ANALYTICAL RESULTS FOR A SINGLE-UNIT SYSTEM SUBJECT TO MARKOVIAN WEAR AND SHOCKS

THESIS

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Abstract

This thesis develops and analyzes a mathematical model for the reliability measures of a single-unit system subject to continuous wear due to its operating environment and randomly occurring shocks that inflict a random amount of damage to the unit. Assuming a Markovian operating environment and shock arrival mechanism, Laplace-Stieltjes transform expressions are obtained for the failure time distribution and all of its moments. Moreover, an analytical expression is derived for the long-run availability of the single-unit system when it is subject to an inspect-and-replace maintenance policy. The analytical results are illustrated, and their results compared with those of Monte Carlo-simulated failure data. The numerical results indicate that the reliability measures may be accurately computed via numerical inversion of the transform expressions in a straightforward manner when the input parameters are known a priori. In stark contrast to the simulation model which requires several hours to obtain the reliability measures, the analytical procedure computes the same measures in only a few seconds.

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Finally, I would like to thank the brave men and women who have served our country. This thesis pales in comparison to the contributions made by the soldiers, sailors, airmen, marines, and coast guardsmen of this and every generation. Their service, sacrifice, and dedication ensures that our nation remains free.

Daniel E. Finkelstein

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1. Introduction

1.1 Background

Throughout its history, the science of reliability has been intertwined with military applications. The formal mathematics of reliability theory was developed during the second World War to examine the high failure rates observed in the military systems of that era. Military applications continued to drive the growth of the field through the 1950's and 1960's. As the cold war between the United States and the Soviet Union intensified, the national defense strategy of mutually assured destruction required the military's nuclear weapons program to obtain unprecedented levels of reliability. To that end, national efforts were made within the military and academic communities to grow the state of the art in reliability theory. Those efforts produced the amalgam of probabilistic and statistical techniques used in modern reliability analysis.

As a vast application area of probability, reliability theory studies, measures and analyzes system failures and repairs in order to improve their operational use [8]. While the pioneering work focused on military applications, today's competitive global economic environment has broadened the scope of reliability research. To remain competitive, organizations must become more efficient while offering superior products. Manufacturers are required to produce reliable products, through reliable processes; a business cannot remain economically viable if its production capabilities are constantly in disrepair. Analogously, the armed forces cannot conduct effective combat operations if their weapon systems are unreliable. For this reason, the modern United States military has mandated that its equipment operate reliably during peace time and combat operations.

For new systems in the acquisition process, this emphasis is reflected in the language contained in the Federal Acquisition Regulation (FAR). Federal law requires acquisition program managers to develop and document a strategy for continuous improvement of product reliability and availability while sustaining readiness [9]. While these regulations only apply to new system acquisitions, the reliability and supportability of legacy systems is becoming an even more important issue. Scarce resources are forcing the military to use systems longer than they were initially intended. The reliability of these aging systems is the main factor in extending their service life. For example, the reliability of the B-52H airframe was the main concern of its service life extension program, and as a result the B-52H, which first flew in 1962, will remain in service until 2040. However, the military cannot ensure their systems will meet these reliability requirements unless adequate methods exist to measure their reliability.

Reliability is defined by Ebeling [9] as the probability that a system will perform a required function for a period of time under normal operating conditions. This definition requires that those conditions, and the function it is expected to perform, be specified. A system's maintainability and availability are associated with its reliability, and collectively, these three measures are referred to as reliability performance measures.

These metrics are very important for one reason: real-world systems fail. Some failures are minor and result in inconveniences, minor damage and small economic loss, while other failures are catastrophic and cause personal and corporate ruin. Recent history provides examples of systems whose failures have caused tremendous economic and personal loss. The Tacoma Narrows bridge fell into the Puget Sound on November 8, 1940 because engineers did not fully understand the effects of metal fatigue on the bridge's reliability [9]. The collapse of the bridge cost millions of dollars, while the 1986 Space Shuttle *Challenger* tragedy cost the lives of all seven astronauts on board. The cause of the explosion was the failure of a simple O-ring [9]. In the modern U.S. military, systems are becoming more complex. Greater emphasis is being placed on fielding reliable and maintainable weapons systems. When a military's systems fail in combat, troops' lives are endangered. Simply put, the best weapon is useless if it cannot be effectively employed because it is unreliable.

Traditionally, the military has favored statistical techniques in measuring weapon system reliability. This method uses historical failure time observations to create empirical measures of a system's reliability. For components that are mass produced at relatively low cost, this approach is feasible because the components can be destructively tested to obtain the required data. Unfortunately, many (if not most) of the systems employed by the U.S. military are not of this nature. An alternative approach is to use probabilistic models to analytically derive reliability measures. This approach defines an abstract model which consists of a collection of mathematical assumptions to evaluate the reliability of any system that meets the assumptions of the model.

The probabilistic approach is more attractive than the statistical approach for two main reasons. First, a system's reliability measures can be determined generally as opposed to those using the statistical approach which are specific to a single system. The probabilistic approach builds a general model that may be used to find the reliability of an entire class of systems. Second, the probabilistic approach facilitates the explicit modelling of the effect of the system's operating environment on its reliability. In contrast, this inherent dependence is not explicitly contained in failure time data used in the statistical approach.

Stochastic modelling is a vital tool when employing the probabilistic approach to reliability analysis. Although the development of such an analytical, stochastic model is difficult, a properly constructed model may accurately assess the system's failure dynamics. In this thesis, the tools of stochastic modelling are used to derive the reliability measures of a system subject to environment-dependent, continuous linear wear and random shocks. Currently, the probabilistic models that exist in the literature do not consider systems subject to this type of failure mechanism.

The single-unit system accumulates damage until the damage exceeds a maximum tolerable level, at which time the system fails. The cumulative damage can be attributed to two separate processes. The first is an environment-dependent wear process and the second is an independent shock process. An environment-dependent wear process is one in which the rate of wear accumulation depends on the state of the operating environment of the system. An example of a environment-dependent wear process is a machine whose normal operating conditions may include two states: a low capacity mode and a full capacity mode. The machine sustains wear at a higher rate when the system operates in the full capacity mode. The wear rates and duration of time spent in each state determine how long the system operates effectively before a failure. The wear process reflects the impact a system's operating environment has on its reliability.

The other contributor to system degradation is an independent shock process. While the system is constantly accumulating damage due to wear, shocks occur at random intervals causing additional damage. The time between shocks is a random variable, and each time a shock occurs, a random amount of damage is inflicted. The total damage caused by shock and wear over time may be modelled as a stochastic process. An example of a system that accrues damage in this manner is the tire on an aircraft landing gear. Consider the tire in two operating environments, high speed (take off and landing) and low speed (taxi). When the aircraft operates at a high speed, the tire wears at a greater rate than when it operates at the slower taxi speed. This represents the state-dependent wear process. Further assume that the instant the plane lands, the tire incurs a random amount of damage due to the shock of landing. Because sortie durations are random, the sequence of landings, and their associated damage magnitudes may, be considered as the shock process. The total cumulative damage to the tire is the sum of the wear and shocks. If the tire must be replaced when the total cumulative damage exceeds a fixed threshold, (i.e. a control limit) then the probability that it must be replaced during some period of time is the reliability of the tire. Currently, there is not a probabilistic method to evaluate the tire's reliability. Using an appropriate failure time model it is possible to determine the frequency with which preventative maintenance should occur, and the appropriate number of spares that should be maintained. Furthermore, if the cost structure of this process is known, it is possible to develop a replacement policy that can maximize availability while minimizing the overall maintenance cost.

In this thesis, the reliability measures for a single-unit system that accumulates damage over time due to the influence of a random environment and the random occurrence of shocks are investigated. More specifically, the failure time distribution, the mean time-to-failure, and all other moments are derived, as well as the long-run availability of such a system when it is maintained under an inspect-and-replace maintenance policy. In application, this research will assist civilian and military analysts to accurately evaluate the reliability and availability of their systems. The ability to compute these reliability measures will ultimately allow decision makers to quantify the risk associated with operating a system over a prescribed time horizon, providing useful insight into meaningful real-world problems and expanding the current knowledge of reliability theory. In the next section the formal problem definition will be provided, as well as a road map for the proposed solution methodology.

1.2 Problem Definition and Methodology

The reliability and long-run availability of a system that is subject to continuous, state-dependent, linear wear and random shocks is considered. Previous research has investigated systems that incur damage caused solely by wear or shocks. Currently there exist analytical expressions for the reliability of these components; however, very little research has been done on systems that simultaneously incur damage from both continuous wear and shocks. Where some results exist, they are complicated multi-dimensional transform solutions that require sophisticated numerical inversion algorithms. These techniques are computationally expensive and make it difficult to efficiently evaluate the reliability and availability of even simple systems. The main contribution of this thesis is to develop a simple closed-form solution in a single transform dimension. The numerical inversion techniques for a one dimensional transform are more expedient and easier to implement. Previous research indicates that one-dimensional inversion may be up to 400 times faster for a even simple system.

The reliability of a system is the probability that it will survive for a given length of time and the complementary probability is the chance it will fail during that time. Computing a system's reliability directly can be difficult and it is often easier to derive the failure time distribution. In this thesis, it will be shown that the failure time distribution satisfies a system of linear, first-order, partial differential equations that may be solved via Laplace transforms. Once the reliability function and failure time moments are derived, an analytical expression for the system's availability under an inspect-and-replace policy is derived.

An inspect-and-replace policy is a maintenance policy used in many real-world systems. It assumes that inspections occur at constant intervals and an inspection reveals if the system is failed or operating. If the system is operating, nothing is done until the next inspection, and if it has failed the system is replaced with a new unit. Replacements are assumed to be instantaneous and inspections are assumed to perfectly diagnose the system's condition. Deriving a measure of availability quantifies the effects repairs will have on the system and will allow analysts to compare competing inspection policies. The results for reliability and availability are confirmed by comparing the analytical solutions to those obtained via simulation. Finally, the numerical results presented in this thesis demonstrate the broad applicability of the models derived herein.

1.3 Thesis Outline

The next chapter includes a review of the previous work done in this field. It begins with an overview of the early works in reliability theory and continues with a detailed look at both wear and shock models, as well as systems that incorporate both damage mechanisms. Chapter 2 provides the reader a frame of reference to understand the contributions of this thesis. It also shows that all existing methods are too cumbersome to implement in practice. Furthermore, this chapter highlights the existing gap in the literature of compound damage models; the gap that this thesis will close.

The formal notation and mathematical model are developed in chapter 3. In the first section an appropriate stochastic model is constructed. Next, the main results of this thesis are obtained by deriving a system of partial differential equations satisfied by the probability distribution of the damage sustained during a time interval. That system is then solved via Laplace transform techniques. Using the transform solution, the failure time distribution is obtained. The third section successfully reduces the two-dimensional result to a single dimension. Using the one-dimensional result the moments of the failure time distribution are derived. This chapter closes by deriving an analytical expression for the long-run availability of the system.

Chapter 4 is dedicated to numerical examples that show the accuracy of the one-dimensional inversion as compared to a Monte Carlo simulation. The examples, drawn from a variety of applications, illustrate the means by which the results of chapter 3 can be used to analyze real-world problems. Most importantly, this chapter demonstrates the enormous disparity in computational effort between the analytical method and Monte Carlo simulation. More specifically, the reliability measures were computed in only a few seconds using the analytical results as compared to roughly four hours for the same measures via simulation. Moreover, a new simulation model (including verification and validation) was needed for each implementation, whereas the analytical results required only a set of new input parameters.

The final chapter concludes the thesis by summarizing the main results and the contributions to the academic community, the United States Air Force and the Department of Defense. Also contained in chapter 5 are recommendations for future research directions.

2. Review of the Literature

In this chapter the literature in the field of reliability is reviewed. The emphasis is on the development of stochastic models to analyze systems that are subject to wear and/or random shocks. The first section is a synopsis of the important works in the field of reliability prior to 1970. The literature and evolution of stochastic shock and wear models will be reviewed in the second and third sections, respectively. In the fourth section the literature for models that combine the effects of random shocks and continuous wear is investigated. The fifth section of this chapter considers some of the literature concerned with the use of stochastic damage models for deriving an optimal replacement policy. The final section connects the literature reviewed to the research effort of this thesis.

2.1 The History of Reliability Theory

The mathematical study of reliability has grown out of the demand of modern technology, and particularly from the experiences of World War II. The early foundation of reliability theory was in the actuarial concepts developed in the insurance industry [9]. While these concepts evolved during the 1930s into the study of structural reliability and fatigue failures, it was not until the second World War that the mathematics of reliability was rigorously studied. The production of materiel and the high failure rates of complex combat systems fueled the rapid growth of the field during the late 1930s and through the end of the war.

The pioneers of reliability theory focused on two main problems. The first, fatigue life, was extensively studied by Weibull [38], who in 1939 introduced a probability distribution to describe the breaking strength of materials. He also proposed using this distribution to describe system lifetimes. Replacement problems were the second area studied during this time. The study of replacement problems introduced the concepts of stochastic modelling to reliability. The mathematics of queueing and renewal theory were used to solve early replacement problems. Lotka's work [25] in 1939 used the theory of renewals to address replacement problems in an industrial setting.

The focus of reliability research for the 1950s was on improving military and civilian aviation systems. Many military systems, such as radar, depended on complex electronic equipment. Highly unreliable vacuum tubes were used in almost all avionic and electronic equipment of the era. The dependence of the aviation industry on these notoriously undependable components led to the formation of Aeronautical Radio, Inc. (ARINC). This group collected and analyzed defective vacuum tubes, and they were able to increase the reliability of numerous types of tubes [3]. ARINC was the first group formed to study the problems of reliability in a real-world setting.

ARINC signaled a shift in how system reliability was studied and improved. Previously, an unreliable system was made more reliable by adding redundant components, but this over-engineering of systems made them larger, heavier, and more complicated. When the United States and the Soviet Union began to develop ballistic missiles in the 1950s, the goal of reliability research was to design more reliable components, eliminating the need for redundancy. The methods used to calculate a system's reliability needed to be changed as well. Prior to this decade, a system's reliability was determined by destructive life testing. This method works well for simple, inexpensive systems, but the method is not acceptable for calculating the reliability of expensive, complex items. Scientists needed to develop more powerful analytical techniques for determining a system's reliability measures [4].

The United States Air Force formed the ad hoc Group on Reliability of Electronic Equipment in December of 1950. The group studied the reliability of Air Force systems and recommended measures to increase reliability and reduce maintenance [3]. The Department of Defense followed the example of the Air Force and established the Advisory Group on Reliability of Electronic Equipment (AGREE) in 1952. In 1957 AGREE published a report entitled "Reliability Versus the Cost of Failure" [32]. This report evaluated the life cycle costs of fielding more reliable systems; it also included acceptance limits and reliability requirements.

The classic view reliability was to consider failures as random occurrences. These occurrences are not actually random, but instead are caused by the physical and chemical interactions of the system. Because the conditions that cause the failures are not always understood, the failure pattern is modelled with a probability distribution [9]. Epstein and Sobel [11] began investigating various distributions used to model failure processes in the 1950s. After discussions with electronic experts they concluded that they should focus on a non-normal distribution with the form

$$f(x, \theta) = \frac{1}{\theta} e^{-x/\theta}, \quad \theta > 0, x \geq 0. \tag{2.1}$$

Their work began the widespread practice of assuming failure times to be exponentially distributed.

The use of the exponential distribution to model random failure times was also studied by Davis [7]. Davis accumulated the results of various lifetime tests and fit different probability distributions to the data. He then performed several goodness-of-fit tests to measure the ability of competing distributions to model the failure data, and found the exponential distribution to be the best. His paper is often cited by subsequent authors who assume failure times to be exponential. The publication of [7] and [11] made the exponential distribution the probability distribution of choice for reliability research.

The exponential distribution gained popularity for another reason: it has properties that simplify the reliability analysis of many systems. Complex systems, like airplanes and cars, are composed of many subsystems. If the failure times of these subsystems are distributed exponentially, then their failure rates can be summed to find the failure rate of the composite system. The exponential distribution also has the memoryless, or Markovian, property. That is, the future lifetime of a component does not depend on how long it has already operated. This property is very important when using a renewal theory approach because exponential failure times simplify the analysis of many systems.

The missile systems and communication networks of the 1960s presented new challenges to reliability researchers. The common practice of assuming an exponential distribution did not accurately model their system reliability. The use of semi-Markov processes to compute system reliability measures was introduced by Weiss [39] in 1956, and two years later Smith [36] published a cogent summation of the known mathematical results in renewal theory. The limiting probability techniques of queuing theory were used at the "Moscow School" of reliability to develop maintenance and repair models [3]. These works helped bridge the gap between the fields of stochastic modelling and reliability.

The emergence of nuclear power plants and the subsequent study of their reliability directed most research in the field to fault tree analysis [3] during the 1970s. Haasl [17] provides the best description of how to construct a fault tree in an engineering setting [3]. However, fault trees of complicated systems needed more powerful algorithms to produce significant reliability results. Research into Boolean and set theoretic combinatorial methods produced the results needed to allow reliability engineers to design safe and effective nuclear power plants [3].

Finally, numerous important shock and wear models were developed from the 1970s through the present. The distinguishing feature of these models is that they are derived by considering stochastic processes that describe the failure generating mechanisms [34]. The dynamic environments in which the systems operate are not modelled with classical techniques, and stochastic models provide the flexibility to describe the failure processes. These models are of great importance to this thesis research, and the associated literature is reviewed in detail in the next three sections.

2.2 Shock Models

Shock models are used to analyze special *stochastic point processes*. A stochastic point process is characterized by isolated events occurring randomly over some domain, normally time [9]. When the random time between events forms a sequence of independent and identically distributed random variables, the stochastic point process is called a *renewal process*. The theory of renewals is a powerful tool used to analyze the reliability of many systems that are subject to damage caused by random shocks.

Shock models can generally be described as follows. A system begins operating at time $t = 0$ in perfect condition, and after a random amount of time τ_1 the system experiences a shock. When the first shock occurs the system incurs some degree of damage Y_1. The magnitude of the shock may be a random variable or a deterministic value. Since the latter is a special case of the former, we will assume Y_i is a random variable. The system continues to operate, and after another random amount of time τ_2, a second shock occurs whose magnitude is given by the random variable Y_2. The random variables τ_i and $\tau_j, i \neq j$ are stochastically equivalent, and the shock arrival process forms a renewal process. The shocks continue until the stopping criterion is met, at which time the system is considered to have failed. The failed system is replaced with an identical unit, and the process renews. Because the replacement unit is identical to the original, the random time until it fails is stochastically equivalent to the original failure time, and another renewal process is formed.

The literature in the area of shock models is concerned with two different failure mechanisms. The first is the *cumulative damage shock model*. In this model, the damage sustained on the ith shock is a random variable, whose distribution $F_Y(\cdot)$ is known, and the total damage the system has experienced up to time t is a random variable $D(t)$. The random variable $D(t)$ depends on the number of shocks that occur up to time t, and the magnitude of those shocks. The number of shocks that occur before time t is the random variable $N(t)$, and the total damage is

$$D(t) = \sum_{i=1}^{N(t)} Y_i. \tag{2.2}$$

In the cumulative damage model the system will fail when the total damage exceeds a threshold value x, and the time of failure T_x is a random variable given by

$$T_x = \inf\{t : D(t) \geq x\}. \tag{2.3}$$

The cumulative shock model is very useful when analyzing systems whose deterioration depends on the total effect of the shocks over time.

The second shock model found in the literature is a *maximum shock model*. Shocks occur according to a renewal process, and the ith shock causes a random amount of damage Y_i. However, the system will function until the magnitude of a single shock exceeds the threshold value x. The random variable S_i denotes the occurrence time of the ith shock, and the dependence of the random failure time T_x on the number of shocks and their magnitude can be characterized as

$$T_x = \inf\{S_i : \max Y_i \geq x\}. \tag{2.4}$$

This model is a special case of the cumulative shock model and is used to compute the reliability of "loaded" systems, where a system experiences random loads and fails when a single load exceeds the system's capacity. The modern body of literature on shock-based failure models stems from the seminal paper of Esary et al. [12] published in 1973.

Esary et al. [12] consider a system subject to random shocks that occur according to a Poisson process. Their paper deserves special attention because it is the first to examine the effects of random shocks and wear. The important shock model results of [12] are reviewed here and the results for wear models are addressed in a subsequent section. The authors define the probability $\widetilde{P}_k$ as the probability the system survives the first $k = 1, 2, ...$ shocks. The probability the system survives beyond time t is then given by

$$\widetilde{H}(t) = \sum_{k=0}^{\infty} \widetilde{P}_k e^{-\lambda t} \frac{(\lambda t)^k}{k!}. \tag{2.5}$$

The authors demonstrate that the various properties of the discrete failure time distribution $\widetilde{P}_k$ are reflected in the corresponding properties of the continuous life distribution $\widetilde{H}(t)$. They then investigate the properties of various models including the cumulative damage and maximum shock models. Finally, the authors investigate the properties of a shock model whose damage threshold may be modelled as a random variable. This model is appropriate to describe systems in which there is significant individual variation in a unit's ability to withstand damage.

Råde [31] introduced a new model, the parallel shock model. A *parallel shock model* is used to describe a system having n identical components, with shocks occurring according to a Poisson renewal process. The first case Råde [31] studies is as follows. When a shock occurs, each component will fail with probability p, independent of all other components. A simple closed-form expression for the Laplace-Stieltjes transform of the failure time distribution and the expected time until system failure is derived.

Råde [31] next considers the case in which shocks occur in the same manner, but the probability of component failure depends on the number of functioning components. This is a more realistic assumption for load-sharing systems. In the second model, the chance a single component fails is p_k, where k is the number of functioning components. Råde's results [31] for the load sharing model were derived using elementary probability arguments and by conditioning on the number of components that fail at the time of the first shock occurence..

Esary et al. [12] and Råde [31] considered systems for which the time between shocks is independent. Shanthikumar and Sumita [33] examined the case when shocks are correlated with the renewal process. Their paper considers a maximum shock model associated with a correlated pair of random variables X_n and Y_n, where X_n is the magnitude of the nth shock and Y_n is the time between the nth and $(n-1)$st shock. The authors developed two related models. In the first model the magnitude of the nth shock depends on the length of the interval since the last shock, and in the second model the magnitude of the nth shock depends on the time until the next shock. The authors analyze a sequence of independently and identically distributed random variables $(X_n, Y_n),\ \ n = 0, 1, 2, ...$ with common joint distribution function. The variates (X_n, Y_n) is independent pairwise, but X_n and Y_n may be correlated for a given n; thus, (X_n, Y_n) are called a correlated pair of renewal sequences.

When X_n and Y_n are independent, and Y_n are identically and exponentially distributed, the general model simplifies to the Poisson shock model of Råde [31], but the Poisson shock model is not appropriate if X_n and Y_n are correlated. There are many such examples in the real world. A *stochastic clearing system* produces inventory, and the inventory grows in quantity over time. The orders are filled at random times, and the random time between shipments is an independent, identically distributed random variable with some general distribution. The shipments form a renewal process and can be viewed as shocks. The amount of cleared product will depend on the time since the last shipment. Shanthikumar and Sumita [33] develop a transform result, an exponential limit theorem, and properties of the associated renewal process of the failure times.

In a subsequent paper, Sumita and Shanthikumar [37] revisit this model, and incorporate a cumulative shock model. Sumita and Shanthikumar [37] derive similar transform results as in their previous paper, as well as asymptotic properties of the system failure time.

Igaki et al. [18] extend Shanthikumar and Sumita [33] by incorporating the influence of an external system. This philosophy allows the shock process to be driven by another independent process. This is the first *state-dependent shock model* and extends the work Çinlar [6] did on state-dependent wear processes. In [18] the system is assumed to occupy some state $i \in S = \{1, 2, ..., K\}$ for an exponentially distributed random amount of time, and then a shock occurs. When the external system is in state i, shocks occur with rate λ_i. The magnitude of the shock is a random variable whose distribution function is $F_i(\cdot)$. When a shock occurs, the system transitions to state $j \in S, i \neq j$.

In the model by Igaki et al. [18] the system changes state after each shock according to a continuous-time Markov chain (CTMC). The joint distribution of (X_n, Y_n) then depends on the transitions of the underlying CTMC. The failure time distribution was derived as a Laplace transform solution for both the cumulative and maximum shock models. Second, expressions for the expected value, second moment, and variance are found for each model. Finally, the authors turn their attention to the limiting behavior by proving a theorem that gives the limit of the failure time distribution as $t \to \infty$.

More recently, Skoilakis [35] generalized the results of Råde [31] and Nakagawa's [27]. In the original models, a system has parallel components that are subject to shocks according to a standard renewal process. Each time a shock occurs, the components fail independently with probability equal to the magnitude of the shock. The shock intensity distribution is temporally homogenous. Skoilakis [35] claims these models are not realistic and extends them by allowing the magnitudes of the shocks to change with time. For example, when a system runs for some length of time it accumulates damage. This damage can be thought of as wear, and a system that is wearing out will be more susceptible to shocks. This can be modelled by allowing the shocks to increase in magnitude and intensity as time passes. Skoilakis' model accounts for system wear out or system repairs.

If the expected value of the inter-shock times is finite, and the jth shock, independent of everything else, has a random magnitude $x \in [0, 1]$, with distribution G_j then the author derives an expression for the mean failure time and the Laplace-Stieltjes transform of the failure time distribution. His second result states that if the inter-shock time distribution is absolutely continuous, then the failure time distribution is also absolutely continuous.

Ebrahimi [10] developed a technique for the comparison of different cumulative shock models. He assumes shocks occur according to a Poisson process, and each shock causes a random amount of damage. As with all cumulative shock models, the system fails when the total damage caused by the shocks exceeds a threshold x. He derives sufficient conditions for the failure rate order and stochastic order to hold between the random lifetimes of two systems whose damage can be described as a cumulative Poisson shock model. His results are important because they allow the direct comparison of two random processes via stochastic ordering.

The literature reviewed thus far has focused on derivation of the failure time distributions and their associated asymptotic properties. Gottlieb [15] does not consider a specific damage process; instead he assumes the device is more likely to fail as the total damage increases. Gottlieb [15] developed sufficient conditions for both the damage process and the device's ability to survive damage that guarantee the lifetime distribution has an increasing failure rate (IFR). He next identifies classes of stochastic damage processes that satisfy these conditions.

Shock models are useful tools to describe the failure mechanism of many systems, but they cannot accurately describe the reliability of systems that are subject to continuous wear. In the next section the relevant wear models and their associated properties are reviewed.

2.3 Wear Models

Shock models assume damage to a system is sustained at discrete times, but this assumption is too restrictive for systems that constantly experience wear. Early wear models used classic shock models and examined the results when the shocks occurred continuously. More sophisticated wear models were quickly developed to analyze reliability measures of systems subject to continuous wear. A general *state-dependent wear model* assumes a system accumulates damage at a rate $r(i), i \in S = \{1, 2, ..., K\}$, which depends on the state the system occupies. The system changes states according to some semi-Markov process until some time when the total amount of damage exceeds a threshold x, and the system ceases to function. Esary et al. [12] defined a general wear process $\{Z(t) : t \geq 0\}$ such that

1. $Z(0) = 0$, and $Z(t + \Delta t) - Z(t) \geq 0$ for all $t, \Delta t \geq 0$ with probability 1,
2. $\{Z(t) : t \geq 0\}$ is a Markov Process, and
3. $P\{Z(t + \Delta t) - Z(t) \leq u | Z(t) = z\}$ is decreasing in z and t for $t, z, \Delta t \geq 0$.

They prove a system subject to continuous wear has an increasing hazard rate average. The proof is accomplished by foregoing the assumption of discrete damage occurring at isolated moments, and instead investigating the properties of a system experiencing continuous shocks. Other results for similar wear models exist in [26] and [5]. Wear processes like the one described in [12] are insufficient to characterize many real systems, and a recent trend has been the development of failure models that consider two processes: one for the traditional wear process and a second for the ambient process that drives the wear process.

Çinlar [5] offers a rigorous mathematical way to describe this failure mechanism by a *Markov additive process* (MAP). A MAP is a bivariate stochastic process $\{(Z(t), X(t)) : t \geq 0\}$. The $Z(t)$ process is an independent CTMC, and the $X(t)$ process is an additive functional of the first. Using a MAP, Çinlar [6] derives results for shock and wear models with this unique structure. He assumes stationary independent increments of the damage process, meaning that the system is temporally homogeneous, and the damage rate depends only on the state of the system. Under this assumption, the author derives the failure time distribution, where the damage threshold is a random variable. Also, he obtains the hazard rate function for the time to failure of a multiplicative killing type failure mechanism. Finally, Çinlar [6] shows that, given a gamma process with a shape parameter that is a function of Brownian motion, the resulting lifetime distribution is Weibull.

Singpurwalla [34] offers a comprehensive review of the results obtained for models whose failures depend on a dynamic environment. In his work, the author examines the four strategies developed to address these systems. The first strategy describes the wear of a system as a diffusion process. The second strategy models the damage mechanism caused by a shock-inflicting process. Third, a response variable that is strongly correlated with the system's lifetime is modelled using a stochastic process. Finally, a covariate process, like the MAP due to Çinlar's [5], is used to link the damage process to the underlying excitation process.

Many wear models exist, but by Çinlar's [6] own admission they are not easily implemented. Much of the recent literature on stochastic shock and wear processes focuses on deriving more readily implementable results. Kharoufeh [19] uses a MAP to model a system that is subject to continuous, state-dependent, linear wear. His main result is a compact transform expression for the failure time distribution. His results are easier to implement numerically than previous works in the field. The current thesis research effort is an extension of his model and uses similar analysis techniques.

Kharoufeh [19] examined a single-unit that accumulates damage via a continuous wear process, $\{X(t) : t \geq 0\}$. The system begins in perfect working condition and accrues wear at various rates which depend on an external random environment. The random environment is modelled as a continuous time stochastic process $\{Z(t) : t \geq 0\}$, and when the ambient process is in state $i \in S = \{1, 2, ..., K\}$, the system accumulates damage at a linear rate $r(i)$. The system continues to function until the instant when the accumulated damage reaches some threshold x. The wear process is assumed to be temporally homogeneous, and the environmental process has a finite state space.

Under these assumptions the author developed a Laplace-Stieltjes expression for the failure time distribution. Using the relationship between the total wear accumulated at time t, $X(t)$, and the random time until failure, T_x, he derived the failure time distribution by solving a system of first order, linear partial differential equations. The moments of the failure time distribution were also derived. This thesis extends his analysis by superimposing random shocks on the wear model. To that end, the literature for systems subject to compound damage processes is reviewed in the next section.

2.4 Compound Damage models

This thesis will extend the results of Kharoufeh [19] by incorporating random shocks into the state-dependent wear process. Currently, there are few models that incorporate the effects of both shocks and wear. Most of these models allow for shocks to occur only at transition epochs. Çinlar's model [6] is aimed at studying systems that are subject to continuous wear and shocks. He assumes shocks occur often, but the magnitude of individual shocks is very small. Define a *compound Poisson process* as one in which shocks occur according to a Poisson process with rate λ, and the magnitudes are identically and independently distributed random variables with distribution ϕ. Next, he defines a gamma process as one in which the damage process has the gamma density. Finally, a compound Poisson process in a random environment is one in which shocks occur with some rate λ_k, depending on the state of the system, k. Çinlar [6] derives the failure time distribution and associated properties for these models. His results, however, are not easily implemented.

Several researchers have examined the effect maintenance has on systems whose reliability is a function of its operating environment. Klutke et. al. [22] examined the availability of an inspected system whose inter-inspection times and wear rates were random. A subsequent paper by Klutke and Yang [23] derives availability results for a system subject to constant degradation, shocks, and a deterministic inspection policy. Kiessler et. al. [21] studied a system with soft failures, whose wear rate depended explicitly on a continuous-time Markov chain. These models encompass the effect a systems operating environment has on its availability, but they do not allow for the system to simultaneously experience state-dependent wear and random shocks.

2.5 Optimal Replacement Literature

One of the most significant applications of shock models is the development of an optimal maintenance policy. In general, an optimal replacement policy balances the cost of preventative replacement with the higher cost of replacing a failed system. An optimal policy minimizes an associated cost function over a given planning horizon.

Nakagawa [27] found an optimal replacement policy for Råde's [31] model. Recall that Råde's model was an n parallel system that was subject to shocks according to a Poisson process. Each component independently has a probability of failure p. The probability of failure is determined by the magnitude of the shock. The magnitude of the shock is a random variable with a known probability distribution. Nakagawa [27] considered the following replacement problem. The entire system is exchanged preventively before system failure if the total number of failed components is greater then some number k. If the system fails, it is replaced at the time of failure. A cost c_F occurs when the system fails and is replaced. A cost c_P is incurred when the system is replaced prior to failure. This problem is only interesting if $c_F > c_P$. Nakagawa [27] derives an expression for k that minimizes the expected cost per unit time in the long run.

Nakagawa's [27] results were valid for Råde's [31] model; however, there are many generalizations of this optimization problem. Feldman [13] considered a more general semi-Markov shock model. In that work a single-unit system is subject to randomly occurring shocks. The magnitude of the shocks is also random. The system will fail when the total accrued damage exceeds some threshold. The probability of failure is a function of the total damage done by previous shocks. This is said to be a semi-Markov process because both the time between shocks and the damage done depend only on the current damage level.

Using this framework Feldman [13] first developed a cost function Ψ to be minimized. To that end, he used a Markov renewal argument to derive the replacement time distribution and the expected replacement time. Next, he turns his attention to the derivative of the cost function. Setting the derivative equal to zero and simplifying provides the optimal result.

In a later paper, Feldman [14] finds an optimal replacement policy for a system that is subject to continuous shocks. His first paper assumed that a finite number of shocks occur in a finite time span. In the later work he relaxes this assumption and allows for a collection of shock times that form a random set containing no isolated points. Feldman [14] uses an embedded MAP to solve the optimal replacement problem. He develops the theory for the optimal stopping of a MAP. Using the optimal stopping theory, he obtains an optimal replacement policy.

Gottlieb [16] creates a more encompassing model and finds an optimal replacement policy. In his model there is no assumption about the monotonicity of the failure rate (previous models required non-decreasing failure rates). Gottlieb [16] also allows for replacement at any time, not just immediately after shocks. Under these conditions he finds the optimal replacement policy is to replace when the item reaches a threshold which depends on the state after the last jump.

Posner and Zuckerman [30] extend all of the previous optimal replacement problems by examining them as control limit policies and applying the theory of Markov decision processes. A *control limit policy* is a replacement rule in which a system is replaced on failure or when the damage level exceeds some threshold value x. Treating the problem in this way allows the authors to examine the structure of the optimal replacement policy and specify sufficient conditions under which the optimal policy possesses the control limit property. The control limit property allows the user to find a threshold value x that triggers a replacement. Using well-known methods of Markov decision processes, the optimal value of x was derived.

2.6 Summary

The study of reliability has evolved during the last century from a minor application of probability to its own mathematical discipline. Modern reliability theory is a vast field incorporating classical probability and renewal theory. The brief review of reliability theory's development in section 2.1 highlighted the unique way this science has evolved over the past century. Today's systems are very complex, and the previous methods of analyzing reliability measures became obsolete. Researchers began to incorporate the techniques of stochastic modelling to help keep pace with the complexity of the systems they were analyzing.

The development of stochastic models provided researchers with powerful tools to analyze the behavior of systems whose failure mechanisms may be characterized by random processes. The relevant shock and wear models were reviewed to develop a historical perspective on the current research. Although these models only consider a single damage mechanism, understanding the analysis used to develop the results is crucial in solving the current problem.

Recent research in this field has aimed at modelling the damage mechanism as a function of some other underlying stochastic process. This technique allows researchers to more accurately model the failure dynamics of complex systems, and will be used in developing the analysis of this thesis. A model that considers a system's reliability without considering the characteristics of its operating environment may lead to an incorrect analysis of the system's reliability. One technique used to capture the dependence of a wear process on some external excitation process is the use of a Markov additive process to model the failure mechanism.

Currently, there are a limited number of models that effectively describe systems that are simultaneously subject to random shocks and wear. The salient models have been reviewed in section 2.4, and it is worth noting that the results that do exist in the literature are not easily implemented. This thesis will partially close this gap in the current literature by investigating a system that is subject to continuous state-dependent wear and random shocks, using a stationary, bivariate stochastic process. The ultimate goal of this research is to contribute a tractable solution for the failure time distribution, moments, and long-run availability of a single-unit system subject to this type of failure mechanism.

3. Formal Model Description

The main analytical results of this thesis are derived in this chapter. The first section describes the mathematical model and notation used throughout. The failure time distribution is derived as a two-dimensional transform in the second section. That solution is reduced to a one-dimensional transform in the third section. The one-dimensional analytical solution is used in section 3.4 to derive the moments of the failure time distribution. Finally, this chapter concludes by investigating the long-run availability of a single-unit system under an inspect-and-replace maintenance policy.

3.1 Mathematical Model

Consider a single-unit system that incurs damage due to state-dependent, continuous, linear wear and random shocks. These two damage mechanisms are independent stochastic processes. The continuous state-dependent wear process and its unique properties are first examined. A state-dependent wear process depends on an external environmental stochastic process. This process is intended to model the normal operating conditions of the system under consideration. It is assumed that the unit's normal operating environment has a finite number of states. The elements of the set $S = \{1, 2, \ldots, K\}$, ($K \geq 2$ is an integer) corresponding to the states in which the system may operate. Without loss of generality, assume the environment begins in state i at time $t = 0$, and remains in state $i \in S$ for a random amount of time, called the sojourn time. Then the environment instantaneously transitions to state $j, j \neq i, j \in S$ and remains there for a random amount of time before transitioning to another state in S. The system continues to evolve randomly over time forming a finite state-space stochastic process.

The random variable $Z(t)$ denotes the state of the ambient environmental process at time t, and it assumes values in the state space $S = \{1, 2, ..., K\}$. The sojourn times in state i are independent and identically distributed exponential random variables with rate $\mu_i, i \in S$. Therefore, $\{Z(t) : t \geq 0\}$ is a special stochastic process, called a finite state-space continuous-time Markov chain (CTMC). Formally, $\{Z(t) : t \geq 0\}$ characterizes the evolution of the ambient environment for all $t \geq 0$. Furthermore, the probability of transitioning from state $i \in S$ to state $j \in S$ at time t is assumed to be time homogeneous and is given by

$$p_{ij}(t) = P\{Z(t) = j|Z(0) = i\}.$$

The Markovian property of a CTMC guarantees that the probability the system will next transition to state j depends only on the current state of the system. Kulkarni [24] provides a cogent discussion of CTMCs and their important properties.

The system experiences wear at a linear rate that depends on the state of the ambient process. Let $R(t)$ be defined as the wear rate of the system at time t and define a positive function $r : S \to \mathbb{R}^+$, where $\mathbb{R}^+$ is the positive real line. The wear-rate process $\{R(t) : t \geq 0\}$ depends explicitly on the surrounding environment and assumes values in the set $\mathcal{D} = \{r(1), \ldots, r(K)\}$. When the system is in state i $(Z(t) = i)$ it experiences wear at rate $r(Z(t)) = r(i)$. Define the $K \times K$ matrix $\mathbf{R}_D$ such that

$$\mathbf{R}_{D_{i,j}} = \begin{cases} r(i), & i = j \\ 0, & \text{otherwise} \end{cases}.$$

Next, define the random variable $W(t)$ to be the total accumulated wear up to time t, so that

$$W(t) = \int_0^t r(Z(u))du.$$

The process $\{W(t) : t \geq 0\}$ is referred to as a state-dependent wear process and is illustrated in Figure 3.1. The random variable T_x is the time at which the magnitude of the accumulated wear exceeds the fixed threshold value x. Formally defined, the lifetime of a system subject to continuous wear only is given by

$$T_x = \inf\{t : W(t) > x\}.$$

State-dependent wear processes have been studied extensively ([12], [6], [26], [19]). In particular, Kharoufeh [19] derived analytical expressions for the failure time distribution and moments for such a system.

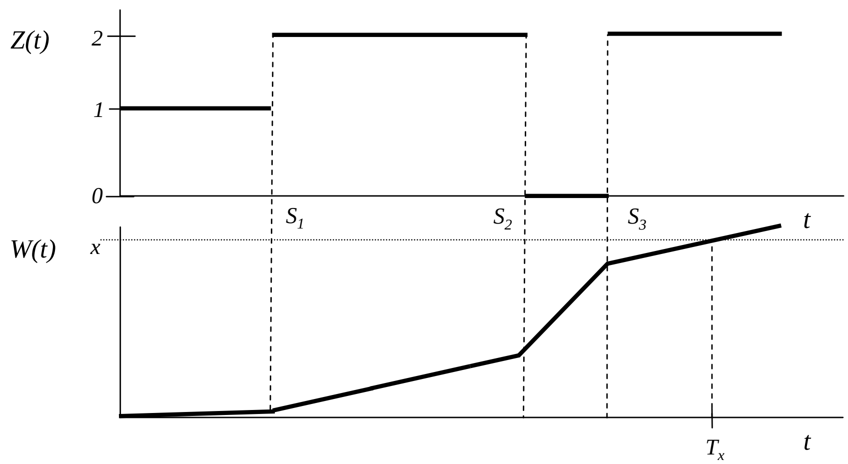

Figure 3.1 Sample paths of cumulative wear and environmental process.

This thesis shall extend the model due to Kharoufeh [19] by superimposing an independent shock process and examining the failure time distribution of a system subject to both continuous wear and random shocks. The shocks in the new model occur according to a Poisson process with intensity λ, implying the times between shocks form a sequence of independent and identically distributed exponential random variables with rate parameter λ. The homogenous Poisson process is denoted $\{N(t) : t \geq 0\}$, where $N(t)$ is the number of shocks that occur in a time interval of length t. The random amount of damage caused by the ith shock is denoted by Y_i. It is assumed that the sequence of random variables, $\{Y_i\}$ is an independent and identically distributed sequence with common CDF $F_Y(y) = P\{Y \leq y\}$. The matrix

$\widetilde{\mathbf{F}}_D(u)$ is defined to be

$$\widetilde{\mathbf{F}}_D(u) = diag\{\widetilde{F}_Y(u)\},$$

where

$$\widetilde{F}_Y(u) = \int_0^\infty e^{-uy} F_Y(dy)$$

denotes the Laplace-Stieltjes transform of F_Y with respect to y. The total amount of damage attributed to random shocks up to time t is the sum of the individual shocks that occur prior to time t, and is given by the random variable,

$$Y(t) = \sum_{i=0}^{N(t)} Y_i$$

so that $\{Y(t) : t \geq 0\}$ forms a compound Poisson process

Define a new random variable $X(t)$ as the total damage accrued by the system up to time t. The total cumulative damage up to t is the sum of the damage caused by the independent wear and shock processes so that

$$X(t) = W(t) + Y(t), \quad t \geq 0.$$

The bivariate stochastic process $\{(X(t), Z(t)) : t \geq 0\}$ completely characterizes the state of the system at time t. The random variable T_x is defined as the failure time of a system subject to both wear and random shocks, so that

$$T_x = \inf\{t : X(t) > x\}. \tag{3.1}$$

A possible sample path of the bivariate process $\{(X(t), Z(t)) : t \geq 0\}$ is illustrated in Figure 3.2.

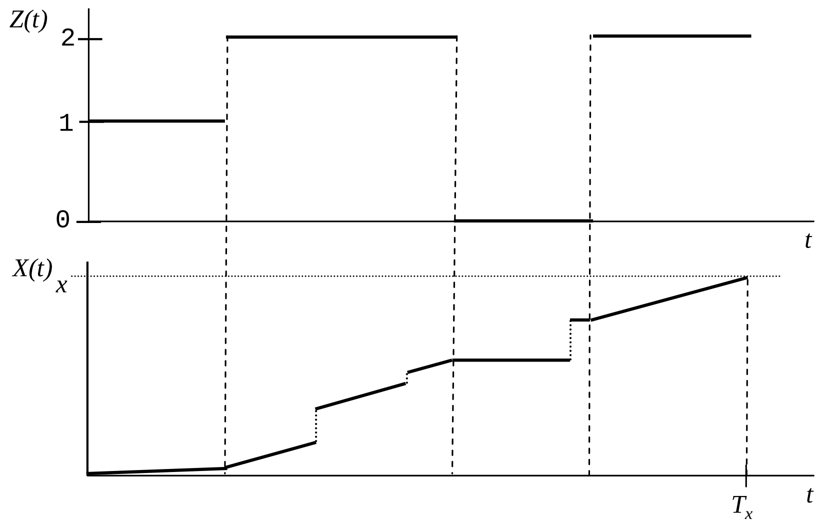

Figure 3.2 Sample paths of the compound damage and environmental process.

This process $\{(X(t), Z(t)) : t \geq 0\}$ characterizes the total damage incurred by the system up to time t, as well as the state of the ambient environment at time t. The failure time distribution is shown to satisfy a system of linear, first-order, partial differential equations. The system is then solved using transform methods, yielding the Laplace-Stieltjes transform of the unconditional failure time distribution for a system subject to state-dependent wear and random shocks.

3.2 Failure Time Distribution

In this section the Laplace-Stieltjes transform of the unconditional failure time distribution $G_x(t) := P\{T_x \leq t\}$ is derived. The failure time distribution is derived by using the unique relationship between the first time to failure and the amount of damage sustained up to time t. If the total amount of damage has not exceeded the threshold x at time t, then the system must fail after time t,

$$\{X(t) \leq x\} \Leftrightarrow \{T_x \geq t\}.$$

This relationship implies that the event, "a failure occurs at or after time t," is equivalent to the event, "the total damage at time t is less than or equal to the threshold x," therefore,

$$P\{T_x \leq t\} = 1 - P\{X(t) \leq x\}. \tag{3.2}$$

Hence, one may compute the failure time distribution if the probability distribution of $X(t)$ is known.

The marginal probability distribution of $X(t)$ is derived by considering the joint probability distribution $\mathbf{V}(x,t)$, where $\mathbf{V}(x,t) = [V_{ij}(x,t)], i,j \in S$. The value $V_{i,j}(x,t)$ is the joint conditional probability that at time t, the degradation of the system is less than the value x, and the environment process is in state j, given the environmental process began in state i,

$$V_{i,j}(x,t) = P\{X(t) \leq x, Z(t) = j | Z(0) = i\}, i,j \in S. \tag{3.3}$$

The joint probability distribution $\mathbf{V}(x,t)$ is computed by examining a small time interval of length ϵ and conditioning on the state of the system at time t. By doing this, the distribution $\mathbf{V}(x,t)$ is proven to satisfy a system of partial differential equations, and the system is solved using well-known methods. In order to solve the system of equations the following transform definitions are needed. First, define the matrix Laplace transform with respect to t to be

$$\mathbf{V}^*(x,s) = \int_0^\infty e^{-st}\mathbf{V}(x,t)dt,$$

and the matrix Laplace-Stieltjes transform with respect to x to be

$$\widetilde{\mathbf{V}}^*(u,s) = \int_0^\infty e^{-ux}\mathbf{V}^*(dx,s).$$

Additionally, the initial probability vector is defined to be $\boldsymbol{\alpha} = [\alpha_i], i \in S$ where $\alpha_i = P\{Z(0) = i\}$. Using these matrix transforms and Equation (3.2) the first main result of this thesis, a closed-form, two-dimensional transform expression for the failure time distribution is derived.

Theorem 3.1 *If the operating environment is a finite, continuous-time Markov chain with infinitesimal generator matrix* $\mathbf{Q}$*, and shocks occur according to a Poisson process with intensity* λ*, then the two-dimensional Laplace-Stieltjes transform of the failure time distribution is given by*

$$\widetilde{G}_u^*(s) = \frac{1}{s} - \boldsymbol{\alpha}\left((s+\lambda)\mathbf{I} + u\mathbf{R}_D - \lambda\widetilde{\mathbf{F}}_D(u) - \mathbf{Q}\right)^{-1}\mathbf{1} \tag{3.4}$$

where $Re(u) > 0$*,* $Re(s) > 0$ *and* $\mathbf{1}$ *is a* $K-$*dimensional column vector of ones.*

Proof. Let $\varepsilon > 0$, then by definition

$$V_{i,j}(x, t+\epsilon) = P\{X(t+\varepsilon) \le x, Z(t+\varepsilon) = j | Z(0) = i\}.$$

Conditioning on the state of the ambient environment at time t by allowing $Z(t) = k, k \in S$ and summing over all possible states in S one may write

$$V_{i,j}(x, t+\epsilon) = \sum_k P\{X(t+\varepsilon) \le x, Z(t+\epsilon) = j | Z(t) = k, Z(0) = i\} P\{Z(t) = k\}.$$

Using the relationship between the joint and conditional probability the expression is rewritten and

$$\begin{aligned} V_{i,j}(x, t+\epsilon) &= \sum_k P\{Z(t+\epsilon) = j | X(t+\epsilon) \le x, Z(t) = k, Z(0) = i\} \\ &\times P\{X(t+\epsilon) \le x | Z(t) = k, Z(0) = i\} P\{Z(t) = k\}. \end{aligned} \tag{3.5}$$

The environmental process $\{Z(t) : t \geq 0\}$ is independent of the damage process $\{X(t) : t \geq 0\}$ implying that

$$P\{Z(t+\epsilon) = j|X(t+\epsilon) \leq x, Z(t) = k, Z(0) = i\} = P\{Z(t+\epsilon) = j|Z(t) = k, Z(0) = i\},$$

and Equation (3.5) becomes

$$\begin{aligned} V_{i,j}(x, t+\epsilon) &= \sum_k P\{Z(t+\epsilon) = j|Z(t) = k, Z(0) = i\} \\ &\times \quad P\{X(t+\epsilon) \leq x|Z(t) = k, Z(0) = i\}P\{Z(t) = k\}. \end{aligned} \tag{3.6}$$

Recalling that $P\{Z(t+\epsilon) = j|Z(t) = k\}$ is denoted $p_{k,j}(\epsilon)$ and $\{Z(t) : t \geq 0\}$ is a CTMC, Equation (3.6) may be written as

$$V_{i,j}(x, t+\epsilon) = \sum_k p_{kj}(\epsilon)P\{X(t+\epsilon) \leq x|Z(t) = k, Z(0) = i\}P\{Z(t) = k\}.$$

Let the random variable $\Upsilon_k(\epsilon)$ be the total damage sustained during the interval $(t, t+\epsilon)$ when $Z(t) = k$. The damage due to wear is a linear function and $\Upsilon_k(\epsilon)$ can be expressed as

$$\Upsilon_k(\epsilon) = r(k)\epsilon + Y(\epsilon).$$

Conditioning on the number of shocks that occur during the interval of length ϵ yields

$$\begin{aligned} V_{i,j}(x, t+\epsilon) &= \sum_k \sum_{n=0}^{\infty} p_{k,j}(\varepsilon)P\{X(t) \leq x - \Upsilon_k(\epsilon), Z(t) = k|Z(0) = i, N(\epsilon) = n\} \\ &\times \quad P\{N(\epsilon) = n\}. \end{aligned}$$

Let $\beta_n \equiv \sum_{i=1}^{n} Y_i$ so that the expression may be rewritten as

$$\begin{aligned} V_{i,j}(x,t+\epsilon) &= \sum_k p_{k,j}(\epsilon)\left(V_{i,k}(x-r(k)\epsilon,t)P\{N(\epsilon)=0\}\right. \\ &+ V_{i,k}(x-r(k)\epsilon-\beta_1,t)P\{N(\epsilon)=1\} \\ &+ V_{i,k}(x-r(k)\epsilon-\beta_2)P\{N(\epsilon)=2\} \\ &+ \left. V_{i,k}(x-r(k)\epsilon-\beta_3,t)P\{N(\epsilon)=3\}+...\right). \end{aligned}$$

Since $\{N(t) : t \geq 0\}$ is a Poisson process (with intensity λ), it is well known [24] that

$$\begin{aligned} P\{N(\epsilon)=0\} &= 1-\lambda\epsilon+o(\epsilon), \\ P\{N(\epsilon)=1\} &= \lambda\epsilon+o(\epsilon), \\ P\{N(\epsilon)\geq 2\} &= o(\epsilon), \end{aligned}$$

where $o(\epsilon)/\epsilon \to 0$ as $\epsilon \to 0$. Substituting for $P\{N(\epsilon) = n\}, n = 0, 1, 2, ...$ and rearranging terms yields

$$\begin{aligned} V_{i,j}(x,t+\epsilon) = \sum_k p_{k,j}(\epsilon)\left[V_{i,k}(x-r(k)\epsilon,t) - \lambda\epsilon V_{i,k}(x-r(k)\epsilon,t)\right. \\ \left. +\lambda\epsilon V_{i,k}(x-r(k)\epsilon-Y_1,t)\right]+o(\epsilon). \end{aligned}$$

The magnitude of subsequent shocks form an independent and identically distributed sequence of random variables, therefore the subscript is subsequently omitted and the magnitude of the first shock is denoted Y. Breaking apart the sum and conditioning on the magnitude of the shock, Y,

$$\begin{aligned} V_{i,j}(x,t+\epsilon) &= \sum_k p_{k,j}(\epsilon)\left[V_{i,k}(x-r(k)\epsilon,t) - \lambda\epsilon V_{i,k}(x-r(k)\epsilon,t)\right] \\ &+ \sum_k \int_0^\infty p_{k,j}(\epsilon)\lambda\epsilon V_{i,k}(x-r(k)\epsilon-y,t)F_Y(dy) + o(\epsilon). \end{aligned}$$

First, the case $k = j$ is removed from the sum leaving

$$\begin{aligned} V_{i,j}(x,t+\epsilon) &= p_{j,j}(\epsilon)\left(V_{i,j}(x-r(j)\epsilon,t) - \lambda\epsilon V_{i,j}(x-r(j)\epsilon,t)\right. \\ &+ \sum_{k\neq j} p_{k,j}(\epsilon)[V_{i,k}(x-r(k)\epsilon,t) - \lambda\epsilon V_{i,k}(x-r(k)\epsilon,t) \\ &+ \int_0^\infty p_{j,j}(\epsilon)\lambda\epsilon V_{i,j}(x-r(j)\epsilon-y,t)F_Y(dy) \\ &+ \sum_{k\neq j}\int_0^\infty p_{k,j}(\epsilon)\lambda\epsilon V_{i,k}(x-r(k)\epsilon-y,t)F_Y(dy). \end{aligned} \tag{3.7}$$

Since $\{Z(t) : t \geq 0\}$ is a continuous-time Markov chain, it is well known [24] that

$$p_{k,j}(\epsilon) = \begin{cases} 1 + q_{kj}\times\epsilon + o(\epsilon), & k = j \\ q_{kj}\times\epsilon + o(\epsilon), & k \neq j. \end{cases} \tag{3.8}$$

Using Equation (3.8), substituting, distributing, and incorporating terms back into Equation (3.7), one obtains

$$\begin{aligned} V_{i,j}(x,t+\epsilon) &= V_{i,j}(x-r(j)\epsilon,t) - \lambda\epsilon V_{i,j}(x-r(j)\epsilon,t) \\ &+ \epsilon\sum_k q_{k,j}\left(V_{i,k}(x-r(k)\epsilon,t) - \lambda\epsilon V_{i,k}(x-r(k)\epsilon,t)\right) \\ &+ \lambda\epsilon\int_0^\infty V_{i,j}(x-r(j)\epsilon-y,t)F_Y(dy) \\ &+ \epsilon^2\lambda\sum_k\int_0^\infty q_{k,j}V_{i,k}(x-r(k)\epsilon-y,t)F_Y(dy) \\ &+ o(\epsilon). \end{aligned}$$

Dividing each side by ϵ and taking the limit as $\epsilon \to 0$, gives

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{V_{i,j}(x,t+\epsilon)}{\epsilon} &= \lim_{\epsilon\to 0}\frac{V_{i,j}(x-r(j)\epsilon,t)}{\epsilon} - \lim_{\epsilon\to 0}\lambda V_{i,j}(x-r(j)\epsilon,t)\\
&+ \lim_{\epsilon\to 0}\sum_k q_{k,j}V_{i,k}(x-r(k)\epsilon,t) - \lim_{\epsilon\to 0}\lambda\epsilon\sum_k q_{k,j}V_{i,k}(x-r(k)\epsilon,t)\\
&+ \lim_{\epsilon\to 0}\lambda\int_0^\infty V_{i,j}(x-r(j)\epsilon-y,t)F_Y(dy)\\
&+ \lim_{\epsilon\to 0}\epsilon\lambda\sum_k\int_0^\infty q_{k,j}V_{i,k}(x-r(k)\epsilon-y,t)F_Y(dy)\\
&+ \lim_{\epsilon\to 0}\frac{o(\epsilon)}{\epsilon}.
\end{aligned}$$

Applying the definition of partial derivatives leaves

$$\begin{aligned}
\frac{\partial V_{i,j}(x,t)}{\partial t} &= -\frac{\partial V_{i,j}(x,t)}{\partial x}r(j) - \lambda V_{i,j}(x,t) + \sum_k q_{k,j}V_{i,k}(x,t)\\
&+ \lim_{\epsilon\to 0}\lambda\int_0^\infty V_{i,j}(x-r(j)\epsilon-y,t)F_Y(dy).
\end{aligned}\tag{3.9}$$

By Lebesgue's bounded convergence theorem, the right-most term of Equation (3.9) may be written as

$$\lim_{\epsilon\to 0}\lambda\int_0^\infty V_{i,j}(x-r(j)\epsilon-y,t)F_Y(dy) = \lim_{n\to\infty}\lambda\int_0^\infty V_{i,j}(x-r(j)\frac{1}{n}-y,t)F_Y(dy)$$

which gives

$$\begin{aligned}
\frac{\partial V_{i,j}(x,t)}{\partial t} &= -\frac{\partial V_{i,j}(x,t)}{\partial x}r(j) - \lambda V_{i,j}(x,t) + \sum_k q_{k,j}V_{i,k}(x,t)\\
&+ \lambda\int_0^\infty V_{i,j}(x-y,t)F_Y(dy).
\end{aligned}\tag{3.10}$$

Recall, $\mathbf{R}_D = diag(r(1), ..., r(K))$ and write Equation (3.10) in matrix form as

$$\frac{\partial \mathbf{V}(x,t)}{\partial t} + \frac{\partial \mathbf{V}(x,t)}{\partial x}\mathbf{R}_D = -\lambda \mathbf{V}(x,t) + \mathbf{V}(x,t)\mathbf{Q} + \lambda \mathbf{V}(\cdot,t) * F_Y(x) \qquad (3.11)$$

where $(*)$ denotes the convolution operator. Next, the Laplace transform, whose operator is denoted $\mathcal{L}^*$, of Equation (3.11) is taken with respect to t. Applying the differentiation property of the Laplace transform,

$$\mathcal{L}^* \left\{ \frac{\partial \mathbf{V}(x,t)}{\partial t} \right\} = s\mathbf{V}^*(x,s) - \mathbf{V}(x,0),$$

yields the following equation,

$$s\mathbf{V}^*(x,s) - \mathbf{I} + \frac{\partial \mathbf{V}^*(x,s)}{\partial x}\mathbf{R}_D = -\lambda \mathbf{V}^*(x,s) + \mathbf{V}^*(x,s)\mathbf{Q} + \lambda \mathbf{V}^*(\cdot,s) * F_Y(x). \qquad (3.12)$$

Next, the Laplace-Stieltjes transform of Equation (3.12) is taken with respect to the spatial dimension,

$$s\widetilde{\mathbf{V}}^*(u,s) - \mathbf{I} + u\widetilde{\mathbf{V}}^*(u,s)\mathbf{R}_D = -\lambda \widetilde{\mathbf{V}}^*(u,s) + \widetilde{\mathbf{V}}^*(u,s)\mathbf{Q} + \lambda \widetilde{\mathbf{V}}^*(u,s)\widetilde{F}_Y(u), \qquad (3.13)$$

where $\widetilde{F}_Y(u)$ is the Laplace-Stieltjes transformation of the cumulative probability distribution of the shock's magnitude,

$$\widetilde{F}_Y(u) = \int_0^\infty e^{-ux} F_Y(dx).$$

Rearranging the terms of Equation (3.13) yields

$$\widetilde{\mathbf{V}}^*(u,s)\left(s\mathbf{I} + u\mathbf{R}_D + \lambda\mathbf{I} - \mathbf{Q} - \lambda\widetilde{\mathbf{F}}_D(u)\right) = \mathbf{I},$$

where $\widetilde{\mathbf{F}}_D(u)$ is a $K \times K$ diagonal matrix with each diagonal element equal to $\widetilde{F}_Y(u)$. Finally, the two-dimensional transform of the joint probability distribution is given by

$$\widetilde{\mathbf{V}}^*(u,s) = \Big((s+\lambda)\mathbf{I} + u\mathbf{R}_D - \lambda\widetilde{\mathbf{F}}_D(u) - \mathbf{Q}\Big)^{-1}, \quad Re(u) > 0, Re(s) > 0.$$

The failure time distribution is obtained as a two-dimensional transform by using the dual relationship of of Equation (3.2) and the initial distribution vector $\boldsymbol{\alpha}$,

$$\begin{aligned} G_x(t) &= 1 - \boldsymbol{\alpha}\mathbf{V}(x,t)\mathbf{1}, \\ G_x^*(s) &= 1/s - \boldsymbol{\alpha}\mathbf{V}^*(x,s)\mathbf{1}, \\ \widetilde{G}_u^*(s) &= 1/s - \boldsymbol{\alpha}\widetilde{\mathbf{V}}^*(u,s)\mathbf{1}, \\ &= 1/s - \boldsymbol{\alpha}\Big((s+\lambda)\mathbf{I} + u\mathbf{R}_D - \lambda\widetilde{\mathbf{F}}_D(u) - \mathbf{Q}\Big)^{-1}\mathbf{1} \end{aligned}$$

with $Re(u) > 0$, $Re(s) > 0$, and $\mathbf{1}$ is a $K-$dimensional column vector of ones.

■

3.3 Dimensional Reduction

The two-dimensional transform solution of Equation (3.4) may be inverted numerically to approximate the solution in the original temporal and spatial domains. However, the existing techniques for two-dimensional numerical inversion are more difficult to implement than one-dimensional inversion algorithms. Furthermore, the two-dimensional techniques are computationally expensive, and unstable, thus, in this section the two-dimensional transform solution is reduced to a single transform dimension. Reducing the solution to a single transform variable will immensely simplify the numerical inversion, making the solution more powerful.

Kharoufeh and Sipe [20] were able to reduce a two-dimensional transform result for a state-dependent wear process (excluding shocks) by converting their original matrix partial differential equation into an ordinary differential equation (ODE), and solving the ODE using an integrating factor. In the spirit of the approach of [20], the one-dimensional solution for our model is derived.

Theorem 3.2 *If the operating environment is a finite, continuous-time Markov chain with infinitesimal generator matrix* $\mathbf{Q}$*, and shocks occur according to a Poisson process with intensity* λ*, then the Laplace-Stieltjes transform of the failure time distribution with respect to* x *is*

$$\widetilde{G}_u(t) \quad = \quad 1 - \boldsymbol{\alpha} \exp\Big((\mathbf{Q} + \lambda(\widetilde{\mathbf{F}}_D(u) - \mathbf{I}) - u\mathbf{R}_D)t\Big)\,\mathbf{1} \tag{3.14}$$

with $Re(u) > 0$*.*

Proof. The result is derived by beginning with the matrix partial differential Equation (3.11),

$$\frac{\partial \mathbf{V}(x,t)}{\partial t} + \frac{\partial \mathbf{V}(x,t)}{\partial x}\mathbf{R}_D = -\lambda \mathbf{V}(x,t) + \mathbf{V}(x,t)\mathbf{Q} + \lambda \mathbf{V}(\cdot,t) * F_Y(x).$$

First, take the Laplace-Stieltjes transform with respect to x, where the symbol $\widetilde{\mathcal{L}}$ denotes the Laplace-Stieltjes transform operator,

$$\widetilde{\mathcal{L}}\left\{\frac{\partial \mathbf{V}(x,t)}{\partial t} + \frac{\partial \mathbf{V}(x,t)}{\partial x}\mathbf{R}_D\right\} \quad = \quad \widetilde{\mathcal{L}}\left\{-\lambda \mathbf{V}(x,t) + \mathbf{V}(x,t)\mathbf{Q} + \lambda \mathbf{V}(\cdot,t) * F_Y(x)\right\}. \tag{3.15}$$

Using the relationship between the Laplace and Laplace-Stieltjes transform, namely

$$\widetilde{\mathcal{L}}\left\{f(x)\right\} = u\mathcal{L}^*\left\{f(x)\right\}, \tag{3.16}$$

it is clear that Equation (3.15) can be written as

$$\frac{d\widetilde{\mathbf{V}}(u,t)}{dt} + u\mathcal{L}^* \left\{ \frac{\partial \mathbf{V}(x,t)}{\partial x} \right\} \mathbf{R}_D = -\lambda u \mathbf{V}^*(u,t) + u\mathbf{V}^*(u,t)\mathbf{Q} + \lambda u \mathbf{V}^*(u,t)\widetilde{F}_Y(u). \tag{3.17}$$

The Laplace transform with respect to x of the partial derivative of the function $\mathbf{V}(x,t)$ is known to be

$$\mathcal{L}^* \left\{ \frac{\mathbf{V}(x,t)}{\partial x} \right\} = u\mathbf{V}^*(u,t) - \mathbf{V}(0,t). \tag{3.18}$$

Using the differentiation property of the Laplace transform in Equation (3.18), Equation (3.17) becomes

$$\frac{d\widetilde{\mathbf{V}}(u,t)}{dt} + u\left(u\mathbf{V}^*(u,t) - \mathbf{V}(0,t)\right)\mathbf{R}_D = -\lambda u \mathbf{V}^*(u,t) + u\mathbf{V}^*(u,t)\mathbf{Q} + \lambda u \mathbf{V}^*(u,t)\widetilde{F}_Y(u).$$

Rearranging terms yields,

$$\frac{d\widetilde{\mathbf{V}}(u,t)}{dt} + u\mathbf{V}^*(u,t)\left(u\mathbf{R}_D + \lambda\mathbf{I} - \mathbf{Q} - \lambda\widetilde{\mathbf{F}}_D(u)\right) = 0,$$

where $\widetilde{\mathbf{F}}_D(u)$ is defined as before. Using the relation in Equation (3.16),

$$\frac{d\widetilde{\mathbf{V}}(u,t)}{dt} + \widetilde{\mathbf{V}}(u,t)\left(u\mathbf{R}_D + \lambda\mathbf{I} - \mathbf{Q} - \lambda\widetilde{\mathbf{F}}_D(u)\right) = 0. \tag{3.19}$$

Define $\mu(t)$ as the integrating factor

$$\begin{aligned} \mu(t) &= \exp\left(\int \left(u\mathbf{R}_D + \lambda\mathbf{I} - \mathbf{Q} - \lambda\widetilde{\mathbf{F}}_D(u)\right) dt\right), \\ &= \exp\left[\left(u\mathbf{R}_D + \lambda\mathbf{I} - \mathbf{Q} - \lambda\widetilde{\mathbf{F}}_D(u)\right) t\right]. \end{aligned} \tag{3.20}$$

For brevity, substitute

$$\mathbf{C} = u\mathbf{R}_D + \lambda\mathbf{I} - \mathbf{Q} - \lambda\widetilde{\mathbf{F}}_D(u),$$

and multiply both sides of Equation (3.19) by the integrating factor of Equation (3.20)

$$\begin{aligned} \frac{d\widetilde{\mathbf{V}}(u,t)}{dt}\exp(\mathbf{C}t) + \widetilde{\mathbf{V}}(u,t)\mathbf{C}\exp(\mathbf{C}t) &= 0, \\ \frac{d(\widetilde{\mathbf{V}}(u,t)\exp(\mathbf{C}t))}{dt} &= 0. \end{aligned} \tag{3.21}$$

The general solution of the ordinary differential equation is obtained by integrating both sides of Equation (3.21) with respect to t,

$$\widetilde{\mathbf{V}}(u,t)\exp(\mathbf{C}t) = \boldsymbol{\psi},$$

where $\boldsymbol{\psi}$ is a matrix of constants of integration. Applying the initial condition $\widetilde{\mathbf{V}}(u,0) = \mathbf{I}$, it is clear that

$$\boldsymbol{\psi} = \mathbf{I}.$$

The specific solution is obtained by reverse substituting for $\mathbf{C}$ and $\boldsymbol{\psi}$,

$$\widetilde{\mathbf{V}}(u,t)\exp\left[\left(u\mathbf{R}_D + \lambda\mathbf{I} - \mathbf{Q} - \lambda\widetilde{\mathbf{F}}_D(u)\right)t\right] = \mathbf{I}. \tag{3.22}$$

Finally, rearranging the terms of Equation (3.22) yields the one-dimensional Laplace-Stieltjes transformation of the joint probability distribution,

$$\widetilde{\mathbf{V}}(u,t) = \exp\left[\left(\mathbf{Q} + \lambda(\widetilde{\mathbf{F}}_D(u) - \mathbf{I}) - u\mathbf{R}_D\right)t\right]. \tag{3.23}$$

Using Equation (3.2) the Laplace-Stieltjes transform of the unconditional failure time distribution is

$$\widetilde{G}_u(t) \;\; = \;\; 1 - \boldsymbol{\alpha}\widetilde{\mathbf{V}}(u,t)\mathbf{1}. \tag{3.24}$$

Equations (3.23) and (3.24) imply the Laplace-Stieltjes transform of the failure time distribution is given by

$$\widetilde{G}_u(t) = 1 - \boldsymbol{\alpha} \exp\left[\left(\mathbf{Q} + \lambda(\widetilde{\mathbf{F}}_D(u) - \mathbf{I}) - u\mathbf{R}_D\right) t\right] \mathbf{1}. \tag{3.25}$$

■

There exists a myriad of methods to numerically invert the one-dimensional Laplace-Stieltjes transform of Equation (3.14). One possible method would be to convert it to a Laplace transform using Equation (3.16) and then use the algorithm of Abate and Witt [1]. In the next section, the one-dimensional Laplace-Stieltjes transform of Equation (3.14) is used to derive a closed-form expression for the nth moment of the failure time distribution.

3.4 Computing Unconditional Moments

In this section an analytical expression for the nth moment of the failure time distribution of a system subject to wear and shocks is derived. The nth moment is computed by evaluating the nth derivative of the Laplace-Stieljes transform of $\widetilde{G}_u(t)$ with respect to t at the value zero. That is,

$$\widetilde{m}_n(u) = (-1)^n \frac{\partial}{\partial s}\left(\widetilde{\mathcal{L}}\left\{\widetilde{G}_u(t)\right\}\right)\bigg|_{s=0}. \tag{3.26}$$

The Laplace-Stieltjes transform of the failure time distribution is

$$\widetilde{G}_u(t) \;=\; 1 - \boldsymbol{\alpha} \exp\left[\left(\mathbf{Q} + \lambda(\widetilde{\mathbf{F}}_D(u) - \mathbf{I}) - u\mathbf{R}_D\right) t\right] \mathbf{1}.$$

Define the function $\boldsymbol{\Phi}_u(t)$ to be

$$\boldsymbol{\Phi}_u(t) = 1 - \exp\left[\left(\mathbf{Q} + \lambda(\widetilde{\mathbf{F}}_D(u) - \mathbf{I}) - u\mathbf{R}_D\right) t\right],$$

and compute the LST of $\boldsymbol{\Phi}_u(t)$ with respect to t. Substitute

$$\mathbf{A} = \mathbf{Q} + \lambda(\widetilde{\mathbf{F}}_D(u) - \mathbf{I}) - u\mathbf{R}_D$$

for brevity to obtain

$$\begin{aligned} \boldsymbol{\Phi}_u(s) &= \int_0^\infty e^{-st} d\boldsymbol{\Phi}_u(t), \\ &= -\mathbf{A} \int_0^\infty e^{-(s\mathbf{I}-\mathbf{A})t} dt. \end{aligned} \tag{3.27}$$

Using fundamental matrix calculus (et. Neuts [29]), the integral of Equation (3.27) is evaluated as

$$\begin{aligned} \boldsymbol{\Phi}_u(s) &= -\mathbf{A}(s\mathbf{I} - \mathbf{A})^{-1}, \\ &= (-s\mathbf{I} + (s\mathbf{I} - \mathbf{A}))(s\mathbf{I} - \mathbf{A})^{-1}, \\ &= -s\mathbf{I}(s\mathbf{I} - \mathbf{A})^{-1} + (s\mathbf{I} - \mathbf{A})(s\mathbf{I} - \mathbf{A})^{-1}, \\ &= \mathbf{I} - s(s\mathbf{I} - \mathbf{A})^{-1}. \end{aligned} \tag{3.28}$$

In the following lemma a closed-form solution for the nth partial derivative of $\boldsymbol{\Phi}_u(s)$ is provided.

Lemma 3.1 *The nth partial derivative of the function $\mathbf{\Phi}_u(s)$ is*

$$\frac{\partial^n}{\partial s^n}\mathbf{\Phi}_u(s) = (-1)^n n!(s\mathbf{I}-\mathbf{A})^{-n}\left(-s(s\mathbf{I}-\mathbf{A})^{-1}+\mathbf{I}\right). \tag{3.29}$$

Proof. The lemma is proven using a mathematical induction argument. First, Equation (3.29) is shown to hold when $n = 0, 1$, and 2. A simple substitution shows the result holds for $n = 0$. The first partial derivative of $\mathbf{\Phi}_u(s)$ is

$$\begin{aligned}\frac{\partial}{\partial s}\mathbf{\Phi}_u(s) &= (s\mathbf{I}-\mathbf{A})^{-2}(\mathbf{I})(s\mathbf{I}) - (s\mathbf{I}-\mathbf{A})^{-1},\\ &= s(s\mathbf{I}-\mathbf{A})^{-2} - (s\mathbf{I}-\mathbf{A})^{-1},\end{aligned}$$

and the second partial derivative

$$\begin{aligned}\frac{\partial^2}{\partial s^2}\mathbf{\Phi}_u(s) &= -2s(s\mathbf{I}-\mathbf{A})^{-3} + (s\mathbf{I}-\mathbf{A})^{-2} + (s\mathbf{I}-\mathbf{A})^{-2}\\ &= -2s(s\mathbf{I}-\mathbf{A})^{-3} + 2(s\mathbf{I}-\mathbf{A})^{-2}.\end{aligned}$$

For the inductive step, the result is assumed to hold for some integer $k > 2$ and is proven true for $k+1$. Using the inductive hypothesis

$$\begin{aligned}\frac{\partial^{k+1}}{\partial s^{k+1}}\mathbf{\Phi}_u(s) &= \frac{\partial}{\partial s}\left((-1)^k k!(s\mathbf{I}-\mathbf{A})^{-k}\left(-s(s\mathbf{I}-\mathbf{A})^{-1}+\mathbf{I}\right)\right)\\ &= \frac{\partial}{\partial s}\left((-1)^{k+1}k!s(s\mathbf{I}-\mathbf{A})^{-(k+1)} + (-1)^k k!(s\mathbf{I}-\mathbf{A})^{-k}\right),\\ &= (-1)^{k+1}k!(s\mathbf{I}-\mathbf{A})^{-(k+1)} + s(-1)^{k+2}(k+1)!(s\mathbf{I}-\mathbf{A})^{-(k+1)-1},\\ &+ k(-1)^{k+1}k!(s\mathbf{I}-\mathbf{A})^{-(k+1)},\\ &= s(-1)^{k+2}(k+1)!(s\mathbf{I}-\mathbf{A})^{-(k+2)} + (k+1)(-1)^{k+1}k!(s\mathbf{I}-\mathbf{A})^{-(k+1)},\\ &= s(-1)^{k+2}(k+1)!(s\mathbf{I}-\mathbf{A})^{-(k+2)} + (-1)^{k+1}(k+1)!(s\mathbf{I}-\mathbf{A})^{-(k+1)},\\ &= (-1)^{k+1}(k+1)!(s\mathbf{I}-\mathbf{A})^{-(k+1)}\left(-s(s\mathbf{I}-\mathbf{A})^{-1}+\mathbf{I}\right).\end{aligned}$$

■

Lemma 3.2 *The nth partial derivative of the function $\mathbf{\Phi}_u(s)$ evaluated at $s = 0$ is*

$$\frac{\partial^n}{\partial s^n}\mathbf{\Phi}_u(s)\bigg|_{s=0} = (-1)^n n!(u\mathbf{R}_D - \mathbf{Q} - \lambda(\widetilde{\mathbf{F}}_D(u) - \mathbf{I}))^{-n}.$$

The proof follows directly from Equation (3.29) by substituting $s = 0$. Next, a closed-form analytical expression for the one-dimensional Laplace-Stieltjes transform of nth moment of the failure time distribution is derived using the results of Lemma 3.1, Lemma 3.2 and Equation (3.26).

Theorem 3.3 *If the operating environment is a finite, continuous-time Markov chain with infinitesimal generator matrix $\mathbf{Q}$, and shocks occur according to a Poisson process with intensity λ, then the Laplace-Stieltjes transform of nth moment of the failure time distribution with respect to x is*

$$\widetilde{m}_n(u) = n!\boldsymbol{\alpha}\left(u\mathbf{R}_D - \mathbf{Q} - \lambda(\widetilde{\mathbf{F}}_D(u) - \mathbf{I})\right)^{-n}\mathbf{1}, \tag{3.30}$$

with $Re(u) > 0$.

Proof. Using Equation (3.26) it is known that

$$\widetilde{m}_n(u) = (-1)^n\, \boldsymbol{\alpha}\frac{\partial^n}{\partial s^n}\mathbf{\Phi}_u(s)\bigg|_{s=0}\mathbf{1}.$$

Using Lemmas 3.1 and 3.2 the final result is obtained after rearranging terms,

$$\begin{aligned}\widetilde{m}_n(u) &= (-1)^{2n}n!\boldsymbol{\alpha}(-\mathbf{A})^{-n}\mathbf{1}, \\ &= n!\boldsymbol{\alpha}\left(u\mathbf{R}_D - \mathbf{Q} - \lambda(\widetilde{\mathbf{F}}_D(u) - \mathbf{I})\right)^{-n}\mathbf{1}.\end{aligned}$$

■

Thus far, only measures of reliability have been addressed. In the next section an inspect-and-replace maintenance policy is defined, and an analytical expression for the system's long-run availability under such a policy is derived using results from renewal theory.

3.5 Availability Measures

The reliability measures derived in the previous sections allow analyst to determine the probability a system will fail at or before some arbitrary time t. However, there remains no tractable way to measure this systems availability. Calculating the instantaneous or average availability of the system is of limited use, in most applications it is sufficient to consider the long-run behavior of the system. The long-run availability of a system is defined to be the limit of the system's average availability, $A(t)$, as the interval tends to infinity,

$$A \;=\; \lim_{t\to\infty} A(t) < \infty.$$

In this section the long-run availability of a system subject to state-dependent wear and random shocks is derived for an inspect-and-replace maintenance policy. An inspect-and-replace maintenance policy is one in which the system is inspected at intervals of τ time units. If upon inspection the system has failed, it is replaced with a new and identical unit; however, if the system is found to be operative, then no action is taken. The inspections are assumed to always correctly diagnose the system's condition.

To derive the long-run availability under these assumptions one must consider the stochastic process $\{\Psi(t) : t \geq 0\}$, where

$$\Psi(t) = \begin{cases} 1 & \text{if } X(t) \leq x. \\ 0 & \text{otherwise} \end{cases}$$

This regenerative process, often called an *up-down machine*, has been studied extensively [2], [24]. A possible sample path of the process is depicted in Figure 3.3.

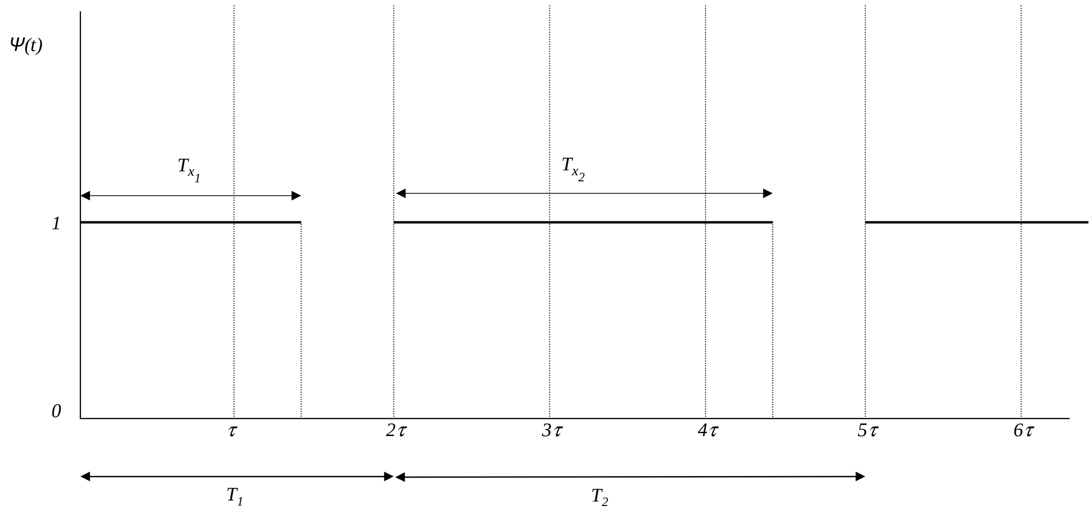

Figure 3.3 Sample path of an up-down machine.

Define the random variable T_{x_i} to be the lifetime of the ith system, so that the sequence of random lifetimes $\{T_{x_1}, T_{x_2}, ...\}$ is independent and identically distributed with a common mean $E[T_x]$. The Laplace-Stieltjes transform of their distribution is given by Equation (3.14). Furthermore, let the random variable T_i be the ith inter-replacement time. The long-run availability of this system is derived using a well-known result from renewal theory, but first, a preparatory lemma, as well as two minor propositions, must be proved.

Proposition 3.1 *The maximum lifetime of a system subject to continuous, state-dependent, linear wear and a Poisson shock process is*

$$\Lambda = \frac{x}{\min\{r(i) : i \in S\}}. \tag{3.31}$$

Proof. Without loss of generality, assume $N(T_x) = 0$ and recall that $N(T_x)$ denotes the random number of shocks that occur in the interval $[0,\ T_x]$. Clearly, the system will operate longer if it accrues damage at the lowest possible rate, $\min\{r(i) :$

$i \in S\}$. Therefore, the sample path that achieves the longest possible lifetime is the one for which the system occupies the most benign environment throughout its life. If the system begins in state j, where $\min\{r(i) : i \in S\} = r(j)$, and remains in state j until failure, then its lifetime is precisely

$$T_x = \frac{x}{r(j)} = \Lambda.$$

In case the system does experience shocks (i.e., $N(T_x) > 0$) the unit's lifetime must be shorter than Λ because shocks are detrimental to the system. ■

The second proposition follows directly from Proposition 3.1.

Proposition 3.2 *For any* $\epsilon \geq 0$

$$G_x(\Lambda + \epsilon) = 1. \tag{3.32}$$

Proof. Proposition 3.1 guarantees that the system will fail at or before $t = \Lambda$, therefore,

$$P\{T_x \leq \Lambda\} = 1 \Rightarrow P\{T_x \leq \Lambda + \epsilon\} = 1. \tag{3.33}$$

■

The needed lemma may now be proven using Propositions 3.1 and 3.2.

Lemma 3.3 *The infinite series* $\{-G_x(k\tau)\}_{k=0}^{\infty}$ *is convergent with sum*

$$\sum_{k=0}^{\infty} -G_x(k\tau) = \gamma - \sum_{k=0}^{\gamma-1} G_x(k\tau), \tag{3.34}$$

where

$$\gamma = \min\{n \geq 1 : n\tau \geq \Lambda\}.$$

Proof. Consider the infinite series

$$\sum_{k=0}^{\infty} -G_x(k\tau) = \int_0^{\tau} dG_x(t) + 2\int_{\tau}^{2\tau} dG_x(t) + 3\int_{2\tau}^{3\tau} dG_x(t) + \cdots . \tag{3.35}$$

The right side of Equation (3.35) may be rewritten in the following way:

$$\begin{aligned} &\sum_{k=0}^{\infty} -G_x(k\tau) = \\ &\quad [G_x(\tau) - G_x(0)] + 2[G_x(2\tau) - G_x(\tau)] + \cdots + \gamma[G_x(\gamma\tau) - G_x((\gamma-1)\tau)] \\ &\quad +(\gamma+1)[G_x((\gamma+1)\tau) - G_x((\gamma)\tau)] + ..., \end{aligned} \tag{3.36}$$

where

$$\gamma = \min\{n \geq 1 : n\tau \geq \Lambda\}. \tag{3.37}$$

Equations (3.32) and (3.37) imply that for all $t \geq \gamma\tau$,

$$G_x(t) = 1. \tag{3.38}$$

Equations (3.36) and (3.38) imply

$$\begin{aligned} &\sum_{k=0}^{\infty} -G_x(k\tau) = \\ &\quad +[G_x(\tau) - G_x(0)] + 2[G_x(2\tau) - G_x(\tau)] + \cdots + \gamma[1 - G_x((\gamma-1)\tau)] \\ &\quad +(\gamma+1)[1-1] + ... \end{aligned} \tag{3.39}$$

Rearranging the terms of Equation (3.39) yields

$$\begin{aligned} \sum_{k=0}^{\infty} -G_x(k\tau) &= -G_x(0) - G_x(\tau) - G_x(2\tau) - ... - (\gamma-1)G_x((\gamma-1)\tau) + \gamma, \\ &= \gamma - \sum_{k=0}^{\gamma-1} G_x(k\tau). \end{aligned} \tag{3.40}$$

■

The main result of this section is now presented.

Theorem 3.4 *If the operating environment is a finite, continuous-time Markov chain with infinitesimal generator matrix* $\mathbf{Q}$*, and shocks occur according to a Poisson process with intensity* λ*, then the long-run availability under an inspect-and-replace maintenance policy where an inspection occurs every* τ *time units is*

$$A = \frac{\widetilde{\mathcal{L}}^{-1}\left\{\boldsymbol{\alpha}\left(u\mathbf{R}_D - \mathbf{Q} - \lambda(\widetilde{\mathbf{F}}_D(u) - \mathbf{I})\right)^{-1}\mathbf{1}\right\}}{\tau\left(\gamma - \sum_{k=0}^{\gamma-1}\widetilde{\mathcal{L}}^{-1}\left\{\widetilde{G}_u(k\tau)\right\}\right)}, \tag{3.41}$$

where $\widetilde{G}_u(t)$ *is the Laplace-Stieltjes transform of Equation (3.14) and* $\widetilde{\mathcal{L}}^{-1}$ *denotes the inverse Laplace-Stieltjes operator.*

Proof.

The long-run availability of an up-down regenerative process is known to be (cf. [2])

$$A = \lim_{t\to\infty} A(t) = \frac{E[T_x]}{E[T]}, \tag{3.42}$$

where $E[T_1] = E[T_2] = \cdots = E[T]$

The Laplace-Stieltjes transform of the mean lifetime for a system subject to state dependent wear and a Poisson shock process is computed by evaluating Equation (3.30) at $n = 1$, therefore

$$E[T_x] = \widetilde{\mathcal{L}}^{-1}\left\{\boldsymbol{\alpha}\left(u\mathbf{R}_D - \mathbf{Q} - \lambda(\widetilde{\mathbf{F}}_D(u) - \mathbf{I})\right)^{-1}\mathbf{1}\right\}. \tag{3.43}$$

An analytical expression for $E[T]$ is next derived.

$$\begin{aligned} E[T] &= E[E[T|T_x]], \\ &= \int_0^\infty E[T|T_x = t] dG_x(t), \end{aligned} \tag{3.44}$$

where $G_x(t) = P\{T_x \leq t\}$. Because a failure will always be found on the first inspection after it occurred, the conditional expectation is given by

$$E[T|T_x = t] = \begin{cases} \tau, & 0 \leq t < \tau \\ 2\tau, & \tau \leq t < 2\tau \\ \vdots & \\ k\tau, & (k-1)\tau \leq t < k\tau \\ \vdots & \end{cases} . \tag{3.45}$$

Using Equations (3.44) and (3.45), one may write

$$\begin{aligned} E[T] &= \tau \int_0^\tau dG_x(t) + 2\tau \int_\tau^{2\tau} dG_x(t) + 3\tau \int_{2\tau}^{3\tau} dG_x(t) + \cdots , \\ &= \tau[G_x(\tau) - G_x(0)] + 2\tau[G_x(2\tau) - G_x(\tau)] + 3\tau[G_x(3\tau) - G_x(2\tau)] + \cdots , \\ &= -\tau(G_x(\tau) + G_x(2\tau) + G_x(3\tau) + \cdots), \\ &= -\tau \sum_{k=0}^{\infty} G_x(k\tau). \end{aligned} \tag{3.46}$$

The infinite series of Equation (3.46) is known to converge by Lemma 3.3, therefore

$$E[T] = \tau \left(\gamma - \sum_{k=0}^{\gamma-1} G_x(k\tau) \right).$$

The Laplace-Stieltjes transform of the failure time distribution is known, therefore $E[T]$ may be computed by evaluating the inverse Laplace-Stieltjes transform

$$G_x(k\tau) = \widetilde{\mathcal{L}}^{-1}\left\{\widetilde{G}_u(k\tau)\right\}.$$

Substituting the appropriate quantities into Equation (3.42) the final result is obtained,

$$A = \frac{\widetilde{\mathcal{L}}^{-1}\left\{\boldsymbol{\alpha}\left(u\mathbf{R}_D - \mathbf{Q} - \lambda(\widetilde{\mathbf{F}}_D(u) - \mathbf{I})\right)^{-1}\mathbf{1}\right\}}{\tau\left(\gamma - \sum_{k=0}^{\gamma-1}\widetilde{\mathcal{L}}^{-1}\left\{\widetilde{G}_u(k\tau)\right\}\right)}, \tag{3.47}$$

■

The Laplace-Stieltjes transform of the long-run availability derived in this section is easily inverted numerically using the algorithm of [1]. Moreover, this result can be used to quickly compare the availability of a system under competing inspection policies, ultimately finding an optimal inter-inspection time that will maximize the long-run availability.

3.6 Summary

This chapter has provided results for the reliability and long-run availability of a system subject to continuous, state-dependent, linear wear and random shocks. Specifically, the failure time distribution of a system subject to this compound damage process was first derived as a two-dimensional Laplace-Stieltjes transform and then reduced to a single dimension. Next, the moments of that distribution were derived using the one-dimensional result of section 3.3. Finally, an expression for the Laplace-Stieltjes transform of the long-run availability under an inspect-and-replace maintenance policy was derived. The utility of these results will be demonstrated in chapter 4.

4. Numerical Results

In this chapter the main results of chapter 3 will be illustrated on five example problems. For each problem, a numerical approximation of the analytical failure time distribution is computed, as are the mean and variance of the failure time distribution. Finally, the long-run availability is computed for each example using an inspect-and-replace maintenance policy. Throughout this chapter, the analytical transform results are inverted numerically using the algorithm of Abate and Whitt [1], and compared to the same measures obtained by simulating the system. The simulation model used in this chapter is explained in the next section

4.1 Simulation

For each problem, a computer simulation of the compound damage process was created using Arena ®, a high-level commercial simulation package. The system is simulated in the following manner: Two entities are created at time $t = 0$. The first entity travels through the environmental process, transitioning from state to state according to the appropriate CTMC. The time spent in each state is recorded, and the wear accumulated for each sojourn time is calculated. After each transition, the total wear is calculated. Meanwhile, the second entity emulates the Poisson shock process. Each time a simulated shock occurs, the magnitude is recorded and the total damage incurred by the shocks is computed. Using the random variables $W(t)$ and $Y(t)$ the simulation calculates the total damage incurred by the system. A simple comparison logic is used to determine if $X(t)$ has exceeded the system's failure threshold value . If the threshold has been breached, then the time the threshold was exceeded is determined; otherwise, the simulation continues.

When the simulated system fails, the failure time is recorded and the replication is terminated. The real system, without maintenance, is assumed to be replaced at failure with a new and identical unit. Therefore, the next replication initializes the system, and runs in the same manner. Using this technique, 100,000 independent replications were conducted. A Student's t-Test determined that 100,000 replications produces a sufficiently small confidence interval about the true reliability measures. Finally, the associated failure times were used to compute an empirical distribution function and other reliability measures.

4.2 Lifetime of an Electric Battery

The first example considers the lifetime of an electric battery used in a diagnostic test set. The process begins when a fully charged battery is placed into service at time $t = 0$. The test set is considered to be operating when a piece of equipment is tested, and idle when no test is being performed. For this process, the test set may be in two distinct states: operating or idle. Hence, the state space is coded as

$$S = \{\text{operating, idle}\} = \{1, 2\}.$$

When the test set is operating, it requires power at a rate of $r(1) = 65/60$ units/hour, and when it is idle it requires power at a rate of $r(2) = 1/4$ units/hour. The battery continuously supplies power at a linear rate, determined by the state of the diagnostic test set. Furthermore, the time to complete a diagnostic test is an exponential random variable with a rate parameter of 25/3 tests/hour, as is the time between tests. Define the random variable $Z(t)$ to be the state of the test set at time t, then the stochastic process $\{Z(t) : t \geq 0\}$ can be characterized by a continuous-time Markov chain (CTMC), whose infinitesimal generator matrix is

$$\mathbf{Q} = \begin{bmatrix} -25/3 & 25/3 \\ 25/3 & -25/3 \end{bmatrix}.$$

Additionally, the battery must supply power to an additional piece of diagnostic equipment. The second piece of equipment is independent of the test set, and requires instantaneous power. The power demands of this piece of equipment can be characterized as a random shock process. The demands occur via a Poisson process with rate 0.5 demands/hour. The magnitude of the power demand is an exponentially distributed random variable with a mean of 0.25 units.

Define the random variable $X(t)$ to be the total power consumption up to time t, and assume the battery has a finite power capacity of 1 unit. Finally, assume the test set begins in state 1 ($Z(0) = 1$) with probability 1. Using the bivariate stochastic process $\{(X(t), Z(t)) : t \geq 0\}$ and the results of chapter 3, the reliability of the battery may be computed. The necessary matrices are:

$$\begin{aligned} \mathbf{R}_D &= \begin{bmatrix} \frac{65}{60} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \\ \widetilde{\mathbf{F}}_D(u) &= \begin{bmatrix} \frac{4}{4+u} & 0 \\ 0 & \frac{4}{4+u} \end{bmatrix}, \end{aligned}$$

and

$$\boldsymbol{\alpha} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The battery will continue to provide power until the total power required by the two pieces of equipment exceeds the total capacity of the battery. Equation (3.14) is used to calculate the Laplace-Stieltjes transform of the failure time distribution,

$$\begin{aligned} \widetilde{G}_u(t) &= 1 - \begin{bmatrix} 1 & 0 \end{bmatrix} \\ &\times \exp\left[\left(\begin{bmatrix} \frac{-25}{3} & \frac{25}{3} \\ \frac{25}{3} & \frac{-25}{3} \end{bmatrix} + \lambda\left(\begin{bmatrix} \frac{4}{4+u} & 0 \\ 0 & \frac{4}{4+u} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) - u\begin{bmatrix} \frac{65}{60} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\right)t\right] \\ &\times \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned} \tag{4.1}$$

The Laplace-Stieltjes transform of Equation (4.1) is converted to a Laplace transform and then approximated at various values of t using the algorithm of [1]. The analytical distribution is shown in Table 4.1, as well as the empirical distribution from the Monte-Carlo simulation. The maximum absolute deviation (MAD) in probability is 0.001492. The quality of the approximation is also depicted in Figure 4.1, where the empirical CDF and the approximated analytical CDF are both plotted against time.

Table 4.1 Analytical versus empirical CDFs for a battery lifetime.

t	Analytical	Simulated	Deviation	t	Analytical	Simulated	Deviation
0.0	0.000000	0.000000	0.000000	1.6	0.821579	0.822560	0.000981
0.1	0.001390	0.001180	0.000210	1.7	0.891184	0.891040	0.000144
0.2	0.003916	0.003670	0.000246	1.8	0.937873	0.936730	0.001143
0.3	0.008151	0.007740	0.000411	1.9	0.966733	0.965800	0.000933
0.4	0.014840	0.013970	0.000870	2.0	0.983266	0.982680	0.000586
0.5	0.024439	0.023700	0.000739	2.1	0.992083	0.991840	0.000243
0.6	0.037958	0.036820	0.001138	2.2	0.996475	0.996420	0.000055
0.7	0.057100	0.056500	0.000600	2.3	0.998522	0.998410	0.000112
0.8	0.083479	0.083160	0.000319	2.4	0.999417	0.999260	0.000157
0.9	0.118717	0.117990	0.000727	2.5	0.999783	0.999700	0.000083
1.0	0.169900	0.170260	0.000360	2.6	0.999924	0.999870	0.000054
1.1	0.247602	0.248730	0.001128	2.7	0.999975	0.999930	0.000045
1.2	0.354030	0.353530	0.000500	2.8	0.999992	0.999990	0.000002
1.3	0.480028	0.479350	0.000678	2.9	0.999998	0.999990	0.000008
1.4	0.609480	0.608730	0.000750	3.0	0.999999	1.000000	0.000001
1.5	0.726675	0.727070	0.000395	3.1	1.000000	1.000000	0.000000

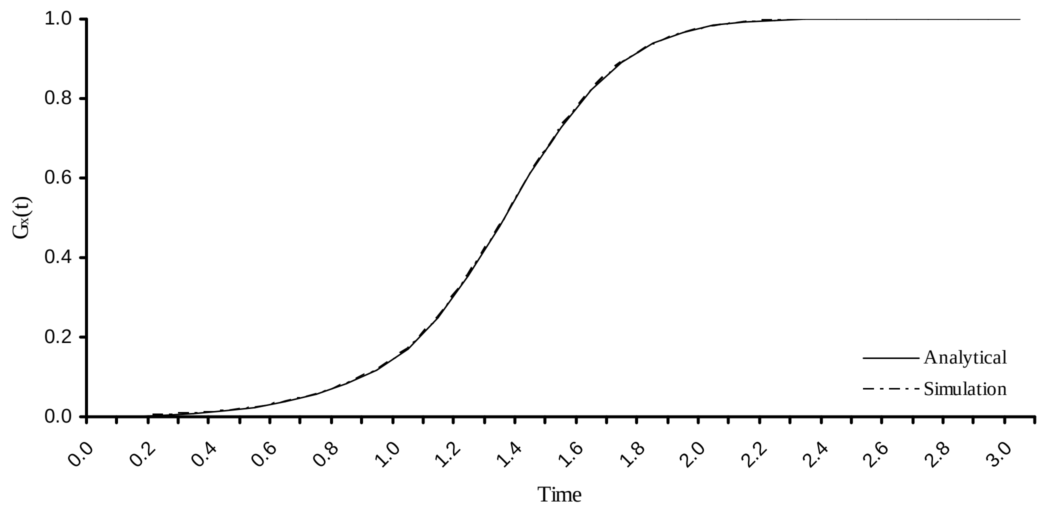

Figure 4.1 Cumulative distribution function of T_x for a battery lifetime.

Next, the analytical mean and variance are compared with those of the simulated data. The mean time to failure (MTTF) of the 100,000 simulated failure times is

$$E[\hat{T}_x] = \frac{1}{100000} \sum_{i=1}^{100000} T_{x_i} = 1.298323,$$

where T_{x_i} is the simulated lifetime of the ith replication. The variance of the failure times, $Var(\hat{T}_x)$, is

$$Var(\hat{T}_x) = \frac{1}{100000} \sum_{i=1}^{100000} \left(T_{x_i} - E[\hat{T}_x]\right)^2 = 0.121089.$$

The same measures are now obtained by numerically inverting the analytical result of Equation (3.30),

$$\begin{aligned} \widetilde{m}_n(u) &= n! \begin{bmatrix} 1 & 0 \end{bmatrix} \\ &\times \left(u \begin{bmatrix} \frac{65}{60} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} - \begin{bmatrix} \frac{-25}{3} & \frac{25}{3} \\ \frac{25}{3} & \frac{-25}{3} \end{bmatrix} - 0.5 \left(\begin{bmatrix} \frac{4}{4+u} & 0 \\ 0 & \frac{4}{4+u} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right)^{-n} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned} \quad (4.2)$$

Again the algorithm of [1] readily inverts Equation (4.2), and the results appear in Table 4.2

Finally, assume battery failures are hidden and that the equipment completes a self-diagnostic test every $\tau = 0.1$ hours. If the battery has failed, then it is instantaneously replaced. If the inspection finds the battery is not failed then no action is taken. The long-run availability of the battery is examined under this inspect-and-replace maintenance policy. Equation (3.31) defines the battery's maximal life to be

$$\Lambda = \frac{x}{\min\{r(i) : i \in S\}} = \frac{1}{0.25}. \tag{4.3}$$

Equations (3.37) and (4.3) imply that $\gamma = 40.0$, therefore using Equation (3.41) the long-run availability is given by

$$\widetilde{A} = \frac{\boldsymbol{\alpha}\left(u\mathbf{R}_D - \mathbf{Q} - \lambda(\widetilde{\mathbf{F}}_D(u) - \mathbf{I})\right)^{-1}\mathbf{1}}{0.1\left(40.0 - \sum_{k=0}^{39}\widetilde{G}_u(0.1k)\right)}, \tag{4.4}$$

The first moment (the numerator) was computed using Equation (4.2). The Laplace-Stieltjes transform of the cumulative distribution function was obtained by Equation (4.1). The numerator and denominator of (4.4) were both inverted numerically using the algorithm of [1]. The numerical approximation of the long-run availability is

$$A = 0.962948.$$

Recall that the long-run availability is

$$\lim_{t\to\infty} A(t) = \frac{E[T_x]}{E[T]}, \tag{4.5}$$

where $E[T]$ is the expected time an inspection will discover a failure. The mean failure and replacement times for the 100,000 replications are 1.298323318 and 1.348338, respectively. Using these values and Equation (4.5), the simulated long-run availability is

$$\hat{A} = \frac{1.298323318}{1.348338} = 0.962906.$$

The analytical and simulated measures of reliability are summarized in Table 4.2

Table 4.2 Comparison of reliability measures for a battery.

Measure	Analytical	Simulation	Deviation
MTTF	1.297626	1.298323	0.000698
Variance	0.121231	0.121089	0.000142
Availability	0.962948	0.962906	0.000042

4.3 Fuel Consumption Model

The second example considers the fuel consumption of an F-16 fighter jet. The F-16 Fighting Falcon is a compact, multirole fighter designed to engage enemy aircraft in air-to-air combat and to attack ground targets. The rate at which the jet consumes fuel depends on its flight profile. The high speeds and constant maneuvering of aerial combat require more fuel than when the F-16 is executing a ground attack mission. Use of the engine's afterburner also causes fuel to be expended. The aircraft can carry fuel internally and in external drop tanks. In this example, the results of this thesis are used to compute the probability that the aircraft will use all of its external fuel before an arbitrary time t.

Define the random variable $Z(t)$ to be the flight environment of the aircraft at time t, so that the sample space is coded as

$$S = \{\text{air-to-ground attack, air-to-air combat, normal flight}\} = \{1, 2, 3\}$$

When the aircraft operates in the normal flight environment it uses fuel at a linear rate of $r(3) = 1.0$ pound/minute. When the F-16 is functioning in a ground attack role the engine consumes fuel at a rate of $r(1) = 3.0$ pounds/minute. Finally, the extreme conditions of air-to-air combat require the engine to burn fuel at a rate of $r(2) = 7.0$ pounds/minute. Furthermore, the time the aircraft spends in flight envelope $i \in S$ is an exponential random variable with intensity μ_i. Therefore, the stochastic process $\{Z(t) : t \geq 0\}$ can be characterized by a continuous-time Markov chain (CTMC), whose infinitesimal generator matrix is

$$\mathbf{Q} = \begin{bmatrix} -10.0 & 5.0 & 5.0 \\ 2.0 & -5.0 & 3.0 \\ 0.4 & 0.6 & -1.0 \end{bmatrix}.$$

Additionally, the pilot will use the afterburner at random intervals, requiring fuel to be dumped into the engine instantaneously. The afterburner can be used in any flight environment and is assumed independent of the flight environment. The use of the afterburner can be thought of as a Poisson shock process with rate 0.01 uses/minute. The magnitude of the fuel needed is a gamma random variable with parameters $\beta = 1$ and $k = 3$, such that the cumulative distribution function is

$$\begin{aligned} F_Y(y) &= 1 - \sum_{r=0}^{k-1} e^{-\beta y} \frac{(\beta y)^r}{r!}, \\ &= 1 - \sum_{r=0}^{2} e^{-y} \frac{(y)^r}{r!}. \end{aligned}$$

The total amount of fuel consumed up to time t is denoted by the random variable $X(t)$. Finally, assume the capacity of the F-16's external fuel tanks is 100 pounds. It is assumed the aircraft has an equal chance of beginning flight in any of the three modes. The random variable T_x denotes the time at which the aircraft must begin using its internal fuel reserves. The probability that the external fuel tanks are insufficient for a sortie of duration t is $G_{100}(t) = P\{T_{100} \leq t\}$. The function

$G_{100}(t)$ can be considered the failure time distribution, and the required matrices are

$$\mathbf{R}_D = \begin{bmatrix} 3.0 & 0.0 & 0.0 \\ 0.0 & 7.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix},$$

$$\widetilde{\mathbf{F}}_D(u) = \begin{bmatrix} \left(\frac{1.0}{1.0+u}\right)^3 & 0.0 & 0.0 \\ 0.0 & \left(\frac{1.0}{1.0+u}\right)^3 & 0.0 \\ 0.0 & 0.0 & \left(\frac{1.0}{1.0+u}\right)^3 \end{bmatrix},$$

and

$$\alpha = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

Using these matrices and Equation (3.14), the Laplace-Stieltjes transform of the failure time distribution was obtained. The approximation of the analytical distribution is shown in Table 4.3 as is the empirical distribution generated by the simulation. The two distributions are also depicted in Figure 4.2.

Next, the numerical approximation of the analytical mean and variance are compared with the those of the simulated data. The mean of the 100,000 simulated failure times is

$$E[\hat{T}_x] = 47.844078,$$

and the variance is

$$Var(\hat{T}_x) = 23.749559.$$

The same measures were calculated numerically by inverting the analytical result of Equation (3.30). The measures appear in Table 4.4.

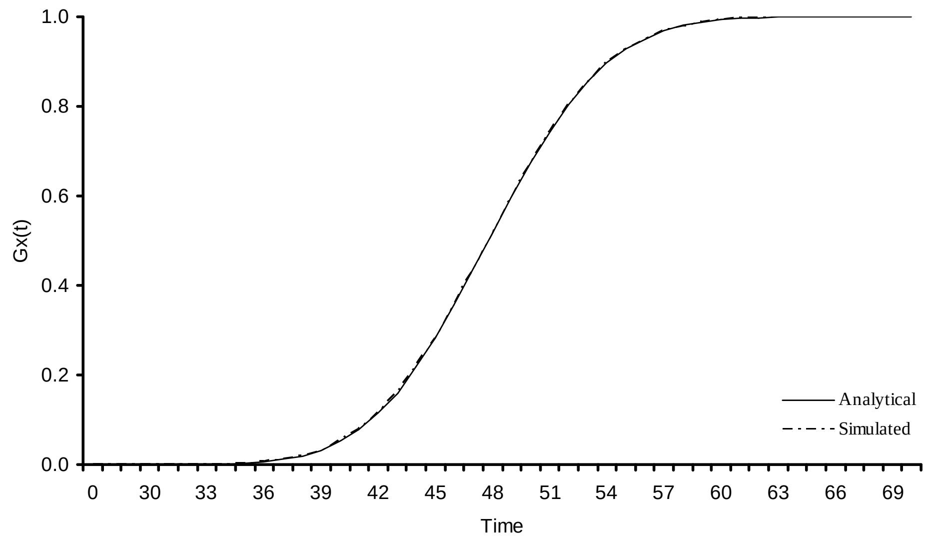

Figure 4.2 Cumulative distribution function of T_x for a fuel consumption model.

The F-16 can carry external fuel tanks to extend its combat capabilities. Consider a sensor that updates the pilot every $\tau = 10.0$ minutes as to which tank the engine is using. If the sensor detects the internal fuel tanks are being used, the pilot is warned and he lands immediately to refuel. The long-run availability in this example can be interpreted as the probability that at an arbitrary time t the aircraft is using its external fuels tanks. The long-run availability is computed using Equation (3.41). The mean analytical failure time was obtained via Equation (3.30) and the long-run availability is

$$A = 0.905818.$$

The same methods employed in the previous example were used to calculate the simulated long-run availability for this and all remaining examples. The results obtained via simulation are compared to the analytical solution in Table 4.4. This example illustrates that the analytical results perform well when applied to a fuel consumption model.

Table 4.3 Analytical versus empirical CDFs for a fuel consumption model.

t	Analytical	Simulated	Deviation	t	Analytical	Simulated	Deviation
28	0.0000053	0.0000000	0.0000053	50	0.6744067	0.6727400	0.0016667
29	0.0000156	0.0000000	0.0000156	51	0.7436441	0.7431400	0.0005041
30	0.0000466	0.0000400	0.0000066	52	0.8041333	0.8041800	0.0000467
31	0.0001250	0.0001000	0.0000250	53	0.8549205	0.8542100	0.0007105
32	0.0003077	0.0002800	0.0000277	54	0.8959107	0.8954700	0.0004407
33	0.0007082	0.0007500	0.0000418	55	0.9277190	0.9267200	0.0009990
34	0.0015347	0.0015200	0.0000147	56	0.9514543	0.9503200	0.0011343
35	0.0031397	0.0033400	0.0002003	57	0.9684865	0.9676000	0.0008865
36	0.0060782	0.0062900	0.0002118	58	0.9802400	0.9794900	0.0007500
37	0.0111620	0.0111300	0.0000320	59	0.9880394	0.9879200	0.0001194
38	0.0194913	0.0190800	0.0004113	60	0.9930157	0.9930700	0.0000543
39	0.0324401	0.0323400	0.0001001	61	0.9960678	0.9962900	0.0002222
40	0.0515734	0.0518300	0.0002566	62	0.9978670	0.9979400	0.0000730
41	0.0784851	0.0787800	0.0002949	63	0.9988859	0.9989500	0.0000641
42	0.1145669	0.1147900	0.0002231	64	0.9994401	0.9994400	0.0000001
43	0.1607379	0.1628500	0.0021121	65	0.9997295	0.9997200	0.0000095
44	0.2171889	0.2188000	0.0016111	66	0.9998745	0.9998900	0.0000155
45	0.2832023	0.2823700	0.0008323	67	0.9999441	0.9999700	0.0000259
46	0.3571021	0.3564800	0.0006221	68	0.9999761	0.9999900	0.0000139
47	0.4363623	0.4359700	0.0003923	69	0.9999902	1.0000000	0.0000098
48	0.5178656	0.5157800	0.0020856	70	0.9999962	1.0000000	0.0000038
49	0.5982695	0.5957300	0.0025395	71	1.0000000	1.0000000	0.0000000

Table 4.4 Comparison of reliability measures for a fuel consumption model.

Measure	Analytical	Simulation	Deviation
MTTF	47.835850	47.844078	0.008228
Variance	23.634982	23.749559	0.114577
Availability	0.905818	0.905740	0.000078

4.4 Tire Tread Wear Model

The useful lifetime of a tire on the M-35 Diesel Engine Driven (DED) vehicle can be characterized by the amount of tread remaining. Furthermore, the total damage done to the tire may be modelled using the compound damage process developed in chapter 3. Define the random variable $Z(t)$ to be the driving conditions of the vehicle at time t. Here the temporal domain is not defined in terms of conventional time. Instead, it is defined in terms of thousands of miles driven, such that 1 unit of "time" is 1,000 miles. The state space of environment may be coded as

$$S = \{\text{transporting cargo, convoy, garrison, off-road, towing}\} = \{1, 2, 3, 4, 5\}.$$

The tread wears at a continuous linear rate which depends solely on the operating conditions of the vehicle. When the M-135 DED operates in state $i \in S$, the tread will wear at rate $r(i)$ mm/1,000 mi. The wear parameters for this problem are in the following vector

$$\mathbf{r} = [\ 1.0 \quad 2.0 \quad 3.0 \quad 4.0 \quad 10.0\ ].$$

Furthermore, the time the vehicle operates in state i is an exponential random variable with rate μ_i. Therefore, the infinitesimal generator matrix is

$$\mathbf{Q} = \begin{bmatrix} -0.500 & 0.125 & 0.125 & 0.125 & 0.125 \\ 0.400 & -2.000 & 0.400 & 0.600 & 0.600 \\ 0.025 & 0.025 & -0.100 & 0.025 & 0.025 \\ 0.050 & 0.050 & 0.050 & -0.200 & 0.050 \\ 1.500 & 1.000 & 1.000 & 1.500 & -5.000 \end{bmatrix}.$$

Additionally, the tread experiences random shocks, possibly caused by harsh breaking or harsh road conditions. This damage process is independent of the wear process. The shock process can be characterized as a Poisson shock process with intensity $\lambda = 0.25$ shocks per 1,000 miles. The magnitude of each shock is distributed according to a gamma probability density function with parameters $\beta = 5$ and $k = 8$. The total damage to the tire is given by the random variable $X(t) = W(t) + Y(t)$, where $W(t)$ is the amount of wear at time t and $Y(t)$ is the total damage incurred by shocks. Finally, assume the tire is considered unsafe to drive after the tread has worn a total of $x = 100$ mm. The initial probability vector for the M-35 DED is

$$\boldsymbol{\alpha} = [\ 0.20 \quad 0.20 \quad 0.20 \quad 0.20 \quad 0.20\ ].$$

The remaining necessary matrices are:

$$\mathbf{R}_D = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 2.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 3.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 4.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 10.0 \end{bmatrix},$$

and

$$\widetilde{\mathbf{F}}_D(u) = \begin{bmatrix} \left(\frac{0.2}{0.2+u}\right)^8 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & \left(\frac{0.2}{0.2+u}\right)^8 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & \left(\frac{0.2}{0.2+u}\right)^8 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & \left(\frac{0.2}{0.2+u}\right)^8 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & \left(\frac{0.2}{0.2+u}\right)^8 \end{bmatrix}.$$

Using these matrices and Equation (3.14), the Laplace-Stieltjes transform of the failure time distribution was obtained. The approximation of the analytical distribution is shown in Table 4.5 as well as the empirical distribution generated by the simulation. The two distributions are also depicted in Figure 4.3. The analytical results closely match those obtained by simulation.

Table 4.5 Analytical versus empirical CDFs for a tread wear model.

t	Analytical	Simulation	Deviation	t	Analytical	Simulation	Deviation
0	0.000000	0.000000	0.000000	19	0.966844	0.967500	0.000656
1	0.006711	0.006860	0.000149	20	0.976395	0.977150	0.000755
2	0.030664	0.030280	0.000384	21	0.983424	0.983870	0.000446
3	0.074461	0.073900	0.000561	22	0.988502	0.988430	0.000072
4	0.136686	0.135750	0.000936	23	0.992140	0.991780	0.000360
5	0.213330	0.211120	0.002210	24	0.994662	0.994310	0.000352
6	0.299173	0.296940	0.002233	25	0.996386	0.996150	0.000236
7	0.388842	0.387790	0.001052	26	0.997571	0.997310	0.000261
8	0.477645	0.475600	0.002045	27	0.998370	0.998200	0.000170
9	0.561752	0.559650	0.002102	28	0.998902	0.998670	0.000232
10	0.638645	0.636260	0.002385	29	0.999268	0.999130	0.000138
11	0.706847	0.704620	0.002227	30	0.999522	0.999470	0.000052
12	0.765582	0.763230	0.002352	31	0.999693	0.999630	0.000063
13	0.815315	0.815150	0.000165	32	0.999810	0.999760	0.000050
14	0.856392	0.856310	0.000082	33	0.999891	0.999840	0.000051
15	0.889820	0.890290	0.000470	34	0.999940	0.999910	0.000030
16	0.916613	0.917310	0.000697	35	0.999968	0.999960	0.000008
17	0.937790	0.938330	0.000540	36	0.999983	0.999980	0.000003
18	0.954269	0.954750	0.000481	37	0.999991	1.000000	0.000009

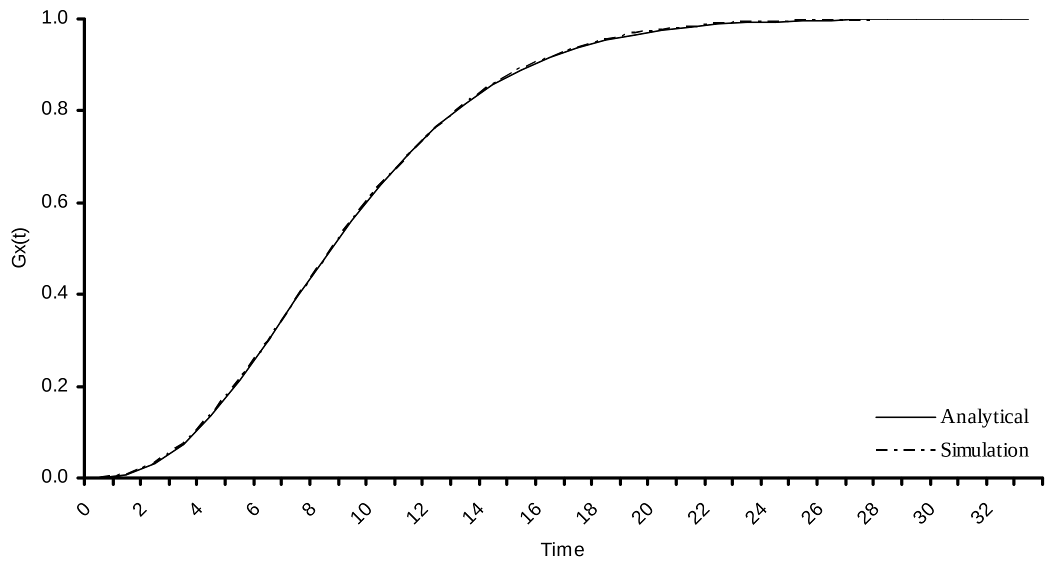

Figure 4.3 Cumulative distribution function of T_x for a tread wear model.

Next, the numerical approximation of the mean and variance are compared with that of the simulated data. The mean simulated time to failure is

$$E[\hat{T}_x] = 8.955105, \tag{4.6}$$

and the variance of the failure times is

$$Var(\hat{T}_x) = 22.103945. \tag{4.7}$$

The same measures are calculated by numerically inverting the analytical result of Equation (3.30). They appear in Table 4.6.

Finally, the long-run availability is considered under the manufacturer's maintenance policy. The tire manufacturer recommends inspecting the tires every 5,000 miles and replacing tires that have exceeded 100 mm of total wear. The amount of time driven on safe tires can be considered the long-run availability of the tire under such a maintenance policy. The long-run availability is computed for this inspect-and-replace policy using Equation (3.41)

$$A = 0.782018.$$

The simulated data was used to calculate the average availability over the duration of the simulation,

$$\hat{A} = 0.782242.$$

It is compared to the analytical solution in Table 4.6.

Table 4.6 Comparison of reliability measures for tire tread.

Measure	Analytical	Simulation	Deviation
MTTF	8.938213	8.955105	0.016892
Variance	22.137441	22.103945	0.033496
Availability	0.782018	0.782242	0.000224

4.5 Lifetime of an Electric Circuit

This section considers an electric circuit whose operating environment consists of 10 states, $S = \{1, 2, ..., 10\}$. The circuit is installed in an unit whose environment at time t, $Z(t)$, can be characterized by a CTMC whose infinitesimal generator matrix is

$$\mathbf{Q} =$$

$$\begin{bmatrix} -1.900 & 0.190 & 0.190 & 0.190 & 0.190 & 0.190 & 0.190 & 0.190 & 0.190 & 0.380 \\ 0.220 & -1.100 & 0.220 & 0.000 & 0.220 & 0.000 & 0.000 & 0.000 & 0.220 & 0.220 \\ 0.027 & 0.000 & -0.900 & 0.603 & 0.000 & 0.270 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.085 & 0.170 & 0.680 & -1.700 & 0.340 & 0.000 & 0.425 & 0.000 & 0.000 & 0.000 \\ 0.080 & 0.000 & 0.000 & 0.000 & -1.600 & 0.320 & 0.080 & 0.240 & 0.320 & 0.560 \\ 0.400 & 0.000 & 0.000 & 0.000 & 0.000 & -2.000 & 0.000 & 0.000 & 0.000 & 1.600 \\ 0.120 & 0.120 & 0.120 & 0.120 & 0.120 & 0.120 & -1.200 & 0.120 & 0.000 & 0.360 \\ 0.100 & 0.000 & 0.000 & 0.150 & 0.000 & 0.000 & 0.000 & -0.500 & 0.100 & 0.150 \\ 1.530 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & -1.700 & 0.170 \\ 0.040 & 0.040 & 0.040 & 0.040 & 0.040 & 0.040 & 0.080 & 0.080 & 0.400 & -0.800 \end{bmatrix}$$

Furthermore, the circuit is known through experimental data to wear at rate $r(i)$/day, when the ambient environment is in state $i \in S$. The appropriate values are contained on the diagonal of $\mathbf{R}_D$. Additionally, the circuit also incurs damage due to the influence of random shocks. These shocks occur according to a Poisson process with intensity $\lambda = 0.25$ shocks/day, and magnitudes that are distributed uniformly over the interval $[0, 5]$, so that the proper $\widetilde{\mathbf{F}}_D$ matrix is

$$\widetilde{\mathbf{F}}_D(u) = \begin{bmatrix} \widetilde{g}(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \widetilde{g}(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \widetilde{g}(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \widetilde{g}(u) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widetilde{g}(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \widetilde{g}(u) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \widetilde{g}(u) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \widetilde{g}(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \widetilde{g}(u) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \widetilde{g}(u) \end{bmatrix},$$

where

$$\widetilde{g}(u) = \frac{1}{5u}(1 - e^{-5u}).$$

Finally, the initial probability vector is

$$\boldsymbol{\alpha} = [\ 0.80 \quad 0.05 \quad 0.00 \quad 0.00 \quad 0.00 \quad 0.00 \quad 0.00 \quad 0.00 \quad 0.00 \quad 0.15\ ].$$

The circuit is assumed to fail when the total damage incurred from wear and shocks exceeds 20 units of damage. Using the appropriate matrices and Equation (3.14) the analytical distribution was approximated using the algorithm of [1]. The analytical CDF was compared to the empirical CDF in Table 4.7 and Figure 4.4.

Table 4.7 Analytical versus empirical CDFs for an electrical circuit.

t	Analytical	Simulation	Deviation	t	Analytical	Simulation	Deviation
0	0.000000	0.000000	0.000000	23	0.848646	0.848710	0.000064
1	0.000000	0.000000	0.000000	24	0.886360	0.885550	0.000810
2	0.000002	0.000000	0.000002	25	0.916610	0.915870	0.000740
3	0.000072	0.000060	0.000012	26	0.940221	0.939950	0.000271
4	0.000329	0.000290	0.000039	27	0.958158	0.958060	0.000098
5	0.001210	0.001160	0.000050	28	0.971418	0.971290	0.000128
6	0.003549	0.003450	0.000099	29	0.980954	0.981150	0.000196
7	0.008523	0.008630	0.000107	30	0.987627	0.987620	0.000007
8	0.017683	0.018140	0.000457	31	0.992169	0.992090	0.000079
9	0.032937	0.033740	0.000803	32	0.995176	0.995410	0.000234
10	0.056102	0.057120	0.001018	33	0.997111	0.997330	0.000219
11	0.088610	0.089870	0.001260	34	0.998321	0.998290	0.000031
12	0.131316	0.133160	0.001844	35	0.999054	0.999030	0.000024
13	0.184232	0.185820	0.001588	36	0.999484	0.999450	0.000034
14	0.246409	0.247350	0.000941	37	0.999728	0.999740	0.000012
15	0.316076	0.317220	0.001144	38	0.999862	0.999840	0.000022
16	0.390865	0.391260	0.000395	39	0.999932	0.999900	0.000032
17	0.468053	0.468300	0.000247	40	0.999968	0.999950	0.000018
18	0.544842	0.544110	0.000732	41	0.999985	0.999980	0.000005
19	0.618626	0.618940	0.000314	42	0.999993	0.999990	0.000003
20	0.687212	0.688090	0.000878	43	0.999997	0.999990	0.000007
21	0.748969	0.748620	0.000349	44	0.999999	0.999990	0.000009
22	0.802903	0.802630	0.000273	45	1.000000	1.000000	0.000000

The mean and variance were computed analytically using Equation(3.30) and are compared to the simulated data in Table 4.8. Next, the long-run availability of the circuit is considered. The electric circuit is assumed to be inspected everyday. If the circuit has failed, then it is replaced with a new component. Otherwise no action is taken. The availability of the system is calculated using Equation (3.41) with the following parameters, $\tau = 1.0$ and $\gamma = 134.0$.

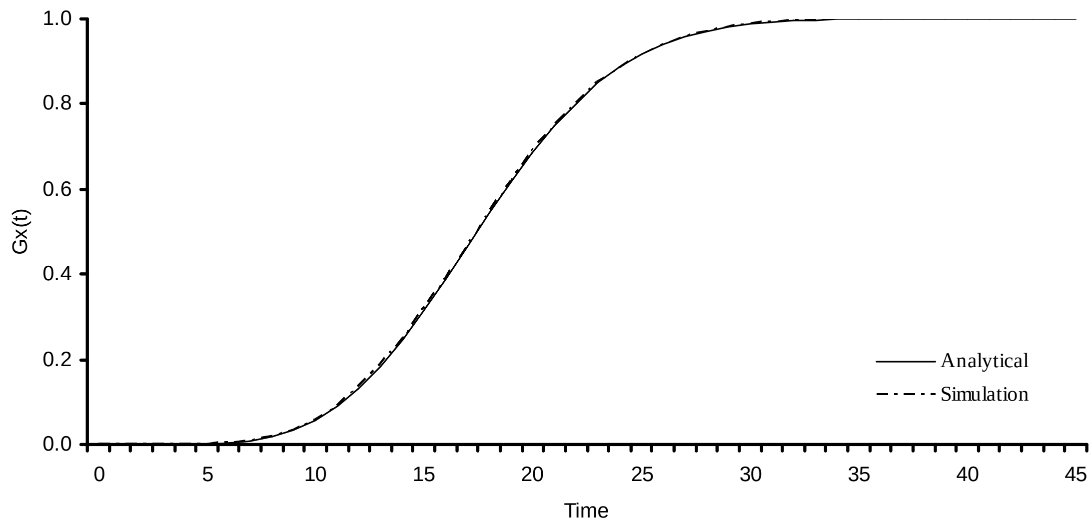

Figure 4.4 Cumulative distribution function of T_x for an electrical circuit.

Table 4.8 Comparison of reliability measures for an electrical circuit.

Measure	Analytical	Simulation	Deviation
MTTF	17.680713	17.712524	0.031811
Variance	25.940322	25.510923	0.429399
Availability	0.972498	0.972494	0.000004

4.6 Failure Dynamics of a Birth-and-Death Process

In this section the operating cost of a manufacturing plant are considered. The manufacturing operation is as follows. Customers place orders with the company according to a stochastic arrival process. The plant has a limited storage and production capacity. Only one order may be processed at a time, and up to an additional 29 orders may be stored in the plant's holding facility. The company receives orders from customers at an effective stationary rate of 0.8 orders/day, and is able to fill those orders at a rate of 1.2 orders/day. The time between successive order arrivals is an exponential random variable, as is the time to process an order. Define the random variable $Z(t)$ to be the number of orders in the system at time t. The $\{Z(t) : t \geq 0\}$ process can be characterized as a birth-and-death process, which is a specific type of CTMC. The infinitesimal generator matrix is a tri-diagonal, 31×31 matrix whose entries are

$$q_{i,j} = \begin{cases} -0.80 & i=0,\, j=0 \\ 0.80 & i=j-1,\, j=1,...,30 \\ 1.20 & i=j+1,\, j=0,...,29 \\ -2.00 & i=j,\, j=1,...,29 \\ -1.20 & i=30,\, j=30 \\ 0.00 & \text{otherwise} \end{cases}.$$

The operating cost of the plant depends on the number of orders in the system and are given in thousands of dollars. It costs $r(i)$ dollars/day to operate the plant if there are $i \in S = \{0, 1, ..., 30\}$ orders in the system. If there are 10 orders or less in the system, then $r(i) = \$0.1, i = 0, 1, ..., 10$. If there are more than 10 orders in the system, then the cost is \$0.2 dollars/day. The $\mathbf{R}_D$ matrix has the following entries

$$\mathbf{R}_{D_{i,j}} = \begin{cases} 0.10 & i=j, \quad j=0,...,10 \\ 0.20 & i=j, \quad j=11,...,30 \\ 0.00 & i \neq j \end{cases}.$$

The plant also incurs other random costs associated with operating a business. These costs can be characterized by a Poisson shock process, where shocks arrive at a rate of 1.0 shock/day and each shock has a random magnitude whose distribution is an exponential random variable with a mean of \$5.00. The $\widetilde{\mathbf{F}}_D$ matrix is a 31×31 diagonal matrix such that

$$\widetilde{\mathbf{F}}_{D_{i,j}}(u) = \begin{cases} \frac{0.2}{0.2+u} & \text{i=j, j=0,...,30} \\ 0, & \text{otherwise} \end{cases}.$$

Finally, the system is assumed, with probability 1, to always begin empty. Hence,

$$\alpha_i = \begin{cases} 1 & i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The time until the cost to operate the plant exceeds the threshold value $x = 1.0$ is a first passage time, and its distribution is calculated using the results derived in chapter 3. Using Equation (3.14) and the preceding matrices, a numeric approximation of the analytical CDF is calculated and displayed in Table 4.9. For comparison, the empirical distribution found by simulation is also in Table 4.9. The two distributions are compared graphically in Figure 4.5.

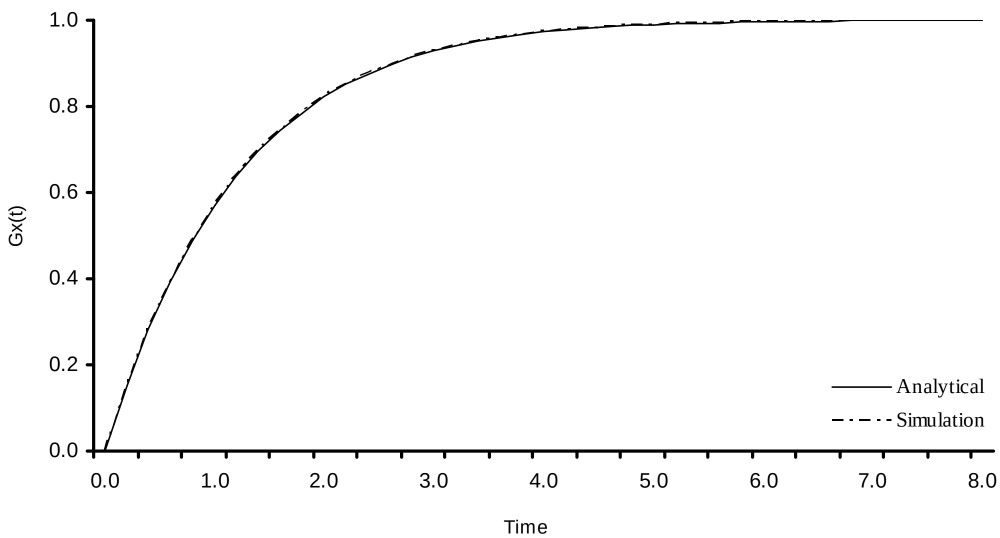

Figure 4.5 Cumulative distribution function of T_x for a birth-and-death process.

Table 4.9 Analytical versus empirical CDFs for a birth-and-death process.

t	Analytical	Simulation	Deviation	t	Analytical	Simulation	Deviation
0.00	0.000000	0.000000	0.000000	5.20	0.991673	0.991810	0.000137
0.20	0.151847	0.151240	0.000607	5.40	0.993117	0.993160	0.000043
0.40	0.281962	0.279870	0.002092	5.60	0.994218	0.994320	0.000102
0.60	0.393193	0.393580	0.000387	5.80	0.995274	0.995340	0.000066
0.80	0.488068	0.487630	0.000438	6.00	0.996285	0.996220	0.000065
1.00	0.568822	0.568610	0.000212	6.20	0.997048	0.996870	0.000178
1.20	0.637418	0.637680	0.000262	6.40	0.997522	0.997320	0.000202
1.40	0.695573	0.696840	0.001267	6.60	0.997897	0.997740	0.000157
1.60	0.744786	0.745450	0.000664	6.80	0.998318	0.998190	0.000128
1.80	0.786357	0.787770	0.001413	7.00	0.998719	0.998460	0.000259
2.00	0.821416	0.822240	0.000824	7.20	0.998983	0.998750	0.000233
2.20	0.850937	0.852230	0.001293	7.40	0.999128	0.999050	0.000078
2.40	0.875750	0.876250	0.000500	7.60	0.999262	0.999230	0.000032
2.60	0.896553	0.897280	0.000727	7.80	0.999429	0.999310	0.000119
2.80	0.913976	0.914760	0.000784	8.00	0.999576	0.999390	0.000186
3.00	0.928594	0.929300	0.000706	8.20	0.999658	0.999470	0.000188
3.20	0.940851	0.941610	0.000759	8.40	0.999702	0.999550	0.000152
3.40	0.951031	0.951620	0.000589	8.60	0.999752	0.999630	0.000122
3.60	0.959413	0.960310	0.000897	8.80	0.999816	0.999710	0.000106
3.80	0.966406	0.966870	0.000464	9.00	0.999866	0.999770	0.000096
4.00	0.972365	0.972850	0.000485	9.20	0.999889	0.999810	0.000079
4.20	0.977366	0.977730	0.000364	9.40	0.999900	0.999850	0.000050
4.40	0.981352	0.981660	0.000308	9.60	0.999919	0.999860	0.000059
4.60	0.984506	0.985040	0.000534	9.80	0.999950	0.999890	0.000060
4.80	0.987219	0.987780	0.000561	10.00	0.999979	1.000000	0.000021
5.00	0.989665	0.990030	0.000365	10.20	1.000000	1.000000	0.000000

A numeric approximation of the mean and variance of the first passage time are calculated using the analytical result in Equation (3.30). The analytical solutions are compared to those measures found via simulation in Table 4.10.

Finally, if the operating costs exceed the threshold, the plant will effectively shut down. The company's accountant reviews the expenditures at the end of each day and if the operating costs have exceeded the threshold, then she immediately pays the outstanding debts and the process renews. Therefore, the necessary parameters are $\tau = 1.0$ and $\gamma = 10.0$. Using the simulated data, the average time the accountant discovers an overdraft is $E[\hat{T}] = 1.72313$ days, and the average time the overdraft occurs is $E[\hat{T}_x] = 1.155588$ days. Equation (4.5) implies the long-run simulated availability is

$$\hat{A} = \frac{1.155588}{1.72313} = 0.670633.$$

Using the analytical results of Equation (3.41), the long-run availability is calculated,

$$A = 0.670967.$$

The simulated long-run availability and numerical approximation of the analytical solution are compared in Table 4.10.

Table 4.10 Comparison of reliability measures for a birth-and-death process.

Measure	Analytical	Simulation	Deviation
MTTF	1.157211	1.155588	0.001623
Variance	1.226178	1.222159	0.004019
Availability	0.670967	0.670633	0.000334

This example demonstrates that the analysis of the reliability model is appropriate for systems whose environmental process includes a large number of states.

4.7 Summary

In this chapter the techniques and results of chapter 3 were applied to compute the relevant reliability measures of various types of systems. Although these examples are somewhat contrived, they demonstrate the wide variety of problems that may be solved using the technique. More importantly, they demonstrate that if the real-world system can be modelled appropriately, the analysis of this thesis provides accurate results without requiring lengthy and time-consuming simulation runs.

More specifically, the numerical approximations obtained with the algorithm of Abate and Whitt [1] closely match the results obtained via simulation. The maximum deviation in probability for these five examples was on the order of 2.5×10^{-3}. However, the computation time needed to obtain the analytical results was orders of magnitude shorter than obtaining the solutions via simulation. The average time required for numerical inversion was only a few seconds, while the average time required to complete the simulation experiment was 4 hours. The computational expedience of the analytical solution makes it possible to quickly compare the reliability of competing maintenance policies.

Although, the analytical solutions of chapter 3 are appealing, they do have minor limitations. The system to be analyzed must be modelled in the manner described in chapter 3. To do this, various parameters (transition rates, degradation rates, etc.) must be known or statistically estimated. Often, it is impossible to know the probability distribution of these parameters. If no information is available, then a hypothesized distribution must be used. Using observational data to estimate the parameters is a valid method but may require a large number of observations to obtain acceptable estimates. However, the same information is required to create a simulation, or any other model used to analyze the system described in this thesis.

The most restrictive assumption of this thesis is that the dynamics of the ambient environment form a CTMC. This assumption requires that state sojourn times be exponentially distributed. For some environments this Markovian assumption is not valid, and the numerical results may not be representative of reality. However, techniques exist to incorporate the Markovian property in a non-Markovian environment. To do this the distributions of the sojourn times must be approximated by phase-type distributions (cf. Neuts [28]). Because phase-type distributions are constructed using phases of exponential distributions the Markovian property is retained. Other possible extensions of this research are discussed in the next and final chapter.

5. Conclusions and Future Research

The field of reliability is concerned with understanding and improving system performance; however, most reliability models fail to capture the effect of a system's operating environment on its useful lifetime. Those models rely on observation of failure times collected in static laboratory environments. These shortcomings lead researchers in the field to develop a new class of state-dependent reliability models which characterize the dependence of the cumulative damage incurred by the unit on the system's operating environment. Incorporating the ambient environment produces an analytical model that more aptly characterizes the system's failure mechanism. However, state-dependent reliability models require more advanced analysis techniques, and often lead to intractable solutions. The existing analysis of compound damage models require computationally expensive and unstable, multidimensional inversion algorithms to provide usable solutions. This thesis provides an analysis of a particular state-dependent reliability model. The main results provide closed-form analytical solutions for a class of compound damage models that may be readily implemented without such cumbersome methods.

An appropriate mathematical model was first constructed and then analyzed. The failure time distribution of a system subject to state-dependent wear and a Poisson shock process was derived as a two-dimensional Laplace-Stieltjes transform. This was accomplished by proving that the joint distribution of the cumulative damage process and the environmental state satisfies a system of partial differential equations, and then solving the system using transform methods. Next, the two-dimensional result was reduced to a one-dimensional Laplace-Stieltjes transform by converting the original system of partial differential equations into a system of ordinary differential equations that are solved using standard techniques. Next, the moments of the distribution were derived using elementary matrix calculus. Finally, the long-run availability of the system under an inspect-and-replace policy was derived using the theory of regenerative processes. All of the results derived in this thesis offer a significant advantage over existing solutions: they may be implemented using simple, efficient one-dimensional numerical inversion algorithms.

The final portion of this research effort illustrated the analytical solutions through numerical examples. First, the analytical results of chapter 3 were used to compute various system reliability measures. For each system, the necessary problem parameters were defined and the analytical solutions were approximated numerically via the one-dimensional inversion algorithm. Approximate solutions were also obtained through computer simulation. The analytical and simulated solutions were compared demonstrating the accuracy of the reliability measures obtained by the analytical result. Moreover, those reliability measures may be obtained more efficiently. In fact, the computation time is orders of magnitude smaller for the analytical results (a few seconds versus several hours).

To implement the techniques of this thesis, one must know the various problem parameters (transition rates, shock rates, shock magnitudes, and wear rates, etc.). If there is no *a priori* information about the distribution of these parameters, then a hypothesized distribution must be used. One may observe the system, noting transition rates, wear rates, shock times, and shock magnitudes. Using this observational data (possibly obtained from sensors) one may estimate the appropriate parameters. Although using observational data is a valid technique, it may require a lengthy observation period. However, other methods used to measure the reliability of the system would be subject to the same limitations. The most restrictive assumption of this thesis, however, is that the dynamics of the ambient environment must be modelled as a continuous-time Markov chain; the analysis cannot be applied directly to a non-Markovian system. However, techniques exist to convert semi-Markov processes into CTMCs using phase-type distributions. It may be possible to use these techniques to apply the analysis of this model to a system with a non-Markovian environmental process.

As with all academic endeavors, this thesis builds on previous work and will serve as a foundation for future research. The main results of this thesis may be extended in several ways. First, performing a parameter analysis would also be a fruitful area of research. If some of the parameters can be controlled in the real system, then a parameter analysis might identify the critical factors of a system's reliability. For example, consider two system designs: one system design reduces the rate at which the systems experiences wear by a factor of 0.5, and the second abates the mean damage done by shocks by a factor of 0.5. A rigorous analysis may determine which system is more reliable. Moreover, the analysis of this thesis only considered a system subject to a single shock process. It is possible to analyze a system subject to multiple shock processes using similar techniques. Perhaps another interesting model would be one in which both the wear process and shock process depend on the state of the ambient environment.

In conclusion, predicting a system's reliability through the analysis of a stochastic reliability model is not a new problem. The classical approach has been to observe previous failures without regard to the system's ambient environment. However, recent developments in the field of reliability theory have emphasized the need to incorporate the effects of the system's operating environment as well. By utilizing the resulting bivariate process, this thesis contributes an analysis of the failure dynamics of a system subject to a state-dependent wear process and a Poisson shock process. However, unlike previous works this analysis produces analytical results that may be easily implemented to solve real-world problem.

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14. ABSTRACT

This thesis develops and analyzes a mathematical model for the reliability measures of a single-unit system subject to continuous wear due to its operating environment and randomly occurring shocks that inflict a random amount of damage to the unit. Assuming a Markovian operating environment and shock arrival mechanism, Laplace-Stieltjes transform expressions are obtained for the failure time distribution and all of its moments. Moreover, an analytical expression is derived for the long-run availability of the single-unit system when it is subject to an inspect-and-replace maintenance policy. The analytical results are illustrated, and their results compared with those of Monte-Carlo simulated failure data. The numerical results indicate that the reliability measures may be accurately computed via numerical inversion of the transform expressions in a straightforward manner when the input parameters are known a priori. In stark contrast to the simulation model which requires several hours to obtain the reliability measures, the analytical procedure computes the same measures in only a few seconds.

15. SUBJECT TERMS

Markov process, wear process, shock process, reliability, availability, stochastic model, compound damage process

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