AFRL-IF-RS-TR-2005-331
Final Technical Report
September 2005

COMBINATORIAL MARKET PROCESSES FOR MULTILATERAL COORDINATION

University of California at Berkeley

Sponsored by
Defense Advanced Research Projects Agency
DARPA Order No. K253

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED.

AIR FORCE RESEARCH LABORATORY
INFORMATION DIRECTORATE
ROME RESEARCH SITE
ROME, NEW YORK

STINFO FINAL REPORT

This report has been reviewed by the Air Force Research Laboratory, Information Directorate, Public Affairs Office (IFOIPA) and is releasable to the National Technical Information Service (NTIS). At NTIS it will be releasable to the general public, including foreign nations.

AFRL-IF-RS-TR-2005-331 has been reviewed and is approved for publication

APPROVED: /s/

RICHARD C. BUTLER, II
Project Engineer

FOR THE DIRECTOR: /s/

WARREN H. DEBANY, JR., Technical Advisor
Information Grid Division
Information Directorate

REPORT DOCUMENTATION PAGE				*Form Approved OMB No. 074-0188*

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	SEPTEMBER 2005	Final Aug 01 – Oct 04

4. TITLE AND SUBTITLE
COMBINATORIAL MARKET PROCESSES FOR MULTILATERAL COORDINATION

5. FUNDING NUMBERS
C - F30602-01-2-0548
PE - 62301E
PR - K253
TA - 04
WU - A1

6. AUTHOR(S)
Rahul Jain and
Pravin Varaiya

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
University of California at Berkeley
Department of Electrical Engineering & Computer Science
271 M Cory Hall
Berkeley California 94720

8. PERFORMING ORGANIZATION REPORT NUMBER
N/A

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Defense Advanced Research Projects Agency	AFRL/IFGA
3701 North Fairfax Drive	525 Brooks Road
Arlington Virginia 22203-1714	Rome New York 13441-4505

10. SPONSORING / MONITORING AGENCY REPORT NUMBER
AFRL-IF-RS-TR-2005-331

11. SUPPLEMENTARY NOTES

AFRL Project Engineer: Richard C. Butler II/IFGA/(315) 330-1888/ Richard.Butler@rl.af.mil

12a. DISTRIBUTION / AVAILABILITY STATEMENT
APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 Words)*
We study a .classical question in a modern context. There are a number of buyers and sellers of a number of distinct goods. Each participant is selfish- It cares more for its own benefit than of the social welfare. Each good is indivisible. It must go completely to one of the participants. Moreover, the participants are not 'passive' as Smith [157] and Walras [167] believed, but 'actively' take actions to further their interest in the spirit of Cournot [27] and Edgeworth [37], and later, von Neumann and Morgenstern [117] and Nash [116]. We are thus interested in the following questions. When does an equilibrium exist in a market with several indivisible goods? And what economic mechanisms yield an allocation that promotes the welfare of the society as a whole? To put it more concretely, we want to examine the existence of competitive equilibrium a combinatorial market, i.e., an exchange economy with several indivisible goods such that consumers have interdependent valuations: A consumer's utility is for a bundle of indivisible goods. Further, we seek auction or market mechanisms that yield social welfare maximizing allocations when participants or agents exercise strategic behavior. Despite this being a long standing question, it has only been incompletely resolved for the setting of interest. The following problems from communication networks and operations research motivated this work.

14. SUBJECT TERMS	15. NUMBER OF PAGES
Combinatorial Markets. Auctions, Bandwidth Allocation, Competitive Equilibrium, Network Topology	66
	16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

Contents

1 Introduction and Summary **1**

1.1 Motivating Problems . . . 1
1.2 Models of Large Markets and Economic Efficiency . . . 3
1.3 Auction Mechanism Design for Combinatorial Markets . . . 5
1.4 Strategic Behavior in Auctions and The Price of Anarchy . . . 7
1.5 Validating Economic Theory through Experiments . . . 9
1.6 Contributions . . . 10

2 Existence of Competitive Equilibrium in Combinatorial Markets **11**

2.1 Introduction . . . 11
2.2 Network Topology and Economic Efficiency . . . 13
2.3 Competitive Equilibrium in the Continuum Model . . . 16
2.4 Approximate Competitive Equilibrium . . . 20
2.5 Chapter Summary . . . 22

3 c-SeBiDA: An Efficient Market Mechanism for Combinatorial Markets **23**

3.1 Introduction . . . 23
3.2 The Combinatorial Sellers' Bid Double Auction . . . 26
3.3 Nash Equilibrium Analysis: c-SeBiDA is Efficient . . . 27
3.4 SeBiDA is Asymptotically Bayesian Incentive Compatible . . . 32
3.5 c-SeBiDA Outcome is Competitive Equilibrium in the Continuum Model . . . 38
3.6 Chapter Summary . . . 41

4 Human-subject Experiments **42**

4.1 Introduction and Literature Review . . . 42
4.2 Combinatorial Auctions . . . 43
4.3 Information and Valuation Structure . . . 44
4.4 Experimental Results . . . 46
4.5 Chapter Summary . . . 48

5 Conclusions and Future Work **49**

Bibliography **51**

List of Figures

2.1 A cyclic network that is not TU. 14
2.2 An acyclic network that is not TU. 15

3.1 The payoff of the buyer as a function of its bid, b, for various cases. 34
3.2 The payoff of the seller as a function of its bid, a, for various cases. 35

List of Tables

4.1 Example of Seller Valuations 45
4.2 Example of Buyer Valuations 45
4.3 Summary of Buyer Percentage Efficiency in Each Round 47
4.4 Summary of Seller Percentage Efficiency in Each Round 47
4.5 Aggregate Average Percentage Shading Factor Per Round 48
4.6 Seller Overbidding Percentage Over Costs 48
4.7 Buyer Underbidding Percentage Over Valuations 48

Chapter 1

Introduction and Summary

1.1 Motivating Problems

> As every individual... intends only his own gain, and he is in this, as in many other cases, led by *an invisible hand* to promote an end which was no part of his intention. Nor is it always the worse for the society that it was no part of it. By pursuing his own interest he frequently promotes that of the society more effectually than when he really intends to promote it.
>
> —Adam Smith, *An Inquiry into the Nature and Causes of the Wealth of Nations* IV.2:9, 1776.

We study a classical question in a modern context. There are a number of buyers and sellers of a number of distinct goods. Each participant is selfish. It cares more for its own benefit than of the social welfare. Each good is indivisible. It must go completely to one of the participants. Moreover, the participants are not 'passive' as Smith [156] and Walras [166] believed, but 'actively' take actions to further their interest in the spirit of Cournot [27] and Edgeworth [37], and later, von Neumann and Morgenstern [117] and Nash [116].

We are thus interested in the following questions. When does an equilibrium exist in a market with several indivisible goods? And what economic mechanisms yield an allocation that promotes the welfare of the society as a whole? To put it more concretely, we want to examine the existence of competitive equilibrium in a *combinatorial market* , i.e., an exchange economy with several indivisible goods such that consumers have interdependent valuations: A consumer's utility is for a bundle of indivisible goods. Further, we seek auction or market mechanisms that yield social welfare maximizing allocations when participants or agents exercise strategic behavior.

Despite this being a long standing question, it has only been incompletely resolved for the setting of interest. The following problems from communication networks and operations research motivated this work.

Wireless Networks. Consider a cellular network. An agency such as the FCC [39] wants to auction spectrum to wireless service providers such as AT&T, Cingular, Sprint and Verizon. The wireless service providers on their part, bid for spectrum in various cells. They aim for widespread coverage for their customers and derive maximum benefit if they can provide service in contiguous cells. Thus, wireless service providers need spectrum in *bundles* of cells. Moreover, the FCC auctions spectrum in some indivisible chunks such as 10 MHz. Thus, spectrum is an *indivisible* good. These two features of spectrum make the allocation problem *combinatorial*. The FCC wants to find allocation mechanisms that determine *uniform* (every unit sells for same price) and *anonymous* (users are not discriminated depending on their identity or ability to pay) prices. Moreover, the mechanisms are required to yield *efficient* allocation, i.e., those that maximize the *social welfare*. We will be more specific later.

Communication Networks. Now, consider a communication network with links $\{1, \cdots, L\}$. There are owners of capacity on links such as AT&T, MCI and Sprint. And there are service-providers such as AOL, Earthlink and Comcast. An owner i owns a certain amount $C_{i,l}$ Mbps of capacity on a particular link and has a reservation cost $c_i(b, l)$ if it were to sell b units on link l. A service-provider j has a reservation utility $v_j(b, R_j)$ for b units of capacity on route R_j, which is a *bundle* of links. As before, the capacity is exchanged in some *indivisible* unit say 10 Mbps. This makes the exchange problem *combinatorial*. We need a market mechanism whose outcome is *efficient*, i.e., maximizes the *trading surplus*.

Electricity Markets. A similar problem arises in power networks. In fact, there is a well established system for trading power on a daily basis. This has "commoditized" power thus making the market more efficient, and ultimately benefiting the consumers. Though the question of mechanisms that achieve full efficiency remains open [127].

Air-slot Allocation. Air-landing and take-off slots are currently allocated to airlines depending on their bids. However, air traffic changes dramatically and this necessitates the need for re-allocation. Currently, this re-allocation is left to the air-traffic controllers with penalties for the errant airlines. However, this reallocation could be determined efficiently through a combinatorial auction [129].

Supply-Chain Management. Similar problems arise in several manufacturing contexts. For example, a car manufacturing unit may bid for x units of an item A *and* item B. It needs both, say, to produce a car. Other car manufacturing units may have similar demands. There could be sellers offering each of items A and B. The exchange could then be determined by a combinatorial auction. Such exchanges are currently determined by bilateral contracts which lead to inefficiencies in the market.

Thus, at an abstract level, the problems that we study are of considerable interest to various areas of engineering, computer science, operations research and management. The solution that we offer is practical. However, in each case, additional technological infrastructure may be necessary. For example, in the context of communication networks, we would need a technology that can establish the routes bought in an automated fashion. This becomes particularly crucial with large numbers of buyers and sellers as in bandwidth exchanges. This study is limited to solving the abstract problem, although it has immediate relevance for these real-world problems.

Thus, the following questions from the above problems motivated our work.

Q.1. When does competitive equilibrium exist in a combinatorial market?

Q.2. What mechanisms achieve outcomes close to competitive equilibrium?

Q.3. Do there exist optimal mechanisms that minimize efficiency loss of any Nash equilibrium when the players act strategically? Does the incomplete information case result in sub-optimal outcomes?

Q.4. How do the theoretical results compare to real world settings when human agents are involved?

In the rest of this summary, we give a high-level description of how we answer these questions. We also discuss how it relates to various research areas, and how it contributes to each of them. In section 1.2, we describe our work on existence of competitive equilibrium in a particular model of large combinatorial markets. In section 1.3, we describe the set-up for combinatorial markets and extant auction mechanism design theory for such markets. Section 1.4 discusses the strategic behavior of agents in an auction and how it may result in Nash equilibrium allocations which are inefficient. In the congestion games literature, this has been called *the price of anarchy*. Section 1.5 presents human-subject experimental results used to verify the game theoretic results that we obtained. Section 1.6 summarizes the contributions.

1.2 Models of Large Markets and Economic Efficiency

First, we investigate whether economically efficient resource allocations are attainable in a large market with independent participants.

Suppose there are N agents and L commodities. All commodities are indivisible. They are often treated as perfectly divisible. This is more for mathematical convenience and may be acceptable when the quantities involved are large but not otherwise. (Even oil which *is* divisible is actually sold in units of barrels.) Thus, throughout, we regard all goods as *indivisible*. Moreover, we will consider all agents to be consumers. That is, it is a *pure exchange* economic system and does not involve firms or production. A consumer i has a consumption set X_i with a preference order $\preceq_i$ on any pair of consumptions in X_i. It is known that for continuous preferences on connected consumption sets, there exists a continuous utility function [30]. With indivisible commodities, preferences are not continuous and the consumption sets are not connected. However, when $\preceq_i$ is a complete order on X_i, it is easy to see that there exists a utility function on X_i. At times, it is more convenient to work directly with utility functions. We will assume that there is a divisible good or currency that circulates as *numeraire* or *money*. We will assign a price p_0 to it as well. The price of all other goods is then obtained in terms of this currency by dividing the price of each good by the price of money.

Given the utility functions of the consumers, the first pertinent question is what allocations are more desirable than others. This has received the attention of economists for a long time. Following Marshall [104], we shall assume that allocations that maximize the social welfare, the sum of utility functions of the consumers is more desirable. One reason for choosing such allocations is that they are *Pareto-efficient* [121]: the allocation cannot be changed to make one agent strictly better off and no other agent worse off.

We will assume that each consumer has a utility function quasi-linear in money. There are no income effects. Moreover, all the goods and the total money available is allocated to the consumers as their initial endowment. We seek a *market equilibrium* wherein a price is assigned to each commodity. And at that price, the demands of all the consumers is such that the market *clears*, i.e., every unit of each commodity gets allocated to some consumer. We will assume that each participant does not anticipate the effect of his actions on price. Such market equilibria are referred to as *general* or *competitive* or *Walrasian equilibria*.

The notion of competitive equilibrium (C.E.) dates back to Walras [166], but it was Wald [165] who laid its modern mathematical foundations and first rigorously proved its existence in competitive markets. This program was carried forward by Arrow and Debreu [4], Gale [47] and McKenzie [110], who proved existence for economies with divisible commodities under the assumption of convex preferences and connected consumption sets. Over the years, this has been improved to the following statement.

Theorem (Arrow-Debreu). *Suppose consumer preferences are continuous, strictly convex and strongly monotone. Suppose there is positive endowment of every commodity and that the excess demand correspondence* $\Phi(\cdot)$ *satisfies the following properties.*
(i) It is continuous.
(ii) It is homogeneous of degree zero.
(iii) $p \cdot \Phi(p) = 0$ *for all* p *(Walras's law).*

(iv) There is an $s > 0$ such that $\Phi_l(p) > -s$ for every commodity l and all p.
(v) If $p^n \to p$, where $p \neq 0$ and $p_l = 0$ for some l, then

$$\max_l \Phi_l(p^n) \to \infty.$$

Then, a competitive equilibrium exists.

A competitive equilibrium is regarded as a desirable outcome because of the *First Theorem of Welfare Economics*: A competitive equilibrium allocation is Pareto-efficient [3]. There are converse theorems. But they require additional conditions on the preferences. For example, the *Second Theorem of Welfare Economics* states: If each consumer holds strictly positive initial endowment of each commodity, the preferences are convex, continuous and strongly monotonic, then there exist prices such that a Pareto-efficient allocation is also a competitive allocation at those prices [30].

Of course, a competitive equilibrium need not always exist and not all Pareto-efficient allocations need be feasible. An allocation should be attainable by actions of a consumer or of a coalition of consumers. Thus, the concept of the *core of an economy* $\mathcal{C}$ is introduced: The set of feasible allocations of the economy such that it cannot be improved upon by any coalition. Clearly, every allocation in the core is Pareto-efficient. Furthermore, the set of all competitive equilibrium allocations $\mathcal{CE}$, is contained in the core. The interesting question then is the equivalence of $\mathcal{C}$ and $\mathcal{CE}$.

A competitive equilibrium need not exist in an economy with indivisible goods. The difficulties primarily lie in the fact that the utility functions are non-concave and discontinuous, and that the consumption sets are totally disconnected. This makes use of any of the standard fixed point theorems such as the Brouwer or the Kakutani fixed point theorems impossible.

Early attempts to deal with indivisible commodities considered "matching models", inspired by the "stable marriage" assignment problem of Gale and Shapley [49]. Shapley and Shubik [152] studied the competitive equilibrium problem in asymmetric markets with buyers and sellers, each participant being one or the other. Shapley and Scarf [151] considered the more general exchange model in which a participant could be both buyer and seller. They focussed on the core and showed that an exchange economy with indivisible goods has a nonempty core. In all this work, it is assumed that each participant buys or sells only one commodity. Thus, the market was *non-combinatorial*.

The problem remains of interest in recent literature as well [95, 100, 17]. However, each of these approaches makes some assumption which restricts general application of the work. For example, [95] assumes that each participant owns at least one indivisible commodity initially. Moreover, utility is also derived from at most one indivisible commodity. A *combinatorial market* is considered in [17] but the agent preferences considered are rather special. Each agent is assumed to have a reservation value for each bundle. Ma [100] considers a general setting but without money, and obtains necessary and sufficient conditions for existence of competitive equilibrium.

Efforts have been made to characterize the limit points of market equilibria of economies with non-convex preferences and indivisibilities as the market grows in size. Debreu and Scarf [31] proposed one model of large economies as a finite economy replicated countably many times. More general models of countable economies were considered in [59, 35]. However, C.E. may not exist even in countable economies with non-convex preferences. Thus, there have been attempts to deal with non-convex preferences in a finite setting by characterizing approximate equilibria. Starr [158] characterized certain approximate competitive equilibrium based on results which state that non-convexities in an aggregate of non-convex sets do not grow in size with the number of sets making up the aggregate. This "averaging" results in non-convexities becoming less important in a large economy.

Aumann introduced a continuum set of participants [9] to model large economies with perfect competition wherein each participant is negligible compared to the overall size of the economy. Unlike [4], he did not assume anything about the valuation of the participants. But the goods are divisible and in such a setting, he showed that competitive equilibrium exists [11]. It was shown by Mas-Colell [105] that Aumann's results do not extend to a continuum economy with indivisible goods without money, i.e., competitive equilibrium need not exist in continuum exchange economies with indivisible commodities. However, Khan and Yamakazi [85] showed that the core of a continuum economy with indivisible goods is non-empty. This raised the hope that some allocations in the core may be decentralized through competitive prices.

In chapter 2, we provide exactly such a result.

We consider an exchange economy with multiple commodities and money. Unlike [17, 95, 100], we consider very general preferences and do not make any assumption on initial endowments. Moreover, we consider a combinatorial market. Our only assumption is that the preferences are continuous and monotonic in money. Our interest is in the perfect competition case, in which each participant is negligible enough that it cannot affect the prices and the allocation. We adopt Aumann's continuum model as our model of perfect competition and obtain the following result.

Theorem 3.2 (C.E. Existence). *Suppose agent preferences are continuous and monotonic in money. There is a positive endowment of every commodity and each consumer has positive endowment of some commodity. Assume that the excess demand correspondence satisfies the following properties.*
(i) $\Phi(p)$ *is homogeneous in* p*.*
(ii) Boundary condition: Suppose $p^\nu \to p^*$*, and* $p_l^* = 0$ *for some* l*. Then,* $z_l^\nu \to \infty$*,* $\forall z^\nu \in \Phi(p^\nu)$*.*
(iii) Walras' Law holds: $p \cdot z = 0, \forall z \in \Phi(p), \forall p \in \Delta^0$*, the relative interior of* Δ*.*
Then, a competitive equilibrium exists in the continuum exchange economy with indivisible commodities and money.

The result is important from a finite economy setting since using the Shapley-Folkman and the Starr theorem [158], one can now show the existence of various approximate competitive equilibrium.

1.3 Auction Mechanism Design for Combinatorial Markets

As noted above a competitive equilibrium is a desirable outcome. Having proved the existence of competitive equilibrium in the continuum economy and various approximate competitive equilibria in the finite economy, the question now is whether there exist mechanisms for combinatorial markets such that it results in a competitive equilibrium with a price assigned to each good.

A simple *market mechanism* that achieves competitive equilibrium for one divisible commodity is the following. Each buyer and each seller reveals his demand as a function of price. The trading price p^* is then determined as the one at which aggregate demand equals aggregate supply. Each buyer receives a quantity of the commodity that he said he demands at the price p^*. Similarly, each seller sells a quantity of the commodity that he said he can supply at the price p^*.

This can be generalized to the case of a combinatorial market with many indivisible goods. While the auction mechanism that we present is for a general combinatorial market, the design is motivated by the communication network resource allocation problem we discussed in section 1.1.

We consider multi-item combinatorial double auctions for resource allocation. Assume that sellers offer "loose" bundles, each with just one type of item (such as a link). For example, if a seller has 5 units of item A and 5 units of item B, he makes two OR offers; one with 5 units of item A and another with 5 units of item B. But then, within each bundle, only a fraction of the units may get sold; say 3 out of 5 units. The buyer's bundles on the other hand are of "all-or-none" kind. If a buyer bids for 5 units of *both* item A and item B, and if this bid is accepted, the buyer must receive all 5 units of each of the two items. As mentioned earlier, this requirement is motivated by situations wherein buyers want to acquire routes on communication networks. The assumption of non-combinatorial "loose" bundles for sellers allows us to set uniform prices on items.

We now describe the mechanism that specifies the 'rules of a game' among buyers and sellers.

Suppose there are L items $(l_1, \cdots, l_L)$, m buyers and n sellers. Buyer i has (true) reservation value v_i per unit for a bundle of items $R_i \subseteq \{l_1, \cdots, l_L\}$, and submits a buy bid of b_i per unit and demands up to δ_i units of the bundle R_i. Thus, the buyers have quasi-linear utility functions of the form $u_i^b(x; \omega, R_i) = \bar{v}_i(x) + \omega$ wherein ω is money and

$$\bar{v}_i(x) = \begin{cases} x \cdot v_i, & \text{for } x \leq \delta_i, \\ \delta_i \cdot v_i, & \text{for } x > \delta_i. \end{cases}$$

Seller j has (true) per unit cost c_j and offers to sell up to σ_j units of l_j at a unit price of a_j. Denote $L_j = \{l_j\}$. Sellers, too, have quasi-linear utility functions of the form $u_j^s(x; \omega, L_j) = -\bar{c}_j(x) + \omega$ wherein ω is money and

$$\bar{c}_j(x) = \begin{cases} x \cdot c_j, & \text{for } x \leq \sigma_j, \\ \infty, & \text{for } x > \sigma_j. \end{cases}$$

The mechanism receives all these bids, and matches some buy and sell bids. The possible matches are described by integers x_i, y_j: $0 \leq x_i \leq \delta_i$ is the number of units of bundle R_i allocated to buyer i and $0 \leq y_j \leq \sigma_j$ is the number of units of item l_j sold by seller j.

The mechanism determines the allocation (x^*, y^*) as the solution of the surplus maximization problem **MIP**:

$$\begin{aligned} \max_{x,y} \quad & \textstyle\sum_i b_i x_i - \sum_j a_j y_j \\ s.t. \quad & \textstyle\sum_j y_j \mathbb{I}(l \in L_j) - \sum_i x_i \mathbb{I}(l \in R_i) \geq 0, \forall l \in [1:L], \\ & x_i \in [0:\delta_i], \forall i, \quad y_j \in [0, \sigma_j], \forall j. \end{aligned} \tag{1.1}$$

MIP is a mixed integer program: Buyer i's bid is matched up to his maximum demand δ_i; Seller j's bid will also be matched up to his maximum supply σ_j. x_i^* is constrained to be integral; y_j^* will be integral due to the demand less than equal to supply constraint.

The settlement price is the highest ask-price among matched sellers,

$$\hat{p}_l = \max\{a_j : y_j^* > 0, l \in L_j\}. \tag{1.2}$$

The payments are determined by these prices. Matched buyers pay the sum of the prices of items in their bundle; matched sellers receive a payment equal to the number of units sold times the price for the item. Unmatched buyers and sellers do not participate. This completes the mechanism description.

Our proposed mechanism called *c-SeBiDA* (combinatorial Sellers' Bid Double Auction) is combinatorial and in a framework that allows us to define uniform and anonymous prices on the links. Such prices are highly desirable from an economic perspective as they yield socially efficient and Pareto-optimal outcomes, but they are achieved by few auction mechanisms.

The analysis of combinatorial auctions is usually very difficult, and even more so for combinatorial double auctions. We thus consider the continuum model and show that the auction outcome is a competitive equilibrium in chapter 3.

Theorem 4.1 (c-SeBiDA outcome is C.E.). *If bid functions of sellers are continuous and non-decreasing, the c-SeBiDA outcome $((x^*, y^*), p^*)$ is a competitive equilibrium in the continuum model.*

While the continuum model is an idealization of the scenario where there are a large number of agents such that no single agent can affect the auction outcome by himself, it suggests that the auction outcome is likely an approximate competitive equilibrium, and hence close to efficient. The methodology used in the proof is novel in that it casts the mechanism in an optimal control framework and appeals to Pontryagin's maximum principle to conclude that the outcome is indeed a competitive equilibrium.

The c-SeBiDA mechanism is similar in spirit to the k-DA mechanism proposed in [145]. However, the two mechanisms are different. In particular, k-DA is non-combinatorial and only for one type of good. It cannot be generalized to the combinatorial case.

In the next section, we discuss other proposals for combinatorial auctions and the properties of c-SeBiDA when the participants are strategic.

1.4 Strategic Behavior in Auctions and The Price of Anarchy

In the discussion so far, we have assumed that the participants do not anticipate that their actions affect the outcome, i.e., they are price-taking. However, in a realistic economic scenario involving a finite number of participants, agents do anticipate how they may affect the outcome and hence act strategically.

Thus, we now focus on how strategic behavior of players affects price when they have complete information. We will assume that players don't strategize over the quantities (namely, δ_i, σ_j), which will be considered fixed in the players' bids. A strategy for buyer i is a buy bid b_i. A strategy for seller j is an ask bid a_j. Let $\theta = ((a_1, \cdots, a_n), (b_1, \cdots, b_n))$ denote a collective strategy. Given θ, the mechanism determines the allocation (x^*, y^*) and the prices $\{\hat{p}_l\}$. So the payoff to buyer i and seller j is, respectively,

$$u_i^b(\theta) = \bar{v}_i(x_i^*) - x_i^* \cdot \sum_{l \in R_i} \hat{p}_l, \tag{1.3}$$

$$u_j^s(\theta) = y_j^* \cdot \sum_{l \in L_j} \hat{p}_l - \bar{c}_j(y_j^*). \tag{1.4}$$

The bids b_i, a_j may be different from the true valuations v_i, c_j, which however figure in the payoffs. Observe that θ really is a function of all the v_i and c_j. Thus, in shorthand, we will also write $\theta(v, c)$ to emphasize this dependence.

When players have complete information about true valuations and costs of the other players, they choose the strategies to maximize their own payoffs given the strategies of others. When they have incomplete information, they maximize $\mathbb{E}[u_i^b(\theta)|b_i]$ (or $\mathbb{E}[u_j^s(\theta)|a_j]$), the expected value of their payoff conditioned on their strategy.

A collective strategy θ^* is a *Nash equilibrium* if no player can increase his payoff by unilaterally changing his strategy. In the case of incomplete information, it is called a *Bayesian-Nash equilibrium*.

We now describe some criteria to evaluate auction mechanisms. In the discussion below, we will drop the superscripts on u.

Individual Rational (IR). A mechanism is *ex post IR* if $u_i(\theta(v, c)) \geq 0$ for all v, c, i.e., the utility derived from any outcome is non-negative. It is *interim IR* if $\mathbb{E}[u_i(\theta(v, c)|v_i] \geq 0$ for all v_i (similarly for c_i), i.e., the expected utility given that it knows its own valuation (or cost) and the distribution of others is non-zero. It is *ex ante IR* if $\mathbb{E}[u_i(\theta(v,c))] \geq 0$, i.e., the expected utility when it only knows the distribution of its own and others valuations (or costs). We assume that the utility derived from non-participation is zero. In this study, we take ex post IR as the desired property.

Incentive Compatible (IC). A mechanism is *IC* if truth-telling is a dominant-strategy Nash equilibrium, i.e., $\theta^* = ((c_1, \cdots, c_n), (v_1, \cdots, v_n))$ is a Nash equilibrium of the auction game. In the incomplete information case, a mechanism with truth-telling as a Bayesian-Nash equilibrium is said to be *Bayesian Incentive Compatible (IC)*. It is pertinent to mention here that when the mechanism is IC or Bayesian IC, truth-telling need not be the only equilibrium.

Efficiency. A mechanism is (allocatively) efficient if it maximizes $\sum_i u_i(\theta(v,c))$ for all v and c.

Budget-balancing. A mechanism is *strong budget-balanced* if the aggregate payments of the buyers equals the aggregate payment of the sellers. It is *weakly budget-balanced* if the aggregate payments of the buyers is greater than or equal to the aggregate payment of the sellers.

Vickrey [162] was the first to realize that despite strategic behavior, there are mechanisms that are IR, IC and efficient. His work was expanded upon by Clark [25] and Groves [51]. It is now well known that the only known positive result in the mechanism design theory is the VCG class of mechanisms [108, 91]. The generalized Vickrey (combinatorial) auction (GVA) (with complete information) is ex post individual rational, dominant strategy incentive compatible and efficient [164]. It is however not budget-balanced. The incomplete information version of GVA (dAGVA) is Bayesian IC, efficient and budget-balanced. It is, however, not ex post IR. Indeed, there exists no mechanism which is efficient, budget-balanced, ex post IR and dominant strategy IC (Hurwicz impossibility theorem) [60]. Moreover, there exists no mechanism which is efficient, budget-balanced, ex post IR and Bayesian IC (Myerson-Satterthwaite impossibility theorem) [115].

The mechanism we provide is a non-VCG combinatorial (market) mechanism which in the complete information case is always efficient, budget-balanced, ex post IR and "almost" dominant strategy IC. In the incomplete information case, it is budget-balanced, ex post IR and asymptotically efficient and Bayesian IC.

Moreover, we show in chapter 3 that *any* Nash equilibrium allocation (say of a network resource allocation game) is always efficient (zero efficiency loss). Specifically,

Theorem 4.2 (Nash equilibria of c-SeBiDA). *(i) A Nash equilibrium exists in the c-SeBiDA game. (ii) Except for the matched seller with the highest bid on each item, it is a dominant strategy for each player to bid truthfully. (iii) Any Nash equilibrium allocation is always efficient.*

In the case of incomplete information, we will show that *any* Bayesian-Nash equilibrium allocation is asymptotically efficient.

Theorem 4.3 (Bayesian-Nash equilibria of c-SeBiDA). *Consider the SeBiDA auction game when both buyers and sellers have ex post individual rationality constraint. Let (α_n, β_n) be a symmetric Bayesian-Nash equilibrium with n buyers and n sellers. Then, (i) $\beta_n(v) = \tilde{\beta}(v) = v \; \forall n \geq 2$, and (ii) $(\alpha_n, \beta_n) \to (\tilde{\alpha}, \tilde{\beta})$ in the uniform topology as $n \to \infty$, i.e., SeBiDA is* asymptotically Bayesian incentive compatible.

Ours is one of few proposals for a *combinatorial double auction* mechanism. It appears to be the only combinatorial market mechanism for strategic agents with unrestricted strategy spaces. We are able to achieve efficient allocations. Furthermore, the mechanism's linear integer program structure makes the computation manageable for many practical applications [77].

This seems to be the only known combinatorial double-auction mechanism with these properties. We now describe relevant literature.

In the classical auction theory literature, most of the attention is focused on one-sided, single-item auctions [86]. There is now a growing body of research devoted to combinatorial auctions [164]. The interplay between economic, game-theoretic and computational issues has sparked interest in algorithmic mechanism design [137]. Some iterative, ascending price combinatorial auctions achieve efficiencies close to the Vickrey auction [12, 33, 112, 141]. But generalized Vickrey auction mechanisms for multiple heterogeneous items may not be computationally tractable [137, 122]. Thus, mechanisms which rely on approximation of the integer program (though with restricted strategy spaces such as "bounded" or "myopic rationality") [122] or linear programming (when there is a particular structure such as "gross" or "agent substitutability") [18] have been proposed.

In [32], one of the first multi-item auction mechanisms is introduced. However, it is not combinatorial and consideration is only given to computation of equilibria among truth-telling agents. An auction for single items is presented in [144]. It is similar in spirit to what we present, but cannot be generalized to multiple items. In [176], a modified Vickrey double auction with participation fees is presented, while [34] considers truthful double auction mechanisms and obtains upper bounds on the profit of any such auction. But the setting in both [34, 144] is non-combinatorial since each bid is for an individual item only.

Our results also relate to recent efforts in the network pricing [40, 78, 94, 153] and congestion games literature [89, 136]. There is an ongoing effort to propose mechanisms for network resource allocation through auctions [79] and to bound the worst case Nash equilibrium efficiency loss (the so-called "price of anarchy" [89]) of such mechanisms when users act strategically [71, 102]. An optimal mechanism that minimizes this efficiency loss has also been proposed in [143], though not extended to the case of multiple items. Most of this literature regards the good (in this case, bandwidth) as divisible, with complete information for all players. The case of indivisible goods or incomplete information case is regarded in the literature as harder.

We considered indivisible goods, combinatorial buy-bids and incomplete information, and showed that the price of anarchy of c-SeBiDA zero.

It is worth noting that a one-sided auction is a special case of a double auction when there is only one seller with zero costs. The network and congestion games [78, 89] are all one-sided auctions.

1.5 Validating Economic Theory through Experiments

It is reasonable to question whether the predictions made by the theory discussed above are accurate predictors of human economic behavior in the real world. The first issue is the assumptions made in developing the theory. The second, even more basic issue, is whether humans make completely rational choices. To incorporate irrational behavior within mathematical models, various bounded rationality models have been proposed. However, the ultimate test for any economic theory is still its success in making good predictions in the marketplace. Thus, pioneered by Vernon L. Smith [155], a methodology of testing economic theory through human subject experiments has been developed. Econometric methods have already revolutionized economics. Roth [137] argues that experimental economics will play the same role in game theory.

Thus, to validate the auction theory that we have developed, we implemented the c-SeBiDA mechanism in a web-based software test-bed [8]. It was then used to conduct human subject experiments to validate the mechanism.

It was observed that as the number of participants was increased, the auction outcome seemed to converge to the efficient allocation. The participants' bids seemed to converge to their true values.

However, considering limitations on the number of participants in a laboratory setting, such a formal conclusion cannot be drawn.

A surprising result was that most participants (except for economic graduate student participants!) seemed to be rather risk-averse. The analysis predicts buyers would bid more than true value. However, this was rarely observed.

Considering that conducting economic experiments is a rather delicate operation, the results reported in chapter 4 should be considered preliminary. However, they do point out the efficacy of such experiments.

1.6 Contributions

We have essentially answered the four questions that we posed in section 1.1.

We showed that a competitive equilibrium exists in a continuum exchange economy with indivisible commodities and money. Surprisingly, this result appears to be apparently unknown in the literature. Our proof involved use of the Lyapunov-Richter theorem for integrals of correspondences. We used the Debreu-Gale-Nikaido lemma instead of the Kakutani fixed point theorem. This implies the existence of some approximate competitive equilibria in finite economies.

We have introduced a combinatorial, sellers' bid, double auction (c-SeBiDA)—a combinatorial market mechanism. We considered the continuum model and showed that within that model, c-SeBiDA outcome is a competitive equilibrium. This suggests that in the finite setting, the auction outcome is close to efficient.

We then considered strategic behavior of players and showed the existence of a Nash equilibrium in the c-SeBiDA auction game with full information. In c-SeBiDA, settlement prices are determined by sellers' bids. We showed that the allocation of c-SeBiDA is efficient. Moreover, truth-telling is a dominant strategy for all players except the highest matched seller for each item. We then considered the Bayesian-Nash equilibrium of the mechanism under incomplete information. We showed that under the ex post individual rationality constraint, symmetric Bayesian-Nash equilibrium strategies converge to truth-telling for the single item auction. Thus, the mechanism is asymptotically Bayesian incentive compatible, and hence asymptotically efficient.

We have shown that, surprisingly, c-SeBiDA has zero "price of anarchy" in the complete information case, and asymptotically zero "price of anarchy" in the incomplete information case.

We have tested the proposed mechanism c-SeBiDA through human-subject experiments.

Chapter 2

Existence of Competitive Equilibrium in Combinatorial Markets

We investigated the existence of competitive equilibrium in combinatorial markets, i.e., markets with several indivisible goods wherein agents have valuations for combinations of various goods. The work was motivated by a resource allocation problem in communication networks with independent and selfish buyers and sellers of bandwidth. We assumed that participants do not anticipate that their demand or supply can affect the allocation. In particular, we adopted Aumann's continuum exchange economy as a model of perfect competition. We first showed how network topology affects the existence of competitive equilibrium. We then showed the existence of competitive equilibrium in a continuum combinatorial market with money. We made minimal assumptions on preferences; only that they are continuous and monotonic in money. We assume that the excess demand correspondence satisfies standard assumptions such as Walras' law. The existence of competitive equilibrium in the continuum combinatorial market was then used to show the existence of various enforceable and non-enforceable approximate competitive equilibria.

2.1 Introduction

We studied the existence of competitive equilibrium in a *combinatorial market* , i.e., a pure exchange economy with several indivisible goods and one divisible good *numeraire* or *money*. Each participant may have interdependent valuations over various goods. This was motivated by the following problem in communication networks.

Consider a network $G = (N, L)$ with a finite set of nodes N, and links L. The transmission capacity (or bandwidth) comes in some integral number of trunks (each trunk being say, 10 Mbps). There are M agents, each with an initial endowment of money and link bandwidth. The allocation of the network resources is determined through a double auction between buyers and sellers. Each buyer specifies the bundle of links (comprising a route), the bandwidth (number of trunks) on each link, and the maximum price it is willing to pay for the bundle; each seller specifies a similar bundle and the minimum price it is willing to accept. We assume that each agent's preferences are monotonic over the bundle (they prefer larger bundles to strictly smaller ones) and continuous in money. Moreover, we assume that buyers insist on getting the same bandwidth on all links in their bundles.

The framework is quite general and can be extended to the case where the network consists of several autonomous systems and their owners are trying to negotiate Service Level Agreements (SLAs) about capacity, access and QoS issues.

We are interested in the following questions: When are Pareto-efficient allocations achievable in a network through a (decentralized) market mechanism? How does efficiency depend on network topology? How does economic efficiency scale with the size of the market? What market mechanisms are available to achieve economic efficiency?

It is well known that competitive equilibrium need not exist in an exchange economy with indivisible goods. The difficulties primarily lie in the fact that the utility functions are non-concave and discontinuous, and that the consumption sets are totally disconnected. This makes use of any of the standard fixed point theorems such as the Brouwer or the Kakutani fixed point theorems to prove existence of competitive equilibrium impossible.

Early attempts to deal with indivisible commodities considered "matching models" inspired by the "stable marriage" assignment problem of Gale and Shapley [49]. Shapley and Shubik [152] studied the competitive equilibrium problem in asymmetric markets when there are buyers and sellers. The commodities are indivisible such as houses [76], but it is assumed that each participant buys or sells only one commodity.

Shapley and Scarf [151] considered the more general exchange model where a participant could be both a buyer and a seller. They focussed on the problem of core and showed that an exchange economy with indivisible goods has a nonempty core. Quinzii [128] studied a similar problem, but considered money as another good, and showed that competitive equilibrium exists and it has a non-empty core. Gale [48] started with slightly different assumptions and also showed that competitive equilibrium exists. In all of the above, it was assumed that utility functions satisfied a "non-transferable" assumption. Yamamoto [175] further generalized this by removing some of these assumptions. All of the above assumed that each participant buys or sells only one commodity. Thus, the market was *non-combinatorial*.

The problem remains of interest in recent literature as well. In [95], van der Laan et al. considered Walrasian equilibrium, but they assumed that each participant owns at least one indivisible commodity initially. Moreover, utility is also derived from at most one indivisible commodity. While Ma [100] considers a more general setup and has a different approach. Necessary and sufficient conditions for existence of competitive equilibrium in an exchange economy with indivisible goods and no money were obtained by considering a coalitional form game and obtaining conditions for it being balanced following [81].

A model incorporating *combinatorial markets* was considered by Bikhchandani and Mamer [17]. They provide necessary and sufficient conditions for existence of competitive equilibrium in an exchange economy with many indivisible goods and money. The market they consider is combinatorial since a consumer wants bundles of commodities. But the agent preferences considered are rather special. Each agent is assumed to have a reservation value for each bundle.

Since a competitive equilibrium may not exist with non-convex preferences and indivisibilities, there have been efforts to characterize the limit points of market equilibria of economies as the market grows in size. Several models of large economies have been proposed. Debreu and Scarf [31] investigated the core of a finite economy replicated countably many times. More general models of countable economies were considered in [35, 59]. However, it is known that C.E. may not exist even in countable economies with non-convex preferences.

Thus, there have been attempts to characterize approximate equilibria with non-convex preferences in a finite setting. Starr [158] characterized certain approximate competitive equilibrium based on results which state that non-convexities in an aggregate of non-convex sets do not grow in size with the number of sets making up the aggregate. This "averaging" results in non-convexities becoming less important in a large economy. Henry [58], Emmerson [38] and Broome [22] extended this work to the case of indivisible goods. As Emmerson noted, indivisibilities do not merely result in non-convex preferences. The consumption sets become totally disconnected as well. This results in the competitive mechanism leading to non Pareto-efficient allocations.

Aumann introduced the continuum model of an economy [9] to model large economies with perfect competition where each participant is negligible compared to the overall size of the economy. Unlike [4, 47], he did not assume anything about the valuation of the participants. But the goods are divisible and in such a setting, he showed that competitive equilibrium exists [11]. It was shown by Mas-Colell [105] that Aumann's results do not extend to a continuum economy with indivisible goods without money, i.e., competitive equilibrium need not exist in continuum exchange economies with indivisible commodities. However, Khan and Yamakazi [85] showed that the core of a continuum economy with indivisible goods is non-empty. This raised the hope that some allocations in the core may be decentralized through competitive prices.

We provide exactly such a result.

We considered an exchange economy with multiple commodities and money. Unlike [17, 95, 100], we considered very general preferences and made no assumption on initial endowments. Moreover, we considered a combinatorial market. Our only assumption was that the preferences are continuous and monotonic in money; a reasonable assumption by any means. Our interest was in the perfect competition case, when each participant is negligible enough that it cannot affect the prices and the allocation.

We first showed that when agents have quasi-linear utility functions, existence of competitive equilibrium, and hence of economically efficient market mechanisms depends on network topology. We showed an example of a finite network with a finite number of agents, for which no competitive equilibrium exists.

We then modeled a perfect competition economy as one with a continuum of agents, each with negligible influence on the final allocation and prices [9]. Such idealized models are used frequently and are helpful in characterizing and finding approximate equilibria that are nearly efficient for finite settings. We showed that a competitive equilibrium exists in a continuum model of a network. This was accomplished using the Debreu-Gale-Nikaido lemma, a useful corollary of Kakutani's fixed point theorem.

The chapter is organized as follows: In section 2.2, we present some examples of finite networks, and show that if bandwidth is indivisible, competitive equilibrium may not exist. Section 2.3 presents existence results for the continuum model of a network. Section 2.4 presents some enforceable and non-enforceable equilibria. Section 3.6 presents conclusions. The proofs of the theorems are technical and presented in the appendices.

2.2 Network Topology and Economic Efficiency

We first prove that a competitive equilibrium exists if the routes that buyers want form a tree and all agents (buyers and sellers) have utilities that are linear in bandwidth and money. Examples are given to show that a competitive equilibrium may not exist if the routes do not form a tree or if utilities are nonlinear.

Links are indexed $j = 1, 2, \cdots$; link j provides C_j trunks of bandwidth (C_j an integer). Its owner, j, can lease $y_j \leq C_j$ trunks and has a per trunk reservation price or cost a_j. Buyer i, $i = 1, 2, \cdots$, wishes to lease x_i trunks on each link j in route R_i. The value to buyer j of one trunk along route R_i

is b_i. Let $A = \{A_{ij}\}$ be the edge-route incidence matrix, i.e. $A_{ij} = 1(0)$, if link $j \in (\notin) R_i$.

With this notation, the allocation (x^*, y^*) with the maximum surplus solves the following integer program:

$$\max_{x,y} \quad \sum_i b_i x_i - \sum_j a_j y_j \tag{2.1}$$

$$s.t. \quad \sum_i A_{ij} x_i \le y_j \le C_j, \ \forall j \tag{2.2}$$

$$x_i, y_j \in \{0, 1, 2, \cdots\}, \ \forall i, j \tag{2.3}$$

The allocation (x^*, y^*) together with a link price vector $p^* = \{p_j^*\}$ is a *competitive equilibrium* if every buyer i maximizes his surplus at x_i^*,

$$\max_{x_i=0,1,\cdots} (b_i - \sum_{j \in R_i} p_j^*) x_i,$$

and every seller j maximizes his profit at y_j^*,

$$\max_{y_j=0,1,\cdots,C_j} (p_j^* - a_j) y_j.$$

A matrix is *Totally Unimodular* (TU) if the determinant of every square submatrix is 0, 1 or -1 [149]. If the routes that buyers want in a network form a tree, its edge-route incidence matrix is TU.

Theorem 2.1. *If A is TU, in particular if the routes form a tree, there is a competitive equilibrium.*

Proof. Consider the relaxed LP version of problem (2.1) in which the integer constraint (2.3) is dropped. Because A is TU, the convex set of allocations (x, y) that satisfy constraint (2.2) has integer-valued vertices. Hence there is an optimal solution (x^*, y^*) to the LP problem which is integer-valued. The Lagrange multipliers $\{p_j^*\}$ associated with the constraint (2.2), together with (x^*, y^*), form a competitive equilibrium, as can be verified from the Duality Theorem of LP.

The proposition has a partial converse: If $(p^*, (x^*, y^*))$ is a competitive equilibrium, (x^*, y^*) is a solution to the relaxed LP problem.

It is well known that a competitive equilibrium exists if every buyer i (seller j) has a utility (cost) function $u_i(x_i)(v_j(y_j))$ that is concave (convex), monotone and continuous (along with some boundary conditions) [5] and *fractional* trunks can be traded. This fact is exploited in [78, 79] to infer existence of competitive equilibrium prices for bandwidth on each link.

Examples 1,2 are non-TU networks that do not have a competitive equilibrium.

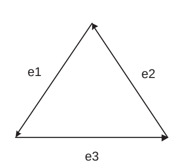

Figure 2.1: A cyclic network that is not TU.

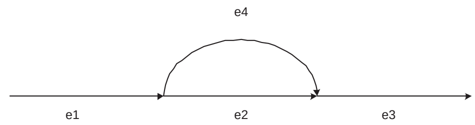

Figure 2.2: An acyclic network that is not TU.

Example 2.1. *Consider the cyclic network in figure 2.1 with buyers* $1, \cdots, 4$*, who want routes* $\{e1, e2\}$*,* $\{e2, e3\}$*,* $\{e3, e1\}$*, and* $\{e_3\}$*, respectively. Buyers* $1, 2, 3$ *receive benefit* $b_i = 1$ *per trunk; buyer 4 receives* $b_4 = \alpha (< 0.5)$*. Sellers own one trunk on each link, and their reservation price* $a_j = 0$ *for all links. The network is not TU, as can be easily checked. Surplus maximization allocates route* $\{e1, e2\}$ *to user 1 and* $\{e3\}$ *to user 4. If prices* p_1, p_2, p_3 *were to support this allocation, they must satisfy the conditions,* $1 = p_1 + p_2 \leq \min(p_2 + p_3, p_1 + p_3)$ *and* $0.5 > \alpha \geq p_3$*, which is impossible. So there is no competitive equilibrium.*

Example 2.2. *Consider the acyclic network of figure 2.2 again with buyers* $1, \cdots, 4$*, desired routes* $\{e1, e2\}, \{e2, e3\}, \{e1, e4, e3\}$*, and* $\{e3\}$ *and benefits* b_i *as before. Each link supports one trunk, and the sellers are as before. Surplus maximization again allocates route* $\{e1, e2\}$ *to user 1 and* $\{e3\}$ *to user 4. Competitive prices supporting this allocation must satisfy* $1 = p_1 + p_2 \leq \min(p_2 + p_3, p_1 + p_4 + p_3)$*,* $0.5 > \alpha \geq p_3$*, and* $p_4 = 0$*, which is impossible.*

Next we see a TU network with nonlinear concave utilities for which there is no competitive equilibrium.

Example 2.3. *Consider a network with two links, each with two trunks of capacity. There are two buyers. Buyer 1 wants a route through both links with bandwidth* x_1 *and has a concave utility function:*
$u_1(x_1; \{l_1, l_2\}) = 1.1x_1, 0 \leq x_1 \leq 1; = x + 0.1, 1 \leq x_1 \leq 2.$
Buyer 2 demands bandwidth x_2 *only on link 2 and has concave utility function:*
$u_2(x; l_2) = 1.5x, 0 \leq x \leq 1; 1.1(x-1) + 1.5, 1 < x \leq 1 + \epsilon; (x - 1 - \epsilon) + 1.6, 1 + \epsilon < x \leq 2,$ *where* $\epsilon = 0.1/1.1$*.*
The sellers have reservation price of 0 on each link. It is easy to check that if fractional trunks can be traded, there is a competitive equilibrium with allocation $x^* = (0.4, 1.6)$ *and prices* $p^* = (0, 1.1)$*. However, if trades must be in integral trunks, there is no competitive equilibrium.*

A competitive equilibrium is efficient, because it maximizes total surplus $\sum_i b_i x_i - \sum_j a_j y_j$. To find an equilibrium, one normally proposes an iterative mechanism (often called 'Walrasian') involving an 'auctioneer', who in the nth round proposes link prices $\{p_j^n\}$, to which agents respond: buyer i places demand x_i^n, seller j offers to supply $y_j \leq C_j$ trunks. The auctioneer calculates the 'excess demand' on link j, $\xi_j^n = \sum A_{ij} x_i^n - y_j^n$, and begins round $n+1$ with price p_j^{n+1} higher or lower than p_j^n, accordingly as ξ_j^n is positive or negative. The equilibrium is reached when $\xi_j \leq 0$ for all j.

Two questions arise: Will the iterations converge? And are such mechanisms practically implementable? Of course, if there is no competitive equilibrium, caused perhaps by indivisibilities, the price-adjustment algorithms will not converge and no practical mechanisms can exist. Thus, in the next section, we study the existence of competitive equilibrium in an ideal model.

2.3 Competitive Equilibrium in the Continuum Model

Consider a combinatorial market with indivisible goods and money. We assume that there is perfect competition in which no single agent can influence the outcome, by considering a continuum of agents (buyers and sellers). The continuum exchange economy, first introduced by Aumann [9], is an idealized model of perfect competition in which no agent has significant 'market power' to be able to alter the outcome. From a practical perspective, the existence of a competitive model in the ideal model can be used to establish the existence of approximate competitive equilibria (which are approximately efficient) in finite economies.

With money

Consider a combinatorial market $\mathcal{G}$ with L indivisible goods $1, \cdots, L$). Let there be C_l units of good l for each l. There is one divisible good 0, called money. There is a continuum of agents indexed $t \in X = [0, M]$, with a given *non-atomic* measure space $(X, \mathcal{B}(X), \mu)$. Suppose there are M possible bundles of indivisible goods and each agent t demands some bundle R_i. For example, all agents $t \in (m, m+1]$ demand bundle R_{m+1}, for $m + 1 \leq M$. Agents' preferences $\succeq_t$ are monotonic and continuous in money. (Monotonicity simply means that if $A \subseteq B$, then $B \succeq_t A$. Continuity means that if $A_n \to A$ and $B \succeq_t A_n$, then $B \succeq_t A$.) As a result, preferences are continuous. A particular example of such preferences is when utility functions are quasi-linear in money, i.e., linear in money. Agent t has an initial endowment ω_t, which is a $L+1$-tuple. Though the following discussion and the results are for any general initial endowment, a particular example is of an auction setting when agent 0 is an auctioneer, endowed with $\omega_0 = (0, C_1, \cdots, C_L)$ (e.g., the whole network) and any other agent $t(> 0)$ has $\omega_t = (m_t, 0, \cdots, 0)$, in which m_t is t's money endowment. Similarly, the price vector $p = (p_0, p_1, \cdots, p_L)$ is a $L+1$-tuple; p_0 is the price of money and p_l is the price of one unit of good l. We will call a system as described above a *continuum combinatorial (exchange) economy* $\mathcal{E}$.

We begin with a few definitions: Let $p \in \Theta = \mathbb{R}_+^{L+1}$ be a price vector. By $p > 0$, we shall mean that all components are non-negative with $p \neq 0$, and by $p \gg 0$, we shall mean that all components are strictly positive.

Commodity space: $\Omega = \mathbb{R}_+ \times \mathbb{Z}_+^L$. Thus, $\omega = (\omega_0, \cdots, \omega_L) \in \Omega$ denotes ω_0 units of money and ω_l units of good l, for $l = 1, \cdots, L$. Note that ω_l for $l > 0$, must be an integer, indicating indivisibility.

Unit price simplex: $\Delta = \{p \in \Theta : \Sigma_0^L p_l = 1\}$. Prices lie in the unit simplex. Later, we can normalize prices so that the price of money, $p_0 = 1$, and we get the prices of other goods in terms of money.

Budget set: $B_t(p) = \{z \in \Omega : p.z \leq p.\omega_t\}$, which gives the allocations that agent t can afford based on its initial endowment at given prices p.

Preference level sets: $\mathcal{P}_t(z) = \{z' \in \Omega : z' \succeq_t z\}$, is the set of allocations preferred by agent t to the allocation z.

Individual demand correspondence: $\psi_t(p) = \{z \in B_t(p) : z \succeq_t z', \forall z' \in B_t(p)\}$. At given prices p, $\psi_t(p)$ is the set of t's most-preferred allocations. There may be more than one most preferred allocation, so ψ_t is a demand correspondence rather than a demand function.

Aggregate excess demand correspondence:

$$\Phi(p) = \int_X \psi_t(p) d\mu - \int_X \omega_t d\mu.$$

We denote the first integral by $\Psi(p)$—the aggregate demand correspondence, and the second integral by $\bar{\omega}$, the total endowment of all agents, or the aggregate supply.

Definition 2.1 (Competitive Equilibrium). *A pair (x^*, p^*) with $p^* \in \Delta$ and $x^* \in \Omega$ is a competitive equilibrium if $x_t^* \in \psi_t(p^*)$ and $0 \in \Phi(p^*)$.*

A competitive equilibrium comprises an allocation and a set of prices such that the prices support an allocation for which aggregate demand equals aggregate supply, or in other words, the aggregate excess demand is zero. Moreover, the allocation to each agent is what it demands at those prices.

We make the following assumptions:

Assumptions

1. $\bar{\omega} \gg 0$ (component-wise positive), and $\omega_t > 0, \forall t$ (component-wise non-negative with some component positive).

2. $\Phi(p)$ is homogeneous in p.

3. Boundary condition: Suppose $p^\nu \to p^*$, and $p_l^* = 0$ for some l. Then, $z_l^\nu \to \infty$, $\forall z^\nu \in \Phi(p^\nu)$.

4. Walras' Law holds: $p.z = 0, \forall z \in \Phi(p), \forall p \in \Delta^0$, the relative interior of Δ.

The first assumption simply states that there is a strictly positive endowment of each good and moreover each agent has a strictly positive endowment of some good. The second assumption ensures that scaling of prices does not alter the competitive allocation if it exists. The third assumption is a boundary condition that holds in the absence of undesirable goods. The fourth assumption, Walras' law, can be shown to hold for the economy under consideration. But we shall assume it without proof. Essentially, it means that if there is positive excess demand for a good at given prices, its price can be reduced still further towards zero.

Now we can show the following:

Theorem 2.2 (Existence). *Under assumptions (1)-(4), a competitive equilibrium exists in the continuum combinatorial economy $\mathcal{E}$.*

The proof relies on Lemma 2.1, which is a corollary of the Kakutani fixed point theorem and the Lyapunov-Richter theorem. It states that the integral of a correspondence with respect to a non-atomic measure is closed and convex-valued [10]. We can set the price of money $p_0 = 1$, and we get the other prices in units of money.

Proof. Consider any non-empty, closed convex subset S of Δ. We will first make some claims about the properties of the aggregate excess demand correspondence [15].

Claim 2.1. *Φ is non-empty and convex-valued on S.*

From assumption 1, Φ is non-empty. Fix $p \in S$. By the Lyapunov-Richter theorem [10] with μ a non-atomic measure on X, and $\psi_t(p)$ a correspondence for each p, $\int_X \psi_t(p) d\mu(t)$ is convex. Hence, Φ is convex.

Claim 2.2. *Φ is compact-valued, hence bounded on S.*

Note that S is compact and for each $p \in S$, $p \gg 0$. Write

$$\psi_t(p) = \bigcap_{z \in B_t(p)} [B_t(p) \cap \mathcal{P}_t(z)].$$

Then, $\mathcal{P}_t(z)$ is closed by continuity of preferences. $B_t(p)$ is closed and bounded for $p \gg 0$. Thus, their intersection is closed. And so is the outer intersection. It is bounded as well. Thus, $\psi_t(p)$ is compact for each $p \gg 0$.

Claim 2.3. *$p \cdot z \leq 0$, $\forall p \in \Delta^0$, $z \in \Phi(p)$.*

Fix $p \in \Delta^0$. By definition,

$$p \cdot z \leq p \cdot \omega_t, \forall z \in \psi_t(p), \forall t \in X.$$

Or, with an abuse of notation:

$$\int_X p \cdot \psi_t(p) d\mu \leq \int_X p \cdot \omega_t d\mu,$$

$$p \cdot \Psi(p) \leq p \cdot \bar{\omega},$$

$$p \cdot \Phi(p) \leq 0,$$

Claim 2.4. *ψ_t is closed and upper semi-continuous (u.s.c.) in S $\forall t \in X$. Hence, Φ is closed and u.s.c. in S.*

Fix $t \in X$. To show ψ_t is closed, we have to show that for any sequences, $\{p^\nu\}, \{z^\nu\}$, $[p^\nu \to p^0, z^\nu \to z^0, z^\nu \in \psi_t(p^\nu)] \implies z^0 \in \psi_t(p^0)$. From the definition of demand correspondence, $p^\nu \cdot z^\nu \leq p^\nu \cdot \omega_t$. Taking limit as $\nu \to \infty$, we get $p^0 \cdot z^0 \leq p^0 \cdot \omega_t$, i.e. $z^0 \in B_t(p^0)$. It remains to show: $z^0 \succeq_t z, \forall z \in B_t(p^0)$.

Consider any $z \in B_t(p^0)$. Then

Case 1: $p^0 \cdot z < p^0 \cdot \omega_t$.

Then, for large enough ν, $p^\nu \cdot z < p^\nu \cdot \omega_t$. This implies that $z \in B_t(p^\nu)$. Now, $z^\nu \in \psi_t(p^\nu)$. Hence, $z^\nu \succeq_t z$. And by continuity of preferences, we get $z^0 \succeq_t z$.

Case 2: $p^0 \cdot z = p^0 \cdot \omega_t$.

Define $z'^\nu := ((1 - 1/\nu)z_0, z_1, \cdots, z_L) \in \Omega$, by divisibility of money. So, $p^0 \cdot z'^\nu < p^0 \cdot \omega_t$. Then, by the same argument as above: $z^0 \succeq_t z'^\nu$. And by continuity of preferences, we get $z^0 \succeq_t z$.

This implies $z^0 \in \psi_t(p)$, i.e. it is closed. Now, to show it is u.s.c., we have to show by proposition 11.11 in [20], that for any sequence $p^\nu \to p^0$, and any $z^\nu \in \psi_t(p^\nu)$, there exists a convergent subsequence $\{z^{\nu_k}\}$ whose limit belongs to $\psi_t(p^0)$.

Now, $p^\nu \to p^0 \gg 0$. Hence, $\exists \nu_0$ s.t. $p^\nu \gg 0, \forall \nu > \nu_0$. Define

$$\pi := \inf\{p_l^\nu : \nu > \nu_0, l = 0, \cdots, L\}.$$

Then, $p^\nu \cdot z^\nu \leq p^\nu \cdot \omega_t$ implies for all $\nu > \nu_0$,

$$0 < z^\nu \leq \frac{p^\nu \cdot \omega_t}{\pi},$$

i.e. the sequence $\{z^\nu\}$ is bounded. By the Bolzano-Weierstrass theorem, there exists a convergent subsequence $\{z^{\nu_k}\}$ converging to say, z^0. Since ψ_t is closed in S, $z^0 \in \psi_t(p^0)$. Thus, it is upper semi-continuous in S.

We now show that Φ is u.s.c (hence closed) as well. Let $p^\nu \to p^0$ in S. Consider $\xi^\nu \in \Psi(p^\nu) = \int_X \psi_t(p^\nu) d\mu$. Then, $\exists z_t^\nu$ s.t. $\xi^\nu = \int_X z_t^\nu d\mu$. Now, ψ_t is compact-valued and u.s.c. in S. Thus, by proposition 11.11 in [20], the sequence $\{z_t^\nu\}$ has a convergent subsequence $\{z_t^{\nu_k}\}$ s.t. $z_t^{\nu_k} \to z_t^0 \in \psi_t(p^0)$. Define $\xi^0 := \int_X z_t^0 d\mu$. Thus,

$$\xi^0 \in \int_I \psi_t(p^0) d\mu = \Psi(p^0).$$

As argued before, Ψ is compact-valued. Hence, by reapplication of the same theorem, it is u.s.c. in S. And so is Φ.

We need the following lemma.

Lemma 2.1 (Debreu-Gale-Nikaido [5, 20]). *Let S be a non-empty closed convex subset in the unit simplex $\Delta \subset \mathbb{R}^n$. Suppose the correspondence $\Phi : \Delta \to \mathcal{P}(\mathbb{R}^n)$ satisfies the following:*
(i) Φ is non-empty, convex-valued $\forall p \in S$,
(ii) Φ is closed,
(iii) $p \cdot z \leq 0, \forall p \in S, z \in \Phi(p)$,
(iv) $\Phi(p)$ is bounded $\forall p \in S$.
Then, $\exists\ p^* \in S$ and $z^* \in \Phi(p)$ s.t. $p \cdot z^* \leq 0$, $\forall p \in S$.

Using this lemma, we get the following proposition.

Proposition 2.1. *For any non-empty, closed convex subset S of Δ^0, $\exists p^0 \in S, z^0 \in \Phi(p^0)$ s.t. $p \cdot z^0 \leq 0, \forall p \in S$.*

Consider an increasing sequence of sets $S^\nu \uparrow \Delta$. Let p^ν, z^ν be those given by the above proposition. Then, $p^\nu \in S^\nu \subset \Delta$, which is compact. Thus, $\exists$ a convergent subsequence $p^{\nu_k} \to p^* \in \Delta$.

Without loss of generality, consider this subsequence as the sequence. Consider any $z^\nu \in \Phi(p^\nu)$. We have the following lower bound on the sequence

$$z^\nu \geq -\bar{\omega}, \forall \nu. \tag{2.4}$$

To get an upper bound, take any $\tilde{p} \gg 0 \in S^\nu$. It exists because $S^\nu \uparrow \Delta$. Using the proposition above, we get

$$\tilde{p} \cdot z^\nu \leq 0, \tag{2.5}$$

for large enough ν. Equations (2.4) and (2.5) imply $\{z^\nu\}(\subset \Omega)$ is bounded. Thus, there exists a convergent subsequence with limit say, z^*.

By assumption 1, $\bar{\omega} \gg 0$. Also, $p^* \in \Delta$. Hence, $p^* \cdot \bar{\omega} > 0$. Further, $p^* \gg 0$ since if $p_l^* = 0$ for some l, $z_l^{\nu_k} \to \infty$, by the boundary condition, which then contradicts the boundedness of the subsequence above. Further, since Φ is closed, $z^* \in \Phi(p^*)$.

This establishes the following lemma.

Lemma 2.2. *$\exists p^* \gg 0 \in \Delta, z^* \in \Phi(p^*)$ s.t. $p^\nu \to p^*, z^\nu \to z^*$, and $p \cdot z^* \leq 0, \forall p \in \Delta$.*

We are now ready to prove the theorem: Walras' law implies $p \cdot z = 0$, $\forall z \in \Phi(p)$, and $\forall p \in \Delta^0$. This implies $p^* \cdot z^* = 0$. From lemma above, $p^* \cdot z^* \leq 0$, and $p^* \gg 0$. This yields $z^* = 0$.

Remarks. In the network resource allocation problems, agents may be indifferent between various bundles of links if they form a route between the same source-destination pair, then theorem 2.2 still holds. And in that case, prices for various alternative routes (given by the sum of link prices along the routes) for a given source-destination pair are the same.

Without money

The role of money is crucial in the above result. The following network example shows that in the absence of money, a competitive equilibrium may not exist, even in a continuum exchange economy.

Example 2.4. *Consider the networks of figure 1, with demands as discussed before in example 3. Now instead of one user of each type demanding a particular route, we have a continuum of users. Let $X = [0, M]$ and let all users in $[0, 1]$, where M is the total number of routes, demand the same route and have identical preferences. We make the same assumption for the other M disjoint intervals of unit length. This reduces the continuum case to the same as example 3, for which a competitive equilibrium does not exist.*

2.4 Approximate Competitive Equilibrium

The continuum economy is a convenient model but still a mathematical fiction. We show that it can be approximated by a large (but finite) economy. Note that in the proof of Theorem 2.2, we have used convexity of the aggregate excess demand correspondence to apply the DGN Theorem. Thus, if we replace $\Phi(p)$ by conv $\Phi(p)$, the following result holds.

Theorem 2.3. *There exists a $p^* \gg 0$, in Δ^0 s.t. $0 \in$ conv $\Phi(p^*)$.*

We can obtain several approximation results using the Shapley-Folkman [5] and Starr theorems [158].

Theorem 2.4 (Non-enforceable approximate equilibria). *(i) If the number of agents m is greater than the number of goods n, then at prices p^*, for which $0 \in$ conv $(\Phi(p^*))$, $\exists x^i \in$ conv $\phi_i(p^*)$ s.t.*

$$\Sigma_i x^i = 0 \text{ and } \#\{i | x^i \notin \phi_i(p)\}/m \leq n/m \to 0.$$

(ii) At prices p^* for which $0 \in$ conv $(\Sigma_i \phi_i(p^*))$, then $\exists x^i \in \phi_i(p^*)$ s.t.

$$\|\Sigma_i x^i\|^2/m \leq R/m \to 0,$$

where $R = \min\{m, n\} \cdot$ greatest $rad^2 \phi_i(p^*)$.

The first result is a straight forward application of the Shapley-Folkman theorem, noting the compactness of the individual demand correspondences from Claim 2. It says that there exists an allocation and prices such that the number of agents who are not happy with their allocation at those prices is bounded by the number of goods. Thus, as the number of agents increases (as in replication), the proportion of unhappy agents becomes arbitrarily small.

The second is an application of the Starr theorem: It says that at prices p^*, the aggregate excess demand per agent becomes arbitrarily small as the number of agents becomes arbitrarily large.

When a set of market-clearing prices do not exist with indivisible goods, it is useful to know whether there exist prices under which demand can be made arbitrarily less than supply as the "size of indivisibility" vanishes while affecting agents' utility only by a small amount.

We show that this is indeed the case. We show this in particular for the market model for networks in [78]. We follow the notation in that paper. We will assume that a unit of bandwidth is small in size compared to demands, or equivalently, the demands are large enough in terms of units of bandwidth.

Consider a network with a set $\mathcal{J}$ of links. For each link $j \in \mathcal{J}$, let $C_j \in \mathbb{Z}_+$ denote the number of available units of bandwidth (i.e., trunks) for this link. Let $\mathcal{R}$ denote the set of possible routes, i.e., a set of subsets of $\mathcal{J}$. A collection of routes $s \subset \mathcal{R}$ connecting a source with a destination is associated with a user who wishes to send traffic through the routes in that collection. His utility $U_s(x_s)$ is assumed to be an increasing, strictly concave function over $\mathbb{R}_+$, the nonnegative reals. The set of all users is denoted by $\mathcal{S}$. The relation of $\mathcal{R}$ in terms of the link set $\mathcal{J}$ is expressed by a 0-1 matrix $A = (A_{jr}; j \in \mathcal{J}, r \in \mathcal{R})$, where A_{jr} is 1(0) if $j \in r$ ($j \notin r$). Likewise we define $H = (H_{sr}; s \in \mathcal{S}, r \in \mathcal{R})$, where H_{sr} is 1(0) if $r \in s$ ($r \notin s$).

In order to study the loss in efficiency as a function of the amount of bandwidth per trunk, we consider a sequence of "discrete" networks indexed by N. For the N-th network, the capacity of link $j \in \mathcal{J}$ in terms of trunks is $C_j^N = NC_j$. Each user is allowed to pick only integral multiples of trunks along his route, thus his utility is a function $U_s^N : \mathbb{Z}_+ \to \mathbb{R}$, with $U_s^N(n) = U_s(n/N)$ for each $s \in \mathcal{S}$. Say that each trunk at link j, costs p_j for each $j \in \mathcal{J}$, then the *cost per trunk* over the path r is $\text{Cost}(r; p) = \sum_{j \in r} p_j$. Similarly, for $s \in \mathcal{S}$ the lowest cost route costs $\text{Cost}(s; p) = \min_{r \in s} \text{Cost}(r; p)$. We now have the following result.

Theorem 2.5. *For each $\epsilon > 0$, there exists an integer $N_0 > 0$ such that $\forall N > N_0$, there exists $(n^N, m^N, p^N) = ((n_s^N)_{s \in \mathcal{S}}, (m_r^N)_{r \in \mathcal{R}}, (p_j^N)_{j \in \mathcal{J}})$ where*

1. *n_s^N maximizes $U_s^N(n_s) - n_s \text{Cost}(s; p^N)$ over $\mathbb{Z}_+$ for every $s \in \mathcal{S}$,*
2. *$Hm^N = n^N$, $Am^N \leq C^N$, $m_r^N \in \mathbb{Z}_+$ and*
3. *$U_s^N(n_s^N) + \epsilon \geq U_s(x_s^*)$, for every $s \in \mathcal{S}$, where $x^* = (x_s^*; s \in \mathcal{S})$, $y^* = (y_r^*; r \in \mathcal{R})$ maximizes $\sum_{s \in \mathcal{S}} U_s(x_s)$ over $x \in \mathbb{R}_+^{|\mathcal{S}|}, y \in \mathbb{R}_+^{|\mathcal{R}|}$ under the constraints $Hy = x$, $Ay \leq C$.*

Proof. Fix $\epsilon > 0$. Without loss of generality, assume $x_s^* > 0$ for every $s \in \mathcal{S}$, and let $p^0 = (p_j^0; j \in \mathcal{J})$ be the equilibrium prices as they are guaranteed to exist for the divisible case by [78]. Now if the users were presented the inflated prices $p = \alpha p^0$ instead, where $\alpha > 1$, each user s will choose x_s so that he maximizes his surplus $U_s(x_s) - x_s \text{Cost}(s; p) = U_s(x_s) - x_s \alpha \text{Cost}(s; p^0)$ over $x_s \geq 0$. So by strict concavity and monotonicity of utilities, he will choose $x_s^\alpha < x_s^*$. Now if we define $y_r^\alpha = y_r^* x_s^\alpha / x_s^*$ for each $r \in s$, then we get $\sum_{r \in \mathcal{R}} H_{sr} y_r^\alpha = x_s^\alpha$ for every s. Also, since all entries of A are nonnegative and there are no all-zero rows, $\sum_{r \in \mathcal{R}} A_{jr} y_j^\alpha < C_j$ for every j. By continuity and strict monotonicity of utilities, we can choose $\alpha > 1$ so that $\sum_{s \in \mathcal{S}} U_s(x_s^\alpha) + 2\epsilon \geq \sum_{s \in \mathcal{S}} U_s(x_s^*)$, and $x_s^\alpha > 0$ for all s.

Now for every $N > 0$, define $p_j^N = p/N$ for each $j \in \mathcal{J}$, and

$$n_s^N = \max \left\{ \arg \max_{n \in \mathbb{Z}_+} U_s^N(n) - n\text{Cost}(s; p) \right\}$$

for each $s \in \mathcal{S}$. By strict concavity we have $|x_s^\alpha - n_s^N/N| \leq 1/N$. Thus it is clear that as $N \to \infty$, part (3) holds, and part (1) holds for every N.

It only remains to show part (2). Since $x_s^\alpha > 0$ for all s, there exists $r \in s$ for which $y_s^\alpha > 0$ and denote it by $r(s)$. Now define $m_{r(s)}^N = n_s^N - \sum_{r \in s \setminus \{r(s)\}} \lfloor N y_r^\alpha \rfloor$ and for each $r \neq r(s)$, $m_r^N = \lfloor N y_r^\alpha \rfloor$. Note that $m_{r(s)}^N / N \to y_r^\alpha(s)$ as $N \to \infty$, so for large enough N, $m_{r(s)}^N > 0$. Also, we have $m_r^N \to y_s^\alpha \geq 0$ as $N \to \infty$ for $r \neq r(s)$. Thus, $Hm^N = n^N$, and $m^N \geq 0$ for large enough N. Furthermore, $Am^n \leq C^N$ for large enough N, since $Ay_s^\alpha < C$.

2.5 Chapter Summary

We studied competitive equilibrium in combinatorial markets. We showed that for finite networks, prices that yield a socially efficient allocation may not exist. We then used a model of perfect competition with a continuum of agents, and showed that with money, it is possible to support the socially efficient allocation with a certain price vector. The key here is the Lyapunov-Richter theorem that enables a convexification of the economy. However, such a result does not hold for countable economies. The main reason is that defining the average of a sequence of correspondences is trickier as the limit may not exist.

The continuum model is useful in showing the existence of enforceable approximate equilibria (when we require that supply exceeds demand) in finite networks. Such approximate equilibria were presented in [66]. It is well-known that the set of the competitive allocations is contained in the *core* (the set of Pareto-optimal allocations). However, it is unknown if the two sets are equal. This is an interesting question and part of future work.

Chapter 3

c-SeBiDA: An Efficient Market Mechanism for Combinatorial Markets

We studied the interaction between buyers and sellers of several indivisible goods (or items). A buyer wants a combination of items, while each seller offers only one type of item. The setting is motivated by communication networks in which buyers want to construct routes using several links and sellers offer transmission capacity on individual links. Agents are strategic and may not be truthful, so a competitive equilibrium may not be realized. To ensure a good outcome among strategic agents, we proposed a combinatorial double auction. We showed that a Nash equilibrium existed for the associated game with complete information, and more surprisingly, the resulting allocation was efficient. In reality, the players may have incomplete information. So, we considered the Bayesian-Nash equilibrium. When there was only one type of item, we showed that the mechanism was asymptotically Bayesian incentive-compatible under the ex post individual rationality constraint and hence asymptotically efficient. Surprisingly, without the ex post individual rationality constraint, the Bayesian-Nash equilibrium strategy for the buyers was to bid more than their true value. We finally considered competitive analysis in the continuum model of the auction setting and showed that the auction outcome was a competitive equilibrium.

3.1 Introduction

We studied the interaction among buyers and sellers of several *indivisible goods* (or *items*). The motivation was to investigate the strategic interaction between internet service providers who lease transmission capacity (or bandwidth) from owners of individual links to form desired routes. Bandwidth is traded in indivisible amounts, say multiples of 100 Mbps. Thus, the buyers want bandwidth on combinations of several links available in multiples of some indivisible unit. This makes the problem *combinatorial*. We considered the interaction in several settings.

The setting of a conventional market economy, in which there is perfect competition, was considered in [66]. It was shown that the interaction among agents results in a competitive equilibrium if their utilities are linear in bandwidth (and money) and they truthfully reveal them, and the desired routes form a tree. The latter requirement is needed for the existence of an equilibrium in the presence of indivisibility.

Strategic agents, however, have an incentive not to be truthful. We proposed a 'combinatorial sellers' bid double auction' (c-SeBiDA) mechanism that achieves a socially desirable interaction among strategic agents. The mechanism requires both buyers and sellers to make bids. It is *combinatorial* because buyers make bids on combinations of items, such as several links that form a route. Each seller, however, offers to sell only a single type of item, e.g., bandwidth on a single link. The mechanism takes all buy and sell bids, solves a mixed-integer program that matches bids to maximize the social surplus, and announces prices at which the matched (i.e., accepted) bids are settled. The settlement price for a link is the highest price asked by a matched seller (hence 'sellers' bid' auction). As a result there is a *uniform* price for each item.

The outcome of strategic behavior in the auction was modelled as a Nash equilibrium. It was shown that under complete information a Nash equilibrium exists. It is not generally a competitive equilibrium. Nevertheless, the Nash equilibrium was efficient. Moreover, it was a dominant strategy for all buyers and for all sellers except the matched seller with the highest-ask price to be truthful.

In an auction setting, players may have incomplete information. Following Harsanyi [57], we considered the Bayesian-Nash equilibrium as the solution concept for the auction game. When there is only one type of item, we showed that if the players use only *ex post Individual Rational (IR)* strategies [108], symmetric Bayesian-Nash equilibrium strategies converge to truth-telling, as the number of players becomes very large.

Following Aumann [9], we then considered the continuum model. It was shown in [66] that a competitive equilibrium exists in a continuum exchange economy with indivisible goods and money (a divisible good). Here we showed that the c-SeBiDA auction outcome is a competitive equilibrium [108] in the continuum model even without money. This was accomplished by casting the mechanism in an optimal control framework and appealing to Pontryagin's maximum principle to conclude existence of competitive prices. This suggests that the auction outcome in a finite setting approximates a competitive equilibrium in the continuum model (see [5] for approximate competitive equilibrium). The proposed mechanism has been implemented in a web-based software testbed and available for use (see `http://auctions.eecs.berkeley.edu`).

Previous Work and Our Contribution

When items are indivisible, a competitive equilibrium may not exist. However, when the utility functions are linear and the demand-supply constraint matrix has a special structure (such as the totally unimodular property [149]), a competitive equilibrium does exist [164]. However, the realization of the competitive equilibrium still requires agents to truthfully report their utilities. But strategic agents (aware of their 'market power') may not be truthful. Thus, many auction mechanisms are designed to elicit truthful reporting following Vickrey's fundamental result [162].

Attention in the auction theory literature has focused on one-sided, single-item auctions [86], but combinatorial bids arise in many contexts, and a growing body of research is devoted to combinatorial auctions [164]. The interplay between economic, game-theoretic and computational issues has sparked interest in algorithmic mechanism design [137]. Some iterative, ascending price combinatorial auctions achieve efficiencies close to the Vickrey auction [12, 33, 112, 141]. It is however, well-known that generalized Vickrey auction mechanisms for multiple heterogeneous items may not be computationally tractable [137, 122]. Thus, mechanisms which rely on approximation of the integer program (though with restricted strategy spaces such as "bounded" or "myopic rationality") [122] or linear programming (when there is a particular structure such as "gross" or "agent substitutability") [18] have been proposed.

In [32], one of the first multi-item auction mechanisms is introduced. However, it is not combinatorial and consideration is only given to computation of equilibria among truth-telling agents. An auction for single items is presented in [144]. It is similar in spirit to what we present, but cannot be generalized to multiple items. In [176], a modified Vickrey double auction with participation fees is presented, while

[34] considers truthful double auction mechanisms and obtains upper bounds on the profit of any such auction. But the setting in both [34, 144] is non-combinatorial since each bid is for an individual item only.

Ours is one of few proposals for a *combinatorial double auction* mechanism. It appears to be the only combinatorial market mechanism for strategic agents with unrestricted strategy spaces. We are able to achieve efficient allocations. Furthermore, the mechanism's linear integer program structure makes the computation manageable for many practical applications [77].

The results here also relate to recent efforts in the network pricing [78, 82, 94, 153] and congestion games literature [89, 136]. There is an on-going effort to propose mechanisms for network resource allocation through auctions [79] and to understand the worst case Nash equilibrium efficiency loss of such mechanisms when users act strategically [71, 102]. An optimal mechanism that minimizes this efficiency loss has also been proposed [143], though not extended to the case of multiple items. Most of this literature regards the good (in this case, bandwidth) as divisible, with complete information for all players. The case of indivisible goods or incomplete information is harder. This chapter considers indivisible goods, combinatorial buy-bids and incomplete information.

The results are significant from several perspectives. It is well known that the only known positive result in the mechanism design theory is the VCG class of mechanisms [108]. The Generalized Vickrey Auction (GVA) (with complete information) is ex post individual rational, dominant strategy incentive compatible and efficient. It is however, not budget-balanced. The incomplete information version of GVA (dAGVA) is Bayesian incentive compatible, efficient and budget-balanced. It is, however, not ex post individual rational. Indeed, there exists no mechanism which is efficient, budget-balanced, ex post individual rational and dominant strategy incentive compatible (Hurwicz impossibility theorem) [60]. Moreover, there exists no mechanism which is efficient, budget-balanced, ex post individual rational and Bayesian incentive compatible (Myerson-Satterthwaite impossibility theorem) [115].

We provide a non-VCG combinatorial (market) mechanism, which in the complete information case, is always efficient, budget-balanced, ex post individual rational and "almost" dominant strategy incentive compatible. In the incomplete information case, it is budget-balanced, ex post individual rational and asymptotically efficient and Bayesian incentive compatible.

Moreover, we showed that *any* Nash equilibrium allocation (say of a network resource allocation game) is always efficient (zero efficiency loss) and *any* Bayesian-Nash equilibrium allocation is asymptotically efficient. This seems to be the only known combinatorial double-auction mechanism with these properties.

It is worth noting that a one-sided auction is a special case of a double auction, when there is only one seller with zero costs. The network and congestion games [78, 89] are all one-sided auctions.

The rest of the chapter is organized as follows. In Section 3.2, we present the combinatorial seller's bid double auction (c-SeBiDA) mechanism. In Section 3.3, we prove that under full information, the auction has a Nash equilibrium that is efficient, although it may not be a competitive equilibrium. In Section 3.4, we show that when the players have incomplete information, the Bayesian-Nash equilibrium strategies for the mechanism with a single item under the ex post individual rationality constraint converge to truth-telling, as the number of players becomes large. Section 3.5 presents a competitive analysis of the c-SeBiDA mechanism in the continuum model. We situate our contribution in relation to existing literature in the conclusion.

3.2 The Combinatorial Sellers' Bid Double Auction

A buyer places buy bids for a bundle of items. A buyer's bid is *combinatorial*: he must receive all items in his bundle or nothing. A buy-bid consists of a buy-price per unit of the bundle and maximum demand; the maximum units of the bundle that the buyer needs. On the other hand, each seller makes *non-combinatorial* bids. A sell-bid consists of an ask-price and maximum supply; the maximum units the seller offers for sale.

The mechanism collects all announced bids, matches a subset of these to maximize the 'surplus' (equation (3.1), below) and declares a settlement price for each item at which the matched buy and ask bids—which we call the winning bids—are transacted. This constitutes the payment rule. As will be seen, each matched buyer's buy bid is larger, and each matched seller's ask bid is smaller than the settlement price, so the outcome respects individual rationality.

There is an asymmetry: buyers make multi-item combinatorial bids, but sellers only offer one type of item. This yields uniform settlement prices for each item.

Players' bids may not be truthful. Players know how the mechanism works and formulate their bids to maximize their individual returns.

A player can make multiple bids. The mechanism treats these as XOR bids. So at most, one bid per player is a winning bid. Therefore, the outcome is the same as if a matched player only makes (one) winning bid. Thus, in the formal description of the *combinatorial Sellers' Bid Double Auction* (c-SeBiDA), each player places only one bid. c-SeBiDA is a 'double' auction because both buyers and sellers bid. It is a 'sellers' bid' auction because the settlement price depends only on the matched sellers' bids, as we will see.

Formal mechanism.

There are L items $l_1, \cdots, l_L$, m buyers and n sellers. Buyer i has (true) reservation value v_i per unit for a bundle of items $R_i \subseteq \{l_1, \cdots, l_L\}$, and submits a buy bid of b_i per unit and demands up to δ_i units of the bundle R_i. Thus, the buyers have quasi-linear utility functions of the form $u_i^b(x; \omega, R_i) = \bar{v}_i(x) + \omega$, where ω is money and

$$\bar{v}_i(x) = \begin{cases} x \cdot v_i, & \text{for } x \leq \delta_i, \\ \delta_i \cdot v_i, & \text{for } x > \delta_i. \end{cases}$$

Seller j has (true) per unit cost c_j and offers to sell up to σ_j units of l_j at a unit price of a_j. Denote $L_j = \{l_j\}$. Again, the sellers have quasi-linear utility functions of the form $u_j^s(x; \omega, L_j) = -\bar{c}_j(x) + \omega$, where ω is money and

$$\bar{c}_j(x) = \begin{cases} x \cdot c_j, & \text{for } x \leq \sigma_j, \\ \infty, & \text{for } x > \sigma_j. \end{cases}$$

The mechanism collects all these bids, and matches some buy and sell bids. The possible matches are described by integers x_i, y_j: $0 \leq x_i \leq \delta_i$ is the number of units of bundle R_i allocated to buyer i and $0 \leq y_j \leq \sigma_j$ is the number of units of item l_j sold by seller j.

The mechanism determines the allocation (x^*, y^*) as the solution of the surplus maximization problem **MIP**:

$$\begin{aligned} \max_{x,y} \quad & \textstyle\sum_i b_i x_i - \sum_j a_j y_j \qquad (3.1) \\ s.t. \quad & \textstyle\sum_j y_j \mathbb{I}(l \in L_j) - \sum_i x_i \mathbb{I}(l \in R_i) \geq 0, \forall l \in [1:L], \\ & x_i \in [0:\delta_i], \forall i, \quad y_j \in [0, \sigma_j], \forall j. \end{aligned}$$

MIP is a Mixed Integer Program: Buyer i's bid is matched up to his maximum demand δ_i; Seller j's bid will also be matched up to his maximum supply σ_j. x_i^* is constrained to be integral; y_j^* will be integral due to the demand less than equal to supply constraint.

The settlement price is the highest ask-price among matched sellers,

$$\hat{p}_l = \max\{a_j : y_j^* > 0, l \in L_j\}. \tag{3.2}$$

The payments are determined by these prices. Matched buyers pay the sum of the prices of items in their bundle. Matched sellers receive a payment equal to the number of units sold times the price for the item. Unmatched buyers and sellers do not participate. This completes the mechanism description.

If i is a matched buyer ($x_i^* > 0$), it must be that his bid $b_i \geq \sum_{l \in R_i} \hat{p}_l$; for otherwise, the surplus (3.1) can be increased by eliminating the corresponding matched bid. Similarly, if j is a matched seller ($y_j^* > 0$), and $l \in L_j$, his bid $a_j \leq \hat{p}_l$; for otherwise the surplus can be increased by eliminating his bid. Thus, the outcome of the auction respects individual rationality.

It is easy to understand how the mechanism picks matched sellers. For each item j, a seller with lower ask bid will be matched before one with a higher bid. So, sellers with bid $a_j < \hat{p}_l$ sell all their supply ($y_j^* = \sigma_j$). At most, one seller with ask bid $a_j = \hat{p}_l$ sells only a part of his total supply ($y_j^* < \sigma_j$). On the other hand, because their bids are combinatorial, the matched buyers are selected only after solving the MIP.

The proposed mechanism resembles the k-double auction mechanism [144]. We designed c-SeBiDA so that its outcome mimics a competitive equilibrium, with a particular interest in the combinatorial case. It was later discovered that the single item version SeBiDA resembles the k-double auction (a special case being called the buyer's bid double auction [145, 167]). But the two mechanisms differ in how the prices are determined. It is not clear what a generalization of the k-double auction would be to the combinatorial case. Moreover, as we will see, SeBiDA has certain incentive-compatibility properties lacking in the k-double auction. This makes the Bayesian-Nash equilibrium analysis simpler.

3.3 Nash Equilibrium Analysis: c-SeBiDA is Efficient

We first focus on how strategic behavior of players affects price when they have complete information. We will assume that players don't strategize over the quantities (namely, δ_i, σ_j), which will be considered fixed in the players' bids. A strategy for buyer i is a buy bid b_i, a strategy for seller j is an ask bid a_j. Let θ denote a collective strategy. Given θ, the mechanism determines the allocation (x^*, y^*) and the prices $\{\hat{p}_l\}$. So the payoff to buyer i and seller j is, respectively,

$$u_i^b(\theta) = \bar{v}_i(x_i^*) - x_i^* \cdot \sum_{l \in R_i} \hat{p}_l, \tag{3.3}$$

$$u_j^s(\theta) = y_j^* \cdot \sum_{l \in L_j} \hat{p}_l - \bar{c}_j(y_j^*). \tag{3.4}$$

The bids b_i, a_j may be different from the true valuations v_i, c_j, which however, figure in the payoffs.

A collective strategy θ^* is a *Nash equilibrium*, if no player can increase his payoff by unilaterally changing his strategy.

Single item, SeBiDA.

We studied the single-item version SeBiDA, of c-SeBiDA. We constructed a Nash equilibrium, and showed it yields a unique and efficient allocation (Theorem 3.1). The proof clarifies the more complex construction in the combinatorial case (Theorem 3.2).

To keep things simple, we will assume that each buyer bids for at most one unit, and each seller sells at most one unit of the item (so δ_i, σ_j equal 1 in (3.3), (3.4)). We will argue later that the results extend to multiple unit bids. There are m buyers and n sellers, whose true valuations and costs lie in $[0, 1]$. To avoid trivial cases of non-uniqueness, assume all buyers have different valuations and all sellers have different costs.

The mechanism finds the allocation (x^*, y^*) that is a solution of the following Integer Program **IP**:

$$\begin{aligned} \max_{x,y} \quad & \textstyle\sum_i b_i x_i - \sum_j a_j y_j \\ s.t. \quad & \textstyle\sum_i x_i \leq \sum_j y_j, \\ & x_i, y_j \in \{0, 1\}. \end{aligned}$$

As in (3.2) the settlement price is

$$\hat{p}(b, a) = \max\{a_j : y_j^* > 0\}.$$

It is easy to find (x^*, y^*): We repeatedly match the highest unmatched buy bid with the lowest unmatched sell bid if the buy bid is greater than the sell bid.

Theorem 3.1. *(i) A Nash equilibrium (b^*, a^*) exists for the SeBiDA game. (ii) Except for the matched seller with the highest bid on each item, it is a dominant strategy for each player to bid truthfully. The highest matched seller bids $\min\{v, c\}$, in which c is the true reservation cost of the unmatched seller with lowest bid and v is the reservation value of the matched buyer with the lowest bid. (iii) The Nash equilibrium is unique. (iv) The equilibrium allocation is efficient.*

Proof. Set $a_0 = c_0 = 0, b_0 = v_0 = 1$. Order the players so $v_1 \geq \cdots \geq v_M$ and $c_1 \leq \cdots \leq c_N$. Let $k = \max\{i \ : \ c_i \leq v_i\}$. We will show that the set of strategies,

$$\forall i, b_i = v_i; \forall j \neq k, a_j = c_j; a_k = \min\{c_{k+1}, v_k\},$$

is a Nash equilibrium.

The first k buyers and sellers are matched and the settlement price is $\hat{p} = a_k$. Consider a matched buyer $i \leq k$. This buyer has no incentive to bid lower, since by doing so he may be able to lower the price, but then he will also become unmatched. Since he is already matched, he certainly will not bid higher.

Consider an unmatched buyer $i > k$. He has no incentive to bid lower, as he will remain unmatched. He can become matched by bidding above a_k, but then, if he does get matched, his payoff will be negative.

Consider an unmatched seller $j > k$. He has no incentive to bid higher, as he will remain unmatched. He can get matched by bidding lower than a_k, but since his cost is $a_j > a_k$, his payoff will be negative.

Consider a matched seller $j < k$. By bidding lower, this seller will not change his payoff. If he bids higher to increase the settlement price, this will happen only if he bids above a_k. But then he will become unmatched.

Lastly, consider the 'marginal' matched seller k. He will not bid lower, as that will decrease his payoff. If he bids more than a_k, his bid will exceed either b_k or a_{k+1}, and in either case he will become unmatched. This proves (i), (ii).

This Nash equilibrium yields the allocation (x^*, y^*) which matches buyers with the highest valuation and sellers with least cost. Hence it is efficient.

We now prove uniqueness.

Suppose $(\tilde{x}, \tilde{y})$ is another Nash equilibrium. Since the two allocations are assumed different, either a buyer or a seller goes from being matched in the first allocation (x^*, y^*), to being unmatched in the second allocation $(\tilde{x}, \tilde{y})$, or vice-versa. Suppose the two Nash equilibria differ in allocation to a buyer who goes from being matched in the first allocation (x^*, y^*), to being unmatched in the second allocation $(\tilde{x}, \tilde{y})$. Then, either there is another buyer who goes from being unmatched to matched, or there is a seller who also goes from being matched to unmatched. Thus, we can see that as we go from (x^*, y^*) to $(\tilde{x}, \tilde{y})$, one of the four cases must occur:

(i) An unmatched buyer i_1 is made matched and a matched buyer i_2 is made unmatched;

(ii) An unmatched seller j_1 is made matched and a matched seller j_2 is made unmatched;

(iii) An unmatched buyer i and unmatched seller j are made matched;

(iv) A matched buyer i and seller j are made unmatched.

Case (i) We must have $v_{i_1} < v_{i_2}$ and the new bids must satisfy $\tilde{b}_{i_2} < \tilde{b}_{i_1}$. But then, either i_1's payoff is negative or i_2 can also bid just above i_1's bid. In either case, $(\tilde{x}, \tilde{y})$ cannot be a Nash equilibrium.

Case (ii) An argument similar to that for case (i) shows that $(\tilde{x}, \tilde{y})$ cannot be a Nash equilibrium.

Case (iii) Since both are unmatched in the first allocation, it must be that $v_i < c_j$. Since both are matched in the second allocation, it must be that $b_i > a_j$, so that one of them must have a negative payoff. Again, $(\tilde{x}, \tilde{y})$ cannot be a Nash equilibrium.

Case (iv) An argument similar to that for case (iii) shows that $(\tilde{x}, \tilde{y})$ cannot be a Nash equilibrium.

The Nash equilibrium is unique and the allocation is efficient. This proves (iv).

Combinatorial case, c-SeBiDA

Above, we constructed a Nash equilibrium for the game described by (3.1)-(3.4), in the case of a single item. The result can be extended to multiple items with single unit bids.

Theorem 3.2. *(i) A Nash equilibrium (b^*, a^*) exists in the c-SeBiDA game. (ii) Except for the matched seller with the highest bid on each item, it is a dominant strategy for each player to bid truthfully. (iii) Any Nash equilibrium allocation is always efficient.*

Proof. For the sake of clarity, we change some of the notation. As before, buyer i demands the bundle R_i with reservation value v_i. Let seller (l, j) be the j-th seller offering item l ($l \in L_j$ in the previous notation) with reservation cost $c_{l,j}$, and assume $c_{l,1} \leq \cdots \leq c_{l,n_l}$, in which n_l is the number of sellers offering item l.

We will iteratively construct a set of strategies to consider as a Nash equilibrium.

Set $a_{l,0} = c_{l,0} = 0, b_0 = v_0 = 1$. Consider the surplus maximization problem (3.1) with true valuations and costs. Let I be the set of matched buyers and k_l the number of matched sellers offering item l determined by the **MIP**. Set $b_i^* = v_i$ for all i; $a_{l,j}^0 = c_{l,j}$; $\gamma_i^t = b_i^* - \sum_{l \in R_i} a_{l,k_l}^t$, the surplus of a matched buyer i at stage $t \geq 0$, and

$$\hat{l} \in \arg\min_{l} \{ \min_{i \in I : l \in R_i} \gamma_i^t \}, \tag{3.5}$$

the item with the smallest surplus among the matched buyers at stage t. Denote the corresponding surplus by $\gamma_{\hat{l}}^t$. Now, define

$$a_{\hat{l},k_{\hat{l}}}^{t+1} := \min\{ a_{\hat{l},k_{\hat{l}}+1}^t, a_{\hat{l},k_{\hat{l}}}^t + \gamma_{\hat{l}}^t \}, \tag{3.6}$$

which is the strategy of seller $(\hat{l}, k_{\hat{l}})$ at the t-th stage. His ask bid is increased to decrease the surplus of the matched buyer with the smallest surplus up to the ask bid of the unmatched seller with the lowest bid. For all other $(l,j) \neq (\hat{l}, k_{\hat{l}})$, the ask bid remains the same, $a_{l,j}^{t+1} = a_{l,j}^t$. This procedure is repeated until the strategies converge. In fact, it is repeated at most, L times. Observe that at each stage, the matches and the allocations from the MIP using the current bids (b^*, a^t) do not change. Let a^* denote the seller ask bids when the procedure converges.

We prove that (b^*, a^*) is a Nash equilibrium, by showing that no player has an incentive to deviate.

First, an unmatched seller offering item l has no incentive to bid lower than a_{l,k_l}^*: Because his reservation cost is higher than that, by bidding lower than his reservation cost, it may get matched, but his payoff will be negative. Next, consider a matched seller $(l,j) \neq (l,k_l)$ offering item l. By bidding higher or lower he cannot change the price of the item, but may end up getting unmatched. Thus, it is the dominant strategy of all sellers, except the 'marginal' seller (l,k_l), to bid truthfully.

Now, consider this marginal matched seller (l,k_l). If he bids lower then a_{l,k_l}^*, his payoff will decrease. He could bid higher, but because of (3.6), either there is an unmatched seller of the item with the same ask bid, or there is a marginal buyer whose surplus has been made zero by (3.6). So, if he bids higher than a_{l,k_l}^*, either he will become unmatched and the first unmatched seller of the item will become matched, or the 'marginal' buyer with zero surplus will become unmatched causing this marginal seller to be unmatched as well. Thus, a_{l,k_l}^* is a Nash strategy of the marginal seller given that all other players (except the marginal sellers of the other items) bid truthfully.

Now, consider the buyers. First, an unmatched buyer i has no incentive to bid lower than b_i^* since he wouldn't match anyway. And if he bids higher, he may become matched, but his payoff will become negative. Next, a matched buyer with a positive payoff has no incentive to bid lower, since by bidding lower he can lower the prices, but only when he becomes unmatched. Also, he certainly has no incentive to bid higher, since by so doing, he will not be able to lower the price. Lastly, consider the 'marginal' matched buyers with zero payoff: Clearly, if they bid higher, their payoff will become negative; and if they bid lower, they will become unmatched. Thus, it is the dominant strategy of all buyers to bid truthfully.

The Nash equilibrium allocation (x^*, y^*) as determined above is efficient since it maximizes (3.1).

We now show that any Nash equilibrium allocation is efficient by extending the arguments in the proof of Theorem 3.1.

Suppose $(\tilde{x}, \tilde{y})$ is another Nash equilibrium which is not efficient. Either there is a buyer or a seller which goes from being matched in (x^*, y^*) to being unmatched in $(\tilde{x}, \tilde{y})$, or vice-versa. If there is a seller that goes from being matched to unmatched then either there is a matched seller in (x^*, y^*) replaced by another seller in $(\tilde{x}, \tilde{y})$ selling the same item (case (i)), or some unmatched sellers in (x^*, y^*) are matched in $(\tilde{x}, \tilde{y})$ with the set of matched sellers in (x^*, y^*) remaining matched. In this case, some unmatched buyer must also become matched (case (ii)). The rest of the cases can be argued similarly. Thus, the two Nash equilibrium allocations would differ in one of the five cases as we go from (x^*, y^*) to $(\tilde{x}, \tilde{y})$.

(i) A matched seller (l, j_1) is made unmatched and a unmatched seller (l, j_2) is made matched;

(ii) An unmatched buyer i demanding R_i is made matched and a set of unmatched sellers J such that $\{l : (l, j_l) \in J\} = R_i$ are made matched;

(iii) A matched buyer i demanding R_i is made unmatched and a set of matched sellers J such that $\{l_j : j \in J\} = R_i$ are made unmatched;

(iv) An unmatched buyer i demanding R_i is made matched and a set of matched buyers J with $j \in J$ demanding R_j such that $\cup_{j \in J} R_j = R_i$ are made unmatched;

(v) A matched buyer i demanding R_i is made unmatched and a set of unmatched buyers J with $j \in J$ demanding R_j such that $\cup_{j \in J} R_j = R_i$ are made matched;

Case (i) We must have $c_{l,j_1} < c_{l,j_2}$ and the new bids must satisfy $\tilde{a}_{l,j_2} < \tilde{a}_{l,j_1}$. But then either (l, j_2)'s payoff is negative or (l, j_1) can also bid just above (l, j_2)'s bid. In either case, $(\tilde{x}, \tilde{y})$ cannot be a Nash equilibrium.

Case (ii) We must have $v_i < \sum_{(l,j_l) \in R_i} c_{l,j_l}$ and the new bids must satisfy $\tilde{b}_i > \sum_{(l,j_l) \in R_i} \tilde{a}_{l,k_l}$ with $\tilde{a}_{l,j_l} < \tilde{a}_{l,k_l}$. This means that either the buyer or at least one seller has a negative payoff. Thus, $(\tilde{x}, \tilde{y})$ cannot be a Nash equilibrium.

Case (iii) The argument for this case is similar to case (ii).

Case (iv) We must have $v_i < \sum_{j \in J} v_j$ and the new bids must satisfy $\tilde{b}_i > \sum_{j \in J} \tilde{b}_j$. But then either i's payoff is negative or any $j \in J$ can bid high enough to outbid i. In either case $(\tilde{x}, \tilde{y})$ cannot be a Nash equilibrium.

Case (v) The argument for this case is similar to case (iv).

Thus, the Nash equilibrium allocation is always efficient. This proves (iii).

It is obvious that if the minimum in step (3.5) is not unique, the Nash equilibrium will not be unique. However, any Nash equilibrium allocation will still be efficient. Furthermore, if there is a unique efficient allocation, the Nash equilibrium is also unique.

A computationally efficient algorithm for the matching problem MIP and for computing the Nash equilibrium is very desirable. However, for most games, it is known to be a computationally hard problem. There is a computationally efficient algorithm for extensive two-person games.

It is interesting to note that:

Theorem 3.3. *With multiple unit buy-bids and single unit sell-bids, i.e., $\sigma_j = 1, \forall j$, the Nash equilibrium allocation and prices $((x^*, y^*), \hat{p})$ are a competitive equilibrium.*

Proof. Consider a matched seller. He supplies exactly one unit at prices $\hat{p}$, while an unmatched, non-marginal seller (l, j) for $j > k_l + 1$, supplies zero units. The unmatched marginal seller (l, k_l) will supply zero units, since $\hat{p}_l \geq a_{l,k_l+1}$. Now, consider a matched buyer i. At prices $\hat{p}$, he demands up to δ_i units of its bundle. If it is the "marginal" matched buyer, its surplus is zero and it may receive anything up to δ_i. If it is a "non-marginal" matched buyer, it receives δ_i units. An unmatched buyer, on the other hand, has zero demand at prices $\hat{p}$. Thus, total demand equals total supply, and the market clears.

The Nash equilibrium need not be a competitive equilibrium if sellers also make multi-unit bids as the following example shows.

Example 3.1. *(1) Consider two buyers, both with $v = 1$, who demand one unit of a good. Suppose there are three sellers owning one unit, each with $c = 0$. Then, the Nash equilibrium price is $\hat{p} = 0$ and it is easy to check it is a competitive price as well.*
(2) Now, consider two buyers, both with $v = 1$, who demand one unit of a good as before, but with one seller owning all three units with $c = 0$. The Nash equilibrium price in this case is $\hat{p} = 1$ which is different from the competitive price of zero.

Thus, the Nash equilibrium may not be a competitive equilibrium, when sellers make multi-unit bids.

Remarks.

1. While we considered single unit bids only, the results extend for multiple unit bids in a straightforward way. In this case, the number of buyers who match and the number of sellers who match will be different, since players ask for and offer multiple units. Still, as in the single unit bid case, there will be a "marginal matched" buyer k_l^b and a "marginal matched" seller k_l^s for each item l. The candidate Nash equilibrium strategies are that all buyers bid truthfully and all sellers bid truthfully, except for the "marginal matched" sellers k_l^s for each l. As before, they bid $a_{l,k_l^s} = \min\{a_{l,k_l^s+1}, b_{k_l^b}\}$. Now, one can check that all the arguments in the proofs of Theorems 3.1 and 3.2 still hold. We only have to consider those "marginal matched" buyers and "marginal matched" sellers whose bids are only partially matched. But it can be argued easily that they too have no incentive to deviate from the said strategies.
2. In our analysis, we have ignored the fact that the players can strategically choose quantities (δ_i, σ_j) that they bid. We have also restricted the players to making one bid each, as opposed to multiple bids, only one of which is accepted. In these cases, the proposed mechanism may yield inefficient Nash equilibria.

3.4 SeBiDA is Asymptotically Bayesian Incentive Compatible

We now consider the incomplete information case. We analyze the SeBiDA market mechanism in the limit of a large number of players. We assume that the number of buyers and the number of sellers is the same, $n \geq 2$. The results can be extended to the case when the number of buyers and sellers are different.

We will consider a Bayesian game to model incomplete information. Suppose nature draws $c_1, \cdots, c_n$ from probability distribution U_1 and draws $v_1, \cdots, v_n$ from probability distribution U_2, which are such that the corresponding pdfs u_1 and u_2 have full support on $[0, 1]$. Each player is then told his own valuation or cost. It is common information that the seller costs are drawn from U_1 and buyer valuations are drawn from U_2. Let $\alpha_j : [0, 1] \to [0, 1]$ denote the strategy of the seller j and $\beta_i : [0, 1] \to [0, 1]$

denote the strategy of the buyer i. Then, the payoff received by the buyers and sellers is as defined by equations (3) and (4). Let $\theta = (\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n)$ denote the collective strategy of the buyers and the sellers. A buyer i chooses strategy β_i to maximize $\mathbb{E}[u_i^b(\theta); \beta_i]$, the conditional expectation of the payoff given its strategy β_i. The seller j chooses strategy α_j to maximize $\mathbb{E}[u_j^s(\theta); \alpha_j]$, the conditional expectation of the payoff given its strategy α_j. The *Bayesian-Nash equilibrium* of the game is then the Nash equilibrium of the Bayesian game defined above.

We consider symmetric Bayesian-Nash equilibria, i.e., equilibria where all buyers use the same strategy β and all sellers use the same strategy α. Let $\tilde{\alpha}(c) := c$ and $\tilde{\beta}(v) := v$ denote the truth-telling strategies. Under strategies α and β, we denote the distribution of ask-bids a and buy-bids b as F and G respectively. We denote $[1 - F(x)]$ by $\bar{F}(x)$. Under $\tilde{\alpha}$ and $\tilde{\beta}$, $F = U_1$ and $G = U_2$. We consider only those bid strategies which satisfy the ex post individual rationality constraint, i.e., $\alpha(c) \geq c$ and $\beta(v) \leq v$. Denote $\mathcal{X} = \{\alpha : \alpha(c) \geq c\}$ and $\mathcal{Y} = \{\beta : \beta(v) \leq v\}$.

We consider single unit bids and assume that a symmetric Bayesian-Nash equilibrium exists.

Theorem 3.4. *Consider the SeBiDA auction game with $(\alpha, \beta) \in \mathcal{X} \times \mathcal{Y}$, i.e., both buyers and sellers have ex post individual rationality constraint. Let (α_n, β_n) be a symmetric Bayesian-Nash equilibrium with n buyers and n sellers. Then, (i) $\beta_n(v) = \tilde{\beta}(v) = v\ \forall n \geq 2$, and (ii) $(\alpha_n, \beta_n) \to (\tilde{\alpha}, \tilde{\beta})$ in the uniform topology as $n \to \infty$, i.e., SeBiDA is* asymptotically Bayesian incentive compatible.

We will first prove two lemmas.

Lemma 3.1. *Consider the SeBiDA auction game with n buyers and n sellers. Suppose the sellers use bid strategy α with $f(a)$, the pdf of its ask-bid under strategy α. Then, the best-response strategy of the buyers β_n satisfies $\beta_n(v) \geq v$ for all $n \geq 2$.*

Proof. Set $a_0 = c_0 = 0, b_0 = v_0 = 1$. Fix a buyer j with valuation v. Suppose sellers use a fixed bidding strategy α and denote the buyers best-response bidding strategy by β_n. Consider the game denoted $\mathcal{G}^{-j}$, where all players except buyer j participate and bid truthfully. Denote the number of matched buyers and sellers by $K = \sup\{k : a_{(k)} \leq b_{(k)}\}$, which is a random variable. Here $a_{(k)}$ denotes the order statistics increasing with k over the ask-bids of the participating sellers and $b_{(k)}$ the order statistics decreasing with k over the buy-bids of the participating buyers. Denote $X = a_{(K)}$, the ask-bid of the matched seller with the highest bid, $Y = a_{(K+1)}$, the ask-bid of the unmatched seller with the lowest bid and $U = b_{(K)}$, the buy-bid of the matched buyer with the lowest bid. It is easy to check that when buyer j also participates and bids $b = \beta(v)$, he gets a positive payoff

$$\pi'_j(b) = \begin{cases} v - X, & \text{if } X < U < b \text{ and } U < Y; \\ v - Y, & \text{if } X < Y < b \text{ and } Y < U. \end{cases} \tag{3.7}$$

The payoff of the buyer as a function of its bid b is shown graphically in figure 3.1. The reader can convince himself that the only relevant quantities for payoff calculation are X, Y and U. Thus, there are only two possible cases: (i) $X < Y < U$ and (ii) $X < U < Y$. Figure 3.1 (i) shows the case of (i) and the payoffs as b varies. As b increases above the dotted line, the payoff changes from zero to $v - b$. Similarly, as b increases above the dotted line in figure 3.1 (ii), the payoff changes from zero to $v - x$.

The expected payoff denoted by $\bar{\pi}'_j$ satisfies the differential equation

$$\frac{d\bar{\pi}'_j}{db} = P^n(A_{b,b})nf(b)(v - b) + \int_0^b P^n(B_{x,b})nf(x)(n-1)g(b)(v - x)dx, \tag{3.8}$$

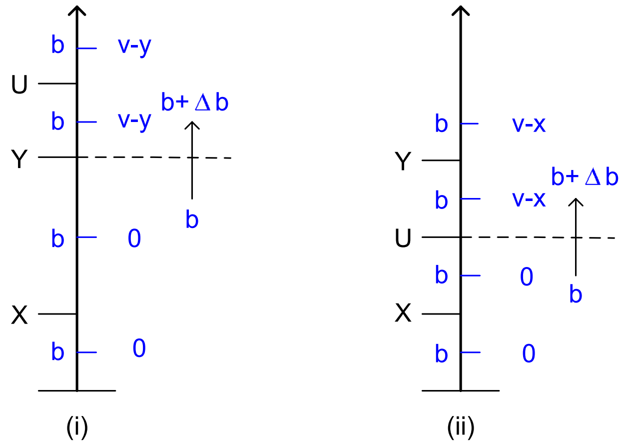

Figure 3.1: The payoff of the buyer as a function of its bid, b, for various cases.

where

$$P^n(A_{x,y}) = \sum_{k=0}^{n-1} \binom{n-1}{k} F^k(x)\bar{F}^{n-1-k}(x) \binom{n-1}{k} \bar{G}^k(y) G^{n-1-k}(y)$$

is the probability of the event that $X = x$ and $Y = y$ with $x < y$, among $n-1$ sellers and $n-1$ buyers. Similarly,

$$P^n(B_{x,y}) = \sum_{k=1}^{n-1} \binom{n-1}{k-1} F^{k-1}(x)\bar{F}^{n-k}(x) \binom{n-2}{k-1} \bar{G}^{k-1}(y) G^{n-1-k}(y)$$

is the probability of the event that $X = x$ and $Y = y$ with $x < y$, among $n-1$ sellers and $n-2$ buyers.

The boundary condition for the differential equation is $\bar{\pi}_j'(0) = 0$. The first term above arises from the change in payoff when b is increased by Δb and $U > Y > b > X$, and $b + \Delta b > Y$ as shown in figure 3.1(i). Similarly, the second term is the change in payoff when $Y > U > b > X$ and $b + \Delta b > U$ as shown in figure 3.1(ii). It is clear from (3.8) that for $b \le v$, $\frac{d\bar{\pi}_j'}{db} > 0$. Given the sellers' play strategy α, the best-response strategy of the buyers β_n is such that $b = \beta_n(v)$ and $\frac{d\bar{\pi}_j'}{db} = 0$. From this it is clear that

$$b = \beta_n(v) \ge v, \ \forall n \ge 2. \tag{3.9}$$

The above conclusion at first glance seems surprising. A buyer's strategy is to bid more than his true value. However, intuitively it makes sense for this mechanism since the prices are determined by the sellers' bids alone, and by bidding higher, a buyer only increases his probability of being matched. Of course, if he bids too high, he may end up with a negative payoff. The result implies that under the ex post individual rationality constraint, the buyer always uses the strategy $\beta_n = \tilde{\beta}$.

Now, we look at the best response strategy of the sellers when the buyers bid truthfully.

Lemma 3.2. *Consider the SeBiDA auction game with n buyers and n sellers and suppose buyers bid truthfully, i.e., $\beta_n = \tilde{\beta}$, and let α_n be the sellers' best-response strategy. Then, $(\alpha_n, \tilde{\beta}) \to (\tilde{\alpha}, \tilde{\beta})$ as $n \to \infty$.*

Proof. Set $a_0 = c_0 = 0, b_0 = v_0 = 1$. Fix a seller i with cost c. Consider the auction game, denoted $\mathcal{G}_{-i}$, in which seller i does not participate and all participating buyers bid truthfully. As before, denote the number of matched buyers and sellers by $K = \sup\{k : a_{(k)} \leq b_{(k)}\}$, $U = b_{(K)}$, the bid of the lowest matched buyer, $W = b_{(K+1)}$, the bid of the highest unmatched buyer, $X = a_{(K)}$, the bid of the highest matched seller, $Y = a_{(K+1)}$, the bid of the lowest unmatched seller, and $Z = a_{(K-1)}$, the bid of the next highest matched seller.

Consider the payoff of the i-th seller when he participates as well. His payoff when he bids $a = \alpha(c)$ is given by

$$\pi_i(a) = \begin{cases} x - c, & \text{if } a < Z < X < W, \text{ or} \\ & Z < a < X < W; \\ a - c, & \text{if } Z < X < a < W, \text{ or} \\ & Z < a < W < X, \text{ or} \\ & Z < W < a < X, \text{ or} \\ & W < Z < a < X; \\ z - c, & \text{if } a < Z < W < X, \text{ or} \\ & a < W < Z < X, \text{ or} \\ & W < a < Z < X. \end{cases} \tag{3.10}$$

The payoff of the seller as his bid a varies is shown graphically in figure 3.2. The reader can convince himself that the only relevant quantities for payoff calculation are X, Z and W. Thus, there are three cases: (i) $Z < X < W$, (ii) $Z < W < X$ and (iii) $W < Z < X$.

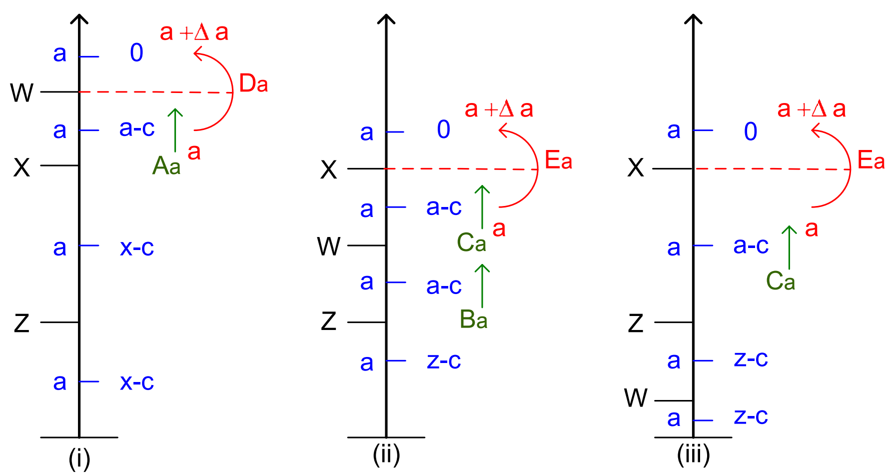

Figure 3.2: The payoff of the seller as a function of its bid, a, for various cases.

The expected payoff denoted by $\bar{\pi}_i$ satisfies the differential equation

$$\begin{aligned} \frac{d\bar{\pi}_i(a)}{da} &= [P^n(A_a) + P^n(B_a) + P^n(C_a)] \\ &\quad -[ng(a)P^n(D_a) + (n-1)f(a)P^n(E_a)](a - c), \end{aligned} \tag{3.11}$$

with the boundary condition $\bar{\pi}_i(1) = 0$ where A_a denotes the event that there are $n-1$ sellers and n buyers and $X < a < W$. As a is increased by Δa, the payoff to the seller also increases by Δa

since seller i is the price-determining seller. Similarly, B_a denotes the event that there are $n-1$ sellers and n buyers and $Z < a < W < X$ and seller i is the price-determining seller. In the same way, C_a denotes the event that there are $n-1$ sellers, n buyers and $\max(Z, W) < a < X$ and seller i is the price-determining seller. D_a denotes the event that there are $n-1$ sellers and $n-1$ buyers, $X < a$ (with the n-th buyer bidding a) and $W \in [a, a+\Delta a]$ so that the seller i becomes unmatched as it increases its bid. Similarly, E_a is the event that there are $n-2$ sellers, n buyers, $W < a$ (with the $(n-1)$-th seller bidding a) and $X \in [a, a+\Delta a]$. And so, as he increases his bid, he becomes unmatched.

Figure 3.2 shows these events graphically. Events A_a, B_a and C_a correspond to various cases when the change in the bid a from a to Δa, causes a change in payoff of Δa. Events D_a and E_a correspond to cases when the change in the bid a from $a + \Delta a$, causes a change in payoff of $-(a-c)$.

The following can then be obtained:

$$\begin{aligned}
P^n(A_a) &= \sum_{k=0}^{n-1} \binom{n-1}{k} F^k(a)\bar{F}^{n-1-k}(a)\binom{n}{k+1}\bar{G}^{k+1}(a)G^{n-(k+1)}(a) \\
P^n(B_a) &= \sum_{k=1}^{n-1} \binom{n-1}{k-1} F^{k-1}(a)\bar{F}^{n-k}(a)\binom{n}{k+1}\bar{G}^{k+1}(a)G^{n-(k+1)}(a) \\
P^n(C_a) &= \sum_{k=1}^{n-1} \binom{n-1}{k-1} F^{k-1}(a)\bar{F}^{n-k}(a)\binom{n}{k}\bar{G}^{k}(a)G^{n-k}(a) \\
P^n(D_a) &= \sum_{k=0}^{n-1} \binom{n-1}{k} F^{k}(a)\bar{F}^{n-1-k}(a)\binom{n-1}{k}\bar{G}^{k}(a)G^{n-1-k}(a) \\
P^n(E_a) &= \sum_{k=1}^{n-1} \binom{n-2}{k-1} F^{k-1}(a)\bar{F}^{n-1-k}(a)\binom{n}{k}\bar{G}^{k}(a)G^{n-k}(a).
\end{aligned} \tag{3.12}$$

Let $a = \alpha_n(c)$ be the best-response strategy of the sellers. Then, $\frac{d\bar{\pi}_i}{da} = 0$ at $a = \alpha_n(c)$. For any $a < c$, $\frac{d\bar{\pi}_i}{da} > 0$ from (3.11). Thus,

$$a = \alpha_n(c) \geq c, \ \forall n \geq 2. \tag{3.13}$$

If $a > c$, setting (3.11) equal to zero and rearranging, we get

$$f(a) = \frac{[P^n(A_a) + P^n(B_a) + P^n(C_a)] - ng(a)P^n(D_a)(a-c)}{(n-1)P^n(E_a)(a-c)} \geq 0,$$

from which we obtain

$$\begin{aligned}
\alpha_n(c) - c &\leq \frac{[P^n(A_a) + P^n(B_a) + P^n(C_a)]}{ng(a)P^n(D_a)} \\
&\leq \frac{1}{g(a)}\left[\frac{\sum_{k=0}^{n-1}\binom{n-1}{k}^2 \frac{z^k}{(k+1)}}{\sum_{k=0}^{n-1}\binom{n-1}{k}^2 z^k}\bar{G} + \frac{\sum_{k=1}^{n-1}\binom{n-1}{k}^2 \frac{z^k}{(n-k)}}{\sum_{k=0}^{n-1}\binom{n-1}{k}^2 z^k}\frac{\bar{G}\bar{F}}{F}\right] \\
&\quad + \frac{1}{g(a)}\left[\frac{\sum_{k=1}^{n-1}\binom{n-1}{k}^2 \frac{kz^k}{(n-k)^2}}{\sum_{k=0}^{n-1}\binom{n-1}{k}^2 z^k}\frac{\bar{F}}{F}\right],
\end{aligned} \tag{3.14}$$

where $z = \frac{F(a)\bar{G}(a)}{\bar{F}(a)G(a)}$. Observe that the terms $\bar{G}(a)$, $\bar{G}(a)\bar{F}(a)$ and $\bar{F}(a)$ in the numerator are upper-bounded by one, and the term $F(a)$ in the denominator is lower-bounded by $F(c)$. It can now be shown that each of the terms converges to zero for all $z > 0$ as $n \to 0$. Thus, $(\alpha_n, \tilde{\beta}) \to (\tilde{\alpha}, \tilde{\beta})$.

The conclusion of this Lemma is what we would expect intuitively. If all buyers bid truthfully, then as the number of sellers increases, increased competition forces them to bid closer and closer to their true costs.

Proof. (Theorem 3.4) By Lemma 3.1 when the sellers use strategy α_n, the buyers under the ex post individual rationality constraint use strategy $\tilde{\beta}$. By Lemma 3.2, when the buyers bid truthfully, sellers' best-response is α_n. Thus, $(\alpha_n, \tilde{\beta})$ is a Bayesian-Nash equilibrium with n players on each side of the market. Further, Lemma 3.2 shows that $(\alpha_n, \beta_n) = (\alpha_n, \tilde{\beta}) \to (\tilde{\alpha}, \tilde{\beta})$ as $n \to \infty$, which is the conclusion we wanted to establish.

Thus, under the ex post IR constraint, SeBiDA is ex ante budget balanced, asymptotically Bayesian incentive compatible and efficient. Unlike in the complete information case, when the mechanism is not incentive compatible, yet the outcome is efficient, in the incomplete information case, the mechanism is only asymptotically efficient.

The mechanism proposed here is related to the Buyer's Bid Double Auction (BBDA) mechanism [144, 145, 167]. While the spirit of the two mechanisms is the same (maximizing the efficiency of trading), the prices and the payments are different. In SeBiDA, the prices are determined by the bids of the sellers only. This makes the market asymmetric. In the complete information case, all buyers have no incentive to bid non-truthfully, but at least one seller does.

In BBDA, the determined price could be either a buyer's bid or a seller's bid. While the claim in Theorem 2.1 of [145] is not correct, in [167], it is simply assumed that the buyers bid truthfully, which need not be true. In fact, we found that for SeBiDA, even though under complete information, it is a dominant strategy for buyers to bid truthfully. This is not the case for incomplete information.

The proof techniques used here are in part inspired by those developed by [24, 144, 145, 167]. The rate of convergence of SeBiDA can be obtained from the analysis in the proof of Lemma 3.2. Strangely, Nash equilibrium analysis was ignored in [145]. Finally, the ex post individual rationality constraint seems restrictive at first glance. However, in two human subject experiments we have conducted using this mechanism, it was observed that all subjects in fact always used strategies that were ex post individual rational [77]. Thus, the predictive power of the result does not seem diminished in real-world settings despite the assumption made. It is also pertinent to mention [146], wherein the authors show that the k-DA class of market mechanisms are worst-case asymptotic optimal, where optimality is measured in how quickly the inefficiency diminishes as the market size increases. The mechanisms are evaluated in the least favorable trading environment.

Bayesian-Nash Equilibrium in a Special Combinatorial Case

We now provide an extension of Theorem 3.4 to a combinatorial case.

Corollary 3.1. *Suppose the buyer valuations and seller costs are uniform over $[0, 1]$, i.e., $U_1 = U_2 = U[0, 1]$. The combinatorial demands of buyers are such that each item is demanded by n buyers and there are n sellers for each item. Then, the claim of Theorem 3.4 still holds, i.e., if both buyers and sellers have ex post individual rationality constraint and (α_n, β_n) is a symmetric Bayesian-Nash equilibrium, then, (i) $\beta_n(v) = \tilde{\beta}(v) = v \; \forall n \geq 2$, and (ii) $(\alpha_n, \beta_n) \to (\tilde{\alpha}, \tilde{\beta})$ in the uniform topology as $n \to \infty$, i.e., c-SeBiDA is* asymptotically Bayesian incentive compatible*.*

The reader can check that the arguments in proof of Theorem 3.4 still hold. We provide an intuitive argument. Suppose a buyer i would present his combinatorial bid as an itemized bid, i.e., a bid for each item in his bundle. Now, for each item in his bundle, it faces the same number of buyers $n-1$ and the same number of sellers n. Suppose all other buyers $j \neq i$ divide their bid equally among all items in their bundle, i.e., if b_j is the bid for the bundle R_j, $b_j/|R_j|$ is the bid for each item in R_j. Then, buyer i has to divide his bid b_i among his items in such a way that his expected payoff is maximized. For given bids of all players, his payoff is zero if he is not matched and non-zero if he is matched. Thus, he has to itemize his bid in a way that he maximizes the probability of being matched. It can be verified that when buyer valuations and seller costs are drawn uniform over $[0,1]$, the probability of his bid being accepted is maximum if the bid is divided equally among all items in his bundle. Thus, he would use the same strategy $\beta_n = b_j/|R_j|$ on each item, which would induce the same distribution of buy-bids G on each item. This is true for all buyers since they are symmetric. Similarly, all sellers will use the same bid strategy α_n, which will induce the distribution of ask-bids F. Now, the game has been reduced to a single-item auction game on each item, and the result follows from Theorem 3.4.

From the Nash equilibrium analysis for the combinatorial case and the Bayesian-Nash analysis for the single item case, it seems plausible that the Bayesian-Nash equilibrium result can be extended to the general combinatorial case. However, the analysis becomes rather messy and is part of future work. Thus, in the next section, we show that the c-SeBiDA outcome when there are a large number of players (as in a continuum model) is a competitive equilibrium.

3.5 c-SeBiDA Outcome is Competitive Equilibrium in the Continuum Model

We now present competitive analysis of the c-SeBiDA mechanism. Since competitive equilibria may not exist for the setting considered, we investigate the behavior of the outcome of the c-SeBiDA auction when the number of players is large enough such that no single player by itself can affect the outcome. An idealization is a continuum of agents. Such a setting was first considered by Aumann [9] in a general equilibrium setting and others have used this approach in the analysis of games [67, 69].

Assume the continuum of buyers is indexed by $t \in [0,1]$, and the continuum of sellers is indexed by $\tau \in [0,1]$. There are m types of buyers and n types of sellers. Let $B_1, \cdots, B_m$ and $S_1, \cdots, S_n$ partition $[0,1]$ so that all buyers in B_i demand the same set of items R_i (corresponding say to a route), and all sellers in S_j offer the same item l_j, $L_j = \{l_j\}$. We assume that the partitions B_i's and S_j's are subintervals.

A buyer $t \in B_i$ has true value $v(t)$, bids $p(t)$ per unit for the set R_i, and demands $\delta(t) \in [0, D]$ units. Suppose $v(t), p(t) \in [0, V]$. A seller $\tau \in S_j$ has true cost $c(\tau)$ and asks $q(\tau)$ for the item(s) L_j with supply $\sigma(\tau) \in [0, S]$ units, with $c(\tau), q(\tau) \in [0, C]$. Let $x(t)$ and $y(\tau)$ be the decision variables, i.e. buyer t's $x(t)$ is 1, if his bid is accepted, 0 otherwise. And similarly, seller τ's $y(\tau)$ is 1 if his offer is accepted, 0 otherwise. We assume that within each partition B_i, the buyers' bid function $b(t)$ is non-increasing, and within each partition S_j, the sellers' bid function $q(\tau)$ is nondecreasing.

Note that while in section 3.2, we assumed that buyers specify a maximum demand and they may be allocated any integral units up to the maximum demand, here we will assume that their bundles are *all-or-none* kind: All demand must be met or none.

Denote the indicator function by $\not\Vdash(\cdot)$ and as before, consider the surplus maximization problem

cLP:

$$
\begin{aligned}
\sup_{x,y} \int_0^1 \sum_{i=1}^{m} x(t)\delta(t)p(t)\mathbb{1}(t \in B_i)dt \quad &- \quad \int_0^1 \sum_{j=1}^{n} y(\tau)\sigma(\tau)q(\tau)\mathbb{1}(\tau \in S_j)d\tau && (3.15)\\
\text{s.t.} \quad & \\
\int_0^1 \sum_{j=1}^{n} y(\tau)\sigma(\tau)\mathbb{1}(l \in L_j, \tau \in S_j)d\tau \quad &- \quad \int_0^1 \sum_{i=1}^{m} x(t)\delta(t)\mathbb{1}(l \in R_i, t \in B_i)dt \geq 0, \\
\forall l \in [1:L] \text{ and } x(t), y(\tau) \quad &\in \quad \{0,1\}, \forall t, \tau \in [0,1].
\end{aligned}
$$

The mechanism determines $((x^*, y^*), \hat{p})$, where (x^*, y^*) is the solution of the above continuous linear integer program and for each $l \in [1:L]$,

$$\hat{p}_l = \sup\{q(\tau) : y(\tau) > 0, \tau \in S_l\}, \tag{3.16}$$

and

$$\check{p}_l = \inf\{q(\tau) : y(\tau) = 0, \tau \in S_l\}. \tag{3.17}$$

The mechanism announces prices $\hat{p} = (\hat{p}_1, \cdots, \hat{p}_L)$; the matched buyers (those for which $x^*(t) = 1$) pay the sum of the prices of the items in their bundle, while the matched sellers (those for which $y^*(\tau) = 1$) get a payment equal to the number of their items sold times the price of the item. When buyers and sellers bid truthfully, the following result holds.

Theorem 3.5. *If the bid function of the sellers $q : [0,1] \to [0, C]$ is continuous and nondecreasing in each partition S_j of $[0,1]$, then (x^*, y^*) is a competitive allocation and $\hat{p}$ is a competitive price.*

Proof. We first show the existence of (x^*, y^*) and $(\lambda_1^*, \cdots, \lambda_L^*)$, the dual variables corresponding to the demand less than equal to supply constraints. We do this by casting the cLP above as an optimal control problem and then appeal to Pontryagin's maximum principle [124]. Define

$$
\begin{aligned}
\dot{\zeta}(t) \quad &:= \quad \Sigma_{i=1}^{m} x(t)\delta(t)p(t)\mathbb{1}(t \in B_i) - \Sigma_{j=1}^{n} y(t)\sigma(t)q(t)\mathbb{1}(t \in S_j), && (3.18)\\
\dot{\xi}_l(t) \quad &:= \quad \sum_{j=1}^{n} y(t)\sigma(t)\mathbb{1}(l \in L_j, t \in S_j) - \sum_{i=1}^{m} x(t)\delta(t)\mathbb{1}(l \in R_i, t \in B_i), && (3.19)\\
\theta(t) \quad &:= \quad (\xi_1(t), \cdots, \xi_L(t), \zeta(t))', && (3.20)
\end{aligned}
$$

where θ is the state of the system, x and y are controls, and $\zeta(t)$ and $\xi(t)$ describe the state evolution as a function of the controls. The objective is to find the optimal control (x^*, y^*) which maximizes $\zeta(1)$. Let

$$\Sigma(t) := \{\dot{\theta}(t) : x_l(0) = 0, \forall l \text{ and } x(t), y(t) \in \{0,1\}, \forall t \in [0,1]\}. \tag{3.21}$$

Observe that $\Sigma(t)$ has cardinality at most 2^{L+1} in $\mathbb{R}^{L+1}$. $\int_0^1 \Sigma(\tau)d\tau$ is the set of reachable states under the set of all allowed control functions, namely, all measurable functions x and y such that $x(\tau), y(\tau) \in \{0,1\}$. Note that $\zeta(1)$ defines our total surplus; i.e., buyer surplus minus seller surplus, and $\xi_l(1)$ defines the excess supply for item l; i.e., total supply minus total demand for item l. Define

$$\Gamma := \{\theta(1) \in \mathbb{R}^{L+1} : \theta(1) \in \int_0^1 \Sigma(\tau)d\tau, \xi_l(1) \geq 0, \forall l\}, \tag{3.22}$$

the set of final reachable states under all control functions such that state evolution happens according to the equations above, and excess supply is non-negative.

Lemma 3.3. Γ *is a compact, convex set.*

Proof. By assumption, $\delta(t), p(t), \sigma(t)$, and $q(t)$ are bounded. By Lyapunov's theorem [10], $\int_0^1 \Sigma(\tau)d\tau$ is a closed and convex set. Since x and y are bounded functions, the integral is bounded as well. Thus, it is also compact. Moreover, $\xi_l(1)$ is a hyperplane, and $\xi(1) \geq 0$ defines a closed subset of $\mathbb{R}^L$. Therefore, $\{\theta(1) : \theta(1) \in \int_0^1 \Sigma(\tau)d\tau\} \bigcap \{\theta(1) : \xi_l(1) \geq 0, l = 1, \cdots, L\}$ is a compact, convex set.

Now, our optimal control problem is: $\sup_{\theta(1)\in\Gamma} \zeta(1)$. But observe that one component of $\theta(1)$ is $\zeta(1)$. Since Γ is compact and convex, the supremum is achieved and an optimal control (x^*, y^*) exists in Γ. By the maximum principle [124], there exist adjoint functions $p_0^*(t)$ and $p_l^*(t)$, $l = 1, \cdots, L$ such that $\dot{p}_0^*(t) = 0$, and $\dot{p}_l^*(t) = 0$, (i.e., $p_l^*(t) = \lambda_l^*$, a constant) for $l = 0, \cdots, L$.

Defining the Lagrangian over the objective function and the demand less than equal to supply constraint

$$L(x, y; \lambda) = \zeta(1) + \sum_{l=1}^{L} \lambda_l \xi_l(1), \tag{3.23}$$

we get from the saddle-point theorem [160],

$$L(x, y; \lambda^*) \leq L(x^*, y^*; \lambda^*) \leq L(x^*, y^*; \lambda). \tag{3.24}$$

We use this saddle point inequality to conclude the existence of a competitive equilibrium.

Lemma 3.4. *If $((x^*, y^*), \lambda^*)$ is a saddle point satisfying the inequality (3.24) above, then the λ^* are competitive equilibrium prices. Moreover, $\hat{p}_l \leq \lambda_l^* \leq \check{p}_l, \forall l = 1, \cdots, L$.*

Proof. Let $((x^*, y^*), \lambda^*)$ be the saddle point satisfying the above inequality. Rewrite the Lagrangian as

$$L(x, y; \lambda) = \sum_{i=1}^{m} \int_{B_i} \delta(t)x(t)(p(t) - \sum_{l\in R_t} \lambda_l)dt + \sum_{j=1}^{n} \int_{S_j} \sigma(\tau)y(\tau)(\lambda_{l(\tau)} - q(\tau))d\tau$$

where $l(\tau)$ is the item offered by seller τ. Now, using the first saddle-point inequality, we get that $x^*(t) = \mathbb{1}(p(t) > \Sigma_{l\in R_t}\lambda_l^*)$ and $y^*(\tau) = \mathbb{1}(q(\tau) < \lambda_{l(\tau)}^*)$, which implies that the Lagrange multipliers are competitive equilibrium prices. To prove the second part, note that by definition, for a given τ, $y(\tau) > 0$ implies that $q(\tau) \leq \lambda_l^*$ for $\tau \in S_l$, which implies the first inequality. Again from definition, we get that $y(\tau) = 0$ implies that $q(\tau) \geq \lambda_l^*$ for $\tau \in S_l$, which implies the second inequality.

To conclude the proof of the theorem, we observe that if q is continuous and non-decreasing in each interval S_j of $[0, 1]$, then $\hat{p}_l = \check{p}_l$ for each l, which then equals λ_l^* by Lemma 3.4.

The implication of this result is that as the number of players becomes large, the outcome of the above auction approximates the competitive equilibria of the associated continuum exchange economy. We will defer discussion of the relationship between the Nash equilibria and the competitive equilibria to the conclusions section.

We now show that the assumption that the sellers' bid function is piecewise continuous and nondecreasing is necessary for the c-SeBiDA's price to be a competitive price.

Example 3.2. *Suppose that there is only one item. Buyers $t \in [0, 0.5]$ have reservation value 3 while buyers $t \in (0.5, 1]$ have reservation value 4. Sellers $t \in [0, 0.5]$ have reservation cost 5 while sellers $t \in (0.5, 1]$ have reservation cost 2. Then, it is clear that the buyers in $(0.5, 1]$ and sellers $(0.5, 1]$ will be matched with surplus $0.5 \times 2 = 1$. Thus, $\hat{p} = 2$ which is not equal to $\check{p} = 3$. As can be easily checked, the competitive price is $\lambda^* = 3$ different from $\hat{p}$.*

3.6 Chapter Summary

We have introduced a combinatorial, Sellers' Bid, Double Auction (c-SeBiDA). It is worth noting that a single-sided auction with one seller and zero costs is a special case of a double auction. We presented three results for c-SeBiDA.

The first result concerned the existence of a Nash equilibrium for c-SeBiDA with full information. In c-SeBiDA, settlement prices are determined by sellers' bids. We showed that the allocation of c-SeBiDA is efficient. Moreover, truth-telling is a dominant strategy for all players except the highest matched seller for each item.

The second result concerned the Bayesian-Nash equilibrium of the mechanism under incomplete information. We showed that under the ex post individual rationality constraint, symmetric Bayesian-Nash equilibrium strategies converge to truth-telling for the single item auction. Thus, the mechanism is asymptotically Bayesian incentive compatible, and hence asymptotically efficient.

The third result concerned the competitive analysis of the c-SeBiDA auction mechanism. We considered the continuum model and showed that within that model c-SeBiDA outcome is a competitive equilibrium. This suggests that in the finite setting, the auction outcome is close to efficient.

In [66], we considered a more general setting and showed that a competitive equilibrium exists in a continuum model of an exchange economy with indivisible items and money (a divisible item). We used the Lyapunov's theorem [10] for convexification of the economy and the Debreu-Gale-Nikaido Lemma [20] to establish existence of a fixed point of the excess demand correspondence. We also showed that there exist non-enforceable competitive equilibria based on the approximation of non-convex sets using the Shapley-Folkman and Starr Theorems [5].

We have tested the proposed mechanism c-SeBiDA through human-subject experiments. Those results can be found elsewhere [77].

We now situate our contribution in the literature that relates Nash and competitive equilibria. The basic idea is that as the economy gets large (in our context the number of buyers and sellers and quantities of items all go to infinity), Nash equilibrium strategies should converge to competitive equilibrium strategies, because 'market power' diminishes.

In [145, 167], it is shown that Bayesian-Nash equilibrium strategies converge to truthful bidding as the market size goes to infinity. The relationship is first investigated in [132]. In a later paper [52], it is shown that under certain regularity conditions, a sufficiently replicated economy has an allocation which is incentive-compatible, individually-rational and ex-post ϵ-efficient. Similarly, [67] shows that the demand functions that an agent might consider based on strategic considerations converge to the competitive demand functions. Further, [69] shows that under certain conditions on beliefs of individual agents, not only do the strategic behaviors of individual agents converge to the competitive behavior, but the Nash equilibrium allocations also converge to the competitive equilibrium allocation. The formulation in [168] is a buyer's bid double auction with a single type of item that maximizes surplus. It is shown that with Bayesian-Nash strategies, the mechanism is asymptotically "incentive efficient," the notion of incentive efficiency being different from that of incentive compatibility and efficiency that we use here. Along a different line of investigation, [50, 145, 140] investigate the rate of convergence of the Nash equilibria to the competitive equilibria for buyer's bid double auction. Finally, implementation and mechanism design in a setting with a continuum of players is discussed in [107].

Chapter 4

Human-subject Experiments

Recent interest in the intersection of economics and engineering has focused on the use of economic theory to deal with allocation of resources under conflicting incentives. Implementing such systems, however, will most likely not yield the desired theoretical outcomes. We propose the use of experimental methods for testing such predictions and better informing the design of such incentive-based systems. We are particularly interested in complex economic environments with complimentary goods/services. We propose the experimental investigation of the use of combinatorial auctions for the allocation of scarce resources in a bandwidth trading market. We performed an experimental study to investigate theoretical results based on the auction mechanism proposed in the previous chapter that has been implemented in a software test bed. It has been used to conduct human-subject experiments to validate the theory developed for it in the previous chapter.

This chapter forms part of the dissertation work of Charis Kaskiris, and appeared in:

C.Kaskiris, R.Jain, R.Rajagopal and P.Varaiya, "Combinatorial auction design for bandwidth trading: An experimental study," *Proc. International Conf. on Experiments in Economic Sciences, Kyoto, Japan*, December 2004.

The chapter completes the auction theory we have developed. It also demonstrates the methods and the efficacy of conducting such human-subject economic experiments.

4.1 Introduction and Literature Review

An introduction to auction theory is provided in [90]. For a broader non-technical survey of auction theory, see [86]. Until the early 1990s, most of the work by economists sought better understanding of the theoretical and strategic properties of traditional auction mechanisms. The Federal Communications Commission auctions of wireless communication licenses were different because of complementarities between the different licenses. These economic environments have been described as combined value auctions and were experimentally investigated in the context of airline slot allocation [129], payloads for NASA's Space Station [13], tracking routes [126], pollution license trading, and spectrum auctions [96, 123, 125]. Further applications are discussed in [164]. The realization that economic models could be used in the design of real-life mechanisms and potentially in the design of market-based control systems in engineering and computer science [26] spurred the investigation of the validity of assumptions made in theoretical contexts and their empirical applicability.

Dealing with complex economic environments with complementarities has proven to be a formidable task for auction theorists. The theoretical properties of different auction formats, such as the simultaneous ascending bid auctions and combinatorial auctions, were poorly understood. Therefore, designers of such systems turned to experimental economics to investigate the properties of such mechanisms. Experimental economics is the application of the laboratory method to test the validity of various economic theories and to test new market mechanisms. Using cash-motivated subjects, economic experiments create real world incentives to help us better understand why markets and other exchange systems work the way they do [44, 73]. Many new auction mechanisms were introduced following improvements in computational power.

4.2 Combinatorial Auctions

Combinatorial Auctions (CA) are studied by economists who investigate the economic rationality of self-interested agents, and by computer scientists dealing with the computational and informational constraints of such auctions. Combinatorial auctions enhance our ability to efficiently allocate multiple resources in complex economic environments because of their generalized bid expression, which allows participants to bid on *packages* of items with related values or costs. They also allow bidders to impose logical constraints that limit the feasible set of auction allocations. They can also handle functional relationships amongst bids or allocations such as budget constraints or aggregation limits. However, this flexibility makes devising an optimal bidding strategy a computationally intensive task for bidders and sellers.

CA allow more expressive bids in which participants can submit package bids with logical constraints that limit allowable outcomes. This type of auction can be useful when participants' values are complementary or when they have production and financial constraints. There are several reasons to prefer to expand the bidding message space. One problem that combinatorial auctions solve is the "exposure" problem, evident in simultaneous ascending auctions [112]. With individual bidding, a bidder is exposed to the risk of winning a few licenses it wants without winning other complementary licenses it wants. Fearing this exposure, a bidder may not bid aggressively, not participate in the auction, or try to collude [28]. Hence, combinatorial bidding appears to have better efficiency in the presence of demand rigidities (i.e. a bidder extracts any valuation only if the whole package is fulfilled).

However, combinatorial auctions are currently rare in practice. The main problems confronted in implementing these auctions stem from the computational uncertainty with regards to winner determination when there are large numbers of items and participants. The auction is also cognitively complex and can lead participants to pursue perverse bidding strategies. They also lead to inefficiencies in cases with the "threshold" effect [23]. The threshold effect is present if aggregating smaller bids would have displaced a larger bid, but the incentives to do so are not aligned.

The computational uncertainty in winner determination comes from the fact that winner determination in combinatorial auctions is equivalent to a Set Packing Problem [164], which is nondeterministic polynomial time complete or hard problem. Bidding in combinatorial auctions is burdensome, both strategically and cognitively, for all participants.

In designing combinatorial auctions, several questions need to be considered: How does the format of the auction withstand the threshold effect? Does iterative bidding allow for strategy building through learning? What is the appropriate level of information feedback to the bidders? What is the computational cost of the algorithms proposed?

It has been observed in the field and during experiments that in complex economic environments, iterative auctions that permit the participants to observe the competition and learn when and how to bid, produce better results than sealed bid auctions. Two frameworks are currently used for iterative procedures. The first is the use of continuous auctions [13] during which bidders may see a set of provisional winning bids, as well as a set of bids to be combined from a standby list. The standby list consists of non-winning bids and these bids are there to signal willingness to combine bids to outbid larger-package bids. The second framework uses multiple rounds of sealed bid formats, which solves repeated integer programming problems. In general, auction systems that provide feedback and allow bidders to revise their bids seem to produce more efficient outcomes [125].

In this chapter, we compare the revenue, efficiency, and bidding properties of a particular combinatorial auction set-up in which there are complementarities among the objects being allocated. Specifically, we conduct laboratory experiments allocating three links with private values and complementarities using the combinatorial auction format under different degrees of complementarity. In the benchmark case, every seller owns all three types of links and every buyer has private values over all subsets of links. In alternative cases, sellers own two types of links with one type being owned by all sellers. Buyers have valuations over bundles of links.

The remainder of the chapter is as follows: Section 4.3 presents the information and valuation structure used in the experiments. Section 4.4 gives an overview of the experimental design and the results of the experiments. Section 4.5 concludes the chapter.

4.3 Information and Valuation Structure

We propose using economic experiments for the design and understanding of mechanisms to allocate resources in engineering and electronic commerce. Even though economic theory has already been applied to engineering problems, most of the models are of theoretical nature. The appropriate use of economic theory in engineering needs to address human participation, and experimental economics provides a way for testing the robustness of such theories. They also provide an environment for formulating new theories and testing improved designs. We investigate different properties of combinatorial auction settings using the combinatorial auction platform we have developed. The platform allows single-sided, double-sided, XOR/OR, combinatorial bidding, and short-selling [63]. Of particular interest is the design of a combinatorial auction system for use in allocating links and trunks in bandwidth trading markets.

A Sellers' Bid Double Auction (SeBiDA) mechanism is proposed in [63], which maximizes total surplus and announces payments based on sellers' bids. The announced allocations and prices form a competitive equilibrium, under certain assumptions on bidding format and valuations. We replicate these conditions and also investigate how robust the mechanism is to differing private valuations. This will necessitate the use of different market environments with different valuations for combinations of goods. In particular, we want to replicate a simple bandwidth trading environment, where the bidding is over links (i.e. goods) and number of trunks (i.e. quantity of goods) on each link. Obtaining different lengths of paths (i.e. combinations of links) provides different valuations for users.

We followed a similar valuation structure used in [113], where users were given valuations over combinations of links and trunks. Subjects were provided with valuation charts over different combinations of goods at different quantities.

Sellers

Each seller owns a combination of links and trunks on those links. Each seller has a cost of operation of each item-trunk pair drawn from a uniform discrete distribution between 5 and 15. The cost of each additional trunk within the same link is uniform. Operation costs are only incurred when a link-trunk

Trunks	link A	link B	link C
1	7	5	6
2	14	10	12
3	21	15	18

Table 4.1: Example of Seller Valuations

Trunks	A	B	C	AB	BC	AC	ABC
1	20	12	24	37	26	35	52
2	39	23	27	72	50	68	101
3	58	34	40	107	74	101	150

Table 4.2: Example of Buyer Valuations

combination is provided to a seller. There is no cost and no benefit associated with unsold link-trunk combinations. Table 4.1 presents an example of a seller endowed with 3 trunks on each link.

In the alternative setup, all sellers own 3 trunks of one type of link, and two sets of two sellers own 3 units of each remaining type of link. Sellers can submit multiple bids (asks) with the restriction that bids cannot be combinatorial. Seller bids are loose, meaning that a single bid may be matched with multiple buyer bids. This would be the case when the seller submits a multi-trunk bid (e.g. $12/trunk for 3 trunks of Link A).

Buyers

Buyers begin each round without owning any links and trunks. They are however, each provided with a chart of private valuations over the all possible subsets and trunks that they may obtain. In the benchmark setup, valuations for each subset of items are generated in the following way:

1. Valuation for each item is generated from a discrete uniform distribution between 10 and 20. For example, item A may be valued at 12.

2. Valuation for each subset of two items is generated by adding the valuation for the two items. Then a number from a uniform distribution between 0 and 5 is added. For example, item A is valued at 12, item B is valued at 14, and the bundle AB is valued at 29.

3. Valuation for having all three items is generated by the maximum additive valuation between combinations of two items and the valuation of the remaining object. Then a number from a uniform distribution between 0 and 2 is added.

4. Each additional trunk for each combination is valued at -1 of the previous single item trunk; -2 of the previous double item combination; -3 of the previous all item combination.

Table 4.2 demonstrates an example of a valuation chart of a buyer based on the procedure described above.

In the alternative setup, each buyer values different sets of links that include only one type of link (i.e. A, AB, AC, ABC). This type of link is never the type of link that all sellers own.

Buyers may bid on combinations of items, but are restricted to have an equal number of trunks (quantity) on each link (item). Buyer bids are not loose and need to be completely satisfied to be matched. All buyer bids are XORed together, hence only one of them can match at each round.

The objective of each bidder would be to improve her endowment position through trading. Initially everyone starts with a level of endowment in goods and money. Users are induced to perform well by only rewarding changes from the initial endowment point. In each round of the experiment, subjects accumulate points based on their buyer/seller surplus they generated. At the end of the experiments, $500 was split in proportion to the total surplus generated. If a participant does nothing s/he will receive nothing at the end of the experiment. Negative balances provide subjects with no payoff (except a showup fee).

4.4 Experimental Results

The experiment consisted of a 3-hour experimental session, which was conducted at the end of July 2004 at the xLab facilities at the University of California Berkeley. Subjects were recruited from the graduate programs in electrical engineering, information management and systems, and economics using e-mail postings. Participants were required to be familiar with basic networking and/or auction understanding. There were two sessions of four rounds each. The subjects were instructed on how to bid using the web-based interface and how the system calculates prices and performs matching. Test runs were conducted so that the subjects could get a feel of how the system worked and how information was displayed.

In the first session (rounds 1-4 in the tables), subjects had a 50% chance of being a buyer or a seller in each of four rounds. Each round had an equal number of sellers and buyers. Two sessions of four rounds each were conducted. During the first session, subjects participated in four rounds of the Combinatorial Seller's Bid Double Auctions using the benchmark setup. During the second session (rounds 5-8 in the tables), subjects participated in four rounds using the same auction format but with the alternative setup valuations.

To ensure that both sessions used the same procedures, we adopted a written protocol which we used on both sessions. In all sessions, the participants were seated in a large room, each sitting at a desk with a laptop computer. They were read instructions and given an opportunity to ask questions. Throughout the session, participants would only communicate through the submission of bids.

The submission of bids in the system was monitored through the server platform and once everyone submitted the bids, they were entered into the combinatorial engine which calculated the price for each link and which buyer and seller matched. The market information, namely the prices of the links, would be posted for everyone to see at the conclusion of the round. The participants who matched would also be notified of which link and what quantity they matched on. Before being asked to bid, participants received a handout depending on whether they were a seller or a buyer. At the conclusion of the experiment, the subjects were paid in private with checks according to their performance during the two sessions.

Results

We present the results for both sessions in terms of efficiency, revenues, and bidding behavior. We pool results from each round and present the average behavior of subjects during each session. We also present how the bidding behavior of subjects changed over consecutive rounds.

Efficiency and Revenue

We begin by considering how each auction setup fared in achieving the efficient allocation of the objects. Buyer efficiency is calculated by dividing the value of the objects actually realized by the bidders by the theoretical maximum obtainable. In the benchmark rounds of session 1, the valuations of the buyers were sufficiently high as to be satisfied by the sellers selling all their links and trunks. Efficiency was

Round	1	2	3	4	5	6	7	8
Mean	56.24	37.65	76.31	81.20	63.19	50.00	55.42	71.19
σ	32.82	11.43	31.93	12.99	33.74	57.74	31.75	33.27

Table 4.3: Summary of Buyer Percentage Efficiency in Each Round

Round	1	2	3	4	5	6	7	8
Mean	55.56	41.67	72.22	75.00	63.89	66.67	58.33	72.22
σ	17.35	22.05	4.81	0.00	17.35	28.87	22.05	25.46

Table 4.4: Summary of Seller Percentage Efficiency in Each Round

then calculated by comparison between the actual outcomes and the maximum possible valuations of the buyers.

Table 4.3 shows the associated standard deviation (σ) below each of the mean figures for efficiency generated during each round of each session. During session 2, given the restricted supply on two links, the valuations were compare on the highest possible combination of valuations given the supply.

For the mechanism to be efficient, it should induce participants with high valuations to be the ones who match. Buyers with the highest valuations for each combination of links and trunks can be identified from their valuation tables. Given our mechanism, the prices should be the highest bids of the sellers with the highest operating costs. This is because of the distribution functions used in assigning valuations to buyers and sellers. What we observe is that even in the case of limited supply, the mechanism performs at about 60% buyer efficiency. What we also observe is that during session 1, subjects have performed better with each new round.

On the seller side efficiency was calculated as percent of items sold given their initial endowments. Table 4.4 shows mean percentage efficiency of the mechanism in distributing the sellers' links and trunks[1].

What we observe is the mechanism performs at about 65% seller efficiency when there are no restrictions on supply or demand, and 61% of seller efficiency when there are restrictions on demand and supply. What we also observe is that during session 1, subjects have performed better with each new session; a result reflected in the buyer efficiency discussion.

Bidding

Another important aspect of investigating a mechanism is with regards to patterns of underbidding or overbidding. This means that we need to pay closer attention to the bidding of participants when their type of bidder (buyer or seller) changes between rounds. The shading factor is a measure of how much lower than the actual valuation of a link-trunk combination has been submitted during each round. The average shading factor for each round is shown in Table 4.5.

What we observed was that sellers tended to overbid above their own costs of operation. What we did not observe however, is bidding with regards to the expected price of each good. Since operating costs were randomly drawn from a discrete uniform distribution between 5 and 15, then the expected operating cost is 10. Suppliers who had operating costs of 5 would bid marginally higher, possibly reflecting high risk-aversive behavior. Table 4.6 shows the overbidding behavior of sellers. The impact

[1] In session 2, round 3, a seller sold one more unit than s/he had, for which s/he got penalized. The efficiency reading on the buyer side was not adjusted for that problem, since the difference would be minimal.

Round	1	2	3	4	5	6	7	8
Mean	23.95	28.79	19.30	31.85	27.00	18.00	13.00	23.00
σ	24.42	26.15	8.59	30.06	26.00	9.00	14.00	21.00

Table 4.5: Aggregate Average Percentage Shading Factor Per Round

Round	1	2	3	4	5	6	7	8
Mean	27.51	14.59	23.44	32.40	8.75	14.25	11.33	30.75
σ	35.14	9.97	2.83	35.10	11.81	12.01	11.50	24.96

Table 4.6: Seller Overbidding Percentage Over Costs

of this bidding behavior is also reflected in the ability of the mechanism to assign the competitive equilibrium prices, since the final uniform price per link is the maximum successful seller ask bid value.

Buyers would underbid in most cases as reflected below. Given the way that prices were determined in the mechanism, specifically, by having the highest accepted seller bid dictate the price, some risk-loving buyers bid with prices exceeding their own valuations in expectation that the actual price paid would be lower. Table 4.7 shows the underbidding behavior of buyers.

Our results show that the mechanism does not induce truth-revelation as a VCG auction [113]. Sellers tend to overbid by 26% and buyers underbid by 20%. Comparisons of this mechanism with the alternative mechanism of simultaneous double auctions of each link or the VCG auction format could shed some light on the comparative efficiency and strategic interactions of this auction mechanism.

4.5 Chapter Summary

Experimental economic approaches can be used to aid engineers in the design of mechanisms for allocation of resources which exhibit complimentary value to users. Such environments include bandwidth trading, spectrum auctions, airport slot planning, hospital staff scheduling, utility pricing, etc. In many cases, the theoretical properties of different allocation mechanisms are unknown. Similarly, implementation challenges with respect to such systems can be identified through the use of experimental economic methods. We have briefly investigated some basic properties of the c-SeBiDA mechanism. In our future investigations, we will examine the efficiency of expanding the bidding space, allowing for buyers to submit both XOR and OR bids. We also want to investigate the impact on efficiency, when we allow the combinatorial auction to be multi-round ascending.

Round	1	2	3	4	5	6	7	8
Mean	20.39	42.98	18.08	31.31	44.50	21.50	14.25	16.00
σ	11.08	30.98	10.08	29.59	24.06	5.32	13.78	14.90

Table 4.7: Buyer Underbidding Percentage Over Valuations

Chapter 5

Conclusions and Future Work

We studied competitive equilibrium in combinatorial markets, i.e., markets with indivisible goods and money where the participants have valuations that depend on bundles of goods. We showed through examples that for finite networks, prices that yield socially efficient allocation may not exist. We then obtained sufficient conditions on network topology for competitive equilibrium to exist when utility functions are linear. Namely, when the network is T.U. (Totally Unimodular, i.e., all the routes lie on a spanning tree). The result is really an observation from network flow theory and there are other sufficient conditions available as well. However, it seems to be the first such observation in the networking literature.

The problem of existence of competitive equilibrium with indivisible goods is a hard one. It has been known that, in general, C.E. does not exist in markets with indivisible goods. However, divisible goods are really an approximation for real goods (which are almost always indivisible in real markets) in markets with a large, even an infinite, number of players. One model of large markets with perfect competition is the continuum model. Such models have been proposed for markets with indivisible goods and it has been shown that the core is non-empty, but competitive equilibrium still may not exist. What is missing in the literature is an analysis of the continuum model with indivisible goods and money. We showed that in such a setting, a competitive equilibrium exists. The key here is the Lyapunov-Richter theorem that enables a convexification of the economy.

However, such a result does not hold for countable economies. The main reason is that defining the average of a sequence of correspondences is trickier as the limit may not exist.

Although a mathematical fiction, the continuum model is very useful in showing the existence of enforceable approximate equilibria (when we require that supply exceeds demand) in finite networks. We presented such approximate equilibria, which follow from theorems about convex approximations of non-convex sets.

It is well-known that the set of the competitive allocations is contained in the *core* (the set of Pareto-optimal allocations). However, it is unknown if the two sets are equal. This is an interesting question and part of future work.

We then introduce a combinatorial market mechanism and present three results. The first result concerned the existence of a Nash equilibrium for the combinatorial, Sellers' Bid, Double Auction (c-SeBiDA) with full information. In c-SeBiDA, settlement prices are determined by sellers' bids. We showed that the allocation of c-SeBiDA is efficient. Moreover, truth-telling is a dominant strategy for all players, except the highest matched seller for each item.

The second result concerned the Bayesian-Nash equilibrium of the mechanism under incomplete information. We showed that under the ex post individual rationality constraint, symmetric Bayesian-Nash equilibrium strategies converge to truth-telling for the single item. Thus, the mechanism is asymptotically Bayesian incentive compatible, and hence asymptotically efficient.

The third result concerned the competitive analysis of the c-SeBIDA auction mechanism. We considered the continuum model and showed that within that model, the c-SeBiDA outcome is a competitive equilibrium. This suggests that in the finite setting, the auction outcome is close to efficient.

What we have essentially shown is a combinatorial market mechanism which has zero "price of anarchy". We have been able to deal with indivisibilities and combinatorial bundles. However, the Nash equilibrium results are for special utility functions, namely "max-linear" functions, i.e., linear up to a maximum and then constant. It is worth noting that the mechanism we proposed is a non-VCG mechanism. The only known mechanisms that are efficient and incentive-compatible are of VCG type. However, VCG mechanisms suffer from computational complexity problems and hence there is a drive to find computationally efficient incentive compatible mechanisms. The mechanism we propose also requires a mixed linear integer program to be solved. However, the real-time complexity is much lower than for VCG mechanisms. Our mechanism seems to work quite well with current linear integer program algorithms for reasonably sized problems. Still, for large problems, the mechanism's matching problem is NP-complete. However, it is our guess that there is a structure in the auction problems of communication networks which will enable us to find efficient matching algorithms. This is part of future work.

Bibliography

[1] C.D.Aliprantis and K.C.Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, Springer-Verlag, 1999.

[2] E. J. Anderson, *Linear Programming in Infinite-dimensional Spaces: Theory and Applications*, Wiley, 1987.

[3] K. J. Arrow, "An extension of the basic theorems of classical welfare economics" in *Proc. Second Berkeley Symposium on Mathematical Statistics and Probability*, (J. Neyman, ed.), pp. 507-532, University of California Press, 1951.

[4] K. J. Arrow and G. Debreu, "Existence of an equilibrium for a competitive economy", *Econometrica* 22(3):265-290, 1954.

[5] K.J. Arrow and F.H. Hahn, *General Competitive Analysis*, Holden-Day, 1971.

[6] K. J. Arrow and L. Hurwicz, "On the stability of the competitive equilibrium I", *Econometrica* 26:522-552, 1958.

[7] J.P Aubin, *Mathematical Methods of Game Theory and Economic Theory*, North-Holland, 1982.

[8] Auctions Theory Group, *http://auctions.eecs.berkeley.edu*, EECS Department, University of California, Berkeley, October 2003.

[9] R.J. Aumann, "Markets with a continuum of traders", *Economterica* 32(1):39-50, 1964.

[10] R.J. Aumann, "Integrals of set-valued functions", *J. Mathematical Analysis and Applications* 12:1-12, 1965.

[11] R.J. Aumann, "Existence of competitive equilibria in markets with a continuum of traders", *Econometrica* 34(1):1-17, 1966.

[12] L.M. Ausubel, "An efficient ascending-bid auction for multiple objects", *American Economic Rev.* Working Paper No. 97-06, University of Maryland, August 2002.

[13] J.S. Banks, J. Ledyard and D.P. Porter, "Allocating uncertain and unresponsive resources: An experimental approach", *The RAND Journal of Economics* 20(1):1-25, 1989.

[14] T. Basar and G.J. Olsder, *Dynamic Noncooperative Game Theory*, Academic Press, 1982.

[15] C.Berge, *Topological Spaces*, Dover Books, 1997.

[16] D.Bertsekas, “Auction algorithms for network flow problems: A tutorial introduction”, LIDS Technical Report, MIT, 1992.

[17] S. Bikhchandani and J. W. Mamer, “Competitive equilibrium in an exchange economies with indivisibilties”, *J. of Economic Theory* 74:385-413, 1997.

[18] S. Bikhchandani and J. Ostroy, “Ascending price Vickrey auctions”, to appear *Games and Economic Behavior*, 2005.

[19] H. Bobzin, *Indivisibilities: Microeconomic Theory with respect to Indivisible Goods and Factors*, Physica-Verlag, 1998.

[20] K.C.Border, *Fixed Point Theorems With Applications to Economics and Game Theory*, Cambridge University Press, 1985.

[21] V. Borokov, *Foundations of Game Theory*, 1984.

[22] J. Broome, “Approximate equilibrium in economies with indivisible commodities”, *J. of Economic Theory* 5:224-249, 1972.

[23] M.M. Bykowsky, R.J. Cull and J.O. Ledyard, "Mutually destructive bidding: The FCC auction design problem", *J. of Regulatory Economics* 17(3):205-28, 2000.

[24] K. Chatterjee and W. Samuelson, “Bargaining under incomplete information”, *Operations Research* 31:835-851, 1983.

[25] E. H. Clark, “Multipart pricing of public goods”, *Public Choice* 2:19-33, 1971.

[26] S.H. Clearwater, *Market-Based Control: A Paradigm for Distributed Resource Allocation*, World Scientific, 1996.

[27] A. Cournot, *Researches into the Mathematical Principles of The Theory of Wealth*. English edition (ed. N. Bacon) Macmillan, 1897.

[28] P. Cramton and J.A. Schwartz, "Collusive bidding: Lessons from the FCC spectrum auctions", *J. of Regulatory Economics* 17(3):229-52, 2000.

[29] P.Dasgupta and E.Maskin, “Efficient auctions”, *Quarterly J. of Economics* 95(2):341-388, 2000.

[30] G. Debreu, *Theory of Value*, Yale University Press, 1959.

[31] G. Debreu and H. Scarf, “A limit theorem on the core of an economy”, *International Economic Review* 4(3):235-246, 1963.

[32] G. Demange, D. Gale and M. Sotomayer, “Multi-item auctions”, *J. Political Economy* 94:843-872, 1986.

[33] C. DeMartini, A. Kwasnica, J. Ledyard and D. Porter, “A new and improved design for multi-object iterative auctions”, *Social Science Working Paper* no. 1054, March 1999.

[34] K. Deshmukh, A.Y. Goldberg, J.D. Hartline and A.R. Karlin, "Truthful and competitive double auctions", *Proc. European Symp. on Algorithms*, 2002.

[35] E. Dierker, *Topological Methods in Walrasian Economics*, Lecture Notes in Economics and Mathematical Systems 92, Springer, 1974.

[36] R.J. Edell, N. McKeown and P. Varaiya "Billing users and pricing for TCP", *IEEE J. on Selected Areas in Communications* 13(7):1162-1175, 1995.

[37] F.Y. Edgeworth *Mathematical Psychics*, Kegan Paul Publishers, London, 1881.

[38] R. D. Emmerson, "Optima and market equilibria with indivisible commodities", *J. of Economic Theory* 5:177-188, 1972.

[39] FCC Auctions Website, *http://www.fcc.gov*.

[40] M. Feldman and J.Chuang, "Hidden-action in multi-hop routing", *Proc. Second Workshop on the Economics of Peer-to-Peer Systems*, 2004.

[41] S. Floyd and V. Jacobson, "Random early detection gateways for congestion avoidance", *IEEE/ACM Trans. on Networking* 1(4):397-413, 1993.

[42] C. Frank, *Production Theory and Indivisible Commodities*, Princeton University Press, 1969.

[43] D. Friedman and J. Rust, *The Double Auction Market: Institutions, Theories, and Evidence*, Addison Wesley, 1993.

[44] D. Friedman and S. Shyam, *Experimental Methods: A Primer for Economists*, Cambridge University Press, 1994.

[45] D. Fudenberg and D.K.Levine, *Theory of Learning in Games*, MIT Press, 1998.

[46] D. Fudenberg and J. Tirole, *Game Theory*, MIT Press, 1991.

[47] D. Gale, "The law of supply and demand", *Mathematica Scandinavica* 3:155-169, 1955.

[48] D. Gale, "Equilibrium in a discrete exchange economy with money", *International J. of Game Theory* 13(1):61-64, 1984.

[49] D. Gale and L. S. Shapley, "College admissions and the stability of marriage", *American Math. Monthly* 69:9-15, 1962.

[50] T. Gresik and M. Satterthwaite, "The rate at which market converges to efficiency as the number of traders increases: An asymptotic result for optimal trading mechanism", *J. Economic Theory* 48:304-332, 1989.

[51] T. Groves, "Incentives in teams", *Econometrica* 41:617-631, 1973.

[52] F. Gul and A. Postlewaite, "Asymptotic efficiency in large exchange economies with assymetric information", *Econometrica* 60(6):1273-1292, 1992.

[53] B. Hajek and G. Gopal, "A framework for studying demand in hierarchical networks (preliminary draft)", preprint, March 2004.

[54] B.Hajek and S.Yao, "Strategic buyers in a sum bid game for flat networks", preprint, March 2004.

[55] P.Hall, "On representatives of sets", *J. London Math. Society* 10:26-30, 1935.

[56] Y.D'Halluin, P.A.Forsyth and K.R.Vetzal, "Managing capacity for telecommunications networks under uncertainity", *IEEE/ACM Trans. on Networking* 10(4):579-588, 2002.

[57] J. Harsanyi, "Games with incomplete information played by Bayesian players", Parts I,II, and III, *Management Science* 14:159-182, 320-334, 486-502, 1967-68.

[58] P. C. Henry, "Indivisibilités andans une economie d'echanges", *Econometrica* 38(3):542-558, 1970.

[59] W.Hildebrand, *Core and Equilibria of a Large Economy*, Princeton University Press, 1974.

[60] L. Hurwicz, "On informationally decentralized systems", in McGuire and Radner eds., *Decision and Organization*, North-Holland, 1972.

[61] M.D. Intrilligator, *Mathematical Optimization and Economic Theory*, Prentice-Hall, Inc., 1971.

[62] R. Jain and P. Varaiya, "Combinatorial bandwidth exchange: Mechanism design and analysis", *Communications in Information and Sciences* 4(3):305-324, 2004.

[63] R. Jain and P. Varaiya, "Efficient bandwidth allocation through auctions", submitted to *INFOCOM 2005*, July 2004. Available at `http://www.eecs.berkeley.edu/~ rjain/papers`.

[64] R. Jain and P. Varaiya, "An efficient incentive-compatible combinatorial market mechanism", *Proc. Allerton Conf.*, 2004.

[65] R. Jain and P. Varaiya, "An asymptotically efficient combinatorial market mechanism", submitted to *Mathematics of Operations Research*, November 2004.

[66] R. Jain, A. Dimakis and P. Varaiya, "On the existence of competitive equilibria in bandwidth markets", *Proc. Allerton Conf.*, 2002.

[67] M. Jackson, "Incentive compatibility and competitive allocations", *Economics Lett.* 40:299-302, 1992.

[68] M.O.Jackson, "Mechanism Theory", in *The Encyclopedia of Life Support Systems*, 2000.

[69] M. Jackson and A. Manelli, "Approximately competitive equilibria in large finite economies", *J. Economic Theory* 77(2):354-376, 1997.

[70] P.Jehiel and B.Moldovanu, "Efficient auction design: The case of interdependent values", preprint, 1998.

[71] R. Johari and J. N. Tsitsiklis, "Efficiency loss in a resource allocation game", *Mathematics of Operations Research* 29(3): 407-435, 2004.

[72] J.Kagel, “Auctions: A survey of experimental research”, in Kagel and Roth, eds. *Handbook of Experimental Economics*, Princeton University Press, 1995.

[73] J.H. Kagel and A.E. Roth, eds., *The Handbook of Experimental Economics*, Princeton University Press, 1995.

[74] D. Kahneman and A. Tversky, *Choices, Values, and Frames*, Cambridge University Press, 2000.

[75] S. Kakutani, “A generalization of Brouwer's fixed-point theorem”, *Duke Mathematical J.*, 8:451-459, 1941.

[76] M. Kaneko, “Housing market with indivisibilities”, *J. of Urban Economics* 13:22-50, 1983.

[77] C.Kaskiris, R.Jain, R.Rajagopal and P.Varaiya, “Combinatorial auction design for bandwidth trading: An experimental study”, *Proc. International Conf. on Experiments in Economic Sciences, Kyoto, Japan*, December 2004.

[78] F.P.Kelly, “Charging and rate control for elastic traffic”, *European Trans. on Telecommunications* 8(1):33-37, 1996.

[79] F.P.Kelly, A.K.Maullo and D.K.H.Tan, “Rate control in communication networks: shadow prices, proportional fairness and stability”, *J. of the Operational Research Soc.* 49(1):237-252, 1998.

[80] F.P.Kelly, “Mathematical modelling of the Internet”, in *Mathematics Unlimited: 2001 and Beyond*, Springer-Verlag, 2001.

[81] A. S. Kelso Jr. and V. P. Crawford, “Job matching, coalition formation, and gross substitutes”, *Econometrica* 50:1483-1504, 1982.

[82] C.Kenyon and G.Cheliotis, “Stochastic models for telecom commodity prices”, *Computer Networks* 36:533-555, 2001.

[83] J.O.Kephart, J.E.Hanson and A.R.Greenwald, “Dynamic pricing by software agents”, *Computer Networks* 32(6):731-752, 2000.

[84] J. M. Keynes, *The General Theory of Employment Interest and Money*, Harcourt Brace, 1936.

[85] M. A. Khan and A. Yamakazi, “On the cores of economies with indivisible commodities and a continuum of traders”, *J. of Economic Theory* 24:218-225, 1981.

[86] P. Klemperer, “Auction theory: A guide to the literature”, *J. Economic Surveys* 13(3):227-286, 1999.

[87] P. Klemperer, “Why every economist should learn some auction theory”, *CEPR Discussion Papers* 2572, 2000.

[88] Y.A. Korilis, A.A. Lazar and A. Orda, “Capacity allocation under noncooperative routing”, *IEEE Trans. Auto. Control* 42(3):309-325, March 1997.

[89] E. Koutsoupias and C. Papadimitriou, “Worst-case equilibria”, *Proc. Symp. on Theoretical Aspects of Computer Science* 16:404-413, 1999.

[90] V.Krishna, *Auction Theory*, Academic Press, 2002.

[91] V.Krishna and M.Perry, "Efficient Mechanism Design", preprint, 1998.

[92] R.Kuckrow, "Bandwidth trading: Down but not out", *Business Communications Rev.*, pp.30-34, April, 2002.

[93] H.Kuhn, "The Hungarian method for the assignment problem", *Naval Research Logistics Quarterly* 2(1):83-97, 1955.

[94] R.J.La and V.Anantharam, "Network pricing using a game theoretic approach", *Proc. Conf. on Decision and Control*, 1999.

[95] G. van der Laan, D. Talman and Z. Yang, "Existence and welfare properties of equilibrium in an exchange economy with multiple divisible and indivisible commodities and linear production technologies", *J. Economic Theory* 103:411-428, 2002.

[96] J.O. Ledyard and D.Porter and A. Ragel, "Experiments testing multi object allocation mechanisms", *J. of Economics and Management Strategy* 6(3):639-75, 1997.

[97] S.H.Low, "A duality model of TCP and queue management algorithms", *IEEE/ACM Trans. on Networking* 11(4):525-536, 2003.

[98] S.H.Low and D.E.Lapsley, "Optimization flow control I: Basic algorithm and convergence", *IEEE/ACM Trans. on Networking* 7(6):861-75, 1999.

[99] S.H.Low and P.P.Varaiya, "A new approach to service provisioning in ATM networks", *IEEE/ACM Trans. on Networking* 1(5):547-553, October 1993.

[100] J. Ma, "Competitive equilibrium with indivisibilities", *J. of Economic Theory* 82:458-468, 1998.

[101] J.K. MacKie-Mason and H. Varian, "Pricing congestible network resources", *IEEE J. Selected Areas in Communication* 13(7):1141-1149, 1995.

[102] R.Maheswaran and T.Basar, "Nash equilibrium and decentralized negotiation in auctioning divisible resources", *J. Group Decision and Negotiation* 12:361-395, 2003.

[103] P.Maille and B.Tuffin, "Multi-bid auctions for bandwidth allocation in communication networks", *Proc. Infocom*, 2002.

[104] A. Marshall, "Principles of Economics", 8th ed., MacMillan, London, 1920.

[105] A. Mas-Colell, "Indivisible commodities and general equilibrium theory", *J. of Economic Theory* 16:443-456, 1977.

[106] A. Mas-Colell, *The Theory of General Economic Equilibrium: A Differentiable Approach*, Cambridge University Press, 1985.

[107] A. Mas-Colell and X. Vives, "Implementation in economies with a continuum of agents", *Rev. Economic Studies* 60(3):613-629, 1993.

[108] A. Mas-Collel, M. Whinston and J. Green, *Microeconomic Theory*, Oxford University Press, 1995.

[109] P. McAfee and J. McMillan, “Analyzing the airwaves auction”, *J. of Economic Perspectives* 10:159-175, 1996.

[110] L. McKenzie, “On the existence of general equilibrium for a competitive market”, *Econometrica* 27:54-71, 1959.

[111] L. McKenzie, “The classical theorem on existence of competitive equilibrium”, *Econometrica* 49(4):819-841, 1981.

[112] P. Milgrom “Putting auction theory to work: The simultaneous ascending auction”, *J. Political Economy* 108(2):245-272, 2000.

[113] J.Morgan, “Combinatorial auctions in the information age: An experimental study”, in *Advances in Applied Microeconomics*, Vol. 11, M. Baye, ed., JAI Press, 2002.

[114] R.B.Myerson, “Optimal auction design”, *Mathematics of Operations Research* 6(1):58-73, 1981.

[115] R. B. Myerson and M. A. Satterthwaite, “Efficient mechanisms for bilateral trading”, *J. of Economic Theory* 28:265-281, 1983.

[116] J.Nash, “Noncooperative Games”, *The Annals of Mathematics* 54(2):286-295, 1951.

[117] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, 1944.

[118] H. Nikaido, “On the classical multilateral exchange problem”, *Metroeconomica* 8:135-145, 1956.

[119] M.J. Osbourne and A. Rubinstein, *A Course in Game Theory*, 1994.

[120] C.H.Papadimitriou, “Algorithms, games, and the internet”, *Proc. STOCS*, 2001.

[121] V. Pareto, *Manuel d’Économie Politique*, Girard & Briere, Paris, 1909.

[122] D. Parkes, *Iterative Combinatorial Auctions: Achieving Economic and Computational Efficiency*, PhD Thesis, University of Pennsylvania, 2001.

[123] C.R. Plott, "Laboratory experimental testbeds: Application to the PCS auction", *J. of Economics and Management Strategy* 6(3):605-38, 1997.

[124] L. Pontryagin, V. Boltyanskii, R. Gamkrelidze and E. Mischenko, *The Mathematical Theory of Optimal Processes*, John Wiley, New York, 1962.

[125] D. Porter, S. Rassenti, A. Roopnarinen and V.L. Smith, Vernon, "Combinatorial auction design", *Proc. of the National Academy of Sciences* 100(19):153-157, 2003.

[126] D. Porter, D.P. Torma and J.O. Ledyard, J.A. Swanson and M. Olson, "The first use of a combined-value auction for transportation services", *Interfaces* 32(5):4-12, 2002.

[127] J. E. Quintero, "Combinatorial electricity auctions", preprint, 2000.

[128] M. Quinzii, "Core and competitive equilibria with indivisibilities", *International J. of Game Theory* 13(1):41-60, 1984.

[129] S. J. Rassenti, V. L. Smith and R. L. Bulfin "A combinatorial auction mechanism for airport time slot allocation", *The Bell Journal of Economics* 13(2):402-417, 1982.

[130] R. Radner, "Competitive equilibrium under uncertainty", *Rev. Economic Studies* 36:31-58, 1968.

[131] E. Rasmusen, *Games and Information*, Basil Blackwell Inc., 1989.

[132] D.Roberts and A.Postlewaite, "The incentives for price-taking behavior in large economies", *Econometrica* 44(1):115-127, 1976.

[133] R.T.Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

[134] A.E. Roth and E. Peranson, "The redesign of the matching market for American physicians: Some engineering aspects of economic design", *American Economic Review* 89(4):748-80, 1999.

[135] A.E. Roth, and M. Sotomayor, *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*, Cambridge University Press, 1990.

[136] T.Roughgarden and E.Tardos, "How bad is selfish routing?", *J. of the ACM* 49(2):236-259, 2002.

[137] A. Ronen, *Solving Optimization Problems Among Selfish Agents*, PhD Thesis, Hebrew University, 2000.

[138] N. Rosen, "Existence and uniqueness of equilibrium points for concave N-person games", Econometrica 33(3):520-534, 1965

[139] H. Royden, *Real Analysis*, Third ed., Prentice-Hall, Inc., 1988.

[140] A. Rustichini, M. Satterthwaite and S. Williams, "Convergence to efficiency in a simple market with incomplete information", *Econometrica* 62(5):1041-1063, 1994.

[141] T. Sandholm, "Algorithm for optimal winner determination in combinatorial auctions", *Artificial Intelligence* 135:1-54, 2002.

[142] T. Sandholm, "Emediator: A next generation electronic commerce server", *Proc. Fourth International Conference on Autonomous Agents*, ACM Press, 341-348, 2000.

[143] S.Sanghavi and B.Hajek, "Optimal allocation of a divisible good to strategic buyers", preprint, March 2004.

[144] M. Satterthwaite and S. Williams, "Bilateral trade with the sealed bid k-double auction: Existence and efficiency", *J. Economic Theory* 48:107-133, 1989.

[145] M. Satterthwaite and S. Williams, "The rate of convergence to efficiency in the buyer's bid double auction as the market becomes large", *Rev. Economic Studies* 56:477-498, 1989.

[146] M. Satterthwaite and S. Williams, "The optimality of a simple market mechanism", *Econometrica* 70(5):1841-1863, 2002.

[147] H. Scarf, "An analysis of markets with a large number of participants", *Recent Advances in Game Theory*, Princeton University Press, 127-155, 1962.

[148] H. Schelhorn, "A reverse convex programming formulation of a combinatorial auction", forthcoming in *Reverse Convex Optimzation: Theory and Applications*, Eds. Moshirvaziri, Amounegar and Jacobsen, Kluwer Academic publishers, 2003.

[149] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley-Interscience, 1986.

[150] A. K. Sen, *Collective Choice and Social Welfare*, Holden-Day, 1970.

[151] L. Shapley and H. Scarf, "On cores and indivisibility", *J. of Mathematical Economics* 1:23-37, 1974.

[152] L. S. Shapley and M. Shubik, "The assignment game I: the core", *International J. of Game Theory* 1:111-130, 1972.

[153] J. Shu and P. Varaiya, "Pricing network services", *Proc. Infocom 2003*, vol. 2, pp. 1221-1230, San Francisco, CA, March 2003.

[154] V. Smith, "An experimental study of competitive market behavior", *J. of Political Economy*, 70:111-137, 1962.

[155] V. Smith, "Experimental auction markets and the Walrasian hypothesis", *J. of Political Economy* 73:387-393, 1965.

[156] A. Smith, *An Inquiry into the Nature and Causes of the Wealth of Nations*, 1776. Reprinted, Liberty Classics, 1981.

[157] R. Srikant, *The Mathematics of Internet Congestion Control*, Birkhauser, 2004.

[158] R. Starr, "Quasi-equilibria in markets with non-convex preferences", *Econometrica* 37:25-38, 1969.

[159] L.-G. Svensson, "Competitive equilibria with indivisible goods", *J. of Economics* 44:373-386, 1972.

[160] P.P. Varaiya, *Lecture Notes on Optimization*, Nostrand-Holland, 1971.

[161] H. R. Varian, *Microeconomic Analysis*, Norton, NY, 1992.

[162] W. Vickrey, "Counterspeculation, auctions, and sealed tenders", *J. Finance* 16:8-37, 1961.

[163] K. Vind, "Edgeworth-Allocations in an exchange economy with many traders", *International Economic Review* 59(2):165-177, 1964.

[164] S. de Vries and R. Vohra, "Combinatorial auctions: A survey", *Informs J. Computing* 15(3):284-309, 2003.

[165] A. Wald "On some systems of equations in mathematical economics", *Econometrica* 19(4):368-403, (Translation of "Ueber einige Gleichungssysteme der mathematischen Oekonomie", *Zeitscrift für Nationalökonomie* 7(5):637-670, 1951.)

[166] L. Walras, *Elements of Pure Economics, or The Theory of Social Wealth*, 1874, English edition (ed. William Jaff), Reprinted by Orion Editions, Philadelphia, PA, 1984.

[167] S. Williams, "Existence and convergence to equilibria in the buyer's bid double auction", *Rev. Economic Studies* 58:351-374, 1991.

[168] R. Wilson, "Incentive efficiency of double auctions", *Econometrica* 53(5):1101-1115, 1985.

[169] P.R. Wurman and M.P. Wellman, "Akba: A progressive, anonymous-price combinatorial auction", *Proc. 2nd ACM Conf. on Electronic Commerce*, ACM Press, 21-29, 2000.

[170] P.R. Wurman, M.P. Wellman and W.E. Walsh, "The Michigan internet auctionbot: A configurable auction server for human and software agents", *Proc. Second International Conf. on Autonomous Agents*, ACM Press, 301-308, 1998.

[171] P. R. Wurman, M. P. Wellman and W. E. Walsh, "A parameterization of the auction design space", *Games and Economic Behavior* 35:304-338, 2001.

[172] M. Xia, J. Stallaert and A.B. Whinston, "Solving the combinatorial double auction problem", *European J. of Operational Research*, 2004.

[173] H.Yaiche, R.R.Mazumdar and C.Rosenberg, "A game theoretic framework for bandwidth allocation and pricing in broadband networks", *IEEE/ACM Trans. on Networking* 8(5):667-678, 2000.

[174] S. Yang and B. Hajek, "An efficient mechanism for allocation of a divisible good (preliminary draft)", November 2004.

[175] Y. Yamamoto, "Competitive equilibria in the market with indivisibility", in *The Computation and Modelling of Economic Equilibria*, eds. A. J. J. Talman and G. van der Laan, North-Holland, 1987.

[176] K. Yoon, "The modified Vickrey double auction", *J. Economic Theory* 101:572-584, 2001.

[177] E. Zacharias and S. R. Williams, "Ex post efficiency in the buyer's bid double auction when demand can be arbitrarily larger than supply", *J. Economic Theory* 97:175-190, 2001.