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Robust Capon Beamforming

Yi Jiang — University of Florida

Petre Stoica — Uppsala University

Zhisong Wang — University of Florida

Jian Li — University of Florida

Outline

- Standard Capon Beamforming (SCB)
- Norm Constrained Capon Beamforming (NCCB)
- Robust Capon Beamforming (RCB)
- Coherent RCB (CRCB)
- Simulation Results
- Conclusions

Standard Capon Beamforming (SCB)

$$\hat{\mathbf{w}}_{SCB} = \arg\min_{\mathbf{w}} \mathbf{w}^* \mathbf{R} \mathbf{w} \quad \text{s.t.} \quad \mathbf{w}^* \mathbf{a}_0 = 1$$

$$\hat{\mathbf{w}}_{SCB} = \frac{\mathbf{R}^{-1}\mathbf{a}_0}{\mathbf{a}_0^* \mathbf{R}^{-1} \mathbf{a}_0}$$

Signal power estimate

$$\hat{\sigma}_0^2 = \mathbf{w}_{SCB}^* \mathbf{R} \mathbf{w}_{SCB} = 1 / \left(\mathbf{a}_0^* \mathbf{R}^{-1} \mathbf{a}_0 \right)$$

Norm Constrained Capon Beamforming (NCCB)

$$\hat{\mathbf{w}}_{NCCB} = \arg\min_{\mathbf{w}} \mathbf{w}^* \mathbf{R} \mathbf{w} \qquad \text{s.t.} \qquad \mathbf{w}^H \mathbf{a}_0 = 1, \quad \|\mathbf{w}\|^2 \leq \zeta$$

Diagonal loading:

$$\hat{\mathbf{w}}_{NCCB} = \frac{(\mathbf{R} + \lambda \mathbf{I})^{-1} \mathbf{a}_0}{\mathbf{a}_0^* (\mathbf{R} + \lambda \mathbf{I})^{-1} \bar{\mathbf{a}}_0}$$

Loading level λ determined by norm constraint.

Recent Robust Beamformers

Directly Address Steering Vector Uncertainties!

- Based on original SCB formulation
  - Robust adaptive beamforming based on worst-case performance optimization
    [Vorobyov, Gershman, Luo, 2001]
  - Robust minimum variance beamforming
    [Lorenz, Boyd, 2001]

Our RCB

Directly Address Steering Vector Uncertainties!

- Based on Covariance Fitting
  - Robust Capon Beamforming [Stoica, Wang, Li, 2002]
  - On Robust Capon Beamforming and Diagonal Loading [Li, Stoica, Wang, 2002]
- New features
  - Steering vector within an uncertainty set
  - Incorporate uncertainty set into formulation directly
  - Computationally most efficient
  - Conceptually simple
  - Scaling ambiguity eliminated

Covariance Fitting

$$\max_{\sigma^2} \sigma^2 \quad \text{s.t.} \quad \mathbf{R} - \sigma^2 \mathbf{a}_0 \mathbf{a}_0^* \geq 0$$

$$\begin{aligned} & \mathbf{R} - \sigma^2 \mathbf{a}_0 \mathbf{a}_0^* \geq 0 \\ \Leftrightarrow \quad & \mathbf{I} - \sigma^2 \mathbf{R}^{-1/2} \mathbf{a}_0 \mathbf{a}_0^* \mathbf{R}^{-1/2} \geq 0 \\ \Leftrightarrow \quad & 1 - \sigma^2 \mathbf{a}_0^* \mathbf{R}^{-1} \mathbf{a}_0 \geq 0 \\ \Leftrightarrow \quad & \sigma^2 \leq \frac{1}{\mathbf{a}_0^* \mathbf{R}^{-1} \mathbf{a}_0} = \hat{\sigma}_0^2 \end{aligned}$$

Same signal power estimate as SCB!

Our Robust *Capon* Beamformer (RCB)

- Incorporate ellipsoidal uncertainty set into covariance fitting

$$\max_{\sigma^2,\mathbf{a}} \sigma^2 \quad \text{s.t.} \;\; \mathbf{R} - \sigma^2 \mathbf{a}\mathbf{a}^* \geq 0$$

$$\forall \; \mathbf{a} \;\in\; \mathbf{a} = \mathbf{B}\mathbf{u} + \bar{\mathbf{a}}, \|\mathbf{u}\| \leq \epsilon$$

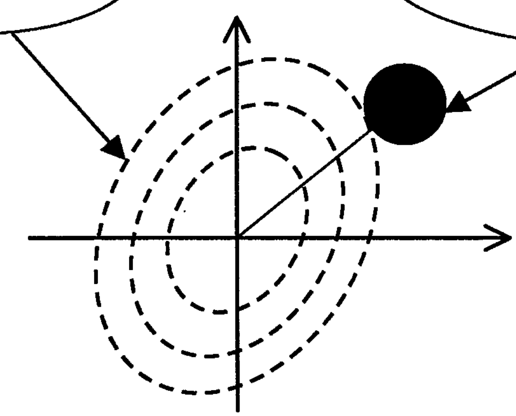
$\mathbf{B} \in \mathcal{C}^{M \times L}, L \leq M$ is of full column rank.

$$\Leftrightarrow \min_{\mathbf{a}} \mathbf{a}^* \mathbf{R}^{-1} \mathbf{a} \quad \text{s.t.} \;\; \mathbf{a} = \mathbf{B}\mathbf{u} + \bar{\mathbf{a}}, \|\mathbf{u}\| \leq \epsilon$$

Our RCB

- Without loss of generality, consider spherical uncertainty set:

$$\min_{\mathbf{a}} \mathbf{a}^* \mathbf{R}^{-1} \mathbf{a} \quad \text{s.t.} \quad \|\mathbf{a} - \bar{\mathbf{a}}\|^2 \leq \epsilon$$



- Solution at boundary of uncertainty set

$$\min_{\mathbf{a}} \mathbf{a}^* \mathbf{R}^{-1} \mathbf{a} \quad \text{s.t.} \quad \|\mathbf{a} - \bar{\mathbf{a}}\|^2 = \epsilon$$

Our RCB

- Use Lagrange multiplier method

$$\hat{\mathrm{a}}_0 = \left(\frac{\mathbf{R}^{-1}}{\lambda} + \mathbf{I}\right)^{-1} \bar{\mathrm{a}}$$

$$= \bar{\mathrm{a}} - (\mathbf{I} + \lambda\mathbf{R})^{-1} \bar{\mathrm{a}}$$

- Obtain Lagrange multiplier $\lambda \geq 0$ by solving

$$g(\lambda) \stackrel{\triangle}{=} \left\|(\mathbf{I} + \lambda\mathbf{R})^{-1} \bar{\mathrm{a}}\right\|^2 = \epsilon$$

via Newton's method (monotonic polynomial -- computationally efficient)

Scaling Ambiguity

- Uncertainty in SOI steering vector cause scaling ambiguity

$(\sigma^2, \mathbf{a})$ and $(\sigma^2/\alpha, \alpha^{1/2}\mathbf{a})$ yield same $\sigma^2\mathbf{a}\mathbf{a}^*$

- Add constraint $||\mathbf{a}_0||^2 = M$ to eliminate ambiguity

$$\hat{\hat{\mathbf{a}}}_0 = \frac{M}{||\hat{\mathbf{a}}_0||}\hat{\mathbf{a}}_0 \qquad \hat{\hat{\sigma}}_0^2 = \hat{\sigma}_0^2||\hat{\mathbf{a}}_0||^2/M$$

Main Steps of Our RCB

- Step 1: $\mathbf{R} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^*$
- Step 2: Obtain Lagrange multiplier λ
- Step 3: $\hat{\mathbf{a}}_0 = \bar{\mathbf{a}} - \mathbf{U}\left(\mathbf{I} + \lambda\mathbf{\Lambda}\right)^{-1}\mathbf{U}^*\bar{\mathbf{a}}$
- Step 4: $\hat{\sigma}_0^2 = \frac{1}{\hat{\mathbf{a}}_0^* \hat{\mathbf{R}}^{-1} \hat{\mathbf{a}}_0} = \frac{1}{\hat{\mathbf{a}}_0^* (\frac{\mathbf{I}}{\lambda} + \hat{\mathbf{R}})^{-1} \bar{\mathbf{a}}}$
- Step 5: $\hat{\hat{\sigma}}_0^2 = \hat{\sigma}_0^2 \|\hat{\mathbf{a}}_0\|^2 / M$

Waveform Estimation

- Obtain weight vector based on $\hat{\mathrm{a}}_0$ or $\hat{\hat{\mathrm{a}}}_0$
- Diagonal loading (spherical constraint)!

$$\hat{\mathbf{w}}_0 = \frac{\mathbf{R}^{-1}\hat{\mathrm{a}}_0}{\hat{\mathrm{a}}_0^*\mathbf{R}^{-1}\hat{\mathrm{a}}_0} = \frac{\left(\mathbf{R}+\frac{1}{\lambda}\mathbf{I}\right)^{-1}\bar{\mathbf{a}}}{\bar{\mathbf{a}}^*\left(\mathbf{R}+\frac{1}{\lambda}\mathbf{I}\right)^{-1}\mathbf{R}\left(\mathbf{R}+\frac{1}{\lambda}\mathbf{I}\right)^{-1}\bar{\mathbf{a}}}$$

- Waveform estimate $\hat{s}_0(n) = \hat{\mathbf{w}}_0^*\mathbf{x}_n$

Advantages of Our RCB

- ❑ Ambiguity elimination obvious for our RCB
  (not considered by others)

- ❑ Computation
  - Our RCB requires $O(M^3)$ flops

    while $O(M^{3.5})$ flops
    for [Vorobyov, Gershman, Luo, 2001]

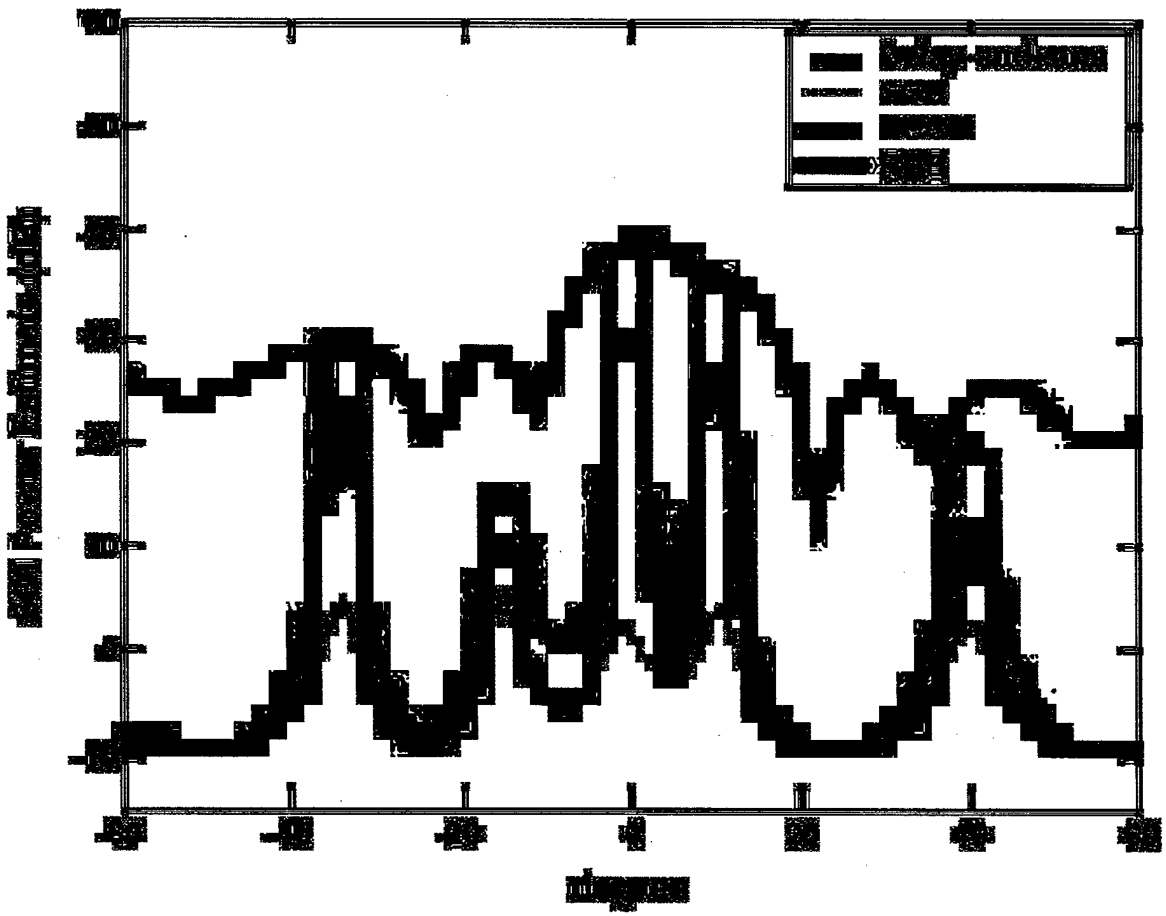
  - More computations needed to determine Lagrange multiplier and polynomial not monotonic for [Lorenz, Boyd, 2001] -- also $O(M^3)$ flops

Numerical Examples

- $M = 10$ sensors
- Uniform linear array with half-wavelength spacing
- Array calibration error exists (independent complex Gaussian random variables added)

Power Estimate vs. Angle

True powers denoted by circles.



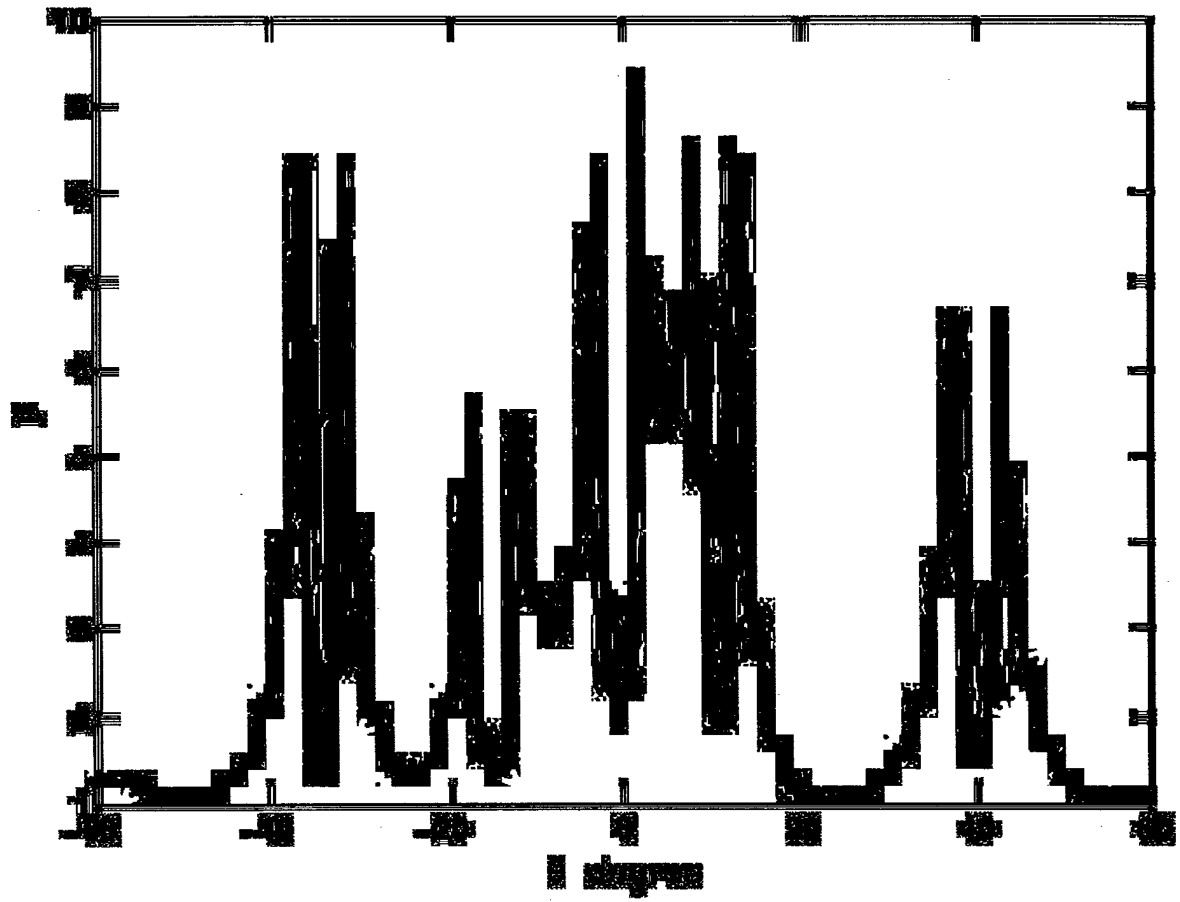
$$\epsilon_0 = 1.0$$

$$\epsilon = 1.0$$

$$\beta = 6.0$$

$$\zeta = \frac{\beta}{M}$$

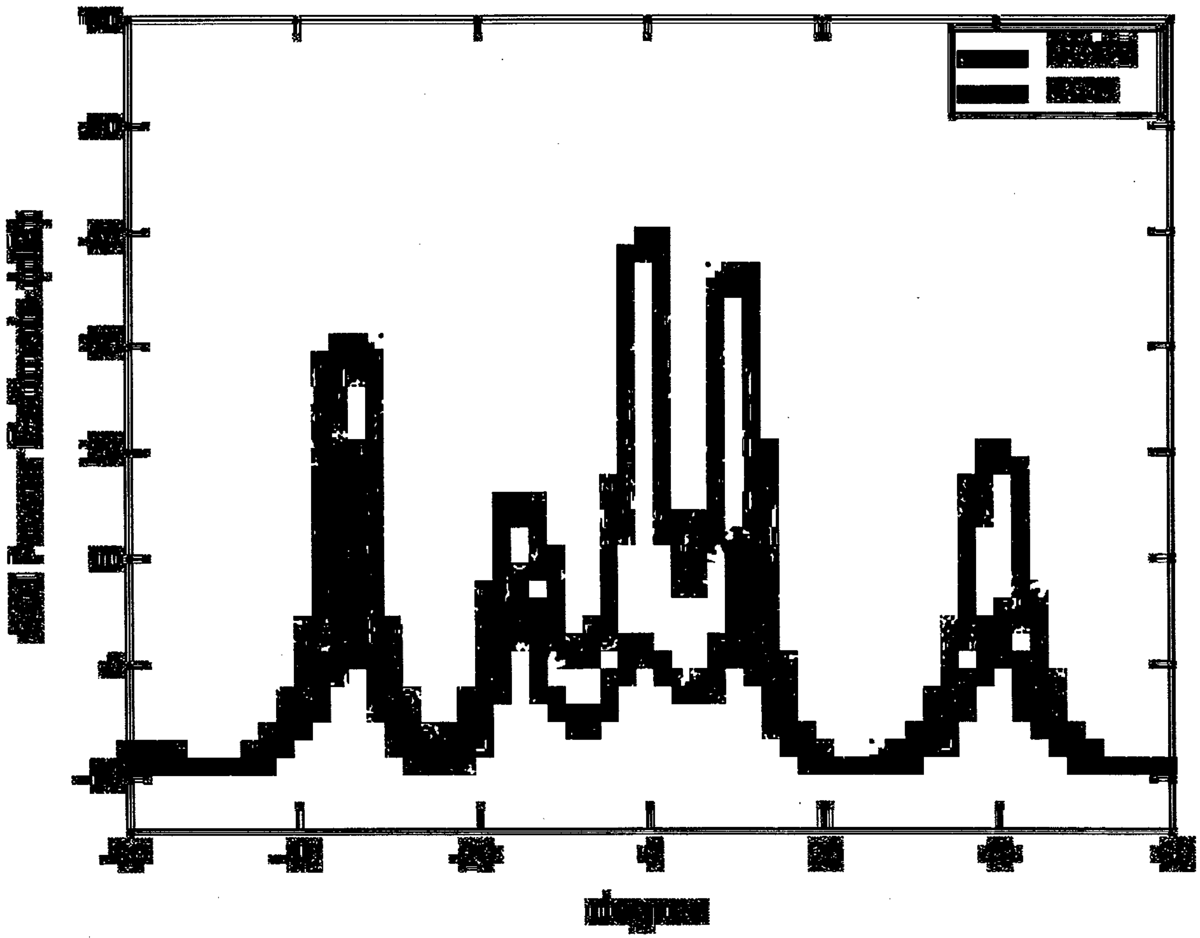
Making NCCB Have Same Diagonal Loading Level As RCB



$\epsilon_0 = 1.0$

$\epsilon = 1.0$

NCCB and RCB Having Same Diagonal Loading Level



$\epsilon_0 = 1.0$

$\epsilon = 1.0$

Coherent RCB (CRCB)

- Motivation – *GPS applications etc.*

From **Multipath Mitigation Performance of Planar *GPS* Adaptive Antenna Arrays for Precision Landing Ground Stations** by J.H. Williams, et al, the MITRE Corporation

- Coherent multipaths exist
- DOAs of multipaths known relative to DOA of SOI

CRCB

- Robust against coherent multipaths as well as steering vector errors.

Steering vector: $\mathbf{a} + \mathbf{V}\mathbf{b}$

$\underline{\mathbf{a}}$: Steering vector of SOI
$\underline{\mathbf{V}}$: Steering vectors of coherent multipaths

- Covariance fitting

$$\max_{\sigma^2, \mathbf{a}, \mathbf{b}} \sigma^2 \quad \text{s.t.} \quad \mathbf{R} - \sigma^2 (\mathbf{a} + \mathbf{V}\mathbf{b})(\mathbf{a} + \mathbf{V}\mathbf{b})^* \geq 0$$

$$\mathbf{a} = \mathbf{B}\mathbf{u} + \bar{\mathbf{a}}, \quad \|\mathbf{u}\|^2 \leq \epsilon$$

Steps of CRCB

- Following similar steps in RCB

$$\Leftrightarrow \min_{\mathbf{a},\mathbf{b}} (\mathbf{a} + \mathbf{V}\mathbf{b})^* \mathbf{R}^{-1} (\mathbf{a} + \mathbf{V}\mathbf{b})$$

$$\text{s.t.} \quad \mathbf{a} = \mathbf{B}\mathbf{u} + \bar{\mathbf{a}}, \ \ \|\mathbf{u}\|^2 \leq \epsilon$$

- Concentrating out $\mathbf{b}$

$$\Leftrightarrow \min_{\mathbf{a}} \mathbf{a}^* \Gamma \mathbf{a}$$

$$\text{s.t.} \quad \mathbf{a} = \mathbf{B}\mathbf{u} + \bar{\mathbf{a}}, \ \ \|\mathbf{u}\|^2 \leq \epsilon$$

with $\Gamma = \mathbf{R}^{-1/2} \mathbf{P}^{\perp}_{\mathbf{R}^{-1/2}\mathbf{V}} \mathbf{R}^{-1/2}$

Insight of CRCB

Let $\mathcal{R}(\mathbf{G}) = \mathcal{N}(\mathbf{V}^*)$

$$\Leftrightarrow \min_{\mathbf{a}} \mathbf{a}^*\mathbf{G}(\mathbf{G}^*\mathbf{R}\mathbf{G})^{-1}\mathbf{G}^*\mathbf{a}$$

$$\text{s.t.} \quad \mathbf{a} = \mathbf{B}\mathbf{u} + \bar{\mathbf{a}}, \ \ \|\mathbf{u}\|^2 \leq \epsilon$$

- Project data to orthogonal subspace of V
- Apply RCB to projected data

Choice of Multipath Subspace

- Error of V causes error of SOI steering vector

  If $\mathrm{G}^*\mathrm{a}_I \neq 0$, it is combined with $\mathrm{G}^*\mathrm{a}_0$

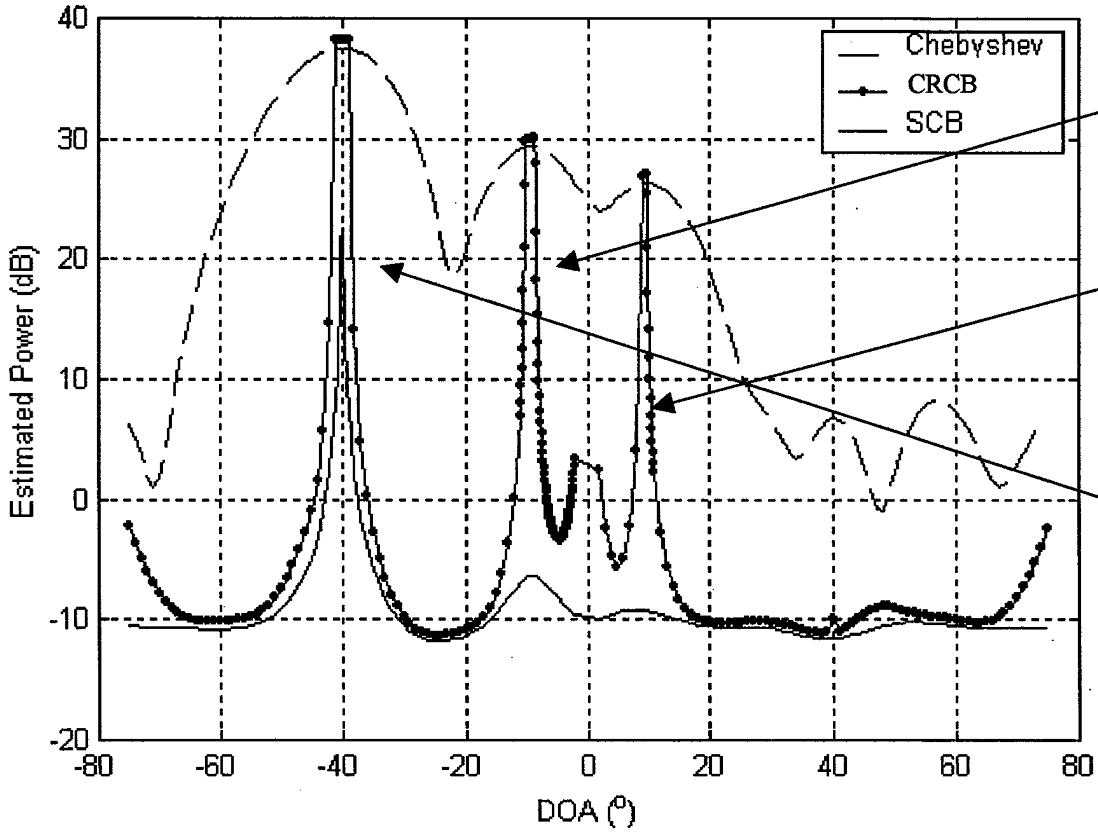
- More columns in V means
  - Better multipath elimination
  - Loss of DOF for interference suppression.

- Doubly RCB is robust against error of V
  - Columns in V should be as independent as possible

Numerical Examples

- M = 10 sensors, 40 snapshots
- Uniform linear array with half-wavelength spacing
- 100 Monte-Carlo trials for average output SINR

Power Estimate vs. Angle



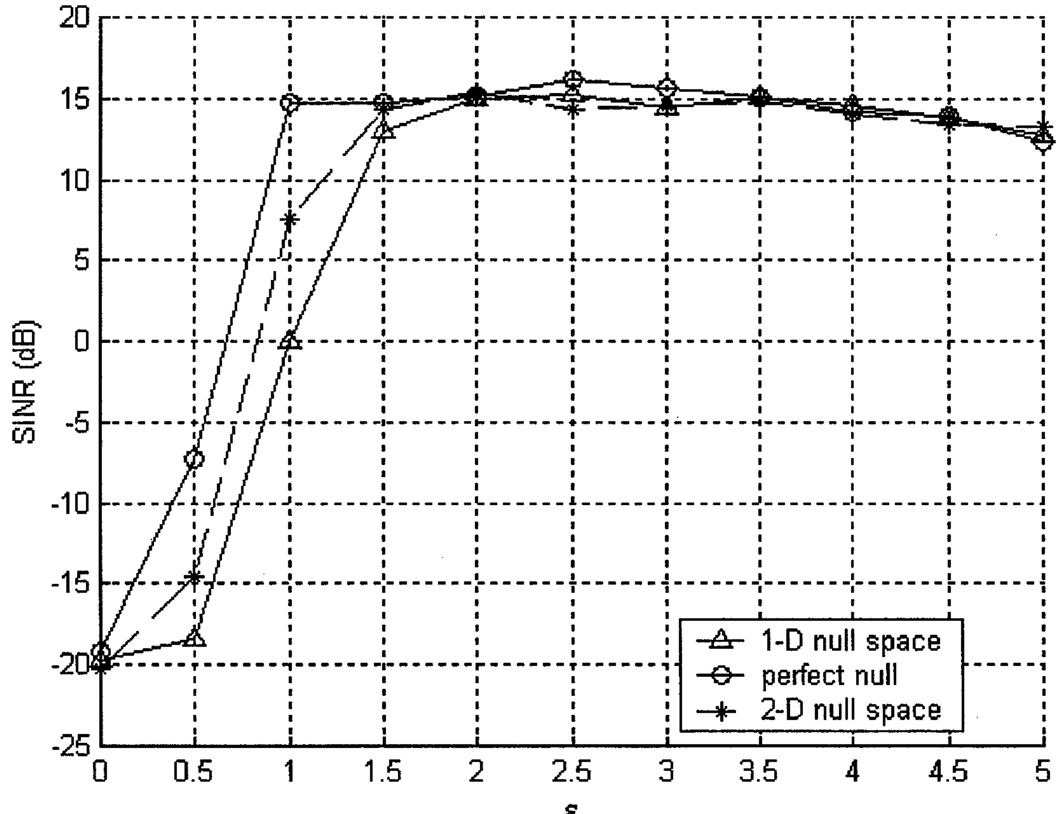
SOI (-10 deg, 30 dB)

Coherent multipath (9 deg, 27 dB)

Non-coherent signal (-40 deg, 40 dB)

Assume $\mathrm{V} = [\mathrm{a}(-\theta_0)]$

Output SINR vs. ϵ



SOI (-20 deg, 30 dB)

Coherent multipath
(19 deg, 27 dB)

Non-coherent signal
(-40 deg, 40 dB)

1-D null space: assume $\mathrm{V} = [\mathrm{a}(-\theta_0)]$

2-D null space: $\mathrm{V} = [\mathrm{a}(-\theta_0 - 0.5^\circ), \mathrm{a}(-\theta_0 + 0.5^\circ)]$

Summary

- Our RCB robust against steering vector errors.
  - Much more accurate SOI power estimate
  - Directly related to uncertainty of steering vector
  - Belongs to (extended) class of diagonal loading approaches
- Much better resolution and interference rejection capability than data-independent beamformers.
- Computationally efficient.
- Can be made robust against coherent interferences (CRCB).

THANK YOU!

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Array Calibration Errors

For small calibration errors

$$(1+\delta_n)e^{j\phi_n} \simeq (1+\delta_n)(1+j\phi_n) \simeq 1+\delta_n+j\phi_n$$

Random amplitude error $\quad \delta_n \sim \mathcal{N}(0,\sigma_\delta^2)$

Random phase error $\quad \phi_n \sim \mathcal{N}(0,\sigma_\phi^2)$

- Array steering vector with calibration errors

$$\tilde{\mathbf{a}}(\theta) = (\mathbf{I}+\mathbf{P})\mathbf{a}(\theta)$$

where $\mathbf{P} = \mathrm{diag}\{\delta_1+j\phi_1, \delta_2+j\phi_2, \cdots, \delta_N+j\phi_N\}$

- $E\{\epsilon_0\} = E\{\|\tilde{\mathbf{a}}(\theta) - \mathbf{a}(\theta)\|^2\} = M(\sigma_\delta^2+\sigma_\phi^2)$