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Robust Control of Bifurcating Nonlinear Systems with Applications

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ROBUST CONTROL OF BIFURCATING NONLINEAR SYSTEMS WITH APPLICATIONS

by

Hsien-Chiarn Lee

Dissertation submitted to the Faculty of the Graduate School
of The University of Maryland in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
1991

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ABSTRACT

Title of Dissertation: Robust Control of Bifurcating Nonlinear Systems with Applications

Hsien-Chiarn Lee, Doctor of Philosophy, 1991

Dissertation directed by: Dr. Eyad H. Abed
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This dissertation addresses issues in the robust control of nonlinear dynamic systems near points of bifurcation, with application to the feedback control of aircraft high angle-of-attack flight dynamics. Specifically, we consider nonlinear control systems for which a nominal equilibrium point loses stability with slight variation of a distinguished system parameter (the "bifurcation parameter"). At such a loss of stability, various static and dynamic bifurcations may occur. These bifurcations often entail the emergence from the nominal equilibrium of new equilibrium points or of periodic solutions. The control laws sought in this work are intended to achieve certain goals related to the stability and/or amplitude of the bifurcated solutions.

An important contribution of this dissertation is the introduction of the so-called "washout filters" into the control of systems undergoing bifurcations. These filters have been used for some time in certain practical control systems. They facilitate attainment of a degree of robustness of the system operating point to control actions and to uncertainty. Here, washout filter–aided feedback stabilization of nonlinear systems is studied in a general framework. Moreover, washout filters are employed in the feedback control of bifurcating systems.

Several critical cases associated with bifurcations are considered. These include cases in which stability is lost through a zero eigenvalue, a pair of pure imaginary eigenvalues, two zero eigenvalues, and two pairs of pure imaginary eigenvalues. Robustness estimates are given for the achieved stabilization.

The foregoing analytical work is complemented with a thorough control study of nonlinear models for the high angle-of-attack longitudinal flight dynamics of an F-8 Crusader aircraft. In this application, we demonstrate the superiority of washout filters in extending the stable high angle-of-attack flight regime. Also, we demonstrate the robustness of the control algorithm by using a fixed controller to stabilize twelve different Hopf bifurcation points in six different aircraft dynamic models. The numerical work employs state-of-the-art software packages for bifurcation analysis.

DEDICATION

To my parents

my wife and my son

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CHAPTER ONE

INTRODUCTION

Bifurcation phenomena, or qualitative changes in a system's qualitative behavior occurring as a parameter is slowly varied, have been observed and analyzed for many real-world systems. Indeed, applications of bifurcation theory are now common in such diverse areas as electronic circuits, electrical power systems, mechanical systems, chemical reaction systems, fluid systems and aircraft systems. Many instances of loss of stability in physical systems have been characterized in terms of elementary local bifurcations. For instance, [1] relates the voltage collapse of a power system to a saddle-node bifurcation; [2], [3] consider loss of stability of axial flow compression systems in terms of local and global bifurcations of equilibria and periodic solutions; and the stall and divergence of aircraft in high incidence flight, as well as other nonlinear aircraft motions, have been linked to bifurcations of the governing dynamic equations [4], [5], [6], [7], [8], [9], [10], [11]. Observation and analysis of bifurcation phenomena are usually useful in understanding nonlinear system behavior. For instance, in [8], through bifurcation analysis, the nonlinear behaviors of aircraft at high angle-of-attack, such as jump to new steady states, oscillations, and hysteresis are predicted and explained.

Although there is a great volume of work on bifurcation analysis both in terms of theory and applications, few general results are available on issues of control of bifurcating systems. Although for some systems bifurcations may be removed by using linear feedback, such controllers may require excessive feedback gains and large bandwidth [12], which may be unfavorable in practical applications.

Several local bifurcation control methods for certain classes of nonlinear bifurcating systems have been developed [13], [14], [15], [16], [17]. Based on these methods, interesting applications [18], [19], [20] have been proposed for satellite stabilization systems and aircraft control systems. These methods focus on the design of stabilizing state feedback control laws at the bifurcation points, as an alternative to removing the bifurcations.

Several authors have focussed on a nonlinear stabilization problem closely related to problems of bifurcation control, without necessarily viewing their work in such terms. The problem here is to stabilize a *critical* equilibrium point of a nonlinear autonomous system, that is an equilibrium at which the system linearization has at least one eigenvalue with zero real part. This being the goal of the studies, the authors usually assume that the critical eigenvalues are *uncontrollable* for the linearized system. Indeed, otherwise, well known results would allow the easy design of a linear state feedback which moves the critical eigenvalues into the open left half of the complex plane, thus stabilizing the equilibrium point. A few examples of such efforts in the stabilization of critical systems are [21] [15] [16] and [17]. The connection between stabilization of critical equilibrium points and control of bifurcations was pointed out in [13]. An equilibrium can undergo a bifurcation only if it is critical at a certain setting of system parameters, and the stability of the critical equilibrium is closely related to that of the bifurcated solutions. In some physical applications, stabilization of critical equilibria is a primary consideration, with little physical consequence of any associated bifurcations. An example of this is [18], wherein a tethered satellite system dynamic model is found to possess two pairs of simple, nonzero imaginary eigenvalues. Only one of these pairs is controllable.On the other hand, [13], [14] consider the local stabilization control for Hopf bifurcating systems and stationary bifurcating systems in which, by the continuity of system dynamics, not only the bifurcation points but also the bifurcated solutions near these bifurcation points are stabilized. For the resulting closed-loop system, in the case of stationary bifurcation, trajectories starting near the nominal equilibrium tend to a bifurcated equilibrium; and, in the case of Hopf bifurcation, trajectories tend toward small-amplitude stable bifurcated periodic orbits.

All of these methods presume that the bifurcation points (critical equilibrium points) are at origin or can be easily transformed to the origin. This implies the need for accurate knowledge of the bifurcation points. Such an assumption is adequate for some applications. For instance, in the tethered satellite system considered in [18], the critical points are clear from the physical problem, and their knowledge does not depend on accurate knowledge of model parameters. However, in many practical applications, either the system model is highly uncertain or the potential operating points are not easy to distinguish. In these cases, accurately determining "where" the bifurcation points are or "when" the control should be applied may become a problem. For instance, in aircraft control systems, there are inherent uncertainties in the dynamic model especially in the high angle-of-attack regime, and there are a broad range of operating conditions in which the aircraft may operate. It is very difficult to accurately locate the bifurcation points and to estimate precisely the parameter values associated with the occurrence of bifurcation. Sometimes, an additional estimation of operating points such as on-line equilibrium computation is necessary to minimize the undesired effects from the bifurcation control to the nominal (noncritical) operating conditions. Nevertheless, the inaccuracy in the estimation as well as in the system model may limit the system capability. (An example of this is given in Section 7.3.3, which concerns the near-stall control of an aircraft model through direct state feedback.) Moreover, even if the system model is uncertainty-free, the knowledge of bifurcation points may still be elusive. In [1], even though the dynamic equations of the power system are given, due to the numerically ill-conditioned nature of the equations, it is very difficult to locate the bifurcation points. Some of the bifurcations in this model are surely not yet detected.

In this dissertation, we derive robust control laws for systems possessing Hopf bifurcations, systems possessing a stationary pitchfork bifurcation, and systems possessing double pairs of pure imaginary eigenvalues. The control algorithms are extended from [13], [14] and [17]. However, the control laws we derive do not depend on accurate knowledge of equilibrium points. Moreover, they preserve the equilibria of the original systems. For the cases where systems possess a Hopf bifurcation and systems with double pairs of pure imaginary eigenvalues in the Jacobian matrix, we focus on the design of a purely nonlinear stabilizing control algorithm such that while the control is applied to any operating point it does not affect its linear stability properties. For applications where the bifurcation points are uncertain and/or the operating points are not unique, our control strategy is to apply the nonlinear stabilizing control over a range of operating points which contains the bifurcation points. Since all the equilibria and the linear stability of each operating point is preserved and since the linear stability dominates the local behavior, the control stabilizes the bifurcation points and the bifurcated solutions, but has little effect on the stable operating points. For cases in which a system or its "embedded system" (system with artificial parameter included) possesses a pitchfork bifurcation, we derive a robust linear control algorithm which also does not depend on accurate knowledge of the operating points and will stabilize the bifurcation point (critical equilibrium point) and both bifurcated equilibrium branches with all the equilibrium branches preserved.

This robustness is achieved by introducing a washout filter to each of the state variable feedback loops. A *washout filter* (or washout circuit [22], or canceller [23]) is a simple high pass filter which allows only the transient signal to pass through. At steady state, the output of the filter is zero. Thus, the equilibrium state of the closed-loop system is the same as that of the open-loop system. Washout filters are used commonly in control systems for electric power systems [24], [25], [26], [27] and aircraft control systems [23], [22], [28], [29]. The main purpose of using these filters is the resulting robustness of the system operating point to the control actions which may be used. In aircraft Dutch roll control, the filter blocks out the steady state yaw rate signal which is fed back to the rudder deflection to increase the Dutch roll damping. The control is initially designed under straight flight conditions. If the yaw rate signal is directly fed back to the rudder, the control has a tendency to eliminate the yawing motion of the aircraft. However, during a steady turn, the aircraft needs to persist at constant yaw rate, which is opposite the effect of the control. Thus, without an additional yaw command to balance the counter effect from the control, the new turning steady state will be different from that of the uncontrolled system. On the other hand, by using a washout filter, the steady state counter effect is removed. The turning steady state (the equilibrium state) is then the same as that of the uncontrolled system. Moreover, at any turning state, the Dutch roll damping is improved. That is, the control is independent of whatever operating point at which it works.

With these advantages of using washout filters in feedback loops, the control laws derived in this dissertation are robust with respect to uncertainty in the equilibrium points (operating points), and preserve all the equilibria of the original systems. Moreover, the freedom in choosing the washout filter time constants is very useful in improving the robustness of the control.

The development of this dissertation is as follows. In Chapter 2, some basic concepts related to bifurcation along with notation and terminology are reviewed. The classification of bifurcations by codimension [30], normal forms for systems near equilibrium points of codimension 1 and 2 [30], and two basic theorems for stationary and Hopf bifurcations [31], [32] are recalled. Then, the use of center manifold reduction [33] and normal form transformation [30] for reducing the system complexity along with three supporting theorems [33], [17] are summarized. Finally, some properties of multilinear functions, involved in the derivation of stabilizability conditions, are reviewed.

In Chapter 3, the model and some basic properties of washout filters are introduced. The advantages, the limitations, and a strategy for using washout filters in feedback control are discussed. The major advantages of using washout filters in feedback loop are that they preserve the equilibria after feedback, they trace (follow) the actual operating points automatically, and that this facilitates the design of a robust controller. However, there are some limitations of their use in stabilizing certain classes of unstable systems. For instance, systems with Jacobian matrix having an odd number of eigenvalues in the open right-half complex plane cannot be stabilized by feedback through the traditional stable washout filters. To alleviate this limitation, an unstable washout filter is also introduced in this chapter.

In Chapter 4, we extend the results of [13] by introducing a washout filter to each of the feedback loops to achieve a robust nonlinear feedback stabilizing control law both for systems possessing a pair of pure imaginary eigenvalues in the Jacobian matrix and for systems undergoing a Hopf bifurcation. Both the case in which the critical mode is controllable and uncontrollable are considered. For the case in which the critical mode is uncontrollable, the sufficient conditions for stabilizability obtained here cover the results in [13]. The control function does not depend on knowledge of the equilibrium points. Thus, it is robust with respect to uncertainty in the system equilibria. The control law preserves equilibrium points and linear stability of the original system. Therefore, it has little effect on the stable noncritical equilibrium points. The control is also robust with respect to general system uncertainty. Two sufficient conditions on the uncertainty range for the existence of a robust controller are discussed in the last section.

Since feedback through washout filters cannot alter the equilibrium points of the original system, it cannot be used to stabilize a pitchfork bifurcating system by shifting the bifurcated branches to the other side of the bifurcation point as was done in [14]. However, by introducing unstable washout filters into the feedback loop, one can change the direction of the *exchange of stability.* A linear stabilizing control algorithm for this purpose is derived in Chapter 3. This algorithm also has the property of preserving equilibria, and it also does not depend on accurate knowledge of the bifurcation point (critical equilibrium point). This algorithm is adequate for systems possessing a pitchfork bifurcation with the zero eigenvalue of the Jacobian matrix controllable. However, with a small modification, an algorithm for stabilizing certain class of critical systems possessing double controllable zero eigenvalues is also derived.

In Chapter 6, we consider deriving a purely nonlinear robustly stabilizing feedback control law for systems possessing two pairs of pure imaginary eigenvalues in their Jacobian matrices. The cases with both critical modes uncontrollable, both critical modes controllable, and only one of critical modes controllable are included. The stability conditions derived by [17] through center manifold reduction and normal form transformation are employed. Through these stability conditions, the results show that stabilization using washout filter–aided feedback control is no more restrictive than that using direct state feedback as derived in [17]. Moreover, with the flexibility of choosing the washout filter time constant, the results are much more robust.

In Chapter 7, applications of Hopf bifurcation control to aircraft high angle-of-attack flight dynamics are given. This chapter involves extensive numerical work, requiring the use of a dedicated bifurcation analysis software package as well as simulation tools. The package AUTO of E. Doedel [34] is used to obtain global bifurcation branches, while the program BIFOR2 of B. Hassard [32] is used to calculate coefficients determining local stability at Hopf bifurcation points. The longitudinal flight dynamic model of an F-8 Crusader of [35] is studied first. By neglecting the effects of varying weight components in body axes, the "pseudo steady-state" ([36], [9], [11], also called the steady state of the fast mode in [37]) possesses an unstable Hopf bifurcation in the region of stall. Both direct state feedback control and feedback through washout filters are employed. The results show that with washout filters, the equilibrium branches are preserved so that the stable operating range is increased more than that with direct state feedback.

In order to reflect more realistic high angle-of-attack dynamics and demonstrate the robustness of the control, six different modified wing lift profiles are employed in the longitudinal dynamics. Through the use of washout filters, two fixed stabilizing controllers are designed. Each one stabilizes all the Hopf bifurcations of six different lift profiles. The one presented last has better response in the bifurcated periodic solutions. It results in a family of stable limit cycles linking two stable Hopf bifurcations which does away with the jumping and hysteresis phenomena in the region of stall. However, it is observed that local indices of the nature of the Hopf bifurcations would seem to indicate the opposite of what is observed for the global bifurcation diagram. This observation is interesting and motivates the possibility of optimization-based design of controllers to achieve desired global characteristics.

The time simulations of the full model in the last section show significant improvement of the aircraft response after control. This also justifies the use of a reduced model (containing only the fast variables) in the analysis and design.

Conclusions and suggestions for further research are collected in Chapter 8.

CHAPTER TWO
MATHEMATICAL PRELIMINARIES

In this chapter, we review some basic results relating to bifurcation theory, center manifold reduction, normal form transformation, and multilinear functions. The properties of codimension one bifurcations along with theorems on stationary and Hopf bifurcations [30], [38], [39], [32], [31] are recalled first. These will be employed in the derivation of stabilizability conditions for systems whose linearization possesses a pair of pure imaginary eigenvalues or a single or double zero eigenvalue. Next, the use of the center manifold theorem [33] and normal form transformation [30], [40] in reducing system complexity is discussed. These methods will be used in deriving stabilizability conditions for systems whose linearization possesses two pairs of pure imaginary eigenvalues. Then, some basic properties of multilinear functions [41] are recalled. Finally, the Fredholm alternative, which will be used for deriving stationary bifurcation formulae in Chapter 5, is recalled.

2.1. Basic bifurcation theory

Bifurcation refers to qualitative changes in the solution structure of dynamical systems occurring with slight variation in system parameters. The parameter values at which the changes occur are called bifurcation values or critical parameter values. Originally, Poincare' used the term "bifurcation" to describe the splitting of equilibrium solutions in a family of differential equations. Bifurcations involving only equilibrium points are known as stationary or static bifurcations. There are also bifurcations, such as Hopf bifurcation, which involve both equilibria and periodic solutions. Consider a system

$$\dot{x} = f_\mu(x), \tag{2.1}$$

where $x \in I\!R^n$ is the system state and $\mu \in I\!R^k$ denotes a k-dimensional parameter; k can be any positive integer. In this work, we restrict k to be 1. The equilibrium solutions are given by the solutions of the equation $f_\mu(x) = 0$. By the implicit function theorem, as μ varies, these equilibria are smooth functions of μ as long as $D_x f_\mu$, the Jacobian derivative of $f_\mu(x)$ with respect to x, does not have zero eigenvalue. The graph of each of these functions of equilibria in (x, μ) space is a *branch* of equilibria of the system. An equilibrium point is called a "*stationary bifurcation point*" if several equilibria join at that point. A necessary condition for an equilibrium (x_0, μ_0) to be a (stationary) bifurcation point is that the Jacobian $D_x f_\mu$ has one or more zero eigenvalues.

Bifurcation are often classified according to the codimension of $D_x f_u$. Using a linear coordinate transformation, $D_x f_\mu$ can be represented in block-diagonal form

$$D_x f' := \begin{pmatrix} A_c & 0 \\ 0 & A_s \end{pmatrix}, \tag{2.2}$$

where A_c is the Jordan block corresponding to the critical modes and A_s denotes the remaining stable modes. Bifurcations from an equilibrium of codimension 1 and 2 require one of the following situations:

Codimension 1 bifurcations:

i) One zero eigenvalue in A_c, that is $A_c = 0$. (This is associated with stationary bifurcation.)

ii) A pair of purely imaginary eigenvalues in A_c. (This is associated with Hopf bifurcation.)

Codimension 2 bifurcations:

i) Double nondiagonalizable zeros in A_c, that is

$$A_c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.3}$$

ii) Double diagonalizable zeros in A_c, that is

$$A_c = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.4}$$

iii) One zero and one pair of purely imaginary eigenvalues in A_c, that is

$$A_c = \begin{pmatrix} 0 & -\omega_c & 0 \\ \omega_c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.5}$$

iv) Two pairs of purely imaginary eigenvalues in A_c, that is

$$A_c = \begin{pmatrix} 0 & -\omega_1 & 0 & 0 \\ \omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega_2 \\ 0 & 0 & \omega_2 & 0 \end{pmatrix}. \tag{2.6}$$

By using the center manifold reduction technique and normal form transformations, we can reduce system (2.1) to a lower order simplified system called the *normal forms*. The normal form preserves the qualitative properties of the solutions near the bifurcation. Analyzing the dynamics of normal forms yields a qualitative picture of the solutions for each type of bifurcation. The normal forms of codimension one bifurcations are summarized as follows:

i) **Saddle-node bifurcation:** The normal form is given by

$$\dot{x} = \mu - x^2. \tag{2.7}$$

The bifurcated solutions exist for $\mu > 0$ and are given by $x = \pm\sqrt{\mu}$. The equilibrium branch $x = \sqrt{\mu}$, is stable while the other branch, $x = -\sqrt{\mu}$, is unstable.

ii) **Transcritical bifurcation:** The normal form is given by

$$\dot{x} = \mu x - x^2. \tag{2.8}$$

The bifurcated solutions, $x = \mu$, exist for both $\mu > 0$ and $\mu < 0$. For $\mu > 0$ (resp. $\mu < 0$), the bifurcated branch is stable (resp. unstable).

iii) **Pitchfork bifurcation:** The normal form (for the *supercritical* case) is given by

$$\dot{x} = \mu x - x^3. \tag{2.9}$$

There are two bifurcated branches, $x = \pm\sqrt{\mu}$, for $\mu > 0$, and they are both stable.

iv) **Hopf bifurcation:** The normal form is given by

$$\begin{aligned} \dot{x} &= -y + x(\mu - (x^2 + y^2)) \\ \dot{y} &= x + y(\mu - (x^2 + y^2)). \end{aligned} \tag{2.10}$$

The associated bifurcated solutions are nontrivial periodic trajectories.

Theorem 2.1. (Stationary Bifurcation Theorem [38]). Suppose f_μ of system (2.1) is sufficiently smooth with respect to both x and μ, $f_\mu(0) = 0$ for all μ, and the Jacobian of f_μ, $A_\mu := D_x f_\mu(x_0(\mu))$, possesses a simple eigenvalue

$\lambda(\mu)$ such that at the critical parameter value $\mu_c = 0$,

$$\lambda'(0) := \left.\frac{d\lambda}{d\mu}\right|_{\mu = 0} \neq 0, \tag{2.11}$$

and all the remaining eigenvalues of A_0 have strictly negative real part. Then:

i) there is an $\epsilon_0 > 0$ and a function

$$\mu(\epsilon) = \mu_1 \epsilon + \mu_2 \epsilon^2 + O(\epsilon^3) \tag{2.12}$$

such that if $\mu_1 \neq 0$, there is a nontrivial equilibrium $x(\mu)$ near $x = 0$ for each $\epsilon \in \{[-\epsilon_0, 0) \cup (0, \epsilon_0]\}$; if $\mu_1 = 0$ and $\mu_2 > 0$ (resp. < 0), there are two equilibrium points $x_\pm(\mu)$ near $x = 0$ for each $\mu \in (0, \epsilon_0]$ (resp. $\mu \in [-\epsilon_0, 0)$).

ii) Exactly one eigenvalue $\beta(\epsilon)$ of the Jacobian evaluated with respect to each of the nontrivial equilibrium points in (i) approaches 0 as $\epsilon \to 0$ and it is given by a real function

$$\beta(\epsilon) = \beta_1 \epsilon + \beta_2 \epsilon^2 + O(\epsilon^3). \tag{2.13}$$

The coefficient β_1 of this function satisfies $\beta_1 = -\lambda'(0)\mu_1$. The nontrivial equilibrium x_- (resp. x_+) is stable (resp. unstable) if $\beta_1 \epsilon < 0$ and is unstable (resp. stable) if $\beta_1 \epsilon > 0$. Nevertheless, the bifurcation point itself is unstable. If $\beta_1 = 0$, then $\beta_2 = -2\lambda'(0)\mu_2$, and the nontrivial equilibria are asymptotically stable if $\beta_2 < 0$ and are unstable if $\beta_2 > 0$.

Theorem 2.2. (C^L-Hopf Bifurcation Theorem [32]). Suppose the system (2.1) satisfies the following conditions:

i) $f_\mu(0) = 0$ for μ in an open interval containing 0, and $0 \in \mathbb{R}^n$ is an isolated equilibrium point of f.

ii) All partial derivatives of the components f_μ^l of the vector f of orders $l \le L+2$ $(L \ge 2)$ (including the partial derivatives with respect to μ) exist and are continuous in x and μ in a neighborhood of $(0,0)$ in $I\!R^n \times I\!R^1$ space,

iii) $A_\mu := D_x f_\mu(0,\mu)$ has a complex conjugate pair of eigenvalues λ and $\bar{\lambda}$ such that $\lambda(\mu) = \alpha(\mu) + j\omega(\mu)$, where $\omega_0 := \omega(0) > 0$, $\alpha(0) = 0$, and

$$\alpha'(0) := \left.\frac{d\alpha}{d\mu}\right|_{\mu = 0} \neq 0. \tag{2.14}$$

iv) The remaining eigenvalues of A_0 have strictly negative real parts.

Then:

i) There exists an $\epsilon_p > 0$ and a C^{L+1} function

$$\mu(\epsilon) = \sum_{i=1}^{[\frac{L}{2}]} \mu_{2i}\epsilon^{2i} + O(\epsilon^{L+1}) \qquad (0 < \epsilon < \epsilon_p) \tag{2.15}$$

such that for each $\epsilon \in (0, \epsilon_p)$ there exists a nonconstant periodic solution $p_\epsilon(t)$ with period

$$T(\epsilon) = \frac{2\pi}{\omega_0}[1 + \sum_{i=1}^{[\frac{L}{2}]} \tau_{2i}\epsilon^{2i}] + O(\epsilon^{L+1}) \qquad (0 < \epsilon < \epsilon_p) \tag{2.16}$$

occurring for $\mu = \mu(\epsilon)$.

ii) There exists a neighborhood η of $x = 0$ and an open interval ϑ containing 0 such that for any $\mu \in \vartheta$, the only nonconstant periodic solutions that lie in η are members of the family $p_\epsilon(t)$.

iii) Exactly two of the Floquet exponents of $p_\epsilon(t)$ approach 0 as $\epsilon \downarrow 0$. One is 0 identically, and the other is a C^{L+1} function

$$\beta(\epsilon) = \sum_{1}^{[\frac{l}{2}]} \beta_{2i}\epsilon^{2i} + O(\epsilon^{L+1}) \qquad (0 < \epsilon < \epsilon_p). \tag{2.17}$$

The periodic solution $p_\epsilon(t)$ is orbitally asymptotically stable if $\beta(\epsilon) < 0$, and is unstable if $\beta(\epsilon) > 0$. If there is a first nonvanishing coefficient μ_{2k}^p, then the first nonvanishing coefficient in Eq. (2.17) is given by

$$\beta_{2k} = -2\lambda'(0)\mu_{2k}. \tag{2.18}$$

Moreover, there is then an $\epsilon_1 \in (0, \epsilon_p)$ such that

$$\mathrm{sgn}[\beta(\mu)] = \mathrm{sgn}[\beta_{2k}] \tag{2.19}$$

for $\mu \in \{\mu | 0 < \mu/\mu_{2k} < \mu(\epsilon_1)/\mu_{2k}\}$. Here, sgn denotes the sign of a real number.

2.2. Center manifold reduction

Center manifold theory is very useful in reducing the order of a system such that the local behavior of solutions of the full system can be determined by the reduced equations on the center manifold. The following terminology and notation are based on [33]. A *local invariant manifold* S for a system (2.1) is a subset of $I\!R^n$ such that for any $x_0 \in S$, the solution $x(t)$ with $x(0) = x_0$ is in S for $|t| < T$ where T is some positive number. If $T = \infty$, then S is said to be an *invariant manifold.*

Consider the nonlinear autonomous system

$$\begin{aligned} \dot{x} &= Ax + f(x, y) \\ \dot{y} &= By + g(x, y) \end{aligned} \tag{2.20}$$

where $x \in I\!R^n$, $y \in I\!R^m$, A and B are constant matrices, f and g are C^2 functions with $f(0,0) = 0$, $g(0,0) = 0$, and the Jacobian matrices of f and g, $f'(0,0) = 0$ and $g'(0,0) = 0$. If all the eigenvalues of A have zero real parts and all the eigenvalues of B have negative real parts, then there exists a *center manifold* for (2.20), $y = h(x)$, $|x| < \delta$, where h is a C^2 function which satisfies the relationship

$$h'(x)[Ax + f(x, h(x))] = Bh(x) + g(x, h(x)), \tag{2.21}$$

where δ is a positive real number. The flow on the center manifold is governed by the $n-$dimensional system

$$\dot{z} = Az + f(z, h(z)). \tag{2.22}$$

Theorem 2.3. ([33], Theorem 2) If the origin, $z = 0$ is a stable (or asymptotically stable) (or unstable) equilibrium of system (2.22), then the origin, $x = 0, y = 0$, is a stable (or asymptotically stable) (or unstable) equilibrium of the full system (2.20). If the origin is a stable equilibrium and $(x(t), y(t))$ is a solution with initial condition $(x(0), y(0))$ sufficiently small, then there exists a solution $z(t)$ of the reduced system such that as $t \to \infty$,

$$\begin{aligned} x(t) &= z(t) + O(e^{-\gamma t}) \\ y(t) &= h(z(t)) + O(e^{-\gamma t}), \end{aligned} \tag{2.23}$$

where $\gamma > 0$ is a constant.

In general, it is very difficult to solve for the function h from the relationship (2.21). However, h can be approximated to any desired order.

Theorem 2.4. ([33], Theorem 3). Let $\phi : I\!R^n \to I\!R^m$ be a C^1 mapping in a neighborhood of the origin with $\phi(0) = 0$ and $\phi'(0) = 0$. Let $M(\phi(x))$ be defined as

$$M(\phi(x)) = \phi'(x)\{Ax + f(x, \phi(x))\} - B\phi(x) - g(x, \phi(x)). \tag{2.24}$$

If $M(\phi(x)) = O(|x|^q)$ as $x \to 0$ for some $q > 1$, then ϕ also satisfies

$$|h(x) - \phi(x)| = O(|x|^q) \tag{2.25}$$

as $x \to 0$.

2.3. Normal form transformation

Center manifold reduction is useful in reducing the system order to the dimension of the center manifold, while normal form reduction simplifies the analytic expression of the reduced system dynamics by successive nonlinear coordinate transformations. The resulting simplified system equations are called *normal forms.* The dynamics for the normal form is qualitatively the same as for the original system.

Consider a system

$$\dot{x} = f(x), \tag{2.26}$$

where $x \in I\!R^n$, f is sufficiently smooth, and 0 is an equilibrium point of the system. Let the coordinate transformation function be $x = y + h(y)$ where h is a purely nonlinear function and $h(0) = 0$. System (2.26) becomes

$$\dot{y} = (I + Dh(y))^{-1} f(y + h(y)). \tag{2.27}$$

The following lemma ensures the preservation of local properties, for instance stability, after this coordinate transformation.

Lemma 2.5. [17] Let h be a smooth mapping from $I\!R^n$ to $I\!R^n$ with $Dh(0) = 0$. Then there exists an open set $D \in I\!R^n$ containing the origin such that the mapping $x = y + h(y)$ from D to D is one-to-one and onto.

Suppose h is a polynomial of degree k. Then $(I+Dh)^{-1} = I-Dh$ modulo terms of degree k and higher, which implies the transformation affects only the nonlinear terms in (2.26) of degree k and higher. Therefore, by successively searching for polynomials h of different degrees to remove the nonessential nonlinear terms, the system equations can be reduced to "normal form."

2.4. Multilinear functions

Multivariate Taylor series expansions are used extensively in our derivation of local stability conditions for bifurcating systems and critical systems. It is very convenient in dealing with these expansions to employ the notation of multilinear functions. The following are some basic definitions and properties of these functions.

Definition 2.1. [41] Given vector spaces over the same field $V_1,\ V_2, \cdots, V_k$ and W, a mapping ϕ from the product space $V_1 \times V_2 \times \cdots \times V_k$ into W is called multilinear if it is linear in each of its arguments. That is, for any v_i, $\tilde{v}_i \in V_i$, $i = 1, \cdots, k$, and any scalars a, b, we have

$$\begin{aligned} \phi(v_1, \cdots, av_i + b\tilde{v}_i + \cdots + v_k) = a\phi(v_1, \cdots, v_i, \cdots, v_k) \\ + b\phi(v_1, \cdots, \tilde{v}_i, \cdots, v_k). \end{aligned} \tag{2.28}$$

The integer k is called the rank (or the degree) of the function ϕ.

Definition 2.2. [42] A $k-$linear function $\phi : V \times V \times \cdots \times V \to W$ is symmetric if it is symmetric in any pair of its arguments. That is,

$$\phi(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = \phi(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k). \tag{2.29}$$

Proposition 2.1. [42] Given a symmetric $k-$linear function $\phi : (I\!R^n)^k \to I\!R^m$ and any vector $y \in I\!R^n$, we have

$$D\phi(x, x, \cdots, x) \cdot y = k\phi(x, x, \cdots, x, y). \tag{2.30}$$

Let Q and C denote symmetric bilinear and trilinear functions respectively. Then we have the following identities:

$$\begin{aligned} C(x - ay, x - ay, x - ay) =& C(x, x, x) - 3aC(x, x, y) \\ &+ 3aC(x, y, y) - a^3C(y, y, y), \end{aligned} \tag{2.31}$$

$$Q(x, y) = \frac{1}{4}[Q(x + y, x + y) - Q(x - y, x - y)]. \tag{2.32}$$

2.5. The Fredholm Alternative

Theorem 2.6. (Solvability Theorem; e.g. [39], [43]). Suppose A is a linear operator defined on the finite dimensional vector space $I\!R^n$ and 0 is an eigenvalue of A. Denote by l the left eigenvector of A corresponding to the zero eigenvalue. Then, given any $y \in I\!R^n$, the equation

$$Ax = y \tag{2.33}$$

is solvable for $x \in I\!R^n$ if and only if $ly = 0$.

CHAPTER THREE
FEEDBACK STABILIZATION THROUGH WASHOUT FILTERS: PRELIMINARIES

This chapter is concerned with the modeling and basic understanding of washout filters in control systems. Both the advantages and limitations of the use of washout filters in feedback controllers are discussed. The advantages of using washout filters are equilibrium preservation, automatic operating point following, and facilitating the design of a robust controller. The limitations are in losing the controllability or stabilizability for some classes of unstable systems or critical systems. Besides the traditional stable washout filters, an unstable washout filter to alleviate some shortcomings in the use of traditional stable washout filters is introduced. Also, a strategy for selecting these washout filters for feedback control is proposed.

3.1. Background and motivation

Our interest here is in the design of stabilizing feedback compensators for uncertain nonlinear systems

$$\dot{x} = f(x, u) \tag{3.1}$$

where the vector field f is uncertain, possibly depending on one or more unknown parameters, u is the scalar input, and the state vector x is available through measurement. Due to the uncertainty in f, there will also be an uncertainty in the equilibrium points (if any) of this system of differential equations. However, despite the uncertainty in the equilibrium, the objective in terms of control design often centers on stabilization of some equilibrium condition. Typically, in the stabilization of equilibrium points of nonlinear system (3.1), one expands the vector field $f(x, u)$ about the equilibrium point of interest, x_e, and then applies linear feedback design techniques to the linearized model. This method, which usually involves static state feedback, does not easily apply to problems in which the dynamics and the equilibrium x_e is uncertain.

If a static state feedback is applied to an uncertain system (3.1), then one expects this feedback to affect the equilibrium points of the system. This is because the feedback would likely be of the form, say, $u = -k(x - x_e')$, where x_e' is a possibly crude approximation of x_e. Dynamic state feedback, on the other hand, can alleviate this difficulty. Of course, this is accomplished at the expense of the increased dimensionality of the controller. A dynamic state feedback which has been employed in problems of this type is the so-called washout filter.

Washout filters are commonly incorporated into control systems for electric power systems [27], [24] and aircraft [23], [28] [22], [29]. The primary benefit of using washout filters is the resulting robustness of the system operating point to control actions which may be employed. In power systems, they are used to remove any steady-state offset voltage which may be produced in the feedback [25], [27]. In the Dutch roll control of an aircraft, a direct state feedback would adversely affect the aircraft's turning capability. By incorporating a washout filter in the control system, the turning capability is maintained.

In this chapter, two types of washout filters, stable washout filter and unstable washout filter, and the general configurations and considerations of their use in the feedback control are discussed. In subsequent chapters, these results are employed to study stabilization algorithms for critical nonlinear systems.

3.2. Washout filter model

A traditional *washout filter* (or washout circuit), as shown in Figure 3.1, is a stable high pass filter with transfer function [22]

$$\begin{aligned} G(s) = \frac{y(s)}{x(s)} &= \frac{s}{(s+d)} \\ &= 1 - \frac{d}{(s+d)}. \end{aligned} \tag{3.2}$$

Here, $d > 0$ is the reciprocal of the filter time constant. With the notation

$$z(s) := \frac{1}{(s+d)} x(s), \tag{3.3}$$

the dynamics of the filter can be written as

$$\dot{z} = x - dz, \tag{3.4a}$$

along with the output equation

$$y = x - dz. \tag{3.4b}$$

At steady-state,

$$z_e = \frac{x_e}{d}, \tag{3.5}$$

the output $y = 0$, and the steady-state input signal x_e is said to have been washed out.

As will be discussed in Section 3.5, there are limitations on the efficacy of traditional stable washout filters in feedback control, and some of these limitations can be alleviated through the use of unstable washout filters.

Unstable washout filters are similar to stable washout filters, except that the reciprocal of the time constant d in Eq. (3.2) is negative. When the input is x_e, the output y is again zero. If the overall closed-loop system is stable, an unstable washout filter behaves as would a stable filter. Figure 3.2 depicts a possible structure of an unstable washout filter.

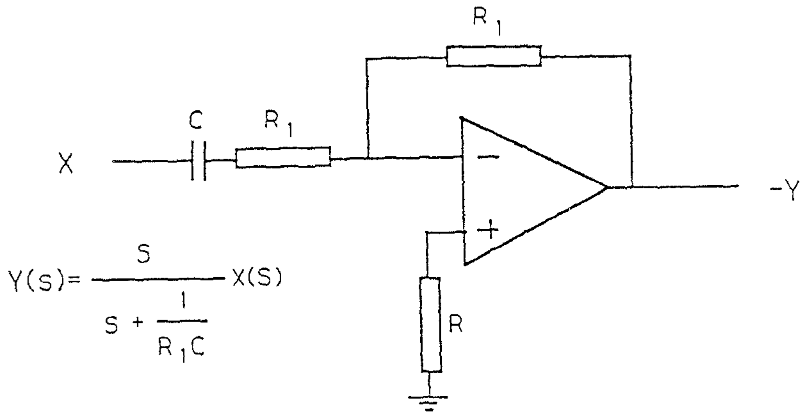


Figure 3.1. Stable washout filter

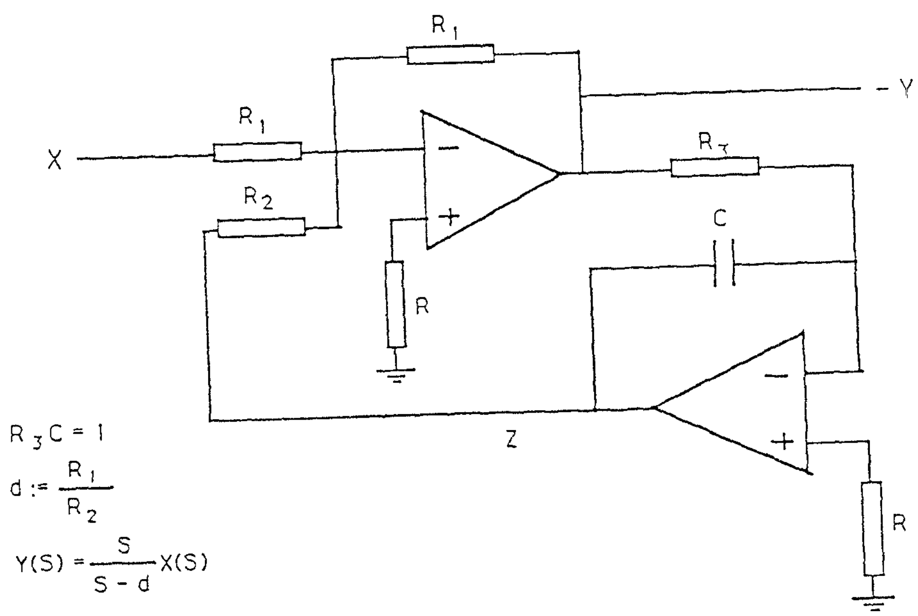


Figure 3.2. Unstable washout filter

3.3. Linear feedback through washout filters

Consider the nonlinear system (3.1) expanded at an equilibrium point x_e

$$\dot{\hat{x}} = A\hat{x} + bu + \phi(\hat{x}, u), \tag{3.6}$$

where $x \in I\!R^n$ is the state vector, u is a scalar input, x_e is an equilibrium, and $\hat{x} := x - x_e$. The function ϕ represents higher order terms, i.e., $\phi(0,0) = 0$ and

$$\frac{\partial \phi(0,0)}{\partial \hat{x}} = 0. \tag{3.7}$$

By incorporating washout filters in feedback loops, as shown in Figure 3.3, one achieves the "washing out" of the feedback signals at steady state.

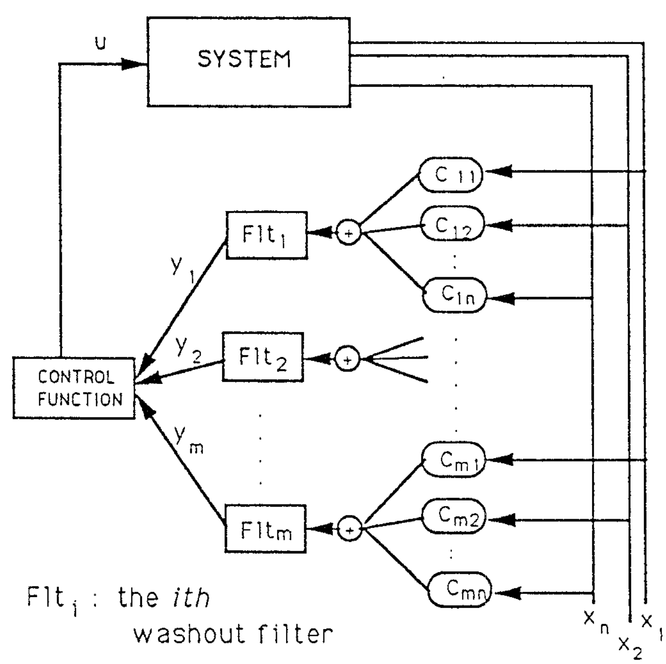


Figure 3.3. Feedback through washout filters

Due to the linearity of washout filters, in Figure 3.3, several different states may feed into one filter, and one state may feed into several different filters, to result in a signal which is a linear function of system states and which vanishes at steady-state. Therefore, the dynamic equations of washout filters used in a feedback loop can be written in general as

$$\begin{aligned}\dot{\hat{z}}_i &= -d_i\hat{z}_i + \sum_{j=1}^{n} c_{ij}\hat{x}_j \\ &= -d_i z_i + \sum_{j=1}^{n} c_{ij}x_j, \end{aligned} \tag{3.8}$$

for $i = 1, \cdots, m$, where m is a positive integer, x_{ei} is the equilibrium state of x_i, z_{ei} is the equilibrium state of the ith washout state z_i, and $\hat{z}_i := z - z_{ei}$. The relation between z_{ei} and x_e is

$$z_{ei} = \frac{1}{d}\sum_{j=1}^{n} c_{ij}x_{ej}. \tag{3.9}$$

The overall system is then

$$\begin{pmatrix}\dot{x} \\ \dot{z}\end{pmatrix} = \begin{pmatrix}A & 0 \\ C & D\end{pmatrix}\begin{pmatrix}\hat{x} \\ \hat{z}\end{pmatrix} + \begin{pmatrix}b \\ 0\end{pmatrix}u + \begin{pmatrix}\phi(\hat{x}, u) \\ 0\end{pmatrix}, \tag{3.10}$$

where C is the $m \times n$ matrix (C_{ij}), $i = 1, \cdots, m$; $j = 1, \cdots, n$, which consists of nonzero row vectors, D is the $m \times m$ diagonal matrix with diagonal entries $d_i, i = 1, \cdots, m$, $z = (z_1, \cdots, z_m)^T$ is the state vector of washout filters, $z_e = (z_{e1}, \cdots, z_{em})^T$ is the equilibrium state vector of z, and $\hat{z} := z - z_e$. The control u then, can be taken as either a linear or nonlinear function of filter outputs, are given by

$$\begin{aligned} y_i &= -d_i z_i + \sum_{j=1}^{n} c_{ij}x_j \\ &= -d_i\hat{z}_i + \sum_{j=1}^{n} c_{ij}\hat{x}_j. \end{aligned} \tag{3.11}$$

There is much freedom in choosing the matrices C and D. The following two lemmas, based on the linear controllability considerations, show two of the basic rules for selecting these two matrices.

Lemma 3.1. If the diagonal entries of the matrix D are not distinct, the linearization of the overall system (3.10) is not controllable even if the subsystem

$$\dot{x} = A\hat{x} + bu \tag{3.12}$$

is controllable.

Proof: From the Popov-Belevitch-Hautus (PBH) rank test [44], the linearized system is controllable if and only if

$$\rho\begin{pmatrix} \lambda I - A & 0 & b \\ C & \lambda I - D & 0 \end{pmatrix} = n + m, \tag{3.13}$$

for each complex number λ. Here, ρ denotes the rank of a matrix. Letting λ_1 be an eigenvalue of D with multiplicity greater than one, then

$$\rho\begin{pmatrix} 0 \\ \lambda_1 I - D \end{pmatrix} < m - 1.$$

Since

$$\rho\begin{pmatrix} \lambda_1 I - A & b \\ C & 0 \end{pmatrix} \leq n + 1, \tag{3.14}$$

we have

$$\begin{aligned} \rho\begin{pmatrix} \lambda_1 I - A & 0 & b \\ C & \lambda_1 I - D & 0 \end{pmatrix} &\leq \rho\begin{pmatrix} \lambda_1 I - A & b \\ C & 0 \end{pmatrix} + \rho\begin{pmatrix} 0 \\ \lambda_1 I - D \end{pmatrix} \\ &< n + m. \end{aligned} \tag{3.15}$$

Thus the linearized system is uncontrollable.

■

Note that the controllability of system (3.10) does not imply that the eigenvalues of system (3.6) can be arbitrarily assigned by feedback through washout filters. However, the uncontrollability of (3.10) does imply that the uncontrollable eigenvalues of system (3.10) will persist. Lemma 3.1 implies that if the eigenvalues (or the time constants) of any two washout filters used in system

(3.10) are the identical, one of these eigenvalues will persist in the overall closed-loop system. This may not be acceptable, especially when two unstable filters with identical eigenvalue are used. In that situation, the system will surely be unstable.

In fact, for linear feedback, with simple transformation, two washout filters with identical time constant can be replaced by one without affecting any other closed-loop eigenvalues. Consider two filters with the same time constant d^{-1}

$$\dot{z}_1 = -dz_1 + \sum_{j=1}^{n} c_{1j}x_j,$$

$$\dot{z}_2 = -dz_2 + \sum_{j=1}^{n} c_{2j}x_j. \tag{3.16}$$

Suppose that the control u is a linear function of the outputs of these two filters

$$\begin{aligned} u =& k_1 y_1 + k_2 y_2 \\ =& \sum_{j=1}^{n}(k_1 c_{1j} + k_2 c_{2j})x_j - d(k_1 z_1 + k_2 z_2). \end{aligned} \tag{3.17}$$

By setting $\tilde{z} = k_1 z_1 + k_2 z_2$, we can construct a new washout filter with identical time constant d^{-1}

$$\dot{\tilde{z}} = -d\tilde{z} + \sum_{j=1}^{n}(k_1 c_{1j} + k_2 c_{2j})x_j. \tag{3.18}$$

The output of this new filter can be written as a linear combination of the outputs of the old filters

$$\tilde{y} = k_1 y_1 + k_2 y_2. \tag{3.19}$$

Thus, we can replace two old filters by simply setting $u = \tilde{y}$. The dimension of the overall closed-loop system is reduced by one. The closed-loop eigenvalues are the same with those of using two washout filters except the uncontrollable one introduced by the redundant washout filter is removed.

Lemma 3.2. Suppose matrices A and D have identical eigenvalue λ_1 and

$$\rho\begin{pmatrix} \lambda_1 I - A & b \\ C & 0 \end{pmatrix} \leq n. \tag{3.20}$$

Then, the overall linearized system (3.10) is not controllable.

Proof: By choosing $\lambda = \lambda_1$,

$$\rho\begin{pmatrix} \lambda I - A & 0 & b \\ C & \lambda I - D & 0 \end{pmatrix} \leq n + m - 1, \tag{3.21}$$

the PBH test fails. Therefore, the overall system is uncontrollable.

∎

Lemma 3.2 shows that to avoid losing the linear controllability, it is better to have all of the eigenvalues of washout filters different from any of the system eigenvalues, otherwise there will be some restrictions in the selection of matrix C. If one of the washout filters does have eigenvalue λ_1 identical to one of system eigenvalues, matrix C should be chosen not only to have the rank in Eq. (3.20) equal to $n+1$ but also to have the column $(0, \cdots, \lambda_1, 0, \cdots)^T$ in

$$\begin{pmatrix} 0 \\ D \end{pmatrix} \tag{3.22}$$

be linearly dependent on the columns in

$$\begin{pmatrix} \lambda_1 I - A & b & 0 \\ C & 0 & \lambda_1 I - D \end{pmatrix}. \tag{3.23}$$

Note that, in purely nonlinear control, we can neglect these rules since all the existing eigenvalues will be preserved after control is applied.

3.4. Advantages of using washout filters

Based on the property of "steady-state washing out ", the advantages of inserting washout filters in a feedback loop can be summarized as equilibrium preservation, automatic equilibrium (operating point) following, and facilitating the design of a robust controller.

3.4.1. Equilibrium Preservation

Equilibrium points represent, in some sense, a system's capability to perform in a certain manner at steady state. There are cases in which such a capability should not be altered by the introduced control strategy. For instance, in the problem of lateral control design for an aircraft, the yaw rate signal is fed back to the rudder to increase the Dutch roll damping factor. The control is usually designed assuming a straight flight condition. That is, one works with a linearized model obtained relative to a straight flight condition. A direct feedback of yaw rate signal has a tendency to eliminate the yaw rate motion of the aircraft so as to increase the yaw damping. Unfortunately, this type of feedback also tends to oppose the aircraft's tendency to turn. For instance, during steady turns, this feedback will generate a steady-state rudder deflection associated with constant yaw rate. This deflection is opposite to the deflection wanted in the turn and means to eliminate the yaw rate. Unless there is an extra yaw command produced by human pilot to compensate this opposition, the aircraft will have a steady turn with less yaw rate (larger sideslip). That is, the aircraft will run into an equilibrium state which is different from the equilibrium without feedback. To remedy this situation, a washout filter is included in the feedback loop. This filter, which rejects steady-state input signals, has the effect of "washing out" the yaw rate signal at steady-state and thus minimizes the tendency opposing a steady turn. In other words, the lateral control designed for level flight does not impact the open-loop equilibrium for turning flight.

The main mechanism which leads to this preservation of equilibrium is that the feedback function goes to zero at any open-loop equilibrium point. Consider a system

$$\dot{x} = f(x, u) \tag{3.24}$$

with

$$f(x_e, 0) = 0, \tag{3.25}$$

where u is the control input and x_e is any equilibrium point for the system with zero input. Let the control input u be a function of $y = (y_1, \cdots, y_m)^T$, y_i, $i = 1, \cdots, m$, defined in Eq. (3.11):

$$u = h(y), \tag{3.26}$$

and let h satisfy

$$h(0) = 0. \tag{3.27}$$

From Eq. (3.9) and (3.11), it is clear that y vanishes at steady-state. Hence

$$f(x_e, h(y_e)) = f(x_e, 0) = 0, \tag{3.28}$$

and x_e remains an equilibrium point of the closed-loop system. Therefore, by incorporating a washout filter in the feedback, the equilibrium points of the original system are preserved. Note that, if the control function h is purely nonlinear, the *linear stability* of each equilibrium point is also preserved.

3.4.2. Automatic equilibrium following

Suppose that the system is designed to operate in a wide range of operating conditions. Let $x_{e,\mu}$ denote the operating point determined by parameter μ. Here, μ can be a scalar or a vector. A typical direct state feedback is to have the control function center at each operating point, that is, to have $u = h(x - x_{e,\mu})$. This implies the need of an on-line computation of operating point $x_{e,\mu}$ for each μ. On the other hand, since the output y of washout filters in Eq. (3.4b) can always be written as

$$\begin{aligned} y &= x - dz \\ &= (x - x_{e,\mu}) - d(z - z_{e,\mu}), \end{aligned} \tag{3.29}$$

where $x_{e,\mu}$ is the actual operating point which system is currently operated at, the control function $u = h(y)$ is guaranteed to center at the correct operating point without any on-line adjustment. For systems which have uncertainty in the operating point or have difficulty in performing the on-line operating point computation, washout filters are very useful.

3.4.3. Facilitating the design of a robust controller

Since the control function h is independent of the operating point, it already provides robustness with respect to uncertainty in the system operating point. In addition to that, it provides better opportunity to estimate the range of other uncertainty that the controller can tolerate. This is because the control function is always centered at the correct operating point. One can concentrate on determining the amount of uncertainty in other system dynamics as if the operating point never changes. To see this, consider applying a nonlinear control, for instance, a quadratic control, on the system (3.6) at operating point x_e. By direct state feedback, the control function will have a form as

$$u = Q_u(x - x_e, x - x_e). \tag{3.30}$$

Should there be any mismatch δx_e in the operating point, the control function would be deformed to

$$u = Q_u(x - x_e, x - x_e) + 2Q_u(\delta x_e, x - x_e) + Q_u(\delta x_e, \delta x_e). \tag{3.31}$$

The induced constant and linear terms in the right-hand side of Eq. (3.31) have to be taken into account during the stability analysis. Due to the dependency on the uncertainty δx_e, these induced terms, while mixed with the uncertainty in other system dynamics after feedback is applied, will either make the stability analysis complicated or make the robustness limited.

On the other hand, by using washout filters in the feedback loop, the pattern of control function remains the same for all operating points. Thus, one can concentrate on analyzing other uncertainty, say, uncertainty in the coefficients of the series expansion of the system dynamics that relates to stability. Therefore, robustness analysis is much easier.

3.5. Limitations of using washout filters

By inserting washout filters in the feedback loop, the outputs of the filters, y_i in Eq. (3.11), instead of arbitrary system states x, are used in the control. Although there is plenty of freedom in selecting the coefficients of d_i and c_{ij}, some capability of direct state feedback is still lost due to the restriction of $d_i \neq 0$. The following lemmas summarize some of the capability limitations while using washout filters in feedback loop. The stability considered there is asymptotical stability.

Lemma 3.3. If a system defined as Eq. (3.6) has an odd number of eigenvalues with positive real part, it cannot be stabilized by feedback through the stable washout filters, even if those eigenvalues are linearly controllable.

Proof: The system is asymptotically stable only if it does not have any eigenvalue with positive real part. Since pure nonlinear control does not affect the eigenvalues, only the linear control and the linearized system are considered in the proof.

Apply a linear feedback $u = Ky$ where K is a $1 \times m$ vector and y is the output of washout filters to the system (3.6). The linear part of the overall system becomes

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{z}} \end{pmatrix} = \hat{A} \begin{pmatrix} \hat{x} \\ \hat{z} \end{pmatrix}, \tag{3.32}$$

where

$$\hat{A} = \begin{pmatrix} A + bKC & bKD \\ C & D \end{pmatrix}, \tag{3.33}$$

and

$$D = \begin{pmatrix} -d_1 & 0 & \cdots & 0 \\ 0 & -d_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -d_m \end{pmatrix}, \tag{3.34}$$

with all $d_i > 0$. The closed-loop characteristic equation is

$\hat{a}(\lambda) := \det(\lambda I - \hat{A})$

$$
\begin{aligned}
&= \det(\lambda I - D)\cdot \det((\lambda I - A - bKC) - bKD(\lambda I - D)^{-1}C)\\
&= \det(\lambda I - D)\cdot \det(\lambda I - A)\cdot\\
&\quad \det(I - bK(I + D(\lambda I - D)^{-1}C(\lambda I - A)^{-1})\\
&= \det(\lambda I - D)\cdot \det(\lambda I - A)\cdot (1 - K(I + D(\lambda I - D)^{-1}C(\lambda I - A)^{-1}b)\\
&= 0. \qquad (3.35)
\end{aligned}
$$

Here, det denotes the determinant of matrix. Since D is a diagonal matrix, we have

$$
\begin{aligned}
I + D(\lambda I - D)^{-1} &= \begin{pmatrix} \frac{\lambda}{\lambda + d_1} & 0 & \cdots & 0 \\ 0 & \frac{\lambda}{\lambda + d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\lambda}{\lambda + d_m} \end{pmatrix}\\
&= \lambda \cdot (\lambda I - D)^{-1}. \qquad (3.36)
\end{aligned}
$$

Let

$$a(\lambda) := \det(\lambda I - A), \qquad (3.37)$$

$$d(\lambda) := \det(\lambda I - D). \qquad (3.38)$$

One can write

$$
\begin{aligned}
(\lambda I - A)^{-1} &= \frac{1}{a(\lambda)} M(\lambda),\\
(\lambda I - D)^{-1} &= \frac{1}{d(\lambda)} N(\lambda), \qquad (3.39)
\end{aligned}
$$

where $M(\lambda)$ and $N(\lambda)$ are polynomials matrices of λ. Substitute Eqs. (3.36)-(3.39) into (3.35), we obtain

$$
\begin{aligned}
\hat{a}(\lambda) &= d(\lambda)\, a(\lambda) + \lambda\ (K\ N(\lambda)\ C\ M(\lambda))\ b\\
&= 0. \qquad (3.40)
\end{aligned}
$$

Since all of the filters are stable, the constant term, $d(0)$, of $d(\lambda)$ is positive. And since there is an odd number of eigenvalues with positive real part, the constant term, $a(0)$, of $a(\lambda)$ is negative. From Eq. (3.40), the constant term, $d(0)a(0)$ of the closed-loop characteristic equation $\hat{a}(\lambda)$ is negative. Thus, the closed-loop system is unstable. That is, any feedback through stable washout filters cannot stabilize the system.

■

Note that negative $a(0)$ implies that the possible way to stabilize the system is to introduce an odd number of unstable washout filters ($d(0) < 0$) to make the constant term of $\hat{a}(\lambda)$ positive. In fact, it is easy to show that systems with odd numbers of controllable eigenvalues having positive real part and remaining eigenvalues having negative real part can be stabilized by repeatedly using feedback through unstable and stable washout filters.

Lemma 3.4. Feedback through washout filters cannot move all the zero eigenvalues of a system. That is, at least one of the zero eigenvalues persist after control is applied.

Proof: Again, since pure nonlinear control does not affect the eigenvalues, only the linear control and the linearized part of the system are considered. Assume that the system (3.6) possesses some zero eigenvalues. We have $\det(A) = 0$. Apply to the system a linear feedback through washout filters, $u = Ky$. The linearization of the overall system will be of the form of Eqs. (3.32)-(3.34). Since the determinant of a matrix is independent of the row operations in the matrix, performing row operations in the determinant of the closed-loop Jacobian matrix, we obtain

$$\begin{aligned}\det \hat{A} &= \det \begin{pmatrix} A + bKC & bKD \\ C & D \end{pmatrix} \\ &= \det \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \\ &= \det(A) \cdot \det(D) \\ &= 0. \end{aligned} \tag{3.41}$$

The nullity of the determinant of the closed-loop Jacobian matrix implies that the closed-loop system possesses a zero eigenvalue.

■

Lemma 3.5. Purely nonlinear feedback through washout filters cannot change the stability of a critical system possessing single zero eigenvalue.

Proof: The proof is based on the fact that the equilibria as well as the type of static bifurcation are not affected by washout feedback. Consider a system

$$\dot{x} = f(x, u) \tag{3.42}$$

with Taylor series expansion at equilibrium point x_e as

$$\begin{aligned} \dot{x} =& A_0(x - x_e) + bu + uA_1((x - x_e) \\ &+ Q_0(x - x_e, x - x_e) + u^2 A_2(x - x_e, x - x_e) \\ &+ C_0(x - x_e, x - x_e, x - x_e) + \ldots \end{aligned} \tag{3.43}$$

Assume that one of the eigenvalues of A_0 is zero and the remaining eigenvalues are in the open left-half plane. The stability of the system can be judged by assuming that there is a pseudo parameter μ such that the zero eigenvalue crosses the origin as μ varies. Since μ is pseudo, the crossing speed of zero eigenvalue with respect to μ can be made nonzero, i.e.

$$\lambda_1'(0) := \frac{\partial \lambda_1(0)}{\partial \mu} \neq 0, \tag{3.44}$$

where $\lambda_1(0) = 0$ is the zero eigenvalue of the original system at $\mu = 0$. From the static bifurcation analysis in [45], if this parametrized system exhibits a *transcritical bifurcation* or a *subcritical pitchfork bifurcation*, the original is unstable. If the parameterized system exhibits a *supercritical pitchfork bifurcation*, the original system is stable. Since purely nonlinear feedback through washout filters preserves the equilibria and the linear stability, it also preserves the type of static bifurcation. Thus, the stability of the system does not change.

■

CHAPTER FOUR

CRITICAL SYSTEM STABILIZATION AND HOPF BIFURCATION CONTROL THROUGH WASHOUT FEEDBACK

In this chapter, a washout filter-aided feedback control law for stabilizing the critical systems possessing a pair of pure imaginary eigenvalues and for controlling the Hopf bifurcated systems is developed. All the remaining eigenvalues of these systems are assumed stable. The control law does not require an accurate knowledge of the system equilibrium points. Thus, it is robust with respect to uncertainty in the equilibrium points. Moreover, the control law preserves all the equilibrium points of the original system. It also ensures the asymptotic stability of the Hopf bifurcation point as well as orbital asymptotic stability for a range of parameter values of the periodic solutions emerging from the bifurcation point.

The control algorithm developed in this chapter extends the work in [13]. However, due to the equilibrium preservation property and the flexibility in choosing the washout filter time constant, the present approach offers the advantage of accommodating the uncertainty in the system dynamics.

4.1. Background and motivation

Local feedback stabilization for critical nonlinear systems for cases in which the system linearization possesses a pair of simple pure imaginary eigenvalues has been studied by Aeyels [15], Abed and Fu [13], and Liaw [17], among others. Aeyels [15] and Liaw [17] approached this problem through center manifold reduction and normal form transformation. Aeyels [15] focused on a class of third-order systems with uncontrollable critical mode. This analysis was extended by Liaw [17] to general finite dimensional systems having either a controllable or an uncontrollable critical mode.

On the other hand, [13] employed the stability formula for Hopf bifurcation ([38]) involving only Taylor series expansion of the vector field and eigenvector computations. They used this formula in addressing the local stabilization problem for critical systems possessing a pair of imaginary eigenvalues. For a system undergoing a Hopf bifurcation, by continuity, the control law stabilizes the Hopf bifurcation point and the bifurcated periodic solutions emerging from this point (locally in parameter space).

Each of the foregoing studies assumed that the critical equilibrium point is known. For convenience, these studies take this equilibrium to be the origin. Indeed, in the discussion in [13] of the stabilization of bifurcated periodic orbits in a parameterized system, it was assumed that the dependence of the equilibrium on the parameter was also known. In many practical applications, equilibrium points may not be easy to obtain due to the model uncertainty or the complexity of the computation. Some equilibria may be crudely approximated within a parameter range of interest. However, as discussed in Chapter 3, the inaccuracy severely limits the robustness to be expected in the controlled system.

In this chapter, by introducing washout filters in the feedback control loop, the need for accurate knowledge of equilibrium points is relieved, and the control laws preserve the equilibrium points of the uncontrolled system. Moreover, by using the flexibility in choosing the time constant for the washout filters, the control function can be much more robust than those previous works not only with respect to the uncertainty in equilibrium points but also with respect to the uncertainty in other system dynamics.

The control law derived here is also based on the stability formula derived by [38] and the control algorithm derived by [13]. Since no center manifold reduction and normal form transformation are necessary, it is much easier to determine the upper limit of uncertainty that the control can tolerate for a system described in a general state variable form.

The problems discussed in this chapter are both local feedback stabilization of critical systems possessing a pair of pure imaginary eigenvalues and the Hopf bifurcation stabilization. The control functions used here are purely nonlinear so that not only the equilibrium points but also the linear stability of the original system are preserved.

4.2. Hopf bifurcation formulae

In this section, a stability criterion for Hopf bifurcation derived by Howard [38], and Abed and Fu [13] is briefly reviewed. The criterion is based on the Taylor series expansion of the vector field and the eigenvector computations, no center manifold transformation and normal form transformation are necessary. Thus, it is very convenient for the robust stabilization control discussed in the Section 4.5.

Consider a nonlinear autonomous system

$$\dot{x} = f_\mu(x), \tag{4.1}$$

where $x \in I\!R^n$ is the state vector, $\mu \in I\!R$ is the system parameter, $f : I\!R^n \to I\!R^n$ is sufficiently smooth in x, μ and $f_\mu(x_{e,\mu}) = 0$, i.e. $x_{e,\mu}$ is an equilibrium point of (4.1) at system parameter μ. Suppose that following hypothesis (called hypothesis (H)) holds: The Jacobian matrix $D_x f_{\mu_c}(x_{e,\mu_c})$ has a simple pair of nonzero pure imaginary eigenvalues $\lambda_1(\mu_c) = j\omega_c$ and $\lambda_2(\mu_c) = -j\omega_c$ with

$\omega_c \neq 0$ and the transversality condition

$$\frac{\partial \mathrm{R_e}[\lambda(\mu_c)]}{\partial \mu} \neq 0 \tag{4.2}$$

satisfied, and all the remaining eigenvalues are in the open left-half plane. Then, the Hopf Bifurcation Theorem asserts the existence of a one-parameter family $\{p_\epsilon, 0 < \epsilon \leq \epsilon_0\}$ of nonconstant periodic solutions of system (4.1) emerging from $x = x_{e,\mu_c}$ at the parameter value μ_c for ϵ_0 sufficiently small. The system parameter μ can be represented as a smooth function of ϵ. The periodic solutions $p_\epsilon(t)$ corresponding to parameter values $\mu(\epsilon)$ have period near $2\pi\omega_c^{-1}$. Exactly one of the characteristic exponents $\beta(\epsilon)$ of the periodic solution p_ϵ governs the asymptotic stability. That is, if $\beta(\epsilon) < 0$, p_ϵ is orbitally asymptotically stable, while if $\beta(\epsilon) > 0$, p_ϵ is unstable. The characteristic exponent $\beta(\epsilon)$ is a real, smooth and even function of ϵ

$$\beta(\epsilon) = \beta_2\epsilon^2 + \beta_4\epsilon^4 + \cdots. \tag{4.3}$$

Moreover, if β_2 in Eq. (4.3) does not equal to zero, the sign of β_2 is sufficient for determining the stability of system (4.1) for $\mu(\epsilon)$ in the sufficiently small neighborhood of μ_c (equivalent to sufficiently small value of ϵ).

An algorithm for computing the "stability parameter", β_2, is as follows

Step 1. Perform a Taylor series expansion on system (4.1) with respect to both x and μ at Hopf bifurcation point, i.e. at $\mu = \mu_c$, $x = x_{e,\mu_c}$.

$$\begin{aligned}\dot{\hat{x}} &= A_0\hat{x} + Q_0(\hat{x},\hat{x}) + C_0(\hat{x},\hat{x},\hat{x}) + \cdots \\ &\quad + \hat{\mu}(A_1\hat{x} + Q_1(\hat{x},\hat{x}) + \cdots) + \hat{\mu}^2 A_2\hat{x} + \cdots,\end{aligned} \tag{4.4}$$

where $\hat{x} = x - x_{e,\mu_c}$, $\hat{\mu} = \mu - \mu_c$, $Q_i(\hat{x}, \hat{x})$ and $C_i(\hat{x}, \hat{x}, \hat{x})$ are the second order (in $\hat{x}$) terms and third order (in $\hat{x}$) terms generated by a vector valued symmetric bilinear form $Q_i(x, y)$ and a vector valued symmetric trilinear form $C_i(x, y, z)$ respectly. With proper rearranging the order of state variables if necessary, let r be the right eigenvector of A_0 with respect to eigenvalue $j\omega_c$ with the first component of r equal to 1. Let l be the left eigenvector of A_0 corresponding to the eigenvalue $j\omega_c$, normalized such that $lr = 1$.

Step 2. Solve the equations

$$A_0 a = -\frac{1}{2} Q_0(r, \bar{r}), \tag{4.5}$$

$$(2j\omega_c I - A_0) b = \frac{1}{2} Q_0(r, r) \tag{4.6}$$

for a and b.

Step 3. The stability coefficient β_2 is given by

$$\beta_2 = 2\mathrm{R_e}\{2lQ_0(r, a) + lQ_0(\bar{r}, b) + \frac{3}{4} C_0(r, r, \bar{r})\}. \tag{4.7}$$

Note that the sign of β_2 not only determines the stability of the periodic solution p_ϵ, but also the stability of the critical equilibrium point x_{e,μ_c}. Therefore, a feedback control law which renders $\beta_2 < 0$ will stabilize both the Hopf bifurcation and the critical equilibrium point.

4.3. Direct state feedback

In this section, a direct state feedback control law for critical system stabilization and Hopf bifurcation control derived by Abed and Fu [13] is recalled. Here, the setting of [13] in which the nominal equilibrium branch is taken as the origin is not assumed. The main advantage of this control law is that it is based on the coefficients in the Taylor series expansion of the original system and on the eigenvector corresponding to the critical mode. Preliminary transformations of variables, such as the center manifold transformation, are not employed. The control law of this section serves as a platform for deriving control laws using washout feedback, and also for comparison with the results obtained subsequently using washout filters.

Consider a one-parameter family of nonlinear autonomous control systems

$$\dot{x} = f_\mu(x, u). \tag{4.8}$$

Equation (4.8) is the natural extension of Eq. (4.1) to allow an additional, smooth dependence on a scalar control input u. Thus, x and μ are as in system (4.1), f_μ is a smooth map from $I\!R^n \times I\!R$ to $I\!R^n$, and u is a scalar input. Assume that hypothesis (H) of section 4.2 holds for $u = 0$ with μ_c a critical value of μ and x_{e,μ_c} the corresponding nominal equilibrium. Set $\mu = \mu_c$. Then Taylor series expansion of (4.8) with respect to x and u at $x = x_{e,\mu_c}$, $u = 0$ gives (here $\hat{x} := x - x_{e,\mu_c}$)

$$\begin{aligned}\dot{\hat{x}} &= A_0\hat{x} + u\gamma + uA_1\hat{x} + Q_0(\hat{x},\hat{x}) \\ &\quad + u^2A_2\hat{x} + uQ_1(\hat{x},\hat{x}) + C_0(\hat{x},\hat{x},\hat{x}) + \cdots .\end{aligned} \tag{4.9}$$

Take the feedback control u to be of the form

$$u = \hat{x}^T Q_u \hat{x} + C_u(\hat{x},\hat{x},\hat{x}), \tag{4.10}$$

where Q_u is a real symmetric $n \times n$ matrix and C_u is a cubic form generated by a scalar-valued symmetric trilinear form. No quartic or higher order terms are included in the feedback since it is clear from the formula (4.7) that these terms do not affect β_2. We also do not include linear terms in the control law (4.10). Any linear feedback used to modify the critical parameter value μ_c is assumed to be accounted in the nominal system (4.8). Of course, linear feedback could conceivably be employed in (4.10) without influencing the value of μ_c. However, such a feedback would influence the associated eigenvectors, along with the linearized stability of other equilibria. To avoid these complications, we choose to exclude linear feedback from the control law (4.10). See, however, Bacciorti and Boier [46] and Abed and Fu [13]. The closed-loop system is

$$\dot{\hat{x}} = A_0\hat{x} + Q_0^*(\hat{x},\hat{x}) + C_0^*(\hat{x},\hat{x},\hat{x}) + \cdots, \tag{4.11}$$

where

$$Q_0^*(\hat{x},\hat{x}) = (\hat{x}^T Q_u \hat{x})\gamma + Q_0(\hat{x},\hat{x}), \tag{4.12}$$

$$C_0^*(\hat{x},\hat{x},\hat{x}) = C_u(\hat{x},\hat{x},\hat{x})\gamma + C_0(\hat{x},\hat{x},\hat{x}) + (\hat{x}^T Q_u \hat{x})A_1\hat{x}. \tag{4.13}$$

We need $Q_0^*(x,y)$, and $C_0^*(x,y,z)$ to be symmetric bilinear and trilinear forms in order to use Eq. (4.7) to calculate the closed-loop β_2. This is automatic in the case of $Q_0^*(\cdot,\cdot)$, since both $x^T Q_u y$ and $Q_0(x,y)$ are symmetric. As for $C_0^*(\cdot,\cdot,\cdot)$, we can *define*

$$\begin{aligned} C_0^*(x,y,z) &= C_u(x,y,z)\gamma + C_0(x,y,z) \\ &= \frac{1}{3}\{(y^T Q_u z)A_1 x + (x^T Q_u y)A_1 z + (z^T Q_u x)A_1 y\}, \end{aligned} \tag{4.14}$$

which is a symmetric trilinear form.

Following the algorithm in Section 4.2, we have

$$\begin{aligned} a^* &= -\frac{1}{2}A_0^{-1}Q_0^*(r,\bar{r}) \\ &= -\frac{1}{2}A_0^{-1}Q_0(r,\bar{r}) - \frac{1}{2}(r^T Q_u r)A_0^{-1}\gamma \end{aligned} \tag{4.15}$$

$$\begin{aligned} b^* &= \frac{1}{2}(2j\omega_c I - A_0)^{-1}Q_0^*(r,r) \\ &= \frac{1}{2}(2j\omega_c I - A_0)^{-1}Q_0(r,r) + \frac{1}{2}(2j\omega_c I - A_0)^{-1}(r^T Q_u r)\gamma, \end{aligned} \tag{4.16}$$

and the closed-loop stability coefficient is

$$\begin{aligned} \beta_2^* &= 2\mathrm{R_e}\{2lQ_0^*(r,a^*) + lQ_0^*(\bar{r},b^*) + \frac{3}{4}lC_0^*(r,r,\bar{r})\} \\ &= \beta_2 + 2\mathrm{R_e}\Delta \end{aligned} \tag{4.17}$$

where Δ is given by

$$\begin{aligned}\Delta = & 2l[r^T Q_u a^* \gamma - Q_0(r, \frac{1}{2}(r^T Q_u \bar{r}) A_0^{-1} \gamma)] \\ & + l[\bar{r}^T Q_u b^* \gamma + Q_0(\bar{r}, \frac{1}{2}(r^T Q_u r)(2j\omega_c I - A_0)^{-1} \gamma)] \\ & + \frac{3}{4} l C_u(r, r, \bar{r})\gamma + \frac{1}{4} l[2(r^T Q_u \bar{r}) A_1 r + (r^T Q_u r) A_1 \bar{r}]. \end{aligned} \tag{4.18}$$

From Eq. (4.17), it is clear that stability of the Hopf bifurcation point and of the local bifurcated periodic solution p_ϵ, (see Section 4.2), can be achieved if $\mathrm{R_e}\Delta$ can be assigned by feedback of the form (4.9). To further analyze the stabilizability, two cases are discussed separately.

4.3.1. Linearly controllable critical mode

From the well known PBH test [44], controllability of the critical mode is equivalent to the requirement $l\gamma \neq 0$. By setting the $Q_u = 0$ in the control law (4.10), Eq. (4.18) becomes

$$\Delta = \frac{3}{4} C_u(r, r, \bar{r}) l\gamma. \tag{4.19}$$

Since the coefficients of $C_u(x, y, z)$ can be arbitrarily assigned, the value of Δ can be assigned arbitrarily in the complex plane. Thus, we have the following result [13].

Theorem 4.1. Suppose the system (4.8) satisfies hypothesis (H) and the critical mode is linearly controllable ($l\gamma \neq 0$). Then there exists a purely nonlinear feedback $u(\hat{x})$ of the form (4.10) which stabilizes the Hopf bifurcation point (critical equilibrium point) and the periodic solutions emerging from that bifurcation point for parameter values μ near the critical parameter value μ_c.

Note that, the equilibrium points extended from the Hopf bifurcation point as the parameter μ is varied need not be preserved unless the control u is taken to be a function of $x - x_{e,\mu}$ instead of $x - x_{e,\mu_c}$.

4.3.2 Linearly uncontrollable critical mode

In this case $l\gamma = 0$, and Eq. (4.18) becomes

$$\Delta = -2lQ_0(4, \frac{1}{2}(r^T Q_u \bar{r})A_0^{-1}\gamma) + lQ_0(\bar{r}, \frac{1}{2}(r^T Q_u r)(2j\omega_c I - A_0)^{-1}\gamma)$$
$$+ \frac{1}{4}l[2(r^T Q_u \bar{r})A_1 r + (r^T Q_u r)Q_1 \bar{r}]. \tag{4.20}$$

Note that only the quadratic terms in the feedback influence Δ. The problem now is to find a condition to ensure the assignability to arbitrary negative values of $\mathrm{R_e}\Delta$ by a quadratic feedback control. To achieve an explicit condition, we employ the following lemma [13].

Lemma 4.2. If r is a right eigenvector of A_0 corresponding to the eigenvalue $j\omega_c$, then there exists a real symmetric matrix Q_u such that

$$\mathrm{I_m} Q_u r = 0 \quad \text{and} \quad \mathrm{R_e} Q_u r \neq 0. \tag{4.21}$$

Based on Lemma 4.2, we can choose a quadratic control u which satisfies condition (4.21).

Let

$$\rho = (\mathrm{R_e} r)^T Q_u (\mathrm{R_e} r). \tag{4.22}$$

Using (4.21),(4.22) in Eq. (4.18), we have

$$\Delta = \rho\{-2lQ_0(r, \frac{1}{2}A_0^{-1}\gamma) + lQ_0(\bar{r}, \frac{1}{2}(2j\omega_c I - A_0)^{-1}\gamma)$$
$$+ \frac{1}{4}l[2A_1 r + A_1 \bar{r}]\}. \tag{4.23}$$

A sufficient condition for the stabilization problem is as follows:

Theorem 4.3. Suppose the system (4.8) satisfies hypothesis (H) and the critical mode is linearly uncontrollable ($l\gamma = 0$). Then there is a quadratic feedback $u(\hat{x})$ with $u(0) = 0$ which stabilizes the Hopf bifurcation point (the critical equilibrium point) and the periodic solutions emerging from that bifurcation point for the parameter values μ near μ_c, provided that

$$\mathrm{R_e}\{-2lQ_0(r, \frac{1}{2}A_0^{-1}\gamma) + lQ_0(\bar{r}, \frac{1}{2}(j\omega_c I - A_0)^{-1}\gamma)$$
$$+ \frac{1}{4}l[2A_1 r + A_1 \bar{r}]\} \neq 0. \tag{4.24}$$

■

As was the case with Theorem 4.1, the equilibria are not necessarily preserved.

4.3.3 Critique of direct state feedback

Recall the control function in Eq. (4.10). The argument of the function u is $\hat{x} = x - x_{e,\mu_c}$. If the equilibrium points $x_{e,\mu}$ are not equal to the critical equilibrium point x_{e,μ_c} for μ near μ_c, the function $f_\mu(x_{e,\mu}, u(x - x_{e,\mu_c}))$ does not necessarily vanish. This means that $x_{e,\mu}$ may no longer be an equilibrium point for parameter values $\mu \neq \mu_c$. This may sometimes be undesirable since an equilibrium point represents a certain capability of the system at steady state. Altering the equilibrium may seriously degrade such a capability. The case of aircraft Dutch roll control as discussed in Chapter 3 is an example of this.

In order to preserve the original equilibrium points while still using direct state feedback, the argument of the control function should be $\hat{x}_\mu := x - x_{e,\mu}$. Clearly this requires knowledge of the whole branch of equilibria within the neighborhood of x_{e,μ_c} of interest, and also requires the control u to depend on the parameter μ. Thus, the need for on-line computation of equilibrium points arises with this type of feedback. For systems exhibiting large uncertainty in the equilibrium points or with mathematic complexity in the computation of equilibrium points, it may be difficult to arrange that the control u be an accurate function of $x - x_{e,\mu}$. Hence, in practice, it may be difficult to preserve the system equilibrium points within the framework of direct (static) state feedback.

Even if the equilibrium points are known accurately, if the system has multiple equilibrium branches, that is, there is at least one equilibrium point $x'_{e,\mu} \neq x_{e,\mu}$ for some parameter values μ, by using the control as a function of $x - x_{e,\mu}$, one still cannot preserve the equilibrium $x'_{e,\mu}$. This defect also is not always acceptable. An example is shown in Chapter 7 in the context of direct state feedback control of aircraft high angle-of-attack flight dynamics. In that example, feedback brings the extraneous equilibria very close to the nominal equilibrium, significantly increasing the stability vulnerability of that equilibrium. This vulnerability occurs because the domain of attraction of the nominal equilibrium is compromised in the direction of a nearby extraneous equilibrium.

On the other hand, by employing the outputs of washout filters (see Chapter 3) as the arguments to the control, the control is rendered independent of the equilibrium points. Hence, on-line computation of equilibrium points is no longer necessary. This type of control is therefore inherently robust to uncertainty in the equilibria. Moreover, all the equilibrium branches, including those extraneous branches, remain where they are before control.

4.4. Feedback through washout filters

Recall the system defined in Eq. (4.8). For each system state variable $x_i, i = 1, \ldots, n$, introduce a washout filter governed by the dynamic equation

$$\dot{z}_i = x_i - dz_i \tag{4.25a}$$

along with output equation

$$y_i = x_i - dz_i. \tag{4.25b}$$

The *extended* system (the system with washout filters added on) becomes

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{z}} \end{pmatrix} = \tilde{A}_0 \begin{pmatrix} \hat{x} \\ \hat{z} \end{pmatrix} + u\tilde{\gamma} + u\tilde{A}_1\hat{x} + \tilde{Q}_0(\hat{x}, \hat{x})$$
$$+ u^2\tilde{A}_2\hat{x} + u\tilde{Q}_1(\hat{x}, \hat{x}) + \tilde{C}_0(\hat{x}, \hat{x}, \hat{x}) + \cdots, \tag{4.26}$$

where z is an n-dimensional vector of washout filter states,

$$z_e := \frac{1}{d} x_{e,\mu_c} \tag{4.27}$$

represents an equilibrium state of z, $\hat{z} := z - z_e$, and

$$\tilde{A}_0 := \begin{pmatrix} A_0 & 0 \\ I & -dI \end{pmatrix}, \tag{4.28}$$

$$\tilde{\gamma} := \begin{pmatrix} \gamma \\ 0 \end{pmatrix}, \tag{4.29}$$

$$\tilde{A}_i := \begin{pmatrix} A_i \\ 0 \end{pmatrix}, \tag{4.30}$$

for $i = 1, 2, \dots$, and

$$\tilde{Q}_j(\zeta, \zeta) = \begin{pmatrix} Q_j(\zeta_1, \zeta_1) \\ 0 \end{pmatrix}, \tag{4.31}$$

$$\tilde{C}_j(\zeta, \zeta, \zeta) = \begin{pmatrix} C_j(\zeta_1, \zeta_1, \zeta_1) \\ 0 \end{pmatrix}, \tag{4.32}$$

for $j = 0, 1, \dots$. Here, $\zeta_1 := \hat{x}$, $\zeta_2 := \hat{z}$, and

$$\zeta := \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \tag{4.33}$$

denotes the extended system state.

Note that only stable washout filters, i.e., those for which $d > 0$, are admissible here, since the eigenvalues of the extended system (4.26), are precisely those of the original system along with those of the washout filters. (Recall that a purely nonlinear feedback is applied.) An unstable washout filter would introduce an unstable eigenvalue, destabilizing the closed-loop system.

For simplicity, n washout filters, one for each system state, are reflected in Eq. (4.26). The order of the closed-loop system is $2n$. In fact, the actual number of washout filters needed, and hence also the resulting increase in system order, depends on the number of states involved in the feedback control. Usually, this will be less than n.

Since all of the eigenvalues are preserved by the (purely nonlinear) feedback, the issue of losing linear controllability discussed in Section 3.3 is not relevant. It is therefore allowed, for instance, to use washout filters all of which have the same eigenvalue $-d$ to simplify the computation.

For simplicity, a single state is taken as input to each washout filter, although the general form presented in Chapter 3 permits several states to be input to the same filter.

Next, in order to investigate the influence of washout filters on the stability parameter β_2, the eigenvectors of the extended system are expressed in terms of the eigenvectors of the original system.Let r and l denote the right and left eigenvectors of the original system corresponding to the eigenvalue $j\omega_c$, respectively, with the first element of r equal to unity and $l \cdot r = 1$. Then we have

$$A_0 r = j\omega_c r, \tag{4.34a}$$

$$l A_0 = j\omega_c l. \tag{4.34b}$$

Denote by $\tilde{r}$ the $2n$-dimensional column vector

$$\tilde{r} = \begin{pmatrix} r \\ \frac{1}{d+j\omega_c} r \end{pmatrix}. \tag{4.35}$$

Since

$$\begin{aligned} \tilde{A}_0 \tilde{r} &= \begin{pmatrix} A_0 & 0 \\ I & -dI \end{pmatrix} \tilde{r} \\ &= \begin{pmatrix} j\omega_c r \\ \frac{j\omega_c}{d+j\omega_c} r \end{pmatrix} \\ &= j\omega_c \tilde{r}, \end{aligned} \tag{4.36}$$

$\tilde{r}$ is a right eigenvector of $\tilde{A}_0$. Moreover, the first element of $\tilde{r}$, being equal to the first element of r, is unity. Denote by $\tilde{l}$ the $2n$-dimensional row vector

$$\tilde{l} = (\, l \quad 0\,). \tag{4.37}$$

We have

$$\begin{aligned}
\tilde{l}\tilde{A}_0 &= (\, l \quad 0\,)\begin{pmatrix} A_0 & 0 \\ I & -dI \end{pmatrix} \\
&= (\, j\omega_c l \quad 0\,) \\
&= j\omega_c \tilde{l}.
\end{aligned} \tag{4.38}$$

Thus, $\tilde{l}$ is a left eigenvector of $\tilde{A}_0$ corresponding to the eigenvalue $j\omega_c$. Note that

$$\tilde{l} \cdot \tilde{r} = l \cdot r = 1. \tag{4.39}$$

Hence, $\tilde{r}$ and $\tilde{l}$ satisfy the normalizations required in computing the stability parameter $\tilde{\beta}_2$ of the overall system by the algorithm of Section 4.2.

Lemma 4.4. The extended system (4.26) has the same stability coefficient as the original system (4.9), that is, $\tilde{\beta}_2 = \beta_2$.

Proof: Following the algorithm of Section 4.2, we have

$$\begin{aligned}
\tilde{a} &= -\frac{1}{2}\tilde{A}_0^{-1}\tilde{Q}_0(\tilde{r},\bar{\tilde{r}}) \\
&= -\frac{1}{2}\begin{pmatrix} A_0^{-1} & 0 \\ \frac{1}{d}A_0^{-1} & -\frac{1}{d}I \end{pmatrix}\begin{pmatrix} Q_0(r,\bar{r}) \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{2}A_0^{-1}Q_0(r,\bar{r}) \\ -\frac{1}{2d}A_0^{-1}Q_0(r,\bar{r}) \end{pmatrix} \\
&= \begin{pmatrix} a \\ \frac{1}{d}a \end{pmatrix},
\end{aligned} \tag{4.40}$$

$$\begin{aligned}
\tilde{b} &= \frac{1}{2}(2j\omega_c I - \tilde{A}_0)^{-1}\tilde{Q}_0(\tilde{r},\tilde{r}) \\
&= \frac{1}{2}\begin{pmatrix} (2j\omega_c I - A_0)^{-1} & 0 \\ \frac{-1}{2j\omega_c+d}(2j\omega_c I - A_0)^{-1} & \frac{1}{2j\omega_c+d}I \end{pmatrix}\begin{pmatrix} Q_0(r,r) \\ 0 \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} \frac{1}{2}(2j\omega_c I - A_0)^{-1} Q_0(r,r) \\ \frac{1}{2(2j\omega_c+d)}(2j\omega_c I - A_0)^{-1} Q_0(r,r) \end{pmatrix}$$

$$= \begin{pmatrix} b \\ \frac{1}{2j\omega_c+d} b \end{pmatrix}. \tag{4.41}$$

From the definitions of $\tilde{Q}_0, \tilde{C}_0, \tilde{l}$, and $\tilde{r}$ in Eqs. (4.31), (4.32), (4.35), (4.37), the stability coefficient becomes

$$\begin{aligned} \tilde{\beta}_2 &:= 2\mathrm{R_e}\{2\tilde{l}\tilde{Q}_0(\tilde{r},\tilde{a}) + \tilde{l}\tilde{Q}_0(\bar{\tilde{r}},\tilde{b}) + \frac{3}{4}\tilde{C}_0(\tilde{r},\tilde{r},\bar{\tilde{r}})\} \\ &= 2\mathrm{R_e}\{2lQ_0(r,a) + lQ_0(\bar{r},b) + \frac{3}{4}C_0(r,r,\bar{r})\} \\ &= \beta_2. \end{aligned} \tag{4.42}$$

■

Next, the *closed-loop* stability parameter β_2^* of the *overall* system (the extended system with control applied) follows from Eqs. (4.15)-(4.16) of Section 4.3:

$$\begin{aligned} \beta_2^* &= \tilde{\beta}_2 + 2\mathrm{R_e}\Delta \\ &= \beta_2 + 2\mathrm{R_e}\Delta, \end{aligned} \tag{4.43}$$

where Δ is given by

$$\begin{aligned} \Delta = {} & 2\tilde{l}[\tilde{r}^T\tilde{Q}_u a^*\tilde{\gamma} - \tilde{Q}_0(\tilde{r}, \frac{1}{2}(\tilde{r}^T\tilde{Q}_u\bar{\tilde{r}})\tilde{A}_0^{-1}\tilde{\gamma})] \\ & + \tilde{l}[\bar{\tilde{r}}^T\tilde{Q}_u b^*\tilde{\gamma} + \tilde{Q}_0(\bar{\tilde{r}}, \frac{1}{2}(\tilde{r}^T\tilde{Q}_u\tilde{r})(2j\omega_c I - \tilde{A}_0)^{-1}\tilde{\gamma})] \\ & + \frac{3}{4}\tilde{l}\tilde{C}_u(\tilde{r},\tilde{r},\bar{\tilde{r}})\tilde{\gamma} + \frac{1}{4}\tilde{l}[2(\tilde{r}^T\tilde{Q}_u\bar{\tilde{r}})\tilde{A}_1\tilde{r} + (\tilde{r}^T\tilde{Q}_u\tilde{r})\tilde{A}_1\bar{\tilde{r}}], \end{aligned} \tag{4.44}$$

with

$$a^* = -\frac{1}{2}\tilde{A}_0^{-1}\tilde{Q}_0(\tilde{r},\bar{\tilde{r}}) - \frac{1}{2}(\tilde{r}^T\tilde{Q}_u\bar{\tilde{r}})\tilde{A}_0^{-1}\tilde{\gamma}, \tag{4.45}$$

$$b^* = \frac{1}{2}(2j\omega_c I - \tilde{A}_0)^{-1}\tilde{Q}_0(\tilde{r},\tilde{r}) + \frac{1}{2}(2j\omega_c I - \tilde{A}_0)^{-1}(\tilde{r}^T\tilde{Q}_u\tilde{r})\tilde{\gamma}, \tag{4.46}$$

and where $\tilde{Q}_u(\zeta,\zeta)$ and $\tilde{C}_u(\zeta,\zeta,\zeta)$ (with ζ as in Eq. (4.33)) are the quadratic and cubic terms of the control function of the overall system. The main difference between direct state feedback and feedback through washout filters is that the control functions $\tilde{Q}_u$ and $\tilde{C}_u$, are now restricted to be functions of the outputs of the washout filters, rather than depending directly on the system states.

Let the control u take the form

$$u = y^T Q_u y + C_u(y,y,y), \tag{4.47}$$

where y is the vector of washout filter outputs

$$\begin{aligned} y &= x - dz && (4.48a)\\ &= \hat{x} - d\hat{z}. && (4.48b) \end{aligned}$$

In (4.47), Q_u is a real symmetric $n \times n$ matrix and C_u is a cubic form generated by a scalar-valued symmetric trilinear form. Substituting y from Eq. (4.48b) into the quadratic and cubic terms in Eq. (4.47), we obtain

$$\begin{aligned} y^T Q_u y &= (\hat{x} - d\hat{z})^T Q_u (\hat{x} - d\hat{z}) \\ &= \begin{pmatrix} \hat{x} & \hat{z} \end{pmatrix} \begin{pmatrix} Q_u & -dQ_u \\ -dQ_u & d^2 Q_u \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{z} \end{pmatrix}, \end{aligned} \tag{4.49}$$

and

$$\begin{aligned} C_u(y,y,y) &= C_u(\hat{x} - d\hat{z}, \hat{x} - d\hat{z}, \hat{x} - d\hat{z}) \\ &= C_u(\hat{x},\hat{x},\hat{x}) - 3dC_u(\hat{x},\hat{x},\hat{z}) + 3d^2 C_u(\hat{x},\hat{z},\hat{z}) - d^3 C_u(\hat{z},\hat{z},\hat{z}). \end{aligned} \tag{4.50}$$

The corresponding quadratic and cubic terms in the control function for the overall system of (4.26) are

$$\tilde{Q}_u := \begin{pmatrix} Q_u & -dQ_u \\ -dQ_u & d^2 Q_u \end{pmatrix}, \tag{4.51}$$

4.4.1. Controllable critical mode

In this case, $l\gamma \neq 0$. Setting the quadratic component of the control to zero and denoting by P the quantity

$$P := \frac{\omega_c^3(\omega_c + jd)}{(d^2 + \omega_c^2)^2}, \tag{4.58}$$

we have

$$\Delta = \frac{3}{4} P C_u(r, r, \bar{r}) l\gamma. \tag{4.59a}$$

Thus

$$\mathrm{R_e}\Delta = \frac{3}{4}\mathrm{R_e}(C_u(r,r,\bar{r})\mathrm{R_e}[Pl\gamma] - \mathrm{I_m}C_u(r,r,\bar{r})\mathrm{I_m}[Pl\gamma]\}. \tag{4.59b}$$

Since $C_u(r, r, \bar{r})$ can be taken to be any complex number, we can find a cubic function $C_u(y, y, y)$ such that $\mathrm{R_e}\Delta$ is sufficiently negative to ensure $\beta_2^* < 0$. Thus, we have the following result.

Theorem 4.5. Suppose system (4.8) satisfies hypothesis (H) and the critical eigenvalues are linearly controllable. Then there is a purely nonlinear washout filter-aided feedback which stabilizes the Hopf bifurcation point (critical equilibrium point) and the periodic solutions emerging from the Hopf bifurcation point for μ the critical value μ_c.

■

Since the feedback function is independent of the equilibrium, no accurate on-line computation of the system equilibria is necessary.

4.4.2. Uncontrollable critical mode

In this case $l\gamma = 0$, and Eq. (4.44) becomes

$$\begin{aligned}\Delta = &-2lQ_0(r, \frac{1}{2}S_2(r^T Q_u \bar{r})A_0^{-1}\gamma) \\ &+ lQ_0(\bar{r}, \frac{1}{2}S_1(r^T Q_u r)(2j\omega_c I - A_0)^{-1}\gamma) \\ &+ \frac{1}{4}l[2S_2(r^T Q_u \bar{r})A_1 r + S_1(r^T Q_u r)A_1 \bar{r}].\end{aligned} \tag{4.60}$$

As in the case of direct state feedback, the cubic control $C_u(y, y, y)$ has no influence on the stability parameter β_2. Following Lemma 4.2, choose a real symmetric matrix Q_u with $\mathrm{I_m} Q_u r = 0$ and $\mathrm{R_e} Q_u r \neq 0$. Denoting

$$\rho := (\mathrm{R_e} r)^T Q_u (\mathrm{R_e} r), \tag{4.61}$$

$$q_1 + j m_1 := l Q_0(r, \frac{1}{2} A_0^{-1} \gamma), \tag{4.62}$$

$$q_2 + j m_2 := l Q_0(\bar{r}, \frac{1}{2} (2 j \omega_c I - A_0)^{-1} \gamma), \tag{4.63}$$

$$\eta_1 + j \theta_1 := l A_1 r, \tag{4.64}$$

$$\eta_2 + j \theta_2 := l A_1 \bar{r}, \tag{4.65}$$

we have

$$\mathrm{R_e} \Delta = \rho [S_2(-2q_1 + \frac{1}{2}\eta_1) + \mathrm{R_e}\{S_1\}(q_2 + \frac{1}{4}\eta_2) - \mathrm{I_m}\{S_1\}(m_2 + \frac{1}{4}\theta_2)]. \tag{4.66}$$

By Eq. (4.66), $\mathrm{R_e}\Delta \neq 0$ is equivalent to the condition

$$d^2 K_3 + d\omega_c K_2 + \omega_c^2 K_1 \neq 0, \tag{4.67}$$

where

$$K_1 := -2q_1 + q_2 + \frac{1}{2}\eta_1 + \frac{1}{4}\eta_2, \tag{4.68}$$

$$K_2 := 2m_2 + \frac{1}{2}\theta_2, \tag{4.69}$$

$$K_3 := -2q_1 - q_2 + \frac{1}{2}\eta_1 - \frac{1}{4}\eta_2. \tag{4.70}$$

Thus, we have the following result.

Theorem 4.6. Suppose that system (4.9) satisfies hypothesis (H) and the critical eigenvalues are uncontrollable. Then there is a quadratic washout filter-aided feedback which stabilizes the Hopf bifurcation point (critical equilibrium point) and the periodic solutions emerging from the Hopf bifurcation point for parameter values μ near μ_c provided that either of the following conditions holds:

$$\text{i)}\quad \mathrm{R_e}\{-2lQ_0(r, \frac{1}{2}A_0^{-1}\gamma) + \frac{1}{2}lA_1 r\} \neq 0 \tag{4.71}$$

$$\text{ii)}\quad 2lQ_0(\bar{r}, \frac{1}{2}(2j\omega_c I - A_0)^{-1}\gamma) + \frac{1}{2}lA_1\bar{r} \neq 0 \tag{4.72}$$

Proof: From the definitions (4.68)-(4.70), condition (i) implies that at least one of K_1 or K_3 does not vanish. Condition (ii) implies that at least one of the following conditions holds:

$$2q_2 + \frac{1}{2}\eta_2 \neq 0, \tag{4.73a}$$

$$2m_2 + \frac{1}{2}\theta_2 \neq 0. \tag{4.73b}$$

If Eq. (4.73a) holds, at least one of K_1 or K_2 does not vanish. If Eq. (4.73b) holds, we have $K_2 \neq 0$. Thus, condition (i) or (ii) implies that at least one of K_1, K_2, K_3 does not vanish.

If $K_3 = 0$ and $K_2 = 0$ but $K_1 \neq 0$, then we have

$$\mathrm{R_e}\Delta = \rho\frac{\omega_c^4}{(d^2+\omega_c^2)^2}K_1 \neq 0. \tag{4.74}$$

If $K_3 = 0$ but $K_2 \neq 0$, by choosing

$$d \neq \frac{-K_1}{K_2}\omega_c, \tag{4.75}$$

we also have $\mathrm{R_e}\Delta \neq 0$.

If $K_3 \neq 0$, by choosing

$$d \neq \frac{K_2 \pm \sqrt{K_2^2 - 4K_3K_1}}{2K_3}\omega_c, \tag{4.76}$$

we still have $\mathrm{R_e}\Delta \neq 0$.

Hence, by proper choosing the washout filter time constant, d, we can make $\mathrm{R_e}\Delta \neq 0$. By choosing the magnitude of ρ sufficiently large, we can ensure the closed-loop stability coefficient $\beta_2^* < 0$ and solve the stabilization problem.

■

Note that, condition $K_1 \neq 0$ is the same with the condition in Theorem 4.3 ([13] Theorem 2) for the case of direct state feedback. Therefore, by using washout filters we have two more flexibility, either $K_2 \neq 0$ or $K_3 \neq 0$, for stabilizing the system.

4.5. Robustness of washout filter-aided feedback control

Since the washout filter-aided feedback control does not require an accurate knowledge of the system equilibrium points, it is robust with respect to uncertainty in the equilibrium points. That is, the same feedback controller can be applied to all the systems having the same Taylor series expansion up to the cubic terms. Moreover, the control function is also robust with respect to other system uncertainty. Consider the system

$$\dot{x} = f_\mu(x,u) + \Delta f_\mu(x,u), \tag{4.77}$$

where $\Delta f_\mu(x,u) \in S$ denotes the uncertain part of the system dynamics, and S is a bounded set within which the system satisfies the hypothesis (H). The problem considered in this section is to specify certain sufficient conditions for S such that, a fixed robust controller for stabilizing all the Hopf bifurcation points and the bifurcated periodic solutions emerging from the Hopf bifurcation points of the system (4.77) with $\Delta f_\mu \in S$ exists.

4.5.1. Controllable critical mode

In this case, both quadratic and cubic control affect the stability coefficient. For simplicity, we consider using the cubic feedback $c_1 y_1^3$ alone. Since the first element of r has been normalized to 1, we have $C_u(r,r,\bar{r}) = c_1$. From Eqs (4.17) and (4.19), the influence on closed-loop stability coefficient is given by

$$\Delta\beta := 2\mathrm{R_e}\Delta$$

$$
\begin{aligned}
&= \frac{3}{2}\mathrm{R_e}[PC_u(r,r,\bar{r})l\gamma] \\
&= \frac{3}{2}c_1[\mathrm{R_e}(P)\mathrm{R_e}(l\gamma) - \mathrm{I_m}(P)\mathrm{I_m}(l\gamma)] \\
&= \frac{3}{2}c_1\frac{\omega_c^3}{(d^2+\omega_c^2)^2}[\omega_c\,\mathrm{R_e}(l\gamma) - d\,\mathrm{I_m}(l\gamma)].
\end{aligned} \tag{4.78}
$$

Lemma 4.7. Suppose the critical mode of system (4.77) is linearly controllable. Then, if either $\mathrm{R_e}(l\gamma)$ does not vanish and does not change sign or $\mathrm{I_m}(l\gamma)$ does not vanish and does not change sign for all $\Delta f_{\mu_c} \in S$, then the robust stabilizing controller given by the form $c_1 y_1^3$ exists. Here, l is the normalized left eigenvector with respect to the critical eigenvalue and γ is the control vector of the system expanded at Hopf bifurcation point.

Proof: Denote S' the set of all the values of $l\gamma$ for $\Delta f_{\mu_c} \in S$. Suppose in S', all the values of $\mathrm{R_e}(l\gamma)$ are not zero and remain the same sign. Set the washout filter time constant d to be

$$
0 < d < \min_{\forall\, l\gamma \in S'}\{\frac{\omega_c|\mathrm{R_e}(l\gamma)|}{|\mathrm{I_m}(l\gamma)|}\} \tag{4.79}
$$

for all $|\mathrm{I_m}(l\gamma)| \neq 0$. Suppose in S', all the values of $\mathrm{I_m}(l\gamma)$ are not zero and remain the same sign. Set

$$
d > \max_{\forall\, l\gamma \in S'}\{\frac{\omega_c\,|\mathrm{R_e}(l\gamma)|}{|\mathrm{I_m}(l\gamma)|}\}. \tag{4.80}
$$

For either case, the term $\omega_c\,\mathrm{R_e}(l\gamma) - d\,\mathrm{I_m}(l\gamma)$ in Eq. (4.79) will remain the same sign for all $\Delta f_\mu \in S$. Choosing c_1 with sufficient magnitude and proper sign, the control $c_1 y_1^3$ stabilizes the system.

■

4.5.2. Uncontrollable critical mode

Since the first element of r is always 1, by choosing

$$
Q_u(y,y) = v_1 y_1^2, \tag{4.81}
$$

we have $\mathrm{I_m}(Q_u r) = 0$ and $\mathrm{R_e}(Q_u r) = v_1$. The influence of the control on the stability coefficient is given by

$$\Delta\beta = v_1 \frac{\omega_c^2}{(d^2+\omega_c^2)^2} C_H, \tag{4.82}$$

where

$$C_H = K_3 d^2 - K_2 d\omega_c + K_1 \omega_c^2, \tag{4.83}$$

and K_1, K_2, K_3 are defined as in Eqs. (4.68)-(4.70).

Lemma 4.8 Suppose the critical mode of system (4.77) is linearly uncontrollable. Then, if for all $\Delta f_\mu \in S$, either K_1 or K_3 does not vanish and does not change sign, then the robust controller of the form $v_1 y_1^2$ exists.

Proof: Case 1. Suppose K_3 does not vanish and remains the same sign, C_H can be rewritten as

$$C_H = K_3\left(d - \frac{K_2 + \sqrt{K_2^2 - 4K_3K_1}}{2K_3}\omega_c\right)\left(d - \frac{K_2 - \sqrt{K_2^2 - 4K_3K_1}}{2K_3}\omega_c\right). \tag{4.84}$$

If $K_2^2 - 4K_3K_3 < 0$, Eq. (4.84) contains a product of a pair of complex conjugate which is always positive. Thus, for any real value of d, C_H and K_3 have the same sign. If $K_2^2 - 4K_3K_1 \geq 0$, by choosing

$$d > \max\left\{\frac{K_2 + \sqrt{K_2^2 - 4K_3K_1}}{2K_3}\omega_c\right\}, \tag{4.85}$$

C_H will have the same sign as K_3.

Case 2. Suppose K_1 does not vanish and remains the same sign, C_H can be rewritten as

$$C_H = K_1\left(\omega_c - \frac{K_2 + \sqrt{K_2^2 - 4K_3K_1}}{2K_1}d\right)\left(\omega_c - \frac{K_2 - \sqrt{K_2^2 - 4K_3K_1}}{2K_1}d\right). \tag{4.86}$$

If $K_2^2 - 4K_1K_3 < 0$, similar to the case of (4.84), C_H and K_1 always have the same sign. If

$$\max\left\{\frac{K_2 + \sqrt{K_2^2 - 4K_1K_3}}{2K_1}\right\} < 0, \tag{4.87}$$

since ω_c is positive, for any positive d, C_H and K_1 also have the same sign. If

$$\max\{\frac{K_2 + \sqrt{K_2^2 - 4K_1K_3}}{2K_1}\} > 0, \tag{4.88}$$

by choosing

$$d < \frac{\min\{\omega_c\}}{\max\{\frac{K_2 + \sqrt{K_2^2 - 4K_1K_3}}{2K_1}\}}, \tag{4.89}$$

the sign of C_H and K_1 will remain the same for all $\Delta f_\mu \in S$.

Either case 1 or case 2 satisfied, we can properly choose the eigenvalue for washout filters to have C_H in (4.82) remain the same sign. Then, by setting v_1 to have sufficient magnitude and proper sign, the robust controller $v_1 y_1^2$ will stabilize the system.

■

CHAPTER FIVE

STABILIZATION OF CRITICAL SYSTEM WITH ONE OR TWO ZERO EIGENVALUES THROUGH WASHOUT FEEDBACK

In this chapter, feedback stabilization through washout filters is studied for critical nonlinear systems for cases in which the system linearization possesses one or two controllable zero eigenvalues. Upon embedding the critical system in a one-parameter family of systems, if the corresponding bifurcations are stationary *transcritical* bifurcations, as discussed in Chapter 3, we find that they cannot be stabilized by state feedback through washout filters. If the corresponding bifurcations are stationary *pitchfork* bifurcations, linear feedback stabilization algorithms through washout filters are developed. For systems with one zero eigenvalue in their Jacobian matrices, the control transforms the corresponding subcritical bifurcation to a supercritical bifurcation by changing the direction of *exchange of stabilities*. This is different from the previous works [14] and [17], where the control changes the direction of bifurcating equilibrium branches. The control function does not depend on the accurate knowledge of the equilibrium point and all the equilibrium points existing before feedback are preserved. The control does depend on the left eigenvector, l, corresponding to the zero eigenvalue. However, the control is robust to uncertainty in the system model. The amount of uncertainty which can be tolerated depends on how the control affects the stable eigenvalues, if l is uncertain. If there is no uncertainty in l, as for classes of systems such as (5.62) (presented later in this chapter), the amount of uncertainty that the control can tolerate can be easily determined.

For systems with two controllable zero eigenvalues in their Jacobian matrices, the control moves one of the zero eigenvalues and stabilizes the other simultaneously. The control is also robust. If the system dynamic model has two diagonal blocks, as in Eqs (5.65)-(5.66) below, it is also easy to determine the degree of uncertainty that the control can tolerate. Otherwise, the impact of the uncertainty in the coordinate transformation on the stable eigenvalues has to be considered.

The control algorithms developed below are based on the stability analysis of the stationary pitchfork bifurcation. Therefore, they can be used to simultaneously control the stabilities of the bifurcated branches of systems exhibiting pitchfork bifurcation without affecting the location of any equilibrium branch.

Similar methods can be applied to critical nonlinear systems whose linearizations possess more than two controllable zero eigenvalues by repeatedly using linear feedback through washout filters to move the zero eigenvalues till one or two zero eigenvalues remain. Then, the algorithms in this chapter can be applied.

5.1. Background and motivation

Feedback stabilization for critical nonlinear systems for cases in which the system linearization possesses a simple zero eigenvalue has been addressed in [14] and [17]. [14] employed the stability property of stationary bifurcating system to stabilize the system by transforming the corresponding bifurcation from a transcritical or subcritical stationary bifurcation to a supercritical bifurcation. The Projection Method, involving only power series expansion of the vector field and eigenvector calculations, was found to be easy to apply to systems with dynamics expressed in general state variable form. The method was also shown to be useful in designing control laws for both the critical nonlinear system and the stationary bifurcated system.

On the other hand, [17] employed center manifold reduction to derive stability conditions for critical systems with a single or a double zero eigenvalue. The stabilization of critical systems with two zero eigenvalues via center manifold reduction was investigated first by Behtash and Sastry [16] for a class of third order systems. Liaw [17] extended this result to more general finite dimensional critical systems. This method allows one to assess stability based on a lower order reduced system. However, the reduction to a center manifold may involve complicated transformations, which makes the determination of robustness difficult.

In this chapter, washout filters are employed in determining a robustified stabilization law for critical systems whose Jacobian matrices possess either one or two zero eigenvalues. This work is motivated by certain results of Chapter 3. In particular, it was found that feedback through washout filters cannot move all the zero eigenvalues, and that purely nonlinear feedback through washout filters cannot stabilize the critical equilibrium point with one zero eigenvalue. However, since washout filter-aided feedback has several advantages, as elaborated in Chapter 3, it is certainly interesting to investigate the possibility of using linear feedback through washout filters to stabilize those systems.

The stability formulae for stationary bifurcating systems derived by [14] are employed in deriving the stabilization algorithms. The systems considered here are restricted to those which undergo pitchfork bifurcations (in the parametrized models), and the critical eigenvalues are assumed controllable. The method is naturally adequate for pitchfork bifurcation control. Systems possessing a transcritical bifurcation are not considered since they can never be stabilized by feedback through washout filters. In fact, even using direct state feedback as in [14] and [17], it is very difficult to stabilize the transcritical case, since an exact nulling of the stability coefficient β_1 (see [14]) is needed.

5.2. Bifurcation formulae

In this section, the stationary bifurcation formulae for stability analysis and computation derived by [38] and [14] are briefly reviewed.

5.2.1. Stability analysis

Consider a one-parameter family of nonlinear autonomous systems

$$\dot{x} = f_{\mu}(x) \tag{5.1}$$

with $f_{\mu}(x_{e,\mu}) = 0$, where $x \in I\!R^n$, μ is a real-valued parameter, f_{μ} is sufficiently smooth in x and μ, and $x_{e,\mu}$ is the nominal equilibrium point of the system as a function of the parameter μ. Suppose the following hypothesis holds:

Hypothesis (S): The Jacobian matrix of system (5.1) at the equilibrium $x_{e,\mu}$ possesses a simple eigenvalue $\lambda_1(\mu)$ with $\lambda_1(0) = 0$,

$$\lambda_1'(0) = \frac{\partial \lambda_1(\mu)}{\partial \mu}\bigg|_{\mu\,=\,0} \neq 0$$

and the remaining eigenvalues $\lambda_2(0), \cdots, \lambda_n(0)$ lie in the open left-half complex plane for μ within a neighborhood of $\mu_c = 0$.

Then near the point $(x_{e,0}, 0)$ of the $(n+1)$-dimensional (x, μ)-space, there exists a parameter ϵ and a locally unique curve of critical points $(x(\epsilon), \mu(\epsilon))$, distinct from $x_{e,\mu}$ and passing through $(x_{e,0}, 0)$. This phenomenon is called stationary bifurcation.

The parameter ϵ may be chosen so that $x(\epsilon), \mu(\epsilon)$ are smooth. The series expansion of $x(\epsilon), \mu(\epsilon)$ can be written as

$$\mu(\epsilon) = \mu_1 \epsilon + \mu_2 \epsilon^2 + \cdots \tag{5.2}$$

$$x(\epsilon) = x_{e,0} + x_1 \epsilon + x_2 \epsilon^2 + \cdots \tag{5.3}$$

If $\mu_1 \neq 0$, the system undergoes a transcritical bifurcation from $x_{e,\mu}$ at $\mu = 0$. That is, there is a second equilibrium point besides $x_{e,\mu}$ for both positive and negative values of μ, $|\mu|$ small. If $\mu_1 = 0$ and $\mu_2 \neq 0$, the system undergoes a pitchfork bifurcation for $|\mu|$ sufficiently small. That is, there are two new equilibrium points for *either* positive *or* negative values of μ, $|\mu|$ small. The new equilibrium points have an eigenvalue β which vanishes at $\mu = 0$. The series expansion of β in ϵ is given by

$$\beta(\epsilon) = \beta_1\epsilon + \beta_2\epsilon^2 + \cdots \tag{5.4}$$

with

$$\beta_1 = -\mu_1\lambda'(0), \tag{5.5}$$

and, in case $\beta_1 = 0$, β_2 is given by

$$\beta_2 = 2\mu_2\lambda'(0). \tag{5.6}$$

Thus, the system exhibits an *exchange of stabilities* at the bifurcation point $x_{e,0}$ (at $\mu = 0$). Moreover, the stability of the bifurcation point $x_{e,0}$ itself is addressed in the following theorem.

Theorem 5.1. Consider a system (5.1) satisfying hypothesis (S). If $\mu_1 \neq 0$, then the bifurcation point $x_{e,0}$ is unstable. If $\mu_1 = 0$ and $\mu_2 \neq 0$, then $x_{e,0}$ is asymptotically stable if $\beta_2 < 0$, but is unstable if $\beta_2 > 0$.

■

5.2.2. Computing β_1 and β_2

The stability coefficients β_1 and β_2 can be determined solely by eigenvector computations and the coefficients of the series expansion of the vector field. This is explicated next.

By assumption, the Jacobian matrix $D_x f_0(x_{e,0})$ of system (5.1) at the critical equilibrium $x_{e,0}$ with $\mu = 0$ possesses a simple zero eigenvalue $\lambda_1(0)$. Let r be a right eigenvector and l a left eigenvector of $D_x f_0(x_{e,0})$ corresponding to $\lambda_1(0)$. Set the first element of r equal to unity, and normalize l so that $l \cdot r = 1$.

From the smoothness assumption on the vector field f, system (5.1) can be rewritten, through series expansion, in the form

$$\begin{aligned}\dot{\hat{x}} =& A_\mu \hat{x} + Q_\mu(\hat{x},\hat{x}) + C_\mu(\hat{x},\hat{x},\hat{x}) + \cdots \\ =& A_0\hat{x} + \mu A_1 \hat{x} + \mu^2 A_2 \hat{x} + \cdots \\ &+ Q_0(\hat{x},\hat{x}) + \mu Q_1(\hat{x},\hat{x}) + \cdots \\ &+ C_0(\hat{x},\hat{x},\hat{x}) + \cdots \end{aligned} \tag{5.7}$$

Here, $\hat{x} := x - x_{e,0}$, A_μ, A_0, A_1, A_2 are $n \times n$ matrices, $Q_\mu(x,x)$, $Q_0(x,x)$, $Q_1(x,x)$ are vector-valued quadratic forms generated by symmetric bilinear forms, and $C_\mu(x,x,x), C_0(x,x,x)$ are vector-valued cubic forms generated by symmetric trilinear forms. Then, it is easy to check [45], [38] that

$$\lambda'(0) = lA_1 r \tag{5.8}$$

Denote by $x'_{e,\mu} = x_{e,\mu} + \chi$ an arbitrary solution to $f_\mu(x,0) = 0$ other than the nominal solution $x_{e,\mu}$, and introduce a parameter

$$\epsilon := l \cdot \chi. \tag{5.9}$$

Expand χ and μ as power series in ϵ:

$$\begin{pmatrix} \chi(\epsilon) \\ \mu(\epsilon) \end{pmatrix} = \sum_{k=1}^{\infty} \epsilon^k \begin{pmatrix} \chi_k \\ \mu_k \end{pmatrix}. \tag{5.10}$$

Next, substitute Eq. (5.10) into Eq. (5.7), and set the right side of Eq. (5.7) to zero. Equating coefficients of ϵ, ϵ^2, and ϵ^3 in the resulting equation yields the following relationships:

$$0 = A_0\chi_1, \tag{5.11}$$

$$0 = A_0\chi_2 + \mu_1 A_1 \chi_1 + Q_0(\chi_1,\chi_1), \tag{5.12}$$

$$\begin{aligned} 0 =& A_0\chi_3 + \mu_1 A_1\chi_2 + \mu_2 A_1 \chi_1 + \mu_1^2 A_2 \chi_1 \\ &+ 2Q_0(\chi_1,\chi_2) + \mu_1 Q_1(\chi_1,\chi_1) \\ &+ C_0(\chi_1,\chi_1,\chi_1). \end{aligned} \tag{5.13}$$

From Eqs. (5.9) and (5.10), we have

$$\begin{aligned} \epsilon &= l \cdot \chi(\epsilon) \\ &= \epsilon l \cdot \chi_1 + \epsilon^2 l \cdot \chi_2 + \cdots . \end{aligned} \tag{5.14}$$

Hence,

$$l \cdot \chi_1 = 1 \quad \text{and} \quad l \cdot \chi_k = 0 \quad \text{for} \quad k \geq 2. \tag{5.15}$$

Based on Eqs. (5.11) and (5.14), we have

$$\chi_1 = r. \tag{5.16}$$

Premultiplying both sides of Eq. (5.12) by l, we obtain

$$\mu_1 = -\frac{1}{\lambda_1'(0)} l Q_0(r, r). \tag{5.18}$$

With conditions of Fredholm Alternative (see Theorem 2.6) now satisfied, we can obtain χ_2 from Eq. (5.12). We have

$$\chi_2 = (R^T R)^{-1} R^T \begin{pmatrix} -\mu_1 A_1 r - Q_0(r, r) \\ 0 \end{pmatrix} \tag{5.18}$$

where

$$R := \begin{pmatrix} A_0 \\ l \end{pmatrix}. \tag{5.19}$$

Substituting χ_1, χ_2, and μ_1 into Eq. (5.13), μ_2 is obtained

$$\mu_2 = \frac{1}{\lambda_1'(0)} \{ \mu_1 l A_1 \chi_2 + \mu_1^2 l A_2 r + 2 l Q_0(r, \chi_2) + \mu_1 l Q_1(r, r) + l C_0(r, r, r) \}. \tag{5.20}$$

Substituting μ_1 from (5.18) into (5.5), we find that β_1 is given by

$$\beta_1 = l Q_0(r, r). \tag{5.21}$$

Similarly, using (5.20) in (5.6) (which applies only in the case $\mu_1 = \beta_1 = 0$), we find that, if $\beta_1 = 0$, then

$$\beta_2 = 2l(2Q_0(r, \chi_2) + C_0(r, r, r)). \tag{5.22}$$

Based on (5.2)-(5.4), (5.21), (5.22) and Theorem 5.1, the stability of the bifurcation point $x_{e,0}$, and that of the bifurcated equilibrium points near $x_{e,0}$, can be determined.

Since β_1, β_2 depend solely on the eigenvectors and the quadratic and cubic terms of the series expansion of the vector field $f_\mu(x_{e,\mu}, 0)$ at $\mu = 0$, these coefficients can also be used to determine the stability of any given critical system at an equilibrium with a simple zero eigenvalue. Thus, the (physical) presence of a stationary bifurcation is not necessary.

5.3. Stabilization in the case of a simple zero eigenvalue

In this section, we present a method for the stabilization of critical systems whose linearization at an equilibrium possesses a simple controllable zero eigenvalue. As usual, the remaining eigenvalues are assumed to lie in the open left-half complex plane. The control employs feedback through washout filters to achieve robustness and equilibrium preservation. The development below employs the stability formulae for stationary bifurcation given in Section 5.2. The resulting control laws are also applicable to stabilization of the bifurcation branches arising from a pitchfork bifurcation, while ensuring the exact preservation of all equilibrium points.

As discussed in Chapter 3, pure nonlinear feedback through washout filters does not change the stability of an equilibrium possessing a simple zero eigenvalue. Moreover, linear washout filter-aided feedback cannot remove the critical zero eigenvalue. Therefore, in the following we employ linear washout filter-aided feedback, achieving stabilization while retaining the critical eigenvalue.

The control law design is performed for one-parameter families of systems undergoing a stationary bifurcation. If given a critical system with no bifurcation involved, we modify the system by introducing an extra μ in such a way that the critical eigenvalue crosses the origin with nonzero speed at $\mu = 0$ and with the remaining eigenvalues restricted to the open left-half complex for plane μ near zero. Then the modified system undergoes a stationary bifurcation, whose nature can be studied using the stability formulae of Section 5.2. If the modified system undergoes a transcritical bifurcation, it cannot be stabilized by any linear feedback through washout filters since a transcritical bifurcation is preserved by any washout filter-aided feedback law. However, if the modified system possesses a subcritical bifurcation and the critical eigenvalue is controllable, then a washout filter-aided stabilizing controller can be designed. This is pursued next.

5.3.1. Pitchfork bifurcation control

Consider a system

$$\dot{x} = f_\mu(x, u) \tag{5.23}$$

with the series expansion at an equilibrium point $x_{e,\mu}$

$$\begin{aligned}\dot{\hat{x}} =& A_0\hat{x} + u\gamma + \mu A_1\hat{x} + u\hat{A}_1 + \mu^2 A_2\hat{x} + \cdots \\ &+ Q_0(\hat{x},\hat{x}) + \mu Q_1(\hat{x},\hat{x}) + u\hat{Q}_1(\hat{x},\hat{x}) + \cdots \\ &+ u^2\hat{A}_2\hat{x} + C_0(\hat{x},\hat{x},\hat{x}) + \cdots \end{aligned} \tag{5.24}$$

where $\hat{x}$, A_i, $Q_i(\hat{x},\hat{x})$, $C_0(\hat{x},\hat{x},\hat{x})$ are as defined for system (5.7), and u is a scalar input. Suppose that with $u = 0$ the system possesses a subcritical pitchfork bifurcation at $\mu = 0$ with stability coefficient $\beta_2 > 0$, and that the zero eigenvalue, $\lambda_1(0)$, is controllable, then from the formulae in Section 5.2.2, we have

$$\begin{aligned}\beta_1 &= lQ_0(r,r) \\ &= 0,\end{aligned} \tag{5.25}$$

and

$$\begin{aligned}\beta_2 &= 2l(2Q_0(r,\chi_2) + C_0(r,r,r)) \\ &> 0,\end{aligned} \tag{5.26}$$

where l, r are the left and the right eigenvectors of the Jacobian matrix A_0 corresponding to the zero eigenvalue. These eigenvectors are normalized as in Section 5.2.2, and χ_2 is defined as in Eq. (5.18).

Introduce into the feedback loop a single washout filter with (scalar) dynamic equation

$$\dot{z}_1 = l \cdot x - dz_1, \tag{5.27}$$

and output equation

$$\begin{aligned} y &= l \cdot x - dz_1 \\ &= l \cdot \hat{x} - d\hat{z}_1, \end{aligned} \tag{5.28}$$

where $\hat{z}_1 := z_1 - z_{1,e}$ and $z_{1,e}$ is the equilibrium state of z_1, given by

$$z_{1,e} = \frac{l \cdot x_{e,0}}{d}. \tag{5.29}$$

Consider a linear control law $u = ky$, and denote $\zeta_1 = \hat{x}$, $\zeta_2 = \hat{z}_1$,

$$\zeta := \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}. \tag{5.30}$$

Then the overall closed-loop system for $\mu = 0$ is

$$\dot{\zeta} = \tilde{A}_0 \zeta + \tilde{Q}_0(\zeta, \zeta) + \tilde{C}_0(\zeta, \zeta, \zeta) + \cdots, \tag{5.31}$$

where

$$\tilde{A}_0 = \begin{pmatrix} A_0 + k\gamma \cdot l & -kd\gamma \\ l & -d \end{pmatrix}, \tag{5.32}$$

$$\tilde{Q}_0(\zeta, \zeta) = \begin{pmatrix} Q_0(\zeta_1, \zeta_1) \\ 0 \end{pmatrix} + \begin{pmatrix} k(l \cdot \zeta_1 - d\zeta_2)\hat{A}_1 x \\ 0 \end{pmatrix}, \tag{5.33}$$

$$\begin{aligned} \tilde{C}_0(\zeta, \zeta, \zeta) = &\begin{pmatrix} C_0(\zeta_1, \zeta_1, \zeta_1) \\ 0 \end{pmatrix} + \begin{pmatrix} [k(l \cdot \zeta_1 - d\zeta_2)]^2 \hat{A}_2 \zeta_1 \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} k(l \cdot \zeta_1 - d\zeta_2)\hat{Q}_1(\zeta_1, \zeta_1) \\ 0 \end{pmatrix}. \end{aligned} \tag{5.34}$$

Symmetric bilinear and trilinear forms $\tilde{Q}_0(\rho_1, \rho_2)$, and $\tilde{C}_0(\rho_1, \rho_2, \rho_3)$ which generate the quadratic and cubic forms $\tilde{Q}_0(\rho, \rho)$ and $\tilde{C}_0(\rho, \rho, \rho)$, respectively, can be chosen as

$$\begin{aligned}\tilde{Q}_0(\rho_1, \rho_2) := & \begin{pmatrix} Q_0(\rho_{1,1}, \rho_{2,1}) \\ 0 \end{pmatrix} \\ & + \frac{1}{2}\{\begin{pmatrix} k(l \cdot \rho_{1,1} - d\rho_{1,2})\hat{A}_1\rho_2 \\ 0 \end{pmatrix} + \begin{pmatrix} k(l \cdot \rho_{2,1} - d\rho_{2,2})\hat{A}_1\rho_1 \\ 0 \end{pmatrix},\end{aligned} \tag{5.35}$$

and

$$\begin{aligned}\tilde{C}_0(\rho_1, \rho_2, \rho_3) := & \begin{pmatrix} C_0(\rho_{1,1}, \rho_{2,1}, \rho_{3,1}) \\ 0 \end{pmatrix} \\ & + \frac{1}{3}\{\begin{pmatrix} [k^2(l \cdot \rho_{1,1} - d\rho_{1,2})(l \cdot \rho_{2,1} - d\rho_{2,2})]\hat{A}_2\rho_{3,1} \\ 0 \end{pmatrix} \\ & + \begin{pmatrix} [k^2(l \cdot \rho_{1,1} - d\rho_{1,2})(l \cdot \rho_{3,1} - d\rho_{3,2})]\hat{A}_2\rho_{2,1} \\ 0 \end{pmatrix} \\ & + \begin{pmatrix} [k^2(l \cdot \rho_{2,1} - d\rho_{2,2})(l \cdot \rho_{3,1} - d\rho_{3,2})]\hat{A}_2\rho_{1,1} \\ 0 \end{pmatrix} \\ & + \begin{pmatrix} k(l \cdot \rho_{1,1} - d\rho_{1,2})\hat{Q}_1(\rho_{2,1}, \rho_{3,1}) \\ 0 \end{pmatrix} \\ & + \begin{pmatrix} k(l \cdot \rho_{2,1} - d\rho_{2,2})\hat{Q}_1(\rho_{1,1}, \rho_{3,1}) \\ 0 \end{pmatrix} \\ & + \begin{pmatrix} k(l \cdot \rho_{3,1} - d\rho_{3,2})\hat{Q}_1(\rho_{1,1}, \rho_{2,1}) \\ 0 \end{pmatrix}\},\end{aligned} \tag{5.36}$$

where ρ_i are $(n+1)$-dimensional column vectors with

$$\rho_i = \begin{pmatrix} \rho_{i,1} \\ \rho_{i,2} \end{pmatrix}, \tag{5.37}$$

$\rho_{i,1} \in I\!R^n$ and $\rho_{i,2} \in I\!R$.

The right and the left eigenvectors, $\tilde{r}$ and $\tilde{l}$, corresponding to the zero eigenvalue of closed-loop Jacobian matrix $\tilde{A}_0$ can be chosen as

$$\tilde{r} = \begin{pmatrix} r \\ d^{-1} \end{pmatrix}, \tag{5.38}$$

$$\tilde{l} = \frac{d}{d - kl \cdot \gamma} \begin{pmatrix} l & -kl \cdot \gamma \end{pmatrix}. \tag{5.39}$$

Then, the first element of $\tilde{r}$ is unity, and $\tilde{l} \cdot \tilde{r} = 1$, as desired. We can now compute the closed-loop stability coefficients β_1^* and β_2^*.

The coefficient β_1^* is zero. To see this, note that

$$\begin{aligned} \tilde{Q}_0(\tilde{r}, \tilde{r}) &= \begin{pmatrix} Q_0(r,r) \\ 0 \end{pmatrix} + \begin{pmatrix} k(l \cdot r - 1)\hat{A}_1 r \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} Q_0(r,r) \\ 0 \end{pmatrix}, \end{aligned} \tag{5.40}$$

and, therefore,

$$\begin{aligned} \beta_1^* &= \tilde{l}\tilde{Q}_0(\tilde{r}, \tilde{r}) \\ &= \frac{d}{d - kl \cdot \gamma} \begin{pmatrix} l & -kl \cdot \gamma \end{pmatrix} \begin{pmatrix} Q_0(r,r) \\ 0 \end{pmatrix} \\ &= \frac{d}{d - kl \cdot \gamma} \beta_1 \\ &= 0. \end{aligned} \tag{5.41}$$

By Eq. (5.22), in order to compute the closed-loop β_2, we first need to compute the closed-loop χ_2. Let $\tilde{\chi}$ be the difference between bifurcated equilibrium point and the nominal equilibrium point $\tilde{x}_{e,\mu}$ of the extended closed-loop system corresponding to χ, the difference of the bifurcated equilibrium point and the nominal equilibrium point $x_{e,\mu}$ of the open-loop system, then

$$\tilde{\chi} = \begin{pmatrix} \chi \\ d^{-1} l \cdot \chi \end{pmatrix}. \tag{5.42}$$

Recall the definitions of ϵ and χ_2 in Eqs. (5.9)-(5.10). The closed-loop parameter $\tilde{\epsilon}$ is

$$\begin{aligned}\tilde{\epsilon} &= \tilde{l} \cdot \tilde{\chi} \\ &= \frac{d}{d - kl \cdot \gamma} \begin{pmatrix} l & -kl \cdot \gamma \end{pmatrix} \cdot \begin{pmatrix} \chi \\ l \cdot \chi d^{-1} \end{pmatrix} \\ &= \epsilon. \end{aligned} \tag{5.43}$$

By Eq. (5.10), the closed-loop equilibrium $\tilde{\chi}$ can be written as a power series in $\tilde{\epsilon}$

$$\begin{aligned}\tilde{\chi} &= \tilde{\epsilon}\tilde{\chi}_1 + \tilde{\epsilon}^2 \tilde{\chi}_2 + \cdots \\ &= \epsilon\tilde{\chi}_1 + \epsilon^2 \tilde{\chi}_2 + \cdots. \end{aligned} \tag{5.44}$$

By Eqs. (5.42)-(5.43), it also can be written as

$$\begin{aligned}\tilde{\chi} &= \begin{pmatrix} \chi \\ d^{-1} l \cdot \chi \end{pmatrix} \\ &= \epsilon \begin{pmatrix} \chi_1 \\ d^{-1} l \cdot \chi_1 \end{pmatrix} + \epsilon^2 \begin{pmatrix} \chi_2 \\ d^{-1} l \cdot \chi_2 \end{pmatrix} + \cdots. \end{aligned} \tag{5.45}$$

Equating the coefficient of ϵ^2 in Eqs. (5.44) and (5.45) we obtain

$$\tilde{\chi}_2 = \begin{pmatrix} \chi_2 \\ d^{-1} l \cdot \chi_2 \end{pmatrix}. \tag{5.46}$$

Substituting $\tilde{\chi}_2$ and $\tilde{r}$ into Eqs. (5.35)-(5.36), we obtain

$$\begin{aligned}\tilde{Q}_0(\tilde{r}, \tilde{\chi}_2) &= \begin{pmatrix} Q_0(r, \chi_2) \\ 0 \end{pmatrix} + \begin{pmatrix} k(l \cdot r - 1)\hat{A}_1 \chi_2 \\ 0 \end{pmatrix} + \begin{pmatrix} k(l \cdot \chi_2 - l \cdot \chi_2)\hat{A}_1 r \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} Q_0(r, \chi_2) \\ 0 \end{pmatrix}, \end{aligned} \tag{5.47}$$

and

$$\begin{aligned}\tilde{C}_0(\tilde{r}, \tilde{r}, \tilde{r}) &= \begin{pmatrix} C_0(r, r, r) \\ 0 \end{pmatrix} + \begin{pmatrix} [k(l \cdot r - l \cdot r)]^2 \hat{A}_2 r \\ 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} k[l \cdot r - l \cdot r]\hat{Q}_1(r, r) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} C_0(r, r, r) \\ 0 \end{pmatrix}. \end{aligned} \tag{5.48}$$

Applying the formula of Eq. (5.22), the closed-loop β_2^* is found to be

$$
\begin{aligned}
\beta_2^* &= 2\tilde{l}\{2\tilde{Q}_0(\tilde{r},\tilde{\chi}_2) + \tilde{C}_0(\tilde{r},\tilde{r},\tilde{r})\} \\
&= 2\frac{d}{d-kl\cdot\gamma}\begin{pmatrix} l & -kl\cdot\gamma \end{pmatrix}\{2\begin{pmatrix} Q_0(r,\chi_2) \\ 0 \end{pmatrix} + \begin{pmatrix} C_0(r,r,r) \\ 0 \end{pmatrix}\} \\
&= \frac{d}{d-kl\cdot\gamma}\beta_2,
\end{aligned}
\tag{5.49}
$$

where β_2 is the corresponding stability coefficient of the original system.

By Theorem 5.1, for stability we need $\beta_2^* < 0$ and that all noncritical eigenvalues including the one introduced by washout filter, be in the open left-half complex plane. Since $\beta_2 > 0$, by Eq. (5.49) we must require

$$
\frac{d}{d-kl\cdot\gamma} < 0 \tag{5.50}
$$

to ensure $\beta_2^* < 0$. As for the closed-loop eigenvalues, consider the characteristic equation of the closed-loop Jacobian matrix $\tilde{A}_0$

$$
\begin{aligned}
\det(\lambda I - \tilde{A}_0) &= \det\begin{pmatrix} \lambda - (A_0 + k\gamma\cdot l) & kd\gamma \\ -l & \lambda + d \end{pmatrix} \\
&= \det\begin{pmatrix} \lambda - A_0 & -k\gamma\lambda \\ -l & \lambda + d \end{pmatrix} \\
&= \det(\lambda - A_0)\cdot(-l(\lambda I - A_0)^{-1}k\gamma\lambda + \lambda + d)
\end{aligned}
\tag{5.51}
$$

Since l is a left eigenvector of A_0 corresponding to the eigenvalue 0, it satisfies $lA_0 = 0$. Rewriting

$$
\begin{aligned}
\lambda l(\lambda I - A_0)^{-1} &= (\lambda l + lA_0)(\lambda I - A_0)^{-1} \\
&= l(\lambda I + A_0)(\lambda I - A_0)^{-1} \\
&= l,
\end{aligned}
\tag{5.52}
$$

Eq. (5.51) becomes

$$
\det(\lambda I - \tilde{A}_0) = \det(\lambda - A_0)(\lambda - kl\cdot\gamma + d). \tag{5.53}
$$

By assumption, one eigenvalue of A_0 is zero and the others are in the open left-half complex plane. To ensure all the noncritical eigenvalues of $\tilde{A}_0$ in the open left-half complex plane, we require

$$d > kl \cdot \gamma. \tag{5.54}$$

Conditions (5.50), (5.54) imply that to stabilize the system, the values of d and k should be chosen such that

$$0 > d > kl \cdot \gamma. \tag{5.55}$$

Note that the condition $d < 0$ implies that an *unstable* washout filter is needed.

Theorem 5.2. Suppose the system (5.23) possesses a pitchfork bifurcation or a critical system defined as Eq. (5.23) with μ fixed to 0 possesses a simple zero eigenvalue. If the zero eigenvalue is controllable and the stability coefficients $\beta_1 = 0$ and $\beta_2 > 0$, then stabilization of the critical point of critical system, and of the bifurcation point and bifurcated branches near bifurcation point for a pitchfork bifurcating system, by using a linear feedback through an unstable washout filter, exists.

■

5.3.2. Geometric meaning of pitchfork bifurcation control

A pitchfork bifurcation is a codimension one bifurcation with normal form

$$\dot{x} = \mu x - ax^3 \tag{5.56}$$

where a is either $+1$ or -1.

If $a = +1$, we have a supercritical bifurcation with bifurcation diagram as shown in Figure 5.1. In this case, the bifurcation point and the bifurcated solutions are stable since for $\mu = 0$,

$$\dot{x} = -x^3, \tag{5.57}$$

and for $\mu > 0$, the bifurcated solutions $x = \pm\sqrt{\mu}$ have eigenvalue $\lambda = -2\mu < 0$. If $a = -1$, we have a subcritical bifurcation with bifurcation diagram as shown in Figure 5.2. In this case, the bifurcation point and the bifurcated solutions are unstable since for $\mu = 0$,

$$\dot{x} = x^3, \tag{5.58}$$

and for $\mu < 0$, the bifurcated solutions $x = \pm\sqrt{-\mu}$ have eigenvalue $\lambda = -2\mu > 0$.

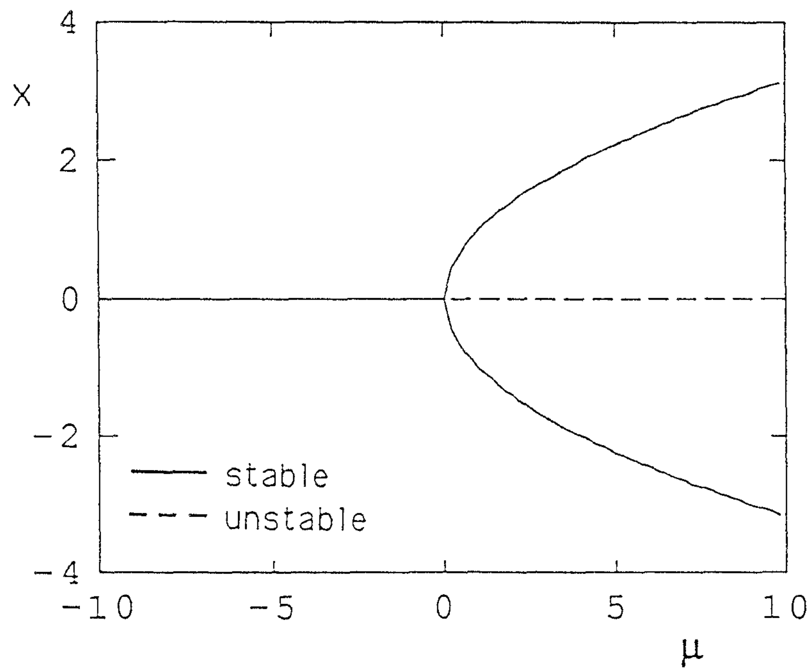


Figure 5.1. Supercritical pitchfork bifurcation

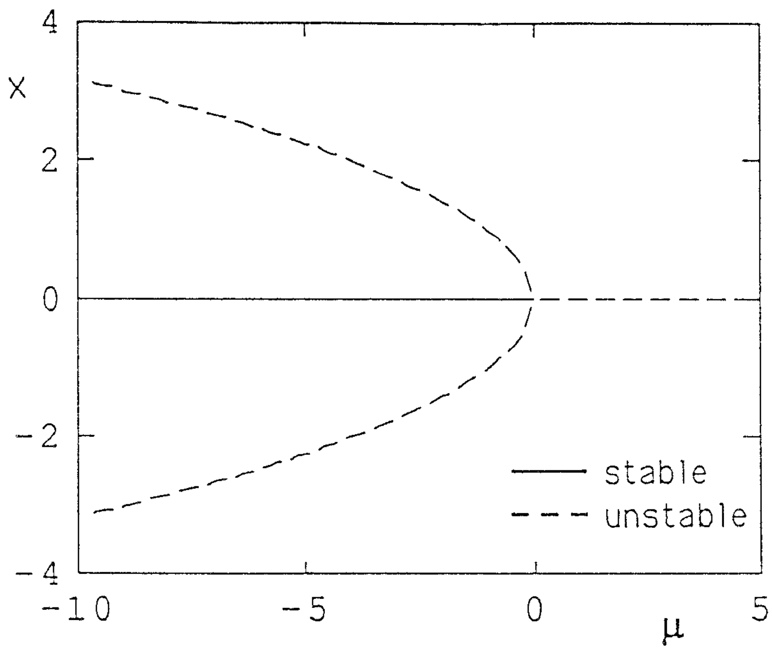


Figure 5.2. Subcritical pitchfork bifurcation

Washout filters preserve all equilibrium points. Hence, if the system possesses a pitchfork bifurcation as in Figure 5.2, then with a washout filter-aided feedback the bifurcation point cannot be stabilized through moving the bifurcated branches to the other side of the bifurcation point (as Figure 5.1) as was done in [14]. In contrast, the stabilization strategy discussed in Section 5.3.1 is to achieve a supercritical bifurcation by changing the linear stability of the nominal branches, as shown in Figure 5.3. That means we need to destabilize the nominal branch which is stable originally. This destabilization is why an unstable washout filter is needed in the feedback loop.

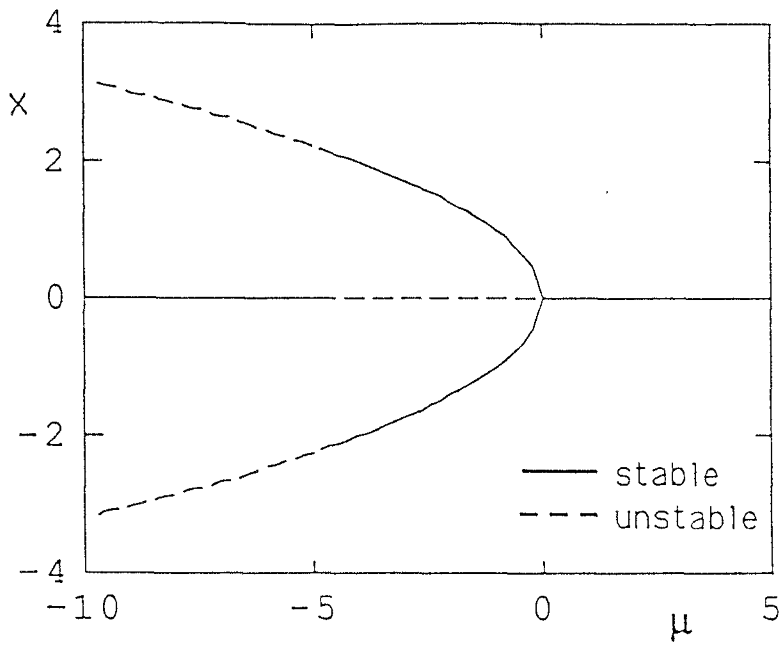


Figure 5.3. Pitchfork bifurcation control through washout filters.

The main property of using this method is that it preserves and stabilizes the bifurcated branches simultaneously without the accurate knowledge of the equilibrium branches. On the contrary, in [14], the bifurcated branches had been moved to the other side of the bifurcation point. Another way of preserving and stabilizing one of the bifurcated branches is to use a linear feedback as a function of $(x - \tilde{x}_{e,\mu})$, where $\tilde{x}_{e,\mu}$ is function of μ denoting the equilibrium point on that bifurcated branch. However, in this case, other branches may not be preserved. Figure 5.4 shows a simple example of the effect of using linear direct state feedback, namely

$$u = k(x - \sqrt{-\mu}), \tag{5.59}$$

to stabilize the system

$$\dot{x} = \mu x + x^3 + u. \tag{5.60}$$

Note that the other two equilibrium branches are severely distored.

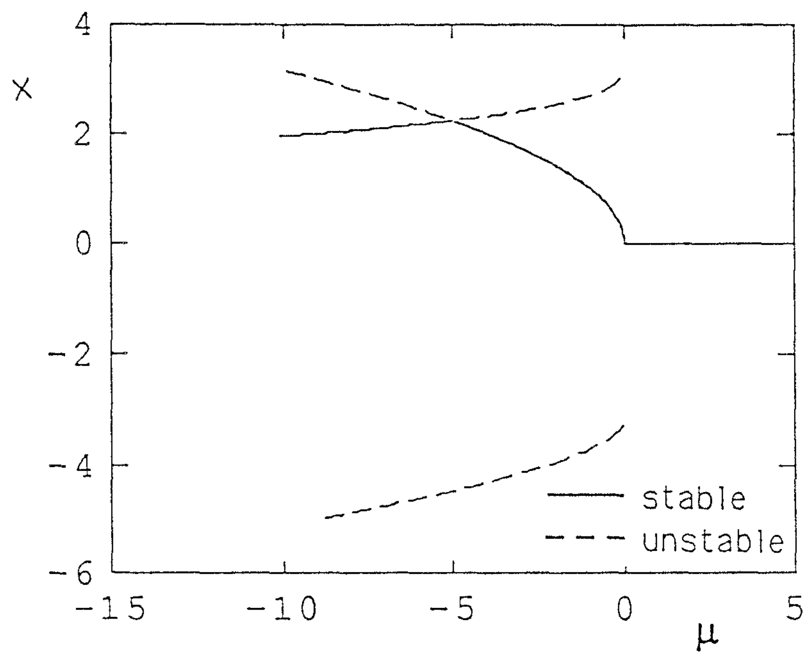


Figure 5.4. Stabilization of bifurcated branch using direct state feedback.

5.3.3. Robustness

The control function $u = ky$ is independent of the equilibrium point, therefore it is robust with respect to uncertainty in the equilibrium points. Moreover, suppose we choose k to make $kl \cdot \gamma$ sufficiently negative so as to make the additional eigenvalue introduced by washout filter be restricted to open left-half complex plane after the control applied. Then, no matter what positive magnitude β_2 may be, the closed-loop β_2^* is negative as long as an unstable washout filter (i.e. $d < 0$) is used. Therefore, the controller is also robust with respect to uncertainty in open-loop β_2 .

The main limitation on the robustness is that the control function depends on the left eigenvalue l (through y). If there is uncertainty in l, condition (5.55) may not ensure the stability of all the noncritical eigenvalues. However, if there is no uncertainty in l, and $l \cdot \gamma$ does not change sign throughout the range of uncertainty, we can choose an arbitrary negative d and a sufficiently large $|k|$ such that

$$\max_{\forall\, l\cdot\gamma\in S'}\{kl\cdot\gamma\} < d \tag{5.61}$$

to stabilize the system. Here S' denotes the set of all possible $l\cdot\gamma$ throughout the range of uncertainty. In another words, with negative value of d and sufficient magnitude of k, we can stabilize an uncertain system for the amount of uncertainty such that within this amount of uncertainty, condition (5.61) holds.

There are classes of systems that l does not vary. For instance, for systems with the Jacobian matrix intrinsically diagonalized as

$$\dot{x} = \begin{pmatrix} 0 & 0 \\ 0 & A_s \end{pmatrix} x + u\gamma + g(x), \tag{5.62}$$

where $g(0) = 0$ and

$$\left.\frac{\partial g(x)}{\partial x}\right|_{x\,=\,0} = 0, \tag{5.63}$$

the left eigenvector l is always $[1, 0, \cdots 0]$.

The main reason of using the left eigenvector l in the control function is, for simplicity, to preserve the stable eigenvalues of the original system (see Eq. (5.51)) during controlling the eigenvalue introduced by washout filter. In fact, for stabilization, we only require that all the noncritical eigenvalues stay in open left-half complex plane. That is, even there is uncertainty in l, the system remains stable as long as noncritical eigenvalues remain in open left-half complex plane. The amount of uncertainty that a fixed controller can tolerate is thus determined by the magnitude of uncertainty that makes any of the noncritical eigenvalues hit the imaginary axis.

From the discussion in Section 5.3.2, the unstable washout filter is used to distablize the nominal equilibrium branch which is originally stable. That is to change the direction of the *exchange of stabilities*. As long as the system does exhibit a pitchfork bifurcation, this change of direction of exchange of stabilities will automatically change the stability of the bifurcation point. Therefore, $\beta_2 > 0$ may not necessary be required.

5.4. Stabilization in the case of double zero eigenvalues

In this section, we consider the stabilization problem for critical systems whose linearization at an equilibrium possesses two controllable zero eigenvalues. The remaining eigenvalues are assumed to lie in the open left-half complex plane. Form Lemma 3.4, we know that feedback through washout filters cannot move both zero eigenvalues. Therefore, in here, with suitable choice of washout filter, we move away one of the zero eigenvalues and stabilize the other by using the result in Section 5.3.

Since both zero eigenvalues are controllable, the systems can be transformed to have two diagonal blocks in their Jacobin matrix with the upper block containing these two zero eigenvalues. Without loss of generality, the system are assumed already in the series expansion form

$$\begin{aligned}\dot{\hat{x}}_1 =& A_0\hat{x}_1 + u\gamma + Q_0(\hat{x},\hat{x}) + C_0(\hat{x},\hat{x},\hat{x}) + uA_1\hat{x} + u^2A_2\hat{x} \\ &+ uQ_1(\hat{x},\hat{x}) + \cdots + O(||\hat{x},u||^4)\end{aligned} \tag{5.64}$$

where $\hat{x} := x - x_e \in I\!R^n$, x_e is the critical equilibrium point,

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \cdot \\ \cdot & \cdot & A_{0s} \end{pmatrix}, \tag{5.65}$$

A_{os} is a $(n-2) \times (n-2)$ matrix with all the eigenvalues in the open left-half complex plane,

$$\gamma = \begin{pmatrix} 0 \\ \gamma_1 \\ \gamma_s \end{pmatrix} \tag{5.66}$$

with γ_1 a nonzero scalar, $\gamma_s \in I\!R^{n-2}$, $A_1, A_2, Q_0(\cdot,\cdot), Q_1(\cdot,\cdot), C_0(\cdot,\cdot,\cdot)$ are as defined in System (5.7), and $O(||\hat{x}, u||^4)$ are those terms with order higher than three.

5.4.1. Control setup

Introduce into the feedback loop a single washout filter with dynamic equations

$$\dot{z}_1 = x_1 + cx_2 - dz_1 \tag{5.67}$$

$$y_1 = x_1 + cx_2 - dz_1 \tag{5.68}.$$

Set $u = ky_1$, the closed-loop system will be in the form of Eq. (5.31) with

$$\tilde{A}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ k\gamma_1 & ck\gamma_1 & 0 & -dk\gamma_1 \\ 0 & 0 & A_{0s} & 0 \\ 1 & c & 0 & -d \end{pmatrix}, \tag{5.69}$$

$$\tilde{Q}_0(\zeta,\zeta) = \begin{pmatrix} Q_0(\zeta_1,\zeta_1) \\ 0 \end{pmatrix} + \begin{pmatrix} ky_1 A_1 \zeta_1 \\ 0 \end{pmatrix}, \tag{5.70}$$

$$\tilde{C}_0(\zeta,\zeta,\zeta) = \begin{pmatrix} C_0(\zeta_1,\zeta_1,\zeta_1) \\ 0 \end{pmatrix} + \begin{pmatrix} (ky_1)^2 A_2 \zeta_1 \\ 0 \end{pmatrix} + \begin{pmatrix} ky_1 Q_1(\zeta_1,\zeta_1) \\ 0 \end{pmatrix}, \tag{5.71}$$

(ζ as defined in Eq. (5.30)).

The characteristic equation of $\tilde{A}_0$ is given by

$$\lambda(\lambda^2 + (d - ck\gamma_1)\lambda + k\gamma_1) \cdot \det|\lambda I - A_{0s}| = 0. \tag{5.72}$$

Since all the eigenvalues of A_{0s} are stable, to move one of the zero eigenvalues of the original system to open left-half complex plane, we need to ensure the closed-loop eigenvalues

$$\lambda = \frac{-d + ck\gamma_1 \pm \sqrt{4k\gamma_1 + (d - ck\gamma_1)^2}}{2} \tag{5.73}$$

having negative real part. The conditions for that are to have $k\gamma_1 < 0$ and $ck\gamma_1 < d$.

The left and right eigenvectors corresponding to the critical eigenvalues can be chosen as

$$l = (0, \frac{-d}{k\gamma_1}, 0, \cdots, d) \tag{5.74}$$

and

$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ d^{-1} \end{pmatrix}. \tag{5.75}$$

Thus, the first element of r is unity, and $l \cdot r = 1$.

It is easy to see that the remaining critical eigenvalue is still controllable since

$$\begin{aligned} l \cdot \begin{pmatrix} \gamma \\ 0 \end{pmatrix} &= -\frac{d}{k} \\ &\neq 0. \end{aligned} \tag{5.76}$$

However, in order to use the results in Section 5.3, we require $l\tilde{Q}_0(r, r) = 0$ so that the corresponding bifurcation is pitchfork instead of transcritical. Denote

$Q_0(x,x)$ and $C_0(x,x,x)$

$$Q_0(x,x) = \begin{pmatrix} q_1(x,x) \\ q_2(x,x) \\ \vdots \\ q_n(x,x) \end{pmatrix}, \tag{5.77}$$

$$C_0(x,x,x) = \begin{pmatrix} c_1(x,x,x) \\ c_2(x,x,x) \\ \vdots \\ c_n(x,x,x) \end{pmatrix}, \tag{5.78}$$

where $q_i(x,x)$ and $c_i(x,x,x)$ are scalar quadratic and cubic functions of x, respectively, with

$$q_i(x,x) := \sum_{l=1}^{n}\sum_{j=l}^{n} q_{i,lj}x_l x_j, \tag{5.79}$$

and

$$c_i(x,x,x) := \sum_{l=1}^{n}\sum_{j=l}^{n}\sum_{k=j}^{n} c_{i,ljk}x_l x_j x_k. \tag{5.80}$$

To have $l\tilde{Q}_0(r,r) = 0$, we require $q_{2,11} = 0$. That is, there is no $\hat{x}^2$ in the dynamic equation of $\hat{x}_2$.

Substituting the closed-loop $\tilde{A}_0$, $\tilde{Q}_0(\cdot,\cdot)$, $\tilde{C}_0(\cdot,\cdot,\cdot)$, l, r into Eqs. (5.18)-(5.19), (5.22) we obtain

$$\chi_2^* = \begin{pmatrix} (-c + d(k\gamma_1)^{-1})q_{1,11} \\ q_{1,11} \\ A_{0s}^{-1} q_{s,11} \\ q_{1,11}(k\gamma_1)^{-1} \end{pmatrix}, \tag{5.81}$$

and

$$\beta_2^* = 2(\frac{-d}{k\gamma_1})\{2(q_{2,12}\, q_{1,11} + q_{2,1s}^T\, A_{0s}^{-1}\, q_{2,11}) + c_{2,111}\}, \tag{5.82}$$

where

$$q_{s,11} := \begin{pmatrix} q_{3,11} \\ q_{4,11} \\ \vdots \\ q_{n,11} \end{pmatrix}, \tag{5.83}$$

$$q_{2,1s} := \begin{pmatrix} q_{2,13} \\ q_{2,14} \\ \vdots \\ q_{2,1n} \end{pmatrix}. \tag{5.84}$$

The sign of β_2 can be arbitrary altered by altering the sign of d. Thus, we have the following sufficient stabilization condition:

Theorem 5.3. Suppose system (5.64) which possesses two controllable zero eigenvalues in the its Jacobian matrix is unstable at equilibrium x_e. Then, it can be stabilized by a linear feedback through a washout filter, provided that

$$i)\quad q_{2,11} = 0 \tag{5.85}$$

$$ii)\quad 2(q_{2,12}q_{1,11} + q_{2,1s}^T A_{0s}^{-1} q_{2,11}) + c_{2,111} \neq 0. \tag{5.86}$$

■

The procedure for selecting the controller are:

Step 1. Set the reciprocal of washout filter time constant d to be in different sign with Eq. (5.86). If d is positive (resp. negative), we require a stable (resp. unstable) washout filter.

Step 2. Select a gain k for feedback to have $k\gamma_1 < 0$.

Step 3. Select a gain c such that $ck\gamma_1 < d$.

5.4.2. Robustness

Suppose the systems intrinsically have a series expansion form of (5.64) with Jacobian matrix having a form of Eq. (5.65), and γ_1 and Eq. (5.86) do not change sign throughout the range of uncertainty. Then, follow the procedure steps 1-3, it is easy to choose c, k and d which satisfy $k\gamma_1 < 0$ and $ck\gamma_1 < d$ for all γ_1 throughout the range of uncertainty. Note that, suppose γ_1 does change sign under certain amount of uncertainty. Within this amount of uncertainty, there are some cases that the zero eigenvalues are uncontrollable, i.e. $\gamma_1 = 0$. This contradicts our assumption that the zero eigenvalues are always controllable.

If any additional coordinate transformation is required to transform the system into the form of (5.64), uncertainty in the transformation matrix has to be taken into account in the robustness consideration.

CHAPTER SIX

STABILIZATION OF CRITICAL SYSTEMS POSSESSING TWO PAIRS OF PURE IMAGINARY EIGENVALUES THROUGH WASHOUT FILTERS

In this chapter, we address feedback stabilization through washout filters of a class of fifth order nonlinear systems whose linearizations possess two pairs of pure imaginary eigenvalues. Three cases are considered: both critical modes uncontrollable; both critical modes controllable; and only one of the critical modes is controllable. For the case in which both critical modes are uncontrollable, the stabilizability conditions we obtain are no more restrictive than those given in [16], [18] and [17], which employed direct state feedback. In addition, the present approach gives flexibility in choosing the washout filter time constants, and the control design is more robust to uncertainty than previous design. In fact, since our control law does not depend on knowledge of the equilibrium points, it is robust with respect to uncertainties in the equilibrium points. For the case in which both critical modes are controllable, we show the existence of a a purely nonlinear robust stabilizing controller. For the case in which only one of the critical modes is controllable, a robust controller which avoids the complexity of involving the states of washout filters is proposed.

6.1. Background and motivation

Feedback stabilization of the class of nonlinear critical systems possessing two pairs of pure imaginary eigenvalues has been studied by Behtash and Sastry [16] and Liaw and Abed [17]. Both works employed center manifold reduction and normal form transformation, to facilitate application of a stability criterion for fourth-order systems in normal form. Behtash and Sastry [16] considered a fifth-order system with two uncontrollable critical modes and a (linearly) controllable stable state. The critical states and the stable state are linearly decoupled. The stability criterion and stabilizability conditions are given in terms of the dynamics of the simplified lower order system. Liaw and Abed [17] extended the work to more general finite dimensional systems with both critical modes controllable or uncontrollable. They also gave a procedure for stating the results in terms of the dynamics of the original system. Behtash and Sastry [16] considered robustness with respect to C^k small perturbations of the vector field. Their rebustness result asserts the local asymptotic stability of a small ball $B_{r(\epsilon)}$ centered at the critical equilibrium point. In each of these works, the critical equilibrium point was taken as the origin. Indeed, even the C^k perturbation considered in [16] does not perturb the equilibrium point.

In this chapter, through the use of washout filters, we extend the results above to the stabilization of systems for which there is uncertainty in both the critical equilibrium point as well as in the system dynamics, and we seek to preserve all system equilibria in spite of the applied control. The stabilizability conditions obtained here are similar to those obtained previously in the direct state feedback setting. In addition, the control functions obtained here are independent of the location of the critical equilibrium point, that is, there is no limitation on the uncertainty in the equilibrium point. Also, under the assumption of the existence of critical eigenvalues a bound on the uncertainty for the existence of certain types of robust controller are determined. Finally, as expected when using washout filters, the equilibrium points of the original system are preserved by the applied feedback.

The system considered here is identical to that studied in [16]. Higher order systems may also be considered, as in [17], and more general system forms and higher degree of degenerate Hopf bifurcation systems may be considered by using bifurcation formulas derivated by Farr, Li and Langford [47], although we do not pursue this here. The stability criterion and the transformation algorithm employed here follow the results of [17]. For cases in which either one or both pairs of critical eigenvalues are controllable, it is clear that linear washout filter-aided feedback can be used first to move the critical eigenvalues into the left half plane. If a critical pair remains (in the case where only one of the two pairs is controllable, say), the Hopf bifurcation control method discussed in Chapter 4 can then be used. We restrict the control to be a purely nonlinear function, so that the linear stability of all equilibrium points is preserved. Thus, the critical eigenvalues will not be moved by the control proposed here. The stability coefficients obtained in this chapter have been verified using the software package MATHEMATICA [48].

6.2. Stability criteria for fourth-order critical systems

The stability criterion derived in [17] for fourth-order systems with all eigenvalues lying on the imaginary axis, which will be used in the derivation of the control law in the following sections, is briefly reviewed here. Consider a fourth-order nonlinear system

$$\dot{x} = \begin{pmatrix} 0 & \Omega_1 & 0 & 0 \\ -\Omega_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_3 \\ 0 & 0 & -\Omega_4 & \end{pmatrix} x + f(x), \tag{6.1}$$

with $\Omega_1\Omega_2 > 0,\ \Omega_3\Omega_4 > 0$, and

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \end{pmatrix}, \tag{6.2}$$

where the $f_i,\ i = 1, \ldots, 4$, are smooth, purely nonlinear scalar functions with power series expansions (for $i = 1, \ldots, 4$)

$f_i(x) = f_{i,11}x_1^2 + f_{i,22}x_2^2 + f_{i,33}x_3^2 + f_{i,44}x_4^2$

$$
\begin{aligned}
&+ f_{i,12}x_1x_2 + f_{i,13}x_1x_3 + f_{i,14}x_1x_4 + f_{i,23}x_2x_3 + f_{i,24}x_2x_4 + f_{i,34}x_3x_4 \\
&+ f_{i,123}x_1x_2x_3 + f_{i,124}x_1x_2x_4 + f_{i,134}x_1x_3x_4 + f_{i,234}x_2x_3x_4 \\
&+ (f_{i,111}x_1 + f_{i,112}x_2 + f_{i,113}x_3 + f_{i,114}x_4)x_1^2 \\
&+ (f_{i,122}x_1 + f_{i,222}x_2 + f_{i,223}x_3 + f_{i,224}x_4)x_2^2 \\
&+ (f_{i,133}x_1 + f_{i,233}x_2 + f_{i,333}x_3 + f_{i,334}x_4)x_3^2 \\
&+ (f_{i,144}x_1 + f_{i,244}x_2 + f_{i,344}x_3 + f_{i,444}x_4)x_4^2 + O(||x^4||). \qquad (6.3)
\end{aligned}
$$

In [17], a two-stages near-identity transformation

$$x = \tilde{x} + P(\tilde{x}) \qquad (6.4)$$

is derived which renders (6.1) in the normal form (6.6)-(6.10) below. Here P is a purely nonlinear function, a quadratic function $P_2(\tilde{x}, \tilde{x})$ and a cubic function $P_3(\tilde{x}, \tilde{x}, \tilde{x})$ for the first and the second stages, respectively. The purpose of the first stage transformation is to remove the quadratic terms of system (6.1). Denote the first-stage transformed version of (6.1)

$$\dot{\tilde{x}} = A\tilde{x} + \tilde{f}(x), \qquad (6.5)$$

where A is identical to the linear part of (6.1) and $\tilde{f}(x) = (\tilde{f}_1(\tilde{x}), \tilde{f}_2(\tilde{x}), \tilde{f}_3(\tilde{x}), \tilde{f}_4(\tilde{x}))^T$. The final transformed version of (6.1) (i.e., the normal form) is given by (here $\tilde{x}$ is replaced, for simplicity, by x):

$$
\begin{aligned}
\dot{x}_1 &= \Omega_1\{x_2 + (\delta_1 x_1 + \epsilon_1 x_2)(x_1^2 + x_2^2) + (\delta_2 x_1 + \epsilon_2 x_2)(x_3^2 + x_4^2)\} \\
&\quad + O(||x^4||), \qquad (6.6) \\
\dot{x}_2 &= \Omega_2\{-x_1 + (\delta_1 x_2 - \epsilon_1 x_1)(x_1^2 + x_2^2) + (\delta_2 x_2 - \epsilon_2 x_1)(x_3^2 + x_4^2)\} \\
&\quad + O(||x^4||), \qquad (6.7) \\
\dot{x}_3 &= \Omega_3\{x_4 + (\delta_3 x_3 + \epsilon_3 x_4)(x_1^2 + x_2^2) + (\delta_4 x_3 + \epsilon_4 x_4)(x_3^2 + x_4^2)\} \\
&\quad + O(||x^4||), \qquad (6.8) \\
\dot{x}_4 &= \Omega_4\{-x_3 + (\delta_3 x_4 - \epsilon_3 x_3)(x_1^2 + x_2^2) + (\delta_4 x_4 - \epsilon_4 x_3)(x_3^2 + x_4^2)\} \\
&\quad + O(||x^4||). \qquad (6.9)
\end{aligned}
$$

The coefficients appearing in (6.6)-(6.9) is given by ([17]p-54):

$$\delta_1 = \frac{\Omega_2(3\tilde{f}_{2,222} + \tilde{f}_{1,122}) + \Omega_1(\tilde{f}_{2,112} + 3\tilde{f}_{1,111})}{3\Omega_1^2 + 2\Omega_1\Omega_2 + 3\Omega_2^2} \tag{6.10}$$

$$\epsilon_1 = \frac{\Omega_1\Omega_2(\tilde{f}_{1,112} - \tilde{f}_{2,122}) + 3\Omega_2^2\tilde{f}_{1,222} - 3\Omega_1^2\tilde{g}_{2,111}}{4\Omega_1\Omega_2(\Omega_1 + \Omega_2)} \tag{6.11}$$

$$\delta_2 = \frac{\Omega_3(\tilde{f}_{1,133} + \tilde{f}_{2,233}) + \Omega_4(\tilde{f}_{1,144} + \tilde{f}_{2,244})}{(\Omega_1 + \Omega_2)\cdot(\Omega_3 + \Omega_4)} \tag{6.12}$$

$$\epsilon_2 = \frac{\Omega_2(\Omega_3\tilde{f}_{1,233} + \Omega_4\tilde{f}_{1,244}) - \Omega_1(\Omega_3\tilde{f}_{2,133} + \Omega_4\tilde{f}_{2,144})}{2\Omega_1\Omega_2(\Omega_3 + \Omega_4)} \tag{6.13}$$

$$\delta_3 = \frac{\Omega_1(\tilde{f}_{3,113} + \tilde{f}_{4,114}) + \Omega_2(\tilde{f}_{3,223} + \tilde{f}_{4,224})}{(\Omega_1 + \Omega_2)\cdot(\Omega_3 + \Omega_4)} \tag{6.14}$$

$$\epsilon_3 = \frac{\Omega_4(\Omega_1\tilde{f}_{3,114} + \Omega_2\tilde{f}_{3,224}) - \Omega_3(\Omega_1\tilde{f}_{4,113} + \Omega_2\tilde{f}_{4,223})}{2\Omega_3\Omega_4(\Omega_1 + \Omega_2)} \tag{6.15}$$

$$\delta_4 = \frac{\Omega_4(3\tilde{f}_{4,444} + \tilde{f}_{3,344}) + \Omega_3(\tilde{f}_{4,334} + 3\tilde{f}_{3,333})}{3\Omega_3^2 + 2\Omega_3\Omega_4 + 3\Omega_4^2} \tag{6.16}$$

$$\epsilon_4 = \frac{\Omega_3\Omega_4(\tilde{f}_{3,334} - \tilde{f}_{4,344}) + 3\Omega_4^2\tilde{f}_{3,444} - 3\Omega_3^2\tilde{f}_{4,333}}{4\Omega_3\Omega_4(\Omega_3 + \Omega_4)}, \tag{6.17}$$

and $\tilde{f}_{i,jkl}$, $i, j, k, l = 1, \ldots, 4$, are the cubic terms jkl of function $\tilde{f}_i$. Next, to analyze stability of the origin for system (6.6)-(6.9), [17] employs a positive definite Liapunov function candidate

$$V = \frac{1}{2}p_1(x_1^2 + \frac{\Omega_1}{\Omega_2}x_2^2) + \frac{1}{2}p_2(x_3^2 + \frac{\Omega_3}{\Omega_4 x_4^2}) \tag{6.18}$$

where $p_1, p_2 > 0$ are determined in the course of the analysis. The time derivative of V along the trajectories of the system defined in Eqs. (6.6) - (6.9) is

$$\begin{aligned}\dot{V} = {} & p_1\Omega_1\delta_1(x_1^2 + x_2^2)^2 + (p_1\Omega_1\delta_2 + p_2\Omega_3\delta_3)\cdot(x_1^2 + x_2^2)\cdot(x_3^2 + x_4^2) \\ & + p_2\Omega_3\delta_4(x_3^2 + x_4^2)^2 + O(||(x_1, x_2, x_3, x_4)||^5).\end{aligned} \tag{6.19}$$

By using Lemma 6A.1 in Appendix 6A, one obtains the following theorem ensuring that $\dot{V}$ is locally negative definite, and thus the local asymptotic stability of the origin.

Theorem 6.1. Let $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$. The origin of system (6.1) is locally asymptotically stable if $\Omega_1\delta_1 < 0, \Omega_3\delta_4 < 0$ and either $\Omega_1\delta_2 \leq 0$ and $\Omega_3\delta_3 \leq 0$, or $\Omega_1\delta_2$ and $\Omega_3\delta_3$ are nonzero and of opposite sign.

A more explicit stability criterion is achieved by rewriting Theorem 6.1 in terms of system coefficients before normal form transformation as follows:

Theorem 6.2. Suppose $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$. The origin of system (6.1) is locally asymptotically stable if $S_1, S_2 < 0$ and $S_3, S_4 \leq 0$ or S_3 and S_4 are nonzero and of opposite sign, where

$$
\begin{aligned}
S_1 = {} & \frac{1}{3\Omega_1^2 + 2\Omega_1\Omega_2 + 3\Omega_2^2}\{\Omega_1[3(\Omega_2 f_{2,222} + \Omega_1 f_{1,111}) + (\Omega_1 f_{2,112} \\
& + \Omega_2 f_{1,122})] + f_{2,22}(\Omega_1 f_{2,12} - 2\Omega_2 f_{1,22}) - f_{1,12}(\Omega_2 f_{1,22} + \Omega_1 f_{1,11}) \\
& + \frac{\Omega_1^2}{\Omega_2} f_{2,11}(f_{2,12} + 2 f_{1,11}) + \frac{\Omega_1}{\Omega_4}[(3\Omega_2 f_{4,22} + \Omega_1 f_{4,11}) f_{2,23} + (3\Omega_1 f_{4,11} \\
& + \Omega_2 f_{4,22}) f_{1,13}] - \frac{\Omega_1}{\Omega_3}[(\Omega_1 f_{3,11} + 3\Omega_2 f_{3,22}) f_{2,24} + (\Omega_2 f_{3,22} \\
& + 3\Omega_1 f_{3,11}) f_{1,14}] + \frac{\Omega_1}{(4\Omega_1\Omega_2 - \Omega_3\Omega_4)\Omega_4}[\Omega_1(\Omega_4 f_{2,14} - 2\Omega_2 f_{2,23}) \\
& + \Omega_2(\Omega_4 f_{1,24} + 2\Omega_1 f_{1,13})] \cdot (\Omega_4 f_{3,12} - 2\Omega_1 f_{4,22}) \\
& - \frac{\Omega_1}{(4\Omega_1\Omega_2 - \Omega_3\Omega_4)\Omega_3}[\Omega_1(2\Omega_2 f_{2,24} + \Omega_3 f_{2,13}) \\
& - \Omega_2(2\Omega_1 f_{1,14} - \Omega_3 f_{1,23})] \cdot (\Omega_3 f_{4,12} - 2\Omega_2 f_{3,22} + 2\Omega_1 f_{3,11})\}, \qquad (6.20) \\
S_2 = {} & \frac{1}{3\Omega_3^2 + 2\Omega_3\Omega_4 + 3\Omega_4^2}\{\Omega_3[3(\Omega_4 f_{4,444} + \Omega_3 f_{3,333}) + (\Omega_3 f_{4,334} \\
& + \Omega_4 f_{3,344})] + f_{4,44}(\Omega_3 f_{4,34} - 2\Omega_4 f_{3,44}) - f_{3,34}(\Omega_4 f_{3,44} + \Omega_3 f_{3,33}) \\
& + \frac{\Omega_3^2}{\Omega_4} f_{4,33}(f_{4,34} + 2 f_{3,33}) + \frac{\Omega_3}{\Omega_2}[(3\Omega_4 f_{2,44} + \Omega_3 f_{2,33}) f_{4,14} + (3\Omega_3 f_{2,33}
\end{aligned}
$$

$$
\begin{aligned}
&+\Omega_4 f_{2,44})f_{3,13}] - \frac{\Omega_3}{\Omega_1}[(\Omega_3 f_{1,33} + 3\Omega_4 f_{1,44})f_{4,24} + (\Omega_4 f_{1,44} \\
&+ 3\Omega_3 f_{1,33})f_{3,23}] + \frac{\Omega_3}{(4\Omega_3\Omega_4 - \Omega_1\Omega_2)\Omega_2}[\Omega_3(\Omega_2 f_{4,23} - 2\Omega_4 f_{4,14}) \\
&+ \Omega_4(\Omega_2 f_{3,24} + 2\Omega_3 f_{3,13})] \cdot (\Omega_2 f_{1,34} - 2\Omega_3 f_{2,33} + 2\Omega_4 f_{2,44}) \\
&- \frac{\Omega_3}{(4\omega_3\Omega_4 - \Omega_1\Omega_2)\Omega_1}[\Omega_3(2\Omega_4 f_{4,24} + \Omega_1 f_{4,13}) \\
&- \Omega_4(2\Omega_3 f_{3,23} - \Omega_1 f_{3,14}] \cdot (\Omega_1 f_{2,34} - 2\Omega_4 f_{1,44} + 2\Omega_3 f_{1,33})\},
\end{aligned}
\tag{6.21}
$$

$$
\begin{aligned}
S_3 = &\frac{\Omega_3}{(\Omega_1 + \Omega_2)\cdot(\Omega_3 + \Omega_4)}\{2f_{1,33}f_{3,23} + \frac{1}{\Omega_3}[2\Omega_4 f_{1,44}f_{4,24} \\
&+ \Omega_1\Omega_4(f_{1,144} + f_{2,244})] + \Omega_1(f_{1,133} + f_{2,233}) \\
&- \frac{2\Omega_2}{\Omega_2} f_{2,33}f_{3,13} + \frac{\Omega_1}{\Omega_3^2\Omega_4}[\Omega_3(\Omega_4 f_{4,44} + (\Omega_3 f_{4,33})(f_{2,23} + f_{1,13}) \\
&- \Omega_4(\Omega_4 f_{3,44} + \Omega_3 f_{3,33})(f_{2,24} + f_{1,14}] - \frac{2\Omega_1\Omega_4}{\Omega_3\Omega_2} f_{2,44}f_{4,14} \\
&+ \frac{1}{\Omega_3\Omega_2}[\Omega_1(\Omega_3 f_{2,33} + \Omega_4 f_{2,44})(f_{2,12} + 2f_{1,11}) \\
&- \Omega_2(\Omega_3 f_{1,33} + \Omega_4 f_{1,44})(f_{1,12} + 2f_{2,22})] \\
&+ \frac{1}{(4\Omega_3\Omega_4 - \Omega_1\Omega_2)\Omega_3}[\Omega_4(\Omega_1 f_{3,14} - 2\Omega_3 f_{3,23}) \\
&+ \Omega_3(\Omega_1 f_{4,13} + 2\Omega_4 f_{4,24})] \cdot (\Omega_1 f_{2,34} - 2\Omega_4 f_{1,44} + 2\Omega_3 f_{1,33}) \\
&- \frac{\Omega_1}{(4\Omega_3\Omega_4 - \Omega_1\Omega_2)\Omega_3\Omega_2}[\Omega(\Omega_2 f_{3,24} + 2\Omega_3 f_{3,13}) \\
&+ \Omega_3(\Omega_2 f_{4,23} - 2\Omega_4 f_{4,14})] \cdot (\Omega_2 f_{1,34} - 2\Omega_3 f_{2,33} + 2\Omega_4 f_{2,44})\},
\end{aligned}
\tag{6.22}
$$

$$
\begin{aligned}
S_4 = &\frac{\Omega_1}{(\Omega_1 + \Omega_2)\cdot(\Omega_3 + \Omega_4)}\{2f_{3,11}f_{1,14} + \frac{1}{\Omega_1}[2\Omega_2 f_{3,22}f_{2,24} \\
&+ \Omega_3\Omega_2(f_{3,223} + f_{4,224})] + \frac{\Omega_3}{\Omega_1^2\Omega_2}[\Omega_1(\Omega_2 f_{2,22} + \Omega_1 f_{2,11})(f_{4,14} + f_{3,13}) \\
&- \Omega_2(\Omega_2 f_{1,22} + \Omega_1 f_{1,11})(f_{4,24} + f_{2,23})] - \frac{2\Omega_3}{\Omega_4} f_{4,11}f_{1,13}
\end{aligned}
$$

$$+ \frac{1}{\Omega_1\Omega_4}[\Omega_3(\Omega_1 f_{4,11} + \Omega_2 f_{4,22})(f_{4,34} + 2f_{3,33})$$

$$- \Omega_4(\Omega_1 f_{3,11} + \Omega_2 f_{3,22})(f_{3,34} + 2f_{4,44})] - \frac{2\Omega_3\Omega_2}{\Omega_1\Omega_4} f_{4,22} f_{2,23}$$

$$+ \frac{1}{(4\Omega_1\Omega_2 - \Omega_3\Omega_4)\Omega_1}[\Omega_2(\Omega_3 f_{1,23} - 2\Omega_1 f_{1,14})$$

$$+ \Omega_1(\Omega_3 f_{2,13} + 2\Omega_2 f_{2,24})] \cdot (\Omega_3 f_{4,12} - 2\Omega_2 f_{3,22} + 2\Omega_1 f_{3,11})$$

$$- \frac{\Omega_3}{(4\Omega_1\Omega_2 - \Omega_3\Omega_4)\Omega_1\Omega_4}[\Omega_2(\Omega_4 f_{1,24} + 2\Omega_1 f_{1,13}) + \Omega_1(\Omega_4 f_{2,14}$$

$$- 2\Omega_2 f_{2,23})] \cdot (\Omega_4 f_{3,12} - 2\Omega_1 f_{4,11} + 2\Omega_2 f_{4,22})$$

$$+ \Omega_3(f_{3,113} + f_{4,114})\}. \tag{6.23}$$

6.3. Problem formulation

The problem considered in this chapter is the feedback stabilization of a critical system possessing two pairs of pure imaginary eigenvalues with the remaining eigenvalues in the open left-half complex plane. For simplicity, the fifth-order system studied in [16] is considered. It is possible, though tedious, to extend the result to more general higher order system, as in [17].

Consider a nonlinear system

$$\begin{pmatrix} \dot{x} \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} x - x_e \\ \zeta - \zeta_e \end{pmatrix} + \begin{pmatrix} f(x - x_e, \zeta - \zeta_e) \\ g(x - x_e, \zeta - \zeta_e) \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} u \tag{6.24}$$

where (x_e, ζ_e) is the critical equilibrium point with input $u = 0$, and

$$A_{11} = \begin{pmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{pmatrix} \tag{6.25}$$

with $\omega_1^2 \neq \alpha\omega_2^2$ for all $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$, and $A_{22} = c$ is a scalar with $c < 0$. Denote the vector function f as

$$f(x, \zeta) = \begin{pmatrix} f_1(x, \zeta) \\ f_2(x, \zeta) \\ f_3(x, \zeta) \\ f_4(x, \zeta) \end{pmatrix} \tag{6.26}$$

with scalar function components $f_i(x,\zeta)$, and similarly for $g(x,\zeta)$. Suppose system (6.24) is smooth enough that each component (say ϕ_i) of f or g may be written in the form

$$\begin{aligned}\phi_i(x,\zeta) = {} & \phi_{i,11}x_1^2 + \phi_{i,22}x_2^2 + \phi_{i,33}x_3^2 + \phi_{i,44}x_4^2 + \phi_{i,\zeta\zeta}\zeta^2 \\ & + \phi_{i,12}x_1x_2 + \phi_{i,13}x_1x_3 + \phi_{i,14}x_1x_4 + \phi_{i,23}x_2x_3 + \phi_{i,24}x_2x_4 + \phi_{i,34}x_3x_4 \\ & + (\phi_{i,1\zeta}x_1 + \phi_{i,2\zeta}x_2 + \phi_{i,3\zeta}x_3 + \phi_{i,4\zeta}x_4)\zeta \\ & + \phi_{i,123}x_1x_2x_3 + \phi_{i,124}x_1x_2x_4 + \phi_{i,134}x_1x_3x_4 + \phi_{i,234}x_2x_3x_4 \\ & + (\phi_{i,12\zeta}x_1x_2 + \phi_{i,13\zeta}x_1x_3 + \phi_{i,14\zeta}x_1x_4 \\ & + \phi_{i,23\zeta}x_2x_3 + \phi_{i,24\zeta}x_2x_4 + \phi_{i,34\zeta}x_3x_4)\zeta \\ & + (\phi_{i,111}x_1 + \phi_{i,112}x_2 + \phi_{i,113}x_3 + \phi_{i,114}x_4 + \phi_{i,11\zeta}\zeta)x_1^2 \\ & + (\phi_{i,122}x_1 + \phi_{i,222}x_2 + \phi_{i,223}x_3 + \phi_{i,224}x_4 + \phi_{i,22\zeta}\zeta)x_2^2 \\ & + (\phi_{i,133}x_1 + \phi_{i,233}x_2 + \phi_{i,333}x_3 + \phi_{i,334}x_4 + \phi_{i,33\zeta}\zeta)x_3^2 \\ & + (\phi_{i,144}x_1 + \phi_{i,244}x_2 + \phi_{i,344}x_3 + \phi_{i,444}x_4 + \phi_{i,44\zeta}\zeta)x_4^2 \\ & + (\phi_{i,1\zeta\zeta}x_1 + \phi_{i,2\zeta\zeta}x_2 + \phi_{i,3\zeta\zeta}x_3 + \phi_{i,4\zeta\zeta}x_4 + \phi_{i,\zeta\zeta\zeta}\zeta)\zeta^2 \\ & + O(||x,\zeta||^4). \end{aligned} \tag{6.27}$$

Let the washout filters used in the feedback loop be governed by the dynamic equations

$$\dot{z}_i = x_i - dz_i \tag{6.28a}$$

and

$$y_i = x_i - dz_i, \tag{6.28b}$$

where $i = 1, \ldots, 4$. Here, the coefficients for states x_i in washout filters have been chosen to be unity, and the time constants (d^{-1}) for the filters are taken to be equal for simplicity. The overall system becomes

$$\begin{pmatrix} \dot{x} \\ \dot{z} \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} A_{11} & 0 & 0 \\ I & -dI & 0 \\ 0 & 0 & A_{22} \end{pmatrix} \begin{pmatrix} \check{x} \\ \check{z} \\ \check{\zeta} \end{pmatrix} + \begin{pmatrix} f(\check{x},\check{\zeta}) \\ 0 \\ g(\check{x},\check{\zeta}) \end{pmatrix} + \begin{pmatrix} b1 \\ 0 \\ b2 \end{pmatrix} u, \tag{6.29}$$

where $\breve{x} := x - x_e, \breve{z} = z - z_e$, and $\breve{\zeta} = \zeta - \zeta_e$. We do not use ζ for control, and there is no washout filter corresponding to ζ, since, as will be discussed in the following sections, ζ does not affect the stability coefficients for the case in which both critical modes are uncontrollable, and for the case in which the critical modes are controllable, x suffices as a measured output for stabilization. Note also that, we must require $d > 0$ to avoid the introduction of unstable eigenvalues.

In order to use the stability criterion in Section 6.2, a block diagonalizing transformation is first applied to decouple states x and z in the linearized dynamics. Let the transformation matrix P be chosen in the form

$$P := \begin{pmatrix} I & 0 & 0 \\ E & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{6.30}$$

where

$$E = \begin{pmatrix} \frac{d}{d^2+\omega_1^2} & -\frac{\omega_1}{d^2+\omega_1^2} & 0 & 0 \\ \frac{\omega_1}{d^2+\omega_1^2} & \frac{d}{d^2+\omega_1^2} & 0 & 0 \\ 0 & 0 & \frac{d}{d^2+\omega_2^2} & -\frac{\omega_2}{d^2+\omega_2^2} \\ 0 & 0 & \frac{\omega_2}{d^2+\omega_2^2} & \frac{d}{d^2+\omega_2^2} \end{pmatrix}. \tag{6.31}$$

Then

$$P^{-1} = \begin{pmatrix} I & 0 & 0 \\ -E & I & 0 \\ 0 & 0 & I \end{pmatrix}. \tag{6.32}$$

The new transformed state is

$$\begin{aligned} \begin{pmatrix} \hat{x} \\ \hat{z} \\ \hat{\zeta} \end{pmatrix} &:= P^{-1} \begin{pmatrix} \breve{x} \\ \breve{z} \\ \breve{\zeta} \end{pmatrix} \\ &= \begin{pmatrix} \breve{x} \\ -E\breve{x} + \breve{z} \\ \breve{\zeta} \end{pmatrix}. \end{aligned} \tag{6.33}$$

The transformed dynamics is

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{z}} \\ \dot{\hat{\zeta}} \end{pmatrix} = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & -dI & 0 \\ 0 & 0 & A_{22} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{z} \\ \hat{\zeta} \end{pmatrix} + \begin{pmatrix} \hat{f}(\hat{x},\hat{\zeta}) \\ 0 \\ \hat{g}(\hat{x},\hat{\zeta}) \end{pmatrix} + \hat{b}u, \tag{6.34}$$

where

$$\hat{b} = \begin{pmatrix} b_1 \\ -Eb_1 \\ b_2 \end{pmatrix}. \tag{6.35}$$

The washout filter outputs can also be expressed in terms of the new state variables as

$$y_1 = \frac{w_1^2}{d^2 + w_1^2}\hat{x}_1 + \frac{dw_1}{d^2 + w_1^2}\hat{x}_2 - d\hat{z}_1, \tag{6.36a}$$

$$y_2 = \frac{-dw_1}{d^2 + w_1^2}\hat{x}_1 + \frac{w_1^2}{d^2 + w_1^2}\hat{x}_2 - d\hat{z}_2, \tag{6.36b}$$

$$y_3 = \frac{w_2^2}{d^2 + w_2^2}\hat{x}_3 + \frac{dw_2}{d^2 + w_2^2}\hat{x}_4 - d\hat{z}_3, \tag{6.36c}$$

$$y_4 = \frac{-dw_2^2}{d^2 + w_2^2}\hat{x}_3 + \frac{w_2^2}{d^2 + w_2^2}\hat{x}_4 - d\hat{z}_4. \tag{6.36d}$$

Since all the eigenvalues of the lower diagonal block of the Jacobian matrix

$$\bar{A}_{22} := \begin{pmatrix} -dI & 0 \\ 0 & A_{22} \end{pmatrix} \tag{6.37}$$

have negative real parts, by the center manifold theorem, there exists a center manifold given by the graph of a function

$$\begin{pmatrix} \hat{z} \\ \hat{\zeta} \end{pmatrix} = h(\hat{x}) = \begin{pmatrix} h_z(\hat{x}) \\ h_\zeta(\hat{x}) \end{pmatrix} \tag{6.38}$$

with

$$h(0) = 0, \qquad Dh(0) = 0 \tag{6.39}$$

and

$$Dh(\hat{x})(A_{11}\hat{x} + \hat{f}(\hat{x}, h_\zeta(\hat{x})) + b_1 u) = \bar{A}_{22}h(\hat{x}) + \begin{pmatrix} 0 \\ \hat{g}(\hat{x}, h_\zeta(\hat{x})) \end{pmatrix} + \begin{pmatrix} -Eb_1 \\ b_2 \end{pmatrix} u. \tag{6.40}$$

For $|x| < \delta$, where δ is some positive number, the local stability of system (6.34) at the origin of $(\hat{x}, \hat{z}, \hat{\zeta})$ space agrees with that for a reduced system

$$\dot{\hat{x}} = A_{11}\hat{x} + \hat{f}(\hat{x}, h(\hat{x})) + b_1 u. \tag{6.41}$$

By conditions (6.39), h can be written as

$$\begin{aligned} h(\hat{x}) = {} & \hat{x}_1^2 h_{11} + \hat{x}_2^2 h_{22} + \hat{x}_3^2 h_{33} + \hat{x}_4^2 h_{44} \\ & + \hat{x}_1\hat{x}_2 h_{12} + \hat{x}_1\hat{x}_3 h_{13} + \hat{x}_1\hat{x}_4 h_{14} + \hat{x}_2\hat{x}_3 h_{23} + \hat{x}_2\hat{x}_4 h_{24} + \hat{x}_3\hat{x}_4 h_{34} \\ & + O(||\hat{x}||^3). \end{aligned} \tag{6.42}$$

Since system (6.41) has the same form as the fourth-order system (6.1), Theorem 6.2 can be used to determine stability.

6.4. Both critical modes uncontrollable

If the vector $b_1 = 0$ in (6.41), both critical modes of system (6.24) are uncontrollable. Take the control u to be a nonlinear function of $\hat{x}$ and $\hat{z}$ of the form:

$$\begin{aligned} u = {} & \sum_{i=1}^{4}\sum_{j=i}^{4} u_{ij}\hat{x}_i\hat{x}_j + \sum_{i=1}^{4}\sum_{j=i}^{4} v_{ij}\hat{z}_i\hat{z}_j \\ & + \sum_{i=1}^{4}\sum_{j=1}^{4} w_{ij}\hat{x}_i\hat{z}_j + U(||\hat{x}, \hat{z}||^3), \end{aligned} \tag{6.43}$$

where u_{ij}, v_{ij}, w_{ij} are scalar constants. Define a 5-dimensional vector function $H(\hat{x})$ (recalling that $\tilde{z} = h_z(\tilde{x})$)

$$\begin{aligned} H(\hat{x}) := {} & \begin{pmatrix} 0 \\ b_2 u \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{g}(\hat{x}, 0) \end{pmatrix} \\ = {} & \hat{x}_1^2 H_{11} + \hat{x}_2^2 H_{22} + \hat{x}_3^2 H_{33} + \hat{x}_4^2 H_{44} + \hat{x}_1\hat{x}_2 H_{12} + \hat{x}_1\hat{x}_3 H_{13} \\ & + \hat{x}_1\hat{x}_4 H_{14} + \hat{x}_2\hat{x}_3 H_{23} + \hat{x}_2\hat{x}_4 H_{24} + \hat{x}_3\hat{x}_4 H_{34} + O(||\hat{x}||^3). \end{aligned} \tag{6.44}$$

Let

$$M := \begin{pmatrix} \bar{A}_{22} & \omega_2 I \\ -\omega_2 I & \bar{A}_{22} \end{pmatrix}. \tag{6.45}$$

Then, by the formulae derived in [17], we can solve

$$h_{34} = -(\bar{A}_{22}^2 + 4\omega_2^2 I)^{-1}(-2\omega_2 H_{44} + 2\omega_2 H_{33} + \bar{A}_{22}H_{34}), \tag{6.46}$$

$$h_{33} = -\bar{A}_{22}^{-1}(H_{33} + \omega_2 h_{34}), \tag{6.47}$$

$$h_{44} = -\bar{A}_{22}^{-1}(H_{44} - \omega_2 h_{34}), \tag{6.48}$$

$$\begin{pmatrix} h_{13} \\ h_{14} \end{pmatrix} = (M^2 + \omega_1^2 I)^{-1}\{M \begin{pmatrix} H_{13} \\ H_{14} \end{pmatrix} - \omega_1 \begin{pmatrix} H_{23} \\ H_{24} \end{pmatrix}\}, \tag{6.49}$$

$$\begin{pmatrix} h_{23} \\ h_{24} \end{pmatrix} = (M^2 + \omega_1^2 I)^{-1}\{M \begin{pmatrix} H_{23} \\ H_{24} \end{pmatrix} + \omega_1 \begin{pmatrix} H_{13} \\ H_{14} \end{pmatrix}\}, \tag{6.50}$$

$$h_{12} = -\{\bar{A}_{22}^2 + 4\omega_1^2 I\}^{-1}\{2\omega_1(H_{11} - H_{22}) + \bar{A}_{22}H_{12}\}, \tag{6.51}$$

$$h_{11} = -\bar{A}_{22}^{-1}(H_{11} + \omega_1 h_{12}), \tag{6.52}$$

$$h_{22} = -\bar{A}_{22}^{-1}(H_{22} - \omega_1 h_{12}). \tag{6.53}$$

Note that, these formulae can be obtained by substituting Eq. (6.42), Eq. (6.43) into Eq. (6.40), and equating coefficients with same powers of $\hat{x}_1\hat{x}_2\hat{x}_3\hat{x}_4$ from both sides to generate a sequence of equality relations.

The reduced system (6.41) becomes

$$\dot{\hat{x}} = A_{11}\hat{x} + \tilde{f}(\hat{x}), \tag{6.54}$$

where

$$\tilde{f}(\hat{x}) := \hat{f}(\hat{x}, h(\hat{x})), \tag{6.55}$$

and $\tilde{f}(\hat{x})$ takes the form as Eqs. (6.2), (6.3) with

$$\tilde{f}_{i,jk} = \hat{f}_{i,jk}, \tag{6.56a}$$

$$\tilde{f}_{i,jjj} = \hat{f}_{i,jjj} + \hat{f}_{i,j\zeta}h_{\zeta,jj}, \tag{6.56b}$$

$$\tilde{f}_{i,jjk} = \hat{f}_{i,jjk} + \hat{f}_{i,j\zeta}h_{\zeta,jk} + \hat{f}_{i,k\zeta}h_{\zeta,jj}, \tag{6.56c}$$

$$\tilde{f}_{i,jkl} = \hat{f}_{i,jkl} + \hat{f}_{i,j\zeta}h_{\zeta,kl} + \hat{f}_{i,k\zeta}h_{\zeta,jl} + \hat{f}_{i,l\zeta}h_{\zeta,jk}, \tag{6.56d}$$

where $i, j, k, l = 1, \ldots, 4$ with $j \neq k \neq l$, and $h_{\zeta,mn}$ with $m, n = 1, \ldots, 4$ is the last element of vector h_{mn}.

6.4.1. Stabilizability condition

From the equations of stability coefficients (6.20)-(6.23), only the quadratic terms $\tilde{f}_{i,mn}$ and cubic terms $\tilde{f}_{i,lmn}$ of $\hat{x}$ in $\tilde{f}$ have influence on coefficients S_1 to S_4. In Eqs. (6.56a)-(6.56d), the quadratic and cubic terms of $\tilde{f}$ is affected by the quadratic terms of $h_\zeta(\hat{x})$. In formulae (6.46)-(6.53), the quadratic terms of $h(\hat{x})$ are affected by the quadratic terms of $H(\hat{x})$. The mapping function, $h_z(\hat{x})$, of $\hat{z}$, is a vector-valued polynomial in $\hat{x}$ of degree greater or equal to 2, and $\hat{z}$ appears in the control function (6.43) in the form of $\hat{z}_i\hat{z}_j$, $\hat{x}_i\hat{z}_j$ or $||\hat{x}, \hat{z}||^3$ which are vector-valued polynomials in $\hat{x}$ of degree greater or equal to 3, and affect only the cubic or higher terms of $H(\hat{x})$, Therefore, state $\hat{z}$ in the nonlinear control function does not affect the stability coefficients S_i. That is, the terms as z_i^2, x_iz_j in feedback function can be ignored. For the similar reason, state ζ in the nonlinear feedback will not affect S_i either. Therefore, ζ is not used in the control function (6.43).

Substituting Eqs. (6.56a) -(6.56d) into formulae of S_i in Eqs. (6.20)-(6.23) with $\Omega_1 = \Omega_2 = \omega_1$ and $\Omega_3 = \Omega_4 = \omega_2$, we have

$$S_1 = \frac{1}{8\omega_1^2}(V_{1,11}u_{11} + V_{1,12}u_{12} + V_{1,22}u_{22} + V_{1,00}), \tag{6.57a}$$

$$S_2 = \frac{1}{8\omega_2^2}(V_{2,33}u_{33} + V_{2,34}u_{34} + V_{2,44}u_{44} + V_{2,00}), \tag{6.57b}$$

$$\begin{aligned} S_3 = \frac{1}{4\omega_1}(&V_{3,13}u_{13} + V_{3,14}u_{14} + V_{3,23}u_{23} + V_{3,24}u_{24} \\ &+ V_{3,33}u_{33} + V_{3,44}u_{44} + V_{3,00}), \end{aligned} \tag{6.57c}$$

$$\begin{aligned} S_4 = \frac{1}{4\omega_2}(&V_{4,13}u_{13} + V_{4,14}u_{14} + V_{4,23}u_{23} + V_{4,24}u_{24} \\ &+ V_{4,11}u_{11} + V_{4,22}u_{22} + V_{4,00}), \end{aligned} \tag{6.57d}$$

where $V_{i,j}$ are shown in Appendix 6.A.

Since the feedback signal is taken from the output of washout filters y_i, if the terms $u_{i,j}\hat{x}_i\hat{x}_j$ in the control function can be generated individually from y_i, then Eqs. (6.57a)-(6.57d) can be used to determine the stabilizability of the system. From the relations of $y := (y_1, y_2, y_3, y_4)^T$ and $\hat{x}$ in Eqs. (6.36a)-(6.36d), we have

$$\hat{x}_1 - \omega_1 d(\omega_1 \hat{z}_1 - d\hat{z}_2) = \frac{1}{\omega_1}\{\omega_1 y_1 - dy_2\} \tag{6.58a}$$

$$\hat{x}_2 - \omega_1 d(d\hat{z}_1 + \omega_1 \hat{z}_2) = \frac{1}{\omega_1}\{dy_1 + \omega_1 y_2\} \tag{6.58b}$$

$$\hat{x}_3 - \omega_2 d(\omega_2 \hat{z}_3 - d\hat{z}_4) = \frac{1}{\omega_2}\{\omega_2 y_3 - dy_4\} \tag{6.58c}$$

$$\hat{x}_4 - \omega_2 d(d\hat{z}_3 + \omega_2 \hat{z}_4) = \frac{1}{\omega_2}\{dy_3 + \omega_2 y_4\} \tag{6.58d}$$

Since $\hat{x}_i\hat{z}_i$ and z_i^2 have no impact on S_i, we can construct the terms $u_{i,j}\hat{x}_i\hat{x}_j$ by using the product of Eqs. (6.58a)-(6.58d) with the terms containing $\hat{z}$ neglected. That is, for instance, we can use the product of $\omega_1 y_1 - dy_2$ and $dy_1 + \omega_1 y_2$ to generate $\hat{x}_1\hat{x}_2$. In this way, there will be some byproduct terms as $Z_1 z_2$, $\hat{x}_1 z_1$, $\hat{x}_1 z_2$, $\hat{x}_2 z_1$, and $\hat{x}_2 z_2$. However, these terms do not affect the stability coefficients, thus can be neglected. Hence, we have the following sufficient condition for stabilizing the system.

Theorem 6.3. Suppose the nonlinear system (6.24) is unstable with both of the critical modes uncontrollable, i.e. $b_1 = 0$. If the rank of either of the following matrices

$$M_1 := \begin{pmatrix} V_{1,11} & V_{1,12} & 0 & 0 & V_{1,22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_{2,33} & V_{2,34} & V_{2,44} \\ 0 & 0 & V_{3,13} & V_{3,14} & 0 & V_{3,23} & V_{3,24} & V_{3,33} & 0 & V_{3,34} \end{pmatrix} \tag{6.59}$$

and

$$M_2 := \begin{pmatrix} V_{1,11} & V_{1,12} & 0 & 0 & V_{1,22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_{2,33} & V_{2,34} & V_{2,44} \\ V_{4,11} & 0 & V_{4,13} & V_{4,14} & V_{4,22} & V_{4,23} & V_{4,24} & 0 & 0 & 0 \end{pmatrix} \tag{6.60}$$

is three, then the system can be stabilized by a quadratic feedback of the form

$$u = \sum_{i=1}^{4} \sum_{j=i}^{4} \hat{u}_{ij} y_i y_j. \tag{6.61}$$

Proof: From the stability criterion in Theorem 6.2, the system can be stabilized if there exists a control to make $S_1 \leq 0, S_2 \leq 0$ and either $S_3 < 0$ or $S_4 < 0$. From Eqs. (6.57a)-(6.57d), if rank of M_1 (resp. M_2) is 3, the values of S_1, S_2, S_3 (resp. S_1, S_2, S_4) can be arbitrary assigned by u_{ij}, the coefficient of $\hat{x}_i \hat{x}_j$ in the feedback function (6.43). Since $\hat{x}_i \hat{x}_j$ can be generated individually by the linear combination of y (with ignoring those terms containing $\hat{z}$), the system can be stabilized by control function (6.61).

Theorem 6.3 involves the tedious computation of all the $V_{i,jk}$ of Eqs. (6.57a)-(6.57d). For some special systems, we can have a simpler stabilizability conditions and a simpler control function.

Corollary 6.4. Assume the nonlinear system defined in (6.24) is unstable with both of the critical modes uncontrollable. If all of the following conditions hold

$i)$ $2(f_{1,1\zeta} - f_{2,2\zeta})\omega_1 - c(f_{1,2\zeta} + f_{2,1\zeta}) \neq 0$

$ii)$ $2(f_{3,3\zeta} - f_{4,4\zeta}\omega_2 - c(f_{3,4\zeta} + f_{4,3\zeta}) \neq 0$

$iii)$ either $f_{1,1\zeta} + f_{2,2\zeta} \neq 0$ or $f_{3,3\zeta} + f_{4,4\zeta} \neq 0$

then the system can be stabilized by a quadratic feedback of y.

Proof: Condition (i) implies $V_{1,12} \neq 0$, condition (ii) implies $V_{2,34} \neq 0$, and condition (iii) implies either $V_{3,33} = V_{3,44} \neq 0$ or $V_{4,11} = V_{4,22} \neq 0$. If

$V_{3,33} = V_{3,44} \neq 0$ (resp. $V_{4,11} = V_{4,22} \neq 0$), u_{33} or u_{44} (resp. u_{11} or u_{22}) can be used to make $S_3 < 0$ (resp. $S_4 < 0$). Although u_{33} or u_{44} might affect S_2 (resp. S_1), it can be overcome by using u_{34} (resp. u_{12}) with sufficient magnitude. By using Eqs. (6.58a)-(6.58d) to construct $\hat{x}_i\hat{x}_j$ in terms of y, the system can be stabilized by a quadratic feedback of y.

■

Note that, as long as the quadratic and cubic terms of $\hat{x}$ can be generated individually from y, the stabilizability conditions for using washout filter-aided feedback will be the same with that for using direct state feedback.

The equations of stability coefficients $S_1 - S_4$ in Eqs. (6.57a)-(6.57d) can be more explicitly written in terms of the coefficients of quadratic control of y. Assume the control u takes the form of Eq. (6.61), S_i becomes

$$S_1 = \frac{1}{8\omega_1^2}(\tilde{V}_{1,11}\tilde{u}_{11} + \tilde{V}_{1,12}\tilde{u}_{12} + \tilde{V}_{1,22}\tilde{u}_{22} + \tilde{V}_{1,00}), \tag{6.62a}$$

$$S_2 = \frac{1}{8\omega_2^2}(\tilde{V}_{2,33}\tilde{u}_{33} + \tilde{V}_{2,34}\tilde{u}_{34} + \tilde{V}_{2,44}\tilde{u}_{44} + \tilde{V}_{2,00}), \tag{6.62b}$$

$$S_3 = \frac{1}{4\omega_1}(\tilde{V}_{3,13}\tilde{u}_{13} + \tilde{V}_{3,14}\tilde{u}_{14} + \tilde{V}_{3,23}\tilde{u}_{23} + \tilde{V}_{3,24}\tilde{u}_{24} + \tilde{V}_{3,33}\tilde{u}_{33} + \tilde{V}_{3,44}\tilde{u}_{44} + \tilde{V}_{3,00}), \tag{6.62c}$$

$$S_4 = \frac{1}{4\omega_2}(\tilde{V}_{4,13}\tilde{u}_{13} + \tilde{V}_{4,14}\tilde{u}_{14} + \tilde{V}_{4,23}\tilde{u}_{23} + \tilde{V}_{4,24}\tilde{u}_{24} + \tilde{V}_{4,11}\tilde{u}_{11} + \tilde{V}_{4,22}\tilde{u}_{22} + \tilde{V}_{4,00}), \tag{6.62d}$$

where $\tilde{V}_{i,jk}$ are shown in Appendix 6.B.

The stabilizability conditions is similar to that in Theorem 6.3 except that elements $V_{i,jk}$ in matrices M_1 and M_2 are replaced by $\tilde{V}_{i,jk}$.

The expression of $\tilde{V}_{i,jk}$ involves the washout filter time constant, d, and is more complicated than $V_{i,jk}$. However, flexibility of choosing d makes the determination of robustness easier.

6.4.2. Robustness

By using washout filters, the control function does not depend on accurate knowledge of the critical equilibrium point, therefore it is robust with respect to uncertainty in the equilibrium point. Moreover, the control is also robust with respect to other modeling uncertainty.

Lemma 6.5. Suppose the nonlinear system defined in (6.24) has bounded uncertainty in its system dynamics, and both of the critical modes are uncontrollable. If through out the region of uncertainty, the element of control vector $b_1 \neq 0$ and does not change sign, and all of the following conditions hold

i) $f_{1,15} + f_{2,25} \neq 0$ and does not change sign, or

$f_{3,35} + f_{4,45} \neq 0$ and does not change sign,

ii) $P_1 := c(f_{1,25} + f_{2,15} - 2(f_{1,15} - f_{2,25})\omega_1) \neq 0$ and does not change sign,

iii) $P_2 := c(f_{3,45} + f_{4,35} - 2(f_{3,35} - f_{4,45})\omega_2) \neq 0$ and does not change sign,

then with proper choice of d, there exists a fixed control function to ensure the stability of the system at its critical equilibrium point. The control function takes the form

$$u = u_{12}y_1y_2 + u_{34}y_3y_4 + u_{kk}y_k^2, \tag{6.63}$$

where k is either 1,2 or 3,4, depending on condition (i).

Proof: If $f_{1,15} - f_{2,25} \neq 0$ (resp. $f_{3,35} - f_{4,45} \neq 0$), we set $k \in \{3,4\}$ (resp. $k \in \{1,2\}$) and choose $|u_{kk}|$ sufficiently large so that $S_3 < 0$ (resp. $S_4 < 0$) throughout the range of uncertainty. From Appendix (6b.2), $\tilde{V}_{1,12}$ can be written as

$$\tilde{V}_{1,12} = \frac{b_2\omega_1^4 P_1(d-\alpha_1)(d-\alpha_2)}{(d^2+\omega_1^2)^2(c^2+4\omega_1^2)}, \tag{6.64}$$

where

$$\alpha_i = \frac{1}{P_1}\{-\omega_1 q_1 \pm \sqrt{q_1^2 + 4P_1^2\omega_1}\} \qquad (i = 1,2) \tag{6.65}$$

and

$$q_1 = c(f_{1,15} - f_{2,25}) + 2\omega_1(f_{1,25} + f_{2,15}). \tag{6.66}$$

By condition (ii), P_1 does not change sign, $\tilde{V}_{1,12}$ will remain the same sign if d is chosen to be larger than all α_i. Similarly, $\tilde{V}_{2,34}$ can be written as

$$\tilde{V}_{2,34} = \frac{b_2\omega_2^4 P_2(d-\alpha_3)(d-\alpha_4)}{(d^2+\omega_2^2)^2(c^2+4\omega_2^2)}, \tag{6.67}$$

where

$$\alpha_i = \frac{1}{P_2}\{-\omega_2 q_2 \pm \sqrt{q_2^2 + 4P_2^2\omega_2}\} \qquad (i=1,2) \tag{6.68}$$

and

$$q_2 = c(f_{3,35} - f_{4,45}) + 2\omega_2(f_{3,45} + f_{4,35}). \tag{6.69}$$

By choosing d large enough, $V_{2,34}$ will remain the same sign. Choose $|\tilde{u}_{1,12}|$ and $|\tilde{u}_{2,34}|$ sufficiently large. We can make S_1 and S_2 negative. Hence, stabilize the system.

■

6.5. Both critical modes controllable

Let the control vector b of system be written as

$$b = \begin{pmatrix} b_{11} \\ b_{12} \\ b_{13} \\ b_{14} \\ b_2 \end{pmatrix}. \tag{6.70}$$

If either b_{11} or b_{12} does not vanish and either b_{13} or b_{14} does not vanish, then both of the critical modes are controllable. A linear control through washout filters can move all the critical eigenvalues from imaginary axis and stabilize the system. However, in some cases, if it is required to preserve other equilibrium points, then a purely nonlinear stabilization control is required.

6.5.1 Stabilizability condition

Recall the stability criterion in Theorem 6.2 , the stability coefficients S_i, are functions of $f_{i,jk}$ and $f_{i,jkl}$. Since both critical modes are linearly controllable, $f_{i,jk}$ and $f_{i,jkl}$ can be affected by the quadratic and the cubic feedback of x, respectively, through b_1. Since our feedback variables are the output y of washout filters, quadratic function of y_i would induce terms as $x_i z_j$ which after mapping to center manifold will affect the closed-loop $f_{i,jkl}$. In order to avoid the complexity of involving state z into stabilizability conditions, a purely cubic feedback of y is considered.

Let the control function u be

$$u = \sum_{i=1}^{4} \sum_{j=i}^{4} \sum_{k=j}^{4} u_{ijk} y_i y_j y_k, \tag{6.71}$$

the stability coefficients become

$$S_1 = V_{1,111} u_{111} + V_{1,112} u_{112} + V_{1,122} u_{122} + V_{1,222} u_{222} + V_{1,000} \tag{6.72a}$$

$$S_2 = V_{2,333} u_{333} + V_{2,334} u_{334} + V_{2,344} V_{344} + V_{2,444} u_{444} + V_{2,000} \tag{6.72b}$$

$$S_3 = V_{3,133} u_{133} + V_{3,144} u_{144} + V_{3,233} u_{233} + V_{3,244} u_{244} + V_{3,000} \tag{6.72c}$$

$$S_4 = V_{4,113} u_{113} + V_{4,114} u_{114} + V_{4,223} u_{223} + V_{4,224} u_{224} + V_{4,000} \tag{6.72d}$$

where

$$V_{1,111} = 3V_{1,122} = \frac{3}{8}[\frac{(b_{11}\omega_1 + b_{12}d)\omega_1^3}{(d^2+\omega_1^2)^2}], \tag{6.73a}$$

$$V_{1,222} = 3V_{1,112} = \frac{3}{8}[\frac{(b_{12}\omega_1 - b_{11}d)\omega_1^3}{(d^2+\omega_1^2)^2}], \tag{6.73b}$$

$$V_{2,333} = 3V_{2,344} = \frac{3}{8}[\frac{(b_{13}\omega_2 + b_{14}d)\omega_2^3}{(d^2+\omega_1^2)^2}], \tag{6.73c}$$

$$V_{2,444} = 3V_{2,334} = \frac{3}{8}[\frac{(b_{14}\omega_2 - b_{13}d)\omega_2^3}{(d^2+\omega_1^2)^2}], \tag{6.73d}$$

$$V_{3,133} = V_{3,144} = \frac{1}{4}[\frac{(b_{11}\omega_1 + b_{12}d)\omega_1\omega_2^2}{(d^2+\omega_1^2)(d^2+\omega_2^2)}], \tag{6.73e}$$

$$V_{3,233} = V_{3,244} = \frac{1}{4}[\frac{(b_{12}\omega_1 - db_{11})\omega_1\omega_2^2}{(d^2+\omega_1^2)(d^2+\omega_2^2)}], \tag{6.73f}$$

$$V_{4,113} = V_{4,223} = \frac{1}{4}[\frac{(b_{13}\omega_1 + b_{14}d)\omega_1\omega_2^2}{(d^2+\omega_1^2)(d^2+\omega_2^2)}], \tag{6.73g}$$

$$V_{4,114} = V_{4,224} = \frac{1}{4}[\frac{(b_{14}\omega_1 - b_{13}d)\omega_1\omega_2^2}{(d^2+\omega_1^2)(d^2+\omega_2^2)}], \tag{6.73h}$$

and $V_{1,000}, V_{2,000}, V_{3,000}$ and $V_{4,000}$ are the same with those $\frac{1}{8\omega_1^2}V_{1,00}$, $\frac{1}{8\omega_2^2}V_{2,00}$, $\frac{1}{4\omega_1}V_{3,00}$, $\frac{1}{4\omega_2}V_{4,00}$ defined in Appendix 6.A.

Since either $b_{11} \neq 0$ or $b_{12} \neq 0$ and either $b_{13} \neq 0$ or $b_{14} \neq 0$, by Eqs. (6.73a)-(6.73h), for each l, $(l = 1, \ldots 4)$, not all $V_{l,ijk}$, $(i, j, k \in \{1, \ldots, 4\})$, are zero. Therefore, the values of stability coefficients S_i can be individually assigned by using cubic control function of y.

Lemma 6.6. If both of the critical modes of a nonlinear system defined in (6.24) are controllable, then there exists a feedback as cubic function of y_i that stabilizes the system.

■

6.5.2. Robustness

As the case in which both critical modes are uncontrollable, the control is robust with respect to uncertainty in critical point and system dynamics. Moreover, the existence of robust controller solely depends on the amount of uncertainty in control vector b.

Lemma 6.7. Consider a nonlinear system defined as (6.24) with bounded uncertainty in its system dynamics. If both of the critical modes are controllable, then there exists a robust control function to stabilize the system at critical equilibrium point provided that either b_{11} or b_{12} does not change sign throughout the range of uncertainty, and either b_{13} or b_{14} does not change sign throughout the range of uncertainty.

Proof: If b_{11} (resp. b_{12}) does not change sign, by choosing d sufficiently large, $V_{1,222}$ (resp. $V_{1,111}$) and $V_{3,233}$ (resp. $V_{3,133}$) will not change sign throughout the uncertainty range. By choosing $|u_{222}|$ and $|u_{233}|$ (resp. $|u_{111}|$ and $|u_{133}|$) sufficiently large, S_1 and S_3 can be set negative throughout the uncertainty range. Similarly, we can set S_2 and S_4 negative throughout the uncertainty range. Thus, the system is stabilized.

■

6.6. Only one controllable critical mode

Without loss of generality, assume that the first pair of eigenvalues $\pm j\omega_1$ of (6.24) are controllable. That is, either $b_{11} \neq 0$ or $b_{12} \neq 0$, and $b_{13} = b_{14} = 0$. It is obvious that a linear feedback through washout filters can be used to move away the first pair of critical eigenvalues, the stabilization method for single pair of purely imaginary eigenvalues discussed in Chapter 4 can then be applied. In here, by the same reason discussed in Section 6.5, the control function is restricted to purely nonlinear.

Since eigenvalues $\pm j\omega_1$, are linear controllable, from the result in Section 6.5, stability coefficients S_1 and S_3 can be controlled by cubic feedback of y_i through b_{11} or b_{12} directly. The main problem of this purely nonlinear control is that S_2 has to be controlled by quadratic feedback due to the lack of linear controllability of eigenvalues $\pm j\omega_2$. These quadratic feedback may influence the stability coefficients S_i, $(i = 1, \ldots, 4)$ directly through b_{11} and b_{12}, and indirectly through b_2 (through the center manifold mapping). This effect makes the equations of S_i much complicated than previous two cases.

In order to reduce the complexity of the computation, the control function is chosen to take the form of

$$\begin{aligned} u :=& \sum_{i=3}^{4}\sum_{j=i}^{4} u_{ij}y_3y_4 + \sum_{i=1}^{2}\sum_{j=i}^{2}\sum_{k=j}^{2} u_{ijk}y_iy_jy_k + \sum_{i=1}^{2}\sum_{j=3}^{4} u_{ijj}y_iy_j^2 \\ =& \sum_{i=3}^{4}\sum_{j=i}^{4} \hat{u}_{ij}\hat{x}_3\hat{x}_4 + \sum_{i=1}^{2}\sum_{j=i}^{2}\sum_{k=j}^{2} \hat{u}_{ijk}\hat{x}_i\hat{x}_j\hat{x}_k + \sum_{i=1}^{2}\sum_{j=3}^{4} \hat{u}_{ijk}x_ix_j^2 \\ &+ \sum_{i=3}^{4}\sum_{j=3}^{4} v_{ij}\hat{x}_i\hat{z}_j + \sum_{i=1}^{2}\sum_{j=i}^{2}\sum_{k=1}^{2}(v_{ijk}\hat{x}_i\hat{x}_j\hat{z}_k + w_{ijk}\hat{x}_k\hat{z}_i\hat{z}_j) + U(||\hat{z}||^3) \end{aligned} \tag{6.74}$$

Note that, quadratic terms y_iy_j, for $i, j \in \{1, 2\}$, do not appear in control function (6.74) since S_1 can always be controlled through the cubic feedback due to the linear controllability. Also, the using these quadratic terms will introduce undesired terms, z_1, z_2 , into S_1 that makes the equation complicated. By using Eq (6.74), the only washout states entering to cubic terms of reduced system (6.41) are z_3 and z_4 . However, they will be converted to cubic functions of $||x_3, x_4||^3$ in $\tilde{f}_1$, and have no influence on S_1.

Define

$$H(\hat{x}) := \begin{pmatrix} \dfrac{-db_{11} + \omega_1 b_{12}}{d^2 + \omega_1^2} \\ \dfrac{-\omega_1 b_{11} - db_{12}}{d^2 + \omega_1^2} \\ 0 \\ 0 \\ b_2 \end{pmatrix} u(\hat{x}) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \hat{g}(x, 0) \end{pmatrix}$$

$$= \hat{x}_1^2 H_{11} + \hat{x}_2^2 H_{22} + \hat{x}_3^2 H_{33} + \hat{x}_4^2 H_{44} + \hat{x}_1\hat{x}_2 H_{12} + \hat{x}_1\hat{x}_3 H_{13}$$
$$+\hat{x}_1\hat{x}_4 H_{14} + \hat{x}_2\hat{x}_3 H_{23} + \hat{x}_2\hat{x}_4 H_{24} + \hat{x}_3\hat{x}_4 H_{34} + O(||\hat{x}||^3), \qquad (6.75)$$

Following the formulae in [17], we can compute vectors h_{ij} through using Eqs. (6.46)-(6.53). The quadratic and cubic terms of the nonlinear function $\tilde{f}(x)$ in the reduced system are given by

$$\tilde{f}_{l,ij} = \begin{cases} \hat{f}_{l,ij} + b_{1l}\hat{u}_{ij} & l \in \{1,2\}, \quad i,j \in \{3,4\} \\ \hat{f}_{l,ij} & \text{otherwise} \end{cases}$$

$$\tilde{f}_{l,iii} = \begin{cases} \hat{f}_{l,iii} + \hat{f}_{l,i\varsigma}h_{\varsigma,ii} + b_{1l}u_{iii} & l,i \in \{1,2\} \\ \hat{f}_{l,iii} + \hat{f}_{l,i\varsigma}h_{\varsigma,ii} + b_{1l}(v_{i3}h_{z_3 ii} & \\ \qquad +v_{i4}h_{z_4,ii}) & l \in \{1,2\}, \quad i \in \{3,4\} \\ \hat{f}_{l,iii} + \hat{f}_{l,i\varsigma}h_{\varsigma,ii} & \text{otherwise} \end{cases}$$

for $i < j$

$$\tilde{f}_{l,ijj} = \begin{cases} \hat{f}_{l,ijj} + \hat{f}_{l,i\varsigma}h_{\varsigma,jj} + \hat{f}_{l,j\varsigma}h_{\varsigma,ij} + b_{1l}u_{ijj} & l,i,j \in \{1,2\} \\ \hat{f}_{l,ijj} + \hat{f}_{l,i\varsigma}h_{\varsigma,jj} + \hat{f}_{l,j\varsigma}h_{\varsigma,ij} & \\ \qquad +b_{1l}[\sum_{k=3}^{4}(v_{ik}h_{z_k,jj} + v_{jk}h_{z_k,ij})] & l \in \{1,2\}\ i,j \in \{3,4\} \\ \hat{f}_{l,ijj} + \hat{f}_{l,ij} + h_{\varsigma,jj} + \hat{f}_{l,j\varsigma}h_{\varsigma,ij} & \text{otherwise} \end{cases}$$

for $i < j < k$

$$\tilde{f}_{l,ijk} = \begin{cases} \hat{f}_{l,ijk} + \hat{f}_{l,i\zeta}h_{\zeta,jk} + \hat{f}_{l,j\zeta}h_{\zeta,ik} + \hat{f}_{l,k\zeta}h_{\zeta,ij} \\ \quad +b_{1l}(\sum_{m=3}^{4}(v_{jm}h_{z_m,ik} + v_{km}h_{z_m,ij})) \\ \qquad\qquad\qquad\qquad\qquad l \in \{1,2\}, \quad i,j,k \in \{1,\ldots,4\} \\ \hat{f}_{l,ijk} + \hat{f}_{l,i\zeta}h_{\zeta,jk} + \hat{f}_{l,j\zeta}h_{\zeta,ik} + \hat{f}_{l,k\zeta}h_{\zeta,ij} \quad \text{otherwise} \end{cases}$$

(Note that, from (6.74), $v_{1,m} = v_{2,m} = 0$).

The stability coefficients become

$$S_1 = \frac{1}{8\omega_1^2}(V_{1,111}u_{111} + V_{1,112}u_{112} + V_{1,122}u_{122} + V_{1,222}u_{222} + V_{1,000}), \tag{6.76a}$$

$$S_2 = \frac{1}{8\omega_2^2}(V_{2,33}u_{33} + V_{2,34}u_{34} + V_{2,44}u_{44} + V_{2,000}), \tag{6.76b}$$

$$S_3 = \frac{1}{4\omega_1}(V_{3,33}u_{33} + V_{3,34}u_{34} + V_{3,44}u_{44} + V_{3,133}u_{133} + V_{3,144}u_{144} + V_{3,233}u_{233} + V_{3,244}u_{244} + V_{3,000}). \tag{6.76c}$$

S_4 is not shown here since controlling S_1, S_2, S_3 is sufficed for stability. All the $V_{i,jk}$, and $V_{i,jkl}$ are shown in Appendix 6.C.

Since S_1, S_3 are controllable, a sufficient condition for stabilizing the system is that not all of $V_{2,ii}$ vanish so that we can control S_2. Two special cases are shown in the following Corollaries.

Corollary 6.8. Suppose the nonlinear system (6.24) has one pair of its critical eigenvalues $\pm j\omega_1$ controllabler and the stable eigenvalue c uncontrollable ($b_2 = 0$). Assume also, for the controllable critical mode, we have $b_{11} \neq 0$ and $b_{12} = 0$. Let

$$\chi_2 := (2f_{4,24} - 2f_{2,22} - f_{1,12})\omega_1^2 + 2(f_{3,14} + f_{4,13})\omega_1\omega_2 + 4(f_{1,22} + 2f_{2,22} - f_{3,23} - f_{4,24})\omega_2^2, \tag{6.77a}$$

$$\chi_1 := (f_{3,24} + f_{4,23})\omega_1 + 2(f_{3,13} - f_{4,14})\omega_2, \tag{6.77b}$$

$$\chi_0 := (-3f_{3,23} - f_{4,24})\omega_1^2 + 2(f_{3,14} + f_{4,13})\omega_1\omega_2 + 8(f_{3,23} + f_{4,24})\omega_2^2, \tag{6.77c}$$

If any of χ_2, χ_1, χ_0 does not vanish, then the system can be stabilized by a feedback u as

$$\begin{aligned} u = b_{11}(&u_{111}y_1^3 + u_{112}y_1^2y_2 + u_{122}y_1y_2^2 + u_{222}y_2^2 + u_{33}y_{33} \\ &+ u_{133}y_1y_3^2 + u_{144}y_1y_4^2 + u_{233}y_2y_3^2 + u_{244}y_2y_4^2). \end{aligned} \tag{6.78}$$

Moreover, within a bounded range of system uncertainty, if either χ_0 or χ_2 does not vanish and does not change sign, then there exists a robust control function for stabilizing the system.

Proof: By Eq. (6.77a)-(6.77c) and Appendix (6c.5),

$$V_{2,33} = -b_{11}\omega_2^2 \frac{\omega_1\chi_2 d^2 + 2\omega_1^2\omega_2\chi_1 d - \omega_2^4\chi_0}{(\omega_1(\omega_1 - 2\omega_2)(\omega_1 + 2\omega_2)(d^2 + \omega_2^2)^2)}. \tag{6.79}$$

If any of χ_2, χ_1, and χ_0 does not vanish, there exists a d such that $V_{2,33}$ is not zero. Thus, S_2 can be made negative. Also, if either χ_0 or χ_2 does not vanish and does not change sign, for d sufficiently large or sufficiently small, $V_{2,33}$ remains the same sign throughout the uncertainty range. A control function with $|u_{33}|$ sufficiently large will make S_2 negative throughout the uncertainty range. As for S_1 and S_3, from (6.76a), (6.76c), we can choose $|u_{ijk}|$ sufficiently large to make them negative. Thus, the system is stabilized.

■

Corollary 6.9. Suppose the stable eigenvalue of the system in Corollary 6.8 is linearly controllable, i.e. $b_2 \neq 0$, Let

$$\gamma_2 := \frac{(f_{1,1\varsigma} + f_{2,2\varsigma})\omega_1}{c(d^2 + \omega_2^2)^2}, \tag{6.80a}$$

$$\gamma_0 := \frac{c^2(3f_{3,3\varsigma} + f_{4,4\varsigma}) + 2c(f_{3,4\varsigma}\omega_2 + f_{4,3\varsigma}\omega_2) + 8\omega_2^2(f_{3,3\varsigma} + f_{4,4\varsigma})}{c(d^2 + \omega_2^2)^2(c^2 + 4\omega_2^2)}, \tag{6.80b}$$

and

$$\beta_2 := -b_{11}\frac{\omega_1\chi_2}{D} - b_2\gamma_2, \tag{6.81a}$$

$$\beta_1 := -b_{11}\frac{2\omega_1\omega_2\chi_1}{D}, \tag{6.81b}$$

$$\beta_0 := b_{11}\frac{\omega_2\chi_0}{D} - b_2\omega_2^4\gamma_0, \tag{6.81c}$$

where

$$D := \omega_1(\omega_1^2 - 4\omega_2^2)(d^2 + \omega_2^2)^2. \tag{6.82}$$

If any of $\beta_2, \beta_1, \beta_0$ is not zero, then the system can be stabilized by a feedback u as in Eq. (6.74). Moreover, within a bounded range of system uncertainty, if either β_0 or β_2 does not vanish and does not change sign, then there exists a robust controller that stabilizes the system.

Appendix 6.A.

Lemma 6A.1. [42] Consider a scalar bivariate function

$$V(x,y) = -ax_1^2 + cx^2y + dx^3 - ey^4 + fxy^3 + gx^2y^2 + hx^3y + kx^4.$$

If all a, c, d, e, f, g, h, k are positive, then V(x, y) is locally negative definite at origin.

$$V_{1,11} = \{\frac{(-3b_2cf_{1,1\varsigma} - b_2cf_{2,2\varsigma})\omega_1{}^2}{c^2+4\omega_1{}^2} + \frac{(-2b_2f_{1,2\varsigma} - 2b_2f_{2,1\varsigma})\omega_1{}^3}{c^2+4\omega_1{}^2} + \frac{(-8b_2f_{1,1\varsigma} - 8b_2f_{2,2\varsigma})\omega_1{}^4}{c(c^2+4\omega_1{}^2)}\} \quad (6a.1)$$

$$V_{1,12} = \{\frac{(-(b_2cf_{1,2\varsigma}) - b_2cf_{2,1\varsigma})\omega_1{}^2}{c^2+4\omega_1{}^2} + \frac{(2b_2f_{1,1\varsigma} - 2b_2f_{2,2\varsigma})\omega_1{}^3}{c^2+4\omega_1{}^2}\} \quad (6a.2)$$

$$V_{1,22} = \{\frac{(-(b_2cf_{1,1\varsigma}) - 3b_2cf_{2,2\varsigma})\omega_1{}^2}{c^2+4\omega_1{}^2} + \frac{(2b_2f_{1,2\varsigma} + 2b_2f_{2,1\varsigma})\omega_1{}^3}{c^2+4\omega_1{}^2} + \frac{(-8b_2f_{1,1\varsigma} - 8b_2f_{2,2\varsigma})\omega_1{}^4}{c(c^2+4\omega_1{}^2)}\} \quad (6a.3)$$

$$V_{2,33} = \{\frac{(-3b_2cf_{3,3\varsigma} - b_2cf_{4,4\varsigma})\omega_2{}^2}{c^2+4\omega_2{}^2} + \frac{(-2b_2f_{3,4\varsigma} - 2b_2f_{4,3\varsigma})\omega_2{}^3}{c^2+4\omega_2{}^2} + \frac{(-8b_2f_{3,3\varsigma} - 8b_2f_{4,4\varsigma})\omega_2{}^4}{c(c^2+4\omega_2{}^2)}\} \quad (6a.4)$$

$$V_{2,34} = \{\frac{b_2\omega_2{}^2(-(cf_{3,4\varsigma}) - cf_{4,3\varsigma} + 2f_{3,3\varsigma}\omega_2 - 2f_{4,4\varsigma}\omega_2)}{c^2+4\omega_2{}^2}\} \quad (6a.5)$$

$$V_{2,44} = \{\frac{(-(b_2cf_{3,3\varsigma}) - 3b_2cf_{4,4\varsigma})\omega_2{}^2}{c^2+4\omega_2{}^2} + \frac{(2b_2f_{3,4\varsigma} + 2b_2f_{4,3\varsigma})\omega_2{}^3}{c^2+4\omega_2{}^2} + \frac{(-8b_2f_{3,3\varsigma} - 8b_2f_{4,4\varsigma})\omega_2{}^4}{c(c^2+4\omega_2{}^2)}\} \quad (6a.6)$$

$$V_{3,33} = \{-\frac{b_2(f_{1,1\varsigma} + f_{2,2\varsigma})\omega_1}{c}\} \quad (6a.7)$$

$$V_{3,44} = \{-\frac{b_2(f_{1,1\varsigma} + f_{2,2\varsigma})\omega_1}{c}\} \quad (6a.8)$$

$$V_{3,13} = b_2\omega_1\{\frac{(c^3 f_{1,3\zeta} + c^2 f_{2,3\zeta}\omega_1 + c f_{1,3\zeta}{\omega_1}^2 + f_{2,3\zeta}{\omega_1}^3)}{D}$$
$$+ \frac{(c^2 f_{1,4\zeta}\omega_2 + 2c f_{2,4\zeta}\omega_1\omega_2 - f_{1,4\zeta}{\omega_1}^2\omega_2)}{D}$$
$$+ \frac{(c f_{1,3\zeta}{\omega_2}^2 - f_{2,3\zeta}\omega_1{\omega_2}^2 + f_{1,4\zeta}{\omega_2}^3)}{D}\} \tag{6a.9}$$

$$V_{3,14} = b_2\omega_1\{\frac{(c^3 f_{1,4\zeta} + c^2 f_{2,4\zeta}\omega_1 + c f_{1,4\zeta}{\omega_1}^2 + f_{2,4\zeta}{\omega_1}^3)}{D}$$
$$+ \frac{(-c^2 f_{1,3\zeta}\omega_2 - 2c f_{2,3\zeta}\omega_1\omega_2 + f_{1,3\zeta}{\omega_1}^2\omega_2)}{D}$$
$$+ \frac{(c f_{1,4\zeta}{\omega_2}^2 - f_{2,4\zeta}\omega_1{\omega_2}^2 - f_{1,3\zeta}{\omega_2}^3)}{D}\} \tag{6a.10}$$

$$V_{3,23} = b_2\omega_1\{-\frac{(-(c^3 f_{2,3\zeta}) + c^2 f_{1,3\zeta}\omega_1 - c f_{2,3\zeta}{\omega_1}^2 + f_{1,3\zeta}{\omega_1}^3)}{D}$$
$$- \frac{(-c^2 f_{2,4\zeta}\omega_2 + 2c f_{1,4\zeta}\omega_1\omega_2 + f_{2,4\zeta}{\omega_1}^2\omega_2)}{D}$$
$$- \frac{(-c f_{2,3\zeta}{\omega_2}^2 - f_{1,3\zeta}\omega_1{\omega_2}^2 - f_{2,4\zeta}{\omega_2}^3)}{D}\} \tag{6a.11}$$

$$V_{3,24} = b_2\omega_1\{-\frac{(-(c^3 f_{2,4\zeta}) + c^2 f_{1,4\zeta}\omega_1 - c f_{2,4\zeta}{\omega_1}^2 + f_{1,4\zeta}{\omega_1}^3 - c^2 f_{2,3\zeta}\omega_2)}{D}$$
$$- \frac{(-2c f_{1,3\zeta}\omega_1\omega_2 - f_{2,3\zeta}{\omega_1}^2\omega_2 - c f_{2,4\zeta}{\omega_2}^2 - f_{1,4\zeta}\omega_1{\omega_2}^2 + f_{2,3\zeta}{\omega_2}^3)}{D}\} \tag{6a.12}$$

$$V_{4,11} = \{-\frac{b_2 (f_{3,3\zeta} + f_{4,4\zeta})\,\omega_2}{c}\} \tag{6a.13}$$

$$V_{4,22} = \{-\frac{b_2 (f_{3,3\zeta} + f_{4,4\zeta})\,\omega_2}{c}\} \tag{6a.14}$$

$$V_{4,13} = b_2\omega_2\{\frac{(c^3 f_{3,1\zeta} + c^2 f_{3,2\zeta}\omega_1 + c f_{3,1\zeta}{\omega_1}^2 + f_{3,2\zeta}{\omega_1}^3 c^2 f_{4,1\zeta}\omega_2)}{D}$$
$$+ \frac{(2c f_{4,2\zeta}\omega_1\omega_2 - f_{4,1\zeta}{\omega_1}^2\omega_2 + c f_{3,1\zeta}{\omega_2}^2 - f_{3,2\zeta}\omega_1{\omega_2}^2 + f_{4,1\zeta}{\omega_2}^3)}{D}\} \tag{6a.15}$$

$$V_{4,14} = b_2\omega_2\{-\frac{\left(-\left(c^3 f_{4,1\zeta}\right) - c^2 f_{4,2\zeta}\omega_1 - c f_{4,1\zeta}{\omega_1}^2 - f_{4,2\zeta}{\omega_1}^3 - 2c f_{3,2\zeta}\omega_1\omega_2\right)}{D}$$

$$- \frac{\left(c^2 f_{3,1\zeta}\omega_2 + - f_{3,1\zeta}{\omega_1}^2\omega_2 - c f_{4,1\zeta}{\omega_2}^2 + f_{4,2\zeta}\omega_1{\omega_2}^2 + f_{3,1\zeta}{\omega_2}^3\right)}{D}\}$$

$$(6a.16)$$

$$V_{4,23} = b_2\omega_2\{\frac{\left(c^3 f_{3,2\zeta} - c^2 f_{3,1\zeta}\omega_1 + c f_{3,2\zeta}{\omega_1}^2 - f_{3,1\zeta}{\omega_1}^3 - 2c f_{4,1\zeta}\omega_1\omega_2\right)}{D}$$

$$+ \frac{\left(c^2 f_{4,2\zeta}\omega_2 - f_{4,2\zeta}{\omega_1}^2\omega_2 + c f_{3,2\zeta}{\omega_2}^2 + f_{3,1\zeta}\omega_1{\omega_2}^2 + f_{4,2\zeta}{\omega_2}^3\right)}{D}\} \quad (6a.17)$$

$$V_{4,24} = b_2\omega_2\{-\frac{\left(-\left(c^3 f_{4,2\zeta}\right) + c^2 f_{4,1\zeta}\omega_1 - c f_{4,2\zeta}{\omega_1}^2 + f_{4,1\zeta}{\omega_1}^3 + 2c f_{3,1\zeta}\omega_1\omega_2\right)}{D}$$

$$- \frac{\left(c^2 f_{3,2\zeta}\omega_2 - f_{3,2\zeta}{\omega_1}^2\omega_2 - c f_{4,2\zeta}{\omega_2}^2 - f_{4,1\zeta}\omega_1{\omega_2}^2 + f_{3,2\zeta}{\omega_2}^3\right)}{D}\} \quad (6a.18)$$

$$V_{1,00} = (-f_{1,11}f_{1,12} - f_{1,12}f_{1,22} + 2f_{1,11}f_{2,11} + f_{2,11}f_{2,12} - 2f_{1,22}f_{2,22}$$

$$+ f_{2,12}f_{2,22})\omega_1 + (3cf_{1,111} + cf_{1,122} + cf_{2,112} + 3cf_{2,222} - 3f_{1,1\zeta}g_{11}$$

$$-f_{2,2\zeta}g_{11} - f_{1,1\zeta}g_{22} - 3f_{2,2\zeta}g_{22})\frac{{\omega_1}^2}{c} + \frac{\left(-\left(cf_{1,2\zeta}g_{12}\right) - cf_{2,1\zeta}g_{12}\right){\omega_1}^2}{c^2 + 4{\omega_1}^2}$$

$$+ \frac{2(-f_{1,2\zeta}g_{11} - f_{2,1\zeta}g_{11} + f_{1,1\zeta}g_{12} - f_{2,2\zeta}g_{12} + f_{1,2\zeta}g_{22} + f_{2,1\zeta}g_{22}){\omega_1}^3}{c^2 + 4{\omega_1}^2}$$

$$+ \frac{\left(4f_{1,1\zeta}g_{11} - 4f_{2,2\zeta}g_{11} - 4f_{1,1\zeta}g_{22} + 4f_{2,2\zeta}g_{22}\right){\omega_1}^4}{c\left(c^2 + 4{\omega_1}^2\right)}$$

$$+ (-3f_{1,14}f_{3,11} - f_{2,24}f_{3,11} - f_{1,14}f_{3,22} - 3f_{2,24}f_{3,22}$$

$$+ 3f_{1,13}f_{4,11} + f_{2,23}f_{4,11} + f_{1,13}f_{4,22} + 3f_{2,23}f_{4,22})\frac{{\omega_1}^2}{\omega_2}$$

$$+ (-f_{1,23}f_{3,11} - f_{2,13}f_{3,11} + f_{1,13}f_{3,12} - f_{2,23}f_{3,12} + f_{1,23}f_{3,22}$$

$$+f_{2,13}f_{3,22} - f_{1,24}f_{4,11} - f_{2,14}f_{4,11} + f_{1,14}f_{4,12} - f_{2,24}f_{4,12} + f_{1,24}f_{4,22}$$

$$+ f_{2,14}f_{4,22})\frac{2{\omega_1}^3}{4{\omega_1}^2 - {\omega_2}^2} + (f_{1,14}f_{3,11} - f_{2,24}f_{3,11} - f_{1,14}f_{3,22} + f_{2,24}f_{3,22}$$

$$- f_{1,13}f_{4,11} + f_{2,23}f_{4,11} + f_{1,13}f_{4,22} - f_{2,23}f_{4,22})\frac{4{\omega_1}^4}{\omega_2\left(4{\omega_1}^2 - {\omega_2}^2\right)}$$

$$+\frac{(f_{1,24}f_{3,12}+f_{2,14}f_{3,12}-f_{1,23}f_{4,12}-f_{2,13}f_{4,12})\omega_1{}^2\omega_2}{4\omega_1{}^2-\omega_2{}^2} \tag{6a.19}$$

$$\begin{aligned}
V_{2,00} = & (-f_{3,33}f_{3,34}-f_{3,34}f_{3,44}+2f_{3,33}f_{4,33}+f_{4,33}f_{4,34}-2f_{3,44}f_{4,44}\\
& +f_{4,34}f_{4,44})\omega_2+(3cf_{3,333}+cf_{3,344}+cf_{4,334}+3cf_{4,444}\\
& -3f_{3,3\zeta}g_{33}-f_{4,4\zeta}g_{33}-f_{3,3\zeta}g_{44}-3f_{4,4\zeta}g_{44})\frac{\omega_2{}^2}{c}\\
& +(3f_{2,33}f_{3,13}+f_{2,44}f_{3,13}-3f_{1,33}f_{3,23}-f_{1,44}f_{3,23}+f_{2,33}f_{4,14}\\
& +3f_{2,44}f_{4,14}-f_{1,33}f_{4,24}-3f_{1,44}f_{4,24})\frac{\omega_2{}^2}{\omega_1}-\frac{(cf_{3,4\zeta}g_{34}+cf_{4,3\zeta}g_{34})\omega_2{}^2}{c^2+4\omega_2{}^2}\\
& +\frac{2(-f_{3,4\zeta}g_{33}-f_{4,3\zeta}g_{33}+f_{3,3\zeta}g_{34}-f_{4,4\zeta}g_{34}+f_{3,4\zeta}g_{44}+f_{4,3\zeta}g_{44})\omega_2{}^3}{c^2+4\omega_2{}^2}\\
& +\frac{(4f_{3,3\zeta}g_{33}-4f_{4,4\zeta}g_{33}-4f_{3,3\zeta}g_{44}+4f_{4,4\zeta}g_{44})\omega_2{}^4}{c(c^2+4\omega_2{}^2)}\\
& +\frac{(-(f_{2,34}f_{3,14})+f_{1,34}f_{3,24}-f_{2,34}f_{4,13}+f_{1,34}f_{4,23})\omega_1\omega_2{}^2}{-\omega_1{}^2+4\omega_2{}^2}\\
& +(f_{1,34}f_{3,13}-f_{1,33}f_{3,14}+f_{1,44}f_{3,14}+f_{2,34}f_{3,23}-f_{2,33}f_{3,24}+f_{2,44}f_{3,24}\\
& -f_{1,33}f_{4,13}+f_{1,44}f_{4,13}-f_{1,34}f_{4,14}-f_{2,33}f_{4,23}+f_{2,44}f_{4,23}\\
& -f_{2,34}f_{4,24})\frac{2\omega_2{}^3}{-\omega_1{}^2+4\omega_2{}^2}\\
& +(-f_{2,33}f_{3,13}+f_{2,44}f_{3,13}+f_{1,33}f_{3,23}-f_{1,44}f_{3,23}\\
& +f_{2,33}f_{4,14}-f_{2,44}f_{4,14}-f_{1,33}f_{4,24}+f_{1,44}f_{4,24})\frac{4\omega_2{}^4}{\omega_1(-\omega_1{}^2+4\omega_2{}^2)}
\end{aligned} \tag{6a.20}$$

$$\begin{aligned}
V_{3,00} = & -f_{1,12}f_{1,33}-f_{1,12}f_{1,44}-2f_{1,33}f_{2,22}-2f_{1,44}f_{2,22}\\
& +2f_{1,11}f_{2,33}+f_{2,12}f_{2,33}+2f_{1,11}f_{2,44}+f_{2,12}f_{2,44}\\
& -2f_{2,33}f_{3,13}+2f_{1,33}f_{3,23}-2f_{2,44}f_{4,14}+2f_{1,44}f_{4,24}\\
& +(cf_{1,133}+cf_{1,144}+cf_{2,233}+cf_{2,244}-f_{1,1\zeta}g_{33}-f_{2,2\zeta}g_{33}\\
& -f_{1,1\zeta}g_{44}-f_{2,2\zeta}g_{44})\frac{\omega_1}{c}
\end{aligned}$$

$$
\begin{aligned}
&+(-f_{1,14}f_{3,33}-f_{2,24}f_{3,33}-f_{1,14}f_{3,44}-f_{2,24}f_{3,44}+f_{1,13}f_{4,33}\\
&+f_{2,23}f_{4,33}+f_{1,13}f_{4,44}+f_{2,23}f_{4,44})\frac{\omega_1}{\omega_2}\\
&+\frac{(c^3f_{1,3\zeta}g_{13}+c^3f_{1,4\zeta}g_{14}+c^3f_{2,3\zeta}g_{23}+c^3f_{2,4\zeta}g_{24})\omega_1}{D}\\
&+\frac{(c^2f_{2,3\zeta}g_{13}+c^2f_{2,4\zeta}g_{14}-c^2f_{1,3\zeta}g_{23}-c^2f_{1,4\zeta}g_{24})\omega_1{}^2}{D}\\
&+\frac{(cf_{1,3\zeta}g_{13}+cf_{1,4\zeta}g_{14}+cf_{2,3\zeta}g_{23}+cf_{2,4\zeta}g_{24})\omega_1{}^3}{D}\\
&+\frac{(f_{2,3\zeta}g_{13}+f_{2,4\zeta}g_{14}-f_{1,3\zeta}g_{23}-f_{1,4\zeta}g_{24})\omega_1{}^4}{D}\\
&+\frac{(c^2f_{1,4\zeta}g_{13}-c^2f_{1,3\zeta}g_{14}+c^2f_{2,4\zeta}g_{23}-c^2f_{2,3\zeta}g_{24})\omega_1\omega_2}{D}\\
&+\frac{(2cf_{2,4\zeta}g_{13}-2cf_{2,3\zeta}g_{14}-2cf_{1,4\zeta}g_{23}+2cf_{1,3\zeta}g_{24})\omega_1{}^2\omega_2}{D}\\
&+\frac{(-(f_{1,4\zeta}g_{13})+f_{1,3\zeta}g_{14}-f_{2,4\zeta}g_{23}+f_{2,3\zeta}g_{24})\omega_1{}^3\omega_2}{D}\\
&+\frac{(cf_{1,3\zeta}g_{13}+cf_{1,4\zeta}g_{14}+cf_{2,3\zeta}g_{23}+cf_{2,4\zeta}g_{24})\omega_1\omega_2{}^2}{D}\\
&+\frac{(-(f_{2,3\zeta}g_{13})-f_{2,4\zeta}g_{14}+f_{1,3\zeta}g_{23}+f_{1,4\zeta}g_{24})\omega_1{}^2\omega_2{}^2}{D}\\
&+\frac{(f_{1,4\zeta}g_{13}-f_{1,3\zeta}g_{14}+f_{2,4\zeta}g_{23}-f_{2,3\zeta}g_{24})\omega_1\omega_2{}^3}{D}\\
&+\frac{(f_{2,34}f_{3,14}-f_{1,34}f_{3,24}+f_{2,34}f_{4,13}-f_{1,34}f_{4,23})\omega_1{}^2}{-\omega_1{}^2+4\omega_2{}^2}\\
&+(-f_{1,34}f_{3,13}+f_{1,33}f_{3,14}-f_{1,44}f_{3,14}-f_{2,34}f_{3,23}+f_{2,33}f_{3,24}\\
&-f_{2,44}f_{3,24}+f_{1,33}f_{4,13}-f_{1,44}f_{4,13}f_{1,34}f_{4,14}+f_{2,33}f_{4,23}-f_{2,44}f_{4,23}\\
&+f_{2,34}f_{4,24})\frac{2\omega_1\omega_2}{-\omega_1{}^2+4\omega_2{}^2}+(f_{2,33}f_{3,13}-f_{2,44}f_{3,13}-f_{1,33}f_{3,23}\\
&+f_{1,44}f_{3,23}-f_{2,33}f_{4,14}+f_{2,44}f_{4,14}+f_{1,33}f_{4,24})\frac{4\omega_2{}^2}{-\omega_1{}^2+4\omega_2{}^2} \qquad (6a.21)
\end{aligned}
$$

$$
V_{4,00}=2f_{1,14}f_{3,11}+2f_{2,24}f_{3,22}-f_{3,11}f_{3,34}-f_{3,22}f_{3,34}
$$

$$
\begin{aligned}
&- 2f_{1,13}f_{4,11} + 2f_{3,33}f_{4,11} - 2f_{2,23}f_{4,22} + 2f_{3,33}f_{4,22} \\
&+ f_{4,11}f_{4,34} + f_{4,22}f_{4,34} - 2f_{3,11}f_{4,44} - 2f_{3,22}f_{4,44} \\
&+ (cf_{3,113} + cf_{3,223} + cf_{4,114} + cf_{4,224} - f_{3,3\zeta}g_{11} \\
&- f_{4,4\zeta}g_{11} - f_{3,3\zeta}g_{22} - f_{4,4\zeta}g_{22})\frac{\omega_2}{c} \\
&+ (f_{2,11}f_{3,13} + f_{2,22}f_{3,13} - f_{1,11}f_{3,23} - f_{1,22}f_{3,23} + f_{2,11}f_{4,14} \\
&+ f_{2,22}f_{4,14} - f_{1,11}f_{4,24} - f_{1,22}f_{4,24})\frac{\omega_2}{\omega_1} \\
&+ (-f_{1,14}f_{3,11} + f_{2,24}f_{3,11} + f_{1,14}f_{3,22} - f_{2,24}f_{3,22} + f_{1,13}f_{4,11} \\
&- f_{2,23}f_{4,11} - f_{1,13}f_{4,22} + f_{2,23}f_{4,22})\frac{4{\omega_1}^2}{4{\omega_1}^2 - {\omega_2}^2} \\
&+ (f_{1,23}f_{3,11} + f_{2,13}f_{3,11} - f_{1,13}f_{3,12} + f_{2,23}f_{3,12} \\
&- f_{1,23}f_{3,22} - f_{2,13}f_{3,22} + f_{1,24}f_{4,11} + f_{2,14}f_{4,11} - f_{1,14}f_{4,12} \\
&+ f_{2,24}f_{4,12} - f_{1,24}f_{4,22} - f_{2,14}f_{4,22})\frac{2\omega_1\omega_2}{4{\omega_1}^2 - {\omega_2}^2} \\
&+ \frac{(-(f_{1,24}f_{3,12}) - f_{2,14}f_{3,12} + f_{1,23}f_{4,12} + f_{2,13}f_{4,12}){\omega_2}^2}{4{\omega_1}^2 - {\omega_2}^2} \\
&+ \frac{(c^3f_{3,1\zeta}g_{13} + c^3f_{4,1\zeta}g_{14} + c^3f_{3,2\zeta}g_{23} + c^3f_{4,2\zeta}g_{24})\omega_2}{D} \\
&+ \frac{(c^2f_{3,2\zeta}g_{13} + c^2f_{4,2\zeta}g_{14} - c^2f_{3,1\zeta}g_{23} - c^2f_{4,1\zeta}g_{24})\omega_1\omega_2}{D} \\
&+ \frac{(cf_{3,1\zeta}g_{13} + cf_{4,1\zeta}g_{14} + cf_{3,2\zeta}g_{23} + cf_{4,2\zeta}g_{24}){\omega_1}^2\omega_2}{D} \\
&+ \frac{(f_{3,2\zeta}g_{13} + f_{4,2\zeta}g_{14} - f_{3,1\zeta}g_{23} - f_{4,1\zeta}g_{24}){\omega_1}^3\omega_2}{D} \\
&+ \frac{(c^2f_{4,1\zeta}g_{13} - c^2f_{3,1\zeta}g_{14} + c^2f_{4,2\zeta}g_{23} - c^2f_{3,2\zeta}g_{24}){\omega_2}^2}{D} \\
&+ \frac{(2cf_{4,2\zeta}g_{13} - 2cf_{3,2\zeta}g_{14} - 2cf_{4,1\zeta}g_{23} + 2cf_{3,1\zeta}g_{24})\omega_1{\omega_2}^2}{D} \\
&+ \frac{(-(f_{4,1\zeta}g_{13}) + f_{3,1\zeta}g_{14} - f_{4,2\zeta}g_{23} + f_{3,2\zeta}g_{24}){\omega_1}^2{\omega_2}^2}{D}
\end{aligned}
$$

$$+\frac{(cf_{3,1\zeta}g_{13}+cf_{4,1\zeta}g_{14}+cf_{3,2\zeta}g_{23}+cf_{4,2\zeta}g_{24}){\omega_2}^3}{D}$$
$$+\frac{(-(f_{3,2\zeta}g_{13})-f_{4,2\zeta}g_{14}+f_{3,1\zeta}g_{23}+f_{4,1\zeta}g_{24})\omega_1{\omega_2}^3}{D}$$
$$+\frac{(f_{4,1\zeta}g_{13}-f_{3,1\zeta}g_{14}+f_{4,2\zeta}g_{23}-f_{3,2\zeta}g_{24}){\omega_2}^4}{D} \tag{6a.22}$$
$$D=(c^2+(\omega_1-\omega_2)^2)(c^2+(\omega_1+\omega_2)^2) \tag{6a.23}$$

Appendix 6.B.

$$\begin{aligned}\tilde{V}_{1,11}=\{&-(c^2d^2f_{1,1\zeta}+3c^2d^2f_{2,2\zeta}+2c^2df_{1,2\zeta}\omega_1-2cd^2f_{1,2\zeta}\omega_1\\&+2c^2df_{2,1\zeta}\omega_1-2cd^2f_{2,1\zeta}\omega_1+3c^2f_{1,1\zeta}{\omega_1}^2-4cdf_{1,1\zeta}{\omega_1}^2\\&+8d^2f_{1,1\zeta}{\omega_1}^2+c^2f_{2,2\zeta}{\omega_1}^2+4cdf_{2,2\zeta}{\omega_1}^2+8d^2f_{2,2\zeta}{\omega_1}^2\\&+2cf_{1,2\zeta}{\omega_1}^3+2cf_{2,1\zeta}{\omega_1}^3+8f_{1,1\zeta}{\omega_1}^4\\&+8f_{2,2\zeta}{\omega_1}^4)\frac{b2\omega_1^4}{c(d^2+{\omega_1}^2)^2(c^2+4{\omega_1}^2)}\}\end{aligned} \tag{6b.1}$$

$$\begin{aligned}\tilde{V}_{1,12}=\{&(cd^2f_{1,2\zeta}+cd^2f_{2,1\zeta}+2cdf_{1,1\zeta}\omega_1-2d^2f_{1,1\zeta}\omega_1\\&-2cdf_{2,2\zeta}\omega_1+2d^2f_{2,2\zeta}\omega_1-cf_{1,2\zeta}{\omega_1}^2+4df_{1,2\zeta}{\omega_1}^2\\&-cf_{2,1\zeta}{\omega_1}^2+4df_{2,1\zeta}{\omega_1}^2+2f_{1,1\zeta}{\omega_1}^3\\&-2f_{2,2\zeta}{\omega_1}^3)\frac{b2{\omega_1}^4}{(d^2+{\omega_1}^2)^2(c^2+4{\omega_1}^2)}\}\end{aligned} \tag{6b.2}$$

$$\begin{aligned}\tilde{V}_{1,22}=\{&-(3c^2d^2f_{1,1\zeta}+c^2d^2f_{2,2\zeta}-2c^2df_{1,2\zeta}\omega_1+2cd^2f_{1,2\zeta}\omega_1\\&-2c^2df_{2,1\zeta}\omega_1+2cd^2f_{2,1\zeta}\omega_1+c^2f_{1,1\zeta}{\omega_1}^2+4cdf_{1,1\zeta}{\omega_1}^2\\&+8d^2f_{1,1\zeta}{\omega_1}^2+3c^2f_{2,2\zeta}{\omega_1}^2-4cdf_{2,2\zeta}{\omega_1}^2+8d^2f_{2,2\zeta}{\omega_1}^2\\&-2cf_{1,2\zeta}{\omega_1}^3-2cf_{2,1\zeta}{\omega_1}^3+8f_{1,1\zeta}{\omega_1}^4\\&+8f_{2,2\zeta}{\omega_1}^4)\frac{b2{\omega_1}^4}{c(d^2+{\omega_1}^2)^2(c^2+4{\omega_1}^2)}\}\end{aligned} \tag{6b.3}$$

$$\tilde{V}_{2,33}=\{-(c^2d^2f_{3,3\zeta}+3c^2d^2f_{4,4\zeta}+2c^2df_{3,4\zeta}\omega_2-2cd^2f_{3,4\zeta}\omega_2$$

$$+ 2c^2df_{4,3\zeta}\omega_2 - 2cd^2f_{4,3\zeta}\omega_2 + 3c^2f_{3,3\zeta}{\omega_2}^2 - 4cdf_{3,3\zeta}{\omega_2}^2$$

$$+ 8d^2f_{3,3\zeta}{\omega_2}^2 + c^2f_{4,4\zeta}{\omega_2}^2 + 4cdf_{4,4\zeta}{\omega_2}^2 + 8d^2f_{4,4\zeta}{\omega_2}^2$$

$$+ 2cf_{3,4\zeta}{\omega_2}^3 + 2cf_{4,3\zeta}{\omega_2}^3 + 8f_{3,3\zeta}{\omega_2}^4$$

$$+ 8f_{4,4\zeta}{\omega_2}^4)\frac{b2{\omega_2}^4}{c(d^2+{\omega_2}^2)^2(c^2+4{\omega_2}^2)}\} \tag{6b.4}$$

$$\tilde{V}_{2,34} = \{(cd^2f_{3,4\zeta} + cd^2f_{4,3\zeta} + 2cdf_{3,3\zeta}\omega_2 - 2d^2f_{3,3\zeta}\omega_2$$

$$- 2cdf_{4,4\zeta}\omega_2 + 2d^2f_{4,4\zeta}\omega_2 - cf_{3,4\zeta}{\omega_2}^2 + 4df_{3,4\zeta}{\omega_2}^2$$

$$- cf_{4,3\zeta}{\omega_2}^2 + 4df_{4,3\zeta}{\omega_2}^2 + 2f_{3,3\zeta}{\omega_2}^3$$

$$- 2f_{4,4\zeta}{\omega_2}^3)\frac{b2{\omega_2}^4}{(d^2+{\omega_2}^2)^2(c^2+4{\omega_2}^2)}\} \tag{6b.5}$$

$$\tilde{V}_{2,44} = \{-(3c^2d^2f_{3,3\zeta} + c^2d^2f_{4,4\zeta} - 2c^2df_{3,4\zeta}\omega_2 + 2cd^2f_{3,4\zeta}\omega_2$$

$$- 2c^2df_{4,3\zeta}\omega_2 + 2cd^2f_{4,3\zeta}\omega_2 + c^2f_{3,3\zeta}{\omega_2}^2 + 4cdf_{3,3\zeta}{\omega_2}^2$$

$$+ 8d^2f_{3,3\zeta}{\omega_2}^2 + 3c^2f_{4,4\zeta}{\omega_2}^2 - 4cdf_{4,4\zeta}{\omega_2}^2 + 8d^2f_{4,4\zeta}{\omega_2}^2$$

$$- 2cf_{3,4\zeta}{\omega_2}^3 - 2cf_{4,3\zeta}{\omega_2}^3 + 8f_{3,3\zeta}{\omega_2}^4 +$$

$$8f_{4,4\zeta}{\omega_2}^4)\frac{b2{\omega_2}^4}{c(d^2+{\omega_2}^2)^2(c^2+4{\omega_2}^2)}\} \tag{6b.6}$$

$$\tilde{V}_{3,33} = \{-\frac{b2(f_{1,1\zeta}+f_{2,2\zeta})\omega_1{\omega_2}^2}{c(d^2+{\omega_2}^2)}\} \tag{6b.7}$$

$$\tilde{V}_{3,44} = \{-\frac{b2(f_{1,1\zeta}+f_{2,2\zeta})\omega_1{\omega_2}^2}{c(d^2+{\omega_2}^2)}\} \tag{6b.8}$$

$$\tilde{V}_{3,13} = \{(c^3d^2f_{2,4\zeta} + c^3df_{1,4\zeta}\omega_1 - c^2d^2f_{1,4\zeta}\omega_1 + c^2df_{2,4\zeta}{\omega_1}^2$$

$$+ cd^2f_{2,4\zeta}{\omega_1}^2 + cdf_{1,4\zeta}{\omega_1}^3 - d^2f_{1,4\zeta}{\omega_1}^3 + df_{2,4\zeta}{\omega_1}^4$$

$$+ c^3df_{2,3\zeta}\omega_2 - c^2d^2f_{2,3\zeta}\omega_2 + c^3f_{1,3\zeta}\omega_1\omega_2 - 2c^2df_{1,3\zeta}\omega_1\omega_2$$

$$+ 2cd^2f_{1,3\zeta}\omega_1\omega_2 + c^2f_{2,3\zeta}{\omega_1}^2\omega_2 - cdf_{2,3\zeta}{\omega_1}^2\omega_2 + d^2f_{2,3\zeta}{\omega_1}^2\omega_2$$

$$+ cf_{1,3\zeta}{\omega_1}^3\omega_2 + f_{2,3\zeta}{\omega_1}^4\omega_2 + c^2df_{2,4\zeta}{\omega_2}^2 + cd^2f_{2,4\zeta}{\omega_2}^2$$

$$+ c^2f_{1,4\zeta}\omega_1{\omega_2}^2 - cdf_{1,4\zeta}\omega_1{\omega_2}^2 + d^2f_{1,4\zeta}\omega_1{\omega_2}^2 + 2cf_{2,4\zeta}{\omega_1}^2{\omega_2}^2$$

$$- 2df_{2,4\zeta}{\omega_1}^2{\omega_2}^2 - f_{1,4\zeta}{\omega_1}^3{\omega_2}^2 + cdf_{2,3\zeta}{\omega_2}^3 - d^2f_{2,3\zeta}{\omega_2}^3$$

$$+ cf_{1,3\zeta}\omega_1{\omega_2}^3 - f_{2,3\zeta}{\omega_1}^2{\omega_2}^3 + df_{2,4\zeta}{\omega_2}^4 + f_{1,4\zeta}\omega_1{\omega_2}^4)$$

$$\frac{b2{\omega_1}^2\omega_2}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)(c^2+{\omega_1}^2-2\omega_1\omega_2+{\omega_2}^2)(c^2+{\omega_1}^2+2\omega_1\omega_2+{\omega_2}^2)}\} \tag{6b.9}$$

$$\tilde{V}_{3,14} = \{(-(c^3d^2f_{2,3\zeta}) - c^3df_{1,3\zeta}\omega_1 + c^2d^2f_{1,3\zeta}\omega_1 - c^2df_{2,3\zeta}{\omega_1}^2$$

$$- cd^2f_{2,3\zeta}{\omega_1}^2 - cdf_{1,3\zeta}{\omega_1}^3 + d^2f_{1,3\zeta}{\omega_1}^3 - df_{2,3\zeta}{\omega_1}^4$$

$$+ c^3df_{2,4\zeta}\omega_2 - c^2d^2f_{2,4\zeta}\omega_2 + c^3f_{1,4\zeta}\omega_1\omega_2 - 2c^2df_{1,4\zeta}\omega_1\omega_2$$

$$+ 2cd^2f_{1,4\zeta}\omega_1\omega_2 + c^2f_{2,4\zeta}{\omega_1}^2\omega_2 - cdf_{2,4\zeta}{\omega_1}^2\omega_2 + d^2f_{2,4\zeta}{\omega_1}^2\omega_2$$

$$+ cf_{1,4\zeta}{\omega_1}^3\omega_2 + f_{2,4\zeta}{\omega_1}^4\omega_2 - c^2df_{2,3\zeta}{\omega_2}^2 - cd^2f_{2,3\zeta}{\omega_2}^2$$

$$- c^2f_{1,3\zeta}\omega_1{\omega_2}^2 + cdf_{1,3\zeta}\omega_1{\omega_2}^2 - d^2f_{1,3\zeta}\omega_1{\omega_2}^2 - 2cf_{2,3\zeta}{\omega_1}^2{\omega_2}^2$$

$$+ 2df_{2,3\zeta}{\omega_1}^2{\omega_2}^2 + f_{1,3\zeta}{\omega_1}^3{\omega_2}^2 + cdf_{2,4\zeta}{\omega_2}^3 - d^2f_{2,4\zeta}{\omega_2}^3$$

$$+ cf_{1,4\zeta}\omega_1{\omega_2}^3 - f_{2,4\zeta}{\omega_1}^2{\omega_2}^3 - df_{2,3\zeta}{\omega_2}^4 - f_{1,3\zeta}\omega_1{\omega_2}^4)$$

$$\frac{b2{\omega_1}^2\omega_2}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)(c^2+{\omega_1}^2-2\omega_1\omega_2+{\omega_2}^2)(c^2+{\omega_1}^2+2\omega_1\omega_2+{\omega_2}^2)}\} \tag{6b.10}$$

$$\tilde{V}_{3,23} = \{-(c^3d^2f_{1,4\zeta} - c^3df_{2,4\zeta}\omega_1 + c^2d^2f_{2,4\zeta}\omega_1 + c^2df_{1,4\zeta}{\omega_1}^2$$

$$+ cd^2f_{1,4\zeta}{\omega_1}^2 - cdf_{2,4\zeta}{\omega_1}^3 + d^2f_{2,4\zeta}{\omega_1}^3 + df_{1,4\zeta}{\omega_1}^4$$

$$+ c^3df_{1,3\zeta}\omega_2 - c^2d^2f_{1,3\zeta}\omega_2 - c^3f_{2,3\zeta}\omega_1\omega_2 + 2c^2df_{2,3\zeta}\omega_1\omega_2$$

$$- 2cd^2f_{2,3\zeta}\omega_1\omega_2 + c^2f_{1,3\zeta}{\omega_1}^2\omega_2 - cdf_{1,3\zeta}{\omega_1}^2\omega_2 + d^2f_{1,3\zeta}{\omega_1}^2\omega_2$$

$$- cf_{2,3\zeta}{\omega_1}^3\omega_2 + f_{1,3\zeta}{\omega_1}^4\omega_2 + c^2df_{1,4\zeta}{\omega_2}^2 + cd^2f_{1,4\zeta}{\omega_2}^2$$

$$- c^2f_{2,4\zeta}\omega_1{\omega_2}^2 + cdf_{2,4\zeta}\omega_1{\omega_2}^2 - d^2f_{2,4\zeta}\omega_1{\omega_2}^2 + 2cf_{1,4\zeta}{\omega_1}^2{\omega_2}^2$$

$$- 2df_{1,4\zeta}{\omega_1}^2{\omega_2}^2 + f_{2,4\zeta}{\omega_1}^3{\omega_2}^2 + cdf_{1,3\zeta}{\omega_2}^3 - d^2f_{1,3\zeta}{\omega_2}^3 - cf_{2,3\zeta}\omega_1{\omega_2}^3$$

$$- f_{1,3\zeta}{\omega_1}^2{\omega_2}^3 + df_{1,4\zeta}{\omega_2}^4 - f_{2,4\zeta}\omega_1{\omega_2}^4)$$

$$\frac{b2{\omega_1}^2\omega_2}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)(c^2+{\omega_1}^2-2\omega_1\omega_2+{\omega_2}^2)(c^2+{\omega_1}^2+2\omega_1\omega_2+{\omega_2}^2)}\}$$

$$(6b.11)$$

$$\tilde{V}_{3,24} = \{-(-(c^3 d^2 f_{1,3\zeta}) + c^3 d f_{2,3\zeta}\omega_1 - c^2 d^2 f_{2,3\zeta}\omega_1 - c^2 d f_{1,3\zeta}{\omega_1}^2$$
$$- c d^2 f_{1,3\zeta}{\omega_1}^2 + c d f_{2,3\zeta}{\omega_1}^3 - d^2 f_{2,3\zeta}{\omega_1}^3 - d f_{1,3\zeta}{\omega_1}^4$$
$$+ c^3 d f_{1,4\zeta}\omega_2 - c^2 d^2 f_{1,4\zeta}\omega_2 - c^3 f_{2,4\zeta}\omega_1\omega_2 + 2c^2 d f_{2,4\zeta}\omega_1\omega_2$$
$$- 2cd^2 f_{2,4\zeta}\omega_1\omega_2 + c^2 f_{1,4\zeta}{\omega_1}^2\omega_2 - c d f_{1,4\zeta}{\omega_1}^2\omega_2 + d^2 f_{1,4\zeta}{\omega_1}^2\omega_2$$
$$- c f_{2,4\zeta}{\omega_1}^3\omega_2 + f_{1,4\zeta}{\omega_1}^4\omega_2 - c^2 d f_{1,3\zeta}{\omega_2}^2 - c d^2 f_{1,3\zeta}{\omega_2}^2$$
$$+ c^2 f_{2,3\zeta}\omega_1{\omega_2}^2 - c d f_{2,3\zeta}\omega_1{\omega_2}^2 + d^2 f_{2,3\zeta}\omega_1{\omega_2}^2 - 2c f_{1,3\zeta}{\omega_1}^2{\omega_2}^2$$
$$+ 2d f_{1,3\zeta}{\omega_1}^2{\omega_2}^2 - f_{2,3\zeta}{\omega_1}^3{\omega_2}^2 + c d f_{1,4\zeta}{\omega_2}^3 - d^2 f_{1,4\zeta}{\omega_2}^3$$
$$- c f_{2,4\zeta}\omega_1{\omega_2}^3 - f_{1,4\zeta}{\omega_1}^2{\omega_2}^3 - d f_{1,3\zeta}{\omega_2}^4 + f_{2,3\zeta}\omega_1{\omega_2}^4)$$
$$\frac{b2{\omega_1}^2\omega_2}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)(c^2+{\omega_1}^2-2\omega_1\omega_2+{\omega_2}^2)(c^2+{\omega_1}^2+2\omega_1\omega_2+{\omega_2}^2)}\}$$

$$(6b.12)$$

$$\tilde{V}_{4,11} = \{-\frac{b2(f_{3,3\zeta}+f_{4,4\zeta}){\omega_1}^2\omega_2}{c(d^2+{\omega_1}^2)}\} \qquad (6b.13)$$

$$\tilde{V}_{4,22} = \{-\frac{b2(f_{3,3\zeta}+f_{4,4\zeta}){\omega_1}^2\omega_2}{c(d^2+{\omega_1}^2)}\} \qquad (6b.14)$$

$$\tilde{V}_{4,13} = \{(c^3 d^2 f_{4,2\zeta} + c^3 d f_{4,1\zeta}\omega_1 - c^2 d^2 f_{4,1\zeta}\omega_1 + c^2 d f_{4,2\zeta}{\omega_1}^2$$
$$+ c d^2 f_{4,2\zeta}{\omega_1}^2 + c d f_{4,1\zeta}{\omega_1}^3 - d^2 f_{4,1\zeta}{\omega_1}^3 + d f_{4,2\zeta}{\omega_1}^4$$
$$+ c^3 d f_{3,2\zeta}\omega_2 - c^2 d^2 f_{3,2\zeta}\omega_2 + c^3 f_{3,1\zeta}\omega_1\omega_2 - 2c^2 d f_{3,1\zeta}\omega_1\omega_2$$
$$+ 2cd^2 f_{3,1\zeta}\omega_1\omega_2 + c^2 f_{3,2\zeta}{\omega_1}^2\omega_2 - c d f_{3,2\zeta}{\omega_1}^2\omega_2 + d^2 f_{3,2\zeta}{\omega_1}^2\omega_2$$
$$+ c f_{3,1\zeta}{\omega_1}^3\omega_2 + f_{3,2\zeta}{\omega_1}^4\omega_2 + c^2 d f_{4,2\zeta}{\omega_2}^2 + c d^2 f_{4,2\zeta}{\omega_2}^2$$
$$+ c^2 f_{4,1\zeta}\omega_1{\omega_2}^2 - c d f_{4,1\zeta}\omega_1{\omega_2}^2 + d^2 f_{4,1\zeta}\omega_1{\omega_2}^2 + 2c f_{4,2\zeta}{\omega_1}^2{\omega_2}^2$$
$$- 2d f_{4,2\zeta}{\omega_1}^2{\omega_2}^2 - f_{4,1\zeta}{\omega_1}^3{\omega_2}^2 + c d f_{3,2\zeta}{\omega_2}^3 - d^2 f_{3,2\zeta}{\omega_2}^3$$
$$+ c f_{3,1\zeta}\omega_1{\omega_2}^3 - f_{3,2\zeta}{\omega_1}^2{\omega_2}^3 + d f_{4,2\zeta}{\omega_2}^4 + f_{4,1\zeta}\omega_1{\omega_2}^4)$$
$$\frac{b2\omega_1{\omega_2}^2}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)(c^2+{\omega_1}^2-2\omega_1\omega_2+{\omega_2}^2)(c^2+{\omega_1}^2+2\omega_1\omega_2+{\omega_2}^2)}\}$$

$$(6b.15)$$

$$\tilde{V}_{4,14} = \{-(c^3 d^2 f_{3,2\zeta} + c^3 d f_{3,1\zeta}\omega_1 - c^2 d^2 f_{3,1\zeta}\omega_1 + c^2 d f_{3,2\zeta}{\omega_1}^2$$
$$+ c d^2 f_{3,2\zeta}{\omega_1}^2 + c d f_{3,1\zeta}{\omega_1}^3 - d^2 f_{3,1\zeta}{\omega_1}^3 + d f_{3,2\zeta}{\omega_1}^4$$
$$- c^3 d f_{4,2\zeta}\omega_2 + c^2 d^2 f_{4,2\zeta}\omega_2 - c^3 f_{4,1\zeta}\omega_1\omega_2 + 2c^2 d f_{4,1\zeta}\omega_1\omega_2$$
$$- 2cd^2 f_{4,1\zeta}\omega_1\omega_2 - c^2 f_{4,2\zeta}{\omega_1}^2\omega_2 + c d f_{4,2\zeta}{\omega_1}^2\omega_2 - d^2 f_{4,2\zeta}{\omega_1}^2\omega_2$$
$$- c f_{4,1\zeta}{\omega_1}^3\omega_2 - f_{4,2\zeta}{\omega_1}^4\omega_2 + c^2 d f_{3,2\zeta}{\omega_2}^2 + c d^2 f_{3,2\zeta}{\omega_2}^2$$
$$+ c^2 f_{3,1\zeta}\omega_1{\omega_2}^2 - c d f_{3,1\zeta}\omega_1{\omega_2}^2 + d^2 f_{3,1\zeta}\omega_1{\omega_2}^2 + 2c f_{3,2\zeta}{\omega_1}^2{\omega_2}^2$$
$$- 2d f_{3,2\zeta}{\omega_1}^2{\omega_2}^2 - f_{3,1\zeta}{\omega_1}^3{\omega_2}^2 - c d f_{4,2\zeta}{\omega_2}^3 + d^2 f_{4,2\zeta}{\omega_2}^3$$
$$- c f_{4,1\zeta}\omega_1{\omega_2}^3 + f_{4,2\zeta}{\omega_1}^2{\omega_2}^3 + d f_{3,2\zeta}{\omega_2}^4 + f_{3,1\zeta}\omega_1{\omega_2}^4)$$
$$\frac{b2\omega_1{\omega_2}^2}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)(c^2+{\omega_1}^2-2\omega_1\omega_2+{\omega_2}^2)(c^2+{\omega_1}^2+2\omega_1\omega_2+{\omega_2}^2)}\}$$

$$(6b.16)$$

$$\tilde{V}_{4,23} = \{(-(c^3 d^2 f_{4,1\zeta}) + c^3 d f_{4,2\zeta}\omega_1 - c^2 d^2 f_{4,2\zeta}\omega_1 - c^2 d f_{4,1\zeta}{\omega_1}^2$$
$$- c d^2 f_{4,1\zeta}{\omega_1}^2 + c d f_{4,2\zeta}{\omega_1}^3 - d^2 f_{4,2\zeta}{\omega_1}^3 - d f_{4,1\zeta}{\omega_1}^4$$
$$- c^3 d f_{3,1\zeta}\omega_2 + c^2 d^2 f_{3,1\zeta}\omega_2 + c^3 f_{3,2\zeta}\omega_1\omega_2 - 2c^2 d f_{3,2\zeta}\omega_1\omega_2$$
$$+ 2cd^2 f_{3,2\zeta}\omega_1\omega_2 - c^2 f_{3,1\zeta}{\omega_1}^2\omega_2 + c d f_{3,1\zeta}{\omega_1}^2\omega_2 - d^2 f_{3,1\zeta}{\omega_1}^2\omega_2$$
$$+ c f_{3,2\zeta}{\omega_1}^3\omega_2 - f_{3,1\zeta}{\omega_1}^4\omega_2 - c^2 d f_{4,1\zeta}{\omega_2}^2 - c d^2 f_{4,1\zeta}{\omega_2}^2$$
$$+ c^2 f_{4,2\zeta}\omega_1{\omega_2}^2 - c d f_{4,2\zeta}\omega_1{\omega_2}^2 + d^2 f_{4,2\zeta}\omega_1{\omega_2}^2 - 2c f_{4,1\zeta}{\omega_1}^2{\omega_2}^2$$
$$+ 2d f_{4,1\zeta}{\omega_1}^2{\omega_2}^2 - f_{4,2\zeta}{\omega_1}^3{\omega_2}^2 - c d f_{3,1\zeta}{\omega_2}^3 + d^2 f_{3,1\zeta}{\omega_2}^3$$
$$+ c f_{3,2\zeta}\omega_1{\omega_2}^3 + f_{3,1\zeta}{\omega_1}^2{\omega_2}^3 - d f_{4,1\zeta}{\omega_2}^4 + f_{4,2\zeta}\omega_1{\omega_2}^4)$$
$$\frac{b2\omega_1{\omega_2}^2}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)(c^2+{\omega_1}^2-2\omega_1\omega_2+{\omega_2}^2)(c^2+{\omega_1}^2+2\omega_1\omega_2+{\omega_2}^2)}\}$$

$$(6b.17)$$

$$\tilde{V}_{4,24} = \{-(-(c^3 d^2 f_{3,1\zeta}) + c^3 d f_{3,2\zeta}\omega_1 - c^2 d^2 f_{3,2\zeta}\omega_1 - c^2 d f_{3,1\zeta}{\omega_1}^2$$
$$- c d^2 f_{3,1\zeta}{\omega_1}^2 + c d f_{3,2\zeta}{\omega_1}^3 - d^2 f_{3,2\zeta}{\omega_1}^3 - d f_{3,1\zeta}{\omega_1}^4$$
$$+ c^3 d f_{4,1\zeta}\omega_2 - c^2 d^2 f_{4,1\zeta}\omega_2 - c^3 f_{4,2\zeta}\omega_1\omega_2 + 2c^2 d f_{4,2\zeta}\omega_1\omega_2$$

$$- 2cd^2 f_{4,2\zeta}\omega_1\omega_2 + c^2 f_{4,1\zeta}{\omega_1}^2\omega_2 - cdf_{4,1\zeta}{\omega_1}^2\omega_2 + d^2 f_{4,1\zeta}{\omega_1}^2\omega_2$$

$$- cf_{4,2\zeta}{\omega_1}^3\omega_2 + f_{4,1\zeta}{\omega_1}^4\omega_2 - c^2 df_{3,1\zeta}{\omega_2}^2 - cd^2 f_{3,1\zeta}{\omega_2}^2$$

$$+ c^2 f_{3,2\zeta}\omega_1{\omega_2}^2 - cdf_{3,2\zeta}\omega_1{\omega_2}^2 + d^2 f_{3,2\zeta}\omega_1{\omega_2}^2 - 2cf_{3,1\zeta}{\omega_1}^2{\omega_2}^2$$

$$+ 2df_{3,1\zeta}{\omega_1}^2{\omega_2}^2 - f_{3,2\zeta}{\omega_1}^3{\omega_2}^2 + cdf_{4,1\zeta}{\omega_2}^3 - d^2 f_{4,1\zeta}{\omega_2}^3$$

$$- cf_{4,2\zeta}\omega_1{\omega_2}^3 - f_{4,1\zeta}{\omega_1}^2{\omega_2}^3 - df_{3,1\zeta}{\omega_2}^4 + f_{3,2\zeta}\omega_1{\omega_2}^4)$$

$$\frac{b2\omega_1{\omega_2}^2}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)(c^2+{\omega_1}^2-2\omega_1\omega_2+{\omega_2}^2)(c^2+{\omega_1}^2+2\omega_1\omega_2+{\omega_2}^2)}\} \tag{6b.18}$$

$$\tilde{V}_{1,00} = V_{1,00} \tag{6b.19}$$

$$\tilde{V}_{2,00} = V_{2,00} \tag{6b.20}$$

$$\tilde{V}_{3,00} = V_{3,00} \tag{6b.21}$$

$$\tilde{V}_{4,00} = V_{4,00} \tag{6b.22}$$

Appendix 6.C.

$$V_{1,111} = \{\frac{3b_{12}d^3{\omega_1}^5}{(d^2+{\omega_1}^2)^3} + \frac{3b_{11}d^2{\omega_1}^6}{(d^2+{\omega_1}^2)^3} + \frac{3b_{12}d{\omega_1}^7}{(d^2+{\omega_1}^2)^3} + \frac{3b_{11}{\omega_1}^8}{(d^2+{\omega_1}^2)^3}\} \tag{6c.1}$$

$$V_{1222} = \{\frac{-3b_{11}d^3{\omega_1}^5}{(d^2+{\omega_1}^2)^3} + \frac{3b_{12}d^2{\omega_1}^6}{(d^2+{\omega_1}^2)^3} - \frac{3b_{11}d{\omega_1}^7}{(d^2+{\omega_1}^2)^3} + \frac{3b_{12}{\omega_1}^8}{(d^2+{\omega_1}^2)^3}\} \tag{6c.2}$$

$$V_{1,112} = \{-\frac{b_{11}d^3{\omega_1}^5}{(d^2+{\omega_1}^2)^3} + \frac{b_{12}d^2{\omega_1}^6}{(d^2+{\omega_1}^2)^3} - \frac{b_{11}d{\omega_1}^7}{(d^2+{\omega_1}^2)^3} + \frac{b_{12}{\omega_1}^8}{(d^2+{\omega_1}^2)^3}\} \tag{6c.3}$$

$$V_{1,122} = \{\frac{b_{12}d^3{\omega_1}^5}{(d^2+{\omega_1}^2)^3} + \frac{b_{11}d^2{\omega_1}^6}{(d^2+{\omega_1}^2)^3} + \frac{b_{12}d{\omega_1}^7}{(d^2+{\omega_1}^2)^3} + \frac{b_{11}{\omega_1}^8}{(d^2+{\omega_1}^2)^3}\} \tag{6c.4}$$

$$V_{2,33} = \{(2b_{12}d^2 f_{1,11} - b_{11}d^2 f_{1,12} + b_{12}d^2 f_{2,12} - 2b_{11}d^2 f_{2,22}$$

$$- 2b_{12}d^2 f_{4,14} + 2b_{11}d^2 f_{4,24})\frac{\omega_2^2}{(d^2+\omega_2^2)^2}$$

$$+ (-\frac{b_2 d^2 f_{1,1\zeta}}{c} - \frac{b_2 d^2 f_{2,2\zeta}}{c})\frac{\omega_1{\omega_2}^2}{(d^2+{\omega_2}^2)^2}$$

$$+(\frac{-3b_2f_{3,3\zeta}}{c}-\frac{b_2f_{4,4\zeta}}{c})\frac{{\omega_2}^6}{(d^2+{\omega_2}^2)^2}$$

$$+\frac{(3b_{12}f_{3,13}-3b_{11}f_{3,23}+b_{12}f_{4,14}-b_{11}f_{4,24}){\omega_2}^6}{\omega_1(d^2+{\omega_2}^2)^2}$$

$$+\frac{(-2b_2f_{3,4\zeta}-2b_2f_{4,3\zeta}){\omega_2}^7}{(d^2+{\omega_2}^2)^2(c^2+4{\omega_2}^2)}$$

$$+(\frac{4b_2f_{3,3\zeta}}{c}-\frac{4b_2f_{4,4\zeta}}{c})\frac{{\omega_2}^8}{(d^2+{\omega_2}^2)^2(c^2+4{\omega_2}^2)}$$

$$+\frac{(-2b_{11}d^2f_{3,14}-2b_{12}d^2f_{3,24}-2b_{11}d^2f_{4,13}-2b_{12}d^2f_{4,23})\omega_1\omega_2^3}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+\frac{(2b_{12}df_{3,14}-2b_{11}df_{3,24}+2b_{12}df_{4,13}-2b_{11}df_{4,23}){\omega_1}^2{\omega_2}^3}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+\frac{(-4b_{12}d^2f_{3,13}+4b_{11}d^2f_{3,23}+4b_{12}d^2f_{4,14}-4b_{11}d^2f_{4,24}){\omega_2}^4}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+\frac{(-4b_{11}df_{3,13}-4b_{12}df_{3,23}+4b_{11}df_{4,14}+4b_{12}df_{4,24})\omega_1{\omega_2}^4}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+\frac{(-2b_{11}f_{3,14}-2b_{12}f_{3,24}-2b_{11}f_{4,13}-2b_{12}f_{4,23}){\omega_2}^7}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+\frac{(-4b_{12}f_{3,13}+4b_{11}f_{3,23}+4b_{12}f_{4,14}-4b_{11}f_{4,24}){\omega_2}^8}{\omega_1(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}\} \tag{6c.5}$$

$$V_{2,34}=\{(2b_{12}df_{1,11}-b_{11}df_{1,12}+b_{12}df_{2,12}-2b_{11}df_{2,22}$$

$$-2b_{12}df_{4,14}+2b_{11}df_{4,24})\frac{{\omega_2}^3}{(d^2+{\omega_2}^2)^2}$$

$$+(-\frac{b_2df_{1,1\zeta}}{c}-\frac{b_2df_{2,2\zeta}}{c})\frac{\omega_1{\omega_2}^3}{(d^2+{\omega_2}^2)^2}$$

$$+(\frac{3b_2df_{3,3\zeta}}{c}+\frac{b_2df_{4,4\zeta}}{c})\frac{{\omega_2}^5}{(d^2+{\omega_2}^2)^2}$$

$$+\frac{(-3b_{12}df_{3,13}+3b_{11}df_{3,23}-b_{12}df_{4,14}+b_{11}df_{4,24}){\omega_2}^5}{\omega_1(d^2+{\omega_2}^2)^2}$$

$$+\frac{(2b_2 d f_{3,4\zeta}+2b_2 d f_{4,3\zeta})\omega_2{}^6}{(d^2+\omega_2{}^2)^2(c^2+4\omega_2{}^2)}$$

$$+(\frac{-4b_2 d f_{3,3\zeta}}{c}+\frac{4b_2 d f_{4,4\zeta}}{c})\frac{\omega_2{}^7}{(d^2+\omega_2{}^2)^2(c^2+4\omega_2{}^2)}$$

$$+\frac{(-(b_{12}d^2 f_{3,14})+b_{11}d^2 f_{3,24}-b_{12}d^2 f_{4,13}+b_{11}d^2 f_{4,23})\omega_1{}^2\omega_2{}^2}{(d^2+\omega_2{}^2)^2(-\omega_1{}^2+4\omega_2{}^2)}$$

$$+\frac{(2b_{11}d^2 f_{3,13}+2b_{12}d^2 f_{3,23}-2b_{11}d^2 f_{4,14}-2b_{12}d^2 f_{4,24})\omega_1\omega_2{}^3}{(d^2+\omega_2{}^2)^2(-\omega_1{}^2+4\omega_2{}^2)}$$

$$+\frac{(-2b_{11}d f_{3,14}-2b_{12}d f_{3,24}-2b_{11}d f_{4,13}-2b_{12}d f_{4,23})\omega_1\omega_2{}^4}{(d^2+\omega_2{}^2)^2(-\omega_1{}^2+4\omega_2{}^2)}$$

$$+\frac{(b_{12}f_{3,14}-b_{11}f_{3,24}+b_{12}f_{4,13}-b_{11}f_{4,23})\omega_1{}^2\omega_2{}^4}{(d^2+\omega_2{}^2)^2(-\omega_1{}^2+4\omega_2{}^2)}$$

$$+\frac{(-4b_{12}d f_{3,13}+4b_{11}d f_{3,23}+4b_{12}d f_{4,14}-4b_{11}d f_{4,24})\omega_2{}^5}{(d^2+\omega_2{}^2)^2(-\omega_1{}^2+4\omega_2{}^2)}$$

$$+\frac{(-2b_{11}f_{3,13}-2b_{12}f_{3,23}+2b_{11}f_{4,14}+2b_{12}f_{4,24})\omega_1\omega_2{}^5}{(d^2+\omega_2{}^2)^2(-\omega_1{}^2+4\omega_2{}^2)}$$

$$+\frac{(2b_{11}d f_{3,14}+2b_{12}d f_{3,24}+2b_{11}d f_{4,13}+2b_{12}d f_{4,23})\omega_2{}^6}{(d^2+\omega_2{}^2)^2(-\omega_1{}^2+4\omega_2{}^2)}$$

$$+\frac{(4b_{12}d f_{3,13}-4b_{11}d f_{3,23}-4b_{12}d f_{4,14}+4b_{11}d f_{4,24})\omega_2{}^7}{\omega_1(d^2+\omega_2{}^2)^2(-\omega_1{}^2+4\omega_2{}^2)}\} \tag{6c.6}$$

$$V_{2,44}=\{(\frac{-3b_2 d^2 f_{3,3\zeta}}{c}-\frac{b_2 d^2 f_{4,4\zeta}}{c}+2b_{12}f_{1,11}-b_{11}f_{1,12}+b_{12}f_{2,12}$$

$$-2b_{11}f_{2,22}-2b_{12}f_{4,14}+2b_{11}f_{4,24})\frac{\omega_2{}^4}{(d^2+\omega_2{}^2)^2}$$

$$+\frac{(3b_{12}d^2 f_{3,13}-3b_{11}d^2 f_{3,23}+b_{12}d^2 f_{4,14}-b_{11}d^2 f_{4,24})\omega_2{}^4}{\omega_1(d^2+\omega_2{}^2)^2}$$

$$+(-\frac{b_2 f_{1,1\zeta}}{c}-\frac{b_2 f_{2,2\zeta}}{c})\frac{\omega_1\omega_2{}^4}{(d^2+\omega_2{}^2)^2}$$

$$+\frac{(-2b_2 d^2 f_{3,4\zeta}-2b_2 d^2 f_{4,3\zeta})\omega_2{}^5}{(d^2+\omega_2{}^2)^2(c^2+4\omega_2{}^2)}$$

$$+ \left(\frac{4b_2 d^2 f_{3,3\zeta}}{c} - \frac{4b_2 d^2 f_{4,4\zeta}}{c}\right)\frac{{\omega_2}^6}{(d^2+{\omega_2}^2)^2(c^2+4{\omega_2}^2)}$$

$$+ \frac{(-2b_{12}df_{3,14} + 2b_{11}df_{3,24} - 2b_{12}df_{4,13} + 2b_{11}df_{4,23}){\omega_1}^2{\omega_2}^3}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+ \frac{(4b_{11}df_{3,13} + 4b_{12}df_{3,23} - 4b_{11}df_{4,14} - 4b_{12}df_{4,24})\omega_1{\omega_2}^4}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+ \frac{(-2b_{11}d^2f_{3,14} - 2b_{12}d^2f_{3,24} - 2b_{11}d^2f_{4,13} - 2b_{12}d^2f_{4,23}){\omega_2}^5}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+ \frac{(-2b_{11}f_{3,14} - 2b_{12}f_{3,24} - 2b_{11}f_{4,13} - 2b_{12}f_{4,23})\omega_1{\omega_2}^5}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+ \frac{(-4b_{12}f_{3,13} + 4b_{11}f_{3,23} + 4b_{12}f_{4,14} - 4b_{11}f_{4,24}){\omega_2}^6}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+ \frac{(-4b_{12}d^2f_{3,13} + 4b_{11}d^2f_{3,23} + 4b_{12}d^2f_{4,14} - 4b_{11}d^2f_{4,24}){\omega_2}^6}{\omega_1(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}\} \qquad (6c.7)$$

$$V_{3,33} = \{(2b_{12}d^2f_{1,11} - b_{11}d^2f_{1,12} + b_{12}d^2f_{2,12} - 2b_{11}d^2f_{2,22}$$

$$\frac{-2b_{12}d^2f_{4,14} + 2b_{11}d^2f_{4,24}){\omega_2}^2}{(d^2+{\omega_2}^2)^2}$$

$$+ \left(-\frac{b_2 d^2 f_{1,1\zeta}}{c} - \frac{b_2 d^2 f_{2,2\zeta}}{c}\right)\frac{\omega_1{\omega_2}^2}{(d^2+{\omega_2}^2)^2}$$

$$+ \frac{(2b_{12}f_{1,11} - b_{11}f_{1,12} + b_{12}f_{2,12} - 2b_{11}f_{2,22} - 2b_{12}f_{3,13} + 2b_{11}f_{3,23}){\omega_2}^4}{(d^2+{\omega_2}^2)^2}$$

$$+ \left(-\frac{b_2 f_{1,1\zeta}}{c} - \frac{b_2 f_{2,2\zeta}}{c}\right)\frac{\omega_1{\omega_2}^4}{(d^2+{\omega_2}^2)^2}$$

$$+ \frac{(-2b_{11}d^2f_{3,14} - 2b_{12}d^2f_{3,24} - 2b_{11}d^2f_{4,13} - 2b_{12}d^2f_{4,23})\omega_1{\omega_2}^3}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+ \frac{(2b_{12}df_{3,14} - 2b_{11}df_{3,24} + 2b_{12}df_{4,13} - 2b_{11}df_{4,23}){\omega_1}^2{\omega_2}^3}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+ \frac{(-4b_{12}d^2f_{3,13} + 4b_{11}d^2f_{3,23} + 4b_{12}d^2f_{4,14} - 4b_{11}d^2f_{4,24}){\omega_2}^4}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+\frac{(-4b_{11}df_{3,13}-4b_{12}df_{3,23}+4b_{11}df_{4,14}+4b_{12}df_{4,24})\omega_1{\omega_2}^4}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+\frac{(2b_{11}f_{3,14}+2b_{12}f_{3,24}+2b_{11}f_{4,13}+2b_{12}f_{4,23})\omega_1{\omega_2}^5}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+\frac{(4b_{12}f_{3,13}-4b_{11}f_{3,23}-4b_{12}f_{4,14}+4b_{11}f_{4,24}){\omega_2}^6}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}\} \tag{6c.8}$$

$$V_{3,34}=\{\frac{(2b_{12}df_{3,13}-2b_{11}df_{3,23}-2b_{12}df_{4,14}+2b_{11}df_{4,24}){\omega_2}^3}{(d^2+{\omega_2}^2)^2}$$

$$+\frac{(-(b_{12}d^2f_{3,14})+b_{11}d^2f_{3,24}-b_{12}d^2f_{4,13}+b_{11}d^2f_{4,23}){\omega_1}^2{\omega_2}^2}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+\frac{(2b_{11}d^2f_{3,13}+2b_{12}d^2f_{3,23}-2b_{11}d^2f_{4,14}-2b_{12}d^2f_{4,24})\omega_1{\omega_2}^3}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+\frac{(-4b_{11}df_{3,14}-4b_{12}df_{3,24}-4b_{11}df_{4,13}-4b_{12}df_{4,23})\omega_1{\omega_2}^4}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+\frac{(b_{12}f_{3,14}-b_{11}f_{3,24}+b_{12}f_{4,13}-b_{11}f_{4,23}){\omega_1}^2{\omega_2}^4}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+\frac{(-8b_{12}df_{3,13}+8b_{11}df_{3,23}+8b_{12}df_{4,14}-8b_{11}df_{4,24}){\omega_2}^5}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+\frac{(-2b_{11}f_{3,13}-2b_{12}f_{3,23}+2b_{11}f_{4,14}+2b_{12}f_{4,24})\omega_1{\omega_2}^5}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}\} \tag{6c.9}$$

$$V_{3,44}=\{(2b_{12}d^2f_{1,11}-b_{11}d^2f_{1,12}+b_{12}d^2f_{2,12}-2b_{11}d^2f_{2,22}$$

$$-2b_{12}d^2f_{3,13}+2b_{11}d^2f_{3,23})\frac{{\omega_2}^2}{(d^2+{\omega_2}^2)^2}$$

$$+(-\frac{b_2d^2f_{1,1\zeta}}{c}-\frac{b_2d^2f_{2,2\zeta}}{c})\frac{\omega_1{\omega_2}^2}{(d^2+{\omega_2}^2)^2}$$

$$+\frac{(2b_{12}f_{1,11}-b_{11}f_{1,12}+b_{12}f_{2,12}-2b_{11}f_{2,22}-2b_{12}f_{4,14}+2b_{11}f_{4,24}){\omega_2}^4}{(d^2+{\omega_2}^2)^2}$$

$$+(-\frac{b_2f_{1,1\zeta}}{c}-\frac{b_2f_{2,2\zeta}}{c})\frac{\omega_1{\omega_2}^4}{(d^2+{\omega_2}^2)^2}$$

$$+\frac{(2b_{11}d^2f_{3,14}+2b_{12}d^2f_{3,24}+2b_{11}d^2f_{4,13}+2b_{12}d^2f_{4,23})\omega_1{\omega_2}^3}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+\frac{(-2b_{12}df_{3,14}+2b_{11}df_{3,24}-2b_{12}df_{4,13}+2b_{11}df_{4,23}){\omega_1}^2{\omega_2}^3}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+\frac{(4b_{12}d^2f_{3,13}-4b_{11}d^2f_{3,23}-4b_{12}d^2f_{4,14}+4b_{11}d^2f_{4,24}t){\omega_2}^4}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+\frac{(4b_{11}df_{3,13}+4b_{12}df_{3,23}-4b_{11}df_{4,14}-4b_{12}df_{4,24})\omega_1{\omega_2}^4}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+\frac{(-2b_{11}f_{3,14}-2b_{12}f_{3,24}-2b_{11}f_{4,13}-2b_{12}f_{4,23})\omega_1{\omega_2}^5}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}$$

$$+\frac{(-4b_{12}f_{3,13}+4b_{11}f_{3,23}+4b_{12}f_{4,14}-4b_{11}f_{4,24}){\omega_2}^6}{(d^2+{\omega_2}^2)^2(-{\omega_1}^2+4{\omega_2}^2)}\} \tag{6c.10}$$

$$V_{3,133}=\{\frac{b_{12}d^3{\omega_1}^2{\omega_2}^2}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)^2}+\frac{b_{11}d^2{\omega_1}^3{\omega_2}^2}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)^2}$$

$$+\frac{b_{12}d{\omega_1}^2{\omega_2}^4}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)^2}+\frac{b_{11}{\omega_1}^3{\omega_2}^4}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)^2}\} \tag{6c.11}$$

$$V_{3,144}=\{\frac{b_{12}d^3{\omega_1}^2{\omega_2}^2}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)^2}+\frac{b_{11}d^2{\omega_1}^3{\omega_2}^2}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)^2}$$

$$+\frac{b_{12}d{\omega_1}^2{\omega_2}^4}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)^2}+\frac{b_{11}{\omega_1}^3{\omega_2}^4}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)^2}\} \tag{6c.12}$$

$$V_{3,233}=\{(-\frac{b_{11}d^3\omega_1^2}{d^2+\omega_1^2}+\frac{b_{12}d^2\omega_1^3}{d^2+{\omega_1}^2})\frac{{\omega_2}^2}{(d^2+{\omega_2}^2)^2}$$

$$+(-\frac{b_{11}d{\omega_1}^2}{d^2+{\omega_1}^2}+\frac{b_{12}{\omega_1}^3}{d^2+{\omega_1}^2})\frac{{\omega_2}^4}{(d^2+{\omega_2}^2)^2}\} \tag{6c.13}$$

$$V_{3,244}=\{-\frac{b_{11}d^3{\omega_1}^2{\omega_2}^2}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)^2}+\frac{b_{12}d^2{\omega_1}^3{\omega_2}^2}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)^2}$$

$$-\frac{b_{11}d{\omega_1}^2{\omega_2}^4}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)^2}+\frac{b_{12}{\omega_1}^3{\omega_2}^4}{(d^2+{\omega_1}^2)(d^2+{\omega_2}^2)^2}\} \tag{6c.14}$$

$$V_{1,000}=V_{1,00} \tag{6c.15}$$

$$V_{2,000} = V_{2,00} \tag{6c.16}$$

$$V_{3,000} = V_{3,00} \tag{6c.17}$$

CHAPTER SEVEN
NONLINEAR BIFURCATION CONTROL OF HIGH ANGLE-OF-ATTACK FLIGHT DYNAMICS

This chapter considers the problem of designing stabilizing control laws for flight over a broad range of angles-of-attack which also serve to signal the pilot of impending stall. The model of the longitudinal dynamics of an F-8 Crusader of Garrard and Jordan [35] is studied. The direct state feedback stabilization derived in [13] and the feedback stabilization through washout filters derived in Chapter 4 are compared. The results show that by using washout filters, the equilibrium of the uncontrolled system is preserved so that the stable operating range can be increased. Second, six different wing lift profiles are introduced to the dynamics to reflect possible uncertainty in modeling the aircraft. By using washout filters, two robustly stabilizing controllers are designed. The latter design results in a window of stable, small amplitude, periodic orbits encircling the equilibrium between two previously unstable Hopf bifurcation points.

7.1. Background and motivation

Several authors have studied the nonlinear phenomena that arise commonly in aircraft flight at high angle-of-attack (alpha). The literature on high alpha flight dynamics, control and aerodynamics has grown at a rapid pace. Of particular relevance here are references [4], [7], [35], [8] and [10]. The direct linkage of aircraft stall and divergence, as well as other nonlinear aircraft motions in high incidence flight, to bifurcations of the governing dynamic equations is a goal of many previous investigations. In particular, both stationary and Hopf bifurcations are reported and/or studied for several aircraft models in [4], [7], and [8]; and a Hopf bifurcation occurring in the lateral dynamics of a slender-wing aircraft has been studied in [10], [7], [19].

In this chapter, we study the stabilization of the operating condition of an aircraft in the neighborhood of stall, in a manner which has negligible effects on the normal flight conditions by preserving the *equilibrium* and the *linear stability* of the normal flight conditions, and provides an *impending stall warning signal* to the pilot. This is done by using a pure nonlinear feedback control to stabilize the unstable Hopf bifurcation in the high alpha flight regime. The stabilized Hopf bifurcation points and the stable periodic solutions serve to extend the useful operating range as well as to warn the pilot of impending stall.

By appealing to singular perturbation theory, the bifurcation analysis and stabilization design are done for the fast mode of the longitudinal model of an F-8 Crusader given in [35]. Since accurate aerodynamic data is, in general, not easy to obtain, we introduce six different wing lift profiles to reflect possible uncertainty in the aerodynamic model. These different wing lift profiles result in different equilibrium profiles for the system. With the help of washout filters, robust controllers are designed to stabilize all the Hopf bifurcations in these equilibrium profiles. The simulation results for both controlled and uncontrolled systems show the significant improvement in reducing the amplitude of post-stall oscillation, which also justifies the use of reduced models in analysis and design.

In all the following figures, a solid line denotes a stable equilibrium, a dashed line denotes an unstable equilibrium, a solid block denotes a stable periodic solution, and a blank block denotes an unstable periodic solution. All the stability coefficients, β_2, are calculated by using the computer program BIFOR2 [32], and all the equilibrium branches and periodic solution branches are calculated by using the computer program AUTO [34].

7.2. Aircraft longitudinal dynamics

Let the body axes of an aircraft be chosen as in Figure 7.1 [28], where the origin of the coordinate is located at the center of gravity of the aircraft, the positive X axis points forward, the positive Y axis points toward the right wing, and the positive Z axis points downward.

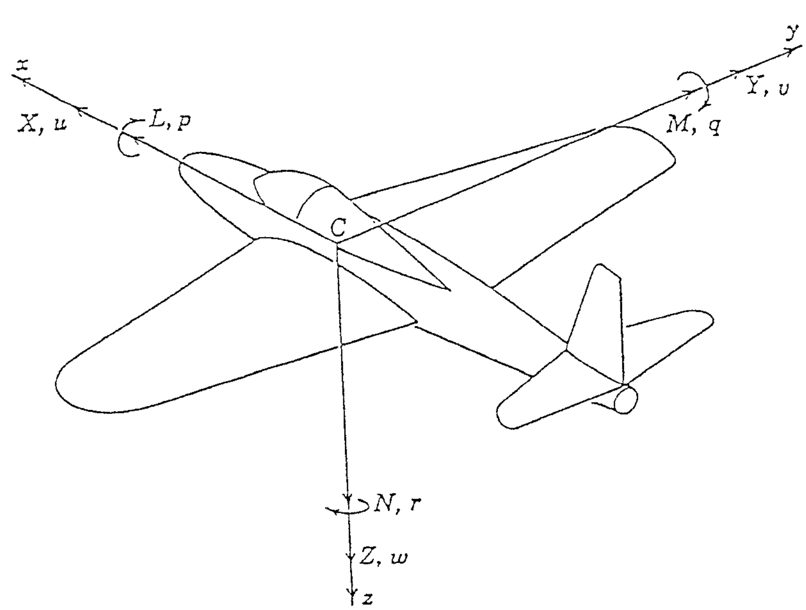


Figure 7.1. Body axis coordinates of an aircraft (Etkin [28])

If the flight condition is symmetric (that is, the velocity V of the aircraft lies in the $X-Z$ plane), the velocity components in the X and Z directions, u, and w, the body attitude θ, the angle-of-attack α, and the principle forces which govern the longitudinal dynamics are as identified in Figure 7.2 [35].

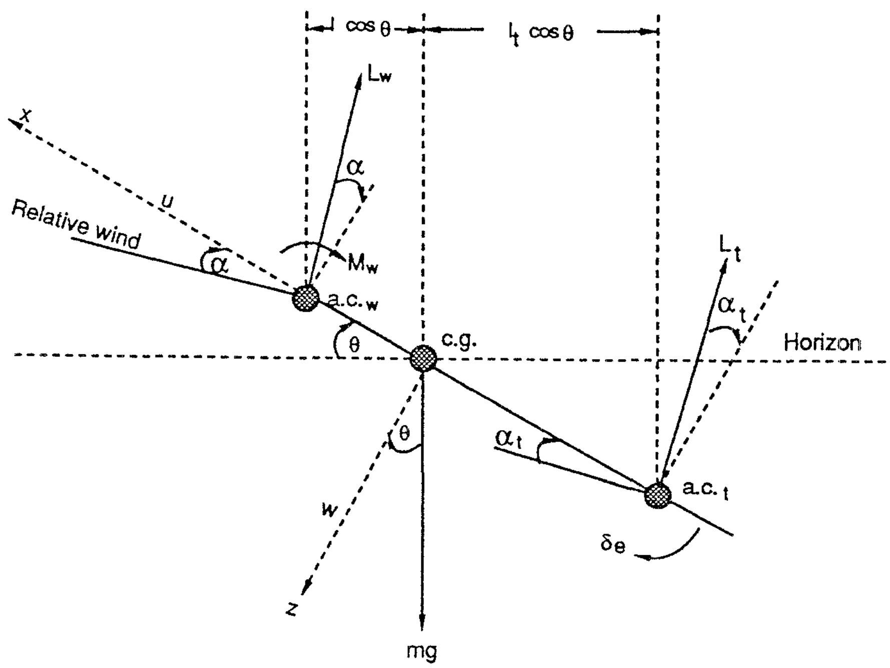


Figure 7.2. Principle longitudinal parameters

The basic equations of motion for longitudinal dynamics with drag and thrust neglected are

$$m(\dot{u} + w\dot{\theta}) = -mg\sin\theta + L_w \sin\alpha + L_t \sin\alpha_t \tag{7.1a}$$

$$m(\dot{w} - u\dot{\theta}) = mg\cos\theta - L_w\cos\alpha - L_t\cos\alpha_t \tag{7.1b}$$

$$I_y\ddot{\theta} = M_w + lL_w\cos\alpha - l_tL_t\cos\alpha_t - c\dot{\theta}. \tag{7.1c}$$

Here,

u : velocity in X direction

w : velocity in Z direction

θ : pitch angle

I_y : moment of inertia about Y axis

L_w : wing lift force

L_t : tail lift force

α : wing angle of attack

α_t : tail angle of attack

M_w : wing moment

m : mass of aircraft

l : distance between wing aerodynamic center and aircraft center of gravity

l_t : distance between tail aerodynamic center and aircraft center of gravity

$c\dot{\theta}$: damping moment

The wing and tail lift forces are

$$L_w = C_L \bar{q} S, \tag{7.2a}$$

$$L_t = C_t \bar{q} S_t, \tag{7.2b}$$

where

C_L : coefficient of the wing lift

C_t : coefficient of the tail lift

$\bar{q}$: dynamic pressure

S : wing area

S_t : horizontal tail area.

By using the relation

$$w = u \tan\alpha, \tag{7.3}$$

the velocity component in the Z direction, w, can be eliminated from Eqs. (7.1a), (7.1b). The equations of motion are then given by

$$\dot{u} = -u\dot{\theta}\tan\alpha - g\sin\theta + \frac{L_w}{m}\sin\alpha + \frac{L_t}{m}\sin\alpha_t \tag{7.4a}$$

$$
\begin{aligned}
\dot{\alpha} = {} & \dot{\theta}\sin^2\alpha + \frac{g}{u}\sin\theta\sin\alpha\cos\alpha - \frac{L_w}{um}\sin^2\alpha\cos\alpha \\
& - \frac{L_t}{um}\sin\alpha\cos\alpha\sin\alpha_t + \dot{\theta}\cos^2\alpha + \frac{g}{u}\cos^2\alpha\cos\theta \\
& - \frac{L_w}{mu}\cos^3\alpha - \frac{L_t}{mu}\cos^2\alpha\cos\alpha_t
\end{aligned} \tag{7.4b}
$$

$$
\ddot{\theta} = \frac{M_w}{I_y} + \frac{lL_w}{I_y}\cos\alpha - \frac{l_t L_t}{I_y}\cos\alpha_t - \frac{c}{I_y}\dot{\theta}. \tag{7.4c}
$$

7.3. High α control: Garrard and Jordan's model

In this section, based on the model in [35], the control of high α longitudinal dynamics for an F-8 crusader is studied. By the singular perturbation reduction theory, the analysis is done using the fast mode of the longitudinal dynamics. The steady state of this reduced model is usually called *pseudo steady-state* (PSS). PSS was originally used by Phillips [36] to analyze the stability of the short-period longitudinal and lateral oscillations. Phillips' analysis predicted some divergence-like motions which would be predicted as an acceptable behavior if the usual linearized stability analysis was used. This method is not suitable in predicting the magnitudes of response peaks [9]. However, it can be used to predict the "jumps" caused by control input [9], [11].

In our analysis of the reduced F-8 model, we find that the aircraft suffers an unstable Hopf bifurcation at α near stall which makes the longitudinal motion diverge past the bifurcation point. For comparison, two types of stabilizing control laws, direct state feedback and feedback through washout filters, are applied to stabilize the Hopf bifurcation point and the periodic solutions emerging from this Hopf bifurcation point.

7.3.1. Garrard and Jordan's simplifying equations

In [35], Garrard and Jordan approximated the wing lift and tail lift coefficients by cubic polynomial functions

$$
C_{Lw} = C_{Lw}^1\alpha - C_{Lw}^2\alpha^3, \tag{7.5}
$$

$$
C_{Lt} = C_{Lt}^1\alpha_t - C_{Lt}^2\alpha_t^3 + a_e\delta_e, \tag{7.6}
$$

where δ_e represents the horizontal tail deflection angle measured clockwise from the X−axis (Figure 7.2), and a_e is the linear approximation of the effect of δ_e on C_{Lt}. The true wing lift coefficient and the cubic approximation are shown in Figure 7.3 [35].

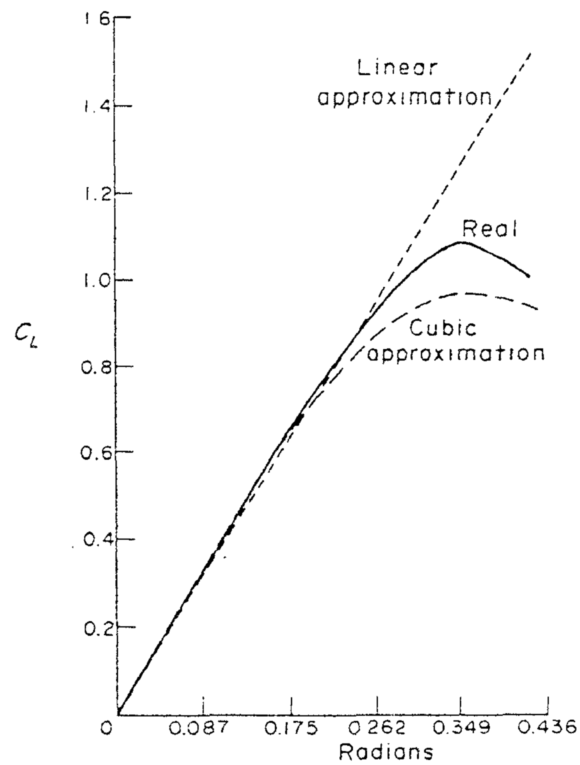


Figure 7.3. C_{L_w} vs. α (Garrard, Jordan [35])

Since the horizontal tail of the F-8 is within the wing wake, the downwash angle, ϵ, has to be included in determining the tail angle-of-attack. With a linear approximation, $\epsilon = a_\epsilon \alpha$, the tail angle-of-attack is given by

$$\begin{aligned} \alpha_t &= \alpha - \epsilon + \delta_e \\ &= (1 - a_\epsilon)\alpha + \delta_e. \end{aligned} \tag{7.7}$$

The aircraft data is given in Table 7.1.

Assuming the aircraft to be in level, unaccelerated ($\dot{u} = 0$) flight at a speed of 0.85 Mach, altitude 30,000 ft, and employ the following trigonometric approximation:

$$\begin{aligned} w &= u \tan \alpha \\ &\approx u(\alpha + \frac{\alpha^3}{3}), \end{aligned} \tag{7.8}$$

$$\begin{aligned} \sin\theta &\approx \theta - \frac{\theta^3}{6} \qquad \sin\alpha \approx \alpha - \frac{\alpha^3}{6} \\ \cos\theta &\approx 1 - \frac{\theta^2}{2} \qquad \cos\alpha \approx 1 - \frac{\alpha^2}{2}. \end{aligned} \tag{7.9}$$

The equations of motion are then simply to ([35], Eqs. (10),(11))

$$\begin{aligned} \dot\alpha =& \dot\theta - \alpha^2\dot\theta - 0.877\alpha\dot\theta - 0.877\alpha + 0.47\alpha^2 + 3.846\alpha^3 - 0.215\delta_e \\ &+ 0.28\delta_e\alpha^2 + 0.47\delta_e^2\alpha + 0.63\delta_e^3 - 0.019\theta^2, \end{aligned} \tag{7.10a}$$

$$\begin{aligned} \ddot\theta =& -0.396\dot\theta - 4.208\alpha - 0.47\alpha^2 - 3.564\alpha^3 - 20.967\delta_e \\ &+ 6.265\delta_e\alpha^2 + 46\delta^2\alpha + 61.4\delta_e^3. \end{aligned} \tag{7.10b}$$

Here, α and δ_e represent the angle-of-attack and the tail deflection angle after the trim terms, $\alpha_0 = 0.044$ radians, $\delta_{e,0} = -0.009$ radians, are subtracted so that $\delta = \alpha = \theta = 0$ is an equilibrium point.

7.3.2. Bifurcation analysis

Since the dynamics of θ and u are the slow *phugoid mode*, comparing to the dynamics of $\dot\theta$ and α which are the fast *short-period mode*, by singular perturbation theory, we may separate the longitudinal dynamics into two time scales and analyze their behavior separately [37]. In this bifurcation analysis, we are interested in the fast mode periodic behavior since it is useful in predicting nonlinear behaviors such as jumps, divergence, etc., and which are believed to have a stronger impact on the pilot. To facilitate this, the velocity is assumed constant (which is already done by [35]) and the θ^2 term in (7.10a) is neglected. This additional simplification implies that the effects of varying weight components in body axes are neglected. The equilibrium state considered here is therefore called a pseudo steady-state ([36], [9], [11], [8]).

In the equations of motion (7.1), the lateral motion is assumed negligible. We denote the angular velocity about the Y axis as $q = \dot{\theta}$. Eqs. (7.10a), (7.10b) become

$$\begin{aligned}\dot{\alpha} =& q - \alpha^2 q - 0.088\alpha q - 0.877\alpha + 0.47\alpha^2 + 3.846\alpha^3 - 0.215\delta_e \\ &+ 0.28\delta_e\alpha^2 + 0.47\delta_e^2\alpha + 0.63\delta_e^3, \qquad (7.11a)\\ \dot{q} =& -0.396q - 4.208\alpha - 0.47\alpha^2 - 3.564\alpha^3 - 20.967\delta_e \\ &+ 6.265\delta_e\alpha^2 + 46\delta_e^2 + 61.4\delta_e^3. \qquad (7.11b)\end{aligned}$$

By treating δ_e as the system parameter, the (pseudo) equilibria for α and q at each command value of δ_e are shown in Figure 7.4a and Figure 7.4b. There is a Hopf bifurcation point on the nominal branch (the branch passing through the origin of the $\alpha - \delta_e$ plane and the $q - \delta_e$ plane) at $\delta_e = -0.064$ where $\alpha = 0.305, q = 0.116$, and a fold point at $\delta_e = -0.1556$ where $\alpha = 0.750, q = -3.36$. At the Hopf bifurcation point, the eigenvalues are given by $\pm j2.212$, and the stability coefficient is $\beta_2 = 3.123$. The sign of this stability coefficient implies instability of the bifurcated periodic solutions. Thus, for $|\delta_e| > 0.064$, transients beginning near equilibrium diverge. This divergence of the uncontrolled system is shown in the simulation of Figure 7.5. Also, the unstable limit cycles emerging from the Hopf bifurcation for $|\delta_e| < 0.064$ limit the attraction region of the nominal stable equilibrium.

The purpose of this control is to extend the useful range of angles-of-attack to increase the safety margin for aircraft operation in the region of stall. Additionally, the nonlinear Hopf bifurcation control provides a better attraction region for these stable operating conditions near the critical bifurcation point by removing the unstable limit cycles encircling them. Also, the stable limit cycles for the post-critical parameter values serve as a warning of impending *post-stall jump*.

Two types of feedback are used in the following sections: direct state feedback and indirect state feedback through washout filters. Direct state feedback is very commonly used in control systems. However, it requires on-line sensing of the parameter value and on-line computation of the equilibrium to prevent the deformation of the nominal operating branch. These may sometimes limit the robustness. Moreover, the deformation of other equilibrium branches may limit global performance. Indirect state feedback has the advantages of relieving the dependency on equilibrium, preserving the equilibria and making the control more robust as discussed in Chapter 3. However, it is accomplished at the expense of the increased dimensionality of the controller.

7.3.3. Hopf bifurcation control: direct state feedback

To remedy the divergent phenomenon for $|\delta_e| > |\delta_{ec}|$ (δ_{ec} denotes the parameter value at the Hopf bifurcation point), one can either use a linear feedback to stabilize the nominal equilibrium condition for a useful range of angle-of-attack, or use a bifurcation control law to render the bifurcated periodic solutions stable and of small amplitude for such a range of angles-of-attack. With the latter design, the aircraft would continually experience an oscillatory pitch motion, which may not be acceptable. With the former, the Hopf bifurcation is delayed to a greater value of angle-of-attack, and operating at higher than that new critical α might result in divergence as well. Thus, we employ a linear-plus-nonlinear feedback. The linear part of the feedback is chosen to delay the Hopf bifurcation, and the nonlinear terms are chosen to stabilize (if necessary) the Hopf bifurcation at the new higher critical angle. The tail deflection angle δ_e which is treated as a system parameter in obtaining the equilibrium curve is also treated as a control signal for stabilization.

For direct state feedback, in order to minimize the deformation of the nominal equilibrium states caused by feedback (to preserve the nominal equilibrium branch in Figures 7.4), the control signal must have a certain dependence on the state, namely

$$\delta_e(x) = \bar{\delta}_e + \{\text{a polynomial in } (x - x_{10}(\bar{\delta}_e)) \text{ and } (x_2 - x_{20}(\bar{\delta}_e))\}. \tag{7.12}$$

Here, x_1 and x_2 are the state variables α and q, respectively, $\bar{\delta}_e$ is the constant commanded value of δ_e, and subscripts 0 indicate equilibrium (trim) values of state variables, which depend on $\bar{\delta}_e$. In this example, we choose a linear function and a cubic function of $\bar{\delta}_e$ to approximate the nominal branch:

$$\alpha_0 = -4.6092\bar{\delta}_e, \tag{7.13a}$$

$$q_0 = 630.8146\bar{\delta}_e^3 - 5.0498\bar{\delta}_e. \tag{7.13b}$$

A linear feedback

$$\delta_e = \bar{\delta}_e + k_1(\alpha - \alpha_0(\bar{\delta}_e)) + k_2(q - q_0(\bar{\delta}_e)) \tag{7.14}$$

with $k_1 = 0.3317$ and $k_2 = 0.0836$ is applied first to move the Hopf bifurcation to $\bar{\delta}_e = -0.109$ where $\alpha = 0.5, q = -0.252$, and the eigenvalues of the linearization are $\pm j2.158$.

The stability coefficient β_2 of the new Hopf bifurcation point is 32.064 which implies instability of the Hopf bifurcation point and the limit cycle emerging from the Hopf bifurcation point. Figures 7.6a and 7.6b show the post-linear feedback equilibria and the unstable limit cycle emerging from the Hopf bifurcation point.

Note that, in Figure 7.6a, the fold point appears at smaller $|\bar{\delta}_e| = 0.1198$ (was 0.1556 for the uncontrolled version) and the nearby equilibrium branch moves toward the nominal branch. The former is due to the inaccuracy in approximating the nominal equilibrium branch by using Eqs. (7.13a),(7.13b). This effect limits the possibility of operating the aircraft at higher angle-of-attack (at those equilibrium points existing in uncontrolled version and beyond the new fold point). The latter is because direct state feedback deforms the equilibrium branches other than the nominal one. This deformation can shrink the attraction region of the stable nominal equilibrium branch.

To stabilize the Hopf bifurcation, and thus result in containment of post-critical trajectories to within a neighborhood of the nominal equilibrium, nonlinear terms are added to the linear feedback above. Since the critical eigenvalues are linearly controllable, by the algorithm in Chapter 4, both quadratic and cubic feedback can be used to render β_2 negative. However, the strategy of choosing the feedback in this case is not only to render β_2 negative but also to make the post-critical stable periodic solutions remain so for as large a range of parameter values as possible. Specifically, we have chosen to add both quadratic and cubic terms to the linear feedback, as follows:

$$\begin{aligned}\delta_e =&\bar{\delta}_e + k_1(\alpha - \alpha_0(\bar{\delta}_e)) + k_2(q - q(\bar{\delta}_e)) + c_1(\alpha - \alpha_0(\bar{\delta}_e))^2 \\ &+ h_1(\alpha - \alpha_0(\bar{\delta}_e))^3 + h_2(q - q(\bar{\delta}_e))^3.\end{aligned} \tag{7.15}$$

Here, $c_1 = h_1 = h_2 = 0.8$, resulting in a bifurcation stability coefficient $\beta_2 = -320.639$. Thus, the Hopf bifurcation for the controlled system is stabilized. Figures 7.7a and 7.7b show the equilibrium branches and the stable periodic solutions emerging from the Hopf bifurcation point toward the fold point. The stable limit cycle becomes a homoclinic orbit and disappears before the fold point. Figure 7.8 shows the convergence of the system trajectory to a stable limit cycle for the post-critical parameter $\bar{\delta}_e = -0.11318$. Figure 7.9 shows a trajectory of the system started near equilibrium for the parameter value $\delta_e = -0.11349$. The trajectory no longer converges to a stable limit cycle, but now diverges.

Note that, in Figures 7.7a and 7.7b, the nearby equilibrium branch are further deformed toward the nominal branch after nonlinear feedback is added. This nearby deformed branch further limits the attraction region of the nominal branch.

7.3.4. Hopf bifurcation control: washout filters

In Section 7.3.3, the closed-loop equilibrium branches are severely deformed partly by inaccuracy of approximating the nominal branch in control function, partly by the nature of using the direct state feedback. Moreover, recall that the original open-loop equilibrium branches are obtained using a cubic approximation of the wing and tail lift coefficients as well as the trigonometric approximation in Eqs. (7.8)-(7.9). Also, more inaccuracy is expected in the control function due to uncertainty in the lift curve, not to mention the possible uncertainty caused by different flight conditions. To remedy this, feedback through washout filters is employed. By using washout filters, no approximation of the nominal equilibrium branch is necessary, and none of the equilibrium branches are deformed. Although the approximation of the lift coefficients and the trigonometric functions are still invoked in the dynamic equations, the equilibrium of the real system will none the less be preserved.

Two washout filters with identical time constant $d^{-1} = 0.1$ are inserted into the feedback loop of aircraft model (7.11) in the following study:

$$\dot{z}_1 = \alpha - dz_1 \tag{7.16a}$$

$$\dot{z}_2 = q - dz_2 \tag{7.16b}$$

with output equations

$$y_1 = \alpha - dz_1 \tag{7.17a}$$

$$y_2 = q - dz_2. \tag{7.17b}$$

The control function is now given by

$$\delta_e = \bar{\delta}_e + k_1 y_1 + k_2 y_2 + c_1 y_1^2 + h_1 y_1^3 + h_2 y_2^3, \tag{7.18}$$

with coefficients $k_1 = 0.317$, $k_2 = 0.0836$, $c_1 = h_1 = h_2 = 0.8$, identical to those used in Sec. 7.3.3 for direct state feedback. The closed-loop equilibrium branches along with the stable periodic solutions are shown in Figures 7.10a and 7.10b. Note that the equilibrium branches coincide exactly with those of the open-loop system.

A Hopf bifurcation occurs at parameter value $\bar{\delta}_e = -0.1085$, for which the angle-of-attack $\alpha = 0.4999$, and $q = -0.2511$. There are stable limit cycles emerging from this Hopf bifurcation point toward higher $|\bar{\delta}_e|$. After $|\bar{\delta}_e| > 0.11403$ (was 0.11349 for direct state feedback), the limit cycles disappear (fold back and become unstable). Figure 7.11 depicts post-critical stable limit cycles for four different values of $\bar{\delta}_e$.

The major advantage of using the washout filters is that the need for approximation of the open-loop equilibrium branches is removed. Also, the preservation of equilibrium branches prevents or delays the occurrence of the homoclinic bifurcation. The occurrence of a homoclinic orbit in the case of direct state feedback was associated with the fold point being moved toward the Hopf bifurcation point. Although the stable limit cycles in this washout filter-aided feedback case also disappear for $|\bar{\delta}_e|$ beyond 0.11403, the preservation of nominal equilibrium branch beyond this point opens the possibility of extending the stable limit cycles to higher values of $\bar{\delta}_e$ by using a different nonlinear stabilizing control law.

7.4. High α control: improved model

In this section, we improve the model employed in Section 7.3 in two ways. First, the approximation of the trigonometric terms in the dynamic equations Eqs. (7.4a)-(7.4c) is not invoked here since the magnitude of the arguments in those trigonometric terms is not small during high α flight. Secondly, the wing lift force coefficient which was approximated by cubic functions of α in Section 7.3 is modified by invoking a sharp *window function* to accelerate drop of lift force after stall. The wing lift coefficient is now given by

$$C_{lw} = (C_{lw}^1 \alpha - c_{lw}^2 \alpha^3)[\frac{1}{1+(\frac{\alpha}{\alpha_p})^m}], \tag{7.19}$$

where m relates to the abruptness of the drop of the lift curve, and α_p is the half-way drop point. This sharp drop of the lift coefficient reflects a more realistic wing lift force vs. α profile.

Six different combinations of m and α_p are chosen for the window function in Eq. (7.19), $\alpha \in \{0.41, 0.45\}$, $m \in \{10, 30, 60\}$, to reflect some possible uncertainty in estimating the lift force. Our goal is to design a fixed feedback controller which stabilizes the aircraft under all these different lift force profiles up to certain useful range of α, and also provides a warning of impending stall by small periodic motion. Figure 7.12 shows the profiles of these six approximations of wing lift force coefficients and the pure cubic approximating coefficient.

Recall the dynamic equations (7.4a)-(7.4c). With the assumption of constant speed, level flight condition and neglecting the effects of varying weight components in body axes, the equations of motion are given by

$$\begin{aligned}\dot{\alpha} =& q\cos^2\alpha - 0.03808\cos^2\alpha - (0.845\alpha\cos\alpha - 2.536\alpha^3\cos^3\alpha)W \\ & - (0.0526\alpha + 0.00987\alpha^3 + 0.2158\delta_e + 0.11843\alpha^2\delta_e + 0.4737\alpha\delta_e^2 \\ & + 0.6316\delta_e^3\cos^2\alpha)\cos^2\alpha\cos(0.25\alpha + \delta_e), \qquad (7.20a)\end{aligned}$$

$$\begin{aligned}\dot{q} =& -0.396q + (0.9319\alpha - 2.796\alpha^3)(\cos\alpha)W - (5.127\alpha + 0.9613\alpha^3 + 21.02\delta_e \\ & + 11.536\alpha^2\delta_e + 46.144\alpha\delta_e^2 + 61.526\delta_e^3)\cos(0.25\alpha + \delta_e), \qquad (7.20b)\end{aligned}$$

where

$$W := \frac{1}{1 + (\frac{\alpha}{\alpha_p})^m}. \qquad (7.20c)$$

Using the computer program BIFOR2 [32], we find that, prior to the application of feedback, the aircraft dynamical model has two subcritical (i.e. *unstable*) Hopf bifurcations for each lift force profile. Table 7.2 lists the Hopf bifurcation points and their corresponding eigenvalues and stability coefficients. Figure 7.13 show the nominal equilibrium branches obtained using the computer program AUTO [34] for these six lift force profiles. The two subcritical Hopf bifurcation branches of each lift force profile turn back and join together after emerging from the Hopf bifurcation points. This implies that, if $\bar{\delta}_e$ goes slightly beyond the first Hopf bifurcation point, the aircraft will jump into a large amplitude oscillation, which is not acceptable.

To remedy this, we give two stabilizing washout filter-aided nonlinear feedback controllers for the system. These controllers do not destroy the Hopf bifurcations discussed above, nor do they shift the critical parameter values for which these Hopf bifurcations occur. Rather, the *direction* of the Hopf bifurcations is reversed, rendering them supercritical, i.e., stable.

7.4.1. The first stabilizing controller

In the first design, we choose q passed through a washout filter as the feedback signal. The output, y_1, of the washout filter is given by

$$y_1 = q - dz_1. \tag{7.21}$$

Here z_1 denotes the state variable introduced by the filter and d is the inverse washout filter time constant. As in Section 7.3, denote by $\bar{\delta}_e$ the constant commanded value of the elevator deflection. Set $d = 1$. Feeding back y_1^3 to the elevator input, we have

$$\delta_e = \bar{\delta}_e + ck_1 y_1^3. \tag{7.22}$$

The reason for choosing y_1^3 as a feedback signal is that it is found to be the most effective among all cubic in making the stability coefficient, β_2, negative. This can be easily checked by using the BIFOR2 program or by comparing the effects of each cubic term on $\mathrm{R_e}\Delta$ of Eq. (4.62b) in Chapter 4. The value of d is chosen for robustness considerations, that is to render the real part of variable P defined in Chapter 4 remaining the same sign at all Hopf bifurcation points under the six different lift force profiles. By doing this, with a suitable choice of the feedback gain, we can have a robust controller to stabilize all the Hopf bifurcations existing for the various lift force profiles.

Set the gain $ck_1 = 20$. Table 7.3 shows the stability coefficients for all the Hopf bifurcations of the six different lift curve profiles. The negativity of all of these stability coefficients implies the stability of all the bifurcation points and the bifurcated periodic solutions emerging from them within a small range of parameter values. In fact, for the lift force profiles with $m = 10$, the control results in a window of small amplitude stable periodic orbits encircling the equilibrium for values of the elevator deflection parameter between the two adjacent Hopf bifurcation points. Figures 7.14 - 7.15 depict these results as produced by the program AUTO.

For the cases where $m = 30$ and 60, the stable limit cycles (shown in Figures 7.16 - 7.19) fold backward and become unstable after emerging from both Hopf bifurcation points for a small range of parameter values. After another small range of parameter values, these unstable limit cycles turn forward, become stable again, and finally join. Although the Hopf bifurcation points and the periodic solutions emerging from these Hopf bifurcation points for parameter values near the critical parameter values are stable, and the amplitudes of these stable periodic solutions are small, for the parameter values slightly beyond this range, there still exists a "jump" to larger amplitude (smaller than that of the uncontrolled system, however) limit cycles. Moreover, there will be a hysteresis and other jumps associated with recovery from those large amplitude oscillations.

To further improve on the above, we design the second washout filter-aided nonlinear feedback controller in the fellowing section.

7.4.2. The second stabilizing controller

Let α passed through another washout filter serve as another feedback signal. The output, y_2, of this washout filter is given by

$$y_2 = \alpha - dz_2. \tag{7.23}$$

Here z_2 denotes the state variable introduced by the new filter and d is the inverse of the new washout filter time constant which is set equal to 1, as for the previous filter. Add the new control term, $y_1^2 y_2$, to the elevator input. The control is now given by

$$\delta_e = \bar{\delta}_e + ck_1 y_1^3 + ck_2 y_1^2 y_2. \tag{7.24}$$

Set the gains $ck_1 = 15$, $ck_2 = 40$. The two unstable periodic branches on top of the two stable branches for the cases in which $m = 30$ and 60 are removed. Now, the outer large amplitude periodic branch is moved down and merged with the two inner stable periodic branches. Thus, the jump and hysteresis phenomena in the previous design are removed and the amplitudes of periodic solutions between the two adjacent Hopf bifurcation points are significantly reduced. Figures 7.20 and 7.23 show the maximum amplitudes of these stable periodic branches. Table 7.4 lists the stability coefficients at each Hopf bifurcation point.

Note that $y_1^2 y_2$ is less effective in rendering β_2 negative than y_1^3. The magnitudes of β_2 in Table 7.4 are smaller than those in Table 7.3 where the first design is applied. However, $y_1^2 y_2$, in this case, is very effective in bringing down the outer periodic branch and eliminating the unstable branches. This is a very important effect as far as the responses of the parameter values between the two adjacent critical parameter values are concerned. This effect may not be easily predicted by the values of β_2 alone since β_2 provides information only on the local behavior near a critical point. Indeed, the amplitudes of the periodic solutions of the first design are smaller than that of the second design for parameter values sufficiently close to the critical values.

7.4.3. Simulation of a third order model

The designs and analysis above are based on the pseudo steady-state of the reduced model, the fast short-period mode. In order to check the model, we consider a more accurate longitudinal model which includes the "slow" gravity effects. This model is Eqs. (7.4b),(7.4c) with u = constant, written in state space form, including the variation of θ through the equation $\dot{\theta} = q$. We perform time domain simulations of this third order model for α, θ, and q for two cases: the uncontrolled case, and the case using the second stabilizing controller. Figures 7.24 - 7.29 show the time simulations of this third order model for $\bar{\delta}_e = -0.088$ and -0.105. For the uncontrolled system, there are large oscillations for parameter values between the two critical values. For the parameter value $\bar{\delta}_e = -0.088$ where the pitch rate q is close to its peak value, there is a slow frequency modulating the periodic solution of the fast mode. This slow frequency comes from the effects of varying weight components in the body axes (due to the θ-variation). For the parameter value $\bar{\delta}_e = -0.105$ where q

is small, the effect of the slow mode is very insignificant. For the preceding two chosen parameter values, the amplitudes of oscillation are significantly reduced after feedback is applied. In fact, the amplitudes of the oscillations can be further reduced if the feedback gains are appropriately increased.

$C^1_{L_w} = 4.0$	$C^2_{L_w} = 12.0$
$C^1_t = 4.0$	$C^2_{L_t} = 12.0$
$a_e = 0.1$	$S = 375 \text{ ft}^2$
$S_t = 93.4 \text{ ft}^2$	$m = 667.7$ slugs
$a_\epsilon = 0.75$	$I_y = 96800.0 \text{ slug} \cdot \text{ft}^2$
$l = 0.189$ ft	$l_t = 16.7$ ft
$M_w = 0.$	$c = 38332.8$
$g = 32.2 \text{ ft/sec}^2$	$\bar{q} = 318.19 \text{ lb/ft}^2$

Table 7.1. Aircraft data

Lift curve window m	α_p	Critical θ_e	α	q	β_2	Critical Eigenvalues
10	0.41	-0.0737	0.327	0.124	3.7	$\pm j2.219$
		-0.1137	0.472	-0.015	11.6	$\pm j2.096$
10	0.45	-0.0796	0.352	0.125	2.1	$\pm j2.202$
		-0.1215	0.504	-0.016	8.3	$\pm j2.063$
30	0.41	-0.0817	0.361	0.133	153.8	$\pm j2.194$
		0.1087	0.450	0.030	202.4	$\pm j2.118$
30	0.45	-0.0888	0.390	0.127	70.9	$\pm j2.171$
		-0.1179	0.488	-0.029	163.9	$\pm j2.093$
60	0.41	-0.0860	0.379	0.131	489.3	± 2.180
		-0.1053	0.435	-0.034	931.7	± 2.202
60	0.45	-0.0936	0.408	0.121	128.4	± 2.155
		0.1150	0.475	0.336	767.0	± 2.093

Table 7.2. Hopf bifurcation data for uncontrolled system

Lift curve window	$m = 10$ $\alpha_p = 0.41$	$m = 10$ $\alpha_p = 0.45$	$m = 30$ $\alpha_p = 0.41$	$m = 30$ $\alpha_p = 0.45$	$m = 60$ $\alpha_p = 0.41$	$m = 60$ $\alpha_p = 0.45$
β_2 of 1st Hopf	-1247	-1263	-1118	-1220	-794	-1176
β_2 of 2nd Hopf	-1348	-1381	-1138	-1210	-397	-596

Table 7.3. Hopf bifurcation data under the first control law applied

Lift curve window	$m = 10$ $\alpha_p = 0.41$	$m = 10$ $\alpha_p = 0.45$	$m = 30$ $\alpha_p = 0.41$	$m = 30$ $\alpha_p = 0.45$	$m = 60$ $\alpha_p = 0.41$	$m = 60$ $\alpha_p = 0.45$
β_2 of 1st Hopf	-994	-1007	-860	-959	-534	-913
β_2 of 2nd Hopf	-1074	-1101	-867	-933	-120	-314

Table 7.4. Hopf bifurcation data under the second control law applied

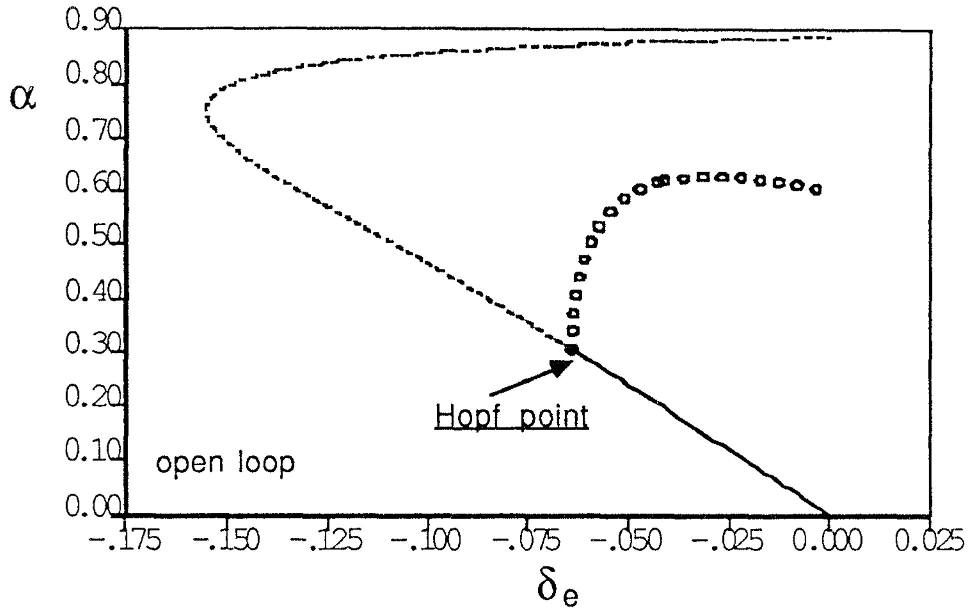


Figure 7.4a. α at open-loop equilibria

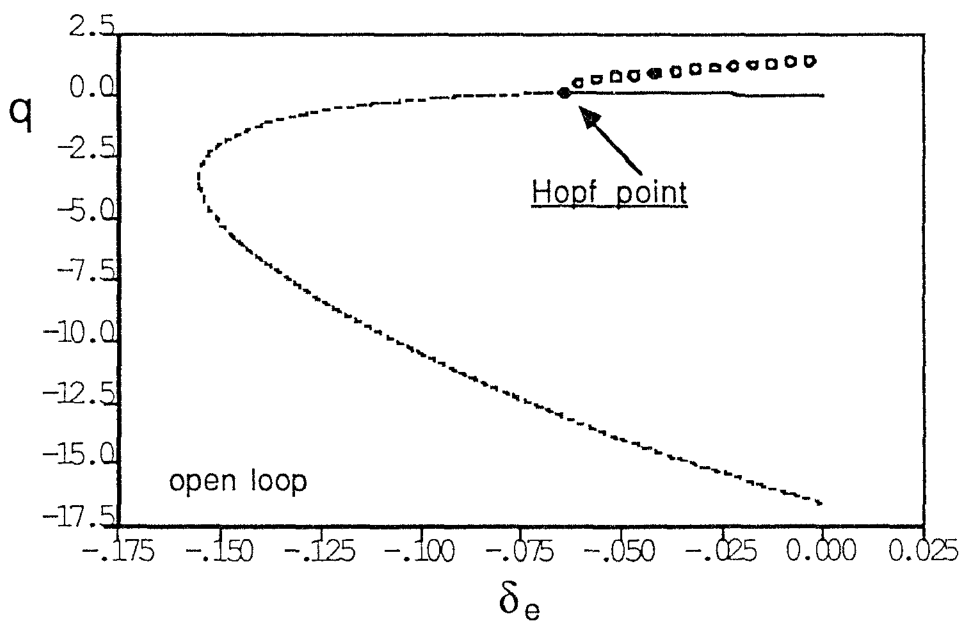


Figure 7.4 b. q at open-loop equilibria

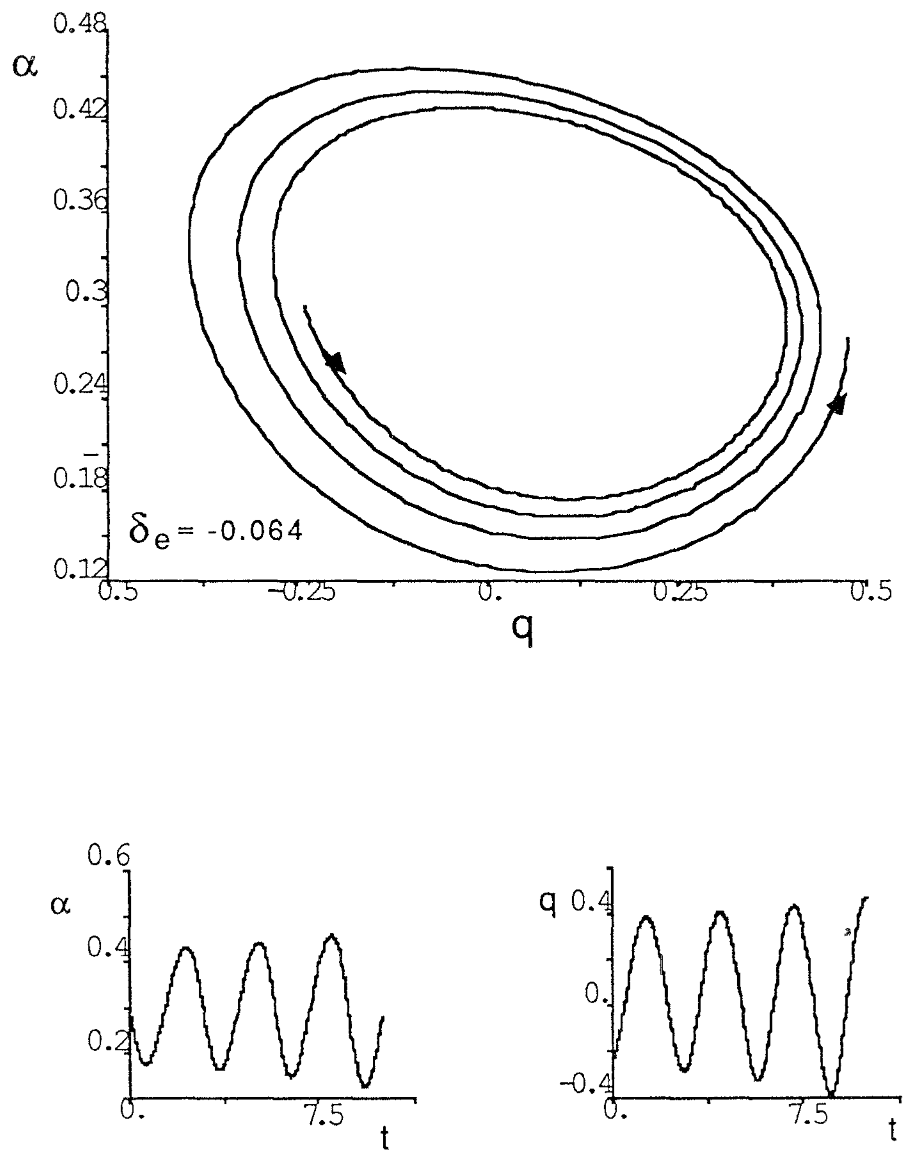


Figure 7.5 . Divergence of uncontrolled system

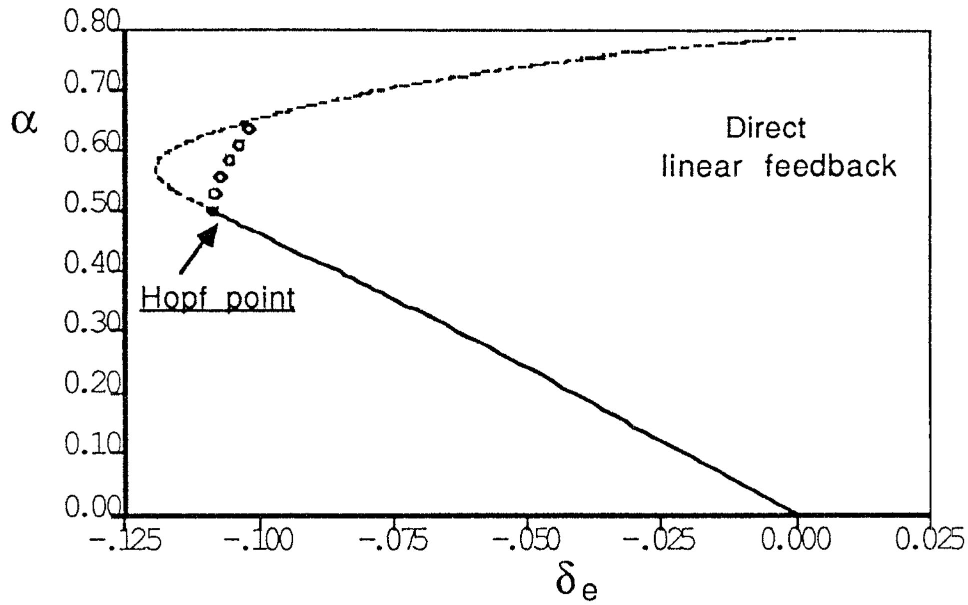


Figure 7.6 a. α at equilibira upon direct linear feedback

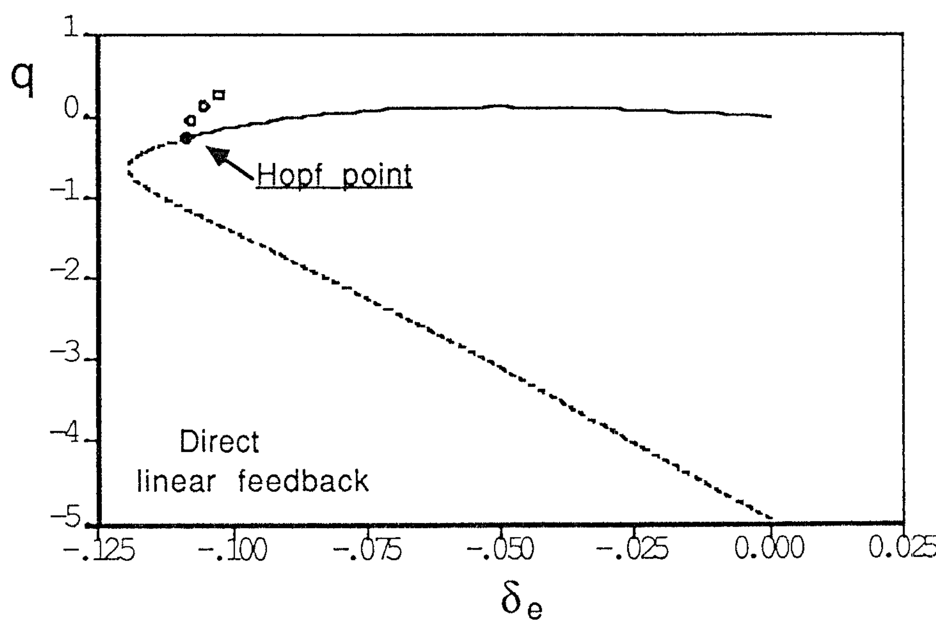


Figure 7.6 b. q at equilibira upon direct linear feedback

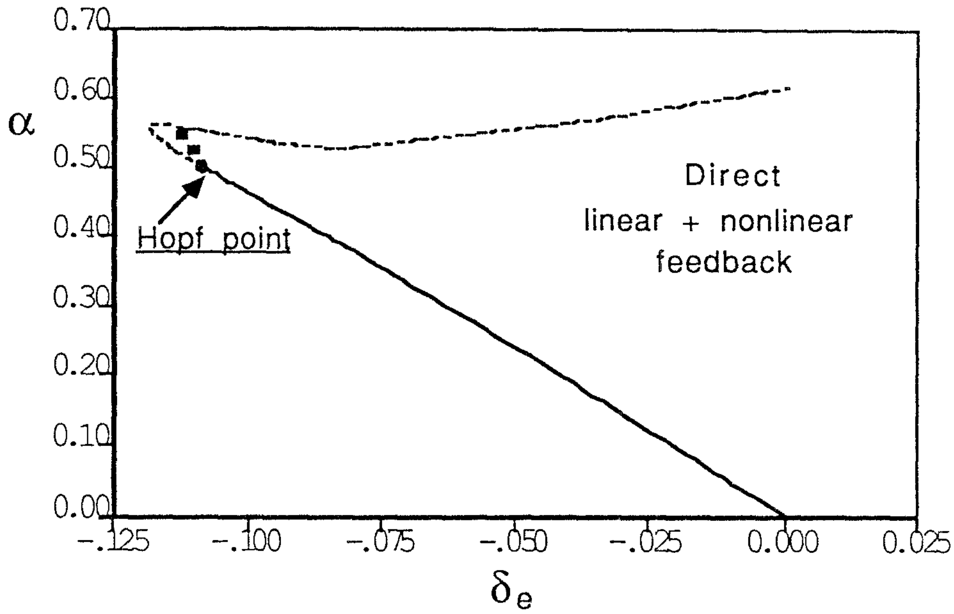


Figure 7.7a. α at equilibria upon direct linear + nonlinear feedback

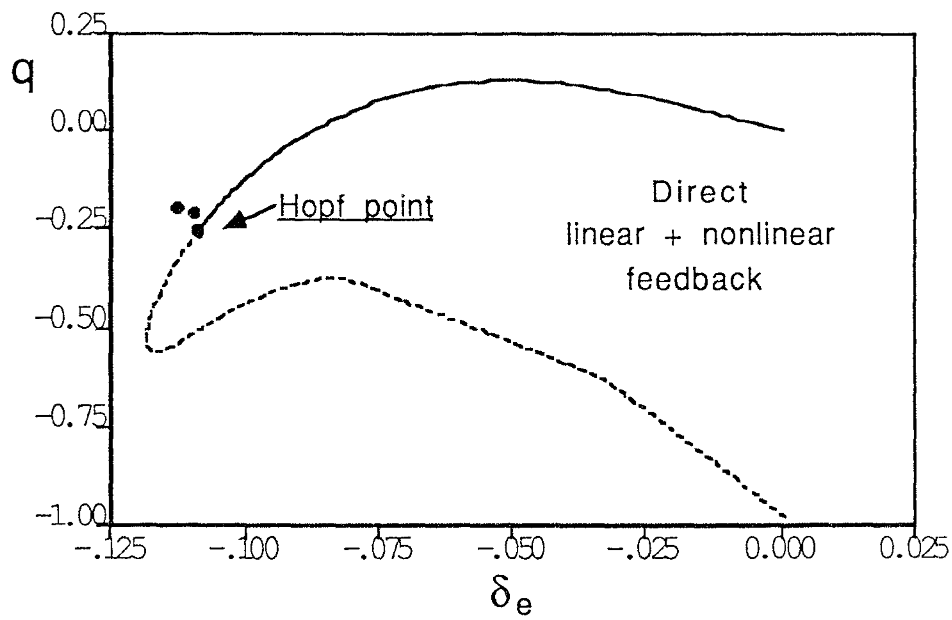


Figure 7.7 b. q at equilibria upon direct linear + nonlinear feedback

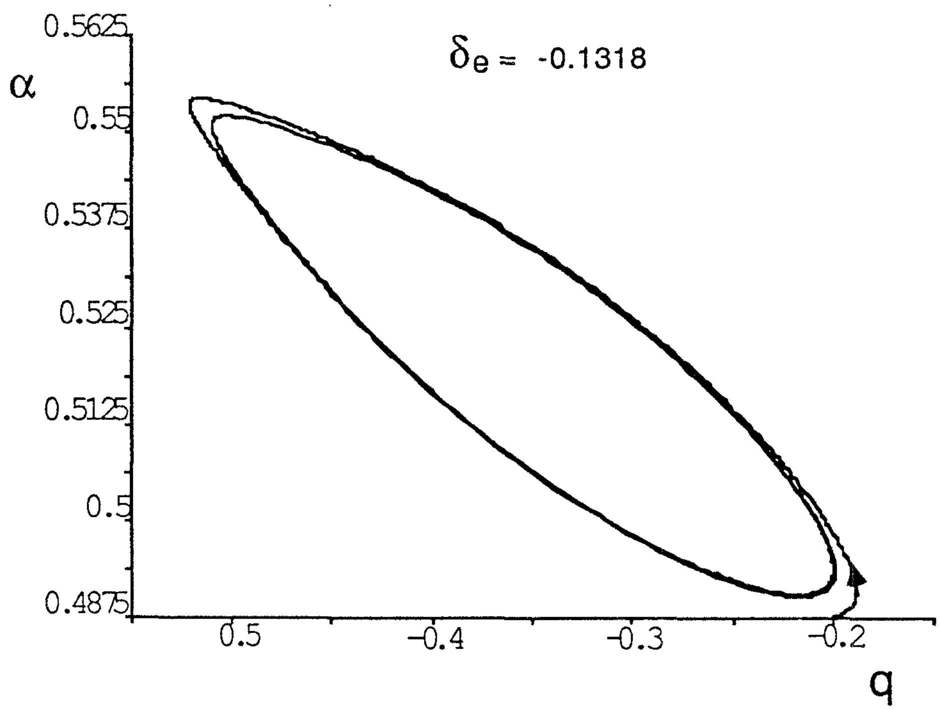


Figure 7.8. Convergence to limit cycle under direct linear + nonlinear feedback

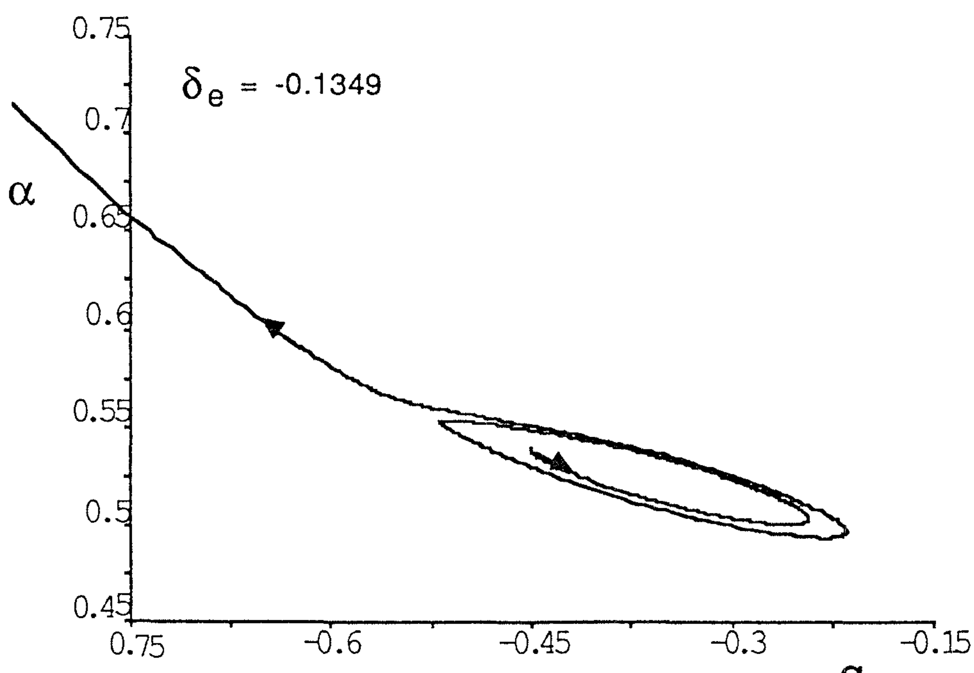


Figure 7.9. Divergence after appearence of homoclinci orbic

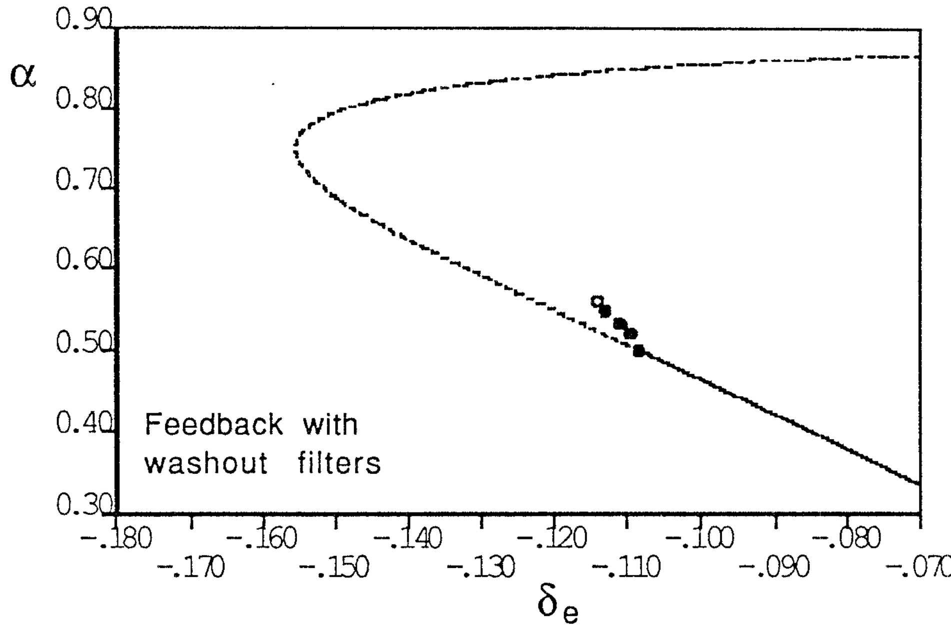


Figure 7.10 a. α at equilibria under washout filter-aided feedback

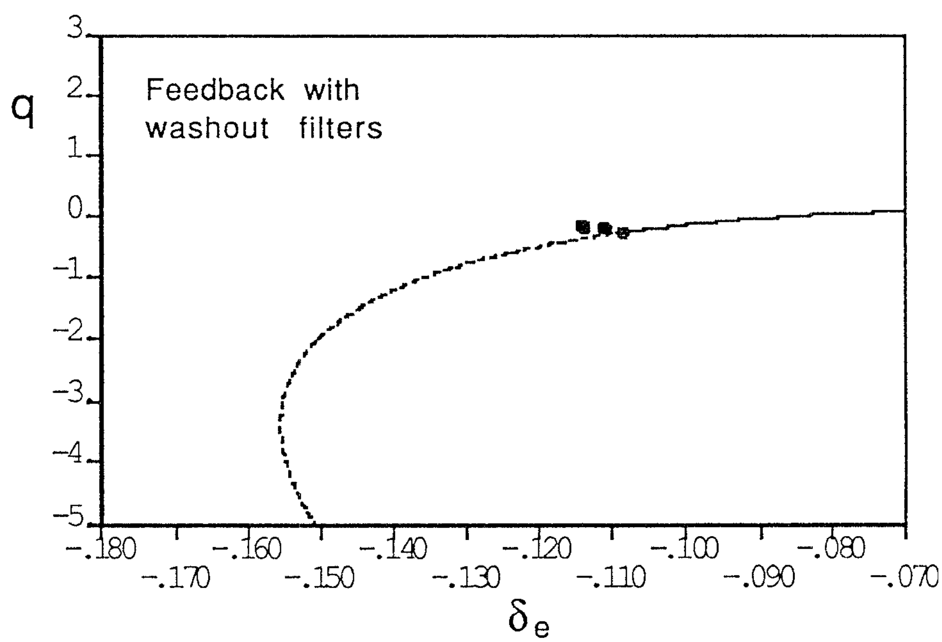


Figure 7.10 b. q at equilibria under washout filter-aided feedback

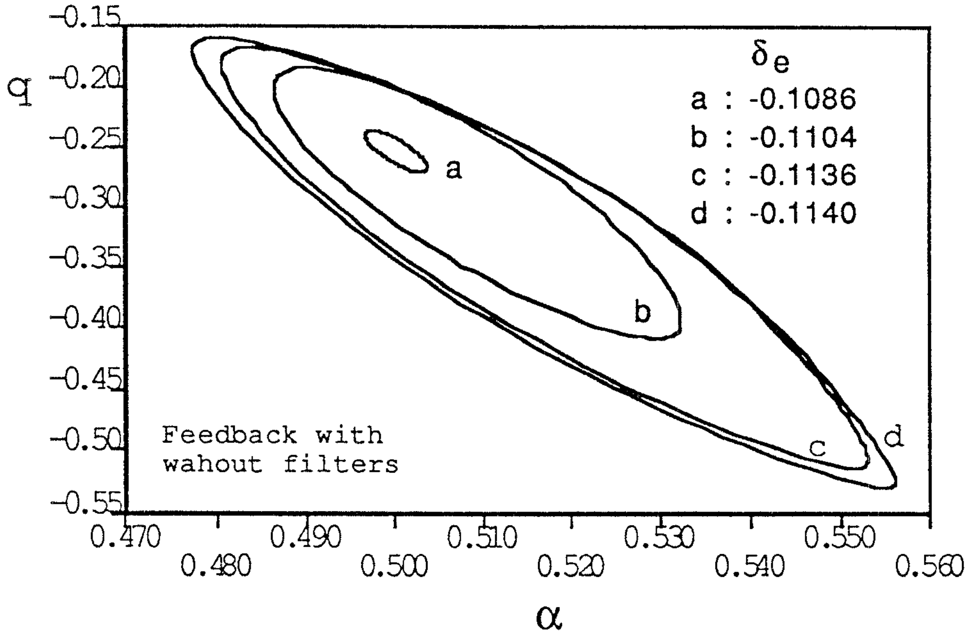


Figure 7.11. Post-critical stable limit cycles under feedback through washout filters

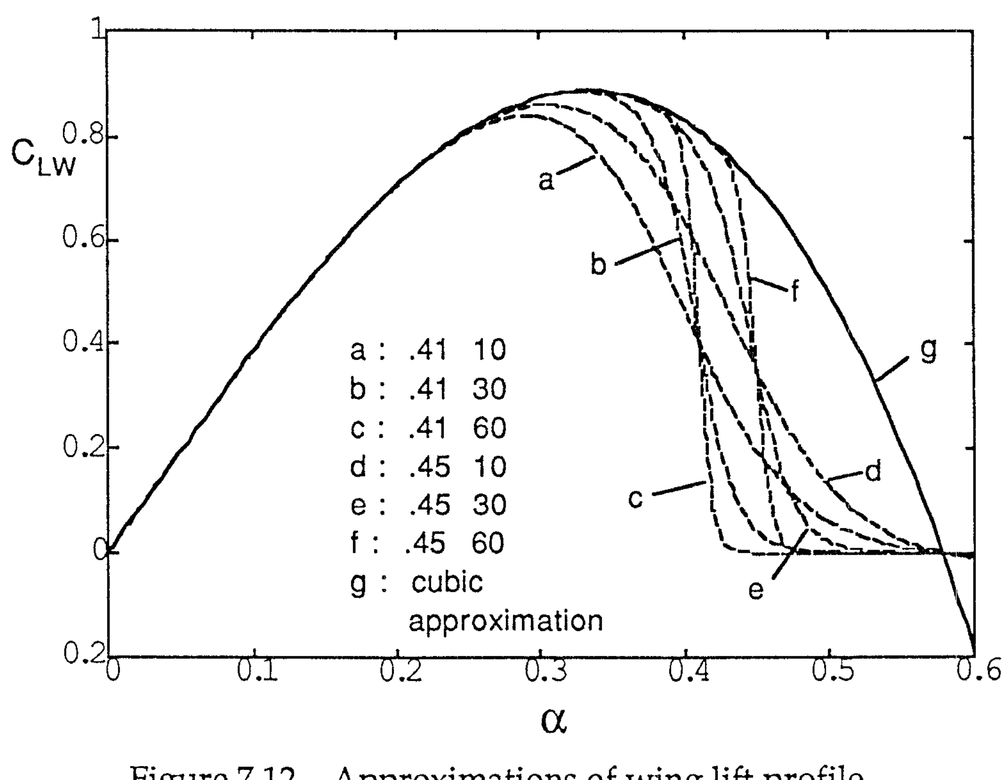


Figure 7.12. Approximations of wing lift profile

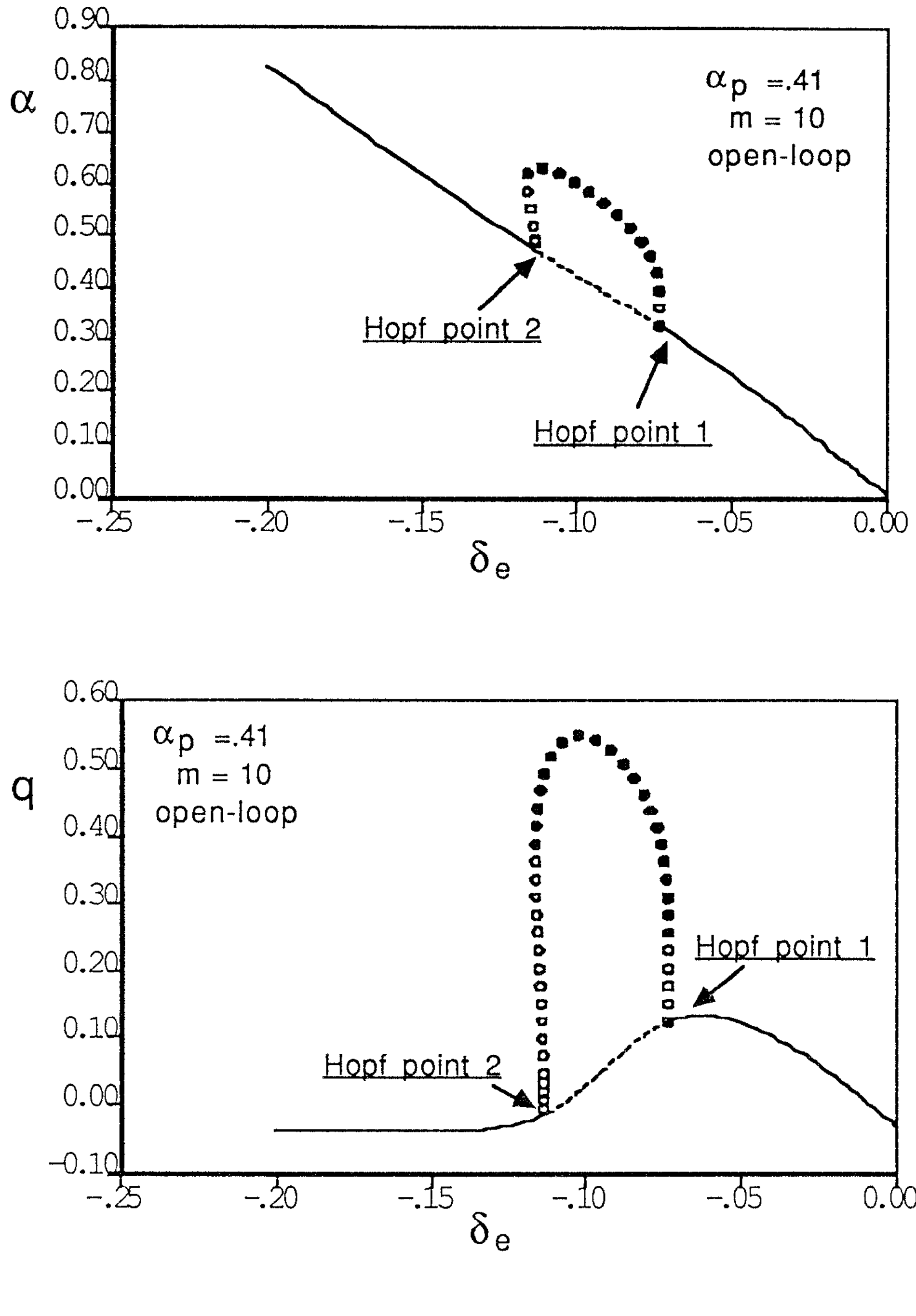


Figure 7.13 a. Open-loop equilibria for the lift curve m=10, α_p =.41

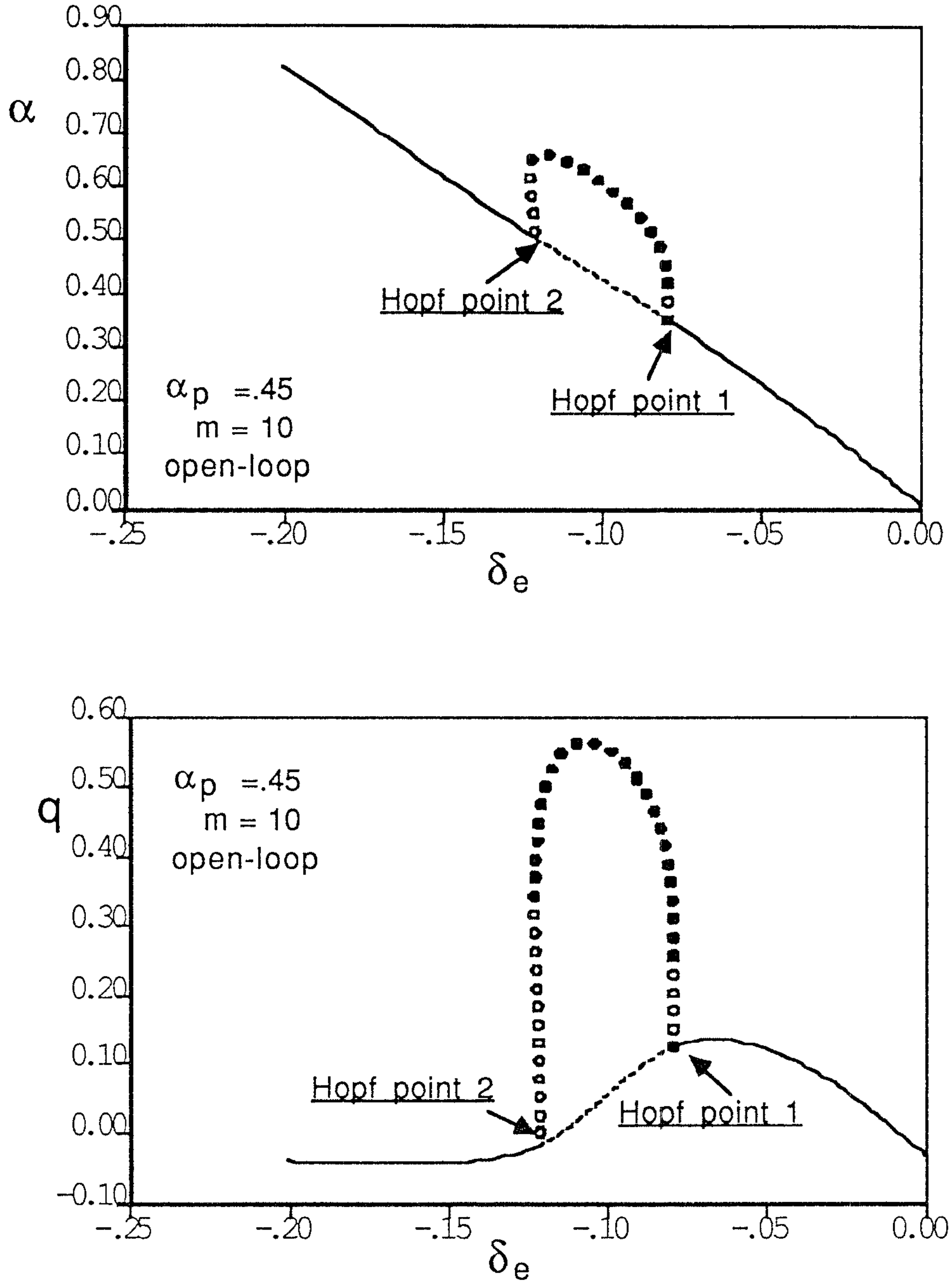


Figure 7.13 b. Open-loop equilibria for the lift curve m=10, α_p =.45

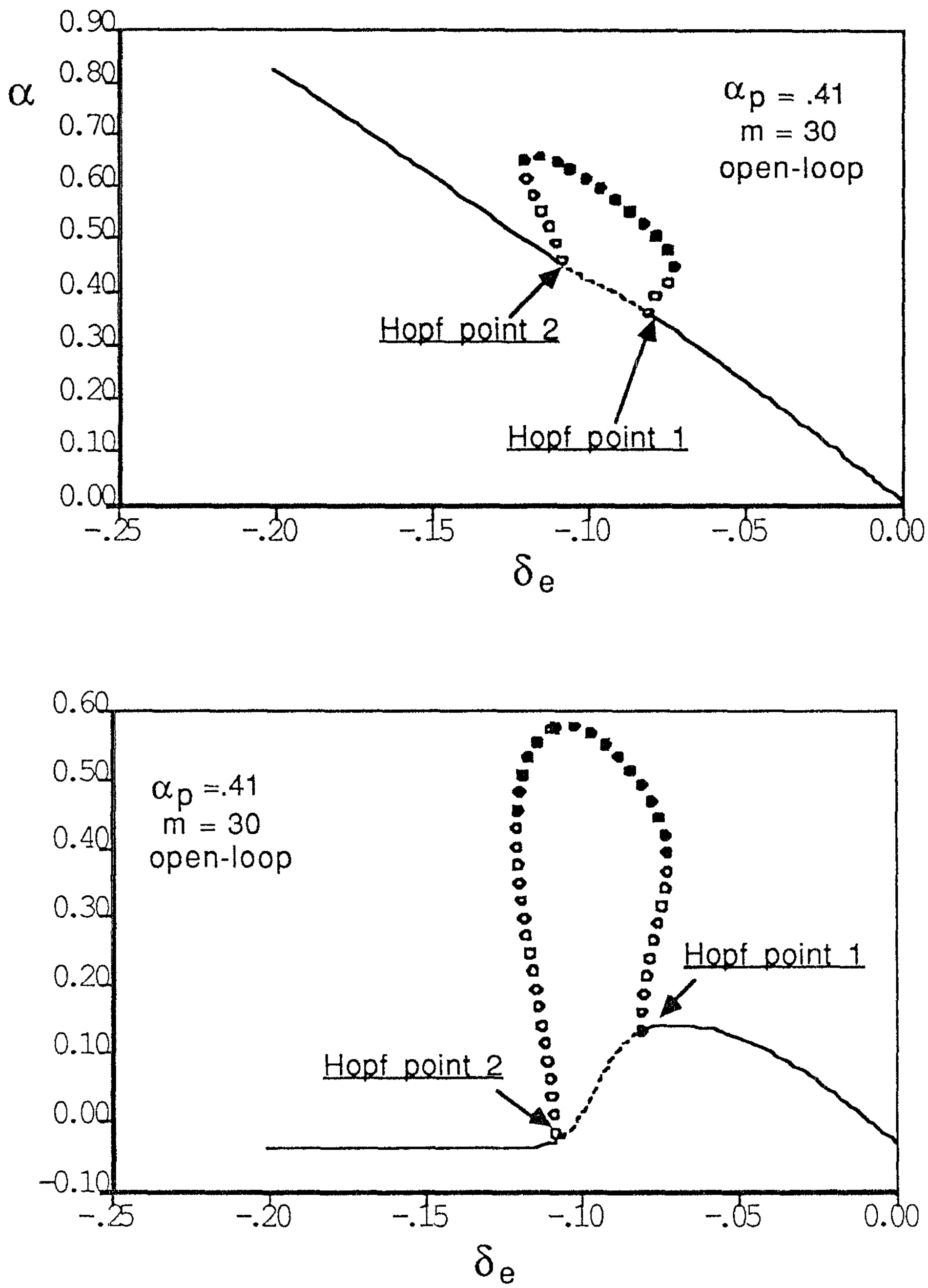
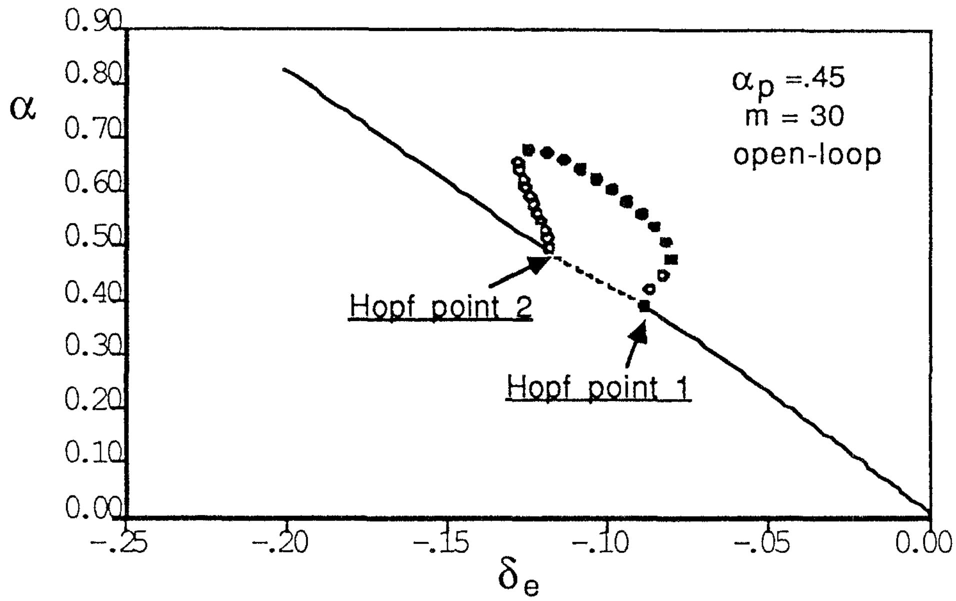


Figure 7.13 c. Open-loop equilibria for the lift curve m=30, α_p =.41



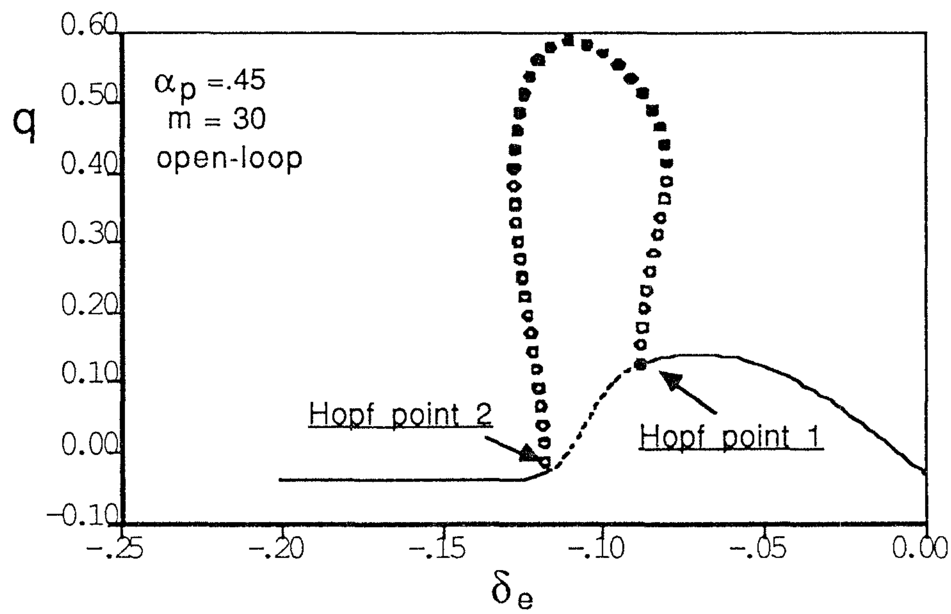


Figure 7.13 d. Open-loop equilibria for the lift curve m=30, α_p =.45

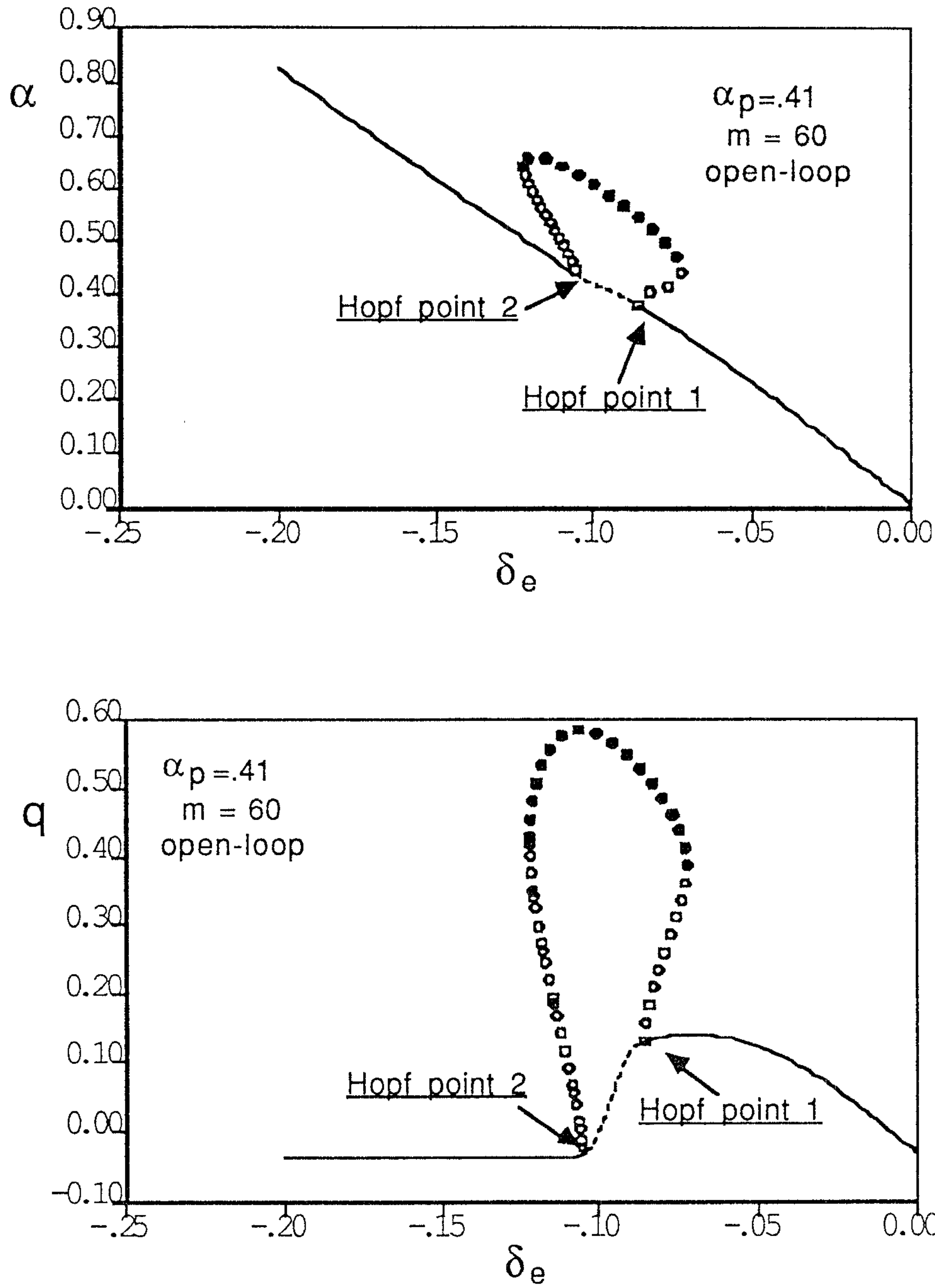


Figure 7.13 e. Open-loop equilibria for the lift curve m=60, α_p =.41

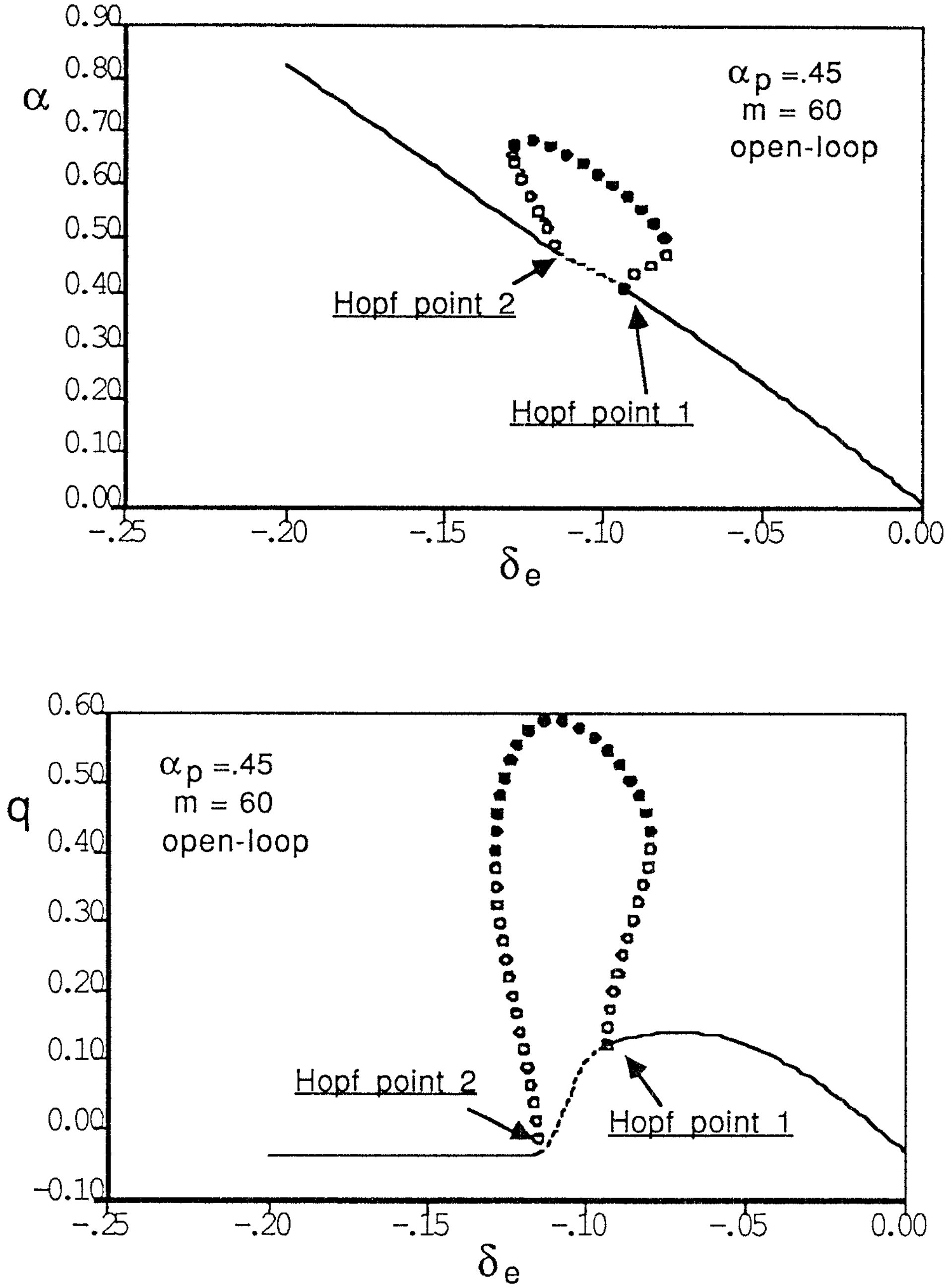


Figure 7.13 f. Open-loop equilibria for the lift curve m=60, α_p =.45

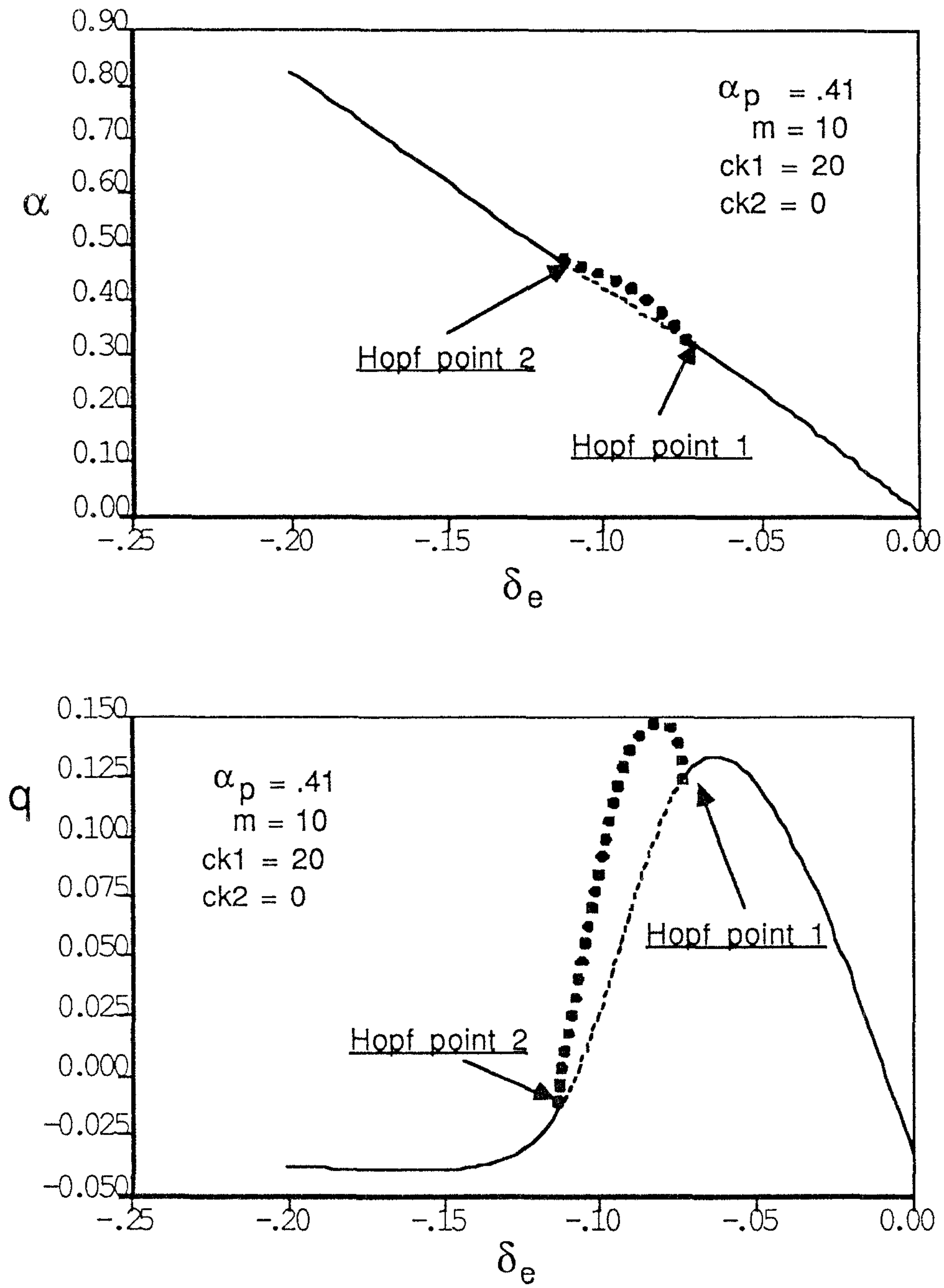


Figure 7.14. Equilibria under the 1st feedback for lift curve m=10 α_p=.41

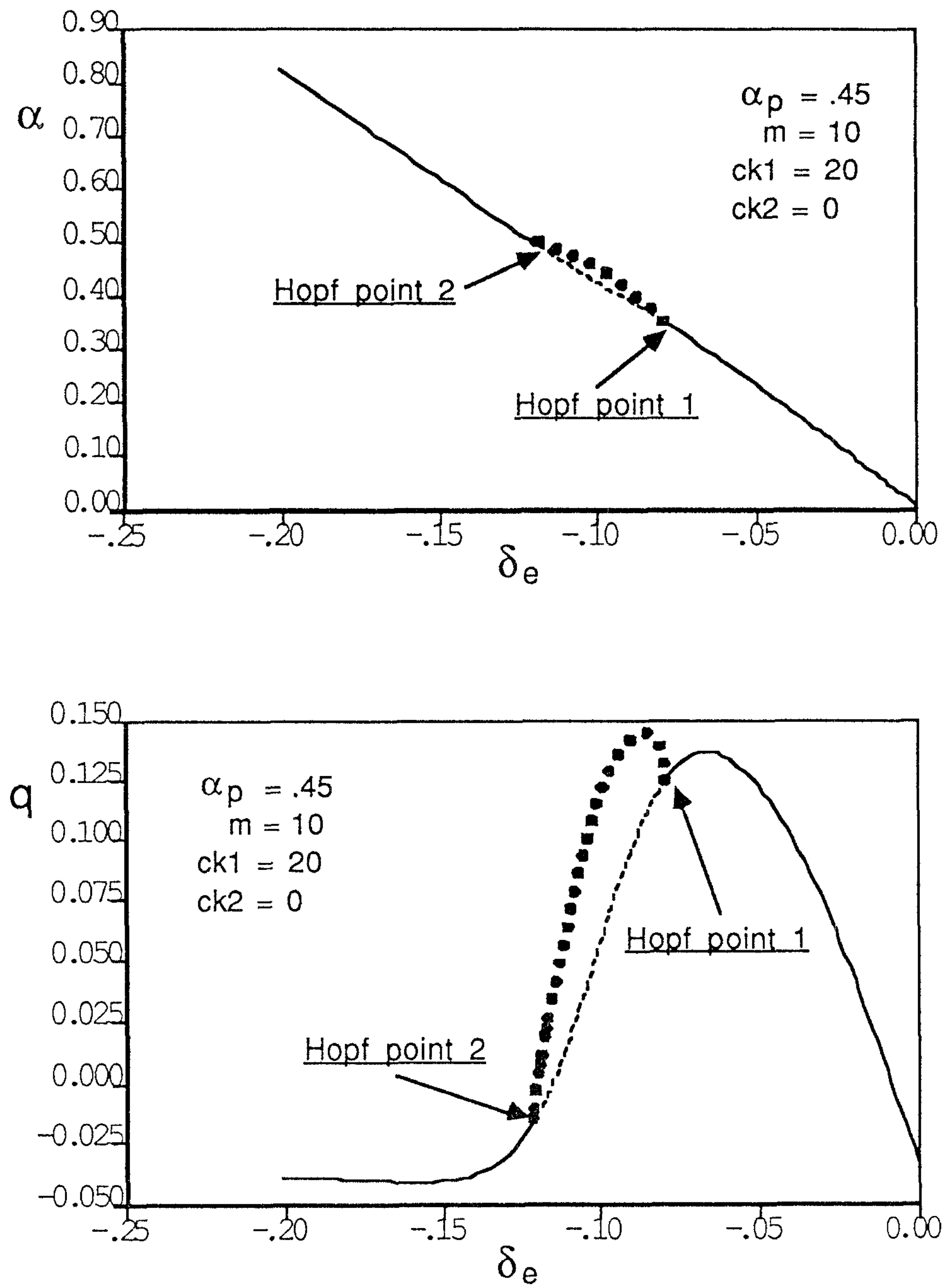


Figure 7.15. Equilibria under the 1st feedback for the lift curve m=10 α_p =.45

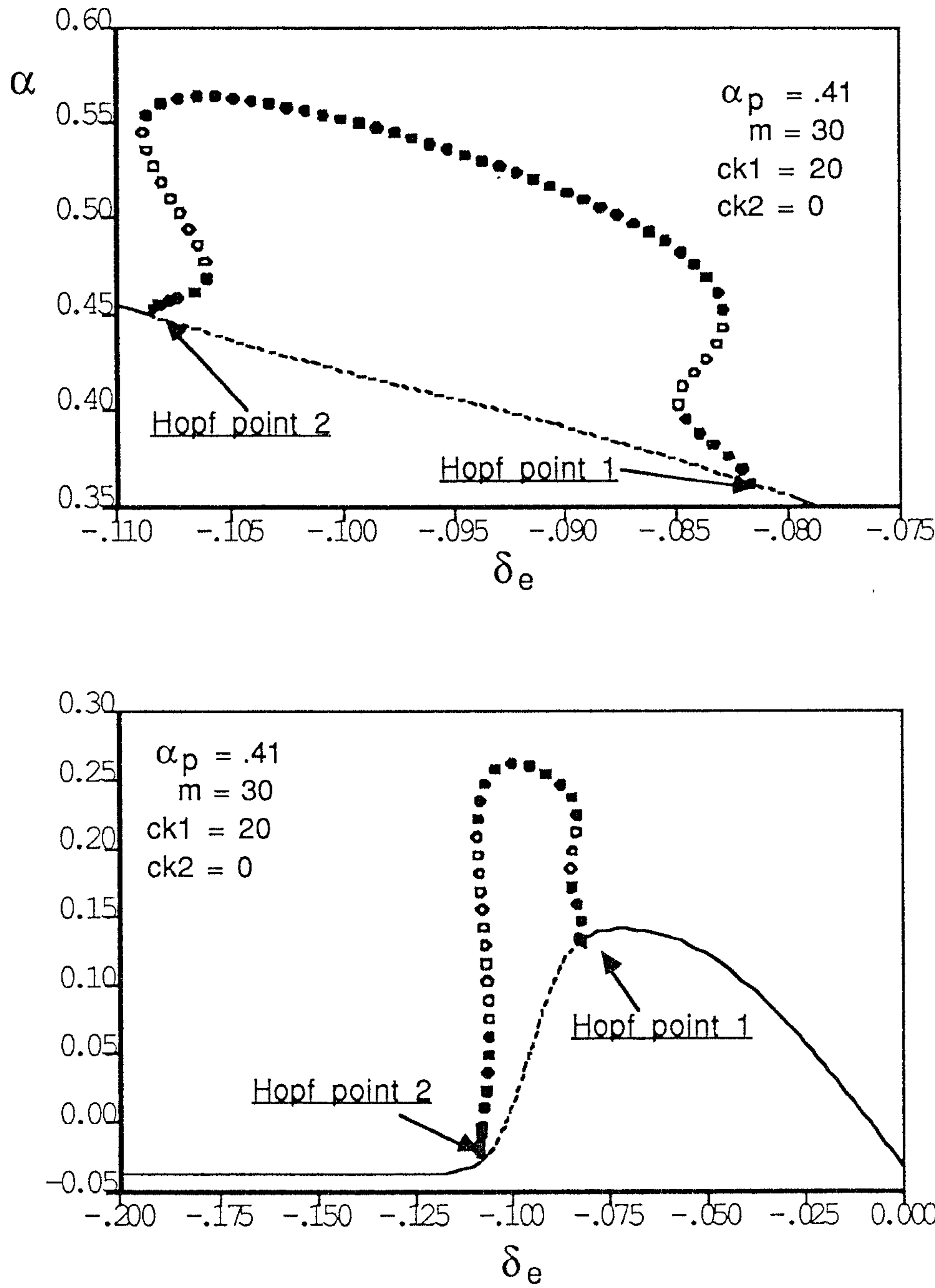
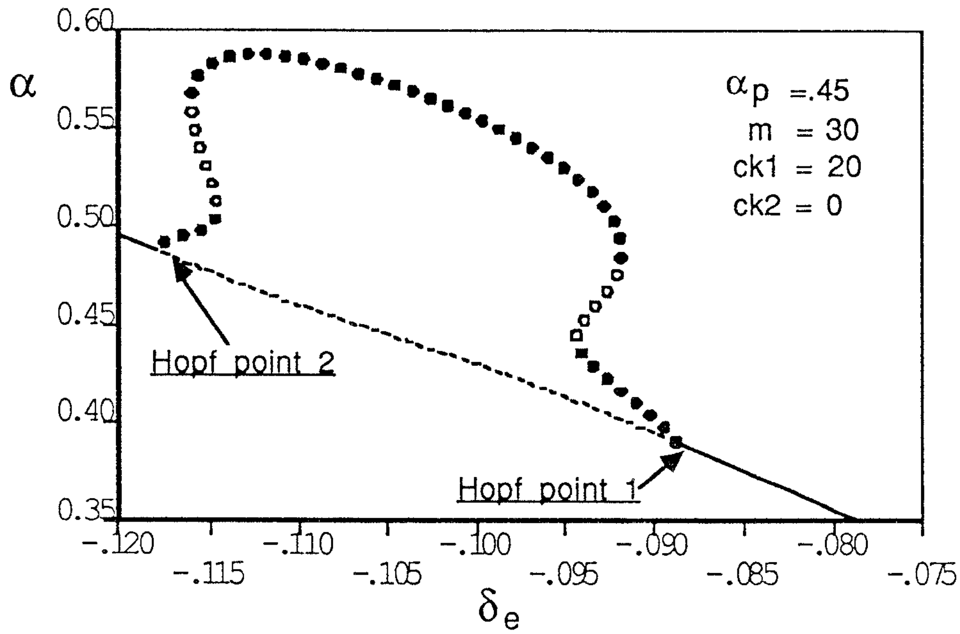


Figure 7.16. Equilibria under the 1st feedback for the lift curve m=30 α_p=.41



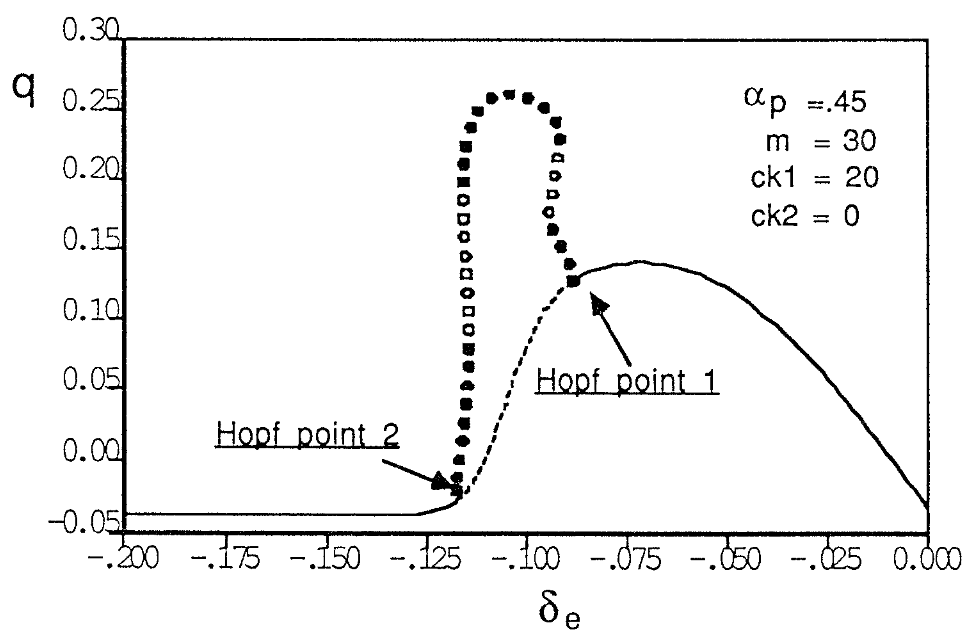


Figure 7.17. Equilibria the 1st feedback for the lift curve m=30 α_p =.45

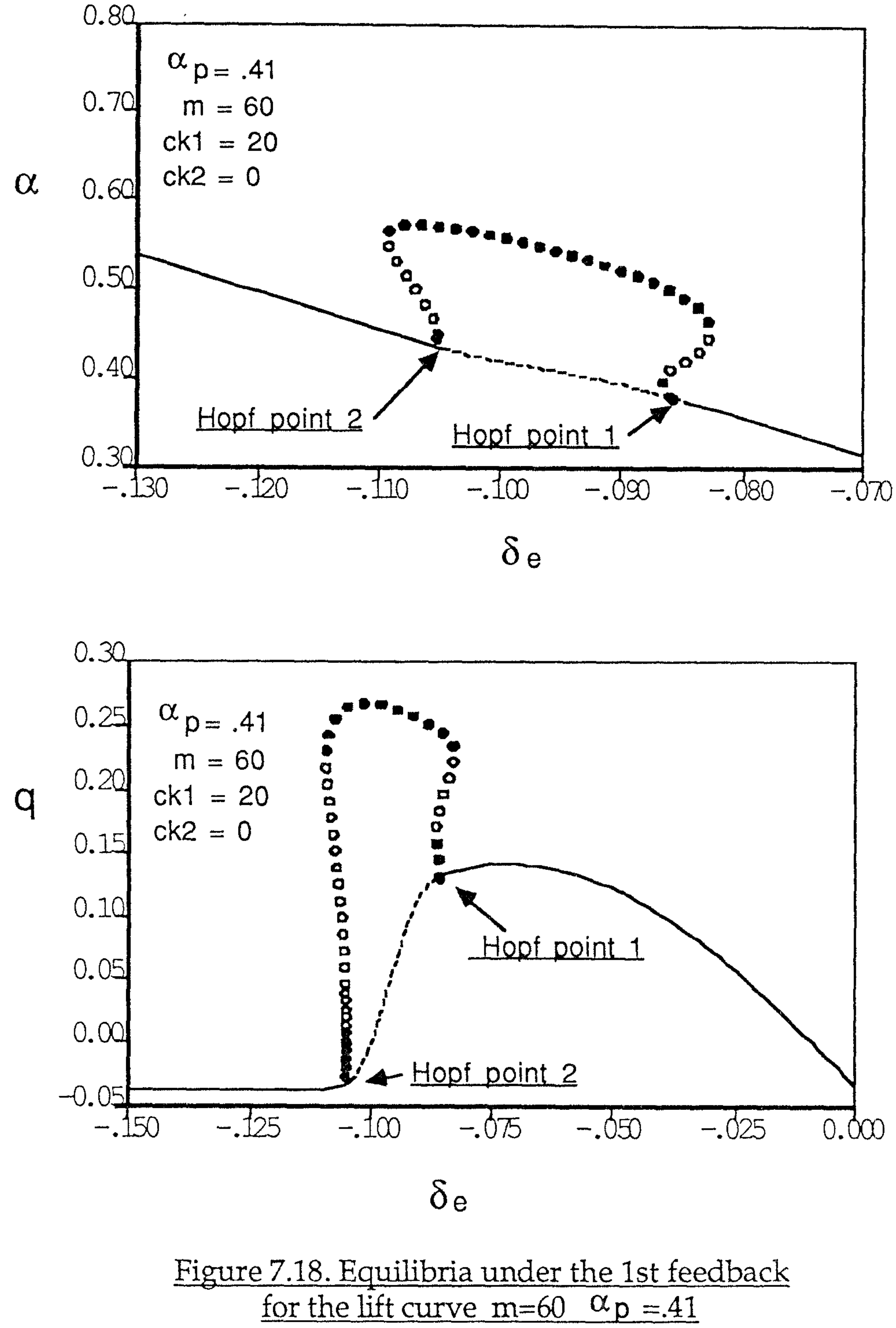


Figure 7.18. Equilibria under the 1st feedback for the lift curve m=60 α_p =.41

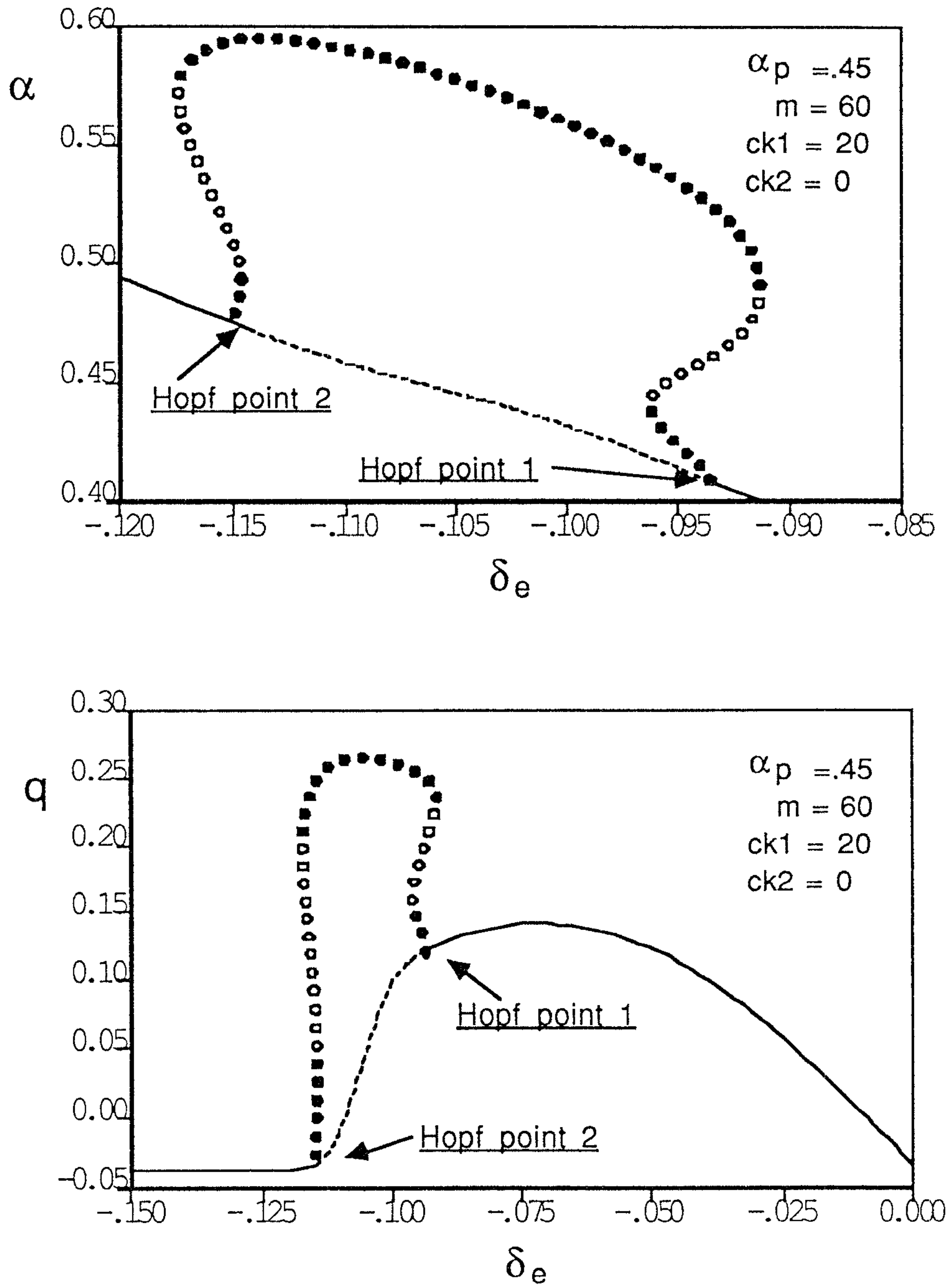


Figure 7.19. Equilibria under the 1st feedback for the lift curve m=60 α_p =.45

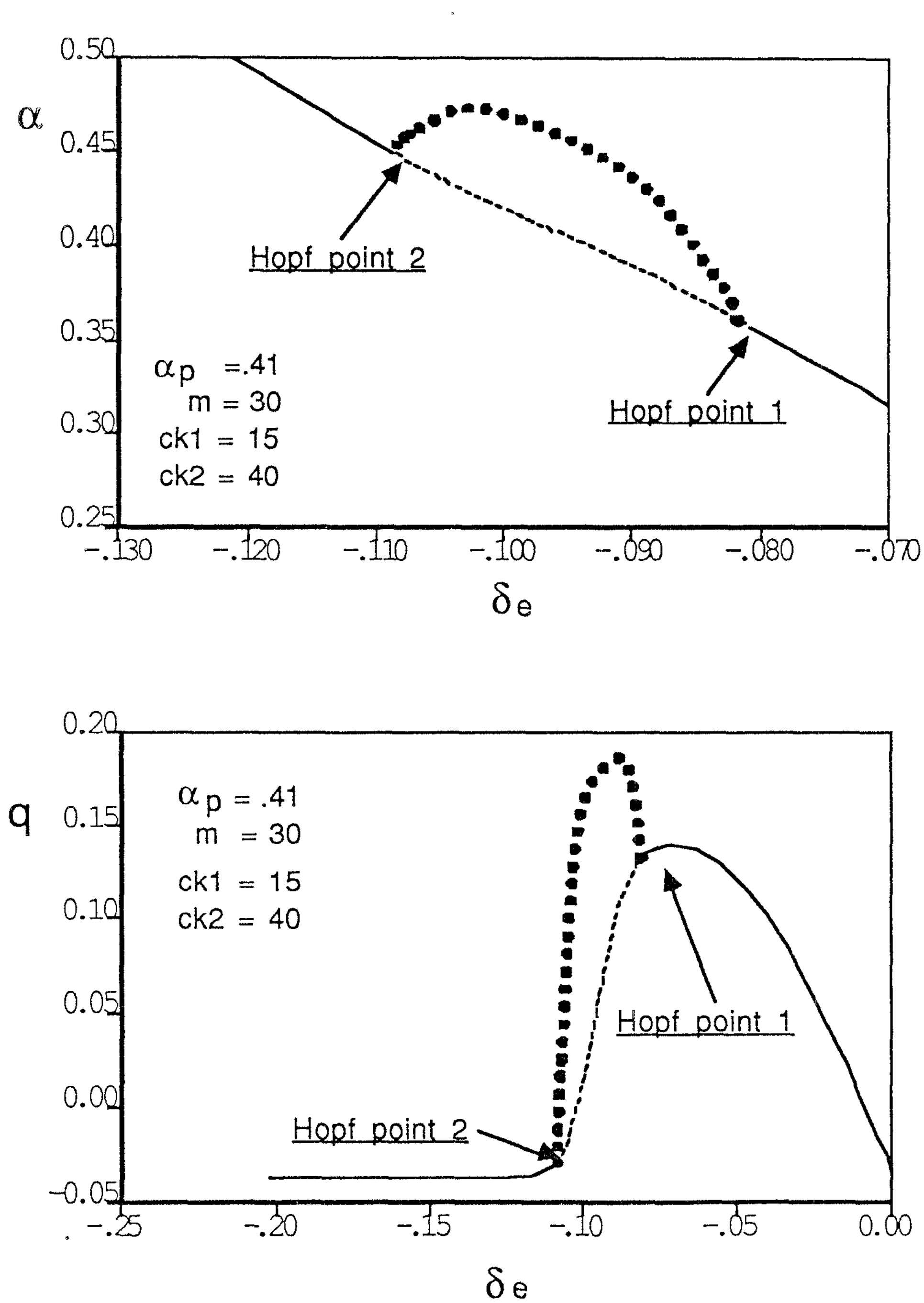


Figure 7.20 . Equilibria under the 2nd feedback for the lift curve m=30 α_p =.41

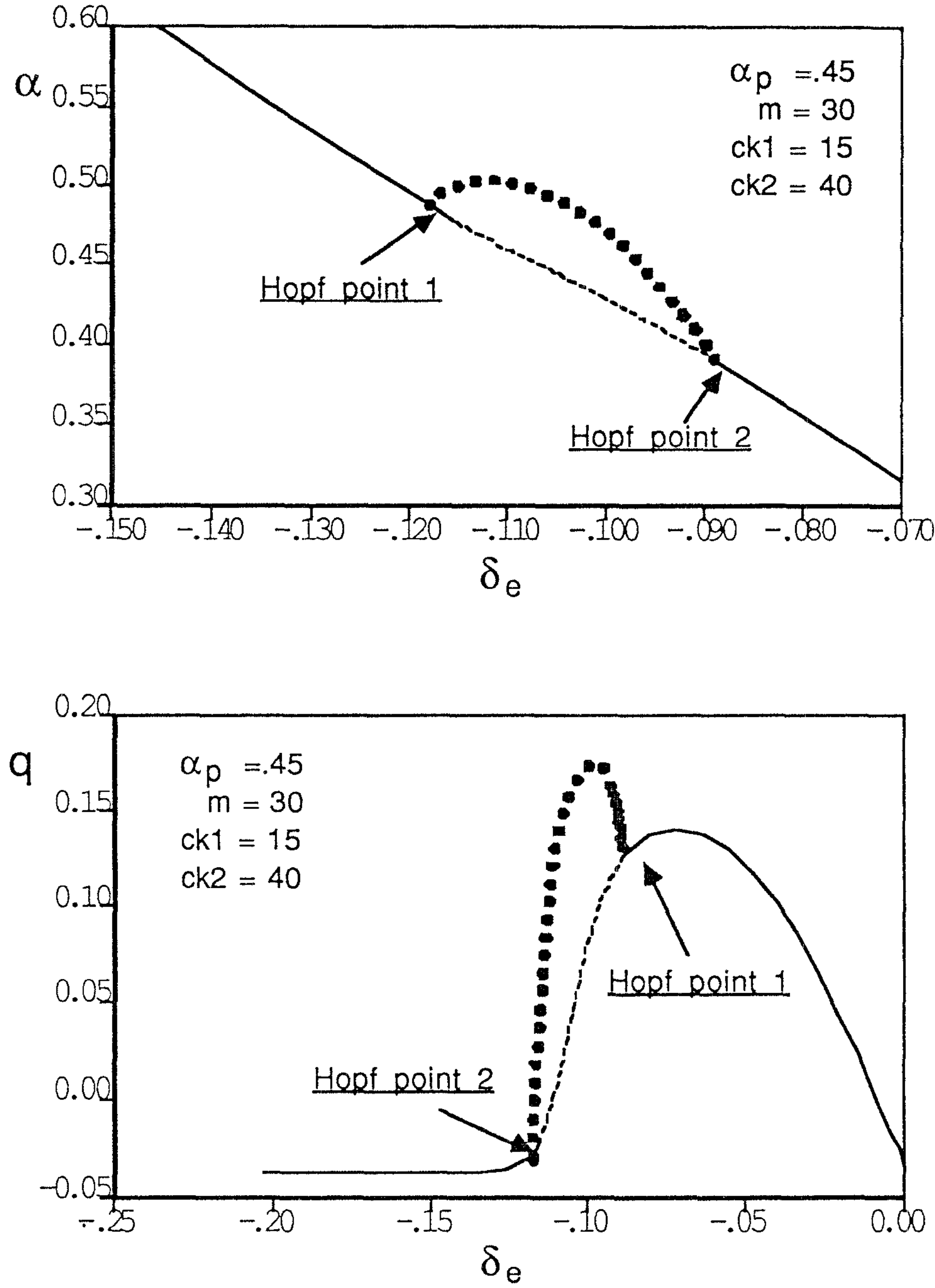


Figure 7.21. Equilibria under the 2nd feedback control for the lift curve m=30 α_p =.45

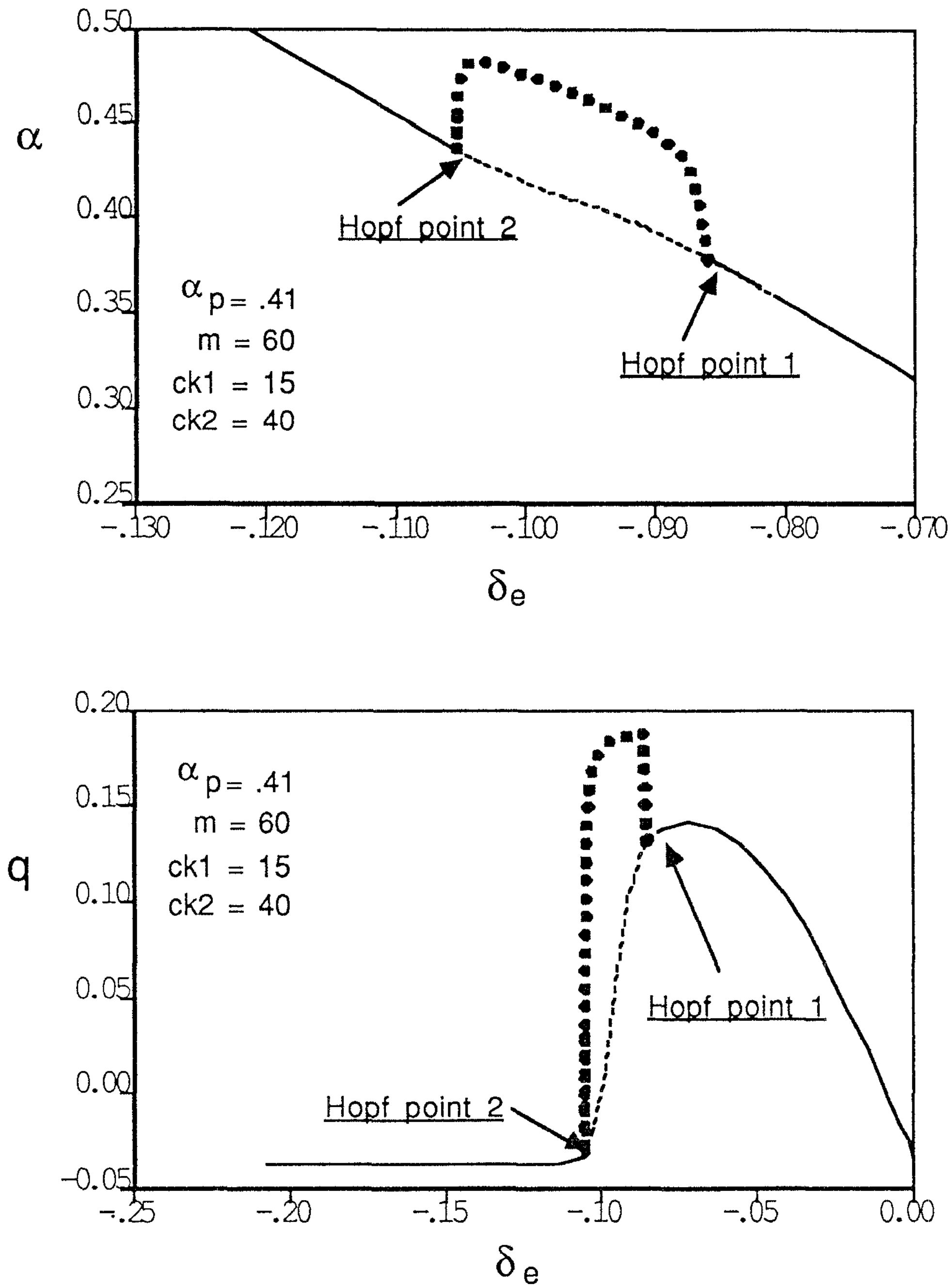


Figure 7.22. Equilibria under the 2nd feedback control for the lift curve m=60 α_p =.41

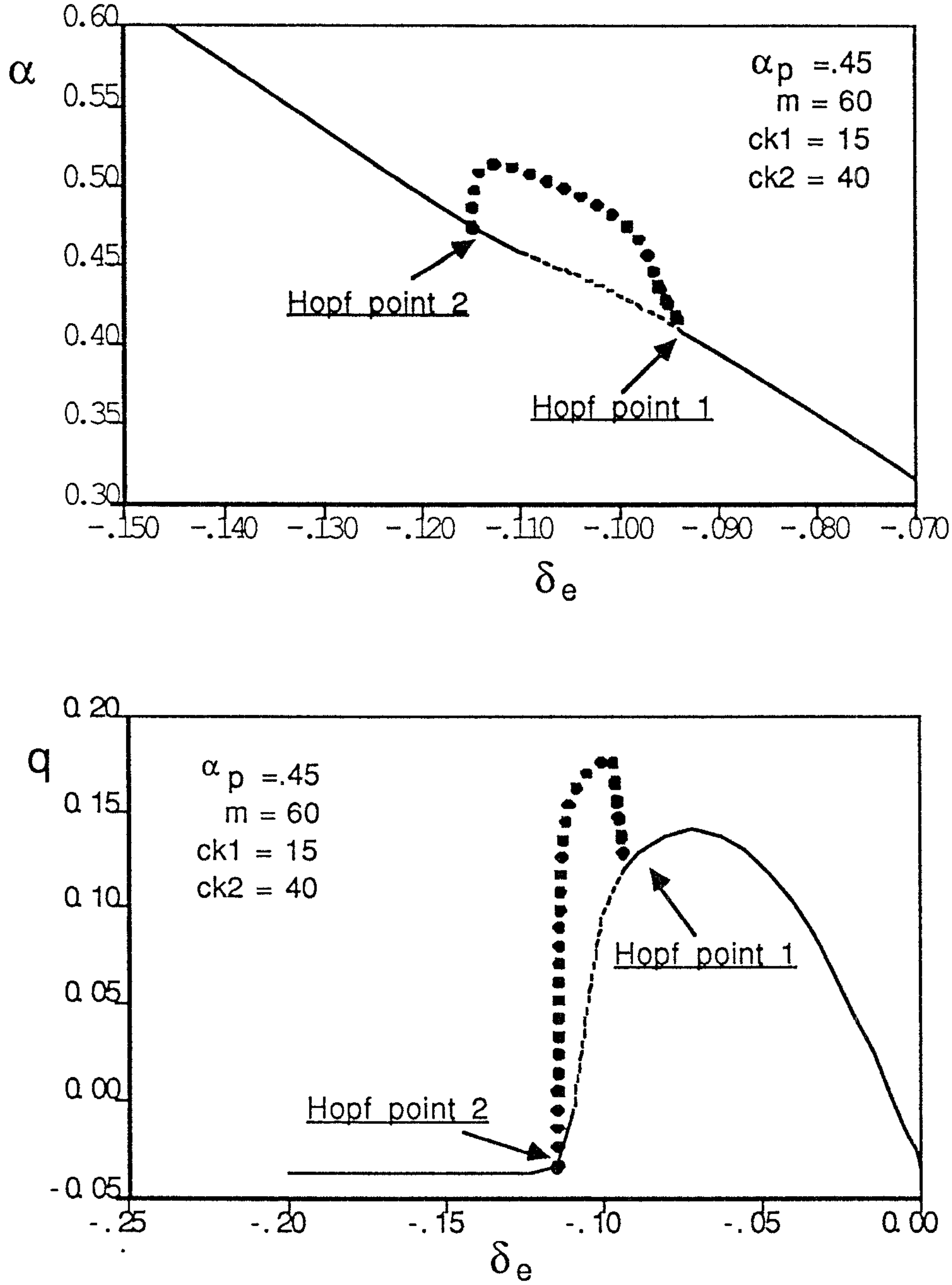
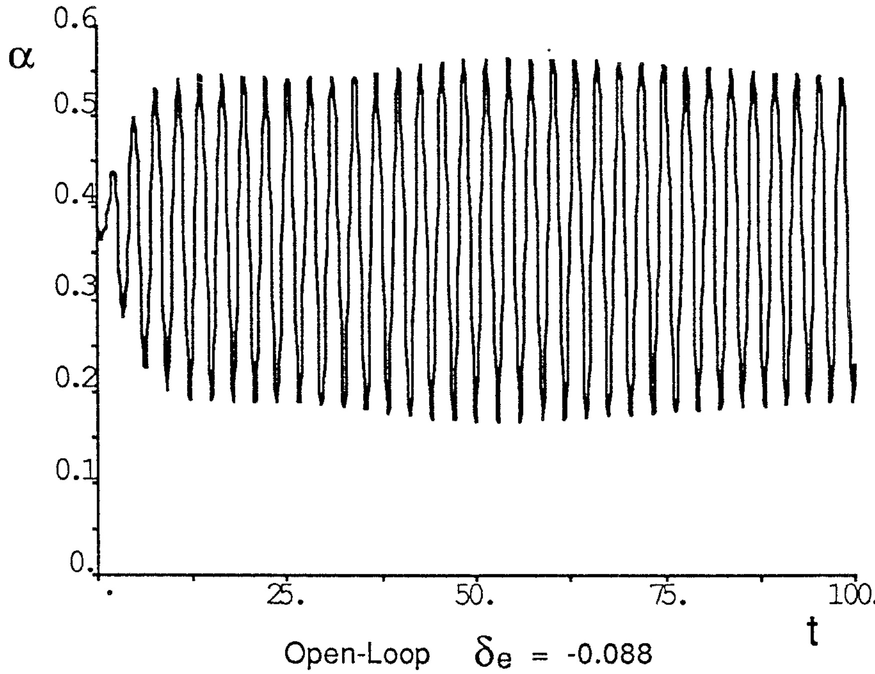


Figure 7.23 Equilibria under the 2nd feedback control for the lift curve m=60 α_p =.45



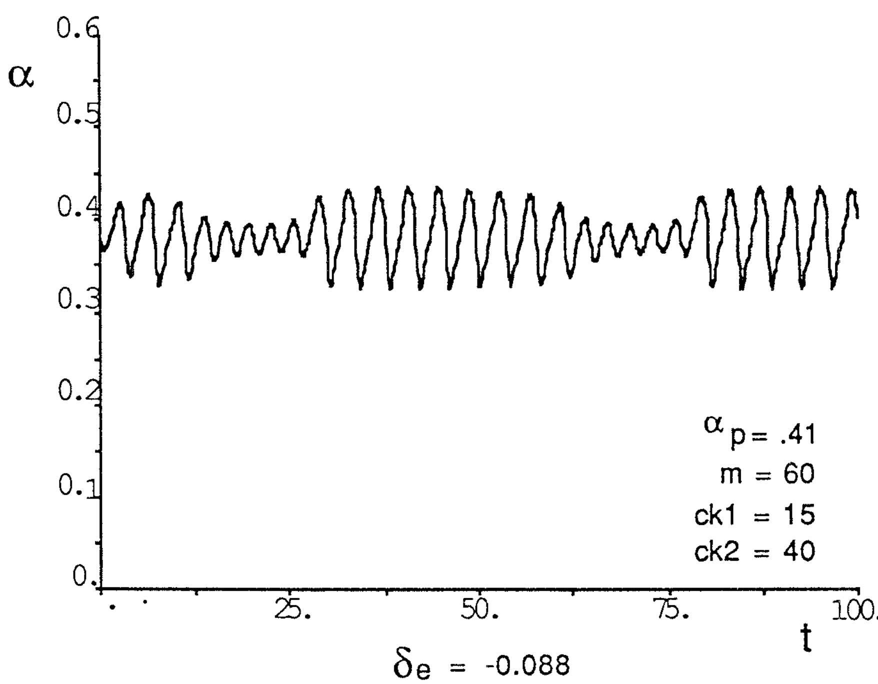


Figure 7.24. α at δ_e =-0.088 for open-loop system and second kind feedback controlled system

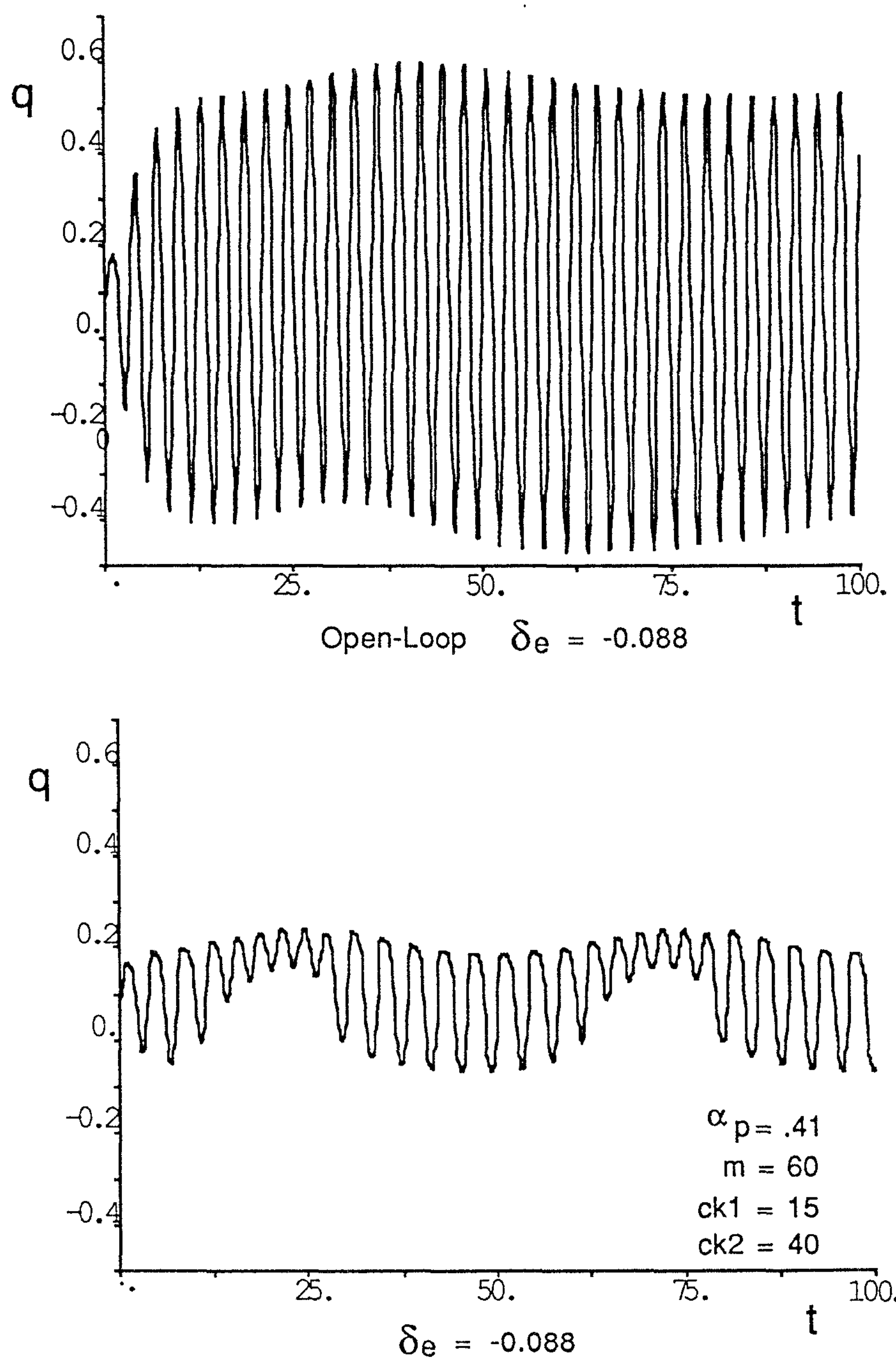
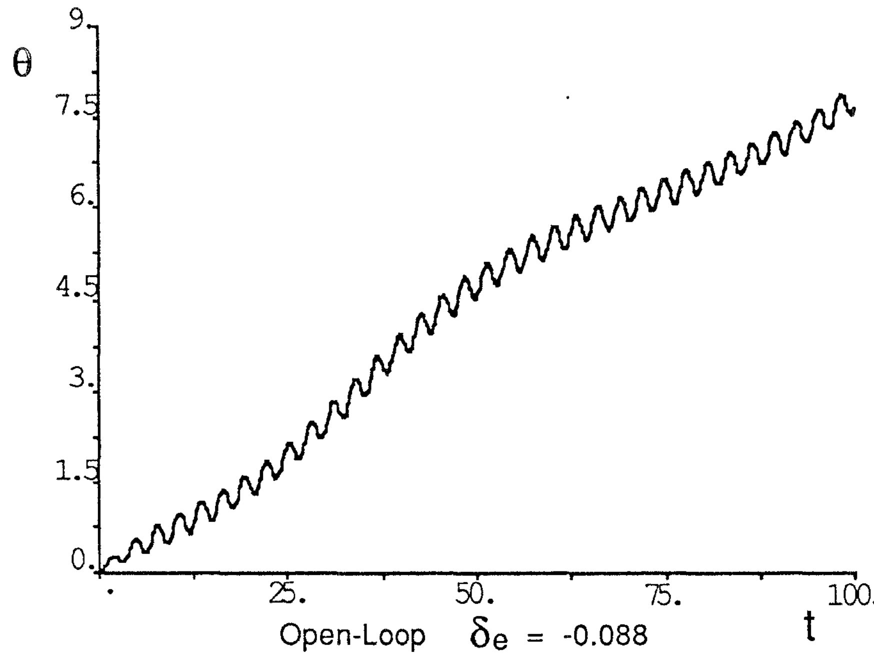


Figure 7.25 . q at δ_e =-0.088 for open-loop system and second kind feedback controlled system



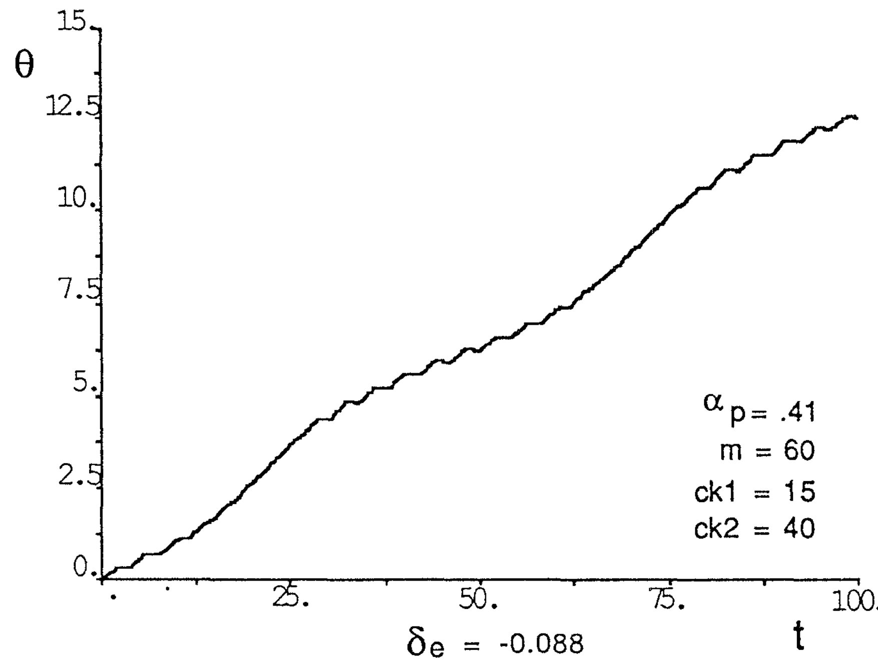


Figure 7.26. θ at δ_e =-0.088 for open-loop system and second kind feedback controlled system

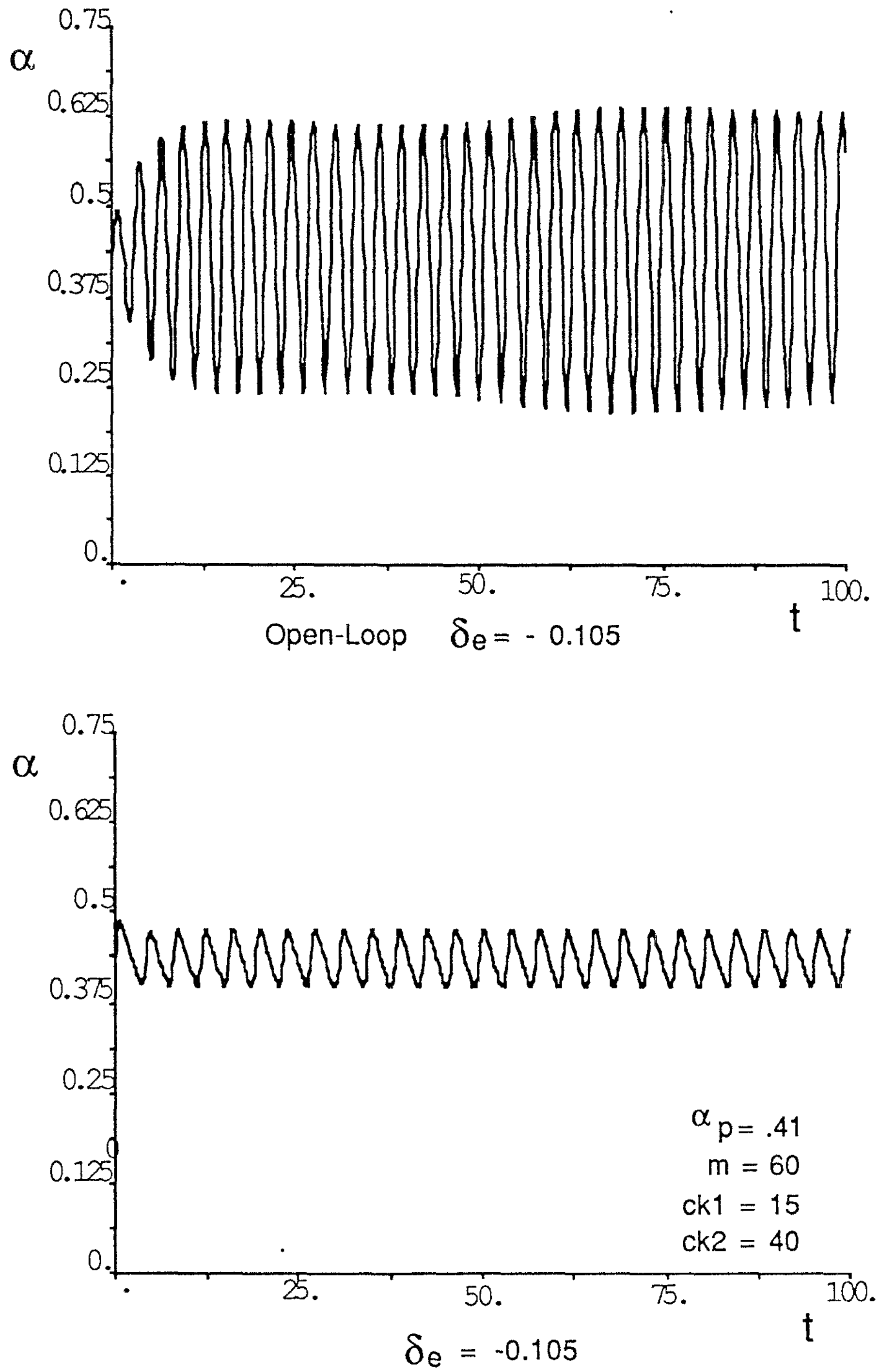
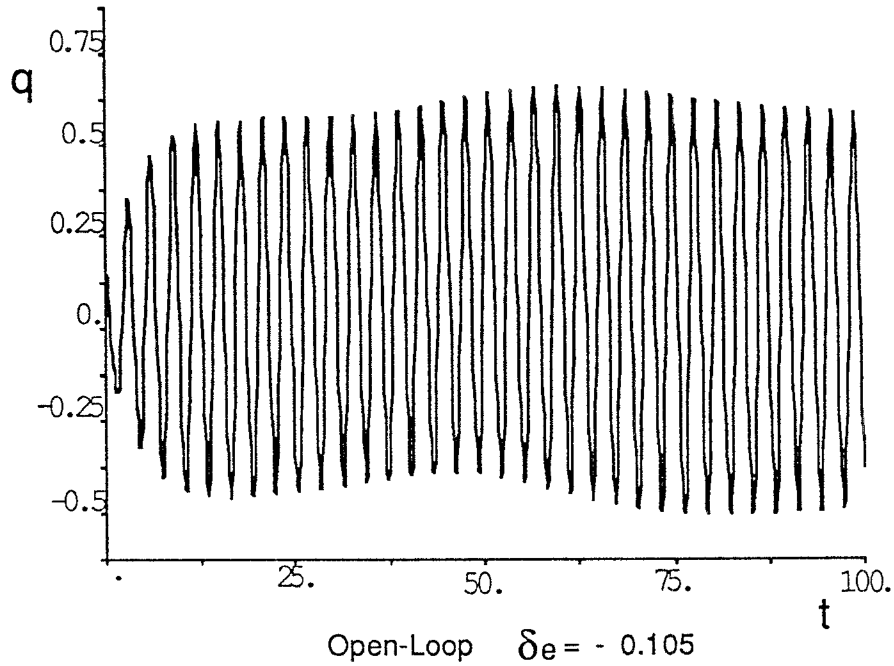


Figure 7.27. α at δ_e =-0.105 for open-loop system and second kind feedback controlled system



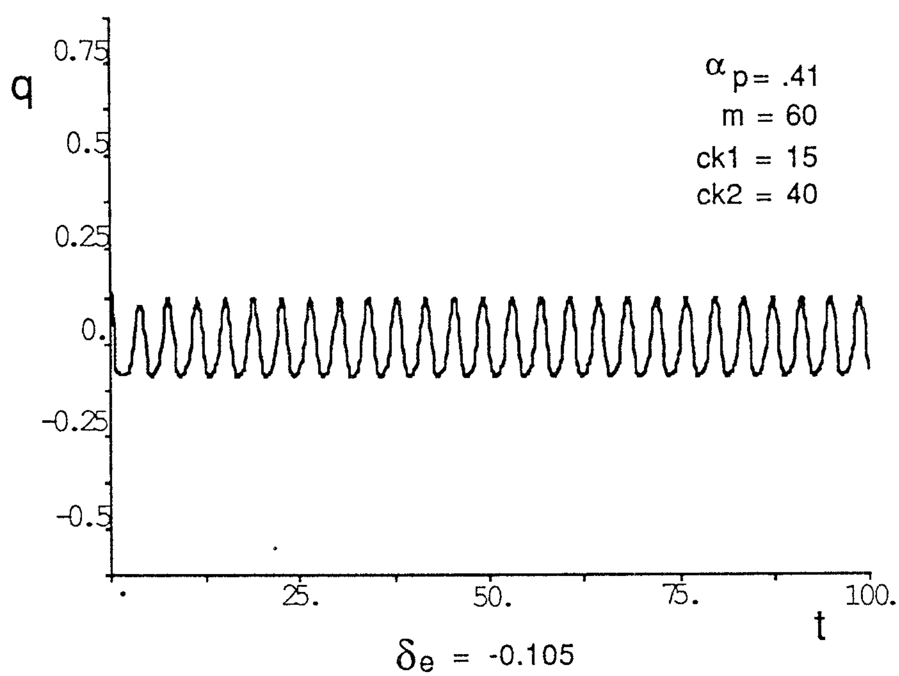


Figure 7.28. q at δ_e =-0.105 for open-loop system and second kind feedback controlled system

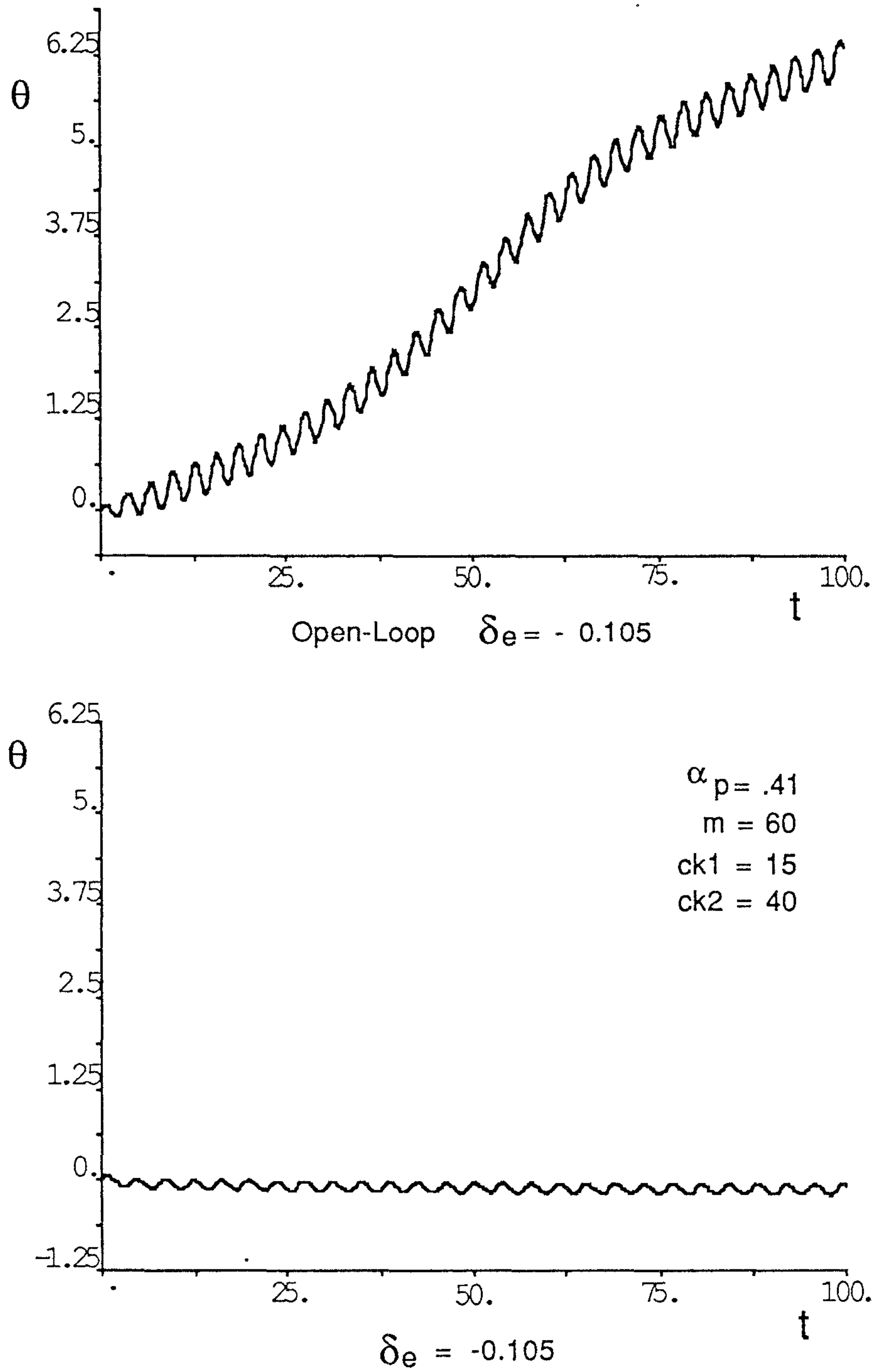


Figure 7.29. θ at δ_e =-0.105 for open-loop system and second kind feedback controlled system

CHAPTER EIGHT

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

We have developed robustly stabilizing feedback control laws for systems exhibiting Hopf bifurcation, stationary pitchfork bifurcation, and systems whose linearization at an equilibrium possesses two pairs of pure imaginary eigenvalues. This is achieved by introducing washout filters into the feedback loops. With these washout filter-aided feedback control laws, the control does not depend on the operating point, and the equilibria of the original system are preserved. Thus, the control is robust to uncertainty in the system operating points.

For systems exhibiting Hopf bifurcation, we derived a purely nonlinear stabilizing control law, preserving the linear stability of the original system. This preservation of linear stability along with the preservation of equilibria permits application of the control over a broad range of operating conditions. Since linear stability dominates the local behavior of the system, this purely nonlinear control will stabilize the Hopf bifurcation point (critical equilibrium point) and the bifurcated periodic solutions, but has little effect on the local behavior of stable noncritical operating points. It is suitable for stabilizing systems which have a broad range of operating conditions such that, within the range of the operating conditions, some of the operating points undergo a Hopf bifurcation but the parameter values for occurrence of these bifurcations are uncertain. Besides the robustness with respect to uncertainty in operating point, the control is also robust with respect to other modeling uncertainty. Since the control derived here is based on the stability formula derived in [38], it only depends on Taylor series expansion of the vector field and eigenvector computations, and no center manifold reduction and normal form transformation are required. Thus, it is feasible to determine the upper limit of uncertainty that the control can tolerate.

For systems undergoing a stationary pitchfork bifurcation, if the critical eigenvalue is controllable, we derived a linear control law to stabilize the bifurcation point. The control employs unstable washout filters to change the direction of *exchange of stabilities*. The bifurcated branches are stabilized. However, the linear stability of the nominal branch is lost. That is, the originally stable nominal branch is now unstable. It is suitable for stabilizing the critical systems whose modified parameterized systems (by adding an artificial parameter to the original systems) undergo a pitchfork bifurcation with the critical eigenvalue controllable. The control is also robust with respect to other system uncertainty (besides that in the system equilibria). However, since the control function depends on the left eigenvector of the critical mode, the amount of uncertainty that the control can tolerate depends on the effect of the left eigenvector uncertainty on the noncritical eigenvalues.

For critical systems possessing a double controllable zero eigenvalue, using washout filter-aided feedback control can only move one of these zero eigenvalues. If after one of the zero eigenvalues is moved, the modified parametrized systems undergo a pitchfork bifurcation, by using the same stabilization algorithm for pitchfork bifurcation, we can apply a linear feedback through the same washout filter which is used to move one zero eigenvalue and stabilize the system simultaneously.

For systems whose linearization possesses two pairs of pure imaginary eigenvalues, we also derived a purely nonlinear stabilizing feedback control law under which the linear stability of the original system is preserved. Similar to the case of Hopf bifurcation, the control is also suitable for systems under a broad range of operating conditions. For the cases in which both critical modes are linearly controllable and both critical modes are linearly uncontrollable, under the stability criterion derived in [17], introducing washout filters into the feedback loop not only does not compromise the stabilizability, but also increases the flexibility in enlarging the bound for robustness.

In the application to aircraft high angle-of-attack Hopf bifurcation control, we showed that using washout filter-aided feedback control is superior to using direct state feedback in extending the range of stable periodic solutions. We also demonstrated the robustness of the use of washout filter-aided feedback by designing two fixed stabilization controllers. Each of these controllers can stabilize twelve Hopf bifurcations for six different profiles of equilibria which were set up to reflect part of the uncertainty in the system model. Also, the amplitudes of the bifurcated periodic solutions are significantly reduced by the control.

To further extend the research covered in this thesis, several possible directions for further study are noted as follows: First, it is important to explore the relation between the nonlinear system performance criteria related to the attraction region, convergence rate, and control energy with the form of the nonlinear controller used and the washout filter time constant. Optimization-based controller design can be pursued in this context. Second, techniques for extending the local bifurcation control algorithms to global bifurcation control or chaotic system control should be investigated. A third possible direction is to extend the washout filter-aided feedback technique to the control of nonlinear slowly varying system.

References

[1] I. Dobson & H.-D. Chiang, "Towards a theory of voltage collapse in electric power systems," *Systems and Control Letters* 13 (1989), 253–262.

[2] E.H. Abed, P.K. Houpt & W.M. Hosny, "Bifurcation analysis of surge and rotating stall in axial-flow compressors," *ASME J. Engineering for Gas Turbines and Power* (to appear, 1991).

[3] D.-C. Liaw, R.A. Adomaitis & E.H. Abed, "Two-parameter bifurcation analysis of axial flow compressor dynamics," in *Proc. 1991 American Control Conference*, Boston, June 1991.

[4] J.V. Carroll & R.K. Mehra, "Bifurcation analysis of nonlinear aircraft dynamics," *J. Guidance* 5 (1982), 529–536.

[5] W.H. Hui & M. Tobak, "Bifurcation analysis of aircraft pitching motions about large mean angles of attack," *J. Guidance* 7 (1984), 113–122.

[6] N. Sri Namachchivaya & S.T. Ariaratnam, "Non-linear stability analysis of aircraft at high angles of attack," *Int. J. Non-Linear Mechanics* 21 (1986), 217–288.

[7] J.E. Cochran, Jr. & C.-S. Ho, "Stability of aircraft motion in critical cases," *J. Guidance* 6 (1983), 272–279.

[8] R.K. Mehra, W.C. Kessell & J.V. Carroll, "Global Stability and Control Analysis of Aircraft at High Angles-of-Attack," U.S. Office of Naval Research, ONRCR-215-248-1, Arlington, VA, June 1977.

[9] A.A. Schy & M.E. Hannah, "Prediction of jump phenomena in roll coupled maneuvers of airplanes," *J. Aircraft* 14 (1977), 375–382 .

[10] A.J. Ross, "Investigation of nonlinear motion experienced on a slender-wing research aircraft," *J. Aircraft* 9 (1972), 625–631.

[11] J.W. Young, A.A. Schy & K.G. Johnson, "Prediction of jump phenomena in aircraft maneuvers, including nonlinear aerodynamic effects," *J. Guidance and Control* 1 (1978), 26–31 .

[12] D. McRuer, D. Johnston & T. Myers, "A perspective on superaugmented flight control: Advantages and problems," *J. Guidance, Control, and Dynamics* 9 (Sep.-Oct. 1986), 447–452.

[13] E.H. Abed & J.-H. Fu, "Local feedback stabilization and bifurcation control, I. Hopf bifurcation," *Systems and Control Letters* 7 (February 1986), 11–17.

[14] E.H. Abed & J.-H. Fu, "Local feedback stabilization and bifurcation control, II. Stationary bifurcation," *Systems and Control Letters* 8 (1987), 467–473.

[15] D. Aeyels, "Stabilization of a class of nonlinear systems by a smooth feedback control," *Systems and Control Letters* 5 (1985), 289–294.

[16] S. Behtash & S.S. Sastry, "Stabilization of nonlinear systems with uncontrollable linearization," *IEEE Trans. Automat. Control* AC-33 (1988), 585–590.

[17] D.-C. Liaw, "Feedback stabilization via center manifold reduction with application to tethered satellites," 1990, Ph.D. dissertation, Dept. of Electrical Engineering, Univ. of Maryland, College Park.

[18] D.-C. Liaw & E.H. Abed, "Stabilization of Tethered Satellites during Station-Keeping," *IEEE Transactions on Automatic Control* 35 (Nov. 1990), 1186–1196.

[19] E.H. Abed, "Nonlinear stabilizing control of high angle-of-attack flight dynamics," in *Proc. AIAA Conf. Guidance, Control and Dynamics*, Boston, August 1989, 532–536.

[20] E.H. Abed & H.-C. Lee, "Nonlinear stabilization of high angle-of-attack flight dynamics using bifurcation control," in *Proc. 1990 American Control Conference*, 1990, 2235–2238.

[21] R.W. Brockett, *Finite Dimensional Linear Systems*, Wiley, New York, 1970.

[22] John H. Blakelock, *Automatic Control of Aircraft and Missiles*, Wiley, New York, 1965.

[23] C.G. Mallery & F.C. Neebe, "Flight test of General Electric self-adaptive control," *Journal of Aircraft* 3 (Sept.-Oct. 1966).

[24] IEEE Committee, "Computer representation of excitation systems," *IEEE Trans. Power Apparatus and Systems* PAS-87 (June, 1968), 1460–1464.

[25] F.P. Demello & C. Concordia, "Concepts of synchronous machine stability as affected by excitation control," *IEEE Trans. Power Apparatus and Systems* PAS-88 (Apr. 1969).

[26] F.R. Schleif, H. D. Hunkins, E. Hattan & W.B. Gish, "Control of rotating exciters for power system damping: Pilot applications and experience," *IEEE Trans. Power Apparatus and Systems* PAS-88 (Aug. 1969).

[27] P.M. Anderson & A.A. Fouad, *Power System Control and Stability*, Iowa State Univ. Press, Ames, IA, 1977.

[28] B. Etkin, *Dynamics of Atmospheric Flight*, Wiley, New York, 1972.

[29] D. McRuer, I. Ashkenas & D. Graham, *Aircraft Dynamics and Automatic Control*, Princeton Univ. Press, Princeton, 1973.

[30] J. Guckenheimer & P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.

[31] J.-H. Fu, *Bifurcation Control and Feedback Stabilization*, Ph.D. Dissertation, Dept. of Elec. Eng., Univ. of Maryland, College Park, 1988.

[32] B.D. Hassard, N.D. Kazarinoff & Y.-H. Wan, *Theory and Applications of Hopf Bifurcation*, Cambridge Univ. Press, Cambridge, 1981.

[33] J. Carr, *Applications of Centre Manifold Theory*, Springer-Verlag, New York, 1981.

[34] E. Doedel, *AUTO 86 User Manual*, Computer Science Dept., Concordia Univ., Montreal, January 1986.

[35] W.L. Garrard & J.M. Jordan, "Design of nonlinear automatic flight control systems," *Automatica* 13 (1977), 497–505.

[36] W.H. Phillips, "Effect of steady rolling on longitudinal stability," NASA TN 1627, 1948.

[37] F.-C Chen & H. K. Khalil, "Two-time-scale longitudinal control of aircraft using singular perturbation," *Journal of Guidance, Control, and Dynamics* 13 (Nov.-Dec. 1990), 952–960.

[38] L.N. Howard, "Nonlinear oscillations," in *Nonlinear Oscillations in Biology*, F.C. Hoppensteadt, ed., Amer. Math. Soc., Providence, R.I., 1979, 1–68.

[39] G. Iooss & D.D. Joseph, *Elementary Stability and Bifurcation Theory*, Springer-Verlag, New York, 1980.

[40] M. Golubitsky & D.G. Schaeffer, *Singularities and Groups in Bifurcation Theory, Part 1*, Springer-Verlag, New York, 1984.

[41] I. Satake, *Linear Algebra*, M. Dekker, New York, 1975.

[42] J.-H. Fu & E.H. Abed, "Families of Liapunov functions for nonlinear systems in critical cases," *IEEE Trans. Automatic Control, (to appear, 1991).*

[43] I. Gohberg & S. Goldberg, *Basic Operator Theory*, Birkhauser, Boston, 1981.

[44] T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1980.

[45] J.E. Marsden & M. McCracken, *The Hopf Bifurcation and Its Applications*, Springer-Verlag, New York, 1976.

[46] A. Bacciotti & P. Boieri, "Linear stabilizability of planar nonlinear systems," *Mathematics of Control, Signals, and Systems* 3 (1990), 183–193.

[47] W.W. Farr, Chengzhi Li, I.S. Labouriau & W.F. Langford, "Degenerate Hopf bifurcation formulas and Hilbert's 16th problem," *SIAM J. Math. Anal.* 20 (1989), 13–30.

[48] S. Wolfram, *Mathematica: A System for Doing Mathematics by Computer* (second edition), Addison-Wesley, Redwood, Calif., 1991.

[49] A.I. Mees & L.O. Chua, "The Hopf bifurcation theorem and its applications to nonlinear oscillations in circuits and systems," *IEEE Trans. Circuits and Systems* CAS-26 (1979), 235–254.

[50] M. Kubicek & M. Marek, *Computational Methods in Bifurcation Theory and Dissipative Structures*, Springer-Verlag, New York, 1983.

[51] E. Hopf, "Bifurcation of a periodic solution from a stationary solution of a system of differential equations," *Ber. Math. Phys. Kl. Sacks. Akad. Wiss.* 94 (1942), 3–22.

[52] M. Vidyasagar, *Nonlinear Systems Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1978.

[53] E.H. Abed, J.-H. Fu, H.-C. Lee & D.-C. Liaw, "Bifurcation control of nonlinear systems," in *New Trends in System Theory*, G. Conte & B.F. Wyman, eds., Birkhäuser, Boston, 1991, to appear.

[54] B. Etkin, *Dynamics of Flight: Stability and Control*, Wiley, New York, 1959.

[55] C.I. Byrnes & A. Lindquist, eds., *Theory and Applications of Nonlinear Control Systems*, North-Holland, Amsterdam, 1986.

[56] R. von Mises, *Theory of Flight*, Dover, New York, 1958.

[57] P. Varaiya, F. Wu & H.-D. Chiang, *Bifurcation and chaos in power systems: a survey*, Electronics Research Laboratory, College of Engineering, Univ. of Ca., Berkeley, Ca., 23 August 1990.

[58] A.J. Ross & L.J. Beecham, "An approximate analysis of the non-linear lateral motion of a slender aircraft (HP 115) at low speeds," Aeronautical Research Council, Ministry of Defence, Reports and Memoranda No. 3674, London, 1971.