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**Spectral Solution of the Helmholtz and Paraxial Wave  
Equations and Classical Diffraction Formulae**

**by Timothy M. Pritchett**

**ARL-TR-3179**

**March 2004**

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**Sensors and Electron Devices Directorate, ARL**

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<b>14. ABSTRACT</b> This report examines the general nonlinear vector wave equation implied by the Maxwell equations in a nonmagnetic, isotropic medium and discusses the various approximations under which this general result reduces to the familiar scalar Helmholtz equation and the paraxial wave equation. We see why the Helmholtz equation may be regarded as a singular perturbation of the paraxial wave equation and how some of the difficulties arising in the solution of the former partial differential equation are related to this fact. Standard integral transform methods are used to obtain general solutions of the Helmholtz equation in a linear medium and of the paraxial wave equation in a linear medium. We show that these solutions are equivalent, respectively, to the exact Rayleigh-Sommerfeld diffraction integral and the Rayleigh-Sommerfeld integral in the Fresnel approximation. We discuss the use of these linear solutions in numerical procedures for treating the corresponding nonlinear beam propagation problems. The general linear solutions are specialized to situations with axial symmetry, and the results are used to treat the example of a clipped Gaussian beam.					
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## 1. Introduction

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To accurately describe the propagation of a light beam through a medium, one must account, roughly speaking, for three phenomena: diffraction, refraction, and absorption. Diffraction refers to the tendency of wave phenomena such as light or sound to deviate from rectilinear propagation when a portion of the wave front is obstructed in some way. Diffractive effects are present to some degree in every real optical system, simply because no real system can be infinite in extent; every real system must necessarily contain an effective aperture of some sort. Refraction is related to the speed with which a wave travels through a medium. It, too, can result in the deviation of a beam from straight-line propagation, as when the refractive properties of the medium depend on position in such a way that different portions of the beam experience different amounts of refraction. Absorption is the process responsible for the change in amplitude of the wave as it propagates through the medium and energy is transferred from the wave to the medium or from the medium to the wave (gain). Both refraction and absorption depend on the amplitude of the propagating wave. In certain media, this dependence is highly nonlinear, while in other materials, such as “slow” (fluence dependent) excited state absorbers ( $I$ ), it is, to be sure, linear in the wave amplitude but reflects the entire history of the interaction between the medium and the propagating beam. In general, calculating the effects of propagation of a high-intensity laser beam through all but the simplest media presents a very difficult problem.

In these “simple” media (isotropic media exhibiting refractive and absorptive responses that are both linear and time independent), beam propagation essentially reduces to pure diffraction. However, even a pure diffraction problem in an uncomplicated geometry can generally not be solved without recourse to various approximations, and these impose limits on the resulting solutions. Nevertheless, solutions do exist.

These solutions are very valuable. The special case of beam propagation (diffraction) in an isotropic, linear, and time-independent medium is extremely important in practice. What is more, the solution of the linear diffraction problem can be employed to attack the more general problem of beam propagation in a nonlinear medium through the use of a numerical procedure that separately computes the effects of diffraction, nonlinear refraction, and nonlinear absorption for each propagation step. This is the “split step” Fourier method introduced by Fleck, Morris, and Feit (2) (sometimes referred to simply as the Beam Propagation Method).

The remainder of this report is organized as follows. Section 2 opens with an examination of the general nonlinear optical wave equation and the conditions under which it reduces to the scalar Helmholtz equation. This is followed by a careful derivation of the paraxial wave equation. Section 3 briefly reviews the salient results of classical scalar diffraction theory. In sections 4 and 5, standard integral transform methods are used to obtain a general solution to the paraxial

wave equation in a linear medium and the Helmholtz equation in a linear medium, respectively. These results underscore the fact (perhaps not widely appreciated) that the exact Rayleigh-Sommerfeld integral is equivalent to the scalar Helmholtz equation, while the Rayleigh-Sommerfeld integral in the Fresnel approximation is equivalent to the paraxial wave equation. The general results of section 4 are specialized to situations with cylindrical symmetry in section 6. In section 7, these cylindrically symmetric results are applied to a practical example—a Gaussian beam clipped by an aperture.

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## 2. Scalar Electromagnetic Wave Equations and Their Validity

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### 2.1 General Wave Equation for Nonlinear Optical Media

The derivation of an electromagnetic wave equation from the Maxwell equations is presented in virtually any textbook on optics or electromagnetic theory. The most general form (3) of the wave equation in a nonmagnetic medium containing neither free charges nor free currents is, in Gaussian units,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E}(\vec{x}, z, t) + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E}(\vec{x}, z, t) = -\frac{4\pi}{c^2} \frac{\partial^2}{\partial t^2} \vec{P}(\vec{x}, z, t). \quad (1)$$

(In anticipation of future developments, we place one of the coordinates,  $z$ , on a separate footing and combine the remaining two coordinates into the two-component vector  $\vec{x}$ , which designates positions in the transverse plane.) In equation 1,  $\vec{P}(\vec{x}, z, t)$  is the total material polarization. It is usually expressed as a power series in the electric field  $\vec{E}(\vec{x}, z, t)$ , with its  $i$ th component given by

$$\begin{aligned} P_i(\vec{x}, z, t) = & \int_{-\infty}^{\infty} dt' \chi_{ij}^{(1)}(\vec{x}, z, t-t') E_j(\vec{x}, z, t') + \\ & \int_{-\infty}^{\infty} dt'' \int_{-\infty}^{\infty} dt' \chi_{ijk}^{(2)}(\vec{x}, z, t-t', t-t'') E_j(\vec{x}, z, t') E_k(\vec{x}, z, t'') + \\ & \int_{-\infty}^{\infty} dt''' \int_{-\infty}^{\infty} dt'' \int_{-\infty}^{\infty} dt' \chi_{ijkl}^{(3)}(\vec{x}, z, t-t', t-t'', t-t''') E_j(\vec{x}, z, t') E_k(\vec{x}, z, t'') E_l(\vec{x}, z, t''') + \\ & \dots \end{aligned}$$

In this expression, the quantities  $\chi_{ij\dots}^{(n)}(\vec{x}, z, t, \dots)$  are components of the  $n$ th-order time-dependent electric susceptibility tensor, and the repeated indices  $j, k, l, \dots$  carry an implied sum (Einstein summation convention). The convolutions in each term of the above expansion for  $P_i(\vec{x}, z, t)$  become simple products upon Fourier transformation.



We consider now the single-frequency component of the electric field oscillating at angular frequency  $\omega$ :

$$\vec{E}(\vec{x}, z, t) = \vec{E}(\vec{x}, z) e^{-i\omega t} + c.c. \quad (2)$$

We employ the well-known vector identity for the curl of a curl and the constitutive relation in the form

$$4\pi\vec{P} = (n^2 - 1)\vec{E}$$

to recast equation 1 as

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} - \frac{\omega^2 n_0^2}{c^2} \vec{E} = \frac{\omega^2}{c^2} (n^2(\vec{x}, z, \omega, \vec{E}; t) - n_0^2) \vec{E}. \quad (3)$$

Here,  $n(\vec{x}, z, \omega, \vec{E}; t)$  is the Fourier component of the refractive index of the medium oscillating at frequency  $\omega$ . Strictly speaking, a quantity in the frequency domain is a function of  $\omega$  only, not of  $t$ . In the present case, however, we write  $n$  with an explicit time dependence to emphasize the fact that each Fourier component may slowly change with time (here, “slowly” means that the time scale of the variation is much longer than  $\omega^{-1}$ ) as the properties of the medium evolve as a result of the interaction with the beam. The value of the refractive index before the arrival of the beam is denoted by  $n_0$ .

If the  $\vec{\nabla}(\vec{\nabla} \cdot \vec{E})$  term can be dropped, equation 3 takes the form of the inhomogeneous Helmholtz equation, which can be solved numerically with the “split step” Fourier scheme alluded to in the previous section. Roughly speaking, the split step procedure treats effects arising from the right-hand side of equation 3 separately from those associated with the left-hand side. One solves the *homogeneous* Helmholtz equation over a short longitudinal step and then accounts separately for the effects over that step of nonlinear (and/or time-dependent linear) refraction and absorption. The homogenous Helmholtz equation, together with the simpler equation to which it reduces in the case of paraxial beams, is the subject of the remainder of this report.

We postpone to a future report a full analysis, using the methods of Lax, Louisell, and McKnight (4), of the implications of dropping the  $\vec{\nabla}(\vec{\nabla} \cdot \vec{E})$  term in equation 3. For the present, we observe only that, since free charge is assumed to be absent, it follows from the Maxwell equation for the electric displacement that  $\vec{\nabla} \cdot (n^2 \vec{E})$  vanishes, or that

$$\vec{\nabla} \cdot \vec{E} = -\frac{1}{2} \vec{E} \cdot \vec{\nabla}(\ln n).$$

## 2.2 Homogeneous Helmholtz Equation

In a linear medium, the divergence of  $\vec{E}$  vanishes. If, in addition, the properties of the medium are independent of time, then the right-hand side of equation 3 vanishes as well and equation 3 reduces to

$$(\nabla^2 + k^2)\vec{E}(\vec{x}, z) = 0. \quad (4)$$

Here,  $k = n_0\omega/c$ , the wave number in the medium. In an absorptive medium,  $n_0$  (and thus  $k$ ) has a positive imaginary part. Equation 4 is the homogeneous vector Helmholtz equation.

We designate the polarization direction of the field with the unit vector  $\hat{e}$  and write  $\vec{E}(\vec{x}, z) = \hat{e}E(\vec{x}, z)$ . If  $\hat{e}$  is constant, equation 4 reduces to a scalar wave equation,

$$(\nabla^2 + k^2)E(\vec{x}, z) = 0, \quad (5)$$

the scalar Helmholtz equation. We now write the electric field as an envelope function times a plane wave propagating in the  $z$ -direction:  $E(\vec{x}, z) = A(\vec{x}, z)e^{ikz}$ . In terms of the envelope function  $A(\vec{x}, z)$ , equation 5 becomes

$$(\partial_z^2 + 2ik\partial_z + \nabla_\perp^2)A(\vec{x}, z) = 0, \quad (6)$$

where  $\nabla_\perp^2$  is the transverse Laplacian, given in Cartesian coordinates by  $\partial_x^2 + \partial_y^2$ .

## 2.3 Paraxial Wave Equation

At this point, we proceed with the *slowly varying envelope approximation* (also known as the paraxial approximation) by introducing suitable “non-dimensionalized” quantities. The problem itself sets a natural length scale  $x_0$  for the transverse coordinates, e.g., the size of an aperture or the spot radius of a focused laser beam. We use this quantity to define the scaled transverse coordinates:  $\vec{X} = \vec{x}/x_0$ . The transverse length scale  $x_0$  and the wave number  $k$  now determine  $z_0$ , the longitudinal length scale or “Rayleigh range,”\* by  $z_0 = kx_0^2$ , and the corresponding scaled longitudinal coordinate is given by  $Z = z/z_0$ . We denote by  $\varepsilon$  the ratio of the transverse and longitudinal length scales:  $\varepsilon = x_0/z_0 = (kx_0)^{-1}$ . (An alternate procedure for non-dimensionalizing the problem, employed by some workers (5), simply expresses all lengths as multiples of the wavelength  $\lambda$ , or equivalently, as multiples of the inverse wave number  $k^{-1}$ . This seemingly very natural choice is not the most convenient for our purposes.)

In terms of these new dimensionless coordinates, equation 6 becomes

$$(\varepsilon^2\partial_Z^2 + 2i\partial_Z + \nabla_\perp^2)A(\vec{X}, Z) = 0. \quad (7)$$

---

\* If one chooses  $x_0$  to be the spot radius at focus of a Gaussian beam, then  $z_0$  is the confocal parameter of the beam, equal to twice the Rayleigh range of the beam as conventionally defined.

In equation 7, the validity of the slowly varying envelope approximation is completely manifest: as long as the transverse length scale  $x_0$  is much greater than the wavelength,  $\varepsilon \ll 1$  and the approximation holds. Equation 7 then reduces to

$$(2i\partial_z + \nabla_{\perp}^2)A(\vec{X}, Z) = 0, \quad (8)$$

the paraxial wave equation. In terms of the original “dimensionful” coordinates  $(\vec{x}, z)$ , the paraxial wave equation reads

$$(2ik\partial_z + \nabla_{\perp}^2)A(\vec{x}, z) = 0. \quad (9)$$

### 3. A Précis of Classical Scalar Diffraction Theory

#### 3.1 History

The intuitive picture presented in 1678 by Christian Huygens (6) forms the basis for the modern theory of diffraction. According to what is now known as the Huygens-Fresnel principle, every point on a wave front may be thought of as the source of spherical secondary wavelets that propagate through space with the same speed and frequency as the primary wave; the wave front at any later time is simply the envelope of the secondary wavelets. In his famous prize paper presented to the Paris Academy in 1818, Augustin Jean Fresnel combined Huygens’s envelope construction with Thomas Young’s ideas about interference. The Huygens-Fresnel principle was placed on a firm mathematical basis in the work of Gustav Kirchoff (7). Kirchoff’s theory, despite its inconsistent assumptions (8, 9) agrees extremely well with experiment in all practical limits. It remained only for Sommerfeld (10) and Rayleigh to remove the inconsistencies from the Kirchoff theory. The mathematical expression of the Huygens-Fresnel principle takes the form of an integral which (after a series of simplifying approximations that impose certain limitations on the relative size of the diffracting aperture, the distance from the aperture to the plane of observation, the size of the observation region on the plane, and the wavelength of the radiation) may be evaluated analytically in a variety of simple aperture geometries (e.g., rectangle, circle, and infinite one-dimensional slit). Conventional textbook treatments (11, 12, 13, 14) typically include several of these approximate calculations.

#### 3.2 Rayleigh-Sommerfeld Diffraction Integral

Although excellent discussions of the conventional integral formulation of diffraction theory are given elsewhere (we find presentation of (14) particularly lucid), in this section we briefly review the standard treatment in order to emphasize several points that surface in our subsequent discussion. We consider the situation at an observation point  $P$  with coordinates  $(\vec{x}, z)$ , as shown in figure 1. (Throughout this discussion, we use the two-component vector  $\vec{x}$  to indicate positions in the transverse plane.) The vector  $\vec{r}$  with components  $(\vec{x} - \vec{x}', z)$  points from an

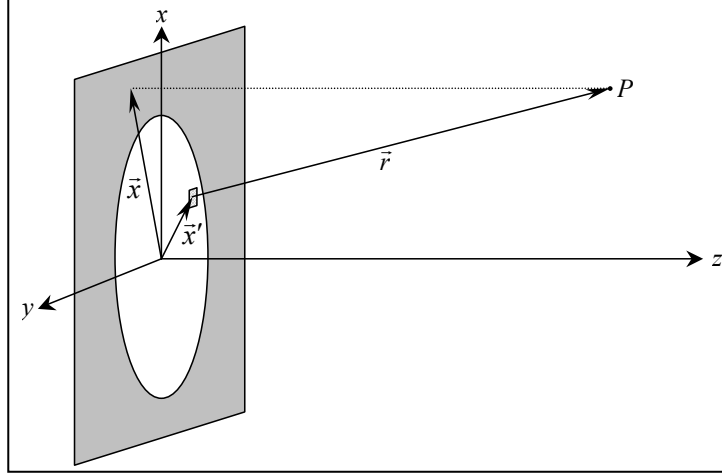


Figure 1. Diffraction by an aperture: geometry of the problem.

infinitesimal source at coordinates  $(\vec{x}', 0)$  to the observation point  $P$ ; the location of the source point ranges over the aperture. The resulting Rayleigh-Sommerfeld diffraction formula is then

$$E(\vec{x}, z) = \frac{1}{2\pi} \iint_{plane} d^2x' E(\vec{x}', 0) \frac{e^{ikr}}{r} \left( \frac{1}{r} - ik \right) \cos(\vec{r}, \hat{z}), \quad (10)$$

where  $r = |\vec{r}| = \sqrt{z^2 + |\vec{x} - \vec{x}'|^2}$  and represents the distance between the infinitesimal source and the observation point, while  $k$  is the wave number, given in terms of the wavelength  $\lambda$  in the medium by  $k = 2\pi/\lambda$ . The integration in equation 10 is formally over the entire  $xy$  plane, but the Kirchoff boundary condition, which sets  $E(\vec{x}, 0) \equiv 0$  for points outside the aperture, restricts the integral to the two-dimensional aperture itself. For the case in which the aperture is illuminated by a normally incident plane wave, it has recently been shown that equation 10 may be rewritten in terms of a one-dimensional parametric integral around the perimeter of the aperture (15).

The direction cosine in equation 10, which has the value  $z/r$ , is known as the *obliquity factor* or *inclination factor*. Fresnel himself recognized the need for a quantity of this kind (12), without which the secondary wavelets envisioned by Huygens would also generate a reverse wave propagating back toward the source illuminating the aperture; no such backward propagating wave is observed experimentally. In the Fresnel-Kirchoff diffraction formula, the obliquity factor is  $\frac{1}{2}(\cos(\vec{r}, \hat{z}) + 1)$ . Interestingly enough, equation 10, including the proper form of the obliquity factor, can be derived entirely from Fourier transform theory, with no reference whatsoever to the Huygens-Fresnel principle (5).

Equation 10 treats electromagnetic radiation as a scalar phenomenon, completely ignoring the fact that the various components of the electric and magnetic field vectors are coupled by Maxwell's equations and thus cannot be treated independently. Within the limitations of a scalar theory, however, equation 10 is considered to be exact throughout the entire diffraction space.

Fortunately, the limitations that a scalar theory places on the macroscopic length scale are not particularly severe; experiments involving the diffraction of microwaves from apertures confirm that the scalar theory is valid, provided the aperture size is much greater than the wavelength of the radiation (16), while a detailed theoretical analysis involving focused Gaussian laser beams shows that the scalar treatment suffices so long as the focal spot size is at least approximately as large as the wavelength (17).

### 3.3 Preliminary Approximations

At distances  $z$  from the aperture on the order of the wavelength, the exact Rayleigh-Sommerfeld diffraction formula, equation 10, can be evaluated by the method of stationary phase, which is described in (11). For  $z$  values of at least several wavelengths, one can take  $(1/r - ik) \approx -ik$ , in which case equation 10 becomes

$$E(\vec{x}, z) = \frac{-i}{\lambda} \iint_{\text{plane}} d^2x' E(\vec{x}', 0) e^{ikr} \frac{z}{r^2}. \quad (11)$$

The highly oscillatory character of the integrand makes the direct numerical evaluation of equation 11 problematic; however, a recent report suggests that present-day desktop computers running commercially available applications software may now be equal to the task, at least for configurations for which the Fresnel number is not too large (18).

### 3.4 Fresnel Approximation

The standard method of attacking the integral in equation 11 relies on the expansion

$$\frac{r}{z} = 1 + \frac{|\vec{x}|^2}{2z^2} - \frac{\vec{x} \cdot \vec{x}'}{z^2} + \frac{|\vec{x}'|^2}{2z^2} + o\left(\frac{|\vec{x} - \vec{x}'|}{z}\right)^4.$$

One approximates the factor of  $r^2$  in the denominator of the integrand with only the zeroth-order term, while for the factor of  $r$  in the phase  $e^{ikr}$ , the three quadratic terms are retained as well.

$$E(\vec{x}, z) = \frac{-i}{\lambda} \frac{e^{ikz}}{z} \iint_{\text{plane}} d^2x' E(\vec{x}', 0) \exp\left[\frac{ik}{2z} |\vec{x} - \vec{x}'|^2\right] \quad (12)$$

This is the Fresnel approximation, whose validity has been the subject of a number of investigations over the years (19, 20).

### 3.5 Fraunhofer Limit

If now the dimensions of the aperture are such that for every point  $(\vec{x}', 0)$  in the aperture,

$$\frac{k |\vec{x}'|^2}{2z} \ll 1,$$

then  $\exp[\frac{i}{2}k|\bar{x}'|^2/z] \approx 1$  and equation 12 reduces to

$$E(\bar{x}, z) = \frac{-i}{\lambda} \frac{e^{ikz}}{z} \exp[\frac{i}{2}k|\bar{x}|^2/z] \iint_{plane} d^2x' E(\bar{x}', 0) \exp\left[-\frac{ik}{z}(\bar{x} \cdot \bar{x}')\right],$$

the Fraunhofer approximation.

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## 4. Solution of the Paraxial Wave Equation

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One can easily verify by direct substitution that equation 12, the diffraction integral in the Fresnel approximation, solves the paraxial wave equation, equation 9, with  $E(\bar{x}, z) = A(\bar{x}, z) e^{ikz}$ , thus demonstrating the equivalence of the Fresnel and paraxial approximations. It is nevertheless illuminating to proceed with the direct solution of equation 7 by standard methods, imposing the “paraxial limit”  $\varepsilon \rightarrow 0$  only when necessary.

### 4.1 Helmholtz Equation as a Singular Perturbation of the Paraxial Wave Equation

We express  $A(\bar{X}, Z)$  in terms of its Fourier transform in the transverse coordinates.

$$A(\bar{X}, Z) = (2\pi)^{-1} \iint_{plane} d^2\kappa \tilde{A}(\bar{\kappa}, Z) e^{i\bar{\kappa} \cdot \bar{X}}$$

The Fourier-transformed field amplitude  $\tilde{A}(\bar{\kappa}, Z)$  satisfies

$$\left(\varepsilon^2 \partial_z^2 + 2i \partial_z - \kappa^2\right) \tilde{A}(\bar{\kappa}, Z) = 0, \quad (13)$$

where  $\kappa = |\bar{\kappa}|$ . Setting  $\tilde{A}(\bar{\kappa}, Z) = \psi(\bar{\kappa}) e^{i\alpha Z}$ , we find that equation 13 is satisfied, provided that

$$\varepsilon^2 \alpha^2 + 2\alpha + \kappa^2 = 0. \quad (14)$$

There are two roots,

$$\alpha_+ = \frac{-1 + \sqrt{1 - \varepsilon^2 \kappa^2}}{\varepsilon^2} = -\frac{1}{2} \kappa^2 + o(\varepsilon^2)$$

and

$$\alpha_- = \frac{-1 - \sqrt{1 - \varepsilon^2 \kappa^2}}{\varepsilon^2} = -2\varepsilon^{-2} + \frac{1}{2} \kappa^2 + o(\varepsilon^2),$$

of which the second,  $\alpha_-$ , is singular in the limit  $\varepsilon \rightarrow 0$ . Returning to the physical field, equation 2,

$$\begin{aligned}\bar{E}(\vec{X}, Z, t) &= \hat{e}(2\pi)^{-1} \iint_{plane} d^2\kappa \psi(\vec{\kappa}) e^{i\vec{\kappa}\cdot\vec{X}} \exp[i(\varepsilon^{-2} + \alpha_{\pm})Z - i\omega t] + c.c. \\ &= \hat{e}(2\pi)^{-1} \iint_{plane} d^2\kappa \psi(\vec{\kappa}) e^{i\vec{\kappa}\cdot\vec{X}} \exp[\pm i \frac{\sqrt{1-\varepsilon^2\kappa^2}}{\varepsilon^2} Z - i\omega t] + c.c.\end{aligned}$$

we see that the root  $\alpha_-$  corresponds to a solution that propagates in the *negative*  $z$ -direction.

Nowhere in the region  $Z > 0$ , either at infinity or elsewhere, is there a source from which such a backward propagating wave could originate, so we are justified in discarding  $\alpha_-$  as unphysical.

Thus, whereas in the integral formulation of diffraction theory, one introduces the obliquity factor in order to suppress troublesome, backward propagating Huygens wavelets, here one must exorcise the demon with an appeal to the boundary condition at  $Z \rightarrow +\infty$ !

In the language of differential equations, one would say that the term of order  $\varepsilon^2$  in equation 7 is a singular perturbation. Such a perturbation can completely alter the character of the solution. In the present case, “turning on” the perturbation introduces a second root in equation 14 where there was only one before and so in some sense, it is not unnatural that this new root should be singular. It corresponds to a rapidly oscillating solution of the differential equation, and this solution can be very troublesome indeed.

## 4.2 Convolution Kernel for the Paraxial Wave Equation

To obtain a solution to the paraxial wave equation, we discard the singular root and approximate the remaining, nonsingular root by the zeroth-order term in its expansion in powers of  $\varepsilon$ :

$$\alpha_+ \approx -\frac{1}{2}\kappa^2.$$

$$A(\vec{X}, Z) = (2\pi)^{-1} \iint_{plane} d^2\kappa \psi(\vec{\kappa}) e^{-\frac{i}{2}\kappa^2 Z} e^{i\vec{\kappa}\cdot\vec{X}} \quad (15)$$

The function  $\psi(\vec{\kappa})$  is determined by the boundary condition on  $A$  at  $Z = 0$ .

$$\psi(\vec{\kappa}) = \tilde{A}(\vec{\kappa}, 0) = (2\pi)^{-1} \iint_{plane} d^2X' A(\vec{X}', 0) e^{-i\vec{\kappa}\cdot\vec{X}'} \quad (16)$$

Substituting equation 16 into 15 and reversing the order of integration, we obtain

$$A(\vec{X}, Z) = \iint_{plane} d^2X' A(\vec{X}', 0) \iint_{plane} \frac{d^2\kappa}{[2\pi]^2} e^{-\frac{i}{2}\kappa^2 Z} e^{i\vec{\kappa}\cdot(\vec{X}-\vec{X}')} . \quad (17)$$

Recalling that  $\kappa^2 = \kappa_x^2 + \kappa_y^2$ , we see that the integral over the  $\kappa$  plane in equation 17 factors into a product of two identical one-dimensional integrals of the form

$$G_1(X - X', Z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq e^{-\frac{i}{2}q^2 Z} e^{iq(X-X')} = \frac{e^{\frac{i(X-X')^2}{2Z}}}{\sqrt{2\pi iZ}} . \quad (18)$$

Equation 17 thus takes the form of a two-dimensional convolution,

$$A(\vec{X}, Z) = \iint_{\text{plane}} d^2 X' A(\vec{X}', 0) G(\vec{X} - \vec{X}', Z), \quad (19)$$

in which the kernel,

$$G(\vec{X} - \vec{X}', Z) = \frac{e^{\frac{i}{2Z}|\vec{X} - \vec{X}'|^2}}{2\pi i Z} = G_1(X - X', Z)G_1(Y - Y', Z), \quad (20)$$

is the square of the one-dimensional convolution kernel in equation 18. Expressed in the “dimensionful” coordinates  $(\vec{x}, z)$ , the two-dimensional convolution kernel in equation 20 is

$$G(\vec{x} - \vec{x}', z) = -\frac{ix_0^2}{\lambda z} \exp\left[\frac{ik}{2z}|\vec{x} - \vec{x}'|^2\right]. \quad (21)$$

Recalling that  $E(\vec{x}, z) = A(\vec{x}, z) e^{ikz}$ , we see that equation 19 is identical to equation 12, the Fresnel approximation of the Rayleigh-Sommerfeld diffraction integral.

### 4.3 A Remark on Convergence

In order to ensure convergence of the integral in equation 18,  $Z = z/kx_0^2$  must have a negative imaginary part. Since  $z$  and  $x_0$  are both real, the requirement that  $\text{Im}[Z] < 0$  is equivalent to the condition  $\text{Im}[k] = \alpha/2 > 0$ , in which  $\alpha$  is the absorption coefficient of the medium; mathematical convergence is thus assured in the case of propagation through an absorptive medium. Of course, the integral in equation 18 also converges for propagation through a lossless medium. In this case, we observe that a perfectly monochromatic field is a mathematical idealization. An actual, physical field must have been turned on at some time in the past, which one may account for by adding a small positive imaginary part to the frequency, so that  $e^{-i\omega t}$ , and thus the physical field, vanishes as  $t \rightarrow -\infty$ . This is equivalent to a positive absorption coefficient.

### 4.4 Numerical Computation of the Fresnel Convolution Integral

In practice, the integral on the right-hand side of equation 19 will generally have to be evaluated numerically. This is most efficiently accomplished through the use of Fourier transform methods using the well known fast Fourier transform (FFT) algorithm. Given a complex-valued function  $f(\vec{X})$  on the plane  $\mathfrak{R}^2$ , we symbolically denote the two-dimensional Fourier transform of  $f(\vec{X})$ , which is a function of  $\vec{\kappa}$ , by

$$\mathbf{F}[f](\vec{\kappa}) = \tilde{f}(\vec{\kappa}) = [2\pi]^{-1} \iint_{\text{plane}} d^2 X f(X) e^{-i\vec{\kappa} \cdot \vec{X}}$$

and the inverse transform of  $\tilde{f}(\vec{\kappa})$ , a function of  $\vec{X}$ , by



$$\mathbf{F}^{-1}[\tilde{f}](\vec{X}) = f(\vec{X}) = [2\pi]^{-1} \iint_{plane} d^2\kappa \tilde{f}(\vec{\kappa}) e^{i\vec{\kappa}\cdot\vec{X}}.$$

Using these relations and the well-known integral representation of the Dirac delta function in terms of complex exponentials, one can write equation 19 symbolically as

$$A(\vec{X}, Z) = 2\pi \mathbf{F}^{-1}[\mathbf{F}[A](\vec{\kappa}, 0) \mathbf{F}[G](\vec{\kappa}, Z)]. \quad (22)$$

Equation 22 is a prescription for calculating  $A(\vec{X}, Z)$ : compute the Fourier transform of  $A(\vec{X}, 0)$ , the field at the boundary,  $Z = 0$ ; multiply this by the Fourier transform of the convolution kernel, equation 20; compute the inverse Fourier transform of the resulting product. Now, the Fourier transform of the convolution kernel in equation 20 is given by

$$\mathbf{F}[G](\vec{\kappa}, Z) = [2\pi]^{-1} e^{-\frac{i}{2}|\vec{\kappa}|^2/Z},$$

so equation 22 is

$$A(\vec{X}, Z) = \mathbf{F}^{-1}\left[\mathbf{F}[A](\vec{\kappa}, 0) e^{-\frac{i}{2}|\vec{\kappa}|^2/Z}\right].$$

This is precisely equation 15, in which  $\psi(\vec{\kappa})$ , given by equation 16, is simply  $\mathbf{F}[A](\vec{\kappa}, 0)$ , the Fourier transform of the boundary condition at  $Z = 0$ . Thus, we see that same procedure that we employed to derive the solution in equation 19 and the corresponding Fresnel kernel in equation 20 also provides the most efficient means of evaluating the integral expression in which the solution is given.

## 5. Solution of the Helmholtz Equation

In this section, we derive the Rayleigh-Sommerfeld result by employing the integral transform methods to solve equation 5, the scalar Helmholtz equation. The calculation proceeds exactly as in the previous section. We take

$$E(\vec{x}, z) = (2\pi)^{-1} \iint_{plane} d^2\kappa \psi(\vec{\kappa}) e^{i\vec{\kappa}\cdot\vec{x}} e^{i\alpha z}, \quad (23)$$

where  $\psi(\vec{\kappa})$  is determined by the boundary condition on  $E$  at  $z = 0$ :

$$\psi(\vec{\kappa}) = (2\pi)^{-1} \iint_{plane} d^2x' E(\vec{x}', 0) e^{-i\vec{\kappa}\cdot\vec{x}'}. \quad (24)$$

The Helmholtz equation is satisfied, provided that

$$\alpha = \pm\sqrt{k^2 - \kappa^2}. \quad (25)$$

The negative root corresponds to a backward propagating solution and, as before, we discard it as unphysical. Substituting equations 24 and 25 into 23 and reversing the order of integration, we obtain

$$E(\bar{x}, z) = (2\pi)^{-2} \iint_{plane} d^2 x' E(\bar{x}', 0) \iint_{plane} d^2 \kappa e^{i\bar{\kappa} \cdot (\bar{x} - \bar{x}')} e^{iz\sqrt{k^2 - \kappa^2}}, \quad (26)$$

or, expressed in the dimensionless variables introduced in section 2.3,

$$E(\bar{X}, Z) = (2\pi)^{-2} \iint_{plane} d^2 X' E(\bar{X}', 0) \iint_{plane} d^2 \kappa e^{i\bar{\kappa} \cdot (\bar{X} - \bar{X}')} e^{iZ\sqrt{1 - \epsilon^2 \kappa^2} / \epsilon^2}.$$

Utilizing the result of Lalor (21),

$$\iint_{plane} d^2 \kappa e^{i\bar{\kappa} \cdot \bar{x}} e^{iz\sqrt{k^2 - \kappa^2}} = -2\pi \frac{\partial}{\partial z} \left( \frac{e^{ik\sqrt{z^2 + |\bar{x}|^2}}}{\sqrt{z^2 + |\bar{x}|^2}} \right),$$

we perform the integration over  $\kappa$  on the right-hand side of equation 26 and so obtain

$$E(\bar{x}, z) = (2\pi)^{-1} \iint_{plane} d^2 x' E(\bar{x}', 0) \frac{e^{ikr}}{r} \left( \frac{1}{r} - ik \right) \frac{z}{r},$$

which is the Rayleigh-Sommerfeld formula, equation 10, with obliquity factor  $z/r$ .

## 6. Axial Symmetry

We now turn our attention to problems exhibiting cylindrical symmetry about the  $z$ -axis. As before,  $x_0$  denotes the transverse length scale characterizing the problem. We employ this length scale to define the dimensionless radial coordinate  $R = r/x_0$  and we use the resulting polar coordinates  $(R, \theta)$  to designate positions in the transverse plane.

### 6.1 Cylindrically Symmetric Restriction of the General Fresnel Kernel

We impose the restriction of cylindrical symmetry on equation 19, the general solution of the paraxial wave equation, and on the associated convolution kernel in equation 20. Cylindrical symmetry requires that the field amplitude  $A(R, Z)$  be independent of the angular coordinate, so without loss of generality, we set  $\theta = 0$ . We use the Law of Cosines to write

$|\bar{X} - \bar{X}'|^2 = R^2 - 2RR' \cos \theta' + R'^2$  and so express the convolution kernel in equation 20 as

$$G(R, \theta, R', \theta', Z)|_{\theta=0} = \frac{1}{2\pi i Z} \exp \left[ \frac{i}{2Z} (R^2 - 2RR' \cos \theta' + R'^2) \right].$$

Inserting the above expression for the convolution kernel into equation 19 with  $d^2 X' = R' dR' d\theta'$ , we employ the well-known integral representation of the Bessel function,

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta' e^{-iu \cos \theta'} = J_0(u),$$

to perform the integration over  $\theta'$  and obtain the following expression for the field amplitude.

$$A(R, Z) = -\frac{i}{Z} \exp\left[\frac{i}{2Z} R^2\right] \int_0^\infty dR' R' A(R', 0) e^{\frac{i}{2Z} R'^2} J_0\left(\frac{RR'}{Z}\right) \quad (27)$$

We write equation 27 in the following form:

$$A(R, Z) = \int_0^\infty dR' R' A(R', 0) G(R, R', Z), \quad (28)$$

in which the factor of  $R'$  appears explicitly in the integrand. The associated kernel is then

$$G(R, R', Z) = \frac{1}{iZ} \exp\left[\frac{i(R^2 + R'^2)}{2Z}\right] J_0\left(\frac{RR'}{Z}\right), \quad (29)$$

the analog of equation 20 for situations involving cylindrical symmetry. (By including the factor of  $R'$  in the integration measure instead of the kernel, we obtain an expression for the kernel that is symmetric in the variables  $R$  and  $R'$ .)

## 6.2 Solution of the Paraxial Wave Equation Via Hankel Transform

It is also possible to obtain the cylindrical symmetry kernel, equation 29, in a somewhat more methodical fashion, namely, by direct solution of the paraxial wave equation in two transverse dimensions under the assumption of cylindrical symmetry. In this case, the transverse Laplacian operator is given by  $\nabla_\perp^2 = \partial_R^2 + \frac{1}{R} \partial_R$  and the paraxial wave equation 8 takes the form

$$(2i\partial_Z + \partial_R^2 + \frac{1}{R} \partial_R) A(R, Z) = 0.$$

The cylindrical symmetry of the problem suggests that we express  $A(R, Z)$  in terms of its Hankel transform,

$$A(R, Z) = \int_0^\infty d\kappa \kappa \tilde{A}(\kappa, Z) J_0(\kappa R). \quad (30)$$

The field amplitude in the transform domain,  $\tilde{A}(\kappa, Z)$ , satisfies the ordinary differential equation  $(i\partial_Z - \frac{1}{2}\kappa^2) \tilde{A}(\kappa, Z) = 0$ . The solution is:

$$\tilde{A}(\kappa, Z) = \tilde{A}(\kappa, 0) e^{-\frac{i}{2}\kappa^2 Z}, \quad (31)$$

where  $\tilde{A}(\kappa, 0)$  is given by the inverse Hankel transform of the boundary condition at  $Z = 0$ .

$$\tilde{A}(\kappa, 0) = \int_0^{\infty} dR' R' A(R', 0) J_0(\kappa R') \quad (32)$$

If we substitute equation 32 into 31, then substitute the result in equation 30 and reverse the order of integration, we obtain

$$A(R, Z) = \int_0^{\infty} dR' R' A(R', 0) \int_0^{\infty} d\kappa \kappa J_0(\kappa R') J_0(\kappa R) e^{-\frac{i}{2}\kappa^2 Z}.$$

Comparing this last expression with equation 28, we recognize that the  $\kappa$  integral is none other than  $G(R, R', Z)$ , the kernel that we seek:

$$G(R, R', Z) = \int_0^{\infty} d\kappa \kappa J_0(\kappa R') J_0(\kappa R) e^{-\frac{i}{2}\kappa^2 Z} = -\frac{i}{Z} e^{\frac{i}{2Z}(R'^2 + R^2)} J_0\left(\frac{RR'}{Z}\right).$$

The integral is given in Watson (22) and reproduces our previous result, equation 29.

### 6.3 Cylindrically Symmetric Fraunhofer Kernel

In the case of circular symmetry, the Fraunhofer condition expressed in dimensionless variables reads  $R'^2 / (2Z) \ll 1$ . In the Fraunhofer limit, the factor  $\exp[\frac{i}{2}R'^2 / Z]$  in the integrand of equation 27 may be replaced by unity, which gives

$$A(R, Z) \approx -\frac{i}{Z} \exp\left[\frac{i}{2Z}R^2\right] \int_0^{\infty} dR' R' A(R', 0) J_0\left(\frac{RR'}{Z}\right). \quad (33)$$

Thus, the Fraunhofer kernel associated with equation 28 is given by

$$G(R, R', Z) \approx \frac{1}{iZ} \exp\left[\frac{iR^2}{2Z}\right] J_0\left(\frac{RR'}{Z}\right).$$

## 7. Example: Clipped Gaussian Beam

We conclude with an example that is of considerable practical importance: the diffraction of a principal-mode Gaussian beam by a coaxial circular aperture that, for simplicity, will be assumed to lie in the focal plane of the beam. This problem is characterized by two radial length scales:  $x_0$ , the radius of the aperture, and  $w_0$ , the half-width of the beam at  $1/e^2$  of its on-axis intensity. Either quantity could be used to define a dimensionless radial coordinate; we choose the former, setting  $R = r/x_0$ . We also define  $W_0 = w_0/x_0$ , the dimensionless quantity characterizing the extent

of the beam. The clipped Gaussian beam is then described by the following Kirchoff boundary condition at  $Z = 0$ .

$$A(R, 0) = \begin{cases} A_0 e^{-R^2/W_0^2}, & R < 1 \\ 0, & R \geq 1 \end{cases} \quad (34)$$

### 7.1 Fresnel Diffraction of a Clipped Gaussian Beam

With the clipped Gaussian boundary condition in equation 34, equation 27 becomes

$$A(R, Z) = -\frac{iA_0}{Z} e^{iR^2/2Z} \int_0^1 dR' R' e^{-\eta R'^2} J_0\left(\frac{RR'}{Z}\right), \quad (35)$$

where  $\eta = W_0^{-2} - i/(2Z)$ . After repeated partial integration and application of the well-known recurrence relation

$$\frac{d}{dx} [x^n J_n(x)] = x^n J_{n-1}(x),$$

equation 35 becomes

$$A(R, Z) = -\frac{iA_0}{2\eta Z} \exp\left[\frac{iR^2}{2Z} - \eta\right] \sum_{n=1}^{\infty} \left(\frac{2\eta Z}{R}\right)^n J_n\left(\frac{R}{Z}\right). \quad (36)$$

Associated with the field amplitude of equation 36 is the intensity profile

$$\frac{I(R, Z)}{I(0, Z)} = |e^\eta - 1|^{-2} \left| \sum_{n=1}^{\infty} \left(\frac{2\eta Z}{R}\right)^n J_n\left(\frac{R}{Z}\right) \right|^2. \quad (37)$$

### 7.2 Fraunhofer Diffraction of a Clipped Gaussian Beam

Substituting equation 34, the boundary condition for a clipped Gaussian beam, into the Fraunhofer result, equation 33, one still obtains equation 35, except that  $\eta = W_0^{-2}$ , instead of  $W_0^{-2} - i/(2Z)$ . We thus obtain the field amplitude of a clipped Gaussian beam in the Fraunhofer limit by simply setting  $\eta = W_0^{-2}$  in our previous result, equation 36.

### 7.3 Limiting Case: Top Hat Beam (Uniformly Illuminated Aperture)

A beam that overfills the aperture ( $W_0 \gg 1$ ) provides illumination that is effectively uniform. In this case,  $\eta \approx -i/(2Z)$  and equation 36, the solution in the Fresnel regime, reduces to

$$A(R, Z) = A_0 e^{\frac{i}{2Z}(R^2+1)} \sum_{n=1}^{\infty} \left(\frac{-i}{R}\right)^n J_n(R/Z).$$

As we saw in section 7.2, the Fraunhofer limit is obtained by setting  $\eta = W_0^{-2}$  in equation 36. In the present case, we have  $W_0 \gg 1$ , so to obtain the Fraunhofer limit, we need only set  $\eta = 0$  in equation 36. Doing so yields

$$A(R, Z) \approx -i A_0 e^{\frac{i}{2Z} R^2} \frac{J_1(R/Z)}{R}.$$

The associated intensity profile in the far field is then given by

$$\frac{I(R, Z)}{I(0, Z)} = \left( \frac{2J_1(R/Z)}{R/Z} \right)^2, \quad (38)$$

the  $\eta \rightarrow 0$  limit of equation 37. Noting that  $R/Z = kx_0 r/z = kx_0 \tan \alpha \approx kx_0 \sin \alpha$ , where  $\alpha$  is the angle that the observation point makes with the  $z$ -axis as measured from the center of the aperture, we recognize that equation 38 is in fact the celebrated result of Airy (23).

## 8. Summary

From the Maxwell equations in a nonmagnetic, isotropic medium, one may derive a general nonlinear vector wave equation. If  $\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) \equiv 0$ , this reduces to the inhomogeneous vector Helmholtz equation, which is generally written with the nonlinearity contained in the source term on the right-hand side. If the direction of the electric field polarization is constant, as is usually the case, one solves the corresponding scalar equation. In a geometry in which the transverse length scale is much larger than the wavelength of the radiation, the Helmholtz equation is well approximated by the paraxial wave equation, which is only first order in the longitudinal variable and thus much easier to solve. The Helmholtz equation may be regarded as a singular perturbation of the paraxial wave equation, and some of the difficulties arising in the solution of the former partial differential equation are related to this fact.

The Huygens-Fresnel principle provides a very intuitive picture of the diffraction of light from an aperture. The mathematical expression of the Huygens-Fresnel principle takes the form of an integral over the aperture: the Rayleigh-Sommerfeld diffraction integral. The oscillatory nature of the integrand makes exact evaluation of the Rayleigh-Sommerfeld integral problematic, so one frequently employs the simplifying Fresnel approximation in evaluating the integral. We have demonstrated how these same results may be obtained without appeal to the Huygens-Fresnel principle by using standard methods to solve the appropriate partial differential equations: the exact Rayleigh-Sommerfeld integral may be obtained from the scalar Helmholtz equation in a linear medium, while the Rayleigh-Sommerfeld integral in the Fresnel approximation results from solution of the paraxial wave equation in a linear medium.

One can employ the solutions of these linear diffraction problems to attack the more general problem of beam propagation in a nonlinear medium through the use of the numerical procedure introduced by Feit and Fleck in the late 1970s. This procedure, known as the “split step” Fourier method or simply as the Beam Propagation Method, separately computes the effects of diffraction, nonlinear refraction, and nonlinear absorption for each propagation step.

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