

Control Theoretic Aspects of Matrix Factorizations

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Contents

- Motivation
- Lie Groups and Lie Algebras
- Control on Lie Groups
- Time - Optimal Control on Lie Groups
- Time - Optimal Factorizations



Motivation

- Quantum Computing
- Quantum Control, Control of Spin Systems
- Control of Numerical Algorithms
- Constructive Controllability, Motion Planning in Robotics



Time-optimal Factorization Problem

- G compact connected Lie group with Lie Algebra \mathfrak{g}
- $\omega := \{\Omega_1^+, \dots, \Omega_r^+, \Omega_1^-, \dots, \Omega_s^-\}$ finite set of LA generators of \mathfrak{g}
- Ω_i^+ : "slow, cost expensive" directions
 Ω_i^- : "fast, cheap" directions
- Given $X \in G$, define

$$T_{\min}(X) = \inf \left\{ \sum_i |t_i^\pm| \mid X = \prod_{\text{finite}} e^{t_i^\pm \Omega_i^\pm} \right\}$$

Problem:

- Is $T_{\min} < \infty$ always? Compute T_{\min} !
- When does there exist a *finite, time-optimal* factorization?

Example


Optimal Condition Numbers

- $G = GL(n)$ general linear group of invertible matrices
- $\omega := \{\Omega_1^+, \dots, \Omega_r^+, \Omega_1^-, \dots, \Omega_s^-\}$ finite set of LA generators of $\mathfrak{gl}(n)$
- Ω_i^+ : "hyperbolic Jacobi rotations"
 Ω_i^- : "standard Jacobi directions"
- Given $X \in G$, define (κ denotes the condition number)


$$T_{\min}(X) = \inf \left\{ \sum_i \kappa(e^{t_i^\pm \Omega_i^\pm}) \mid X = \prod_{\text{finite}} e^{t_i^\pm \Omega_i^\pm} \right\}$$

Problem:

- This factorization task with minimal total condition number!
- Does there exist factorization with better condition numbers than for X ?



Lie Groups & Lie Algebras



Intermezzo: Lie Groups and Lie Algebras

Example. General linear group of invertible $n \times n$ matrices

$$GL(n, \mathbb{R}) := \{X \in \mathbb{R}^{n \times n} \mid \det X \neq 0\}.$$

Definition. A matrix *Lie group* is any subgroup $G \subset GL(n, \mathbb{R})$ that is also a (locally closed) submanifold of $\mathbb{R}^{n \times n}$.



Intermezzo: Lie Groups and Lie Algebras

Examples, cont'd:

(a) The *real orthogonal group*

$$O(n) := \{X \in \mathbb{R}^{n \times n} \mid XX^T = I_n\}$$

(b) The *special unitary group*

$$SU(n) := \{X \in \mathbb{C}^{n \times n} \mid XX^* = I_n, \det X = 1\}$$

(c) The *Euclidean group*

$$E(n) := \left\{ \left[\begin{array}{c|c} R & p \\ \hline 0 & 1 \end{array} \right] \mid R \in O(n), p \in \mathbb{R}^n \right\}.$$

The first two examples are compact groups, while the third is not.

Intermezzo: Lie Groups and Lie Algebras

Definition. A vector space V with a bilinear operation $[,] : V \times V \rightarrow V$ satisfying

- (i) $[x, y] = -[y, x]$
- (ii) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ (Jacobi Identity)

is called a *Lie Algebra*.



Intermezzo: Lie Groups and Lie Algebras

- Lie algebras are the tangent spaces of Lie groups.
- **Theorem.** Let $G \subset GL(n, \mathbb{R})$ be a matrix Lie group. Then the tangent space $\mathfrak{g} := T_I G$ at the identity matrix is a Lie algebra with commutator as the Lie bracket:

$$[X, Y] = XY - YX.$$



Intermezzo: Lie Groups and Lie Algebras

Examples

(a) The Lie algebra of $O(n)$ is

$$\mathfrak{o}(n) := \{\Omega \in \mathbb{R}^{n \times n} \mid \Omega^\top = -\Omega\}.$$

(b) The Lie algebra of $SU(n)$ is

$$\mathfrak{su}(n) := \{\Omega \in \mathbb{C}^{n \times n} \mid \Omega^* = -\Omega, \text{tr}\Omega = 0\}$$

(c) The Lie algebra of $E(n)$ is

$$\mathfrak{e}(n) := \left\{ \left[\begin{array}{cc} \Omega & v \\ 0 & 0 \end{array} \right] \mid \Omega^\top = -\Omega, v \in \mathbb{R}^n \right\}.$$



Control on Lie Groups



Control on Lie Groups

- G Lie Group with Lie Algebra \mathfrak{g} .
- Bilinear control system on G

$$(\Sigma) \quad \dot{X}(t) = \left(A_d + \sum_{j=1}^m u_j(t) A_j \right) X(t), \quad X(0) = I,$$

where $A_d, A_1, \dots, A_m \in \mathfrak{g}$.

- Reachable Set at time $T > 0$

$$\mathcal{R}(T) = \{X_F \in G \mid \exists u_1, \dots, u_m \text{ and } s \leq T : X(s) = X_F\}$$

- Reachable Set

$$\mathcal{R} = \cup_T \mathcal{R}(T)$$

Control on Lie Groups

Definition

- **Accessibility:** The reachable set $\mathcal{R}(T)$ has an interior point
- **Local Controllability:** The identity $I \in \mathcal{R}(T)$ is an interior point
- **Controllability:** For any $X_F \in G$ there exist controls $u_1(\cdot), \dots, u_m(\cdot)$ and $T > 0$ s.t. the solution of (Σ) satisfies $X(0) = I, X(T) = X_F$.



Control on Lie Groups

Problem 1 (Accessibility)

- Definition (*System Lie Algebra*)

$\mathcal{L} :=$ smallest Lie subalgebra of \mathfrak{g} , containing A_1, \dots, A_m, A_d

Generators: ($[A, B] = AB - BA$)

$$A_d, A_1, \dots, A_m, [A_d, A_i], [A_i, A_j], [A_d, [A_i, A_j]], \dots$$

- Theorem. (Σ) is accessible if and only if the system Lie algebra is $\mathcal{L} = \mathfrak{g}$.

Control on Lie Groups

- Theorem (Lian et al. 1994) Suppose

(i) For some constant controls u_1, \dots, u_m

$$(\Sigma_{const}) \quad \dot{X} = (A_d + \sum_j u_j A_j) X$$

is weakly positively Poisson stable.

(ii) The system Lie algebra \mathcal{L} satisfies $\mathcal{L} = \mathfrak{g}$.

Then the bilinear control system is controllable.

Accessibility + Poisson Stability \Rightarrow Controllability

Control on Lie Groups

Definition (Poisson Stability)

Flow of (Σ_{const}) : $\Phi : G \times \mathbb{R} \rightarrow G$; $(z, t) \mapsto \Phi(z, t)$

- (Σ_{const}) is **Weakly Positively Poisson Stable** if for all $z \in G$, any neighborhood $B(z)$ of z and all $T > 0$, there exists $t > T$ such that $\Phi(U_z, t) \cap B(z) \neq \emptyset$.

Examples: a swing (no damping), satellite attitude, ball rolling in a bowl.



Control on Lie Groups

- Theorem (Jurdjevic-Sussmann) Assume:
 - (i) There exist constant controls such that $A_d + \sum_j u_j A_j$ lies in a **compact** subalgebra \mathfrak{k} of \mathfrak{g} .
 - (ii) The system Lie algebra \mathcal{L} satisfies $\mathcal{L} = \mathfrak{g}$.

Then the system (Σ) is controllable.



Control on Lie Groups

- Corollary

Let G be a **compact** connected Lie group. Then (Σ) is controllable if and only if

$$\mathcal{L} = \mathfrak{g}.$$





Time-Optimal Control on Lie Groups



Time-Optimal Control on Lie Groups

General Notation:

- Let G be a compact Lie Group with Lie algebra \mathfrak{g} ; $K \subset G$ a compact connected Lie subgroup with LA \mathfrak{k} . Consider the bilinear control system on G

$$(\Sigma) \quad \dot{X} = \left(A_d + \sum_{j=1}^m u_j A_j \right) X, \quad X(0) = I$$

with $A_d \in \mathfrak{g}$, $A_1, \dots, A_m \in \mathfrak{k}$.

- Assumption:
 - Σ is controllable, i.e. $\mathfrak{g} = \text{LA}$ generated by A_d, A_1, \dots, A_m
 - $\mathfrak{k} = \text{LA}$ generated by A_1, \dots, A_m

Time-Optimal Control on Lie Groups

- Given: Initial state $X_0 = I$, Final state $X_F \in G$
- Problem 1. Find controls $u_1(\cdot), \dots, u_m(\cdot)$ s.t. the corresponding solution $X(t)$ of (Σ) satisfies

$$X(0) = X_0, \quad X(T) = X_F \quad \text{for some } T > 0$$

- Problem 2. If problem 1 has at least one solution, then find a time-optimal one, i.e. one with *minimal* $T = T_{\text{opt}}(X_F)$.
- Problem 1 is always solvable, provided (Σ) is controllable!



Time-Optimal Control on Lie Groups

Fast versus slow directions

- A_d is called the *drift term*, A_1, \dots, A_m the *fast directions*
- **Fact 1.** If $A_d = 0$ and (Σ) controllable, then can control to X_F in *arbitrarily small time*: $T_{\text{opt}}(X_F) = 0$, always!
- **Fact 2.** The presence of drift term $A_d \neq 0$ is responsible for $T_{\text{opt}} > 0$.
- **Idea:** Factor out fast directions!

Time-Optimal Control on Lie Groups

Quotient System and Equivalence Principle

- Consider the quotient space

$$G/K := \{Kg \mid g \in G\}$$

of left co-sets Kg , $K = \exp(\mathfrak{k})$ Lie Group generated by fast controls.

- G/K is a smooth manifold



Time-Optimal Control on Lie Groups

Example: (NMR)

- For the NMR Schrödinger Equation on $G = SU(2^N)$

$$\dot{X} = -i \left(H_d + \sum_{j=1}^{2N} u_j H_j \right) X, \quad X(0) = I$$

$\mathfrak{k} :=$ LA generated by iH_1, \dots, iH_{2N}

$K := \exp(\mathfrak{k})$ compact, connected Lie subgroup of $SU(2^N)$,
generated by $\exp(itH_j), t \in \mathbb{R}, j = 1, \dots, 2N$.

One verifies $K = SU(2) \otimes \dots \otimes SU(2)$

- For $N = 1 : K = SU(2) = G$
- For $N = 2 : K = SU(2) \otimes SU(2) \simeq SO(4) \subset SU(4)$

Time-Optimal Control on Lie Groups

Quotient System and Equivalence Principle

- The *quotient system* of

$$(\Sigma) \quad \dot{X} = \left(A_d + \sum_{j=1}^m u_j A_j \right) X, \quad X(0) = I, \quad X(T) = X_F$$

is the control system on G/K

$$(\Sigma/K) \quad \dot{P} = \text{Ad}_{U(t)}(A_d)P, \quad P(0) = K, \quad P(T) = KX_F$$

$\text{Ad}_g(A_d) = gA_dg^{-1}$, $g \in K$. The control functions for (Σ/K) are arbitrary L^1_{loc} functions $t \mapsto U(t) \in K$.

Time-Optimal Control on Lie Groups

Quotient System and Equivalence Principle

- Theorem (Equivalence Principle).

(Σ) is controllable on G iff (Σ/K) is controllable on G/K .
Moreover, the optimal times on G and G/K coincide.

$$T_{\text{opt}}^G(X_F) = T_{\text{opt}}^{G/K}(KX_F)$$

Proof: PhD thesis by Khaneja

- The optimal time $T_{\text{opt}}^{G/K}$ has an interpretation within Sub-Riemannian Geometry.

Time-Optimal Control on Lie Groups

Sub-Riemannian Geometry

- Let M be a Riemannian manifold, $E \subset TM$ a constant dimensional subbundle that satisfies the *Hörmander Condition*

For any $p \in M$, the LA of the sections of E evaluated in p is equal to T_pM (controllability cond.)

- For any two points $x, y \in M$, the *Sub-Riemannian distance* is

$$d(x, y) := \inf \left\{ \int_0^1 \|\dot{\alpha}(t)\| dt \mid \alpha(0) = x, \alpha(1) = y, \dot{\alpha}(t) \in E_{\alpha(t)} \right\}.$$

- **Example:** $M = G/K$, $E_p := \text{span}\{kA_dk^{-1} \mid k \in K\}P$, $P \in M$ satisfies the Hörmander Cond. (Equivalence principle)
- **NMR:** $M = SU(2^N)/SU(2) \otimes \dots \otimes SU(2)$ Sub-Riemannian space

Time-Optimal Control on Lie Groups

Sub-Riemannian Geometry

- Theorem.

$$T_{\text{opt}}^{G/K}(KX_F) = d(K, KX_F)$$

Sub-Riemannian distance

- Remark. The Sub-Riemannian distance $d(x, y)$ is greater than or equal the Riemannian distance on G/K :

$$d(x, y) \geq \text{geodesic distance between } x, y$$

- There is one case where these distances are equal: *Riemannian symmetric spaces*.

Time-Optimal Control on Lie Groups

Sub-Riemannian Geometry

- Theorem. If G/K is a Riemannian Symmetric Space, then

$T_{\text{opt}}(X_F) = \text{length of a geodesic in } G/K \text{ that connects } K \text{ with } KX_F$

- Main Advantage: Riemannian distances (i.e. lengths of geodesics) are much easier to compute than Sub-Riemannian distances.



Time-Optimal Control on Lie Groups

- Theorem. The homogenous space G/K is a Riemannian symmetric space, provided $(\mathfrak{g}, \mathfrak{k})$ is a Cartan-pair, i.e. \mathfrak{g} is semisimple and

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{p} := \mathfrak{k}^\perp$$

satisfies

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$$



Time-Optimal Control on Lie Groups

Riemannian Symmetric Spaces

- $SU(n)/SO(n)$ is a Riemannian Symmetric Space
- $SU(4)/SU(2) \otimes SU(2)$ is a Riemannian Symmetric Space (good! 2-Spin Case)
- $SU(8)/SU(2) \otimes SU(2) \otimes SU(2)$ is *NOT* a Riemannian Symmetric Space (bad!)





Time-Optimal Factorization



Time-optimal Factorization

- Let G be a connected, compact Lie group with Lie algebra \mathfrak{g} .
- Let $K \subset G$ be a connected compact subgroup with Lie algebra \mathfrak{k} .
- Let $\Delta \in \mathfrak{g}$ be a drift term s.t. $\langle \Delta, \mathfrak{k} \rangle_L = \mathfrak{g}$.
- Consider the discrete control System:

$$(\Sigma_d) \quad X_{n+1} = K_n e^{t_n \Delta} L_n X_n, \quad X_0 = I \quad K_n, L_n \in K, t_n \geq 0.$$

For $X \in G$ let $T_{\text{opt}}^d(X) :=$

$$\inf \left\{ \sum_{n=1}^{\infty} t_n \mid \exists (K_n, L_n, t_n) : \prod_{n=1}^{\infty} K_n e^{t_n \Delta} L_n = X \right\}.$$

Time-optimal Factorization

Problem:

- Is (Σ_d) controllable, i.e. does $T_{\text{opt}}^d(X) < \infty$ hold for all $X \in G$?
- Determine the “minimal” time $T_{\text{opt}}^d(X)$ for $X \in G$.



Time-optimal Factorization

Generalized Version (multiple drifts)

- G compact connected Lie group with LA \mathfrak{g}
- $\omega := \{\Omega_1^+, \dots, \Omega_r^+, \Omega_1^-, \dots, \Omega_s^-\}$ finite set of LA generators of \mathfrak{k}
- Ω_i^+ : "slow, cost expensive" directions
 Ω_i^- : "fast, cheap" directions
- Given $X \in G$, define

$$T_{\min}(X) = \inf \left\{ \sum_i |t_i^{\pm}| \mid X = \prod_{\text{finite}} e^{t_i^{\pm} \Omega_i^{\pm}} \right\}$$



Time-optimal Factorization

Problem

- Is $T_{\min} < \infty$ always? Compute T_{\min} !
- When does there exist a *finite, time-optimal* factorization?



Time-optimal Factorization

Example 1 (Euler Angles)

- $SO(3)$, $\omega = \{\Omega_1^+, \Omega_1^-\}$,

$$\Omega_1^+ := \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \Omega_1^- := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- *Euler Angles:*

$$X = e^{\theta_1 \Omega_1^-} e^{\theta_2 \Omega_1^+} e^{\theta_3 \Omega_1^-}, \quad \theta_i \in [-\pi, \pi]$$

- We will show: Euler Angles are time-optimal and

$$T_{\min} = |\theta_2| \in [0, \pi]$$

Time-optimal Factorization

Example 2 (Euler Angles)

- $SO(3)$, $\omega = \{\Omega_1^+, \Omega_2^+\}$, $\Omega_2^+ := \Omega_1^-$
- Then Euler angles are i.g. *NOT* time-optimal:

$$T_{\min} < \theta_1 + \theta_2 + \theta_3 ! \quad (\text{Mittenhuber})$$



Time-optimal Factorization

Equivalence Principle

- Let G be a connected, compact Lie group with Lie algebra \mathfrak{g} .
- Let $\mathfrak{k} := \langle A_1, \dots, A_m \rangle_L$, $K := \exp \mathfrak{k}$.
- Let $\Delta \in \mathfrak{g}$ be a drift term such that $\langle \Delta, \mathfrak{k} \rangle_L = \mathfrak{g}$.
- Theorem.
 - (a) The discrete control system (Σ_d) on G is controllable and thus $T_{\text{opt}}^d(X) < \infty$
 - (b) For any $X \in G$ the minimal times $T_{\text{opt}}^d(X) = T_{\text{opt}}(X)$ coincide, where $T_{\text{opt}}(X)$ is the minimal time for the control problem

$$\dot{X} = \left(\Delta + \sum_{j=1}^m u_j A_j \right) X, \quad X(0) = I, X(T) = X$$

Time-optimal Factorization

- Problem: I.g. time optimal factorizations are infinite

Under what conditions on the drift term Δ are they *finite*?

- Definition [Haselgrove, Nielsen, Osborne]: A drift term Δ is called *lazy*, if there exists $\varepsilon > 0$ such that

$$T_{\text{opt}}(e^{t\Delta}) < t \quad \text{for all } t \in (0, \varepsilon). \quad (**)$$

If Δ is not lazy, we call it *fast*.



Time-optimal Factorization

- Theorem. If Δ is lazy, there are no finite, time optimal factorizations for any element $X \in G - K$.



Time-optimal Factorization

- Conjecture 1: There exists a finite, time optimal factorization for all $X \in G$ iff Δ is fast.
- Conjecture 2: Δ fast $\iff [\Delta, \Delta^\perp] = 0$.
- Remark: Conjecture 2 implies Conjecture 1.



Computation of Optimal Time

Theorem (Khaneja). Let $(\mathfrak{g}, \mathfrak{k})$ be a Cartan pair. Let Δ^\perp be the orthogonal projection of Δ onto \mathfrak{p} and let \mathfrak{a} be a maximal abelian subalgebra of \mathfrak{p} that contains Δ^\perp . Then:

- Each $X \in G$ has a decomposition of the form

$$X = U\Sigma V \quad \text{with } U, V \in K \text{ and } \Sigma \in \exp \mathfrak{a}.$$

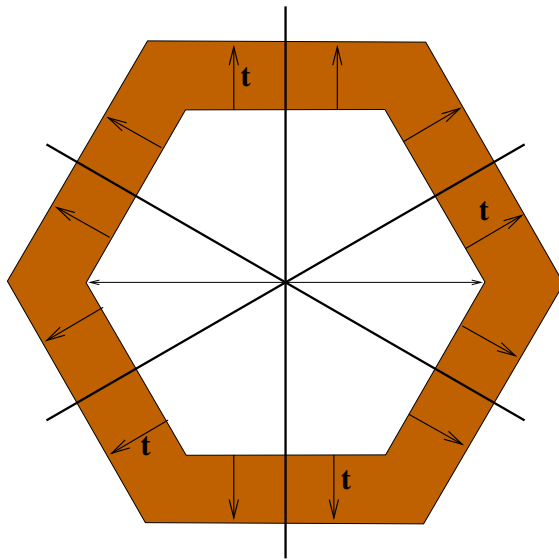
- The minimal time is given by

$$T_{\text{opt}}(X) = \min \left\{ t \geq 0 \mid \left(t \cdot \text{conv } \mathcal{W}(\Delta^\perp) \right) \cap \exp^{-1}(\Sigma) \neq \emptyset \right\},$$

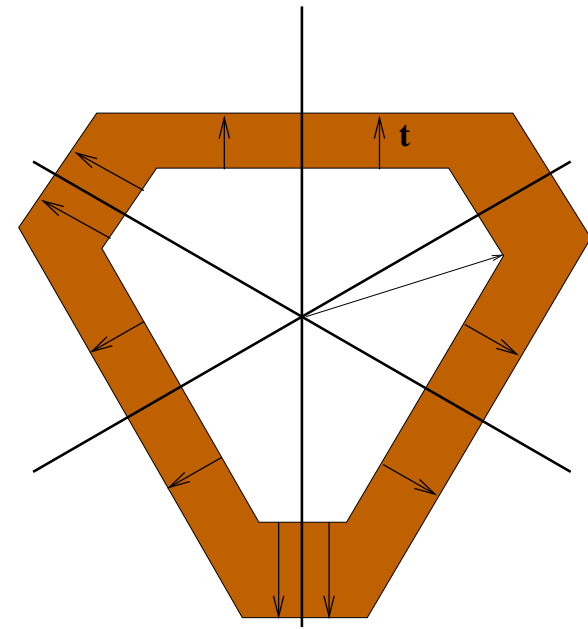
where $X = U\Sigma V$ is an arbitrary factorization of the above type and $\mathcal{W}(\Delta^\perp)$ denotes the Weyl orbit of Δ^\perp .



Computation of Optimal Time



Convex hull of the Weyl Orbit of a "symmetric" drift term Δ



Convex hull of the Weyl Orbit of an arbitrary Δ .



Computation of Optimal Time

Example 1, cont'd:

- $G := \text{SO}(3)$ and $\mathfrak{g} := \mathfrak{so}(3)$,

$$\Omega_1 := \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \Omega_2 := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- $\Delta := \alpha\Omega_1 + \beta\Omega_2$, $\mathfrak{k} := \langle \Omega_2 \rangle$

Euler Angles: $X = e^{\theta_1\Omega_2}e^{\theta_2\Omega_1}e^{\theta_3\Omega_2}$, $\theta_i \in [-\pi, \pi]$

- $T_{\text{opt}}(X) = \alpha^{-1}|\theta_2|$,
- Δ fast $\iff \beta = 0$.

Computation of minimal time

Example: (NMR cont'd)

- NMR-Schrödinger equation on $SU(4)$

$$\dot{X} = -2\pi i \left(H_d + \sum_{i=1}^4 u_i H_i \right), \quad X(0) = I,$$

where $H_d := \sigma_z \otimes \sigma_z$, $H_1 := I_2 \otimes \sigma_x$, $H_2 := I_2 \otimes \sigma_y$, $H_3 := \sigma_x \otimes I_2$,
and $H_4 := \sigma_y \otimes I_2$.

- $K = SU(2) \otimes SU(2)$.
- $\Delta = -2\pi i H_d$ and $\mathfrak{a} := i \langle \sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_z \rangle$.

Computation of minimal time

Example: (NMR cont'd)

Theorem. For all $X = U\Sigma V \in SU(4)$ and $U, V \in K$, and $\Sigma \in \exp \mathfrak{a}$ fixed it holds

- $T(X) = \min \left\{ \sum_{n=1}^3 |t_n| \left| e^{t_1 2\pi i (\sigma_x \otimes \sigma_x)} e^{t_2 2\pi i (\sigma_y \otimes \sigma_y)} e^{t_3 2\pi i (\sigma_z \otimes \sigma_z)} = \Sigma \right. \right\}$

- $T(X) \leq \frac{3}{2}$



Computation of minimal time

Optimization Algorithm (NMR cont'd)

Let $X(t, u) = U(u_1, \dots, u_6)\Sigma(t_1, t_2, t_3)V(u_7, \dots, u_{12})$,

$$U(u_1, \dots, u_6) = e^{-i2\pi u_1 H_1} e^{-i2\pi u_2 H_2} e^{-i2\pi u_3 H_1} e^{-i2\pi u_4 H_3} e^{-i2\pi u_5 H_4} e^{-i2\pi u_6 H_3}$$

$$V(u_7, \dots, u_{12}) = e^{-i2\pi u_7 H_1} e^{-i2\pi u_8 H_2} e^{-i2\pi u_9 H_1} e^{-i2\pi u_{10} H_3} e^{-i2\pi u_{11} H_4} e^{-i2\pi u_{12} H_3}$$

$$\Sigma = e^{t_1 2\pi i (\sigma_x \otimes \sigma_x)} e^{t_2 2\pi i (\sigma_y \otimes \sigma_y)} e^{t_3 2\pi i (\sigma_z \otimes \sigma_z)}$$

To compute the minimal time $T(X)$, we combine simulated annealing with gradient methods to solve the nonlinear optimization problem:

$$\begin{aligned} \min \quad & f(t, u) := |t_1| + |t_2| + |t_3|, \\ \text{subject to} \quad & g(t, u) := 4 - \text{Re tr}(X_F^* X(t, u)) = 0 \end{aligned}$$

where $t = [t_1, t_2, t_3]$, $u = [u_1, u_2, \dots, u_{12}] \in [-1, 1]^{12}$

Computation of Time-optimal Pulse Sequences

Consists of two sub-problems:

- Given $T \geq 0$, solve

$$\begin{aligned} \min_{t,u} \quad & g(t, u), \\ \text{subject to} \quad & f(t, u) \leq T, \\ & t \geq 0. \end{aligned}$$

- Let $V(T)$ be the global optimal value of $g(t, u)$, associated with a given $T \geq 0$.

$$\begin{aligned} \text{Minimize} \quad & T \\ \text{subject to} \quad & V(T) = 0, \\ & T \geq 0. \end{aligned}$$

Computation of Time-optimal Pulse Sequences

Example

$$X_F = e^{-\frac{i\pi}{4}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

$$T(X_F) = 1.499996$$

$$t = [0.499993 \mid 0.500017 \mid 0.499986]$$

