Least Squares Algorithms for Constant-Acceleration Target Tracking

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***Abstract*—A unified treatment of several least squares (LS) algorithms is presented for bearings-only tracking of a target moving at constant acceleration. The close link between the maximum likelihood (ML) estimator and other nonlinear and "linearized" LS algorithms is explored under the assumption of Gaussian bearing noise. In this context, a new asymptotically unbiased closed-form instrumental variables (IV) algorithm is derived. Reduced-bias total least squares (TLS) and constrained TLS (CTLS) algorithms are developed. The equivalence of the ML algorithm to the structured TLS (STLS) algorithm is established. Simulation examples are provided to demonstrate the improved performance of the IV and TLS estimators *vis-à-vis* the pseudolinear estimator.**

I. Introduction

The paper presents a unified treatment of several classical and new LS algorithms for bearings-only tracking of a target moving at constant acceleration. Tracking of constant-velocity targets and stationary target localization are special cases of constant-acceleration target tracking. The key to target tracking is the estimation of target trajectory from noisy bearing and own-ship location measurements. For targets moving at constant acceleration, the target trajectory is defined by the target motion parameters, viz., position, velocity and acceleration. Target tracking has many civilian and military applications, such as air traffic control, surveillance, etc.

Firstly we derive the classical ML estimator for the target motion parameters. The ML estimator becomes a nonlinear LS estimator for zero-mean Gaussian distributed bearing noise. While the ML estimator is optimal, it does not have a closed-form solution because of the nonlinear relationship between the measurements and the unknown parameters. Based on the classical ML estimator, a linearized weighted LS estimator is derived under the assumption of small bearing noise, which yields a closed-form solution. This algorithm is simply an extension of the target localization algorithm developed by Stansfield in [1] to tracking of moving targets. A pseudolinear estimator is obtained from the Stansfield estimator by removing the weighting matrix. An alternative derivation for the pseudolinear estimator based on orthogonal vector representation yields the equivalent orthogonal vectors (OV) estimator. The linear LS estimators are known to be biased [2]. A new closed-form asymptotically unbiased IV estimator is proposed in the paper. The instrumental variables are obtained from another estimate such as LS or TLS. Unlike the algorithms in [2, 3], no recursive computations are required for the new IV estimator. TLS and CTLS estimators are developed to improve the performance of the OV estimator by attempting to mitigate errors in both the system matrix and the data vector. A structured TLS estimator is also formulated and the equivalence between the STLS and ML estimators is established. The performance improvement achieved by the TLS and IV estimators compared with the OV estimator is demonstrated with simulation examples. The simulations also show the capability of the IV estimator to outperform the ML estimator.

II. Problem Formulation

The two-dimensional target tracking problem using bearing measurements only is depicted in Fig. 1. The objective of bearings-only target tracking is to identify the target location $\boldsymbol{p}_k$ from noisy bearing and observer position measurements over a finite time interval $0 \le k \le N-1$.

In Fig. 1, the relation between the bearing angle, observer position and target location is given by the following nonlinear equation

$$\theta_k = \tan^{-1}\frac{\Delta y_k}{\Delta x_k}, \quad k = 0, \ldots, N-1 \qquad (1)$$

where $\Delta y_k \triangleq p_{y,k} - r_{y,k}$, $\Delta x_k \triangleq p_{x,k} - r_{x,k}$, $\boldsymbol{p}_k = [p_{x,k}, p_{y,k}]^T$ is the target location vector and $\boldsymbol{r}_k = [r_{x,k}, r_{y,k}]^T$ is the observer position at k.

We make the following assumptions about the target tracking problem:

- The target is moving at constant acceleration. Let $\boldsymbol{p}_0$ and $\boldsymbol{v}_0$ denote the target position and velocity vector at $k = 0$, and $\boldsymbol{a}$ be the constant target acceleration vector. Assuming that the bearing and observer position measurements are taken at regular time instants $t_k = kT$ where T is the sampling interval, the target location at time t_k, $k =$

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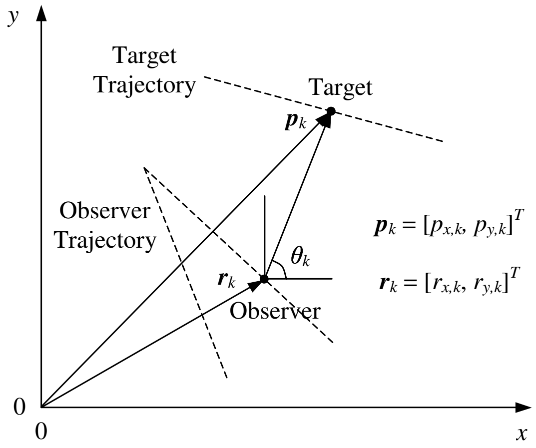

Fig. 1. Two-dimensional bearings-only target tracking geometry.

$0, 1, \ldots, N-1$ is given by [4]

$$\boldsymbol{p}_k = \boldsymbol{p}_0 + t_k \boldsymbol{v}_0 + \frac{t_k^2}{2}\boldsymbol{a} \tag{2a}$$

$$= \boldsymbol{M}_k \boldsymbol{\xi} \tag{2b}$$

where

$$\boldsymbol{M}_k = \begin{bmatrix} 1 & 0 & t_k & 0 & \frac{1}{2}t_k^2 & 0 \\ 0 & 1 & 0 & t_k & 0 & \frac{1}{2}t_k^2 \end{bmatrix}$$

and

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{v}_0 \\ \boldsymbol{a} \end{bmatrix}$$

is the 6×1 target motion parameter vector to be estimated. Given an estimate of $\boldsymbol{\xi}$, the target locations can be obtained from (2).

- The bearing measurements are subject to independent zero-mean Gaussian noise:

$$\tilde{\theta}_k = \theta_k + n_k, \quad n_k \sim \mathcal{N}(0, \sigma_{n_k}^2) \tag{3}$$

where the $\tilde{\theta}_k$, $k = 0, \ldots, N-1$, are the bearing measurements available for target tracking, and n_k is a Gaussian random variable with zero mean and variance $\sigma_{n_k}^2$.

- The observer position measurements are subject to independent bivariate Gaussian noise:

$$\tilde{\boldsymbol{r}}_k = \boldsymbol{r}_k + \boldsymbol{w}_k, \quad \boldsymbol{w}_k \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{C}) \tag{4}$$

where the covariance matrix $\boldsymbol{C}$ is diagonal, i.e., the errors in x and y coordinates of the observer position measurements are independent.

- The observer trajectory is such that the target is observable.

III. LS Tracking Algorithms

A. ML Estimator

Assuming that the observer position errors are zero, that is, $\boldsymbol{w}_k = \boldsymbol{0}$ in (4), and using the Gaussianity assumption for bearing measurement noise, the likelihood function for the bearing measurements can be written as

$$p(\tilde{\boldsymbol{\theta}}|\boldsymbol{\xi}) = \frac{1}{(2\pi)^{N/2}|\boldsymbol{K}|^{1/2}} \times \exp\Big\{-\frac{1}{2}(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}(\boldsymbol{\xi}))^T \boldsymbol{K}^{-1}(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}(\boldsymbol{\xi}))\Big\}$$

where

$$\tilde{\boldsymbol{\theta}} = [\tilde{\theta}_0, \tilde{\theta}_2, \ldots, \tilde{\theta}_{N-1}]^T$$

is the $N \times 1$ vector of noisy bearing measurements,

$$\boldsymbol{\theta}(\boldsymbol{\xi}) = [\theta_0, \theta_2, \ldots, \theta_{N-1}]^T$$

is the $N \times 1$ vector of bearing angles which are dependent on the target locations $\boldsymbol{p}_k$ and the target motion parameters $\boldsymbol{\xi}$ through (1) and (2), $\boldsymbol{K} = \operatorname{diag}(\sigma_{n_0}^2, \ldots, \sigma_{n_{N-1}}^2)$ is the $N \times N$ diagonal covariance matrix of the bearing measurement errors, and $|\boldsymbol{K}|$ is the determinant of $\boldsymbol{K}$.

The maximum likelihood estimator of the target motion parameters $\hat{\boldsymbol{\xi}}_{\text{ML}}$ is obtained from maximization of the likelihood function $p(\tilde{\boldsymbol{\theta}}|\boldsymbol{\xi})$ over all possible $\boldsymbol{\xi}$ [5]. To simplify the maximization problem, the log-likelihood function is used:

$$\ln p(\tilde{\boldsymbol{\theta}}|\boldsymbol{\xi}) = -\frac{1}{2}\ln((2\pi)^N|\boldsymbol{K}|) - \frac{1}{2}(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}(\boldsymbol{\xi}))^T \boldsymbol{K}^{-1}(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}(\boldsymbol{\xi})).$$

Noting that the first term of the log-likelihood function is independent of $\boldsymbol{\xi}$, the maximization of the log-likelihood function can be achieved by minimizing the second term only after sign inversion, leading to

$$\hat{\boldsymbol{\xi}}_{\text{ML}} = \arg\min_{\boldsymbol{\xi} \in \mathbb{R}^6} J_{\text{ML}}(\boldsymbol{\xi}) \tag{5}$$

where $J_{\text{ML}}(\boldsymbol{\xi})$ is the ML cost function given by

$$J_{\text{ML}}(\boldsymbol{\xi}) = \frac{1}{2}(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}(\boldsymbol{\xi}))^T \boldsymbol{K}^{-1}(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}(\boldsymbol{\xi})). \tag{6}$$

The minimization of $J_{\text{ML}}(\boldsymbol{\xi})$ over $\boldsymbol{\xi}$ is in fact a nonlinear LS problem. The solution to this minimization problem satisfies

$$\left.\frac{\partial J_{\text{ML}}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}}\right|_{\boldsymbol{\xi}=\hat{\boldsymbol{\xi}}_{\text{ML}}} = \boldsymbol{0}. \tag{7}$$

A closed-form solution to (7) does not exist. This requires the use of numerical gradient-based search techniques. The gradient-based search techniques will yield a unique solution for $\hat{\boldsymbol{\xi}}_{\text{ML}}$ if the ML cost function is convex with a unique global minimum. The convexity of the ML cost function was stated in [6] without proof.

B. Stansfield Estimator

Assuming that the bearing errors are very small, i.e., $\tilde{\theta}_k - \theta_k \approx 0$, we can write [7]

$$\tilde{\theta}_k - \theta_k \approx \sin(\tilde{\theta}_k - \theta_k).$$

Substituting this approximation into (6) yields the following cost function

$$J_S(\boldsymbol{p}) = \sum_{k=0}^{N-1} \frac{1}{2\sigma_{n_k}^2} \sin^2(\tilde{\theta}_k - \theta_k) \tag{8a}$$

$$= \frac{1}{2}(\boldsymbol{F\xi} - \boldsymbol{b})^T \boldsymbol{W}^{-1}(\boldsymbol{F\xi} - \boldsymbol{b}) \tag{8b}$$

where $\boldsymbol{F}$ is the $N \times 6$ matrix defined by

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{a}_0^T \boldsymbol{M}_0 \\ \boldsymbol{a}_1^T \boldsymbol{M}_1 \\ \vdots \\ \boldsymbol{a}_{N-1}^T \boldsymbol{M}_{N-1} \end{bmatrix}_{N\times 6}, \quad \boldsymbol{a}_k = \begin{bmatrix} \sin\tilde{\theta}_k \\ -\cos\tilde{\theta}_k \end{bmatrix}, \tag{9}$$

$\boldsymbol{b}$ is the $N \times 1$ vector

$$\boldsymbol{b} = \begin{bmatrix} \boldsymbol{a}_0^T \boldsymbol{r}_0 \\ \boldsymbol{a}_1^T \boldsymbol{r}_1 \\ \vdots \\ \boldsymbol{a}_{N-1}^T \boldsymbol{r}_{N-1} \end{bmatrix}, \tag{10}$$

and $\boldsymbol{W}$ is the $N \times N$ diagonal weighting vector

$$\boldsymbol{W} = \begin{bmatrix} d_0^2\sigma_{n_0}^2 & & & \boldsymbol{0} \\ & d_1^2\sigma_{n_1}^2 & & \\ & & \ddots & \\ \boldsymbol{0} & & & d_{N-1}^2\sigma_{n_{N-1}}^2 \end{bmatrix}. \tag{11}$$

Here $d_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$ denotes the target range from the observer location at time instant k. The weighting matrix is therefore dependent on $\boldsymbol{\xi}$.

To obtain a linear LS solution to the bearings-only target tracking problem, the dependence of $\boldsymbol{W}$ on the target motion parameter vector $\boldsymbol{\xi}$ is ignored by using the true range values between the observer and the target, assuming that this information is available. This simplification allows us to treat (8b) as a weighted LS problem with a closed-form solution given by

$$\hat{\boldsymbol{\xi}}_S = \arg\min_{\boldsymbol{\xi}\in\mathbb{R}^6} J_S(\boldsymbol{\xi}) \tag{12a}$$

$$= (\boldsymbol{F}^T\boldsymbol{W}^{-1}\boldsymbol{F})^{-1}\boldsymbol{F}^T\boldsymbol{W}^{-1}\boldsymbol{b}. \tag{12b}$$

This estimator is an extension of the Stansfield target location estimator [1] to moving targets. We assume that the matrix $\boldsymbol{F}$ is full-rank as required by the observability assumption.

C. OV Estimator

In (3) we formulated the target tracking problem by referring to a statistical model for the bearing measurement errors. Alternatively we can establish an orthogonal vector sum relationship between the true and measured bearing vectors as shown in Fig. 2 where $\tilde{\boldsymbol{s}}_k$ is the noisy (measured) bearing vector emanating from the observer position $\boldsymbol{r}_k$ and makes an angle of $\tilde{\theta}_k$ with the x-axis, $\boldsymbol{s}_k$ is the bearing vector between $\boldsymbol{r}_k$ and the target $\boldsymbol{p}_k$, and $\boldsymbol{e}_k$ is the orthogonal error vector (note that $\boldsymbol{e}_k^T\tilde{\boldsymbol{s}}_k = 0$).

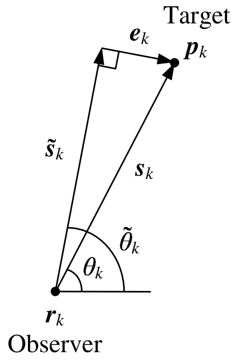

Fig. 2. Relationship between noisy bearing vector $\tilde{\boldsymbol{s}}_k$ and the target location $\boldsymbol{p}_k$.

From Fig. 2 we have

$$\boldsymbol{p}_k = \boldsymbol{r}_k + \boldsymbol{s}_k \tag{13a}$$

$$= \boldsymbol{r}_k + \tilde{\boldsymbol{s}}_k + \boldsymbol{e}_k \tag{13b}$$

where the orthogonal vector $\boldsymbol{e}_k$ is defined by

$$\boldsymbol{e}_k = \|\boldsymbol{s}_k\|_2 \sin(\tilde{\theta}_k - \theta_k) \begin{bmatrix} \sin\tilde{\theta}_k \\ -\cos\tilde{\theta}_k \end{bmatrix} \tag{14}$$

$$= d_k \sin n_k \, \boldsymbol{a}_k. \tag{15}$$

Here $d_k = \|\boldsymbol{s}_k\|_2$ is the range between the observer vector $\boldsymbol{r}_k$ and the target $\boldsymbol{p}_k$ defined after (11), $n_k = \tilde{\theta}_k - \theta_k$ is the Gaussian bearing noise in (3), and $\boldsymbol{a}_k$ is the unit vector orthogonal to $\tilde{\boldsymbol{s}}_k$ defined in (9).

To eliminate the noisy bearing vector $\tilde{\boldsymbol{s}}_k$ from the equation, multiply (13b) through with the transpose of the orthogonal unit vector $\boldsymbol{a}_k$, yielding

$$\boldsymbol{a}_k^T\boldsymbol{p}_k = \boldsymbol{a}_k^T\boldsymbol{r}_k + \eta_k \tag{16}$$

where $\eta_k = d_k \sin n_k$ is a zero-mean nonlinear Gaussian noise. Concatenating (16) for $k = 0, \ldots, N-1$, we get

$$\boldsymbol{F\xi} = \boldsymbol{b} + \boldsymbol{\eta} \tag{17}$$

where $\boldsymbol{F}$ and $\boldsymbol{b}$ were defined in (9) and (10), respectively, and $\boldsymbol{\eta} = [\eta_0, \ldots, \eta_{N-1}]^T$. An LS solution to (17) is given by

$$\hat{\boldsymbol{\xi}}_{LS} = \arg\min_{\boldsymbol{\xi}\in\mathbb{R}^6} \|\boldsymbol{F\xi} - \boldsymbol{b}\|_2^2 \tag{18a}$$

$$= (\boldsymbol{F}^T\boldsymbol{F})^{-1}\boldsymbol{F}^T\boldsymbol{b} \tag{18b}$$

which is referred to as the orthogonal vectors (OV) estimator or the pseudolinear estimator [2].

Under the small bearing error assumption (i.e., $n_k \approx 0$), the errors η_k can be approximated by $\eta_k \approx d_k n_k$

which suggests a Gaussian η_k with variance $d_k^2\sigma_{n_k}^2$. To handle errors η_k with different variances, a weighted LS solution can be constructed:

$$\arg\min_{\boldsymbol{\xi}\in\mathbb{R}^6} \|\boldsymbol{W}^{-1/2}(\boldsymbol{F}\boldsymbol{\xi}-\boldsymbol{b})\|_2^2$$

which is identical to the Stansfield solution in (12).

The diagonal weighting matrix $\boldsymbol{W}$ aims to improve the LS solution by adjusting the contributions of equations concatenated in (17) according to their noise variances. The diagonal entries of $\boldsymbol{W}^{-1/2}$ (i.e., the weights) are determined from the reciprocal of noise standard deviations. If the range information d_k is not available, the OV estimator is preferred over the Stansfield estimator as it does not require the use of the weighting matrix. In practical applications, the range information for the target is not usually available. In the absence of this, the weighting matrix is typically replaced by an identity matrix (assuming identical bearing noise variances), which reduces the Stansfield estimator to the OV estimator.

D. IV Estimator

The mean of the OV estimate is

$$E\{\hat{\boldsymbol{\xi}}_{\text{LS}}\} = \boldsymbol{\xi} - E\{(\boldsymbol{F}^T\boldsymbol{F})^{-1}\boldsymbol{F}^T\boldsymbol{\eta}\}$$

where $-E\{(\boldsymbol{F}^T\boldsymbol{F})^{-1}\boldsymbol{F}^T\boldsymbol{\eta}\}$ is the estimation bias. Even though we have $E\{\boldsymbol{\eta}\} = \boldsymbol{0}$, the estimation bias is generally non-zero because $\boldsymbol{F}$ and $\boldsymbol{\eta}$ are correlated (the bearing noise n_k appears in both $\boldsymbol{F}$ and $\boldsymbol{\eta}$) [8]. The OV solution is obtained from the normal equations

$$\boldsymbol{F}^T\boldsymbol{F}\hat{\boldsymbol{\xi}}_{\text{LS}} = \boldsymbol{F}^T\boldsymbol{b}.$$

Since the reason for the bias is the correlation between $\boldsymbol{F}$ and $\boldsymbol{\eta}$, we can consider the following modified normal equations:

$$\boldsymbol{G}^T\boldsymbol{F}\hat{\boldsymbol{\xi}}_{\text{IV}} = \boldsymbol{G}^T\boldsymbol{b}$$

where $\boldsymbol{G}$ is the matrix of instrumental variables. If $\boldsymbol{G}$ is chosen such that $E\{\boldsymbol{G}^T\boldsymbol{F}\}$ is nonsingular and $E\{\boldsymbol{G}^T\boldsymbol{\eta}\} = \boldsymbol{0}$, then $\hat{\boldsymbol{\xi}}_{\text{IV}}$ will be asymptotically unbiased, i.e., $E\{\hat{\boldsymbol{\xi}}_{\text{LS}}\} = \boldsymbol{\xi}$ as $N \to \infty$.

A simple and appealing choice for the instrumental variable matrix $\boldsymbol{G}$ is first to obtain an estimate for the target motion parameters using one of the LS algorithms. In view of its improved estimation performance, we favour the TLS solution $\hat{\boldsymbol{\xi}}_{\text{TLS}}$ which is discussed in the next section. The target location estimates $\hat{\boldsymbol{p}}_k = [\hat{p}_{x,k}, \hat{p}_{y,k}]^T$ are then obtained from (2), using $\hat{\boldsymbol{\xi}}_{\text{TLS}}$. Given the $\hat{\boldsymbol{p}}_k$, we can construct the instrumental variables matrix:

$$\boldsymbol{G} = \begin{bmatrix} \hat{\boldsymbol{a}}_0^T\boldsymbol{M}_0/\hat{d}_0^2\sigma_{n_0}^2 \\ \vdots \\ \hat{\boldsymbol{a}}_{N-1}^T\boldsymbol{M}_{N-1}/\hat{d}_{N-1}^2\sigma_{n_{N-1}}^2 \end{bmatrix}, \quad \hat{\boldsymbol{a}}_k = \begin{bmatrix} \sin\hat{\theta}_k \\ -\cos\hat{\theta}_k \end{bmatrix}$$

where $\hat{d}_k = \|\hat{\boldsymbol{p}}_k - \boldsymbol{r}_k\|_2$ and $\hat{\theta}_k = \tan^{-1}\frac{\hat{p}_{y,k}-r_{y,k}}{\hat{p}_{x,k}-r_{x,k}}$. The asymptotically unbiased IV estimator is then given by

$$\hat{\boldsymbol{\xi}}_{\text{IV}} = (\boldsymbol{G}^T\boldsymbol{F})^{-1}\boldsymbol{G}^T\boldsymbol{b}.$$

E. TLS Estimator

As common with all LS solutions, the OV estimator implicitly assumes that only $\boldsymbol{b}$ is subject to error. In fact, because of the presence of errors in bearing measurements, the system matrix $\boldsymbol{F}$ is also subject to error. To improve the accuracy of the OV estimator, the concept of *total least squares* (TLS) can be invoked to mitigate errors in both $\boldsymbol{F}$ and $\boldsymbol{b}$ [9]. Formally, the TLS estimate $\hat{\boldsymbol{\xi}}_{\text{TLS}}$ is given by the solution of the following constrained optimization problem [10, 11]:

$$\min_{(\boldsymbol{F}+\boldsymbol{\Delta})\hat{\boldsymbol{\xi}}_{\text{TLS}}=\boldsymbol{b}+\boldsymbol{\delta}} \|\boldsymbol{L}[\boldsymbol{\Delta}, -\boldsymbol{\delta}]\boldsymbol{T}\|_F \tag{19}$$

where $\boldsymbol{L} = \text{diag}(l_1, \ldots, l_N)$ is an $N \times N$ diagonal weighting matrix, $\boldsymbol{T} = \text{diag}(t_1, \ldots, t_7)$ is a 7×7 diagonal weighting matrix, and $\|\cdot\|_F$ denotes the Frobenius norm. According to the optimization problem in (19), the TLS solution is obtained by adding minimal perturbations to $\boldsymbol{F}$ and $\boldsymbol{b}$ so that the perturbed matrix equation is consistent.

The TLS solution can be obtained from a singular value decomposition (SVD) of the augmented $N \times 7$ matrix $\boldsymbol{L}[\boldsymbol{F}, -\boldsymbol{b}]\boldsymbol{T}$:

$$\boldsymbol{L}[\boldsymbol{F}, -\boldsymbol{b}]\boldsymbol{T} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T \tag{20a}$$

$$= \sum_{i=1}^{7}\sigma_i\boldsymbol{u}_i\boldsymbol{v}_i^T \tag{20b}$$

where $\boldsymbol{U} = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_7]$ is an $N \times 7$ unitary matrix, $\boldsymbol{\Sigma} = \text{diag}(\sigma_1, \ldots, \sigma_7)$ is the 7×7 diagonal matrix of ordered singular values (i.e., $\sigma_1 \geq \cdots \geq \sigma_7 > 0$), and $\boldsymbol{V} = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_7]$ is a 7×7 unitary matrix. The TLS estimate is given by

$$\hat{\boldsymbol{\xi}}_{\text{TLS}} = \frac{1}{t_7 v_{77}} \begin{bmatrix} t_1 v_{17} \\ \vdots \\ t_6 v_{67} \end{bmatrix} \tag{21}$$

where $\boldsymbol{v}_7 = [v_{17}, \ldots, v_{77}]^T$ is the seventh column of $\boldsymbol{V}$. Equation (21) assumes that the smallest singular value is unique, i.e., $\sigma_6 > \sigma_7$. This assumption may be violated on rare occasions. If this happens, alternative solutions can be obtained (see [11] for details).

The TLS solution in general exhibits smaller estimation bias than the LS solution [12]. In the absence of bearing and observer location errors, we get $\boldsymbol{F}_o\boldsymbol{\xi} = \boldsymbol{b}_o$ where $\boldsymbol{F}_o$ and $\boldsymbol{b}_o$ are obtained from true bearing angles and observer locations. The noisy matrix $\boldsymbol{F}$ and vector $\boldsymbol{b}$ are related to $\boldsymbol{F}_o$ and $\boldsymbol{b}_o$ through $\boldsymbol{F} = \boldsymbol{F}_o + \boldsymbol{\Delta}_o$ and $\boldsymbol{b} = \boldsymbol{b}_o + \boldsymbol{\delta}_o$ where, for $n_k \approx 0$,

$$\boldsymbol{\Delta}_o \approx \begin{bmatrix} \boldsymbol{\nu}_0^T\boldsymbol{M}_0 \\ \vdots \\ \boldsymbol{\nu}_{N-1}^T\boldsymbol{M}_{N-1} \end{bmatrix}, \quad \boldsymbol{\nu}_k = n_k \begin{bmatrix} \cos\theta_k \\ \sin\theta_k \end{bmatrix}$$

$$\boldsymbol{\delta}_o \approx \begin{bmatrix} \boldsymbol{\nu}_0^T\boldsymbol{r}_0 \\ \vdots \\ \boldsymbol{\nu}_{N-1}^T\boldsymbol{r}_{N-1} \end{bmatrix}.$$

The weighting matrices can be chosen by taking into account the error variances in $\boldsymbol{F}$ and $\boldsymbol{b}$. If we set $\boldsymbol{L} = \boldsymbol{I}$ and $\boldsymbol{T} = \mathrm{diag}(1, 1, \epsilon)$, where $\epsilon > 0$, and let $\epsilon \to 0$, the TLS solution converges to the LS solution.

F. CTLS Estimator

The perturbed system matrix $\boldsymbol{F} + \boldsymbol{\Delta}$ and data vector $\boldsymbol{b} + \boldsymbol{\delta}$ that result from TLS do not retain the structure of the original system matrix and data vector defined in (9) and (10). The accuracy of the TLS estimate can be improved if the TLS perturbations are forced to obey the structure of the system matrix and data vector. The imposition of these constraints on the TLS solution leads to a constrained TLS (CTLS) problem that can be formulated as

$$\min_{\boldsymbol{\Delta},\, \boldsymbol{\delta}} \|\boldsymbol{L}[\boldsymbol{\Delta}, -\boldsymbol{\delta}]\boldsymbol{T}\|_F \tag{22a}$$

subject to

$$(\boldsymbol{F} + \boldsymbol{\Delta})\hat{\boldsymbol{\xi}}_{\text{CTLS}} = \boldsymbol{b} + \boldsymbol{\delta} \tag{22b}$$

$$\boldsymbol{F} + \boldsymbol{\Delta} = \begin{bmatrix} \check{\boldsymbol{a}}_0^T & \boldsymbol{0} & \boldsymbol{0} \\ \check{\boldsymbol{a}}_1^T & t_1\check{\boldsymbol{a}}_1^T & \frac{t_1^2}{2}\check{\boldsymbol{a}}_1^T \\ & \vdots & \\ \check{\boldsymbol{a}}_{N-1}^T & t_{N-1}\check{\boldsymbol{a}}_{N-1}^T & \frac{t_{N-1}^2}{2}\check{\boldsymbol{a}}_{N-1}^T \end{bmatrix} \tag{22c}$$

$$\|\check{\boldsymbol{a}}_i\|_2 = 1, \quad i = 0, \ldots, N-1 \tag{22d}$$

$$\boldsymbol{b} + \boldsymbol{\delta} = \begin{bmatrix} \check{\boldsymbol{a}}_0^T \boldsymbol{r}_0 \\ \vdots \\ \check{\boldsymbol{a}}_{N-1}^T \boldsymbol{r}_{N-1} \end{bmatrix} \tag{22e}$$

A detailed treatment of CTLS can be found in [9].

A closed-form solution to (22) is not available because of the nonlinearity of constraints. We therefore seek an iterative numerical solution based on nonlinear programming and the method of successive projections [9]. The idea behind successive projections is to obtain the TLS solution and then to project it to vectors satisfying the given constraints using a least-squares criterion, and repeat the whole procedure until TLS converges to a solution satisfying all the constraints.

The method of successive projections was shown to produce a vector sequence that always contains a subsequence that converges to a vector satisfying the constraints of the optimization problem under some mild conditions on the projections [13].

G. STLS Estimator

The CTLS algorithm attempts to maintain the structure of the system matrix and the data vector while perturbing them to achieve consistency. The constrained optimization problem can be recast as

$$\min_{\boldsymbol{\delta}} \|\boldsymbol{L}\boldsymbol{\delta}\|_2^2 \tag{23a}$$

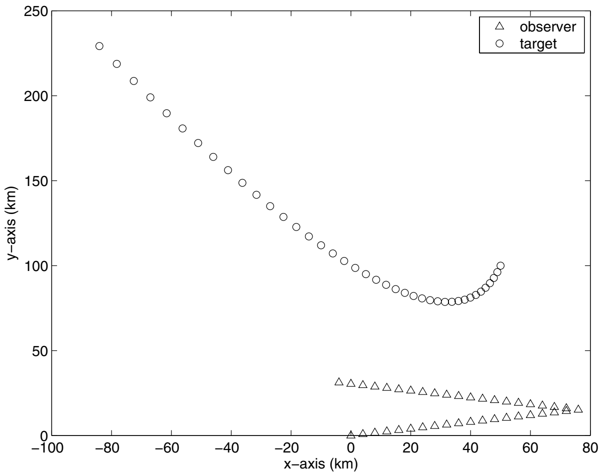

Fig. 3. Simulated target tracking geometry.

subject to

$$\boldsymbol{F}(\tilde{\boldsymbol{\theta}} + \boldsymbol{\delta})\hat{\boldsymbol{\xi}}_{\text{STLS}} = \boldsymbol{b}(\tilde{\boldsymbol{\theta}} + \boldsymbol{\delta}) \tag{23b}$$

where $\boldsymbol{L}$ is an $N \times N$ diagonal weighting matrix and

$$\boldsymbol{F}(\boldsymbol{\phi}) = \begin{bmatrix} \boldsymbol{a}^T(\phi_0)\boldsymbol{M}_0 \\ \vdots \\ \boldsymbol{a}^T(\phi_{N-1})\boldsymbol{M}_{N-1} \end{bmatrix} \tag{24a}$$

$$\boldsymbol{b}(\boldsymbol{\phi}) = \begin{bmatrix} \boldsymbol{a}^T(\phi_0)\boldsymbol{r}_0 \\ \vdots \\ \boldsymbol{a}^T(\phi_{N-1})\boldsymbol{r}_{N-1} \end{bmatrix} \tag{24b}$$

with $\boldsymbol{\phi} = [\phi_0, \ldots, \phi_{N-1}]^T$ and $\boldsymbol{a}(\phi_i) = [\sin\phi_i, -\cos\phi_i]^T$. Equation (24) makes explicit the nonlinear parameterization of $\boldsymbol{F}$ and $\boldsymbol{b}$ in terms of N bearing angles $\boldsymbol{\phi}$. The constrained optimization problem in (23) is known as the *structured total least squares* problem, and its solution has been addressed in [14, 15].

The STLS problem is in fact identical to the ML problem. The equivalence between the STLS and ML solutions can be seen by setting $\boldsymbol{L} = \boldsymbol{K}^{-1/2}$, which reduces (23a) to the minimization of the ML cost function in (6), and by rewriting (1) as

$$\frac{\sin\theta_k}{\cos\theta_k} = \frac{p_{y,k} - r_{y,k}}{p_{x,k} - r_{x,k}}$$

which shows that the ML solution satisfies the consistency constraint in (23b).

IV. Simulation Examples

In the simulation examples, we use the target tracking geometry shown in Fig. 3. The observer trajectory consists of two constant velocity legs. The target moves at a constant acceleration with motion parameters $\boldsymbol{p}_0 = [50, 100]^T$, $\boldsymbol{v}_0 = [-2, -8]^T$ and $\boldsymbol{a} = [-0.5, 1.5]^T$. The bearing

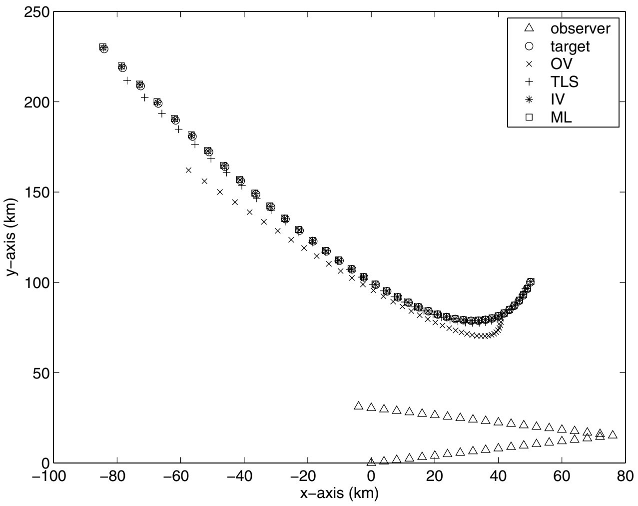

Fig. 4. Average of target track estimates.

TABLE I MSE ($\sigma_w = 0.01$)

σ_n	OV	TLS	IV	ML
0.1°	4.8	3.5	3.0	3.0
0.3°	122.0	33.4	28.1	28.2
0.5°	554.9	99.1	79.2	79.8
0.7°	1267.1	191.5	157.6	159.6
0.9°	2128.5	332.8	294.4	302.7

TABLE II Bias of $\boldsymbol{p}_0$ Estimate ($\sigma_w = 0.01$)

σ_n	OV	TLS	IV	ML
0.1°	1.18	0.17	0.01	0.01
0.3°	9.64	1.44	0.11	0.16
0.5°	22.08	3.67	0.38	0.55
0.7°	34.17	5.72	1.21	1.46
0.9°	44.77	8.97	1.21	1.60

measurements are taken at $t_k = kT$, where $T = 0.5$ and $k = 0, \ldots, N-1$. For the moving target, $N = 40$ bearing measurements are collected by the observer at marked locations in Fig. 3. The bearing measurements are subject to i.i.d. zero-mean Gaussian noise with variance σ_n^2. The observer location measurements are corrupted by i.i.d. zero-mean bivariate Gaussian noise with covariance matrix $\boldsymbol{C} = \text{diag}(\sigma_w^2, \sigma_w^2)$. The target motion parameters were estimated using the OV, TLS, IV and ML estimators. The results of 1000 Monte Carlo simulations for mean-squared error (MSE) and initial velocity estimation bias are listed in Tables I and II for different bearing noise standard deviations σ_n with the observer position error standard deviation fixed at $\sigma_w = 0.01$km. Fig. 4 shows the average of target track estimates for different algorithms at $\sigma_n = 0.5°$. For the target-observer encounter depicted in Fig. 3, the IV estimator yields the best accuracy narrowly outperforming the ML estimator. The TLS estimator appears to have a much improved performance compared with the OV estimator.

V. References

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