

Wavelets and Affine Distributions

A Time-Frequency Perspective

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OUTLINE

- The notion of time-frequency analysis
- Linear and quadratic time-frequency analysis
- Short-time Fourier transform and wavelet transform; spectrogram and scalogram
- Constant-bandwidth analysis vs. constant-Q analysis
- The affine class
- Affine time-frequency smoothing
- Hyperbolic time-frequency localization

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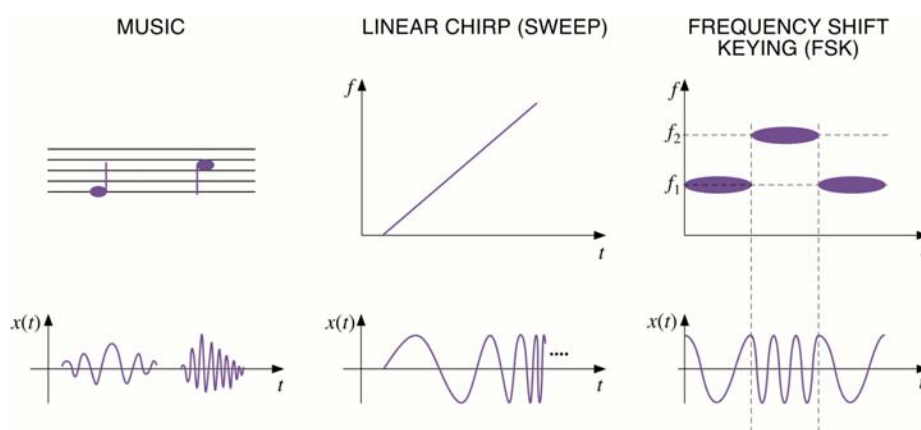
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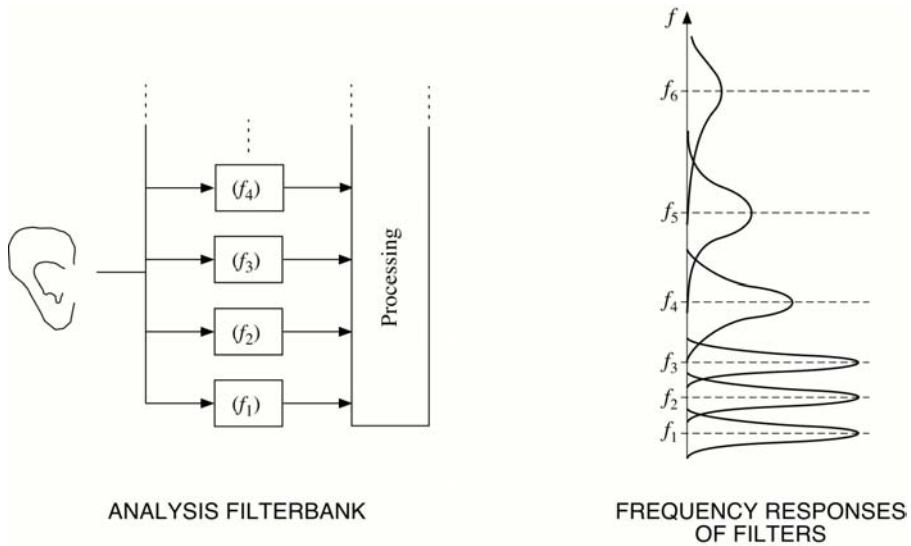
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The notion of time-frequency (TF) analysis



Auditory perception as TF analysis

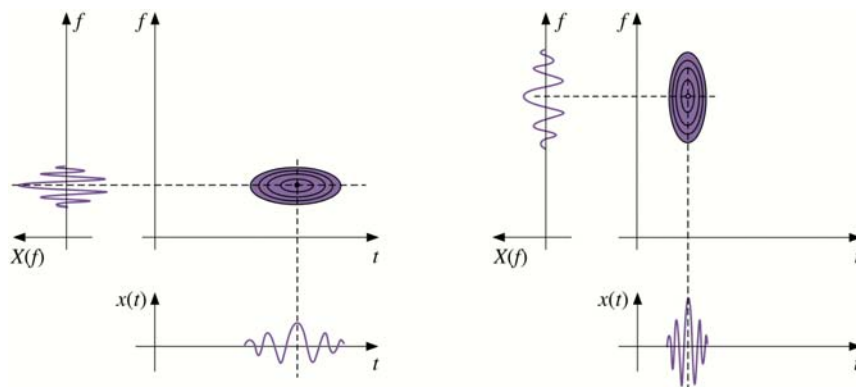


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The TF plane

- Visualize time-frequency location/concentration of signal $x(t)$:



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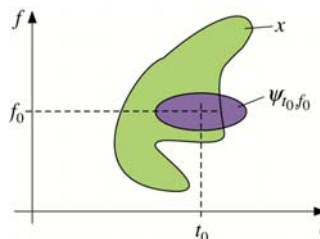
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Linear TF analysis

- **TF analysis:** Measure contribution of TF point (t_0, f_0) to signal $x(t)$
- **General approach:** Inner product of $x(t)$ with “test signal” or “sounding signal” $\psi_{t_0, f_0}(t)$ located about (t_0, f_0) :

$$\text{LTFR}_x(t_0, f_0) := \langle x, \psi_{t_0, f_0} \rangle = \int_{-\infty}^{\infty} x(t) \psi_{t_0, f_0}^*(t) dt$$

LTFR = Linear TF Representation



Linear TF synthesis

- **TF synthesis (inversion of LTFR):** Recover (“synthesize”) signal $x(t)$ from $LTFR_x(t_0, f_0)$

- **General approach:**

$$x(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} LTFR_x(t_0, f_0) \tilde{\psi}_{t_0, f_0}(t) dt_0 df_0$$

$x(t)$ is represented as superposition of TF localized signal components, weighted by “TF coefficient function” $LTFR_x(t_0, f_0)$

- **Problem:** How to construct test (analysis) functions $\psi_{t_0, f_0}(t)$ and synthesis functions $\tilde{\psi}_{t_0, f_0}(t)$?

Quadratic TF analysis

- **TF analysis:** Measure “energy contribution” of TF point (t_0, f_0) to signal $x(t)$

- **Simple approach:**

$$\begin{aligned} QTFR_x(t_0, f_0) &:= |LTFR_x(t_0, f_0)|^2 = |\langle x, \psi_{t_0, f_0} \rangle|^2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t_1) x^*(t_2) \psi_{t_0, f_0}^*(t_1) \psi_{t_0, f_0}(t_2) dt_1 dt_2 \end{aligned}$$

QTFR = Quadratic TF Representation

- Want QTFR to distribute signal energy E_x over TF plane:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} QTFR_x(t, f) dt df = E_x \quad \text{“TF energy distribution”}$$

- **Problem:** How to construct test (analysis) functions $\psi_{t_0, f_0}(t)$?

Construction of analysis/synthesis functions

- **Problem:** Construct family of analysis functions $\{\psi_{t_0, f_0}(t)\}$ such that $\psi_{t_0, f_0}(t)$ is localized about TF point (t_0, f_0)
- **Systematic approach:** $\psi_{t_0, f_0}(t)$ derived from “prototype function” $\psi(t)$ via unitary “TF displacement operator” U_{t_0, f_0} :

$$\psi_{t_0, f_0}(t) := (U_{t_0, f_0} \psi)(t)$$

- Same for synthesis functions $\{\tilde{\psi}_{t_0, f_0}(t)\}$:

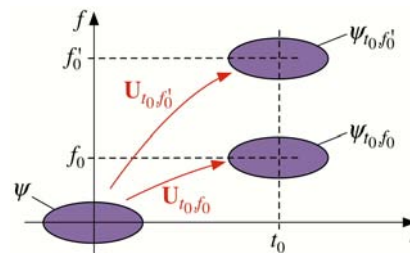
$$\tilde{\psi}_{t_0, f_0}(t) := (U_{t_0, f_0} \tilde{\psi})(t)$$

- **Two classical definitions of U_{t_0, f_0} :**
 - TF shift
 - TF scaling (compression/dilatation) + time shift

Two classical definitions of operator U

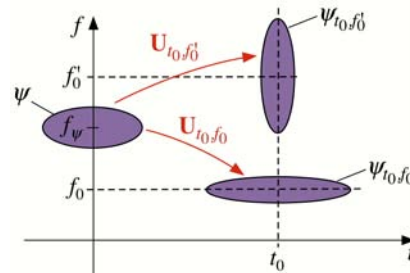
- **TF shift:**

$$\begin{aligned} \psi_{t_0, f_0}(t) &= (U_{t_0, f_0} \psi)(t) \\ &= \psi(t - t_0) e^{j2\pi f_0 t} \end{aligned}$$



- **TF scaling + time shift:**

$$\begin{aligned} \psi_{t_0, f_0}(t) &= (U_{t_0, f_0} \psi)(t) \\ &= \sqrt{\frac{f_0}{f_\psi}} \psi\left(\frac{f_0}{f_\psi}(t - t_0)\right) \\ &= \frac{1}{\sqrt{a}} \psi\left(\frac{t - t_0}{a}\right) \Big|_{a=f_\psi/f_0} \end{aligned}$$



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Short-Time Fourier Transform (STFT)

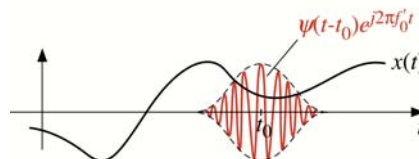
- Recall TF shift:

$$\psi_{t_0, f_0}(t) = (\mathbf{U}_{t_0, f_0} \psi)(t) = \psi(t - t_0) e^{j2\pi f_0 t}$$

- \Rightarrow LTFR = **STFT**:

$$\text{STFT}_x(t_0, f_0) = \langle x, \mathbf{U}_{t_0, f_0} \psi \rangle = \int_{-\infty}^{\infty} x(t) \psi^*(t - t_0) e^{-j2\pi f_0 t} dt$$

STFT = FT of local (windowed) segment of $x(t)$:



STFT signal synthesis

- Recall STFT analysis:

$$\text{STFT}_x(t_0, f_0) = \langle x, \mathbf{U}_{t_0, f_0} \psi \rangle = \int_{-\infty}^{\infty} x(t) \psi^*(t-t_0) e^{-j2\pi f_0 t} dt$$

- STFT signal synthesis:

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{STFT}_x(t_0, f_0) (\mathbf{U}_{t_0, f_0} \tilde{\psi})(t) dt_0 df_0 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{STFT}_x(t_0, f_0) \tilde{\psi}(t-t_0) e^{j2\pi f_0 t} dt_0 df_0 \end{aligned}$$

$x(t)$ is weighted superposition of TF shifted versions of $\tilde{\psi}(t)$

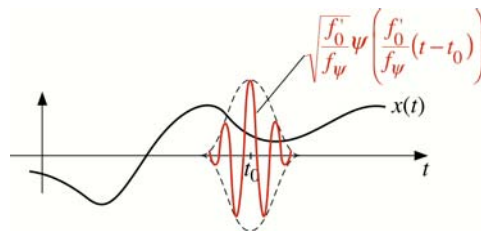
Wavelet Transform (WT)

- Recall TF scaling + time shift:

$$\psi_{t_0, f_0}(t) = (\mathbf{U}_{t_0, f_0} \psi)(t) = \sqrt{\frac{f_0}{f_\psi}} \psi\left(\frac{f_0}{f_\psi}(t-t_0)\right)$$

- \Rightarrow LTFR = WT:

$$\text{WT}_x(t_0, f_0) = \langle x, \mathbf{U}_{t_0, f_0} \psi \rangle = \int_{-\infty}^{\infty} x(t) \sqrt{\frac{f_0}{f_\psi}} \psi^*\left(\frac{f_0}{f_\psi}(t-t_0)\right) dt$$



WT signal synthesis

- Recall WT analysis:

$$\text{WT}_x(t_0, f_0) = \langle x, \mathbf{U}_{t_0, f_0} \psi \rangle = \int_{-\infty}^{\infty} x(t) \sqrt{\frac{f_0}{f_\psi}} \psi^* \left(\frac{f_0}{f_\psi} (t - t_0) \right) dt$$

- WT signal synthesis:

$$\begin{aligned} x(t) &= \int_0^{\infty} \int_{-\infty}^{\infty} \text{WT}_x(t_0, f_0) (\mathbf{U}_{t_0, f_0} \tilde{\psi})(t) dt_0 df_0 \\ &= \int_0^{\infty} \int_{-\infty}^{\infty} \text{WT}_x(t_0, f_0) \sqrt{\frac{f_0}{f_\psi}} \tilde{\psi} \left(\frac{f_0}{f_\psi} (t - t_0) \right) dt_0 df_0 \end{aligned}$$

$x(t)$ is weighted superposition of TF scaled and time shifted versions of $\tilde{\psi}(t)$

Spectrogram and scalogram

- Recall LTFR \rightarrow QTFR:

$$\text{QTFR}_x(t_0, f_0) = |\text{LTFR}_x(t_0, f_0)|^2 = |\langle x, \mathbf{U}_{t_0, f_0} \psi \rangle|^2$$

- STFT \rightarrow spectrogram:

$$\text{SPEC}_x(t_0, f_0) := |\text{STFT}_x(t_0, f_0)|^2 = \left| \int_{-\infty}^{\infty} x(t) \psi^*(t - t_0) e^{-j2\pi f_0 t} dt \right|^2$$

- WT \rightarrow scalogram:

$$\text{SCAL}_x(t_0, f_0) := |\text{WT}_x(t_0, f_0)|^2 = \left| \int_{-\infty}^{\infty} x(t) \sqrt{\frac{f_0}{f_\psi}} \psi^* \left(\frac{f_0}{f_\psi} (t - t_0) \right) dt \right|^2$$

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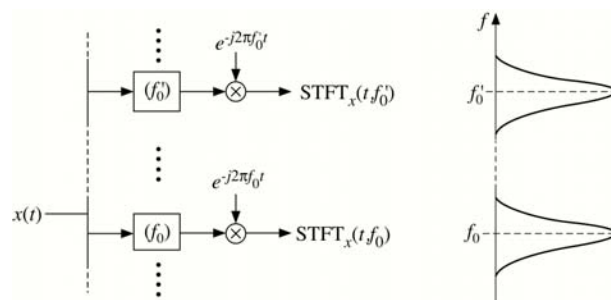
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STFT and constant-BW filterbank: analysis

- STFT analysis as convolution:

$$\begin{aligned} \text{STFT}_x(t_0, f_0) &= \int_{-\infty}^{\infty} x(t) \psi^*(t-t_0) e^{-j2\pi f_0 t} dt \\ &= [x(t) * \psi^*(-t) e^{j2\pi f_0 t}] \cdot e^{-j2\pi f_0 t} \Big|_{t=t_0} \end{aligned}$$

- \Rightarrow Filterbank interpretation/implementation:



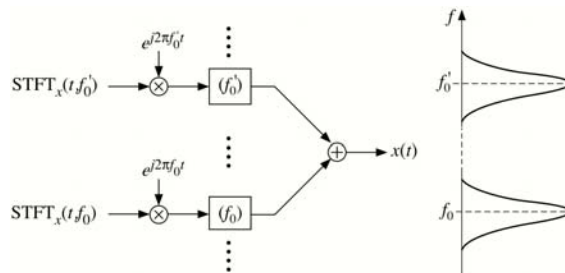
STFT and constant-BW filterbank: synthesis

- STFT synthesis as convolution:

$$x(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{STFT}_x(t_0, f_0) \tilde{\psi}(t-t_0) e^{j2\pi f_0 t} dt_0 df_0$$

$$= \int_{-\infty}^{\infty} \left[\text{STFT}_x(t_0, f_0) e^{j2\pi f_0 t_0} * \tilde{\psi}(t-t_0) e^{j2\pi f_0 t_0} \right]_{t_0=t} df_0$$

- \Rightarrow Filterbank interpretation/implementation:

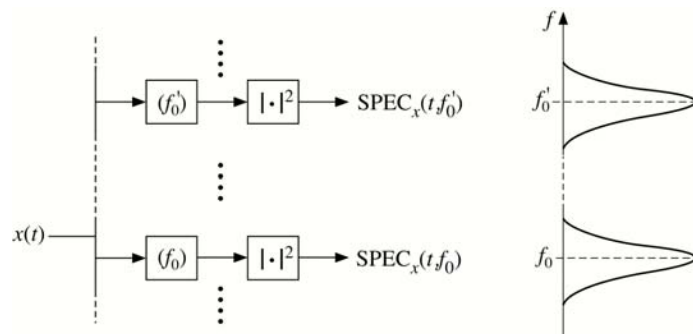


Spectrogram analysis as constant-BW filterbank

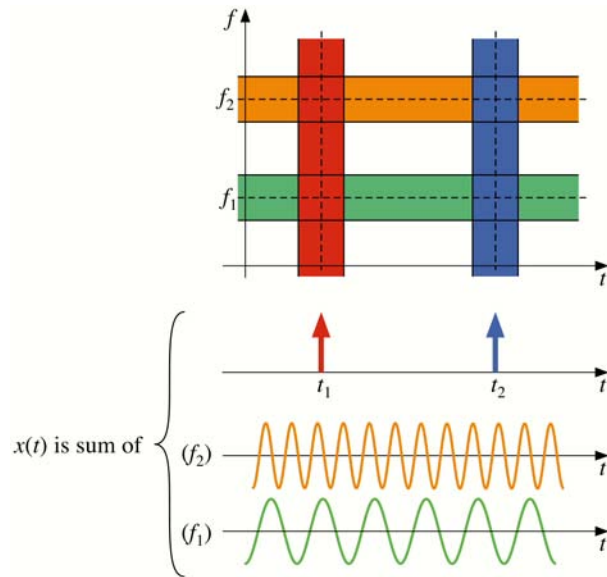
- Spectrogram analysis as convolution:

$$\text{SPEC}_x(t_0, f_0) = |\text{STFT}_x(t_0, f_0)|^2 = \left| [x(t) * \psi^*(-t) e^{j2\pi f_0 t}]_{t=t_0} \right|^2$$

- \Rightarrow Filterbank interpretation/implementation:



STFT / spectrogram: example



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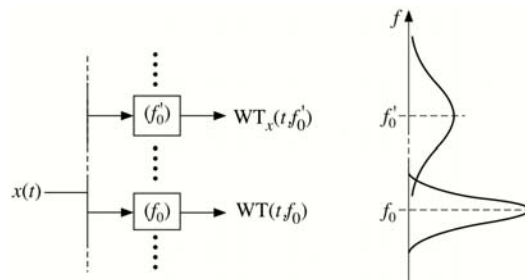
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WT and constant-Q filterbank: analysis

- WT analysis as convolution:

$$\begin{aligned} \text{WT}_x(t_0, f_0) &= \int_{-\infty}^{\infty} x(t) \sqrt{\frac{f_0}{f_\psi}} \psi^* \left(\frac{f_0}{f_\psi} (t - t_0) \right) dt \\ &= \left[x(t) * \sqrt{\frac{f_0}{f_\psi}} \psi^* \left(-\frac{f_0}{f_\psi} t \right) \right]_{t=t_0} \end{aligned}$$

- \Rightarrow Filterbank interpretation/implementation:



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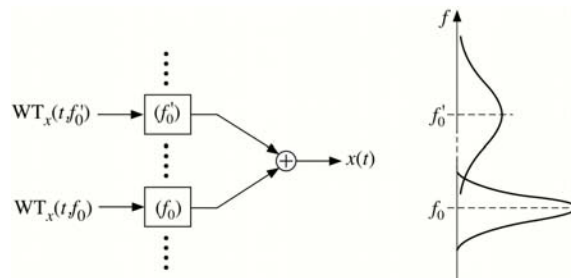
WT and constant-Q filterbank: synthesis

- WT synthesis as convolution:

$$x(t) = \int_0^{\infty} \int_{-\infty}^{\infty} \text{WT}_x(t_0, f_0) \sqrt{\frac{f_0}{f_\psi}} \tilde{\psi}\left(\frac{f_0}{f_\psi}(t-t_0)\right) dt_0 df_0$$

$$= \int_0^{\infty} \left[\text{WT}_x(t_0, f_0) * \sqrt{\frac{f_0}{f_\psi}} \tilde{\psi}\left(\frac{f_0}{f_\psi} t_0\right) \right]_{t_0=t} df_0$$

- \Rightarrow Filterbank interpretation/implementation:

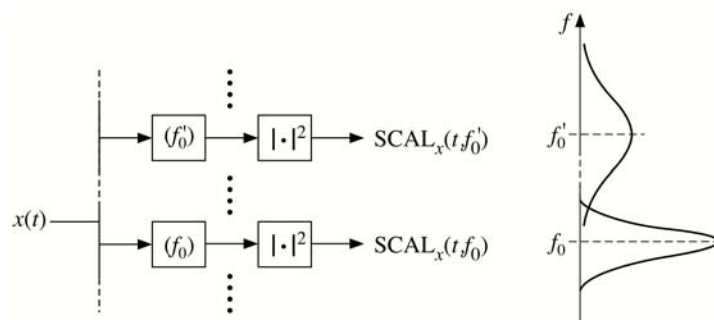


Scalogram analysis as constant-Q filterbank

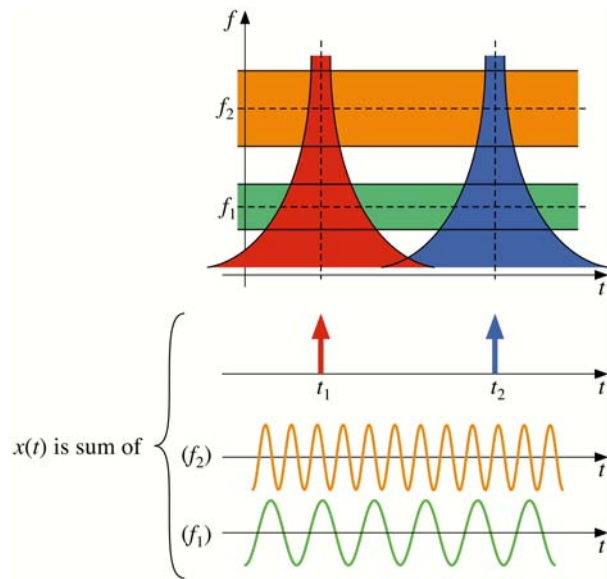
- Scalogram analysis as convolution:

$$\text{SCAL}_x(t_0, f_0) = |\text{WT}_x(t_0, f_0)|^2 = \left| \left[x(t) * \sqrt{\frac{f_0}{f_\psi}} \psi^*\left(-\frac{f_0}{f_\psi} t\right) \right]_{t=t_0} \right|^2$$

- \Rightarrow Filterbank interpretation/implementation:



WT / scalogram: example

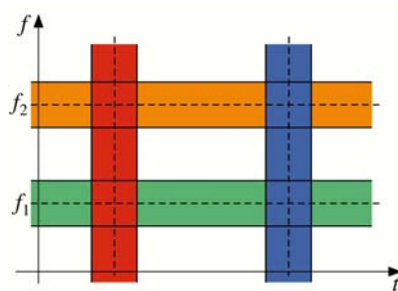


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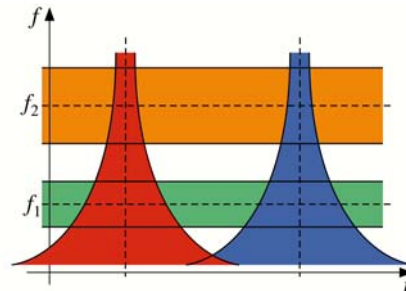
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STFT / spectrogram vs. WT / scalogram

STFT / spectrogram



WT / scalogram



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Good-bye and hello

- Good-bye to:
 - STFT
 - spectrogram
 - constant-BW analysis
- Hello to:
 - affine class of QTFRs
 - Wigner distribution and Bertrand distribution
 - hyperbolic TF localization

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Axiomatic (covariance-based) definition of WT

- Generic LTFR expression:

$$\text{LTFR}_x(t, f) = \int_{-\infty}^{\infty} x(t') K(t'; t, f) dt'$$

- **Covariance** of LTFR to TF scalings + time shifts:

$$y(t) = \frac{1}{\sqrt{a}} x\left(\frac{t-\tau}{a}\right) \leftrightarrow Y(f) = \sqrt{a} X(af) e^{-j2\pi\tau f}$$

$$\Rightarrow \text{LTFR}_y(t, f) = \text{LTFR}_x\left(\frac{t-\tau}{a}, af\right)$$

- Can show that **covariant LTFRs are given by WT**

$$\text{WT}_x(t, f) = \int_{-\infty}^{\infty} x(t') \sqrt{\frac{f}{f_\psi}} \psi^*\left(\frac{f}{f_\psi}(t'-t)\right) dt' = \sqrt{f} \int_{-\infty}^{\infty} x(t') \phi(f(t'-t)) dt'$$

Axiomatic (covariance-based) definition of the affine class of QTFRs

- Generic QTFR expression:

$$\text{QTFR}_x(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t_1) x^*(t_2) K(t_1, t_2; t, f) dt_1 dt_2$$

- **Covariance** of QTFR to TF scalings + time shifts:

$$y(t) = \frac{1}{\sqrt{a}} x\left(\frac{t-\tau}{a}\right) \leftrightarrow Y(f) = \sqrt{a} X(af) e^{-j2\pi\tau f}$$

$$\Rightarrow \text{QTFR}_y(t, f) = \text{QTFR}_x\left(\frac{t-\tau}{a}, af\right)$$

- Can show that **covariant QTFRs are given by**

$$\text{AC}_x(t, f) = f \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t_1) x^*(t_2) \phi(f(t_1-t), f(t_2-t)) dt_1 dt_2$$

AC = Affine Class

The affine class of QTFRs

- Affine class of QTFRs:

$$AC_x(t, f) = f \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t_1) x^*(t_2) \phi(f(t_1-t), f(t_2-t)) dt_1 dt_2$$

- 2-D “kernel” $\phi(\alpha_1, \alpha_2)$ specifies QTFR of the AC
- Scalogram is a member of the AC; its kernel is *separable*:

$$\phi(\alpha_1, \alpha_2) = \frac{1}{f_\psi} \psi^*\left(\frac{\alpha_1}{f_\psi}\right) \psi\left(\frac{\alpha_2}{f_\psi}\right)$$

- Expression of AC QTFRs in terms of signal's FT:

$$AC_x(t, f) = \frac{1}{f} \int_0^{\infty} \int_0^{\infty} X(f_1) X^*(f_2) \Phi\left(\frac{f_1}{f}, \frac{f_2}{f}\right) e^{j2\pi(f_1-f_2)t} df_1 df_2$$

Affine class and affine group

- TF scaling + time shift:

$$(\mathbf{U}_{a,\tau} x)(t) = \frac{1}{\sqrt{a}} x\left(\frac{t-\tau}{a}\right) = \sqrt{\beta} x(\beta t + \gamma) =: (\tilde{\mathbf{U}}_{\beta,\gamma} x)(t)$$

- Affine time transformation $t \rightarrow \beta t + \gamma$ (“clock change”)
- Composition of clock changes is another clock change:

$$\tilde{\mathbf{U}}_{\beta_2,\gamma_2} \tilde{\mathbf{U}}_{\beta_1,\gamma_1} = \tilde{\mathbf{U}}_{\beta_1\beta_2,\gamma_1+\beta_1\gamma_2}$$

- $\Rightarrow \tilde{\mathbf{U}}_{\beta,\gamma}$ is unitary representation of the **affine group**:

- Set: $(\beta, \gamma) \in \mathbb{R}^+ \times \mathbb{R}$
- Group operation: $(\beta_1, \gamma_1) \circ (\beta_2, \gamma_2) = (\beta_1\beta_2, \gamma_1 + \beta_1\gamma_2)$
- Neutral element: $(\beta_0, \gamma_0) = (1, 0)$

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The Wigner-Ville Distribution (WVD)

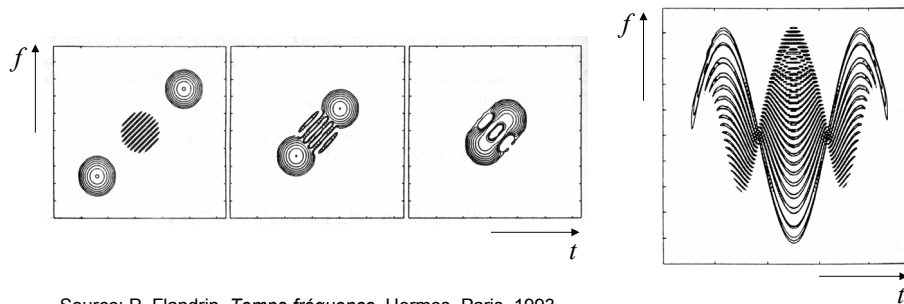
- Prominent member of the AC: the WVD

$$\text{WVD}_x(t, f) = \int_{-\infty}^{\infty} x\left(t + \frac{\tau}{2}\right) x^*\left(t - \frac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau$$

- Properties of the WVD:
 - Covariant to TF scaling and time shift (of course)
 - Covariant to frequency shift \Rightarrow **not constant-Q**
 - Real for any (real or complex) signal $x(t)$
 - Marginal properties: e.g., $\int_{-\infty}^{\infty} \text{WVD}_x(t, f) dt = |X(f)|^2$
 - Localization properties: e.g., $\text{WVD}_x(t, f) = \delta(f - f_0)$ for $x(t) = e^{j2\pi f_0 t}$
 - Many more...

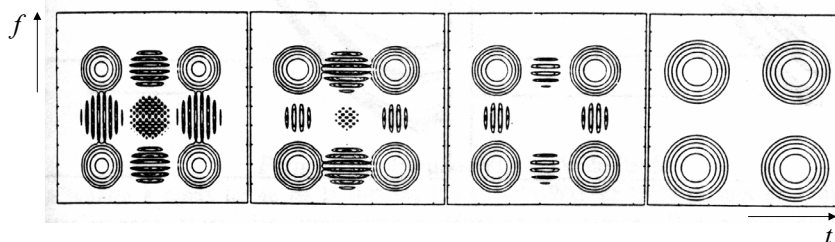
Interference terms in the WVD

$$\begin{aligned} \text{WVD}_{x_1+x_2}(t, f) &= \text{WVD}_{x_1}(t, f) + \text{WVD}_{x_2}(t, f) \\ &\quad + \underbrace{2 \operatorname{Re}\{\text{WVD}_{x_1, x_2}(t, f)\}}_{\text{Interference/cross term}} \end{aligned}$$



Source: P. Flandrin, *Temps-fréquence*. Hermes, Paris, 1993

Constant-BW smoothing of the WVD



Smaller/less interference terms
 →
 Poorer TF resolution

Source: P. Flandrin, *Temps-fréquence*. Hermes, Paris, 1993

AC expression in terms of WVD

- Any QTFR of the AC can be expressed in terms of the WVD:

$$AC_x(t, f) = \int_0^{\infty} \int_{-\infty}^{\infty} WVD_x(t', f') \sigma\left(f(t'-t), \frac{f'}{f}\right) dt' df'$$

where $\sigma(\alpha, \beta)$ is related to $\phi(\alpha_1, \alpha_2)$ and $\Phi(\beta_1, \beta_2)$ by FTs

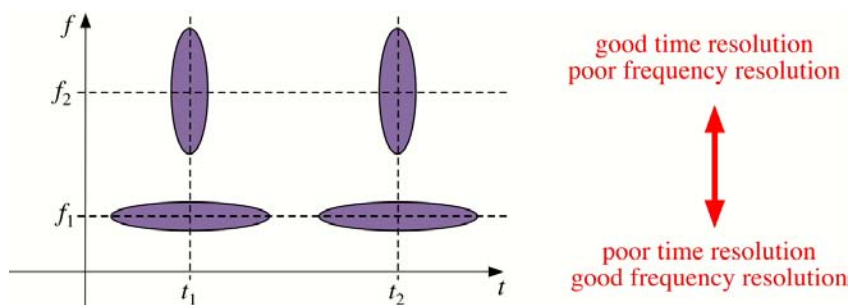
- If $\sigma(\alpha, \beta)$ is a *smooth* function, then $AC_x(t, f)$ is a **smoothed version** of $WVD_x(t, f)$
- Smoothing causes...**
 - smaller/less interference terms
 - poorer TF resolution

Affine (constant-Q) smoothing, different from constant-BW smoothing shown on previous slide!

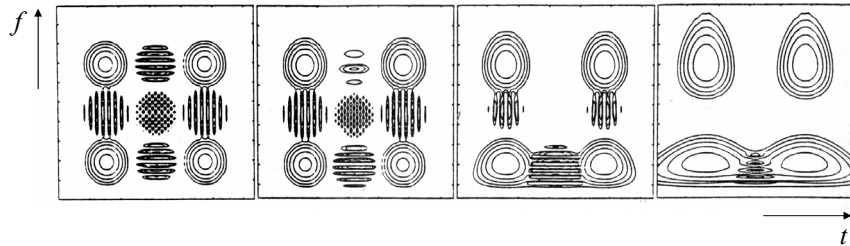
Affine (constant-Q) smoothing of the WVD

- Recall: $AC_x(t, f) = \int_0^{\infty} \int_{-\infty}^{\infty} WVD_x(t', f') \underbrace{\sigma\left(f(t'-t), \frac{f'}{f}\right)}_{\text{Smoothing function}} dt' df'$

- Smoothing function $\sigma\left(f(t'-t), \frac{f'}{f}\right)$ at various TF positions:



Affine smoothing: example



Smaller/less interference terms

Poorer TF resolution

Source: P. Flandrin, *Temps-fréquence*. Hermes, Paris, 1993

Scalogram as smoothed WVD

- Recall scalogram:

$$\text{SCAL}_x(t, f) := |\text{WT}_x(t, f)|^2 = \left| \int_{-\infty}^{\infty} x(t') \sqrt{\frac{f}{f_\psi}} \psi^* \left(\frac{f}{f_\psi} (t' - t) \right) dt' \right|^2$$

- Expression of scalogram as smoothed WVD:

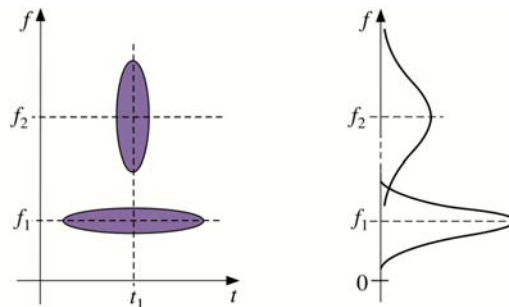
$$\text{SCAL}_x(t, f) = \int_0^{\infty} \int_{-\infty}^{\infty} \text{WVD}_x(t', f') \underbrace{\text{WVD}_\psi \left(\frac{f}{f_\psi} (t' - t), \frac{f'}{f/f_\psi} \right)}_{\text{Smoothing function is WVD of wavelet:}} dt' df'$$

$$\sigma(\alpha, \beta) = \text{WVD}_\psi \left(\frac{\alpha}{f_\psi}, f_\psi \beta \right)$$

Affine WVD smoothing and constant-Q analysis

- Scalogram as smoothed WVD:

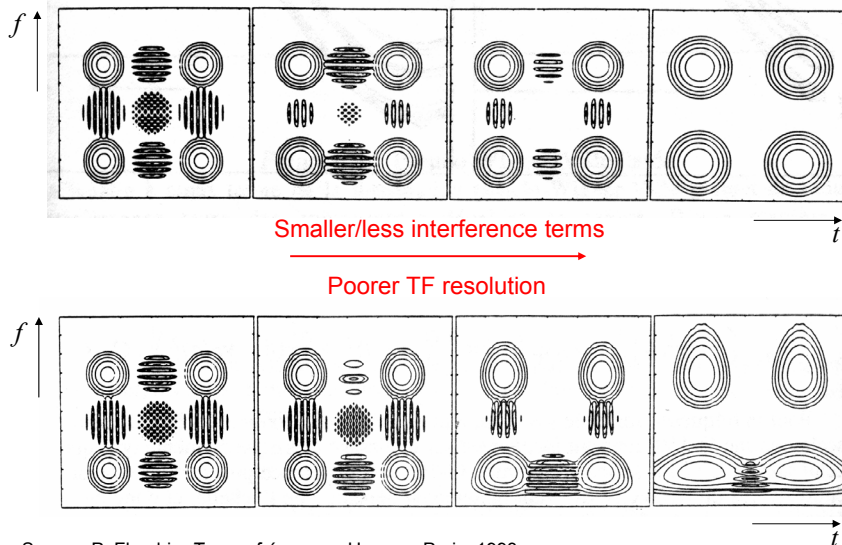
$$\begin{aligned} \text{SCAL}_x(t, f) &= \int_0^\infty \int_{-\infty}^\infty \text{WVD}_x(t', f') \text{WVD}_\psi\left(\frac{f}{f_\psi}(t'-t), \frac{f'}{f/f_\psi}\right) dt' df' \\ &= \left| \int_{-\infty}^\infty x(t') \sqrt{\frac{f}{f_\psi}} \psi^*\left(\frac{f}{f_\psi}(t'-t)\right) dt' \right|^2 \end{aligned}$$



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Constant-BW vs. affine (constant-Q) smoothing



Source: P. Flandrin, *Temps-fréquence*. Hermes, Paris, 1993

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OUTLINE

- The notion of time-frequency analysis
- Linear and quadratic time-frequency analysis
- Short-time Fourier transform and wavelet transform; spectrogram and scalogram
- Constant-bandwidth analysis vs. constant-Q analysis
- The affine class
- Affine time-frequency smoothing
- **Hyperbolic time-frequency localization**

Doppler-tolerant signals

- TF scaling / Doppler effect:

$$(C_a x)(t) = \frac{1}{\sqrt{a}} x\left(\frac{t}{a}\right) \leftrightarrow \sqrt{a} X(af)$$

- “Doppler-tolerant” signal = eigenfunction of C_a :

$$(C_a x)(t) = \lambda_a x(t)$$

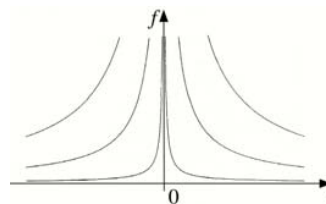
- Solution: “hyperbolic impulse”

$$X(f) = H_c(f) = \frac{1}{\sqrt{f}} e^{-j2\pi c \ln(f/f_r)}, \quad f > 0, \quad c \in \mathbb{R}$$

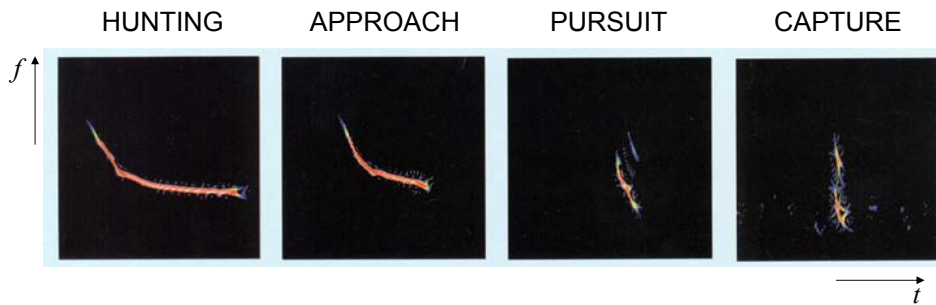
- Group delay:

$$\tau(f) = -\frac{1}{2\pi} \frac{d}{df} \arg\{H_c(f)\} = \underbrace{\frac{c}{f}}$$

Hyperbola in the TF plane



Example: Bat sonar signals



Source: P. Flandrin

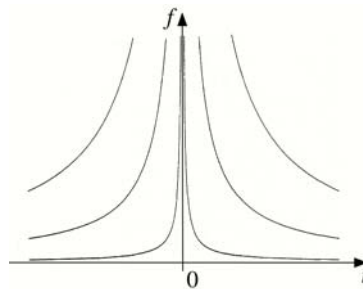
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Hyperbolic TF localization

- Want AC QTFR to satisfy **hyperbolic TF localization property**:

$$X(f) = \frac{1}{\sqrt{f}} e^{-j2\pi c \ln(f/f_r)} \Rightarrow AC_x(t, f) = \frac{1}{f} \delta\left(t - \frac{c}{f}\right)$$



- *Not* satisfied by WVD !

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The Bertrand P_0 distribution

- The hyperbolic TF localization property is satisfied by the (unitary) **Bertrand P_0 distribution**

$$\text{BER}_x(t, f) = f \int_{-\infty}^{\infty} X(f \lambda(u)) X^*(f \lambda(-u)) e^{j2\pi f t u} \mu(u) du, \quad f > 0$$

with

$$\lambda(u) = \frac{e^{u/2} u/2}{\sinh(u/2)}, \quad \mu(u) = \frac{u/2}{\sinh(u/2)}$$

- The Bertrand P_0 distribution is a **central member of the AC**. It satisfies several important properties (besides the hyperbolic TF localization property).

Bertrand P_0 distribution as generator of the AC

- Any QTFR of the AC can be expressed in terms of the Bertrand P_0 distribution:

$$\text{AC}_x(t, f) = \int_0^{\infty} \int_{-\infty}^{\infty} \text{BER}_x(t', f') \tilde{\sigma}\left(f(t'-t), \frac{f'}{f}\right) dt' df',$$

where $\tilde{\sigma}(\alpha, \beta)$ is related to $\sigma(\alpha, \beta)$

- Special case: **scalogram**

$$\text{SCAL}_x(t, f) = \int_0^{\infty} \int_{-\infty}^{\infty} \text{BER}_x(t', f') \underbrace{\text{BER}_\psi\left(\frac{f}{f_\psi}(t'-t), \frac{f'}{f/f_\psi}\right)}_{\text{Smoothing function is BER of wavelet}} dt' df'$$

$$\tilde{\sigma}(\alpha, \beta) = \text{BER}_\psi\left(\frac{\alpha}{f_\psi}, f_\psi \beta\right)$$

Mellin transform and hyperbolic marginals

- Recall *hyperbolic impulse* $H_c(f) = \frac{1}{\sqrt{f}} e^{-j2\pi c \ln(f/f_r)}$

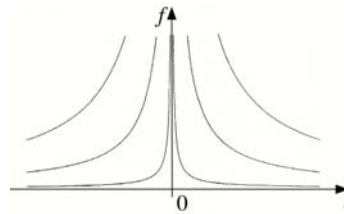
- Mellin transform:

$$M_x(c) = \langle X, H_c \rangle = \int_0^{\infty} X(f) e^{j2\pi c \ln(f/f_r)} \frac{df}{\sqrt{f}}$$

- Hyperbolic marginal property:

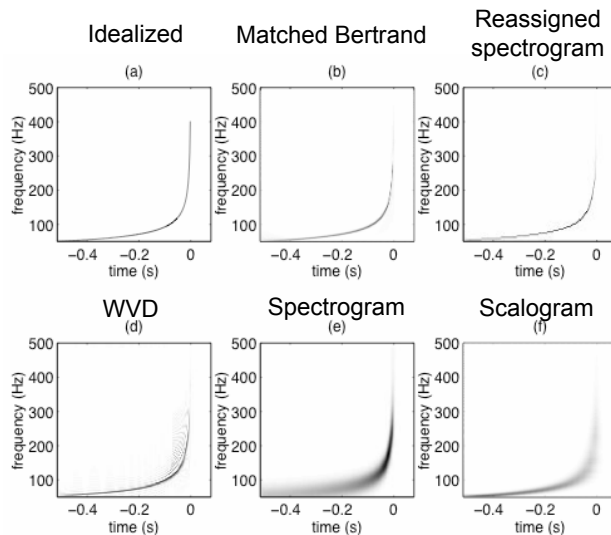
$$\int_0^{\infty} \underbrace{AC_x\left(\frac{c}{f}, f\right)}_{\text{Integrate } AC_x(t,f) \text{ over TF hyperbola } t=cf} \frac{df}{f} = |M_x(c)|^2$$

Integrate $AC_x(t, f)$ over TF hyperbola $t=cf$



- Not satisfied by WVD... but satisfied by Bertrand P_0 distribution !

Application: TF analysis of gravitational wave



Source: E. Chassande-Mottin and P. Flandrin, On the time-frequency detection of chirps. *Appl. Comp. Harm. Anal.*, 6(9): 252-281, 1999.

Conclusion

- Linear and quadratic TF analysis
- Short-time Fourier transform and spectrogram
- Wavelet transform and scalogram
- Filterbank interpretation: constant-BW analysis versus constant-Q analysis
- Scaling/shift covariance and affine class of QTFRs
- Wigner-Ville distribution and affine smoothing
- Doppler tolerance and hyperbolic impulses
- Hyperbolic TF localization and Bertrand P_0 distribution
- Mellin transform and hyperbolic marginal property

WARNING

YOU ARE LEAVING THE
TIME-FREQUENCY PLANE