrg., submitted.

23. V. Varadan and V. Varadan, *Acoustic, electromagnetic, and elastodynamic fields*, in Field Representations and Introduction to Scattering, V. Varadan and V. Varadan, eds., North-Holland, Amsterdam, 1991.