The Impact Of Accommodation Coefficient On Concentric Couette Flow

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Abstract. Rarefied Couette flow between concentric rotating cylinders is investigated using Maxwell's slip-velocity boundary treatment. The study provides an independent reassessment of the velocity inversion process and demonstrates that the occurrence of an inverted velocity profile depends solely on the accommodation coefficient of the stationary outer cylinder. From the analysis, criteria are derived for the accommodation coefficient of the outer cylinder that can be used to predict whether an inverted, partially-inverted or normal velocity profile will occur. The present results are in close agreement with previous analytical solutions and in good qualitative agreement with available DSMC data.

INTRODUCTION

Couette flow between concentric rotating cylinders is a classical fluid dynamics problem that can be found in many textbooks. However, recent analytical and DSMC studies [1-3] have suggested that under certain conditions of rarefaction (in particular, when the accommodation coefficient is small), the velocity profile between the cylinders reverses direction so that the gas moves faster near the stationary outer cylinder. This anomalous behavior has been described as an 'inverted velocity profile' because the velocity of the gas increases with distance from the rotating inner cylinder. The effect is completely non-intuitive and contrary to the normal velocity profile expected within a cylindrical Couette flow.

The phenomenon of velocity inversion was first predicted by Einzel, Panzer and Liu (EPL) [1] who recognized the importance of accounting for surface curvature and developed a generalized slip-boundary condition for flows over curved surfaces. EPL did not specifically consider a rarefied gas and instead analyzed the flow using the concept of a *slip-length*. They showed that the velocity profile within a concentric Couette flow would become inverted for large values of slip-length.

Tibbs *et al*. [2] have subsequently recast EPL's formulation so that it can be applied to a rarefied gas. This was achieved by defining the slip-length, $\zeta_0 = a\,(2/\sigma - 1)\lambda$ where λ is the mean free path of the gas molecules, $a \approx 1.15$ and σ is the tangential momentum accommodation coefficient (TMAC) which can vary from zero (for specular reflection) up to unity (for complete or diffuse accommodation). Tibbs *et al*. found that the velocity inversion process only occurs for small values of TMAC. They also presented results from a direct simulation Monte Carlo (DSMC) approach and showed that the velocity profiles predicted by the EPL formulation were in good agreement with the DSMC results. The phenomenon of velocity inversion has also been investigated by Aoki *et al*. [3] using two alternative approaches: an asymptotic analytical solution at low Knudsen numbers, and a direct numerical solution of the Boltzmann equation at higher Knudsen numbers. It was found that the occurrence of a velocity inversion depended upon the ratio of accommodation coefficient to Knudsen number.

This paper provides an independent reassessment of the velocity inversion process using Maxwell's original slip-velocity boundary condition [4]. Unlike previous investigations which have generally assumed the accommodation coefficients at the inner and outer cylinder walls are identical, the present study considers the influence of both the inner and outer accommodation coefficients. The analysis shows that the velocity inversion phenomenon is only dependent on the accommodation coefficient of the outer stationary cylinder. From the analysis, criteria are derived for the accommodation coefficient that can be used to predict whether an inverted, partially-inverted or normal velocity profile will occur.

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CYLINDRICAL COUETTE FLOW

The present study considers isothermal, rarefied Couette flow between two concentric cylinders. The inner cylinder has a radius, R_1, and rotates at a constant angular velocity, ω, while the outer cylinder of radius, R_2, is stationary. For cylindrical Couette flow in a polar co-ordinate (r,θ) reference frame, the circumferential momentum expression in the Navier-Stokes equations reduces to (see for example, Schlichting [5]):

$$\frac{d^2 u_\theta}{dr^2} + \frac{d}{dr}\left(\frac{u_\theta}{r}\right) = 0\ , \tag{1}$$

where u_θ is the tangential velocity component and r is the radius. Maxwell's slip-velocity boundary condition [4] can be written in cylindrical coordinate form as follows:

$$u_{\theta\ (gas)} - u_{\theta\ (wall)} = \pm \frac{(2-\sigma)}{\sigma}\lambda\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right)\bigg|_{wall}\ . \tag{2}$$

The additional term, u_θ/r, on the right hand side of Eq. (2) ensures that the slip-velocity is directly proportional to the wall shear stress. Unfortunately, this term is often ignored in the analysis of slip-flows over non-planar surfaces, leading to incorrect slip-flow behavior. Although the misapplication of Maxwell's slip-velocity boundary condition, Eq. (2), is widespread, there are several instances where curved boundaries have been treated correctly [3,6,7]. Applying Eq. (2) to the inner and outer cylinder walls gives

$$u_\theta\big|_{r=R_1} = \omega R_1 + \frac{(2-\sigma_1)}{\sigma_1}\lambda\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right)\bigg|_{r=R_1}\ , \tag{3}$$

and

$$u_\theta\big|_{r=R_2} = -\frac{(2-\sigma_2)}{\sigma_2}\lambda\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right)\bigg|_{r=R_2}\ , \tag{4}$$

where σ_1 and σ_2 are the tangential momentum accommodation coefficients of the inner and outer cylinders, respectively.

Adopting the notation proposed by Einzel *et al*. [1], the tangential velocity profile can be written as

$$u_\theta(r) = a\,r + \frac{b}{r}\ , \tag{5}$$

where

$$a = \frac{A}{(A-B)}\omega \qquad \text{and} \qquad b = \frac{1}{(B-A)}\omega\ . \tag{6}$$

It can readily be shown that the parameters, A and B, are given by

$$A = \frac{1}{R_2^2}\left(1 - \frac{(2-\sigma_2)}{\sigma_2}\frac{2\lambda}{R_2}\right) \qquad \text{and} \qquad B = \frac{1}{R_1^2}\left(1 + \frac{(2-\sigma_1)}{\sigma_1}\frac{2\lambda}{R_1}\right). \tag{7}$$

For convenience, the velocity profile in Eq. (5) can be non-dimensionalized with respect to the circumferential velocity of the rotating inner cylinder, giving

$$u_\theta^* = \frac{u_\theta}{\omega R_1} = \frac{A}{(A-B)}\frac{r}{R_1} + \frac{1}{(B-A)\,R_1\,r} = \frac{1}{(A-B)\,R_1}\left(Ar - \frac{1}{r}\right). \tag{8}$$

RESULTS AND DISCUSSION

For compatibility with the results presented by Tibbs *et al*. [2], the radii of the inner and outer cylinders are chosen to be 3λ and 5λ, respectively. Initially the accommodation coefficients are assumed to be equal at the inner and outer cylinders. Figure 1 presents a comparison between the analytical solution (Eq. (8)) and the DSMC results obtained by Tibbs *et al*. [2] over a range of accommodation coefficients from $\sigma = 1$ down to $\sigma = 0.1$. The quantitative agreement between the DSMC data (symbols) and the analytical predictions (lines) is not particularly close but this is to be expected since the separation distance between the cylinder walls is only two mean free paths, implying a Knudsen number (based upon the annular clearance) of 0.5. At such a high Knudsen number, continuum-based flow models are likely to be at the limit of their applicability. Nevertheless, the results show that the analytical formulation and the DSMC data follow the same basic trends and predict an inverted velocity profile for an accommodation coefficient of 0.1.

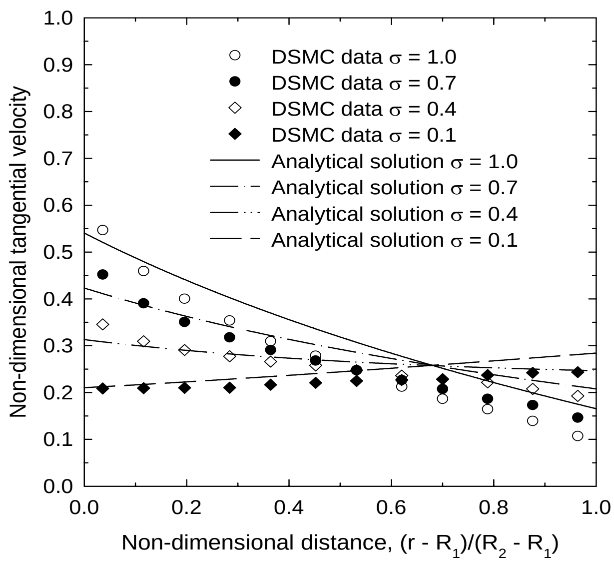

FIGURE 1. Non-dimensional velocity profiles for $\sigma_1 = \sigma_2 = \sigma$.

Criteria for velocity inversion

Although previous studies have reported that the phenomenon of velocity inversion is associated with low values of accommodation coefficient, specific criteria governing the formation of an inverted velocity profile have never been published. In the present work, we identify a set of limiting analytical criteria that will predict whether a normal or an inverted velocity profile will occur. We also introduce the concept of a *partially-inverted* velocity profile to describe the case when the velocity initially decreases away from the inner rotating cylinder but then increases towards the outer stationary cylinder. The study also considers the influence of the both the inner and outer accommodation coefficients.

The velocity gradient can be used as a criterion to judge whether the profile is increasing or decreasing with radial distance, *r*. Differentiating the non-dimensionalized velocity distribution, Eq. (8), with respect to *r* yields

$$\frac{du_\theta^*}{dr} = \frac{1}{(A-B)\,R_1}\left(A + \frac{1}{r^2}\right). \tag{9}$$

For clarity, the symbol denoting the non-dimensionalized velocity (*) will be omitted from subsequent equations. It can be shown from Eq. (7) that

$$A - B = -\frac{1}{R_1^2\,R_2^2}\left[\left(R_2^2 - R_1^2\right) + R_1^2\,\frac{(2-\sigma_2)}{\sigma_2}\frac{2\lambda}{R_2} + R_2^2\,\frac{(2-\sigma_1)}{\sigma_1}\frac{2\lambda}{R_1}\right], \tag{10}$$

and since $R_1 < R_2$, the term $(A-B)$ will always be *negative*.

A fully-inverted velocity profile will occur when the velocity increases monotonically from the inner to the outer cylinder. This implies that du_θ/dr must be greater than zero throughout the range $R_1 \leq r \leq R_2$. From Eq. (9), this requirement can only be satisfied when $A + r^{-2} < 0$. However,

$$A + R_2^{-2} \;\leq\; A + r^{-2} \;\leq\; A + R_1^{-2}\,, \tag{11}$$

and therefore, the requirement to satisfy $A + r^{-2} < 0$ can be achieved by satisfying $A + R_1^{-2} < 0$. This condition yields the following criterion:

$$A + \frac{1}{R_1^2} = \frac{1}{R_1^2 R_2^2}\left[R_1^2 + R_2^2 - \frac{(2-\sigma_2)}{\sigma_2}\frac{2\lambda R_1^2}{R_2} \right] < 0\,, \tag{12}$$

which is true if

$$\sigma_2 < \; 2\left(1 + \frac{\left(R_1^2 + R_2^2\right) R_2}{2\lambda R_1^2} \right)^{-1}. \tag{13}$$

Equation (13) defines the limiting criterion for the occurrence of a fully-inverted velocity profile. It should be noted that this equation is independent of σ_1 and therefore the phenomenon of velocity inversion is dependent only upon the value of the accommodation coefficient of the stationary outer cylinder. For the specific case when the inner and outer cylinder radii are 3λ and 5λ, respectively, Eq. (13) provides an upper bound for a fully-inverted velocity profile, given by $\sigma_2 < 9/47 \approx 0.1915$. This criterion is illustrated in Fig. 2, which shows a set of inverted velocity profiles for $\sigma_2 < 9/47$.

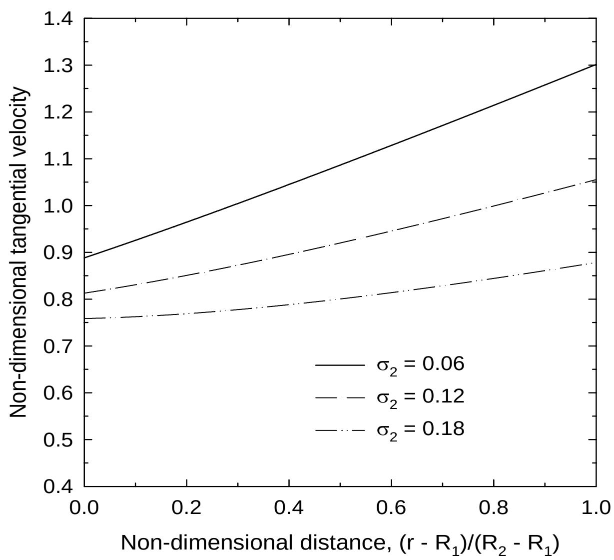

FIGURE 2. Fully-inverted velocity profiles for $\sigma_1 = 1$ and $\sigma_2 < 9/47$.

Conversely, if $du_\theta/dr < 0$ throughout the range $R_1 \leq r \leq R_2$, then the velocity profile will simply be a decreasing function of r and no velocity inversion will occur. Using Eq. (9), this condition requires that $A + r^{-2} > 0$. From Eq. (11), if $A + R_2^{-2} > 0$ is true, then $A + r^{-2} > 0$ will always be satisfied. We therefore need to satisfy the following expression for there to be no velocity inversion:

$$A + \frac{1}{R_2^2} = \frac{1}{R_2^2}\left[2 - \frac{(2-\sigma_2)}{\sigma_2}\frac{2\lambda}{R_2} \right] > 0\,, \tag{14}$$

which can be rearranged to give

$$\sigma_2 > 2\left(1+\frac{R_2}{\lambda}\right)^{-1}. \tag{15}$$

Inspection of Eq. (15) reveals that the criterion for the occurrence of a normal velocity profile is again independent of σ_1 and is purely a function of the accommodation coefficient of the outer cylinder. For the specific geometry considered by Tibbs *et al*. [2], with $R_1 = 3\lambda$ and $R_2 = 5\lambda$, the limiting value for there to be no velocity inversion is $\sigma_2 > 1/3$. Thus, high values of accommodation coefficient will always lead to a normal (non-inverted) velocity profile.

The final case is when the accommodation coefficient of the outer cylinder lies between the two constraints defined in Eqs. (13) and (15), i.e.

$$2\left(1+\frac{\left(R_1^2+R_2^2\right)R_2}{2\lambda R_1^2}\right)^{-1} < \sigma_2 < 2\left(1+\frac{R_2}{\lambda}\right)^{-1}. \tag{16}$$

This case can be described as being a *partially-inverted* velocity profile since the velocity initially decreases away from the inner cylinder but then increases towards the outer cylinder, as shown in Fig. 3. It can be seen that the position of minimum velocity moves towards the outer cylinder as σ_2 increases. The location of the minimum velocity occurs when $du_\theta/dr = 0$, which implies that

$$A+\frac{1}{r^2}=0. \tag{17}$$

Substituting for *A* from Eq. (7) into Eq. (17) gives the position as

$$r=\sqrt{\frac{R_2^2}{\frac{(2-\sigma_2)}{\sigma_2}\frac{2\lambda}{R_2}-1}}. \tag{18}$$

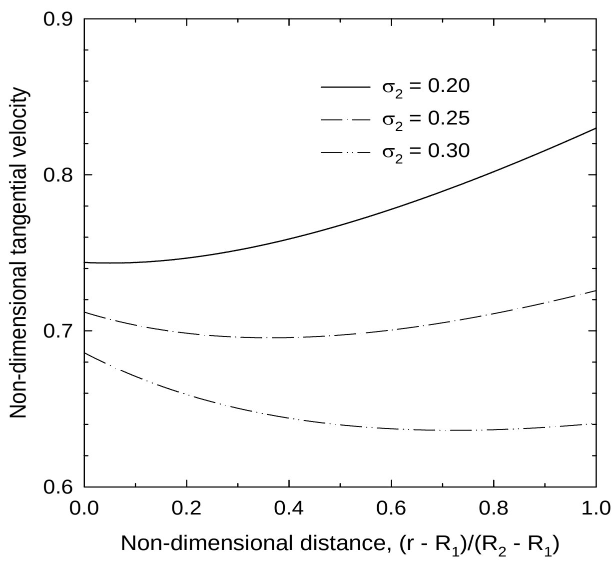

FIGURE 3. Partially-inverted velocity profiles for $\sigma_1 = 1$ and $9/47 < \sigma_2 < 1/3$.

CONCLUDING REMARKS

The Navier-Stokes equations, with boundary conditions derived from Maxwell's slip-flow treatment, have been used to develop an analytical model for isothermal rarefied Couette flow between concentric rotating cylinders. Following previous studies, the present investigation considers the important case of a stationary outer cylinder and a rotating inner cylinder. The study provides an independent reassessment of the velocity inversion process and demonstrates that the occurrence of an inverted velocity profile depends solely on the accommodation coefficient of the outer cylinder. The present results are in close agreement with previous analytical solutions and in good qualitative agreement with available DSMC data. Exact quantitative agreement between the analytical solution and the DSMC data is not expected, since the annular clearance between the cylinders is only two mean free paths, implying a Knudsen number of 0.5. At this degree of rarefaction, continuum fluid models are likely to be at, or beyond, their limit of applicability. Nevertheless, the analytical approach provides a method of estimating limiting criteria that can be used to predict whether an inverted, partially-inverted or normal velocity profile will occur.

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