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In this report we study, design and evaluate the performance of space-time systems over fading wireless channels. We present a new method to derive closed-form expressions for the exact bit error probability for optimum combining (OC). Our method differs from previous approaches in that it starts from the decision metrics of OC instead of the moment generating function. This approach facilitates obtaining closed-form expressions. With these expressions for BEP, evaluating the performance of OC is easy and accurate.

We propose a noncoherent detection scheme in which the channel gain of the desired signal is unknown to the receiver. The maximum likelihood decision statistic is derived for the detector and its performance is demonstrated by analysis and simulation. To reduce the computation complexity of the decision statistic, we present a sub-optimum decision feedback algorithm for iterative symbol detection. We also develop another noncoherent detection scheme, where the only required channel information is the channel amplitude of the interference. It is shown that when the interference level is high, this detection technique can achieve good performance.

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Joint Space-Time Coded Modulation and Channel Coding over Fading Channels with Cochannel Interference

Final Technical Report

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Chapter 1

Executive Summary

This report summarizes the work on "Joint Space-Time Coded Modulation and Channel Coding over Fading Channels with Cochannel Interference " carried out at the NJIT from July 1999 to August 2002. The main contribution of this work was to develop algorithms for adaptive arrays for wireless communications and analyze their performance in fading channels with cochannel interference. The research covers several topics that are summarized below.

We first studied the performance of coherent detection scheme for adaptive arrays over wireless channel. For adaptive arrays, optimum combining (OC) is an excellent processing technique to combat multipath fading and suppress cochannel interference. In this report we present new methods to derive closed-form expressions for the exact bit error probability (BEP) for optimum combining ([1], [2]). Our method differs from previous ones in that it starts from the decision metrics of OC instead of the moment generating function. This approach facilitates obtaining closed-form expressions. The final expression is for multiple interferers and multiple reception branches, with the number of interferers less than the number of reception branches. We assume that the modulation is binary phase shift keying (BPSK), and the channels are independent Rayleigh fading channels. With these expressions for BEP, evaluating the performance of OC is easy and accurate.

OC is a coherent detection scheme. To implement it, the following channel information is required: the channel gain (including amplitude and phase) of the desired signal, the covariance matrix of the interference plus noise. For communication systems where channel phase is very difficult or impossible to recover, OC is not practical and noncoherent detection is necessary. We propose a noncoherent detection scheme for adaptive arrays ([3], [4]). The scheme is called multiple symbol differential detection (MSDD), in which the channel gain of the desired signal is unknown to the receiver, but the covariance matrix of the interference plus noise is known. The maximum likelihood decision statistic is derived for the detector and its performance is demonstrated by analysis and simulation. To reduce the computation complexity of the MSDD decision statistic, we present a sub-optimum decision feedback algorithm for iterative symbol detection. This sub-optimum algorithm achieves performance that is very close to that of optimum algorithm. A closed-form approximation to the union bound of pairwise error probability is shown to provide a good approximation to the bit error probability. We show analytically and numerically that with an increasing observation interval, the performance of this kind of MSDD approaches that of OC with differential encoding. We also develop MSDD for another kind of noncoherent detection where the only required channel information is the channel amplitude of the interference. It is shown that when the interference level is high, this MSDD technique can achieve good performance. This work was presented in [5].

Chapter 2

Introduction

2.1 Motivation

In modern commercial wireless communication systems such as code division multiple access (CDMA) systems and time division multiple access (TDMA) systems, the cellular concept is widely applied to increase system capacity [6]. Under this concept, the entire service area is divided into small areas called cells. Several cells comprise a cell cluster. A cell cluster uses all the available resource. For TDMA the resource is frequency channels, while for CDMA the resource is codes. For TDMA systems, each cell can use a portion of the available frequency channels, but cells in the neighborhood use different frequency channels. The cells that use the same frequency channels are at least a cell away from each other. In this way the limited precious frequency resource could be reused, hence the system capacity can be increased infinitely (at least in theory), while at the same time the interference is kept to a minimum. What should be kept in mind is, though the interference is minimized, it still exists due to the same frequency channel used by different cells. Actually the interference becomes one of the factors that limit the performance of the wireless communication systems.

The cellular concept is illuminated in Fig. 2.1. A_i to G_i (i =0, 1, ..., 6) are cells that form a cell cluster. Cells A_i for i =0, 1, ..., 6 use the same frequency channels, so do cells B_i to G_i. The user $s_{I,i}$ in cells A_i (i =1, 2, ..., 6) can interfere the user s in cell A_0.

Another factor that constraints the performance is multipath fading. In the wireless environment, due to reflection, diffraction and scattering, the transmitted signal may fade greatly and reach the receive antenna through more than one path (shown in Fig. 2.2). Since the locations of the transmitter, the obstacles and the reflectors are random, the transmit paths are random as well. The received signal from a different path may add up constructively, or destructively. The total effect of this summation is a random attenuation of the transmitted signal. When the attenuation is deep, the received signal is so weak that the receiver won't be able to recover the transmitted signal. To resolve this problem, diversity is introduced. By using diversity technique, several replicas of the same information signal are transmitted over independently fading channels. The probability that all the signal components reaching the receiver will fade simultaneously is reduced considerably.

Three of the techniques that can achieve more than one independently fading version of the same information-bearing signal are [7, Chapter 14]:

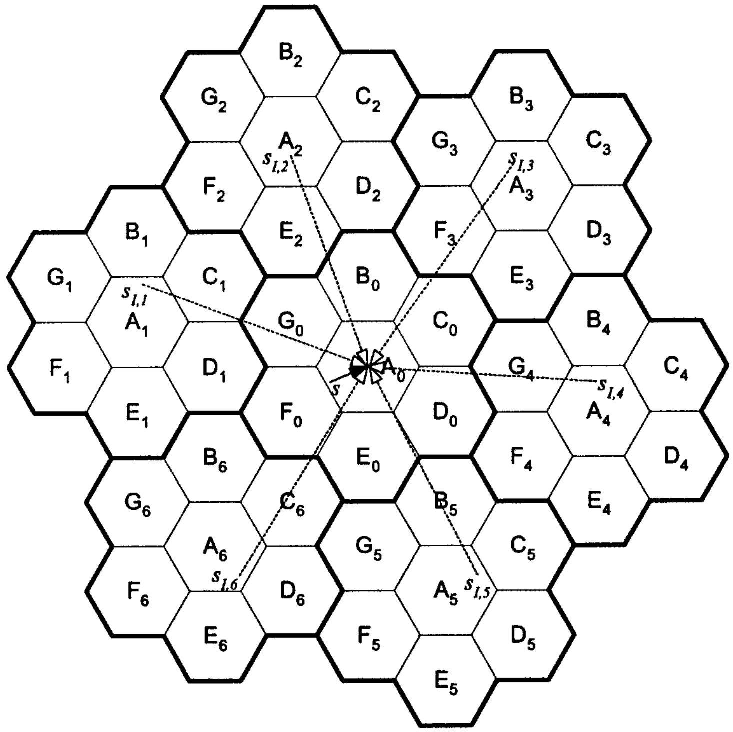

Figure 2.1: The cellular concept. Since cells A_i for i = 1, ..., 6 use the same frequency channels as cell A_0, the user $s_{I,i}$ (i =1, 2, ..., 6) may interfere the user s in cell A_0

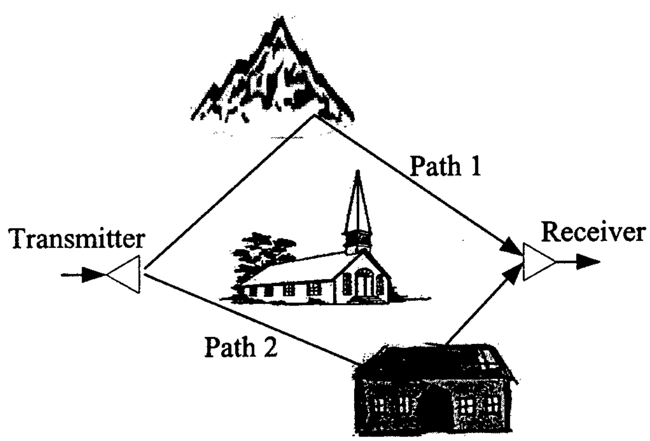

Figure 2.2: Multipath. The transmitted signal reaches the receiver through path 1 and path 2.

- Temporal diversity: the same information bearing signal is transmitted in more than one time slot, where the separation between successive time slots equals or exceeds the coherence time of the channel.

- Frequency diversity: the same information bearing signal is transmitted on more than one carrier frequency, where the separation between successive carrier frequencies equals or exceeds the coherence bandwidth of the channel.

- Spatial diversity: more than one transmit and/or receive antenna are employed. The antennas are spaced sufficiently far apart that the multipath components in the signal have independent fading.

Since spatial diversity does not require expansion of the bandwidth, it is desirable for a bandwidth-limited system when cost and size permitted. And as pointed out in [8], spatial diversity could be used to cancel interference as well as to combat fading. Capacity of a system with spatial diversity has been proven to increase with the number of antennas [9].

It is for these advantages that communication systems with spatial diversity has been an appealing research area. Reception adversity has been implemented in base stations. For example, in the second generation IS-136 TDMA base station [10], two receive antennas are deployed. Technology has been developed for deploying 4 receive antennas at the base station.

Currently most of the research on spatial diversity is focused on space-time codes ([11], [12], [13]), which employ transmit diversity. While space-time codes can provide some coding gain as well as spatial diversity, and could be the future application, this work focuses on a more practical problem for now: performance analysis of communication systems with reception diversity but without transmit diversity and coding. To more easily understand the topic, the following presents the basic system model used in this work.

2.2 System Model

We consider communication system with reception diversity but without transmit diversity. All the signals are baseband signals. As shown in Fig. 2.3, there is one transmit antenna, L reception branches and N_I interferers in the system. The sampled output of the matched filter for the ℓ-th branch at time k is expressed as

$$r_{k,\ell} = \sqrt{P_s} c_\ell s_k + \sum_{i=1}^{N_I} \sqrt{P_I} c_{i,\ell} s_{i,k} + n_{k,\ell}, \ \ell = 1, \cdots, L \tag{2.1}$$

where the parameters in (2.1) are defined as:

P_s : power of the desired signal.

c_ℓ : channel gain of the ℓ-th branch for the desired signal.

s_k : desired transmitted symbol.

P_I : power of the interferers.

$c_{i,\ell}$: channel gain of the ℓ-th branch for the i-th interferer.

$s_{i,k}$: interference signal.

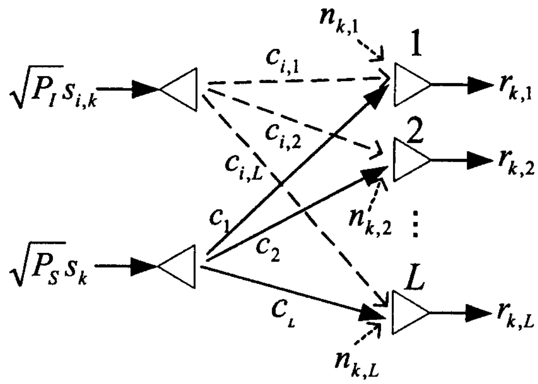

Figure 2.3: Diagram for systems with L reception diversity branches. s_k is the desired signal. There could be more than one interferer $s_{i,k}$ (only one is shown in the figure).

$n_{k,\ell}$: complex white Gaussian noise.

s_k and $s_{i,k}$ could be multiple phase shift keying (M-PSK) symbols, differential multiple phase shift keying (M-DPSK) symbols, or Gaussian distributed signals. We will define them more specifically in later chapters.

The signal model in vector notation is

$$\mathbf{r}_k = \sqrt{P_s}\mathbf{c}s_k + \sqrt{P_I}\sum_{i=1}^{N_I}\mathbf{c}_i s_{i,k} + \mathbf{n}_k \tag{2.2}$$

where $\mathbf{r}_k = [r_{k,1}, r_{k,2}, \cdots, r_{k,L}]^T$, and the superscript T denotes vector transposition; $\mathbf{c}, \mathbf{c}_i$ and $\mathbf{n}_k$ are vectors that are similarly defined as $\mathbf{r}_k$.

The channel gains c_ℓ and $c_{i,\ell}$ are assumed to be independently and identically distributed (i.i.d.), zero-mean, circularly symmetric, complex Gaussian random variables (Rayleigh fading), with variance $\Omega_\ell/2$ (for c_ℓ) and $1/2$ (for $c_{i,\ell}$) per dimension. For future use, we also define the fading matrix for the interferers as $\mathbf{c}_I = [\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_{N_I}]$.

Define the interference plus noise vector as

$$\mathbf{z}_k = \sqrt{P_I}\sum_{i=1}^{N_I}\mathbf{c}_i s_{i,k} + \mathbf{n}_k \tag{2.3}$$

then (2.2) becomes

$$\mathbf{r}_k = \sqrt{P_s}\mathbf{c}s_k + \mathbf{z}_k \tag{2.4}$$

In the analysis, we often treat $\mathbf{c}_I$ as constant vector first, then treat it as random vector. When we treat $\mathbf{c}_I$ as constant vector, the covariance matrix of $\mathbf{z}_k$ is

$$\mathbf{R}_z = P_I\sum_{i=1}^{N_I}\mathbf{c}_i\mathbf{c}_i^H + \text{diag}\ \left(\sigma_1^2, \sigma_2^2, \cdots, \sigma_L^2\right) \tag{2.5}$$

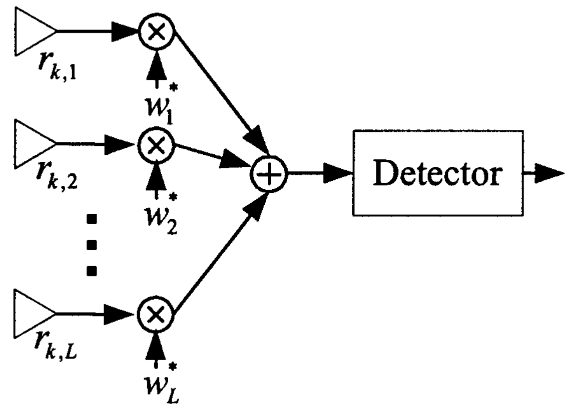

Figure 2.4: Diagram of the optimum combining detector.

where the superscript H denotes the Hermitian transposition and $(\sigma_1^2, \sigma_2^2, \cdots, \sigma_L^2)$ is the power profile of the noise.

2.3 Background

For wireless communication systems with reception diversity, Optimum combining (OC) is a well-known approach to combat fading and suppress cochannel interference. Its structure is shown in Fig. 2.4.

In Fig. 2.4, the received signal $\mathbf{r}_k = [r_{k,1}, r_{k,2}, \cdots, r_{k,L}]^T$ is weighted and combined. The weight vector $\mathbf{w} = [w_1, w_2, \cdots, w_L]^T$ is chosen in such that the output of combiner (which is the input to the detector) achieves the maximum signal-to-interference-plus-noise ratio (SINR). The decision rule for OC is

$$\widehat{s}_k = \arg\max_{s_k} p\left(\mathbf{r}_k | s_k, \mathbf{c}, \mathbf{R}_z\right) \tag{2.6}$$

where $p\left(\mathbf{r}_k | s_k, \mathbf{c}, \mathbf{R}_z\right)$ is the probability of $\mathbf{r}_k$ conditioned on $s_k, \mathbf{c}$, and $\mathbf{R}_z$. A simplified version of this decision rule will be shown in Chapter 3.

One of our efforts is to derive closed-form expressions for bit error probability (BEP) for OC. Those kinds of expressions have only been obtained for some special cases. Some related work about OC is summarized in Chapter 3.

OC is a coherent detection scheme. To construct the weight vector $\mathbf{w}$, the following channel information is required: $\mathbf{c}$, which is the channel gain (include amplitude and phase) of the desired signal and $\mathbf{R}_z$, the covariance matrix of the interference plus noise. For communication systems where channel phases are very difficult or impossible to recover, OC is not practical. Under this circumstance, noncoherent detection scheme must be considered.

One kind of noncoherent detection scheme is differential detection, in which the transmitted signals are differentially encoded. For conventional differential detection, two received signals are used at the observation interval to make decision about the transmitted signal. The recovery of channel phase is not required. The decision rule for conventional differential detection is

$$(\widehat{s}_{k-1}, \widehat{s}_k) = \arg \max_{s_{k-1}, s_k} p\left(\mathbf{r}_{k-1}, \mathbf{r}_k | s_{k-1}, s_k, \mathbf{R}_z\right), \tag{2.7}$$

where $p\left(\mathbf{r}_{k-1}, \mathbf{r}_k | s_{k-1}, s_k, \mathbf{R}_z\right)$ is the probability of $\mathbf{r}_{k-1}, \mathbf{r}_k$ conditioned on s_{k-1}, s_k and $\mathbf{R}_z$. Conventional differential detection suffers performance penalty comparing to coherent detection.

Multiple sysmbol differential detection (MSDD) achieves performance between that of conventional differential detection and coherent detection. In MSDD, more than two symbols are used in the observation interval. It was shown that with the increase of symbols in the observation interval, the performance of differential detection can be improved significantly. The decision rule for MSDD is

$$\widehat{\mathbf{s}}_k = \arg \max_{\mathbf{s}_k} p\left(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{R}_z\right), \tag{2.8}$$

where $\mathbf{s}_k \triangleq \left[s_{k-(K-1)}, \cdots, s_{k-1}, s_k\right]^T$ is a sequence of K ($K > 2$) symbols, $\underline{\mathbf{r}}_k = \left[\mathbf{r}_{k-(K-1)}, \cdots, \mathbf{r}_k\right]^T$ is a vector of all the received signal in the observation interval, and $p\left(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{R}_z\right)$ is the probability of $\underline{\mathbf{r}}_k$ conditioned on $\mathbf{s}_k$ and $\mathbf{R}_z$. Some related work about MSDD is summarized in Chapter 5.

In this work, we applied MSDD to communication systems with interference. The only required channel information for that kind of MSDD is the covariance matrix of the interference plus noise $\mathbf{R}_z$. By simulation and analysis results, we demonstrate that MSDD could achieve performance close to that of OC with differential encoding.

We also develop MSDD for another kind of noncoherent detection where the only required channel information is the channel amplitude of the interference. Its decision rule is

$$(\widehat{\mathbf{s}}_k, \widehat{\mathbf{s}}_{I,k}) = \arg \max_{\mathbf{s}_k, \mathbf{s}_{I,k}} p\left(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, |\mathbf{c}_I|\right), \tag{2.9}$$

where $\mathbf{s}_k$ and $\underline{\mathbf{r}}_k$ are defined as mentioned before; $\mathbf{s}_{I,k} \triangleq \left[s_{I,k-(K-1)}, \cdots, s_{I,k-1}, s_{I,k}\right]^T$ is the sequence of interference symbols; $|\mathbf{c}_i|$ is the channel amplitude of the interference, and $p\left(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, |\mathbf{c}_i|\right)$ is the probability of $\underline{\mathbf{r}}_k$ conditioned on $\mathbf{s}_k$, $\mathbf{s}_{I,k}$, and $|\mathbf{c}_i|$.

It is shown that when the interference level is high, this MSDD technique can achieve better performance than detectors using optimum combining (with differential encoding).

2.4 Outline of the Report

The main topics of this work are:

- Performance analysis of OC.
- Derivation of the decision statistic and performance analysis of MSDD.
- Performance comparison of OC and MSDD.

In Chapter 3, We present a new method to derive the closed-form expressions for the exact BEP for OC. We first derive the BEP conditioned on the fading of the interference, then derive the unconditional BEP. The expressions are for multiple interferers and multiple reception branches, with the number of interferers less than that of reception branches. The modulation is BPSK and the channels are independent Rayleigh fading channels.

In Chapter 4, We derive simpler asymptotic expressions for BEP of OC for M-PSK modulation and one interferer.

In Chapter 5, we develop a detector exploiting MSDD technique. The channel gain of the desired signal are assumed unknown. M-DPSK modulation is employed. The decision statistic for the detector is derived based on the principle of maximum likelihood sequence detection (MLSD). The performance of the detector is demonstrated by simulation results and analysis results.

In Chapter 6, another kind of MSDD is presented to suppress cochannel interference in communication systems. The channel gain of the desired signal and the channel phase of the interference are assume unknown, but the channel amplitude of interference is assumed known at the receiver. The interference signal is assumed to have the same M-DPSK modulation as the desired signal. A maximum likelihood sequence detector is developed for detecting both the desired signal and the interference signal. It is a kind of multiuser detector employing MSDD.

Summary and future work are presented in Chapter 7.

Chapter 3

Performance Analysis for OC with BPSK Modulation and Multiple Interferers

3.1 Introduction

As mentioned in Chapter 2, for wireless communication systems with reception diversity, optimum combining (OC) is a well-known approach to combat fading and suppress cochannel interference. It combines the output of the reception branches in an optimum way and achieves the maximum output signal-to-interference-plus-noise ratio (SINR).

Performance analysis of OC has been an active research area. A lot of effort has been focused on obtaining closed-form expressions for BEP. For the case of only one interferer existing in the system, some work has been done and useful expressions have been derived. [8] and [14] gave out expressions for BEP when the interference is not fading and the modulation is BPSK. The non-closed-form expression for fading interference was presented in [15], and closed-from expression for BEP was obtained in [16]. A non-closed-from expression of BEP for M-PSK was shown in [17].

For the case where the number of interferers is equal to or greater than the number of reception branches and thermal noise is negligible, BEP could be expressed in closed form [18].

When there are more than one interferers and the noise is not negligible, no closed-from expression for the exact BEP is available until now, but the performance has been studied extensively through simulation [8], upper bounds[19], approximate expression [20] and non closed-from expressions ([21], [22]). The related work about OC was summarized in [17].

The conventional way of deriving the expression for BEP is to first derive the probability density function of the SINR conditioned on the fading of the interference. The conditional BEP is expressed as a function of SINR. The unconditional BEP (which is what we want) is obtained by averaging the conditional BEP first over SINR, and then over the fading of the interference. Since it involves two steps of averaging, a closed-form expression for BEP is difficult to obtain.

In this chapter, we present a new method to derive BEP. We first derive the BEP conditioned on the fading of the interference directly, then derive the unconditional BEP by averaging the conditional BEP over the fading of the interference. This approach involves only one averaging. Though complicated, this method provides closed-form expression at the end. The basic configuration for our derivation is BPSK modulation, Rayleigh fading channel. We assume the number of interferers is less than the number of reception branches. We should point out that the method developed here could be applied to the case where the number of interferers is equal to or greater than the number of reception branches.

3.2 System Model

The system model used in this chapter is the same as that mentioned in Section 2.2, which is represented by the following expression:

$$r_{k,\ell} = \sqrt{P_s} c_\ell s_k + \sum_{i=1}^{N_I} \sqrt{P_I} c_{i,\ell} s_{i,k} + n_{k,\ell}, \; \ell = 1, \cdots, L \tag{3.1}$$

where the parameters are defined in Section 2.2. We assume the number of independent reception branches L is greater than the number of interferers N_I. s_k is the desired transmitted BPSK symbol. The interference signal $s_{i,k}$ is assumed to be Gaussian distributed, with zero mean and unit variance.

The channel gains c_ℓ of the desired signal and $c_{i,\ell}$ of the interference are assumed to be independently and identically distributed (i.i.d.), zero-mean, circularly symmetric, complex Gaussian random variables (Rayleigh fading), with variance 1/2 per dimension.

As mentioned in Section 2.2, the signal model in vector notation is

$$\mathbf{r}_k = \sqrt{P_s}\mathbf{c} s_k + \sqrt{P_I} \sum_{i=1}^{N_I} \mathbf{c}_i s_{i,k} + \mathbf{n}_k. \tag{3.2}$$

3.3 Derivation of Conditional BEP

As shown in Fig. 2.4, for OC detector, the received signal $\mathbf{r}_k$ are weighted and combined to get an output signal. The weighted vector that yields the largest SINR is [8]

$$\mathbf{w} = \mathbf{R}_z^{-1}\mathbf{c}. \tag{3.3}$$

The output of the combiner is $\mathbf{w}^H\mathbf{r}_k$. For BPSK modulation, the decision rule of the detector is: if $\mathrm{Re}(\mathbf{w}^H\mathbf{r}_k) \geqslant 0$, the decision is made that 1 is transmitted; otherwise the decision is made that -1 is transmitted.

From now on, we assume the transmitted symbol is $s_k = 1$, then from (3.2), the received signal is

$$\mathbf{r}_k = \sqrt{P_s}\mathbf{c} + \sqrt{P_I} \sum_{i=1}^{N_I} \mathbf{c}_i s_{i,k} + \mathbf{n}_k. \tag{3.4}$$

Define random variable

$$D = 2\text{Re}\left(\mathbf{w}^H \mathbf{r}_k\right) = \mathbf{w}^H \mathbf{r}_k + \left(\mathbf{w}^H \mathbf{r}_k\right)^*. \tag{3.5}$$

Note that D is random due to that channels $\mathbf{c}$ and $\mathbf{c}_I$, interference $s_{i,k}$ and noise $\mathbf{n}_k$ are random. According to the decision rule, when $D < 0$, the decision is made that -1 is transmitted and an error occurs. Therefore the BEP is $P_e = \Pr(D < 0)$. Our purpose is to expression P_e as a closed-from function of the SNR, SIR, number of reception branches L and number of interferers N_I.

It is very difficult to derive the expression for P_e directly. We derive it in two steps. In this section, we treat the fading of the interference $\mathbf{c}_I$ as constant (which is actually random). More specifically, in this section, $\mathbf{c}_I = [\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_{N_I}]$ is considered as non-random, $\mathbf{r}_k$ (and consequently D) is random only due to the randomness of $\mathbf{c}$, $s_{i,k}$ and $\mathbf{n}_k$. Based on this assumption, we derive the conditional BEP $P\left(e|\mathbf{c}_I\right) = \Pr(D < 0|\mathbf{c}_I)$. In the next section, we will consider the random effect of $\mathbf{c}_I$ and derive the unconditional BEP P_e by averaging the conditional BEP $P\left(e|\mathbf{c}_I\right)$ over $\mathbf{c}_I$.

When the fading of the interference $\mathbf{c}_I$ is treated as constant, the covariance matrix of the interference plus noise $\mathbf{z}_k$ is

$$\mathbf{R}_z = P_I \sum_{i=1}^{N_I} \mathbf{c}_i \mathbf{c}_i^H + \sigma^2 \mathbf{I}_L, \tag{3.6}$$

where the superscript H denotes the Hermitian transposition, σ^2 is the power of the noise and $\mathbf{I}_L$ is an identity matrix of rank L.

Let $\Phi_D(j\omega)$ be the characteristic function of D conditioned on $\mathbf{c}_I$. Using similar procedure as in [7, Appendix B], it can be shown that the conditional BEP is

$$\begin{aligned} P\left(e|\mathbf{c}_I\right) &= \Pr(D < 0|\mathbf{c}_I) \\ &= -\frac{1}{2\pi j} \int_{-\infty+j\varepsilon}^{\infty+j\varepsilon} \frac{\Phi_D(j\omega)}{\omega} d\omega \\ &= -\sum_{\text{Im}(\omega_m)>0} \text{Res}\left[\frac{\Phi_D(j\omega)}{\omega}; \omega_m\right], \end{aligned} \tag{3.7}$$

where ε is a small positive number; $\text{Res}\left[\frac{\Phi_D(j\omega)}{\omega}; \omega_m\right]$ denotes the residue of function $\frac{\Phi_D(j\omega)}{\omega}$ at pole ω_m. The summation is taken over the poles in the upper half of the complex plane.

The task at hand is to evaluate the characteristic function $\Phi_D(j\omega)$. To that end, we try to express D in (3.5) as a quadratic form of complex-valued Gaussian random variables and apply the results in [7, Appendix B].

Substitute (3.3) into (3.5),

$$D = \mathbf{c}^H \mathbf{R}_z^{-1} \mathbf{r}_k + \left(\mathbf{c}^H \mathbf{R}_z^{-1} \mathbf{r}_k\right)^*. \tag{3.8}$$

Diagonalize $\mathbf{R}_z$ as $\mathbf{R}_z = \mathbf{U}_z \Lambda_z \mathbf{U}_z^H$, where $\Lambda_z = \text{diag}\left(\lambda_1, \cdots, \lambda_L\right)$, $\lambda_1, \cdots, \lambda_L$ are the eigenvalues of $\mathbf{R}_z$, and $\mathbf{U}_z$ is the unitary matrix whose columns are the eigenvectors of $\mathbf{R}_z$. According to [22], we can assume that the L eigenvalues of $\mathbf{R}_z$ satisfy: $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant$

$\lambda_{N_I} \geqslant \sigma^2$ and $\lambda_m = \sigma^2$ for $m = N_I+1, N_I+2, \cdots, L$. Define the vector of the non-zero eigenvalues as $\lambda = (\lambda_1, \cdots, \lambda_{N_I})$.

From $\mathbf{R}_z = \mathbf{U}_z \Lambda_z \mathbf{U}_z^H$ we have

$$\mathbf{R}_z^{-1} = \mathbf{U}_z \Lambda_z^{-1} \mathbf{U}_z^H. \tag{3.9}$$

Substitute the above expression into (3.8),

$$\begin{aligned} D &= \mathbf{c}^H \mathbf{U}_z \Lambda_z^{-1} \mathbf{U}_z^H \mathbf{r}_k + \left(\mathbf{c}^H \mathbf{U}_z \Lambda_z^{-1} \mathbf{U}_z^H \mathbf{r}_k\right)^* \\ &= \left(\mathbf{U}_z^H \mathbf{c}\right)^H \Lambda_z^{-1} \left(\mathbf{U}_z^H \mathbf{r}_k\right) + \left[\left(\mathbf{U}_z^H \mathbf{c}\right)^H \Lambda_z^{-1} \left(\mathbf{U}_z^H \mathbf{r}_k\right)\right]^*. \end{aligned} \tag{3.10}$$

Define the whitened interference-plus-noise vector $\mathbf{x} = [x_1, x_2, \cdots, x_L]^T$ and modified channel vector $\mathbf{g} = [g_1, g_2, \cdots, g_L]^T$ as

$$\mathbf{x} = \mathbf{U}_z^H \mathbf{r}_k \tag{3.11}$$

$$\mathbf{g} = \mathbf{U}_z^H \mathbf{c}. \tag{3.12}$$

Note that since $\mathbf{U}_z$ is unitary, the vectors $\mathbf{g}$ and $\mathbf{c}$ have the same distribution.

Substitute (3.11) and (3.12) in (3.10),

$$\begin{aligned} D &= \mathbf{g}^H \Lambda_z^{-1} \mathbf{x} + \left(\mathbf{g}^H \Lambda_z^{-1} \mathbf{x}\right)^* \\ &= \sum_{m=1}^{L} \lambda_m^{-1} \left(g_m^* x_m + g_m x_m^*\right). \end{aligned} \tag{3.13}$$

In (3.13), D is a quadratic form of complex-valued random vectors $\mathbf{x}$ and $\mathbf{g}$. Applying the results in [7, Appendix B] to (3.13) (see Appendix A in this report for details), we obtain the characteristic function of D as

$$\Phi_D(j\omega) = \prod_{m=1}^{2L} \frac{1}{1 - j\mu_m \omega} \tag{3.14}$$

where

$$\mu_m = \begin{cases} \frac{\sqrt{P_s}}{\lambda_m} + \sqrt{\frac{P_s}{\lambda_m^2} + \frac{1}{\lambda_m}} & m = 1, \cdots, N_I \\ \frac{\sqrt{P_s}}{\lambda_{m-N_I}} - \sqrt{\frac{P_s}{\lambda_{m-N_I}^2} + \frac{1}{\lambda_{m-N_I}}} & m = N_I+1, \cdots, 2N_I \\ \mu_{00} & m = 2N_I+1, \cdots, 2N_I + (L - N_I) \\ \mu_{01} & m = 2N_I + (L - N_I) + 1, \cdots, 2L \end{cases} \tag{3.15}$$

and

$$\mu_{00} \triangleq \frac{\sqrt{P_s}}{\sigma^2} + \sqrt{\frac{P_s}{\sigma^4} + \frac{1}{\sigma^2}} \tag{3.16}$$

$$\mu_{01} \triangleq \frac{\sqrt{P_s}}{\sigma^2} - \sqrt{\frac{P_s}{\sigma^4} + \frac{1}{\sigma^2}}. \tag{3.17}$$

Notice that with regard to the fading of the interference $\mathbf{c}_I$, μ_{00} and μ_{01} are constants. They only change with P_s and σ^2. Define the signal to noise ratio (SNR) γ as

$$\gamma = \frac{P_s}{\sigma^2}. \tag{3.18}$$

Then μ_{00} and μ_{01} in (3.16) and (3.17) could be written as

$$\mu_{00} = \frac{1}{\sqrt{P_s}}\left(\gamma + \sqrt{\gamma^2+\gamma}\right) \tag{3.19}$$

$$\mu_{01} = \frac{1}{\sqrt{P_s}}\left(\gamma - \sqrt{\gamma^2+\gamma}\right). \tag{3.20}$$

Using (3.15) and (3.14), we have

$$\frac{\Phi_D(j\omega)}{\omega} = \frac{1}{\omega}\left[\prod_{m=1}^{2N_I}\frac{1}{(1-j\mu_m\omega)}\right]\frac{1}{(1-j\mu_{00}\omega)^{L-N_I}}\frac{1}{(1-j\mu_{01}\omega)^{L-N_I}}. \tag{3.21}$$

Since $\mu_m < 0$ $(m = N_I+1, N_I+2, \cdots, 2N_I)$ and $\mu_{01} < 0$, for $\Phi_D(j\omega)/\omega$ in (3.21), the poles $-j/\mu_m$ $(m = N_I+1, N_I+2, \cdots, 2N_I)$ and pole $-j/\mu_{01}$ are in the upper half of the complex plane. Therefore (3.7) becomes

$$P(e|\mathbf{c}_I) = -\sum_{m=N_I+1}^{2N_I}\text{Res}\left[\frac{\Phi_D(j\omega)}{\omega}; -j\frac{1}{\mu_m}\right] - \text{Res}\left[\frac{\Phi_D(j\omega)}{\omega}; -j\frac{1}{\mu_{01}}\right]. \tag{3.22}$$

Substituting (3.21) into (3.22), and carrying out the calculation of the residues (Appendix B), we get the conditional BEP as

$$\begin{aligned}
&P(e|\mathbf{c}_I)\\
=& -\sum_{m=N_I+1}^{2N_I}\left[\mu_m^{2L-1}\left(\prod_{n=1}^{m-1}\frac{1}{\mu_n-\mu_m}\right)\left(\prod_{n=m+1}^{2N_I}\frac{1}{\mu_n-\mu_m}\right)\right.\\
&\left.\frac{1}{(\mu_{00}-\mu_m)^{L-N_I}}\frac{1}{(\mu_{01}-\mu_m)^{L-N_I}}\right]\\
&-\frac{1}{(\mu_{00}\mu_{01})^{L-N_I}}\frac{1}{(L-N_I-1)!}(-1)^{L-N_I-1}\sum_{\ell=0}^{L-N_I-1}\frac{(L-N_I+\ell-1)!}{\ell!}\\
&\left[\mu_{01}^{L-N_I-\ell}+\sum_{n=1}^{2N_I}\frac{(\mu_n\mu_{01})^{L-N_I-\ell}}{(\mu_n-\mu_{01})^{L-N_I-\ell}}\mu_n^{2N_I-1}\left(\prod_{i=1}^{n-1}\frac{1}{\mu_i-\mu_n}\right)\left(\prod_{i=n+1}^{2N_I}\frac{1}{\mu_i-\mu_n}\right)\right]\\
&\frac{(\mu_{00}\mu_{01})^{L-N_I+\ell}}{(\mu_{00}-\mu_{01})^{L-N_I+\ell}}.
\end{aligned} \tag{3.23}$$

$P(e|\mathbf{c}_I)$ is a function of μ_m $(m = 1, 2, \cdots, 2N_I)$. Since μ_m's $(m = 1, 2, \cdots, 2N_I)$ are functions of eigenvalues λ_m's $(m = 1, 2, \cdots, N_I)$, $P(e|\mathbf{c}_I)$ is a function of $\lambda = (\lambda_1, \cdots, \lambda_{N_I})$, which is related to the fading of interference $\mathbf{c}_I$.

Special case: no interference

We can use the method detailed above to derive the BEP for the case where there is no interference. In this case, OC becomes maximum ratio combining. Since the power of interference $P_I = 0$, from (3.6) we have the covariance matrix $\mathbf{R}_z = \sigma^2 \mathbf{I}_L$. The eigenvalues of $\mathbf{R}_z$ would be $\lambda_m = \sigma^2$ for $m = 1, 2, \cdots, L$. Referring to (3.15), (3.16) and (3.17), we have $\mu_m = \mu_{00}$ for $m = 1, 2, \cdots, N_I$; $\mu_m = \mu_{01}$ for $m = N_I + 1, N_I + 2, \cdots, 2N_I$. (3.21) becomes

$$\frac{\Phi_D(j\omega)}{\omega} = \frac{1}{\omega} \frac{1}{(1 - j\mu_{00}\omega)^L} \frac{1}{(1 - j\mu_{01}\omega)^L}. \tag{3.24}$$

Substitute the above expression in (3.7), we get the BEP as

$$P_e = -\text{Res}\left[\frac{\Phi_D(j\omega)}{\omega}; -j\frac{1}{\mu_{01}}\right]. \tag{3.25}$$

Since there is no interference, the BEP is unconditional BEP. After carrying out the calculation of the residues, we get

$$P_e = \frac{(-\mu_{01})^L}{(\mu_{00} - \mu_{01})^L (L-1)!} \left[\sum_{\ell=0}^{L-1} \frac{(L-1+\ell)!}{\ell!} \frac{\mu_{00}^{\ell}}{(\mu_{00} - \mu_{01})^{\ell}}\right]. \tag{3.26}$$

(3.26) is the expression of BEP for without interference. We can express it in a more convenient form.

Substitute (3.19) and (3.20) in (3.26), we have

$$P_e = \left[\frac{1}{2}\left(1 - \sqrt{\frac{\gamma}{1+\gamma}}\right)\right]^L \frac{1}{(L-1)!} \left\{\sum_{\ell=0}^{L-1} \frac{(L-1+\ell)!}{\ell!} \left[\frac{1}{2}\left(1 + \sqrt{\frac{\gamma}{1+\gamma}}\right)\right]^{\ell}\right\}. \tag{3.27}$$

Define

$$\mu = \sqrt{\frac{\gamma}{1+\gamma}}. \tag{3.28}$$

Then (3.27) becomes

$$P_e = \left[\frac{1}{2}(1-\mu)\right]^L \left\{\sum_{\ell=0}^{L-1} \binom{L-1+\ell}{\ell} \left[\frac{1}{2}(1+\mu)\right]^{\ell}\right\}, \tag{3.29}$$

which is the same as Eq. (14-4-15) in [7]. We derive (3.29) without the need of integration. In [7], (3.29) was obtained by integration.

When SNR $\gamma \gg 1$, $\mu = \left(1 + \frac{1}{\gamma}\right)^{-\frac{1}{2}} \approx 1 - \frac{1}{2\gamma}$. (3.29) could be approximated as

$$\begin{aligned} P_e &\approx \frac{1}{4^L \gamma^L} \sum_{\ell=0}^{L-1} \binom{L-1+\ell}{\ell} \\ &= \frac{1}{4^L \gamma^L} \binom{2L-1}{L}. \end{aligned} \tag{3.30}$$

3.4 Derivation of Unconditional BEP

The unconditional BEP P_e is obtained by averaging the conditional BEP $P(e|\mathbf{c}_I)$ over the fading of the interference $\mathbf{c}_I$. Since $P(e|\mathbf{c}_I)$ is a function of the vector of eigenvalues λ, and λ depends on $\mathbf{c}_I$, averaging $P(e|\mathbf{c}_I)$ over $\mathbf{c}_I$ is equivalent to averaging it over λ. Hence

$$P_e = \int P(e|\mathbf{c}_I) p_\lambda(\lambda) d\lambda \tag{3.31}$$

where $p_\lambda(\lambda)$ is the joint probability density function of λ given out in [22] as

$$\begin{aligned} p_\lambda(\lambda) &= K_0 \frac{1}{P_I^{N_I}} \left[\prod_{i=1}^{N_I} \exp\left(-\frac{\lambda_i - \sigma^2}{P_I}\right) \left(\frac{\lambda_i - \sigma^2}{P_I}\right)^{L-N_I} \right] \\ &\times \left[\prod_{1 \le i < j \le N_I - 1} \left(\frac{\lambda_i - \sigma^2}{P_I} - \frac{\lambda_j - \sigma^2}{P_I} \right)^2 \right], \end{aligned} \tag{3.32}$$

where

$$K_0 = \frac{1}{\prod_{i=1}^{N_I} (L-i)! \prod_{i=1}^{N_I} (N_I - i)!}. \tag{3.33}$$

In (3.23), the conditional BEP $P(e|\mathbf{c}_I)$ is a function of μ_m's. As shown in (3.15), μ_m's are irrational functions of eigenvalues λ_m's. Since integration of irrational function is very difficult to carry out, we try to convert the conditional BEP $P(e|\mathbf{c}_I)$ into a rational function of some other variables and then averaging it over those variables. For this purpose, we define constant (with respect to $\mathbf{c}_I$) η as

$$\eta = \sqrt{\frac{\sigma^2}{P_s} + 1}, \tag{3.34}$$

and variables y_m as

$$y_m = \sqrt{\frac{\lambda_m}{P_s} + 1}, \quad m = 1, 2, \cdots, N_I. \tag{3.35}$$

Also define vector $\mathbf{y} = [y_1, y_2, \cdots, y_{N_I}]$. Since $\lambda = (\lambda_1, \cdots, \lambda_{N_I})$ is a random vector, so is $\mathbf{y}$. From (3.34) and (3.35) we have

$$\sigma^2 = P_s(\eta^2 - 1) \tag{3.36}$$

$$\lambda_m = P_s(y_m^2 - 1) \quad m = 1, 2, \cdots, N_I. \tag{3.37}$$

Using (3.34), (3.16) and (3.17), we get

$$\mu_{00} = \frac{\sqrt{P_s}}{\sigma^2}(1 + \eta) \tag{3.38}$$

$$\mu_{01} = \frac{\sqrt{P_s}}{\sigma^2}(1 - \eta). \tag{3.39}$$

From (3.35) and (3.15), we have

$$\mu_m = \frac{\sqrt{P_s}}{\lambda_m}(1+y_m) \tag{3.40}$$

$$\mu_{m+N_I} = \frac{\sqrt{P_s}}{\lambda_m}(1-y_m) \tag{3.41}$$

for $m = 1, 2, \cdots, N_I$.

Substituting (3.38), (3.39), (3.40) and (3.41) into (3.23), after some straightforward manipulations, we get the conditional BEP as a function of the variables y_m's,

$$\begin{aligned} P(e|\mathbf{y}) &= -\sum_{m=1}^{N_I} f_m(\mathbf{y}) - \frac{1}{(L-N_I-1)!}(-1)^{L-N_I-1} \\ &\sum_{\ell=0}^{L-N_I-1} \frac{(L-N_I+\ell-1)!}{\ell!}\left[1+\sum_{m=1}^{N_I} g_{m,\ell}(\mathbf{y})\right]\frac{(1-\eta)^{L-N_I-\ell}}{(2\eta)^{L-N_I+\ell}}\left(-\frac{1}{\gamma}\right)^{\ell} \end{aligned} \tag{3.42}$$

where

$$f_m(\mathbf{y}) = \frac{1-y_m}{2y_m}\frac{(1-\eta^2)^{L-N_I}}{(y_m^2-\eta^2)^{L-N_I}}\left\{\prod_{n=1}^{m-1}\frac{1-y_n^2}{y_m^2-y_n^2}\right\}\left\{\prod_{n=m+1}^{N_I}\frac{1-y_n^2}{y_m^2-y_n^2}\right\} \tag{3.43}$$

and

$$\begin{aligned} g_{m,\ell}(\mathbf{y}) &= (-1)^{L-N_I-\ell}\frac{(1+\eta)^{L-N_I-\ell}}{2y_m}\frac{1}{(y_m^2-\eta^2)^{L-N_I-\ell}} \\ &F_{L-N_I-\ell}(y_m)\left\{\prod_{n=1}^{m-1}\frac{1-y_n^2}{y_m^2-y_n^2}\right\}\left\{\prod_{n=m+1}^{N_I}\frac{1-y_n^2}{y_m^2-y_n^2}\right\} \end{aligned} \tag{3.44}$$

with

$$F_{L-N_I-\ell}(y_m) = -(1+y_m)(\eta-y_m)^{L-N_I-\ell} + (1-y_m)(\eta+y_m)^{L-N_I-\ell}. \tag{3.45}$$

Obviously, conditional BEP $P(e|\mathbf{y})$ is a rational function of the random variables y_m's. The joint probability density function of $\mathbf{y}$ is

$$\begin{aligned} p_{\mathbf{y}}(\mathbf{y}) &= p_{\lambda}(\lambda)\frac{d\lambda_1}{dy_1}\cdots\frac{d\lambda_{N_I}}{dy_{N_I}}\bigg|_{\lambda=\mathbf{y}} \\ &= K_1\left\{\prod_{i=1}^{N_I}\exp\left[-\beta\left(y_i^2-\eta^2\right)\right]\left(y_i^2-\eta^2\right)^{L-N_I}\right\} \\ &\times\left[\prod_{1\le i<j\le N_I-1}\left(y_i^2-y_j^2\right)^2\right]y_1y_2\cdots y_{N_I} \end{aligned} \tag{3.46}$$

for $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_{N_I} \geqslant \eta$, where

$$\beta = P_s/P_I \tag{3.47}$$

is the signal to interference ratio (SIR) and

$$K_1 = \frac{2^{N_I}}{\left[\prod_{i=1}^{N_I}(L-i)!\right]\left[\prod_{i=1}^{N_I}(N_I-i)!\right]}\beta^{LN_I}. \tag{3.48}$$

The unconditional BEP P_e is obtained by averaging the conditional $P(e|\mathbf{y})$ over the random vector $\mathbf{y}$,

$$\begin{aligned} P_e &= \int P(e|\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y})d\mathbf{y} \\ &= -\sum_{m=1}^{N_I}\int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y})d\mathbf{y} - \frac{1}{(L-N_I-1)!}(-1)^{L-N_I-1} \\ &\quad \sum_{\ell=0}^{L-N_I-1}\frac{(L-N_I+\ell-1)!}{\ell!}\left(-\frac{1}{\gamma}\right)^{\ell} \\ &\quad \left[1+\sum_{m=1}^{N_I}\int g_{m,\ell}(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y})d\mathbf{y}\right]\frac{(1-\eta)^{L-N_I-\ell}}{(2\eta)^{L-N_I+\ell}}. \end{aligned} \tag{3.49}$$

In order to carry out the averaging of $g_{m,\ell}(\mathbf{y})$, we need to express it in a more convenient way. Specifically, for easy to integrate, we express the function $F_{L-N_I-\ell}(y_m)$ in $g_{m,\ell}(\mathbf{y})$ (shown in (3.44) and (3.45)) as a sum of $(y_m^2-\eta^2)$ to integer power. From Appendix C, we have

$$F_{L-N_I-\ell}(y_m) = 2y_m \sum_{t=0}^{[(L-N_I-\ell)/2]} a_{L-N_I-\ell,t}\left(y_m^2-\eta^2\right)^t, \tag{3.50}$$

where $[(L-N_I-\ell)/2]$ denotes the largest integer that is equal to or less than $(L-N_I-\ell)/2$, and $a_{L-N_I-\ell,t}$ is evaluated as:

$$\begin{aligned} a_{L-N_I-\ell,t} &= \left[\binom{L-N_I-\ell-1-t}{t}(1-\eta) - 2\eta\binom{L-N_I-\ell-1-t}{t-1}\right] \\ &\quad (2\eta)^{L-N_I-\ell-1-2t}. \end{aligned} \tag{3.51}$$

When calculating $a_{L-N_I-\ell,t}$, we assume $\binom{m}{n}=0$ for $m<n$ or $n<0$.

The first sum of multiple-fold integrals $\sum_{m=1}^{N_I}\int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y})d\mathbf{y}$ in (3.49) is evaluated in Appendix D. The final results is:

$$\begin{aligned} \sum_{m=1}^{N_I}\int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y})d\mathbf{y} &= \left(-\frac{1}{\gamma}\right)^{L-N_I}\beta^{L-N_I+1}\sum_{p=0}^{N_I-1}\sum_{q=0}^{N_I-1}(-1)^q H(p,q) \\ &\quad \frac{1}{\gamma^p}\left(B_q - \frac{1}{2}\frac{q!}{\beta^{q+1}}\right)\beta^{p+q}, \end{aligned} \tag{3.52}$$

where B_q $(q = 0, 1, 2, \cdots)$ is a series that could be evaluated as follows:

$$B_0 = \sqrt{\frac{\pi}{\beta}} \exp\left(\beta\eta^2\right) Q\left(\sqrt{2\beta}\eta\right) \tag{3.53}$$

$$B_1 = \frac{\eta}{2\beta} + \sqrt{\frac{\pi}{\beta}} \left[\frac{1}{2\beta} - \eta^2\right] \exp\left(\beta\eta^2\right) Q\left(\sqrt{2\beta}\eta\right) \tag{3.54}$$

$$B_q = \left[\frac{(2q-1)}{2\beta} - \eta^2\right] B_{q-1} + \frac{(q-1)}{\beta}\eta^2 B_{q-2}, \tag{3.55}$$

and when $N_I = 1$, $H(p,q) = \frac{1}{(L-1)!}$; when $N_I > 1$,

$$\begin{aligned} & H(p,q) \\ = & \frac{1}{\left[\prod_{i=1}^{N_I} (L-i)!\right] \left[\prod_{i=1}^{N_I} (N_I - i)!\right]} \\ & \sum_{\substack{m_1+m_2+\cdots+m_{L-1}=L-1-p \\ m_i \in \{0,1\}}} \sum_{\substack{n_1+n_2+\cdots+n_{L-1}=L-1-q \\ n_i \in \{0,1\}}} \det \mathbf{W}, \end{aligned} \tag{3.56}$$

where $\det \mathbf{W}$ is the determinant of a $(N_I - 1) \times (N_I - 1)$ matrix whose element on the i-th row, j-th column is

$$W_{i,j} = (m_j + n_j + L - N_I + i + j - 2)!. \tag{3.57}$$

The summations in (3.56) are taken over all sets of indices meeting the stated conditions. $H(p,q)$ depends on L, N_I, p and q, and is independent of SNR γ and SIR β. Therefore we can calculate it for once and then use it for all value of SNR and SIR.

Similarly as in Appendix D, we can get the second sum of multiple-fold integrals in (3.49) as:

$$\begin{aligned} \sum_{m=1}^{N_I} \int g_{m,\ell}(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} = & (-1)^{L-N_I-\ell} (1+\eta)^{L-N_I-\ell} \beta^{L-N_I+1} \\ & \sum_{t=0}^{[(L-N_I-\ell)/2]} a_{L-N_I-\ell,t} \frac{1}{\beta^{\ell+t+1}} \\ & \sum_{p=0}^{N_I-1} \sum_{q=0}^{N_I-1} (-1)^q H(p,q) (\ell+t+q)! \left(\frac{\beta}{\gamma}\right)^p. \end{aligned} \tag{3.58}$$

(3.49) is the expression for the exact BEP of OC for BPSK over Rayleigh fading channels. From it and other related expressions we can calculate BEP for any given number of branches L , number of interferers N_I (with $N_I < L$), SNR $\gamma = P_s/\sigma^2$ and SIR $\beta = P_s/P_I$.

Based on (3.49), we can derive simpler asymptotic expression for the special case SNR $\gamma \gg 1$ and SIR $\beta \ll 1$.

Special case: SIR $\beta \ll 1$ and SNR $\gamma \gg 1$

Notice that (3.52) and (3.58) all contain the term β^{L-N_I+1}. Since $\beta \ll 1$ and $L > N_I$, $\beta^{L-N_I+1} \approx 0$. We conclude that

$$\sum_{m=1}^{N_I} \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \approx 0 \tag{3.59}$$

$$\sum_{m=1}^{N_I} \int g_{m,\ell}(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \approx 0. \tag{3.60}$$

From (3.49) we have

$$P_e \approx -\frac{1}{(L-N_I-1)!}(-1)^{L-N_I-1} \sum_{\ell=0}^{L-N_I-1} \frac{(L-N_I+\ell-1)!}{\ell!} \left(-\frac{1}{\gamma}\right)^{\ell} \frac{(1-\eta)^{L-N_I-\ell}}{(2\eta)^{L-N_I+\ell}}. \tag{3.61}$$

Since $\gamma \gg 1$,from (3.34),

$$\eta = \sqrt{\frac{1}{\gamma}+1} \approx 1 + \frac{1}{2\gamma}. \tag{3.62}$$

Substitute (3.62) in (3.61),

$$\begin{aligned} P_e &\approx -\frac{1}{(L-N_I-1)!}(-1)^{L-N_I-1} \\ &\quad \sum_{\ell=0}^{L-N_I-1} \frac{(L-N_I+\ell-1)!}{\ell!} \left(-\frac{1}{\gamma}\right)^{\ell} \frac{\left(-\frac{1}{2\gamma}\right)^{L-N_I-\ell}}{(2)^{L-N_I+\ell}} \\ &= \frac{1}{4^{L-N_I}\gamma^{L-N_I}} \frac{1}{(L-N_I-1)!} \sum_{\ell=0}^{L-N_I-1} \frac{(L-N_I+\ell-1)!}{\ell!} \\ &= \frac{1}{4^{L-N_I}\gamma^{L-N_I}} \binom{2(L-N_I)-1}{(L-N_I)}. \end{aligned} \tag{3.63}$$

Comparing (3.63) with (3.30), we can see that the performance of a system with L branches and N_I large interferers is equivalent to that of a system with $(L-N_I)$ branches but without interference.

3.5 Numerical Results

As mentioned in the introduction section, the performance of OC has been evaluated in several papers (without using closed-form expressions). We don't want to repeat that. We only use numerical results to show that the analysis results from the closed-form expressions match the simulation results..

Fig. 3.1 to Fig. 3.4 show BEP versus SNR for different SIR. Fig. 3.1, 3.2 and 3.3 are for $L = 4$ branches, and $N_I = 1, 2, 3$ interferers, respectively. Fig. 3.4 is for $L = 8$ and $N_I = 5$.

In all the figures, the analysis results are very close to the simulation results. That proves the validity of the analytical expression for BEP.

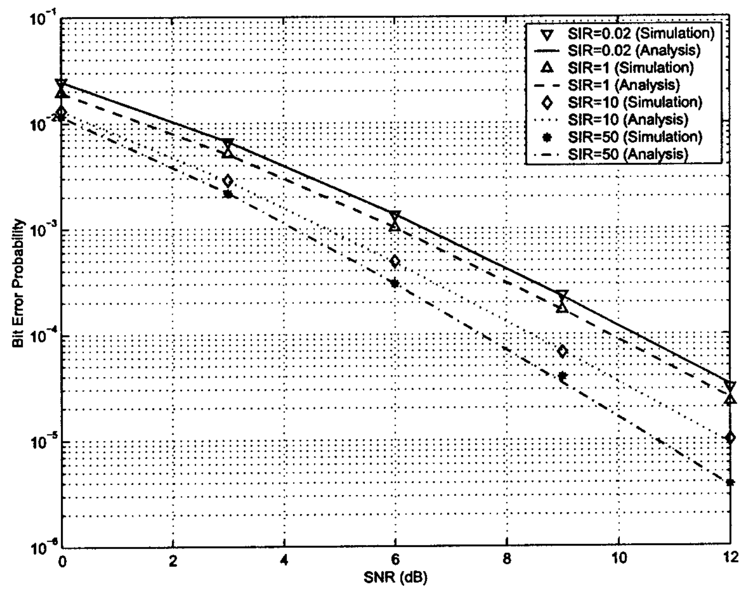

Figure 3.1: BEP versus SNR for $L = 4$ branches, $N_I = 1$ interferer.

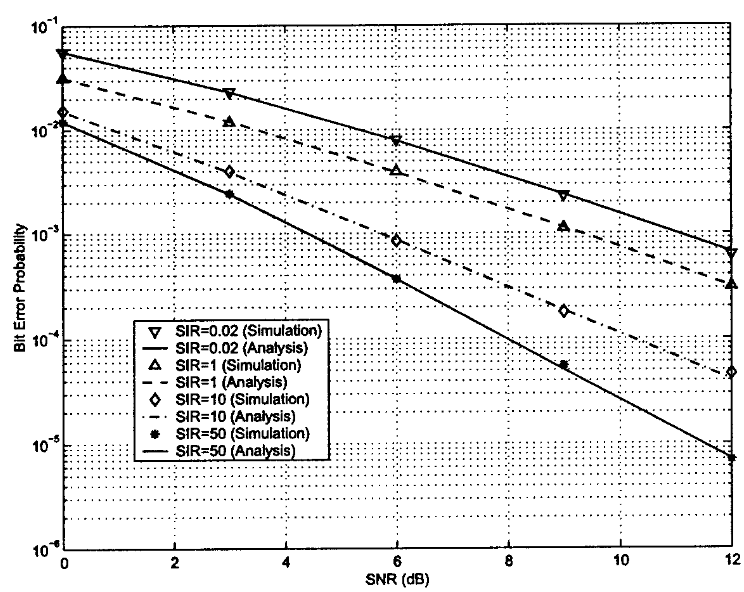

Figure 3.2: BEP versus SNR for $L = 4$ branches, $N_I = 2$ interferers.

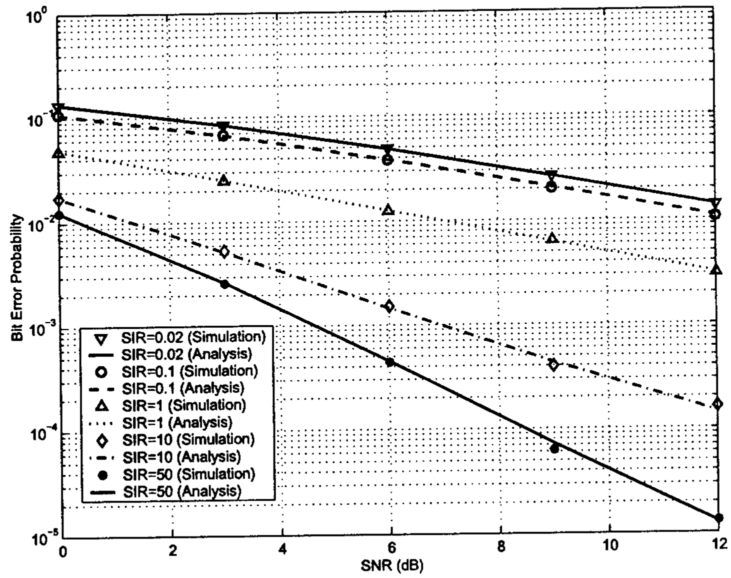

Figure 3.3: BEP versus SNR for $L = 4$ branches, $N_I = 3$ interferers.

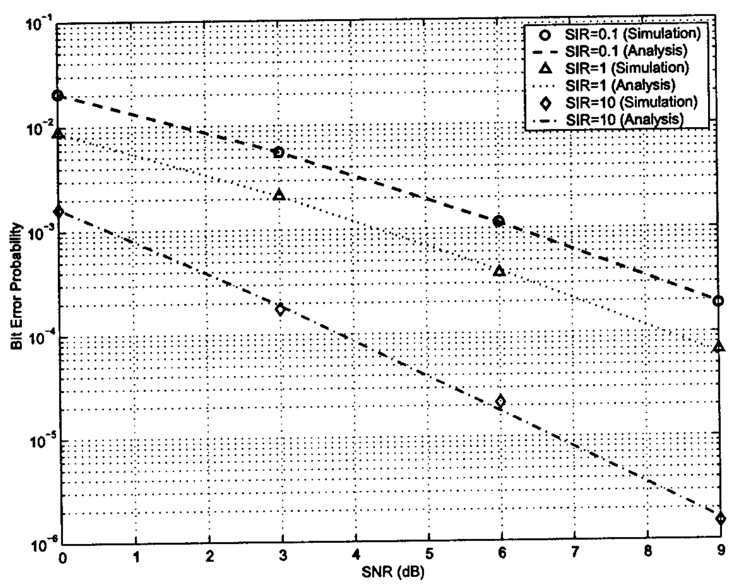

Figure 3.4: BEP versus SNR for $L = 8$ branches, $N_I = 5$ interferers.

Chapter 4

Asymptotic BEP for OC with M-PSK Modulation and One Interferer

4.1 Introduction

In Chapter 3, we derived closed-form expressions of BEP for OC. The expressions were for BPSK modulation, Rayleigh fading channels, multiple interferers with the number of interferers less than the number of reception branches.

An expression was given out for the BEP with M-PSK modulation and one interferer in [17]. It involved integration. In this chapter, we will derive closed-form expressions of BEP for asymptotically high SNR. The asymptotic expressions give intuitive inside of OC and are easy to calculate. We need these expressions to compare the performance of OC with that of MSDD later.

Expressions were given out for the BEP of OC with M-PSK and multiple interferers in [22]. Future work needs to be done to get asymptotic expressions from that.

4.2 Existing Expressions

For M-PSK signals, the conditional symbol error probability (SEP) for OC is [17, Eq. 10.29] (some notations have been changed to agree with this report)

$$P_{\text{s,M-PSK}}(E|\lambda_1) = \frac{1}{\pi}\int_0^{(M-1)\pi/M}\left[M_{\gamma_t|\lambda_1}(s)\Big|_{s=-\frac{\sin^2(\pi/M)}{\sin^2\theta}}\right]d\theta, \tag{4.1}$$

where function $M_{\gamma_t|\lambda_1}(s)$ is the moment generating function of SINR γ_t conditioned on λ_1, the largest eigenvalue of the interference plus noise covariance matrix $\mathbf{R}_z$, and integer M is the number of symbols in M-PSK modulation. From [17, Eq. 10.11], we have

$$\begin{aligned} & M_{\gamma_t|\lambda_1}(s)\Big|_{s=-\frac{\sin^2(\pi/M)}{\sin^2\theta}} \\ = \; & M_{\gamma_t|\lambda_1}\left(-\frac{\sin^2(\pi/M)}{\sin^2\theta}\right) \\ = \; & \left(\frac{\sin^2\theta}{\sin^2\theta+\sin^2(\pi/M)\gamma}\right)^{L-1}\left(\frac{\sin^2\theta}{\sin^2\theta+\sin^2(\pi/M)\frac{P_s}{\lambda_1}}\right). \end{aligned} \tag{4.2}$$

The unconditional SEP $P_{\text{s,M-PSK}}$ is obtained by averaging the conditional SEP over the random eigenvalue λ_1,

$$P_{\text{s,M-PSK}} = \int_{\sigma^2}^{\infty} P_{\text{s,M-PSK}}(E|\lambda_1)p_{\lambda_1}(\lambda_1)d\lambda_1, \tag{4.3}$$

where $p_{\lambda_1}(\lambda_1)$ is the probability density function of λ_1. From (3.32) in Chapter 3, for where there is only one interferer (i.e., $N_I = 1$), we have

$$p_{\lambda_1}(\lambda_1) = \frac{1}{(L-1)!}\frac{1}{P_I}\exp\left(-\frac{\lambda_1-\sigma^2}{P_I}\right)\left(\frac{\lambda_1-\sigma^2}{P_I}\right)^{L-1}. \tag{4.4}$$

For Gray coding and large SNR, the relation between SEP $P_{\text{s,M-PSK}}$ and BEP $P_{\text{b,M-PSK}}$ is

$$P_{\text{b,M-PSK}} \approx \frac{1}{\log_2 M}P_{\text{s,M-PSK}}. \tag{4.5}$$

From (4.5), (4.3) and (4.1), the BEP is

$$P_{\text{b,M-PSK}} \approx \frac{1}{\log_2 M}\frac{1}{\pi}\int_{\sigma^2}^{\infty}\left[\int_0^{(M-1)\pi/M} M_{\gamma_t|\lambda_1}\left(-\frac{\sin^2(\pi/M)}{\sin^2\theta}\right)d\theta\right]p_{\lambda_1}(\lambda_1)d\lambda_1. \tag{4.6}$$

(4.6) is the expression for the general cases. The approximation is due to the transformation from SEP to BEP.

Next we derive the asymptotic BEP of M-PSK for high SNR $\gamma \gg 1$, for the cases of no interference and SIR $\ll 1$.

4.3 Asymptotic Expression for No Interference

When there is no interference, $P_I = 0$. The OC is equivalent to maximum ratio combining (MRC). In this case, λ_1 is a constant, $\lambda_1 = \sigma^2$. Substitute it into (4.2),

$$M_{\gamma_t|\lambda_1}\left(-\frac{\sin^2(\pi/M)}{\sin^2\theta}\right) = \left(\frac{\sin^2\theta}{\sin^2\theta+\sin^2(\pi/M)\gamma}\right)^{L}. \tag{4.7}$$

(4.6) becomes

$$P_{\text{b,M-PSK}} \approx \frac{1}{\log_2 M}\frac{1}{\pi}\int_0^{(M-1)\pi/M} M_{\gamma_t|\lambda_1}\left(-\frac{\sin^2(\pi/M)}{\sin^2\theta}\right)d\theta. \tag{4.8}$$

For high SNR $\gamma \gg 1$,

$$M_{\gamma_t|\bar{\lambda}_1}\left(-\frac{\sin^2(\pi/M)}{\sin^2\theta}\right) \approx \frac{1}{\gamma^L \sin^{2L}(\pi/M)} \sin^{2L}\theta. \tag{4.9}$$

Substitute (4.9) into (4.8),

$$P_{\text{b,M-PSK}} \approx \frac{1}{\log_2 M} \frac{1}{\pi} \frac{1}{\gamma^L \sin^{2L}(\pi/M)} \int_0^{(M-1)\pi/M} \sin^{2L}\theta d\theta. \tag{4.10}$$

The above integration is treated separately for $M = 2$ and $M \geqslant 4$.

4.3.1 BPSK, $M = 2$

Using Eq. (2.513) in [23], we have

$$\frac{1}{\pi}\int_0^{\pi/2} \sin^{2L}\theta d\theta = \frac{1}{2^{2L+1}}\binom{2L}{L}. \tag{4.11}$$

Since

$$\begin{aligned}
\binom{2L}{L} &= \frac{(2L)!}{L!L!} \\
&= \frac{2L}{L}\frac{(2L-1)!}{(L-1)!L!} \\
&= 2\binom{2L-1}{L-1},
\end{aligned} \tag{4.12}$$

$$\frac{1}{\pi}\int_0^{\pi/2} \sin^{2L}\theta d\theta = \frac{1}{2^{2L}}\binom{2L-1}{L-1}. \tag{4.13}$$

Substituting (4.13) into (4.10), we get the BEP for BPSK as

$$P_{\text{b,BPSK}} \approx 2\binom{2L-1}{L-1}\frac{1}{4^L\gamma^L}. \tag{4.14}$$

4.3.2 M-PSK, $M \geqslant 4$

When $M = 4$ (QPSK), define a function $\rho(L)$ such that

$$\frac{1}{\pi}\int_0^{3\pi/4} \sin^{2L}\theta d\theta = \rho(L)\frac{1}{2^{2L}}\binom{2L-1}{L-1}. \tag{4.15}$$

The relation between $\rho(L)$ and L is shown in Fig. 4.1 and the following table.

L	2	3	4	5	6	10	20	80
$\rho(L)$	1.92	1.97	1.99	1.99	2.00	2.00	2.00	2.00

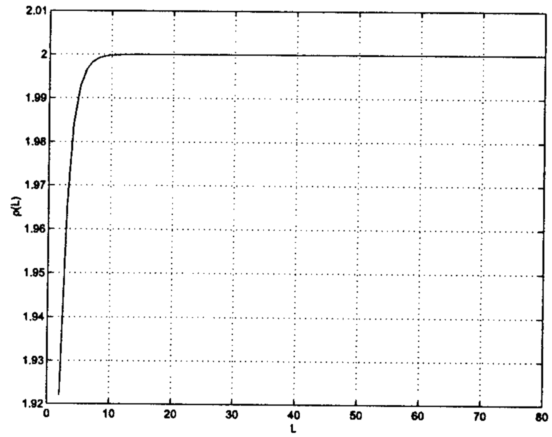

Figure 4.1: $\rho(L)$ versus L.

The table above and Fig. 4.1 show that $\rho(L) \approx 2$ for $L \geq 2$. Then (4.15) becomes

$$\frac{1}{\pi}\int_0^{3\pi/4} \sin^{2L}\theta d\theta \approx 2\frac{1}{2^{2L}}\binom{2L-1}{L-1}. \tag{4.16}$$

When $M \geqslant 4$, since $\frac{3\pi}{4} \leqslant \frac{(M-1)\pi}{M} < \pi$ and $\sin^{2L}\theta > 0$,

$$\frac{1}{\pi}\int_0^{3\pi/4} \sin^{2L}\theta d\theta \leqslant \frac{1}{\pi}\int_0^{(M-1)\pi/M} \sin^{2L}\theta d\theta < \frac{1}{\pi}\int_0^{\pi} \sin^{2L}\theta d\theta. \tag{4.17}$$

Since

$$\begin{aligned}\frac{1}{\pi}\int_0^{\pi} \sin^{2L}\theta d\theta &= \frac{1}{\pi}\int_0^{\pi/2} \sin^{2L}\theta d\theta + \frac{1}{\pi}\int_{\pi/2}^{\pi} \sin^{2L}\theta d\theta \\ &= \frac{1}{\pi}\int_0^{\pi/2} \sin^{2L}\theta d\theta + \frac{1}{\pi}\int_{-\pi/2}^{0} \sin^{2L}(\theta+\pi)\, d(\pi+\theta) \\ &= 2\frac{1}{\pi}\int_0^{\pi/2} \sin^{2L}\theta d\theta, \end{aligned} \tag{4.18}$$

using (4.13), we have

$$\frac{1}{\pi}\int_0^{\pi} \sin^{2L}\theta d\theta = 2\frac{1}{2^{2L}}\binom{2L-1}{L-1}. \tag{4.19}$$

It follows that in (4.17), $\frac{1}{\pi}\int_0^{(M-1)\pi/M} \sin^{2L}\theta d\theta$, for $M \geqslant 4$, is 'sandwiched' between two quantities approximately equal to $2\frac{1}{2^{2L}}\binom{2L-1}{L-1}$, hence

$$\frac{1}{\pi}\int_0^{(M-1)\pi/M} \sin^{2L}\theta d\theta \approx 2\frac{1}{2^{2L}}\binom{2L-1}{L-1}. \tag{4.20}$$

Substitute (4.20) into (4.10), we get the BEP for $M \geqslant 4$ as

$$\begin{aligned} P_{\text{b,M-PSK}} &\approx \frac{1}{\log_2 M} \frac{1}{\gamma^L \sin^{2L}(\pi/M)} 2 \frac{1}{2^{2L}} \binom{2L-1}{L-1} \\ &= \frac{2}{\log_2 M} \binom{2L-1}{L-1} \frac{1}{4^L \gamma^L \sin^{2L}(\pi/M)}. \end{aligned} \tag{4.21}$$

In the numerical section, it is proved that for high SNR $\gamma \gg 1$, the approximate BEP expressions in (4.14) and (4.21) are very close to the BEP yielded by (4.8).

4.4 Asymptotic Expressions for SIR $\ll 1$

In [17, Chapter 10], it is shown that the unconditional SEP in (4.6) can be very closely approximated as

$$P_{\text{s,M-PSK}} \approx P_{\text{s,M-PSK}}(E|\lambda_1)|_{\lambda_1=\overline{\lambda}_1} = \frac{1}{\pi} \int_0^{(M-1)\pi/M} M_{\gamma_t|\overline{\lambda}_1}\left(-\frac{\sin^2(\pi/M)}{\sin^2\theta}\right) d\theta, \tag{4.22}$$

where $\overline{\lambda}_1$ is the expectation value of $\lambda_1, \overline{\lambda}_1 = LP_I + \sigma^2$. Substitute $\overline{\lambda}_1$ into $M_{\gamma_t|\lambda_1}(\cdot)$ (from (4.2)),

$$M_{\gamma_t|\overline{\lambda}_1}\left(-\frac{\sin^2(\pi/M)}{\sin^2\theta}\right) = \left(\frac{\sin^2\theta}{\sin^2\theta + \sin^2(\pi/M)\gamma}\right)^{L-1} \left(\frac{\sin^2\theta}{\sin^2\theta + \sin^2(\pi/M)\frac{P_s}{LP_I+\sigma^2}}\right). \tag{4.23}$$

For SIR $\ll 1$, $P_s \ll P_I$, hence $P_s \ll LP_I + \sigma^2$. Applying this to (4.23), we obtain

$$M_{\gamma_t|\overline{\lambda}_1}\left(-\frac{\sin^2(\pi/M)}{\sin^2\theta}\right) \approx \frac{1}{\gamma^{L-1}\sin^{2(L-1)}(\pi/M)} \sin^{2(L-1)}\theta. \tag{4.24}$$

The difference between the MGF $M_{\gamma_t|\overline{\lambda}_1}(\cdot)$ for the case of SIR $\ll 1$ in (4.24) and the MGF without interference in (4.9) is the loss of a diversity degree of freedom in (4.24). We conclude that the BEP for this case is obtained by replacing L in (4.14) and (4.21) with $L-1$. Therefore for optimum combining in the presence of an interference source, when SNR $\gg 1$ and SIR $\ll 1$,

$$P_{\text{b,BPSK}} \approx \binom{2(L-1)-1}{(L-1)-1} \frac{1}{4^{L-1}\gamma^{L-1}} \quad \text{for BPSK} \tag{4.25}$$

and

$$P_{\text{b,M-PSK}} \approx \frac{2}{\log_2 M} \binom{2(L-1)-1}{(L-1)-1} \frac{1}{4^{L-1}\gamma^{L-1}\sin^{2(L-1)}(\pi/M)} \quad \text{for M-PSK.} \tag{4.26}$$

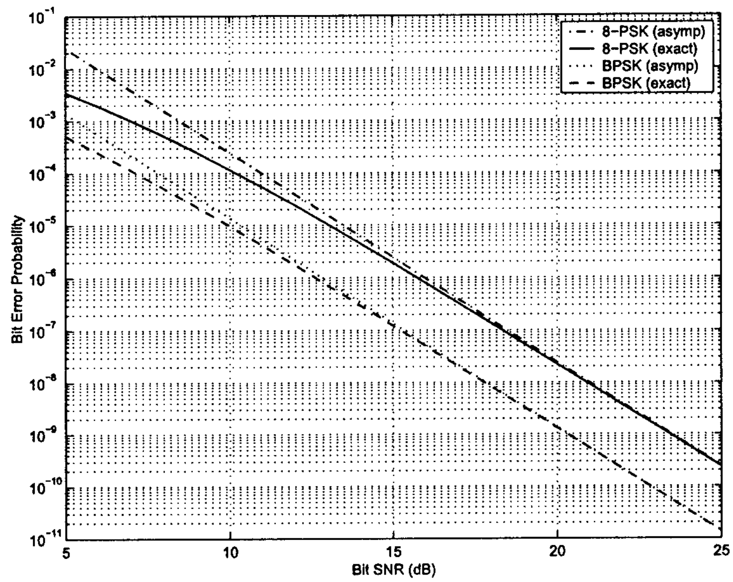

Figure 4.2: Comparison of asymptotic results and exact results, no interference, $L = 4$.

4.5 Numerical Results

In this section we provide some numerical results to show how close the asymptotic expressions are to the non-asymptotic expressions (which we call 'exact' expressions).

Since the BEP for QPSK is very close to that of BPSK, we only show BEP for BPSK and 8-PSK. All the results are for $L = 4$ reception branches.

Fig. (4.2) show that when there is no interference, the asymptotic results yielded by (4.14) (for BPSK) and (4.21) (for 8-PSK) are very close to the exact results yielded by (4.8) for SNR > 15 dB.

Fig. 4.3 and (4.4) show the results for the case with interference. The asymptotic results are yielded by (4.25) (for BPSK) and (4.26) (for 8-PSK). The 'exact' results are yielded by (4.22) [1]. For SIR $= 0$ dB and BPSK in Fig. 4.3, the asymptotic results are not very close to the exact results since SIR is not much less than 1. For all other cases shown, the asymptotic results are very good approximation to exact results for SNR > 15 dB.

[1]The difference between (4.22) and (4.6) is so small that it could be neglected. Hence in here we call results from (4.22) exact results.

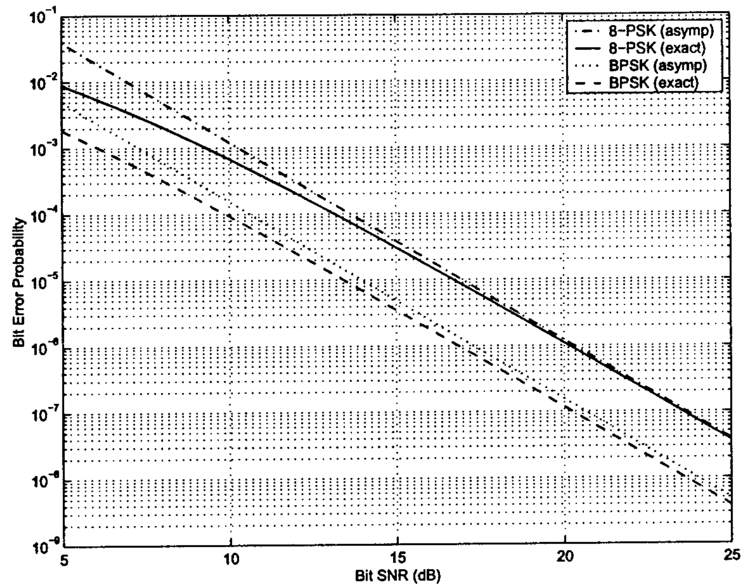

Figure 4.3: Comparison of asymptotic results and exact results, $L = 4$ branches, SIR $= -0$ dB.

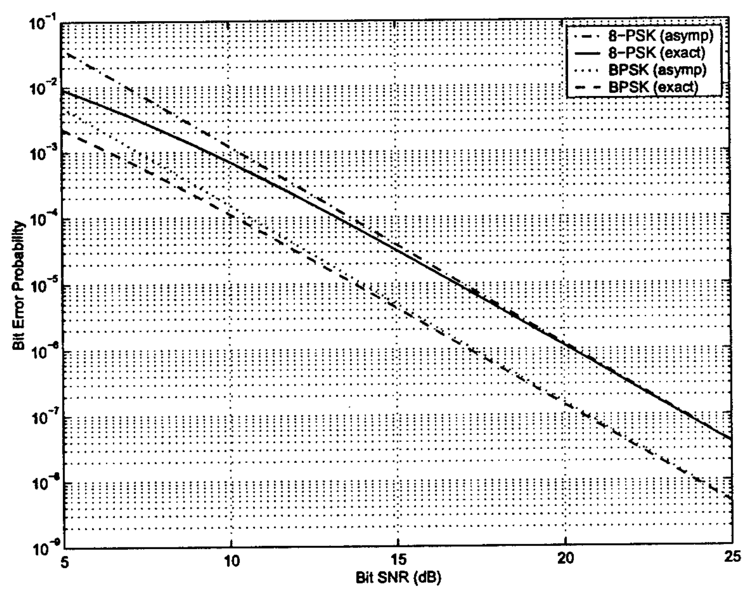

Figure 4.4: Comparison of asymptotic results and exact results, $L = 4$ branches, SIR $= -7$ dB.

Chapter 5

Multiple-Symbol Differential Detection with Known Covariance Matrix of Interference Plus Noise

5.1 Introduction

In the previous two chapters, we analyzed the performance of OC. OC is a coherent detection technique which requires the channel phase of the desired signal. In this and the next chapter, we present a noncoherent detection scheme which is called multiple symbol differential detection (MSDD).

MSDD was first proposed for detecting M-PSK signals transmitted over an additive white Gaussian noise (AWGN) channel [24]. The main advantage of MSDD is that it does not require a coherent phase reference at the receiver (it does require however, the ability to measure relative phase difference).

MSDD performs maximum likelihood detection of a sequence of information symbols based on a finite observation interval. The method was presented as a bridge of the gap between the performance of coherent detection of M-PSK and conventional differential detection of M-ary differential phase shift keying (M-DPSK). The channel phase was assumed to be constant over multiple symbol intervals and was unknown to the receiver. In [24] it was shown that for a long observation interval, the performance (in terms of the required SNR for a given BEP) of MSDD approached that of coherent detection (with differential encoding at the transmitter).

MSDD was extended to trellis coded M-PSK in [25]. MSDD for the fading channel was analyzed in [26] and for correlated fading in [27]. MSDD application to multiuser CDMA was considered in [28]. Performance of MSDD with narrow-band interference over nonfading channel was discussed in [29]. A system with MSDD and reception diversity was formulated in [30] and [31], while [32] considered MSDD with transmit diversity.

In this chapter, we derive an extension to MSDD for communication in the presence of a single interference source. The channel of the desired signal is a diversity Rayleigh channel with multiple outputs. The channel realizations at each output are mutually independent, constant over the observation interval and unknown to the receiver. The Gaussian assumption is made with respect to the aggregate of interference plus noise. The covariance matrix of the interference plus noise is assumed known. The MSDD decision statistic is derived based on the principle of maximum likelihood sequence detection (MLSD). A closed-form expression for the pairwise error probability (PEP) is derived. A closed-form expression for the BEP is intractable, however one is obtained for an approximation to the union bound. The approximation utilizes only dominant terms in the union bound and it is shown to be a good approximation of the BEP. We show that with an increasing number of symbols in the observation interval, the performance of MSDD approaches that of OC (with differential encoding at the transmitter).

In the course of designing simulations for evaluating MSDD, we realized that there was no efficient MSDD algorithm available for MSDD with diversity. The computational complexity of direct computation of the decision statistic grows exponentially with the number of symbols in the observation interval. For single channel MSDD, an optimum algorithm was proposed in [33]. Sub-optimal decision feedback algorithms for the single channel case were suggested in [34], [35], and [36]. In this chapter, we modify the sub-optimal decision feedback algorithm in [36] for application to MSDD with diversity. The main improvement over published algorithms is the introduction of iterations for symbol detection.

5.2 System Model

For convenience, we repeat the system model which was initially described in Section 2.2, since there are some difference and some additional assumptions for MSDD.

Consider a wireless communications system operating over L independent reception branches. the sampled output of the matched filter corresponding to time k and the ℓ-th branch is

$$r_{k,\ell} = \sqrt{P_s} c_\ell s_k + z_{k,\ell}, \quad \ell = 1, 2, \cdots, L, \tag{5.1}$$

where P_s is the power of the desired signal, c_ℓ is the channel gain of the ℓ-th branch, s_k is the transmitted M-DPSK symbol, and $z_{k,\ell}$ is Gaussian correlated noise.

The transmitted signals can be expressed as $s_k = e^{j\theta_k}$, $\theta_k = 2\pi (i_k - 1)/M$, $i_k = 1, 2, \cdots, M$. The transmitted symbols are differentially encoded, i.e., $\theta_k = \theta_{k-1} + \Delta\theta_k$, where $\Delta\theta_k$ is the phase representing the transmitted information at time k.

The signal model in vector notation is

$$\mathbf{r}_k = \sqrt{P_s} \mathbf{c} s_k + \mathbf{z}_k, \tag{5.2}$$

where $\mathbf{r}_k = [r_{k,1}, \cdots, r_{k,L}]^T$, $\mathbf{c}$ and $\mathbf{z}_k$ are vectors defined similar to $\mathbf{r}_k$.

The channel gains c_ℓ's are assumed to be independent and identically distributed (i.i.d.), zero-mean, circularly symmetric, complex Gaussian random variables (Rayleigh fading), with variance $\Omega_\ell/2$ per dimension. The correlated noise term $\mathbf{z}_k$ is the aggregate of an interference source and AWGN and it is assumed to be complex-valued, zero-mean, circularly symmetric, and governed by a Gaussian distribution with covariance matrix $\mathbf{R}_z = E\left[\mathbf{z}_k \mathbf{z}_k^H\right]$. For a single interference source and AWGN, the covariance matrix can be expressed as

$$\mathbf{R}_z = E\left[\mathbf{z}_k \mathbf{z}_k^H\right] = P_I \mathbf{c}_I \mathbf{c}_I^H + \text{diag}\ \left(\sigma_1^2, \sigma_2^2, \cdots, \sigma_L^2\right), \tag{5.3}$$

where P_I is the interference power, $\mathbf{c}_I$ is the interference channel vector, and $(\sigma_1^2, \sigma_2^2, \cdots, \sigma_L^2)$ is the power profile of the AWGN.

Consider a sequence of K symbols running from time $k-(K-1)$ to k. Assume the channel is static over the duration of this sequence. Using vector notation,

$$\underline{\mathbf{r}}_k = \sqrt{P_s}\mathbf{H}\mathbf{s}_k + \underline{\mathbf{z}}_k, \tag{5.4}$$

where $\underline{\mathbf{r}}_k = \left[\mathbf{r}_{k-(K-1)}, \cdots, \mathbf{r}_k\right]^T$, $\mathbf{s}_k = \left[s_{k-(K-1)}, \cdots, s_k\right]^T$ and $\underline{\mathbf{z}}_k$ is a vector defined similar as $\underline{\mathbf{r}}_k$, and $\mathbf{H} = \mathbf{I}_K \otimes \mathbf{c}$ is the channel matrix for the signal of interest, where $\otimes$ denotes the Kronecker product, and $\mathbf{I}_K$ is the identity matrix of rank K.

5.3 Decision Statistic

We formulate the decision statistic for a symbol sequence $\mathbf{s}_k = \left[s_{k-(K-1)}, \cdots, s_k\right]^T$ based on an observation interval consisting of length K as embodied in the vector $\underline{\mathbf{r}}_k$. Assume covariance matrix of the interference plus noise $\mathbf{R}_z$ is known, the maximum likelihood detector for the sequence $\mathbf{s}_k$ is given by

$$\widehat{\mathbf{s}}_k = \arg\max_{\mathbf{s}_k} p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{R}_z), \tag{5.5}$$

where $p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{R}_z)$ is the likelihood of the observed data $\underline{\mathbf{r}}_k$ given the transmitted symbol sequence $\mathbf{s}_k$ and the covariance matrix $\mathbf{R}_z$. Under the Gaussian assumption for the aggregate of interference and noise, the observation $\underline{\mathbf{r}}_k$ conditioned on the transmitted sequence $\mathbf{s}_k$, the covariance matrix $\mathbf{R}_z$ and channel $\mathbf{c}$ has a multivariate Gaussian distribution. The conditional probability $p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{R}_z, \mathbf{c})$ can then be expressed as

$$p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{R}_z, \mathbf{c}) = \pi^{-KL} |\mathbf{R}_z|^{-K} \exp\left\{ -\sum_{i=0}^{K-1} (\mathbf{r}_{k-i} - \sqrt{P_s}\mathbf{c}s_{k-i})^H \mathbf{R}_z^{-1} (\mathbf{r}_{k-i} - \sqrt{P_s}\mathbf{c}s_{k-i}) \right\} \tag{5.6}$$

Diagonalize the interference plus noise covariance matrix $\mathbf{R}_z$ as $\mathbf{R}_z = \mathbf{U}_z \Lambda_z \mathbf{U}_z^H$, where $\Lambda_z = \text{diag}\ (\lambda_1, \cdots, \lambda_L)$, $\lambda_1, \cdots, \lambda_L$ are the eigenvalues of $\mathbf{R}_z$, and $\mathbf{U}_z$ is a unitary matrix whose columns are the eigenvectors of $\mathbf{R}_z$. It follows that (5.6) can be written as

$$p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{R}_z, \mathbf{g}) = \pi^{-KL} |\mathbf{R}_z|^{-K} \exp\left\{ -\sum_{i=0}^{K-1} (\mathbf{x}_{k-i} - \sqrt{P_s}\mathbf{g}s_{k-i})^H \Lambda_z^{-1} (\mathbf{x}_{k-i} - \sqrt{P_s}\mathbf{g}s_{k-i}) \right\}, \tag{5.7}$$

where

$$\mathbf{x}_{k-i} = \mathbf{U}_z^H \mathbf{r}_{k-i} \tag{5.8}$$
$$\mathbf{g} = \mathbf{U}_z^H \mathbf{c}. \tag{5.9}$$

$\mathbf{x}_{k-i}$ is the whitened received signal vector and $\mathbf{g}$ is the modified channel vector. Note that since $\mathbf{U}_z$ is unitary, the modified channel vector $\mathbf{g}$ has the same distribution as the original channel vector $\mathbf{c}$. Let the components of the modified channel vector $\mathbf{g}$ be expressed as $g_\ell = \alpha_\ell e^{j\phi_\ell}$, $\ell = 1, \ldots, L$. Likewise, let the ℓth component of $\mathbf{x}_{k-i}$ be $x_{k-i,\ell}$. Expanding the exponent in (5.7) and grouping terms that do not depend on $\mathbf{g}$ or s_{k-i}, we obtain

$$\begin{aligned} p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{R}_z, \mathbf{g}) &= \pi^{-KL}|\mathbf{R}_z|^{-K} \exp\{-C_0\} \\ &\prod_{\ell=1}^{L} \exp\left\{-KP_s\lambda_\ell^{-1}\alpha_\ell^2 + 2\sqrt{P_s}\lambda_\ell^{-1}\left|y_\ell(\mathbf{s}_k)\right|\alpha_\ell \cos\left(\phi_\ell - \theta_\ell(\mathbf{s}_k)\right)\right\}, \end{aligned} \tag{5.10}$$

where

$$C_0 = \sum_{i=0}^{K-1}\sum_{\ell=1}^{L} \lambda_\ell^{-1}\left|x_{k-i,\ell}\right|^2 \tag{5.11}$$

and

$$\begin{aligned} y_\ell(\mathbf{s}_k) &= \left(\sum_{i=0}^{K-1} x_{k-i,\ell}s_{k-i}^*\right) \\ &= \left|y_\ell(\mathbf{s}_k)\right| e^{j\theta_\ell(\mathbf{s}_k)}. \end{aligned} \tag{5.12}$$

Note that $y_\ell(\mathbf{s}_k)$ is a function of both the transmitted sequence $\mathbf{s}_k$ and the observed sequence $\underline{\mathbf{r}}_k$.

Recalling that the components of the modified channel vector $\mathbf{g}$ have the same distribution as the components of the channel vector $\mathbf{c}$, it follows that α_ℓ is Rayleigh with $E[\alpha_\ell^2] = \Omega_\ell$ and ϕ_ℓ is uniformly distributed in the interval $[0,\ 2\pi)$. To average the conditional distribution $p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{R}_z, \mathbf{g})$ over the modified channel $\mathbf{g}$, we need to evaluate the integral

$$p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{R}_z) = \int p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{R}_z, \mathbf{g}) p_{\mathbf{g}}(\mathbf{g})\, d\mathbf{g} \tag{5.13}$$

where $p_{\mathbf{g}}(\mathbf{g})$ is the probability density function of $\mathbf{g}$. Assume the channels are independent to each other, then

$$p_{\mathbf{g}}(\mathbf{g}) = \prod_{\ell=1}^{L} p_{\alpha_\ell}(\alpha_\ell)p_{\phi_\ell}(\phi_\ell) \tag{5.14}$$

where $p_{\alpha_\ell}(\alpha_\ell)$ and $p_{\phi_\ell}(\phi_\ell)$ are the probability density functions of α_ℓ and ϕ_ℓ respectively. For Rayleigh fading channels,

$$p_{\alpha_\ell}(\alpha_\ell) = \frac{2\alpha_\ell}{\Omega_\ell}\exp\left(-\frac{\alpha_\ell^2}{\Omega_\ell}\right) \quad 0 \le \alpha_\ell < \infty \tag{5.15}$$

$$p_{\phi_\ell}(\phi_\ell) = \frac{1}{2\pi} \qquad 0 \le \phi_\ell < 2\pi. \tag{5.16}$$

Substitute (5.10), (5.14), (5.15) and (5.16) into (5.13), and separate the integrations,

$$p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{R}_z) = \pi^{-KM}|\mathbf{R}_z|^{-K} \exp\{-C_0\} \prod_{\ell=1}^{L} \left\{ \frac{2}{\Omega_\ell} \int_0^\infty \exp\left\{ -\left(KP_s\lambda_\ell^{-1} + \frac{1}{\Omega_\ell} \right) \alpha_\ell^2 \right\} \alpha_\ell d\alpha_\ell \right.$$
$$\left. \frac{1}{2\pi} \int_0^{2\pi} \exp\left\{ 2\sqrt{P_s}\lambda_\ell^{-1} |y_\ell(\mathbf{s}_k)| \alpha_\ell \cos(\phi_\ell - \theta_\ell) \right\} d\phi_\ell \right\}. \quad (5.17)$$

After the averaging over the uniform distribution of ϕ_ℓ is carried out,

$$p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{R}_z) = \pi^{-KL}|\mathbf{R}_z|^{-K} \exp\{-C_0\}$$
$$\prod_{\ell=1}^{L} \frac{2}{\Omega_\ell} \int_0^\infty \exp\left\{ -\left(KP_s\lambda_\ell^{-1} + \frac{1}{\Omega_\ell} \right) \alpha_\ell^2 \right\} \quad (5.18)$$
$$I_0(2\sqrt{P_s}\lambda_\ell^{-1} |y_\ell(\mathbf{s}_k)| \alpha_\ell)\alpha_\ell d\alpha_\ell, \quad (5.19)$$

where $I_0(x)$ is the zeroth order modified Bessel function of the first kind. Using the integration expression of Bessel function $I_0(x)$ in Appendix F, we obtain,

$$p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{R}_z) = \pi^{-KL}|\mathbf{R}_z|^{-K} \exp\{-C_0\} \left(\prod_{\ell=1}^{L} \frac{\lambda_\ell}{KP_s\Omega_\ell + \lambda_\ell} \right)$$
$$\exp\left\{ P_s \sum_{\ell=1}^{L} \frac{\Omega_\ell |y_\ell(\mathbf{s}_k)|^2}{\lambda_\ell (KP_s\Omega_\ell + \lambda_\ell)} \right\}. \quad (5.20)$$

In (5.20), only the argument of the exponential function is dependent on the transmitted sequence $\mathbf{s}_k$ since only the terms $y_\ell(\mathbf{s}_k)$'s are functions of $\mathbf{s}_k$. Due to the monotonicity of the exponential function, maximizing $p(\underline{\mathbf{r}}_k|\mathbf{s}_k)$ with respect to $\mathbf{s}_k$ is equivalent to maximizing the following decision statistic:

$$\eta(\mathbf{s}_k) = \sum_{\ell=1}^{L} \frac{\Omega_\ell |y_\ell(\mathbf{s}_k)|^2}{\lambda_\ell (KP_s\Omega_\ell + \lambda_\ell)}. \quad (5.21)$$

From (5.5), the corresponding MSDD decision rule is

$$\hat{\mathbf{s}}_k = \arg\max_{\mathbf{s}_k} \eta(\mathbf{s}_k). \quad (5.22)$$

The detector searches through sequences $\mathbf{s}_k$ and chooses the sequence that has the largest decision metric $\eta(\mathbf{s}_k)$. A diagram of the MSDD receiver is shown in Fig. 5.1.

From the previous relation, it follows that the optimum multiple symbol differential detector for multiple channel branches and in the presence of interference, is a weighted sum of correlations of whitened observations and hypothesis symbols. Note that this decision statistic does not require knowledge of the signal channel vector.

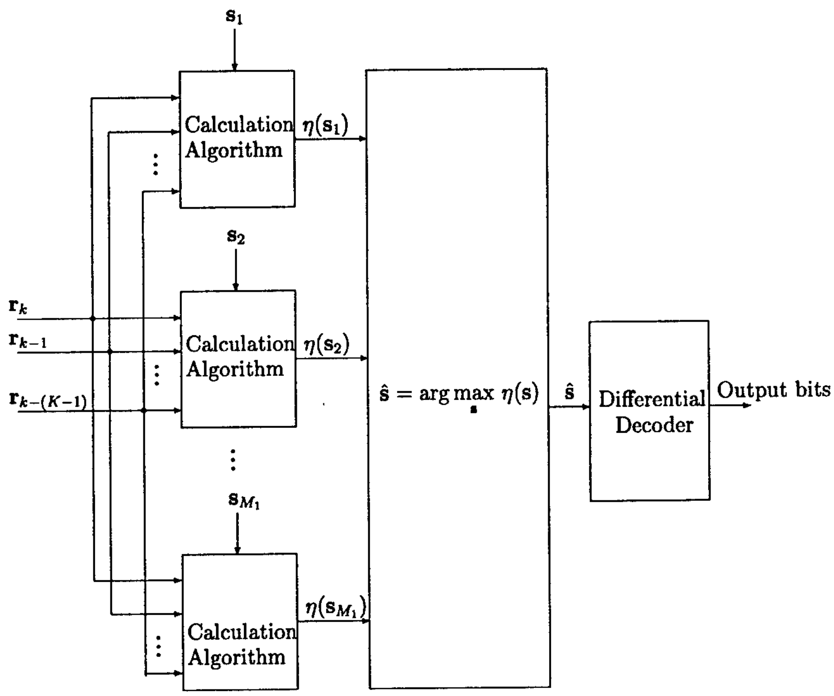

Figure 5.1: The diagram of multiple symbol differential detector. For M-DPSK with M symbols, the number of symbol sequences that need to be tried is $M_1 = M^{K-1}$.

The decision statistic in (5.21) provides multiple symbol differential detection for a M-DPSK sequence transmitted over multiple, independent fading channels in the presence of correlated Gaussian noise.

The decision statistic is ambiguous with respect to an arbitrary phase θ'. Indeed, let $\mathbf{s}'_k = e^{j\theta'}\mathbf{s}_k$, then

$$\begin{aligned} |y_\ell(\mathbf{s}'_k)| &= \left|y_\ell(e^{j\theta'}\mathbf{s}_k)\right| \\ &= \left|\sum_{i=0}^{K-1} x_{k-i,\ell}\left(e^{j\theta'}s_{k-i}\right)^*\right| \\ &= |y_\ell(\mathbf{s}_k)|\,. \end{aligned} \tag{5.23}$$

Differential encoding at the transmitter is required to resolve this ambiguity.

5.3.1 Iterative Decision Feedback Algorithm

The complexity of MSDD for M-DPSK with a K symbols observation interval increases with M^{K-1}. For large K, this makes simulations impractical. To overcome this difficulty, a practical sub-optimal algorithm that uses decisions feedback was implemented. The basic idea of the algorithm is to make symbol by symbol decisions rather than testing the full sequence of symbols simultaneously. The algorithm proceeds from symbol to symbol along the sequence of K symbols; at symbol i it maximizes a decision statistic assuming that the

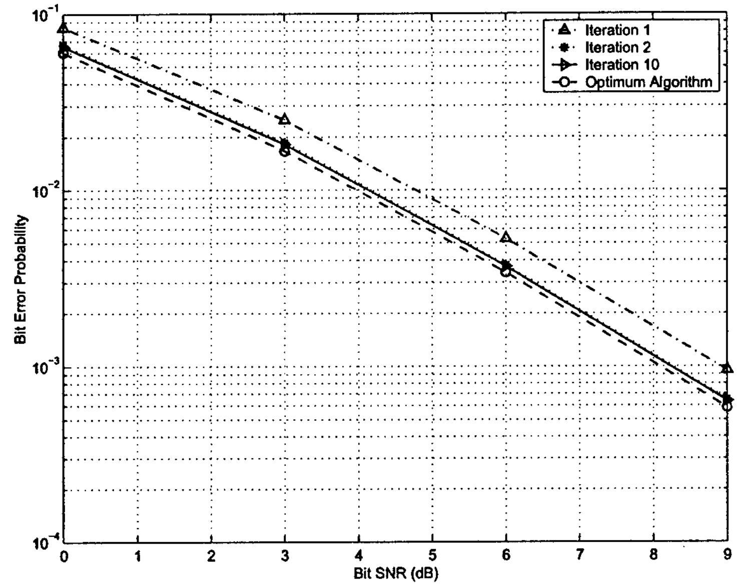

Figure 5.2: Comparison of optimum algorithm and iterative decision feedback algorithm for $L = 4$ branches, DPSK modulation, SIR $= -6$ dB.

other $(K-1)$ symbols have been detected and are known. Several iterations can be carried out to improve performance. The algorithm was implemented as the following procedure:

1. Initialization:

   (a) Initialize iteration index, $m = 0$.

   (b) Initialize $\mathbf{s}_k^{(m)} \triangleq \left[1, s_{k-(K-2)}^{(m)}, s_{k-(K-3)}^{(m)}, \cdots, s_k^{(m)}\right]^T = [1, 0, 0, \cdots, 0]^T$.

   (c) Initialize time index $i = K - 2$.

2. Increase iteration index $m + 1 \rightarrow m$.

3. For $i = (K-2)$ to 0,

   Evaluate $s_{k-i}^{(m)} = \arg\max_{s_{k-i}^{(m)}} \eta\left(\left[1, s_{k-(K-2)}^{(m)}, \cdots, s_{k-i}^{(m)}, s_{k-i+1}^{(m-1)}, \cdots, s_k^{(m-1)}\right]^T\right).$

   End loop i.

4. If m is not equal to the required iteration number (which is determined empirically), go back to step 2.

5. Differentially decode $\mathbf{s}_k^{(m)}$ to get the final output.

To demonstrate the performance of this sub-optimal decisions feedback algorithm, Fig. 5.2 compares the sub-optimal and optimal (based on (5.22)) algorithms. The comparison is for the case of $L = 4$ diversity branches, DPSK modulation, and SIR = −6 dB. For an observation interval of $K = 12$ symbols, with just 2 iterations, the performance of the sub-optimal decisions feedback algorithm is within just 0.2 dB of that of the optimum algorithm. The advantage of the sub-optimal decisions feedback algorithm is, of course, that it takes much less time to run than the optimum algorithm. From the figure, it can also be observed that iterations are beneficial to the performance of decisions feedback. The second iteration provides about 0.5 dB gain relative to that of without iteration (iteration 1). Additional iterations do not seem to improve the performance.

5.3.2 Special Case

Some special cases provide insights into the operation of MSDD. For a channel with a flat gain profile $\Omega_\ell = 1$, and a flat AWGN profile $\sigma_\ell^2 = \sigma^2$ for $\ell = 1, 2, \cdots, L$, (5.21) can be expressed as

$$\eta(\mathbf{s}_k) = \sum_{\ell=1}^{L} \frac{|y_\ell(\mathbf{s}_k)|^2}{\lambda_\ell (KP_s + \lambda_\ell)}. \tag{5.24}$$

We further specialize (5.24) to the following special cases.

No Interference

For this case, $P_I = 0$, the noise covariance matrix $\mathbf{R}_z = \sigma^2 \mathbf{I}_L$, eigenvalues $\lambda_\ell = \sigma^2$, $\ell = 1, 2, \cdots, L$, and $\mathbf{U}_z = \mathbf{I}_L$. Then (5.12) simplifies to

$$y_\ell(\mathbf{s}_k) = \sum_{i=0}^{K-1} r_{k-i,\ell} s_{k-i}^*. \tag{5.25}$$

The decision statistic in (5.24) becomes

$$\eta(\mathbf{s}_k) = \frac{1}{\sigma^2 (KP_s + \sigma^2)} \sum_{\ell=1}^{L} |y_\ell(\mathbf{s}_k)|^2 . \tag{5.26}$$

Since the term outside the sum is independent of $\mathbf{s}_k$, the above decision statistic is equivalent to

$$\eta(\mathbf{s}_k) = \sum_{\ell=1}^{L} |y_\ell(\mathbf{s}_k)|^2 = \sum_{\ell=1}^{L} \left| \sum_{i=0}^{K-1} r_{k-i,\ell} s_{k-i}^* \right|^2 . \tag{5.27}$$

This decision statistic is the same as that in [31, equation (8)]. Indeed (5.21) is the generalization of [31, equation (8)] to MSDD in the presence of interference.

Interference $\gg$ Noise

For a uniform AWGN power profile, the eigenvalues of the interference plus noise covariance matrix (5.3) are $\lambda_1 = P_I \sum_{\ell=1}^{L} |c_{I,\ell}|^2 + \sigma^2$, $\lambda_2 = \ldots = \lambda_L = \sigma^2$. For a high interference to noise ratio, $\lambda_1 \gg \lambda_\ell$, $\ell \neq 1$. It follows that the decision statistic in (5.24) can be approximated by the expression

$$\eta(\mathbf{s}_k) \approx \sum_{\ell=2}^{L} \frac{|y_\ell(\mathbf{s}_k)|^2}{\sigma^2 (KP_s + \sigma^2)}. \tag{5.28}$$

The interpretation of this result is that for a strong interference source, the decision statistic is similar to that of MSDD without interference and one fewer degree of freedom. This result will be further demonstrated in the ensuing error probability analysis.

5.4 Error Probability Analysis

An exact expression for the BEP for differential detection can be obtained only for DPSK modulation and $K = 2$ symbols. The exact error analysis is intractable for the general case of MSDD with M-DPSK modulation over diversity channels and in the presence of interference. The alternative approach is to obtain an analytical approximate upper bound. In this section, we first derive an exact expression for the PEP. Then, using this expression, we derive the union bound of the BEP. From the union bound an approximate upper bound is derived. The approximate upper bound consists of relatively simple algebraic expressions. Even simpler expressions are obtained for the asymptotically large SNR and small signal to interference ratio (SIR). In the numerical results section, it is shown that the approximate upper bound is very close to the BEP obtained by simulation.

5.4.1 PEP Analysis

In the derivation of the PEP, we assume a uniform flat power profile for the desired signal channel, $\Omega_\ell = 1$, and a flat AWGN profile with $\sigma_\ell^2 = \sigma^2$ for $\ell = 1, 2, \cdots, L$. The PEP is developed for correlated noise characterized by the covariance matrix in (5.3).

In general, the interference source is subject to effects of the fading channel (similar to the desired source). It follows that analysis using the covariance matrix $\mathbf{R}_z$ in (5.3) is conditional on the interference random channel $\mathbf{c}_I$. Results obtained from such analysis need to be averaged over the distribution of $\mathbf{c}_I$. Fortunately, this complication can be avoided recognizing that when the detector acts to suppress the interference, there is only a small penalty in using in the analysis the average value of the interference power $P_I E\left[\mathbf{c}_I^H \mathbf{c}_I\right]$ in lieu of the instantaneous power $P_I \mathbf{c}_I^H \mathbf{c}_I$ (see [15]). Assuming that the interference channel $\mathbf{c}_I$ is complex-valued, zero mean and with variance $\Omega_{I,\ell}/2 = 1/2$ per dimension, it follows that the average eigenvalues of $\mathbf{R}_z$ are

$$\begin{aligned} \lambda_1 &= P_I E\left[\mathbf{c}_I^H \mathbf{c}_I\right] + \sigma^2 \\ &= LP_I + \sigma^2 \end{aligned} \tag{5.29}$$

and $\lambda_\ell = \sigma^2$ for $\ell = 2, 3, \cdots, L$.

Let $\mathbf{s}_k$ and $\mathbf{s}'_k$ denote two sequences each containing K M-DPSK symbols. The PEP that $\mathbf{s}_k$ is transmitted but $\mathbf{s}'_k$ is detected ($\mathbf{s}'_k \neq s_0 \mathbf{s}_k$, where s_0 is an arbitrary M-PSK symbol) is denoted as $P\left(\mathbf{s}_k \to \mathbf{s}'_k\right)$. An error event occurs when $\eta(\mathbf{s}_k) < \eta(\mathbf{s}'_k)$. Define random variable D,

$$D = \eta(\mathbf{s}_k) - \eta(\mathbf{s}'_k). \tag{5.30}$$

Note that D is random due to both the random noise and the random channels. We seek to evaluate the probability that $D < 0$.

Using steps similar to [7, Appendix B], it can be shown that

$$\begin{aligned} P\left(\mathbf{s}_k \to \mathbf{s}'_k\right) &= -\frac{1}{2\pi j} \int_{-\infty + j\varepsilon}^{\infty + j\varepsilon} \frac{\Phi_D(j\omega)}{\omega} d\omega \\ &= -\sum_{\mathrm{Im}(\omega_\ell) > 0} \mathrm{Res}\left[\frac{\Phi_D(j\omega)}{\omega}; \omega_\ell\right], \end{aligned} \tag{5.31}$$

where ε is a small positive number; $\Phi_D(j\omega)$ is the characteristic function of D; $\mathrm{Res}\left[\frac{\Phi_D(j\omega)}{\omega}; \omega_\ell\right]$ denotes the residue of $\frac{\Phi_D(j\omega)}{\omega}$ at pole ω_ℓ; the summation is taken over the poles in the upper half of the complex plane.

In Appendix G, the following expression is derived for the characteristic function of the random variable D:

$$\Phi_D(j\omega) = \frac{1}{(1 - j\mu_1\omega)} \frac{1}{(1 - j\mu_2\omega)} \frac{1}{(1 - j\mu_3\omega)^{L-1}} \frac{1}{(1 - j\mu_4\omega)^{L-1}}, \tag{5.32}$$

where

$$\mu_i = \begin{cases} \frac{1}{2} b_1^2 \left(\zeta P_s \pm \sqrt{\zeta^2 P_s^2 + 4\left(K P_s + \lambda_1\right) \zeta \lambda_1}\right) & i = 1, 2 \\ \frac{1}{2} b_2^2 \left(\zeta P_s \pm \sqrt{\zeta^2 P_s^2 + 4\left(K P_s + \lambda_2\right) \zeta \lambda_2}\right) & i = 3, 4 \end{cases} \tag{5.33}$$

and

$$b_1 = \sqrt{\frac{1}{\lambda_1 \left(K P_s + \lambda_1\right)}} \tag{5.34}$$

$$b_2 = \sqrt{\frac{1}{\lambda_2 \left(K P_s + \lambda_2\right)}} \tag{5.35}$$

$$\zeta = K^2 - \left|v(\mathbf{s}_k, \mathbf{s}'_k)\right|^2. \tag{5.36}$$

In (5.33), the plus sign is taken for $i = 1$ and 3, while the minus sign is taken for $i = 2$ and 4. In (5.36), $v(\mathbf{s}_k, \mathbf{s}'_k) = \mathbf{s}'^H_k \mathbf{s}_k$ is the correlation coefficient between the transmitted sequence $\mathbf{s}_k$ and the detected sequence $\mathbf{s}'_k$. Note that $0 \leq v(\mathbf{s}_k, \mathbf{s}'_k) \leq K$, with equality on the right hand side when $\mathbf{s}'_k = \mathbf{s}_k$.

Substituting (5.32) into (5.31), the PEP is obtained as

$$\begin{aligned} &P\left(\mathbf{s}_k \rightarrow \mathbf{s}'_k\right) \\ &= -\sum_{\operatorname{Im}(\omega_\ell)>0} \operatorname{Res}\left[\frac{1}{\omega}\frac{1}{(1-j\mu_1\omega)}\frac{1}{(1-j\mu_2\omega)}\frac{1}{(1-j\mu_3\omega)^{L-1}}\frac{1}{(1-j\mu_4\omega)^{L-1}};\omega_\ell\right]. \end{aligned} \tag{5.37}$$

Since μ_1, $\mu_3 > 0$ and μ_2, $\mu_4 < 0$, only the poles $-j/\mu_2$ and $-j/\mu_4$ are in the upper half of the complex plane. Eq. (5.37) becomes

$$\begin{aligned} &P\left(\mathbf{s}_k \rightarrow \mathbf{s}'_k\right) \\ &= -\operatorname{Res}\left[\frac{1}{\omega}\frac{1}{(1-j\mu_1\omega)}\frac{1}{(1-j\mu_2\omega)}\frac{1}{(1-j\mu_3\omega)^{L-1}}\frac{1}{(1-j\mu_4\omega)^{L-1}};-j\frac{1}{\mu_2}\right] \\ &\quad -\operatorname{Res}\left[\frac{1}{\omega}\frac{1}{(1-j\mu_1\omega)}\frac{1}{(1-j\mu_2\omega)}\frac{1}{(1-j\mu_3\omega)^{L-1}}\frac{1}{(1-j\mu_4\omega)^{L-1}};-j\frac{1}{\mu_4}\right]. \end{aligned} \tag{5.38}$$

After some cumbersome, but straightforward manipulations (which are similar to the manipulations shown in Appendix B), it can be shown that the PEP is

$$\begin{aligned} &P\left(\mathbf{s}_k \rightarrow \mathbf{s}'_k\right) \\ &= -\frac{(\mu_2)^{2L-1}}{(\mu_1-\mu_2)(\mu_3-\mu_2)^{L-1}(\mu_4-\mu_2)^{L-1}} - \frac{(-1)^L\mu_4^{L-1}}{\mu_3^{L-1}}\frac{1}{(L-2)!}\sum_{k=0}^{L-2}\frac{(L-2+k)!}{k!} \\ &\left[1-\frac{\mu_1}{(\mu_1-\mu_2)}\frac{\mu_1^{L-1-k}}{(\mu_1-\mu_4)^{L-1-k}}+\frac{\mu_2}{(\mu_1-\mu_2)}\frac{\mu_2^{L-1-k}}{(\mu_2-\mu_4)^{L-1-k}}\right]\frac{\mu_3^{L-1+k}}{(\mu_3-\mu_4)^{L-1+k}}. \end{aligned} \tag{5.39}$$

The former expression is the *exact* PEP of MSDD with diversity branches and a rank one interference source. The PEP is conditioned on the transmitted sequence $\mathbf{s}_k$ and is a function of the detected sequence $\mathbf{s}'_k$. Note that the expression in (5.39) already incorporates statistical information on the channel and interference. This form of the PEP is quite complicated and does not afford much insight. It is of interest to obtain simpler expressions for special cases. In the ensuing analysis, the symbol signal to noise ratio SNR is $\gamma = P_s/\sigma^2$, and signal to interference ratio is SIR $= P_s/P_I$.

No Interference

For this case $P_I = 0$, $\lambda_1 = \lambda_2 = \sigma^2$. From (5.33) we have:

$$\mu_i = \begin{cases} \frac{1}{2P_s}\frac{\gamma^2}{(K\gamma+1)}\left(\zeta+\sqrt{\zeta^2+4\left[K+\frac{1}{\gamma}\right]\zeta\frac{1}{\gamma}}\right) & i=1,3 \\ \frac{1}{2P_s}\frac{\gamma^2}{(K\gamma+1)}\left(\zeta-\sqrt{\zeta^2+4\left[K+\frac{1}{\gamma}\right]\zeta\frac{1}{\gamma}}\right) & i=2,4 \end{cases}. \tag{5.40}$$

Since $\mu_1 = \mu_3$, and $\mu_2 = \mu_4$, it follows that the PEP in (5.38) can be rewritten as

$$P\left(\mathbf{s}_k \rightarrow \mathbf{s}_k'\right) = -\mathrm{Res}\left[\frac{1}{\omega}\frac{1}{\left(1-j\mu_1\omega\right)^L}\frac{1}{\left(1-j\mu_2\omega\right)^L}; -j\frac{1}{\mu_2}\right] \tag{5.41}$$

It is not too difficult to show that this results in the following expression for the PEP:

$$P\left(\mathbf{s}_k \rightarrow \mathbf{s}_k'\right) = \frac{\left(-\mu_2\right)^L}{(\mu_1-\mu_2)^L(L-1)!}\left[\sum_{\ell=0}^{L-1}\frac{(L-1+\ell)!}{\ell!}\frac{\mu_1^\ell}{(\mu_1-\mu_2)^\ell}\right]. \tag{5.42}$$

It could be shown numerically that (5.42) is equivalent to the PEP developed in [31]. However, (5.42) has the advantage that it provides the PEP in closed form without the need of integration.

The case of no interference can be further simplified for large signal to noise ratio $\gamma \gg 1$. In this case, (5.40) simplifies to

$$\mu_i \approx \begin{cases} \gamma\zeta/KP_s & i = 1,3 \\ -1/P_s & i = 2,4 \end{cases}. \tag{5.43}$$

Substituting these results in (5.42) and noticing that $\mu_1 \gg \mu_2$, we have

$$\begin{aligned} P\left(\mathbf{s}_k \rightarrow \mathbf{s}_k'\right) &\approx \frac{(1/P_s)^L}{(\gamma\zeta/KP_s)^L(L-1)!}\sum_{\ell=0}^{L-1}\frac{(L-1+\ell)!}{\ell!} \\ &= \binom{2L-1}{L}\frac{1}{(\frac{\zeta}{K}\gamma)^L}. \end{aligned} \tag{5.44}$$

This expression clearly exhibits the L-order diversity of the system.

SIR $\ll 1$, SNR $\gg 1$

By assumption $P_I \gg P_s \gg \sigma^2$, therefore $\lambda_1 \gg P_s \gg \sigma^2$. From (5.34), we have $b_1 \approx 1/\lambda_1$. After some simple manipulations, it follows from (5.33),

$$\mu_{1,2} \approx \pm\frac{\sqrt{\zeta}}{\lambda_1}. \tag{5.45}$$

To evaluate μ_3 and μ_4, we approximate $b_2 \approx 1/\sqrt{KP_s\sigma^2}$ and substitute in (5.33) to obtain $\mu_3 \approx \zeta/\left(K\sigma^2\right)$ and $\mu_4 \approx -1/P_s$. Substituting these approximate values into (5.39) and keeping only the dominant term, after some manipulations, we have

$$P\left(\mathbf{s}_k \rightarrow \mathbf{s}_k'\right) \approx \binom{2\,(L-1)-1}{(L-1)}\frac{1}{\left(\frac{\zeta}{K}\gamma\right)^{L-1}}. \tag{5.46}$$

Comparing (5.46) with (5.44), we can see the PEP for systems with diversity L and a large interference is equal to the PEP for systems with diversity $(L-1)$ and without interference. This result is well known for interference suppression using OC. This analysis proves that the loss of degree of freedom due to interference suppression carries over to MSDD over a diversity Rayleigh channel.

5.4.2 BEP Approximate Upper Bound

The sequence $\mathbf{s}_k$ of M-DPSK symbols corresponds to $(K-1)\log_2 M$ information bits (with differential encoding, the first symbol is known). Let $\mathbf{u}_k$ be the sequence of $(K-1)\log_2 M$ information bits encoded as $\mathbf{s}_k$, and let $\mathbf{u}'_k$ be the sequence of information bits which results from the detection of $\mathbf{s}'_k$. The pairwise BEP associated with transmitting a sequence $\mathbf{u}_k$ and detecting another sequence $\mathbf{u}'_k$ is given by

$$P_b\left(\mathbf{s}_k \to \mathbf{s}'_k\right) = \frac{1}{(K-1)\log_2 M} h(\mathbf{u}_k, \mathbf{u}'_k) P\left(\mathbf{s}_k \to \mathbf{s}'_k\right), \tag{5.47}$$

where $h(\mathbf{u}_k, \mathbf{u}'_k)$ denotes the Hamming distance between $\mathbf{u}_k$ and $\mathbf{u}'_k$.

The BEP that $\mathbf{s}_k$ is transmitted, but an error sequence (any error sequence) is detected, is upper bounded by the union of all pairwise bit error events. Since $\mathbf{s}_k$ can be any input sequence (e.g., the null sequence $\mathbf{s}_k = [1,0,0,\cdots,0]^T$), we drop the dependency on $\mathbf{s}_k$ from the notation. The union bound on the BEP can then be written as

$$\begin{aligned} P_b &\leq \sum_{\mathbf{u}'_k \neq \mathbf{u}_k} P_b(\mathbf{s}_k \to \mathbf{s}'_k) \\ &= \frac{1}{(K-1)\log_2 M} \sum_{\mathbf{u}'_k \neq \mathbf{u}_k} h(\mathbf{u}_k, \mathbf{u}'_k) P\left(\mathbf{s}_k \to \mathbf{s}'_k\right), \end{aligned} \tag{5.48}$$

where the summation is taken over all the sequences $\mathbf{u}'_k$'s which are different from the transmitted sequence of information bits $\mathbf{u}_k$.

Direct application of (5.48) does not shed light on the mechanisms affecting MSDD performance. A clearer picture is obtained by developing an approximation to the union bound. Note that the union bound in (5.48) is a function of the PEP's, which in turn are determined by μ_1, μ_2, μ_3 and μ_4 (see (5.39)). From (5.33), μ_1, μ_2, μ_3 and μ_4 are functions of the quantity $|v(\mathbf{s}_k, \mathbf{s}'_k)|^2$ through the relation $\zeta = K^2 - |v(\mathbf{s}_k, \mathbf{s}'_k)|^2$. In [24], it is shown that on the AWGN channel, for large SNR, the dominant terms in the BEP occur for sequences for which the quantity $|v(\mathbf{s}_k, \mathbf{s}'_k)|^2$ is maximum. Carrying over the same approach to the fading channel, keeping only the dominant terms and noticing that $P\left(\mathbf{s}_k \to \mathbf{s}'_k\right)$ is constant if $|v\left(\mathbf{s}_k, \mathbf{s}'_k\right)|$ is constant, we obtain the following approximation to the union bound

$$\begin{aligned} A &= \frac{1}{(K-1)\log_2 M} \left[\sum_{\substack{\mathbf{u}'_k \neq \mathbf{u}_k \\ |v(\mathbf{s}_k, \mathbf{s}'_k)| = |v(\mathbf{s}_k, \mathbf{s}'_k)|_{\max}}} h(\mathbf{u}_k, \mathbf{u}'_k) \right] \times \\ &\quad \left[P\left(\mathbf{s}_k \to \mathbf{s}'_k\right) \Big|_{|v(\mathbf{s}_k, \mathbf{s}'_k)| = |v(\mathbf{s}_k, \mathbf{s}'_k)|_{\max}} \right]. \end{aligned} \tag{5.49}$$

The maximum value of $|v(\mathbf{s}_k, \mathbf{s}'_k)|$ was shown in [24, Eq. (38)] to be

$$|v\left(\mathbf{s}_k, \mathbf{s}'_k\right)|_{\max} = \sqrt{(K-1)^2 + 2(K-1)\left(1 - 2\sin^2\frac{\pi}{M}\right) + 1}. \tag{5.50}$$

Also from [24, Appendix B], for sequences such that $|v(\mathbf{s}_k, \mathbf{s}'_k)| = |v(\mathbf{s}_k, \mathbf{s}'_k)|_{\max}$, the accumulated Hamming distances are

$$\sum_{\substack{\mathbf{u}'_k \neq \mathbf{u}_k \\ |v(\mathbf{s}_k,\mathbf{s}'_k)| = |v(\mathbf{s}_k,\mathbf{s}'_k)|_{\max}}} h(\mathbf{u}_k, \mathbf{u}'_k) = \begin{cases} 1, & K = 2 \\ 2(K-1), & K > 2 \end{cases} \tag{5.51}$$

for binary modulation, $M = 2$ and

$$\sum_{\substack{\mathbf{u}'_k \neq \mathbf{u}_k \\ |v(\mathbf{s}_k,\mathbf{s}'_k)| = |v(\mathbf{s}_k,\mathbf{s}'_k)|_{\max}}} h(\mathbf{u}_k, \mathbf{u}'_k) = \begin{cases} 2, & K = 2 \\ 4(K-1), & K > 2 \end{cases} \tag{5.52}$$

for multilevel modulation, $M \geqslant 4$.

Strictly speaking, (5.49) is not an upper bound of the BEP. Numerical results, however, show that it is very close to or larger than the BEP obtained by simulation. Therefore we will use (5.49) to study the performance of MSDD in the presence of interference.

Next, we evaluate the approximate upper bound for differential binary PSK (DPSK) and M–DPSK ($M \geq 4$) modulations.

DPSK ($M = 2$)

For this case, from (5.50) we have

$$|v(\mathbf{s}_k, \mathbf{s}'_k)|_{\max} = K - 2. \tag{5.53}$$

For conventional differential detection, the observation interval is $K = 2$ symbols, $|v(\mathbf{s}_k, \mathbf{s}'_k)|_{\max} = 0$. In this case, there is only one error sequence, therefore the PEP is also the BEP,

$$P_b = P(\mathbf{s}_k \to \mathbf{s}'_k)\big|_{|v(\mathbf{s}_k,\mathbf{s}'_k)|=0}. \tag{5.54}$$

Substituting $P(\mathbf{s}_k \to \mathbf{s}'_k)$ from (5.42) into (5.54), we obtain the exact BEP for DPSK over L diversity fading channels without interference. For high SNR $\gg 1$, using (5.44), we get

$$P_{b,\mathrm{DPSK}} \approx \binom{2L-1}{L} \frac{1}{(2\gamma)^L}. \tag{5.55}$$

This expression is the same as the one in [7, Eq. (14-4-28)], and it demonstrates that familiar expressions for differential detection can be obtained as a special case of the general case treated in this report.

For a longer observation interval $K > 2$, substitute (5.53) and (5.51) into (5.49) to obtain

$$A_{\mathrm{DPSK}} = 2\,P(\mathbf{s}_k \to \mathbf{s}'_k)\big|_{|v(\mathbf{s}_k,\mathbf{s}'_k)|=K-2}. \tag{5.56}$$

This expression is for the approximate BEP upper bound for DPSK over slow-fading Rayleigh diversity channels with interference. Next, we compute some special cases for $K > 2$ and SNR $\gamma \gg 1$, which result in simplified expressions.

No Interference, SNR $\gg 1$ Using (5.44) in (5.56), the approximate upper bound is

$$
\begin{aligned}
A_{\text{DPSK}} &= 2\binom{2L-1}{L}\frac{1}{(\frac{\zeta}{K}\gamma)^L}\Bigg|_{|v(\mathbf{s}_k,\mathbf{s}'_k)|=K-2} \\
&= 2\binom{2L-1}{L}\frac{1}{4^L\gamma^L\left(1-\frac{1}{K}\right)^L}. \qquad (5.57)
\end{aligned}
$$

SIR$\ll 1$, SNR $\gg 1$ Substituting (5.46) in (5.56) we obtain a result similar to (5.57), except L is substituted with $(L-1)$,

$$A_{\text{DPSK}} = 2\binom{2(L-1)-1}{L-1}\frac{1}{4^{L-1}\gamma^{L-1}\left(1-\frac{1}{K}\right)^{L-1}}. \qquad (5.58)$$

M-DPSK $(M \geqslant 4)$

For M-DPSK, substitute (5.52) and (5.50) into (5.49), we obtain the following approximate upper bound:

$$A_{\text{M-DPSK}} = \frac{2}{\log_2 M}\, P\left(\mathbf{s}_k \to \mathbf{s}'_k\right)\Big|_{\left|v\left(\mathbf{s}_k,\mathbf{s}'_k\right)\right|=2\left|\cos\frac{\pi}{M}\right|} \qquad (5.59)$$

for $K=2$ symbols and

$$A_{\text{M-DPSK}} = \frac{4}{\log_2 M}\, P\left(\mathbf{s}_k \to \mathbf{s}'_k\right)\Big|_{\left|v\left(\mathbf{s}_k,\mathbf{s}'_k\right)\right|=\sqrt{(K-1)^2+2(K-1)\left(1-2\sin^2\frac{\pi}{M}\right)+1}} \qquad (5.60)$$

for observation intervals of size $K>2$.

Simplified expressions for special cases are computed below.

No Interference, SNR $\gg 1$ Using (5.44), we obtain from (5.59) and (5.60) the approximate upper bound

$$A_{\text{M-DPSK}} = \frac{2}{\log_2 M}\binom{2L-1}{L}\frac{1}{2^L\gamma^L\sin^{2L}(\pi/M)} \qquad (5.61)$$

for $K=2$ and

$$A_{\text{M-DPSK}} = \frac{4}{\log_2 M}\binom{2L-1}{L}\frac{1}{4^L\gamma^L\sin^{2L}(\pi/M)\left(1-\frac{1}{K}\right)^L} \qquad (5.62)$$

for $K>2$.

SIR $\ll 1$, SNR $\gg 1$ Substituting (5.46) in (5.59) and (5.60) we obtain a result similar to (5.61) and (5.62), except that L is substituted with $(L-1)$,

$$A_{\text{M-DPSK}} = \frac{2}{\log_2 M}\binom{2(L-1)-1}{L-1}\frac{1}{2^{L-1}\gamma^{L-1}\sin^{2(L-1)}(\pi/M)} \tag{5.63}$$

for $K=2$ and

$$A_{\text{M-DPSK}} = \frac{4}{\log_2 M}\binom{2(L-1)-1}{L-1}\frac{1}{4^{L-1}\gamma^{L-1}\sin^{2(L-1)}(\pi/M)\left(1-\frac{1}{K}\right)^{L-1}} \tag{5.64}$$

for $K>2$.

5.5 Comparison with OC

In this section, we compare the BEP of MSDD with that of OC analytically. Since the expressions of the BEP for general cases are very complex, we can only compare the performance analytically for the cases of small SIR (relative large interference) and no interference. Since both cases yield similar results, we only do that for the case of small SIR. In next section, we will compare the performance of MSDD and OC numerically.

From (4.25) and (4.26) in Chapter 4, for SNR $\gamma \gg 1$ and SIR $\ll 1$, the BEP for OC is approximated by the expressions

$$P_{\text{b,BPSK}} \approx 2\binom{2(L-1)-1}{(L-1)-1}\frac{1}{4^{L-1}\gamma^{L-1}} \tag{5.65}$$

and

$$P_{\text{b,M-PSK}} \approx \frac{4}{\log_2 M}\binom{2(L-1)-1}{(L-1)-1}\frac{1}{4^{L-1}\gamma^{L-1}\sin^{2(L-1)}(\pi/M)}. \tag{5.66}$$

The above two equations are for OC with M-PSK modulation. Since MSDD uses M-DPSK modulation. We want to compare them on the same basis.

For OC, the exact expression of the BEP for M-DPSK is very difficult to obtain. But judging from Eq. (4.200) in [37], the BEP for M-DPSK is about twice that of M-PSK except for very small SNR. That can be demonstrated by simulation. Therefore, the BEP for M-DPSK is

$$P_{\text{b,M-DPSK}} \approx 2P_{\text{b,M-PSK}}. \tag{5.67}$$

Substitute (5.65) and (5.66) in (5.67), we obtain the BEP for OC using M-DPSK as

$$P_{\text{b,DPSK}} \approx 2\binom{2(L-1)-1}{(L-1)-1}\frac{1}{4^{L-1}\gamma^{L-1}} \tag{5.68}$$

and

$$P_{\text{b,M-DPSK}} \approx \frac{4}{\log_2 M}\binom{2(L-1)-1}{(L-1)-1}\frac{1}{4^{L-1}\gamma^{L-1}\sin^{2(L-1)}(\pi/M)}. \tag{5.69}$$

The ratio of the BEP of OC and the approximate upper bound of MSDD (for $K > 2$) is given by the ratios of (5.68) to (5.58) and (5.69) to (5.64), respectively

$$\frac{P_{\text{b,DPSK}}}{A_{\text{DPSK}}} = \frac{P_{\text{b,M-DPSK}}}{A_{\text{M-DPSK}}} = \left(1 - \frac{1}{K}\right)^{L-1}. \tag{5.70}$$

The ratio approaches 1 as $K \to \infty$. We conclude that in the presence of interference, when the observation interval of MSDD increases to infinity, i.e., $K \to \infty$, the performance of MSDD approaches that of OC with differential encoding.

According to (5.67), the BEP of OC with differential encoding is about twice that of OC without differential encoding. Therefore for MSDD of large K, the BEP is only about twice that of OC without differential encoding.

5.6 Numerical Results

Numerical results presented in this section include Monte Carlo simulation results and analysis results. In all cases, the channel branches and noise power profiles are assumed to be uniform, i.e., $\Omega_\ell = 1$ and $\sigma_\ell^2 = \sigma^2$ for $\ell = 1, 2, \cdots, L$. The bit SNR is $(P_s/\log_2(M))/\sigma^2$. For comparison purposes, we also provide BEP curves for OC with differential encoding. All the figures are for $L = 4$ diversity branches.

Fig. 5.3 shows the BEP versus SNR for DPSK at SIR $= -6$ dB. Curves labeled 'Simulation' represent simulation results, while curves labeled 'Analysis' show analytical results as yielded by the approximate upper bounds (5.54) (for $K = 2$) and (5.56) (for $K > 2$). In all cases, PEP's were exact as computed by (5.39). The interference plus noise term was generated such that its covariance matrix followed (5.3). The OC curve was generated by simulation. It can be observed that analysis results are very close to simulation results. It is also observed that the performance of MSDD approaches that of OC with differential encoding as K, the number of symbols in observation interval increases. For example, at BEP $= 2 \times 10^{-3}$, when $K = 2$, the SNR difference between MSDD and OC is about 2.2 dB. When $K = 7$, the difference is about 1.0 dB. At $K = 40$, the difference becomes an insignificant 0.2 dB.

Fig. 5.4 and Fig. 5.5 are for DQPSK and 8-DPSK respectively. The curves in these figures follow the same trends as in Fig. 5.3.

The results shown in Fig. 5.6 to 5.8 are all analytical results. In these figures, bit error probabilities are represented by their approximate upper bounds. The approximate upper bound is computed based on the exact PEP expression in (5.39) except for Fig. 5.8.

Fig. 5.6 shows the BEP of MSDD as a function of the number of symbols in the observation interval, K. It is evident that for both DPSK (binary modulation) and for 8-DPSK ($M = 8$), the performance of MSDD approaches that of OC as the observation interval increases.

Fig. 5.7 shows the BEP versus SIR, for bit SNR $= 10$ dB, and for the cases of $K = 2$ and $K = 40$ symbols. It is observed that when $K = 40$, MSDD achieves performance close to that of OC with differential encoding regardless of the SIR.

Fig. 5.8 is intended to verify the asymptotic large SNR approximation to the PEP. The signal modulation is DQPSK. Curves labeled 'asymp' represent asymptotic results computed by applying (5.63) (for $K = 2$) and (5.64) (for $K = 40$); curves labeled 'exact' represent exact results from (5.59) (for $K = 2$) and (5.60) (for $K = 40$). It is observed that for most SNR of interest (SNR > 10), the approximate upper bound based on asymptotic PEP is very close to the approximate upper bound based on the exact PEP.

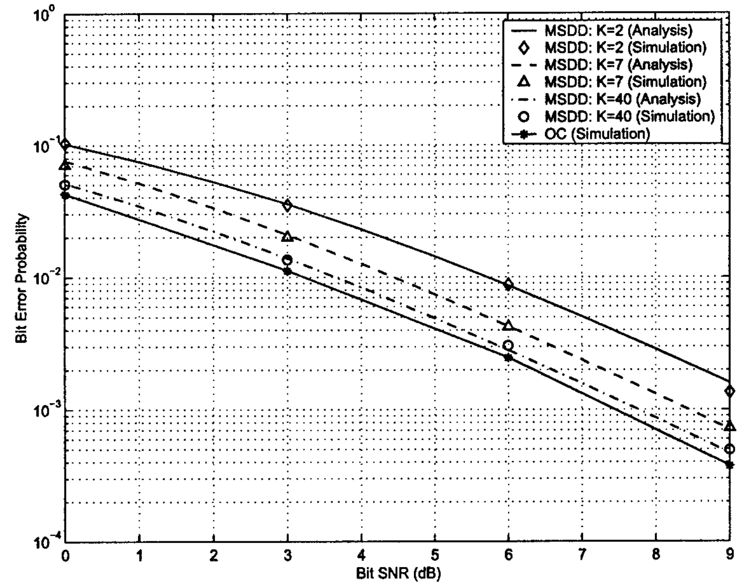

Figure 5.3: BEP versus SNR for $L = 4$ branches, DPSK modulation, SIR $= -6$ dB.

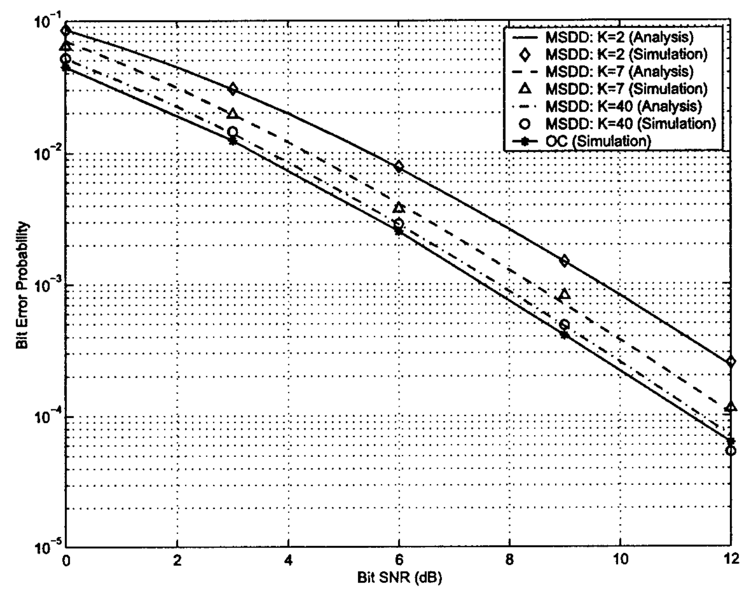

Figure 5.4: BEP versus SNR for $L = 4$ branches, DQPSK modulation, SIR $= -6$ dB.

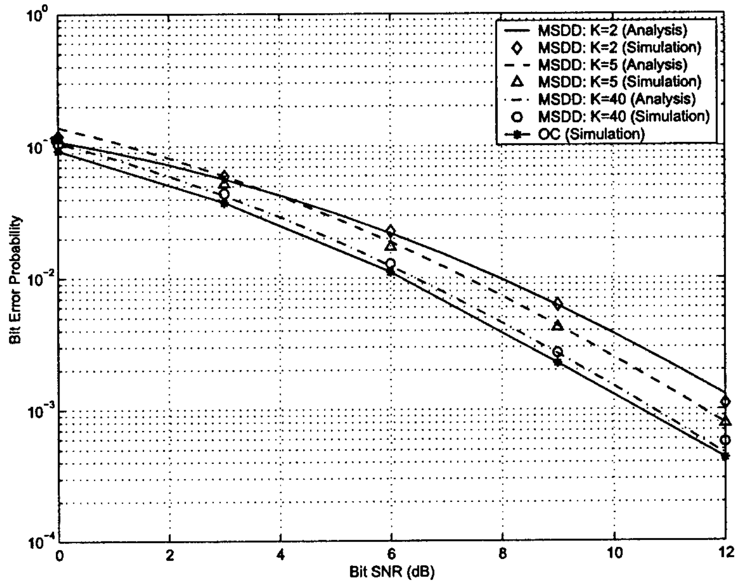

Figure 5.5: BEP versus SNR for $L = 4$ branches, 8-DPSK modulation, SIR $= -6$ dB.

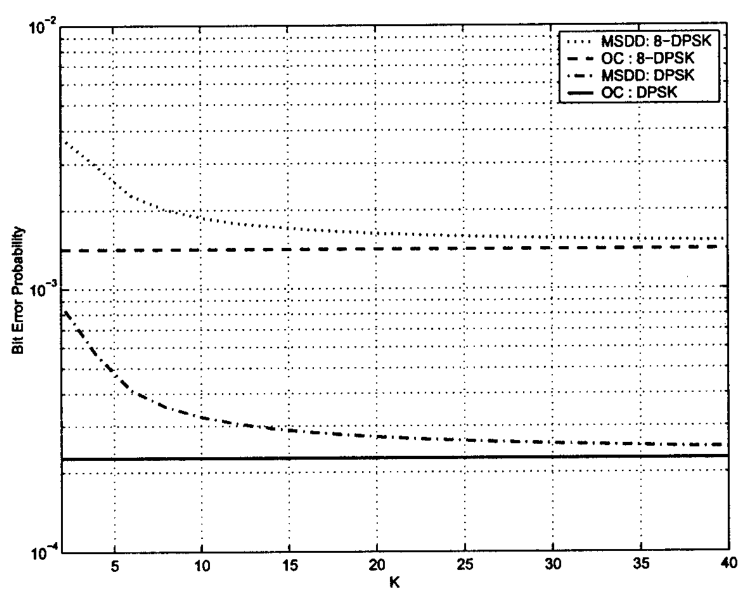

Figure 5.6: BEP versus the number of symbols in the observation interval K for $L = 4$ branches, SIR $= -6$ dB.

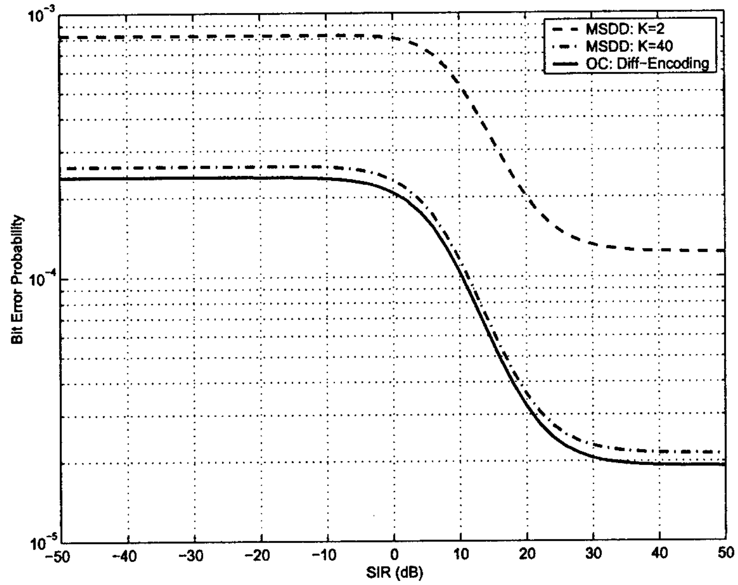

Figure 5.7: BEP versus SIR for $L = 4$ branches, DQPSK modulation, bit SNR = 10 dB.

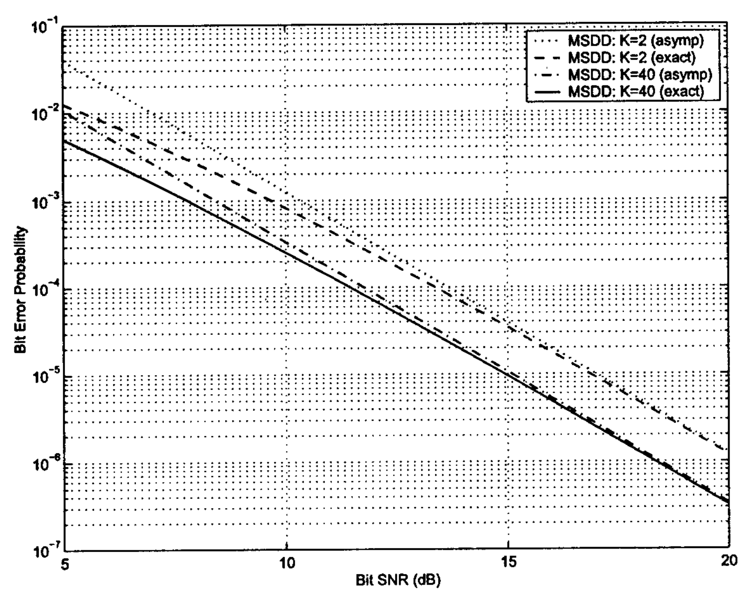

Figure 5.8: Comparison of asymptotic results and exact results for $L = 4$ branches, DQPSK modulation, SIR $= -6$ dB.

Chapter 6

Multipe-Symbol Differential Detection with Known Channel Information of Interference

6.1 Introduction

Until now, we have discussed two kinds of detectors for communication systems with reception diversity in the presence of white Gaussian noise and interference source. In Chapter 3, we discussed detector using OC. To implement OC, the channel gain of the desired signal and the covariance matrix of the interference plus noise must be available to the receiver. In Chapter 5, we discussed MSDD for the case where the channel gain of the desired signal was assumed to be unknown, but the covariance matrix of the interference plus noise was assumed to be known. Both detectors show the ability to suppress interference.

It would be desirable to be able to suppress the interference without requiring any information of the interference. Unfortunately that is impossible. In Chapter 3, the interference plus noise is modeled as

$$\mathbf{z}_k = \sqrt{P_I}\sum_{i=1}^{N_I} \mathbf{c}_i s_{i,k} + \mathbf{n}_k \tag{6.1}$$

If we don't have any information about $\mathbf{c}_i$ (which is assumed to be Gaussian distributed), the interference term $\sqrt{P_I}\sum_{i=1}^{N_I} \mathbf{c}_i s_{i,k}$ would be the same as Gaussian noise and could not be distinguished from the white Gaussian noise $\mathbf{n}_k$. Therefore at least some information about the interference is required.

In this Chapter, we develop detector for the case where the only required channel information is the amplitude of the channels of the interference. The scenario is similar to that in chapter 5. But in addition to assuming that the channel gain of the desired signal is unknown, the phase of the channel of the interference is assumed to be unknown as well. The channel amplitude of the interference is assumed known. Moreover, the interference is assumed to have the same MDPSK modulation as the desired signal. A maximum likelihood sequence detector (MLSD) is formulated for the joint detection of the desired signal and the interference. Simulation is performed for DPSK modulation. Simulation results in terms of

BEP versus SNR are given out and compared with the results obtained by other detection schemes.

It is shown that when the interference level is high, this MSDD technique can achieve better performance than detectors using OC (with differential encoding).

6.2 System Model

The system model used in this chapter is similar to that mentioned in Section 2.2, except that now we assume there is only one interferer. The output of the match filter is

$$r_{k,\ell} = \sqrt{P_s} c_\ell s_k + \sqrt{P_I} c_{I,\ell} s_{I,k} + n_{k,\ell},\ \ell = 1, 2, \cdots, L. \tag{6.2}$$

The definitions of the variables are in Section 2.2. Both the desired signal s_k and the interference source $s_{I,k}$ are assumed to be M-DPSK symbols.

The signals in vector notation are

$$\mathbf{r}_k = \sqrt{P_s}\mathbf{c}s_k + \sqrt{P_I}\mathbf{c}_I s_{I,k} + \mathbf{n}_k. \tag{6.3}$$

We assume both c_ℓ and $c_{I,\ell}$ are zero-mean complex Gaussian random variables (Rayleigh fading), and they are mutual independent. For convenience, we define $c_\ell = \alpha_\ell e^{j\phi_\ell}, c_{I,\ell} = \alpha_{I,\ell} e^{j\phi_{I,\ell}}$, vectors $\alpha = [\alpha_1, \alpha_1, \cdots, \alpha_L]$. And similarly as α, define vector ϕ, α_I and ϕ_I. In this paper, α_I is assumed to be known, but α, ϕ and ϕ_I are assumed to be unknown.

Assume the covariance matrix of $\mathbf{n}_k$ is

$$\mathbf{R}_n = E\left[\mathbf{n}_k \mathbf{n}_k^H\right] = \text{diag}\left[\sigma_1^2, \sigma_2^2, \cdots, \sigma_L^2\right], \tag{6.4}$$

where σ_ℓ^2 $(\ell = 1, \cdots, L)$ is the power of the noise on the ℓ-th branch.

Consider a sequence of K symbols running from time $k - (K-1)$ to k. Assume the channels are static within the duration of this sequence. Using vector notation,

$$\underline{\mathbf{r}}_k = \sqrt{P_s}\mathbf{H}\mathbf{s}_k + \sqrt{P_I}\mathbf{H}_I \mathbf{s}_{I,k} + \underline{\mathbf{n}}_k, \tag{6.5}$$

where $\underline{\mathbf{r}}_k = \left[\mathbf{r}_{k-(K-1)}, \mathbf{r}_{k-(K-2)}, \cdots, \mathbf{r}_k\right]^T$. The channel matrix $\mathbf{H} = \mathbf{I}_K \otimes \mathbf{c}$, where $\otimes$ denotes the Kronecker product, and $\mathbf{I}_K$ is the identity matrix of rank K. $\mathbf{H}_I = \mathbf{I}_K \otimes \mathbf{c}_I$, $\mathbf{s}_k = \left[s_{k-(K-1)}, \cdots, s_k\right]^T$, $\mathbf{s}_{I,k} = \left[s_{I,k-(K-1)}, \cdots, s_{I,k}\right]^T$, $\underline{\mathbf{n}}_k = \left[\mathbf{n}_{k-(K-1)}, \cdots, \mathbf{n}_k\right]^T$.

Some other assumptions of the signals and channels will be given out in the derivation of the decision statistic.

6.3 Decision Statistic

In this section, we derive the decision statistic for MSDD with known channel amplitude of the interference α_I. MSDD is a form of MLSD. The decision is made after K symbols are transmitted and received. Conditioned on the channels $\mathbf{c}$ and $\mathbf{c}_I$, the coherent decision criterion for sequence detection is given by

$$(\widehat{\mathbf{s}}_k, \widehat{\mathbf{s}}_{I,k}) = \arg \max_{\mathbf{s}_k, \mathbf{s}_{I,k}} p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}, \mathbf{c}_I). \tag{6.6}$$

Note that we perform detection of both the desired and the interference symbols. The pair $(\mathbf{s}_k, \mathbf{s}_{I,k})$ that maximizes $p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}, \mathbf{c}_I)$ is chosen as the detected symbols. When the only known channel information is α_I, the decision rule for MLSD would be

$$(\widehat{\mathbf{s}}_k, \widehat{\mathbf{s}}_{I,k}) = \arg \max_{\mathbf{s}_k, \mathbf{s}_{I,k}} p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{s}_{I,k}, \alpha_I). \tag{6.7}$$

Next we try to find the expression for $p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{s}_{I,k}, \alpha_I)$.

At time k, the probability of $\mathbf{r}_k$ conditioned on $s_k, s_{I,k}, \mathbf{c}$, and $\mathbf{c}_I$ is

$$\begin{aligned} & p(\mathbf{r}_k|s_k, s_{I,k}, \mathbf{c}, \mathbf{c}_I) \\ = \; & \pi^{-L}|\mathbf{R}_n|^{-1} \exp\left\{-\left(\mathbf{r}_k - \sqrt{P_s}\mathbf{c}s_k - \sqrt{P_I}\mathbf{c}_I s_{I,k}\right)^H \mathbf{R}_n^{-1}\right. \\ & \left.\left(\mathbf{r}_k - \sqrt{P_s}\mathbf{c}s_k - \sqrt{P_I}\mathbf{c}_I s_{I,k}\right)\right\} \qquad (6.8) \\ = \; & \pi^{-L}|\mathbf{R}_n|^{-1} \exp\left\{-\sum_{\ell=1}^{L} \frac{\left|r_{k,\ell} - \sqrt{P_s}c_\ell s_k - \sqrt{P_I}c_{I,\ell}s_{I,k}\right|^2}{\sigma_\ell^2}\right\}. \qquad (6.9) \end{aligned}$$

Assume $s_k, s_{I,k}$ and $\mathbf{n}_k$ are independent in the time domain, the conditional probability of all the received signal $\underline{\mathbf{r}}_k$ would be

$$\begin{aligned} & p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}, \mathbf{c}_I) \\ = \; & \prod_{i=k-(K-1)}^{k} p(\mathbf{r}_i|s_i, s_{I,i}, \mathbf{c}, \mathbf{c}_I) \\ = \; & \pi^{-LK}|\mathbf{R}_n|^{-K} \exp\left\{-\sum_{i=k-(K-1)}^{k} \sum_{\ell=1}^{L} \frac{\left|r_{i,\ell} - \sqrt{P_s}c_\ell s_i - \sqrt{P_I}c_{I,\ell}s_{I,i}\right|^2}{\sigma_\ell^2}\right\}. \qquad (6.10) \end{aligned}$$

Denoting the exponent in (6.10) as A, and expanding it,

$$\begin{aligned} A \triangleq \; & -\sum_{i=k-(K-1)}^{k} \sum_{\ell=1}^{L} \frac{\left|r_{i,\ell} - \sqrt{P_s}c_\ell s_i - \sqrt{P_I}c_{I,\ell}s_{I,i}\right|^2}{\sigma_\ell^2} \\ = \; & -\sum_{\ell=1}^{L} \frac{1}{\sigma_\ell^2} \sum_{i=k-(K-1)}^{k} |r_{i,\ell}|^2 - \sum_{\ell=1}^{L} \frac{1}{\sigma_\ell^2}\left[KP_I\alpha_{I,\ell}^2 - \sqrt{P_I}y_{I,\ell}^* c_{I,\ell} - \right. \\ & \left.\sqrt{P_I}y_{I,\ell}c_{I,\ell}^*\right] - \sum_{\ell=1}^{L} \frac{1}{\sigma_\ell^2}\left[KP_s\alpha_\ell^2 + \left(\sqrt{P_sP_I}yc_{I,\ell}^* - \sqrt{P_s}y_\ell^*\right)c_\ell + \right. \\ & \left.\left(\sqrt{P_sP_I}y^* c_{I,\ell} - \sqrt{P_s}y_\ell\right)c_\ell^*\right], \qquad (6.11) \end{aligned}$$

where

$$y_\ell = \sum_{i=k-(K-1)}^{k} r_{i,\ell} s_i^* \tag{6.12}$$

$$y_{I,\ell} = \sum_{i=k-(K-1)}^{k} r_{i,\ell} s_{I,i}^* \tag{6.13}$$

$$y = \sum_{i=k-(K-1)}^{k} s_i s_{I,i}^*. \tag{6.14}$$

Note that $y_\ell, y_{I,\ell}$ and y are functions of $\mathbf{s}_k$ and/or $\mathbf{s}_{I,k}$.

Define

$$C_0 = \sum_{\ell=1}^{L} \frac{1}{\sigma_\ell^2} \sum_{i=k-(K-1)}^{k} |r_{i,\ell}|^2 \tag{6.15}$$

$$C_\ell = \frac{1}{\sigma_\ell^2} \left[K P_I \alpha_{I,\ell}^2 - \sqrt{P_I} y_{I,\ell}^* c_{I,\ell} - \sqrt{P_I} y_{I,\ell} c_{I,\ell}^* \right] \tag{6.16}$$

$$x_\ell = |x_\ell| e^{i\psi_\ell} = \sqrt{P_s P_I} y c_{I,\ell}^* - \sqrt{P_s} y_\ell^*, \tag{6.17}$$

then (6.11) becomes

$$A = -C_0 - \sum_{\ell=1}^{L} C_\ell - \sum_{\ell=1}^{L} \frac{1}{\sigma_\ell^2} \left[K P_s \alpha_\ell^2 + 2 |x_\ell| \alpha_\ell \cos(\phi_\ell + \psi_\ell) \right]. \tag{6.18}$$

Substitute it into (6.10), then

$$p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}, \mathbf{c}_I) \tag{6.19}$$

$$= C_1 \exp\left\{ -\sum_{\ell=1}^{L} C_\ell \right\} \tag{6.20}$$

$$\exp\left\{ -\sum_{\ell=1}^{L} \frac{1}{\sigma_\ell^2} \left[K P_s \alpha_\ell^2 + 2 |x_\ell| \alpha_\ell \cos(\phi_\ell + \psi_\ell) \right] \right\}, \tag{6.21}$$

where

$$C_1 = \pi^{-LK} |\mathbf{R}_z|^{-K} \exp\{-C_0\}. \tag{6.22}$$

As mentioned above, the channel $\mathbf{c}$ is assumed unknown. We eliminate $\mathbf{c}$ in $p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}, \mathbf{c}_I)$ by integrating over it,

$$p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}_I)$$

$$= \int p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}, \mathbf{c}_I) p_c(\mathbf{c})\, d\mathbf{c}$$

$$= \int \int p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}, \mathbf{c}_I) p_\alpha(\alpha)\, p_\phi(\phi)\, d\alpha d\phi, \tag{6.23}$$

where $p_{\mathbf{c}}(\mathbf{c})$, $p_\alpha(\alpha)$ and $p_\phi(\phi)$ are the probability density functions of $\mathbf{c}$, α and ϕ respectively.

Since the channels are independent to each other,

$$p_\alpha(\alpha)\, p_\phi(\phi) = \prod_{\ell=1}^{L} p_{\alpha_\ell}(\alpha_\ell) p_{\phi_\ell}(\phi_\ell), \tag{6.24}$$

where $p_{\alpha_\ell}(\alpha_\ell)$ and $p_{\phi_\ell}(\phi_\ell)$ are the probability density functions of α_ℓ and ϕ_ℓ respectively. For Rayleigh fading channels,

$$\begin{aligned} p_{\alpha_\ell}(\alpha_\ell) &= \frac{2\alpha_\ell}{\Omega_\ell} \exp\left(-\frac{\alpha_\ell^2}{\Omega_\ell}\right) \quad 0 \le \alpha_\ell < \infty & (6.25) \\ p_{\phi_\ell}(\phi_\ell) &= \frac{1}{2\pi} \qquad 0 \le \phi_\ell < 2\pi, & (6.26) \end{aligned}$$

where $\Omega_\ell = E\left[\alpha_\ell^2\right]$ is the mean square of the amplitude of the ℓth channel.

Substitute (6.24), (6.25) and (6.26) into (6.23), after some straightforward manipulations,

$$\begin{aligned} & p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}_I) \\ = \; & C_1 \prod_{\ell=1}^{L} \frac{\sigma_\ell^2}{KP_s\Omega_\ell + \sigma_\ell^2} \exp\left(\frac{\Omega_\ell |x_\ell|^2}{\sigma_\ell^2 (KP_s\Omega_\ell + \sigma_\ell^2)} - C_\ell\right). \end{aligned} \tag{6.27}$$

Expanding the exponent in (6.27) and collecting the terms that contain $\alpha_{I,\ell}$ and $\phi_{I,\ell}$,

$$\begin{aligned} & \frac{\Omega_\ell |x_\ell|^2}{\sigma_\ell^2 (KP_s\Omega_\ell + \sigma_\ell^2)} - C_\ell \\ = \; & D_{\ell,1} - \frac{KP_I}{\sigma_\ell^2}\alpha_{I,\ell}^2 + D_{\ell,2}\alpha_{I,\ell}^2 + 2\left|D_{\ell,3}\right| \alpha_{I,\ell} \cos\left(\phi_{I,\ell} - \varphi_\ell\right), \end{aligned} \tag{6.28}$$

where

$$\begin{aligned} D_{\ell,1} &= \frac{\Omega_\ell P_s \left|y_\ell^*\right|^2}{\sigma_\ell^2 (KP_s\Omega_\ell + \sigma_\ell^2)} & (6.29) \\ D_{\ell,2} &= \frac{\Omega_\ell P_s P_I |y|^2}{\sigma_\ell^2 (KP_s\Omega_\ell + \sigma_\ell^2)} & (6.30) \\ D_{\ell,3} &= \left|D_{\ell,3}\right| e^{i\varphi_\ell} \\ &= \frac{(KP_s\Omega_\ell + \sigma_\ell^2)\sqrt{P_I}}{\sigma_\ell^2 (KP_s\Omega_\ell + \sigma_\ell^2)} y_{I,\ell} - \frac{\Omega_\ell P_s \sqrt{P_I}}{\sigma_\ell^2 (KP_s\Omega_\ell + \sigma_\ell^2)} y y_\ell. & (6.31) \end{aligned}$$

Substitute (6.28) into (6.27),

$$\begin{aligned} & p(\underline{\mathbf{r}}_k | \mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}_I) \\ = \; & C_1 \prod_{\ell=1}^{L} \frac{\sigma_\ell^2}{KP_s\Omega_\ell + \sigma_\ell^2} \\ & \exp\left\{D_{\ell,1} - \frac{KP_I}{\sigma_\ell^2}\alpha_{I,\ell}^2 + D_{\ell,2}\alpha_{I,\ell}^2 + 2\left|D_{\ell,3}\right| \alpha_{I,\ell} \cos\left(\phi_{I,\ell} - \varphi_\ell\right)\right\}. \end{aligned} \tag{6.32}$$

In (6.32), the probability of $\underline{\mathbf{r}}_k$ is dependent on $\phi_{I,\ell}$ ($\ell = 1, \cdots, L$) (which is included in $\mathbf{c}_I$). We can cancel this dependence by integration,

$$p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{s}_{I,k}, \alpha_I) = \int p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{s}_{I,k}, \mathbf{c}_I) p_{\phi_I}(\phi_I)\, d\phi_I, \tag{6.33}$$

where $p_{\phi_I}(\phi_I)$ is the probability density function of ϕ_I. For independent Rayleigh fading channels,

$$p_{\phi_I}(\phi_I) = \left(\frac{1}{2\pi}\right)^L \quad 0 \le \phi_{I,1}, \cdots, \phi_{I,L} < 2\pi \tag{6.34}$$

Substitute (6.32) and (6.34) into (6.33) and carry out the integration, we have

$$\begin{aligned} & p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{s}_{I,k}, \alpha_I) \\ = \; & C_1 \left\{ \prod_{\ell=1}^{L} \frac{\sigma_\ell^2}{KP_s\Omega_\ell + \sigma_\ell^2} \right\} \exp\left\{ -\sum_{\ell=1}^{L} \frac{KP_I}{\sigma_\ell^2} \alpha_{I,\ell}^2 \right\} \eta_{LSDD}\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right), \end{aligned} \tag{6.35}$$

where

$$\eta_{MSDD}\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right) = \prod_{\ell=1}^{L} \exp\left\{D_{\ell,1} + D_{\ell,2}\alpha_{I,\ell}^2\right\} I_0\left(2\left|D_{\ell,3}\right| \alpha_{I,\ell}\right), \tag{6.36}$$

and $I_0(x)$ is the zeroth order modified Bessel function of the first kind. Note that $\eta_{MSDD}\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right)$ is a function of $\mathbf{s}_k$ and $\mathbf{s}_{I,k}$.

In (6.35), only $\eta_{MSDD}\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right)$ is dependent on $\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right)$. Maximizing $p(\underline{\mathbf{r}}_k|\mathbf{s}_k, \mathbf{s}_{I,k}, \alpha_I)$ with respect to $\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right)$ is equivalent to maximizing $\eta_{MSDD}\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right)$. Therefore $\eta_{MSDD}\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right)$ can be used as decision statistic for the MSDD detector. The corresponding decision rule is

$$\left(\widehat{\mathbf{s}}_k, \widehat{\mathbf{s}}_{I,k}\right) = \arg\max_{\mathbf{s}_k, \mathbf{s}_{I,k}} \eta_{MSDD}\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right). \tag{6.37}$$

The MSDD detector searches through all possible $\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right)$ and chooses the pair that has the largest $\eta_{MSDD}\left(\mathbf{s}_k, \mathbf{s}_{I,k}\right)$ as the detected output.

To complete this section, we briefly review maximum likelihood (ML) detector. It is used in the simulation results section for comparison with MSDD. ML detector is a kind of coherent detection technique that requires the channel gain of both the desired signal and interference. It makes symbol-by-symbol detection instead of sequence detection for MSDD. The ML decision rule is given by

$$\left(\widehat{s}_k, \widehat{s}_{I,k}\right) = \arg\max_{s_k, s_{I,k}} p(\mathbf{r}_k|s_k, s_{I,k}, \mathbf{c}, \mathbf{c}_I). \tag{6.38}$$

From (6.38) and $p(\mathbf{r}_k|s_k, s_{I,k}, \mathbf{c}, \mathbf{c}_I)$ shown in (6.9), we can get the equivalent ML decision rule as

$$\left(\widehat{s}_k, \widehat{s}_{I,k}\right) = \arg\max_{s_k, s_{I,k}} \eta_{ML}\left(s_k, s_{I,k}\right), \tag{6.39}$$

where the decision statistic

$$\eta_{ML}\left(s_k, s_{I,k}\right) = \sum_{\ell=1}^{L} \frac{\left|r_{k,\ell} - \sqrt{P_s} c_\ell s_k - \sqrt{P_I} c_{I,\ell} s_{I,k}\right|^2}{\sigma_\ell^2}. \tag{6.40}$$

6.4 Simulation Results

The communication systems simulated had 4 diversity branches. DPSK modulation was employed. The channels of both the desired signal and the interference were assumed to be uniform, i.e., $\Omega_\ell = 1$, $E\left[\alpha_{I,\ell}^2\right] = 1$ and $\sigma_\ell^2 = \sigma^2$ for $\ell = 1, 2, \cdots, L$.

In the figures, SNR is the signal to noise ratio defined as SNR $= P_s/\sigma^2$, while SIR is the signal to interference ratio defined as SIR $= P_s/P_I$. Fig. 6.1 and Fig. 6.2 were generated for SIR $= 10$ dB and SIR $= -10$ dB, respectively.

The curves labeled "MSDD($K = 2$)" and "MSDD($K = 7$)" are the results for the MSDD detector developed in this chapter. It is observed that the performance improves with the increase in K, which is the number of symbols in the observation interval. For example, in Fig. 6.2, at BEP $= 2 \times 10^{-3}$, the required SNR for $K = 2$ is about 8.5 dB; for $K = 7$ it is 5.5 dB. That means increasing the observation interval from $K = 2$ to $K = 7$ symbols results in a 3 dB SNR improvement.

The curves labeled "OC" are the results for OC. The curves labeled "MSDD(known cov, $K = 13$)" are for the MSDD detector discussed in Chapter 5, which was developed for known covariance matrix of the interference plus noise. MSDD ($K = 7$) has about the same computation complexity as MSDD (known cov, $K = 13$).

In Fig. 6.1, the BEP of MSDD ($K = 7$) is larger than that of MSDD (known cov, $K = 13$) and OC. In, Fig. 6.2, the BEP of MSDD ($K = 7$) is less than that of MSDD (known cov, $K = 13$) and OC. We conclude that at high interference level, MSDD ($K = 7$) has better performance than MSDD (known cov, $K = 13$) and OC. That can be explained as following. MSDD ($K = 7$) detects the interference signal $\mathbf{s}_{I,k}$ as well as the desired signal $\mathbf{s}_k$, but the MSDD (known cov, $K = 13$) and the OC detector only detect the desired signal. MSDD developed in this chapter is a kind of multiuser detection that gets better performance with the increase in interference power. These results are reflected in Fig. 6.3, which shows the difference of the required SNR of MSDD and OC at BEP$= 10^{-3}$.

In both Fig. 6.1 and Fig. 6.2, the performance of MSDD ($K = 7$) is not as good as that of maximum likelihood detector (curves labeled "ML"). But the difference is small for a high interference level. In Fig. 6.2, at BEP $= 2 \times 10^{-4}$, the difference of the required SNR is about 1.6 dB. These results are reflected in Fig. 6.4, which shows the difference of the required SNR of MSDD and ML at BEP$= 10^{-3}$. We can expect that this difference will decrease with the increase of K.

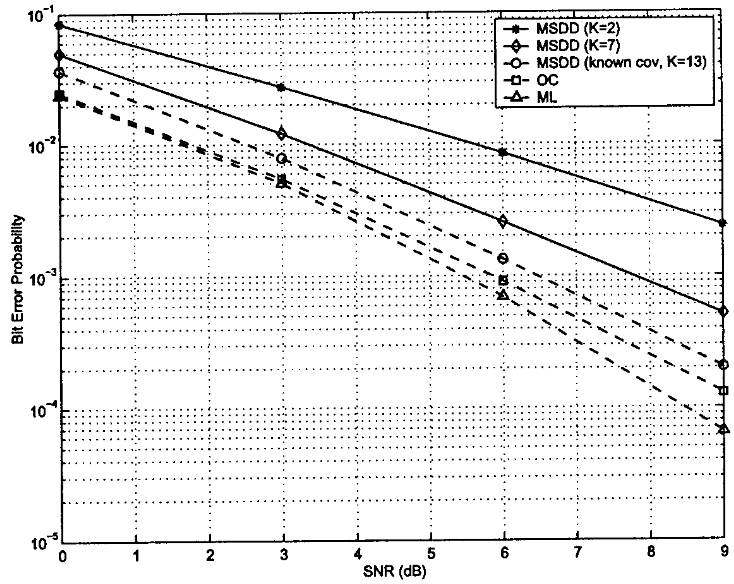

Figure 6.1: BEP versus SNR for MSDD, OC and ML. For $L = 4$ branches, SIR = 10 dB. 'MSDD ($K = 2$)' and 'MSDD ($K = 7$)' are BEP for the MSDD developed in this chapter, while 'MSDD (known cov, $K = 13$)' is for MSDD derived in Chapter 5

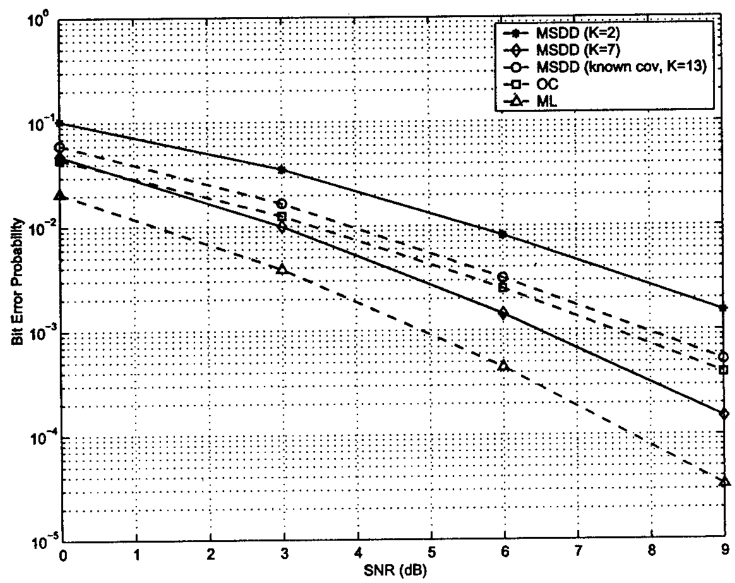

Figure 6.2: BEP versus SNR for MSDD, OC and ML. For $L = 4$ branches, SIR = −10 dB. 'MSDD ($K = 2$)' and 'MSDD ($K = 7$)' are BEP for the MSDD developed in this chapter, while 'MSDD (known cov, $K = 13$)' is for MSDD derived in Chapter 5

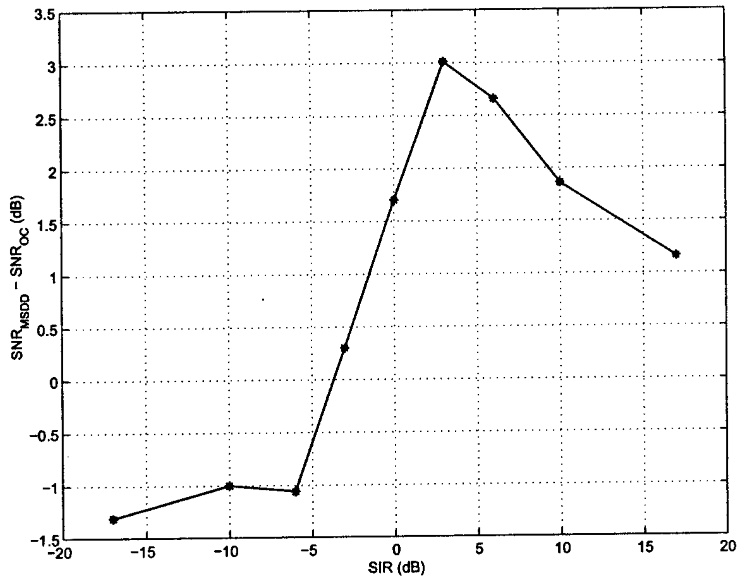

Figure 6.3: The difference of the required SNR (at BEP= 10^{-3}) of MSDD and OC versus SIR

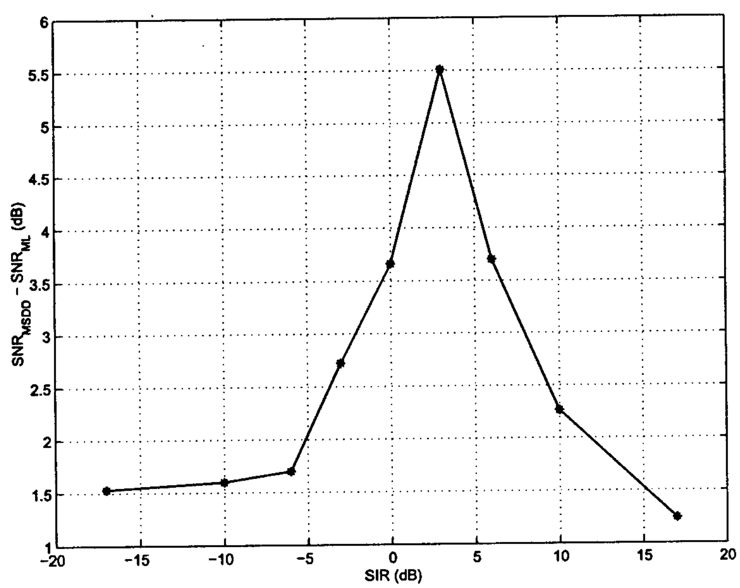

Figure 6.4: The difference of the required SNR (at BEP= 10^{-3}) of MSDD and ML versus SIR

Chapter 7

Summary and future work

7.1 Summary

In this report, we present the following work we have done:

- Obtained closed-form expressions of the exact BEP of OC for BPSK, multiple interferers, with the number of interferers less than the number of reception branches.
- Formulated simpler asymptotic expressions of BEP of OC for M-PSK, one interferer.
- Developed the decision statistic of MSDD for communication system with one interferer. The performance of this detection scheme was analyzed. Through analysis results and simulation results, we proved that with an increasing observation interval, the performance of MSDD approached that of OC with differential encoding.
- Evaluated the performance of MSDD for the case when the channel information of the interference was known.

7.2 Future Work

Some future work includes:

- Derive the closed-form expression of the exact BEP of OC for BPSK, multiple interferers, with the number of interferers is equal to or greater than that of the reception branches.
- Try to derive closed-form expressions of the exact BEP of OC for M-PSK, multiple interferers.
- Derive simpler approximate expressions of BEP for OC.
- Apply the above MSDD scheme (for known covariance matrix of interference-plus-noise matrix) to multiple interferers. Analyze and compare its performance with that of OC.

Appendix A

Derivation of the Characteristic Function for OC

In this appendix, we derive the expression (3.14) for the characteristic function $\Phi_D(j\omega)$ of the test statistic D.

Define

$$d_m = \lambda_m^{-1} g_m^* x_m + \lambda_m^{-1} g_m x_m^*, \tag{A.1}$$

then from (3.13) we have

$$D = \sum_{m=1}^{L} d_m. \tag{A.2}$$

From the signal model in Section 3.2, the definition of the whitened interference-plus-noise vector $\mathbf{x}$ and modified channel vector $\mathbf{g}$ in (3.11) and (3.12), after some algebra, the covariance matrix of $\mathbf{x}$ and $\mathbf{g}$ can be evaluated as

$$\mathbf{R}_{xx} = E\left[\mathbf{x}\mathbf{x}^H\right] = P_s \mathbf{I}_L + \Lambda_z \tag{A.3}$$

$$\mathbf{R}_{gg} = \mathbf{I}_L \tag{A.4}$$

$$\mathbf{R}_{xg} = \mathbf{R}_{gx} = \sqrt{P_s}\mathbf{I}_L. \tag{A.5}$$

To use the results in [7, Appendix B], we identify the following quantities using the notation in the reference: $X_m = x_m$, $Y_m = g_m$. Then from (A.3) to (A.5), we have in the notation of the reference

$$\overline{X}_m = \overline{Y}_m = 0 \tag{A.6}$$

$$\mu_{xx,m} = \frac{1}{2}(P_s + \lambda_m) \tag{A.7}$$

$$\mu_{yy,m} = \frac{1}{2} \tag{A.8}$$

$$\mu_{xy,m} = \mu_{yx,m} = \frac{1}{2}\sqrt{P_s}. \tag{A.9}$$

Substitute the above equations into Eq. (B-6) and Eq. (B-5) in [7, Appendix B], together with $A_m = B_m = 0$ and $C_m = \lambda_m^{-1}$. After some straightforward manipulations, we get the characteristic function of d_m as

$$\phi_{d_m}(j\omega) = \frac{1}{\left(1 - j\omega\frac{1}{\nu_{1,m}}\right)\left(1 + j\omega\frac{1}{\nu_{2,m}}\right)}, \tag{A.10}$$

where

$$\nu_{1,m} = \sqrt{P_s + \lambda_m} - \sqrt{P_s} \tag{A.11}$$

$$\nu_{1,m} = \sqrt{P_s + \lambda_m} + \sqrt{P_s}. \tag{A.12}$$

It follows that the characteristic function of D is

$$\Phi_D(j\omega) \tag{A.13}$$

$$= \prod_{m=1}^{L} \phi_{d_m}(j\omega) \tag{A.14}$$

$$= \prod_{m=1}^{L} \frac{1}{\left(1 - j\omega\frac{1}{\nu_{1,m}}\right)\left(1 + j\omega\frac{1}{\nu_{2,m}}\right)}.$$

$$= \prod_{m=1}^{L} \frac{1}{\left[1 - j\omega\left(\frac{\sqrt{P_s}}{\lambda_m} + \sqrt{\frac{P_s}{\lambda_m^2} + \frac{1}{\lambda_m}}\right)\right]\left[1 - j\omega\left(\frac{\sqrt{P_s}}{\lambda_m} - \sqrt{\frac{P_s}{\lambda_m^2} + \frac{1}{\lambda_m}}\right)\right]} \tag{A.15}$$

For ease to manipulate later, we define μ_m as (notice that $\lambda_m = N_0$ for $m = N_I + 1, N_I + 2, \cdots, L$)

$$\mu_m = \begin{cases} \frac{\sqrt{P_s}}{\lambda_m} + \sqrt{\frac{P_s}{\lambda_m^2} + \frac{1}{\lambda_m}} & m = 1, \cdots, N_I \\ \frac{\sqrt{P_s}}{\lambda_{m-N_I}} - \sqrt{\frac{P_s}{\lambda_{m-N_I}^2} + \frac{1}{\lambda_{m-N_I}}} & m = N_I + 1, \cdots, 2N_I \\ \mu_{00} & m = 2N_I + 1, \cdots, 2N_I + (L - N_I) \\ \mu_{01} & m = 2N_I + (L - N_I) + 1, \cdots, 2L \end{cases} \tag{A.16}$$

where

$$\mu_{00} \triangleq \frac{\sqrt{P_s}}{N_0} + \sqrt{\frac{P_s}{N_0^2} + \frac{1}{N_0}} \tag{A.17}$$

$$\mu_{01} \triangleq \frac{\sqrt{P_s}}{N_0} - \sqrt{\frac{P_s}{N_0^2} + \frac{1}{N_0}}. \tag{A.18}$$

Then $\Phi_D(j\omega)$ in (A.15) could be expressed as

$$\Phi_D(j\omega) = \prod_{m=1}^{2L} \frac{1}{1 - j\mu_m\omega}. \tag{A.19}$$

Appendix B

Evaluation of the Residues for OC

In this appendix, we evaluate the residue in (3.22).

According to the complex variables theory [38], if a function $f(\omega)$ has a pole ω_0 of order N, the residue of $f(\omega)$ at ω_0 is

$$\operatorname{Res}\left[f(\omega);\omega_0\right] = \frac{1}{(N-1)!}\frac{d^{(N-1)}}{d\omega^{(N-1)}}\left[(\omega-\omega_0)^N f(\omega)\right]\Big|_{\omega=\omega_0}. \tag{B.1}$$

$\frac{\Phi_{D|s_t}(j\omega)}{\omega}$ in (3.21) can be expressed as

$$\begin{aligned}\frac{\Phi_D(j\omega)}{\omega} &= \frac{1}{(-j)^{2L}}\left(\prod_{n=1}^{2N_I}\frac{1}{\mu_n}\right)\frac{1}{(\mu_{00}\mu_{01})^{L-N_I}} \\ &\quad \frac{1}{\omega}\left(\prod_{n=1}^{2N_I}\frac{1}{\omega+j\frac{1}{2\mu_n}}\right)\frac{1}{\left(\omega+j\frac{1}{\mu_{00}}\right)^{L-N_I}}\frac{1}{\left(\omega+j\frac{1}{\mu_{01}}\right)^{L-N_I}}.\end{aligned} \tag{B.2}$$

B.1 Evaluate the residues at poles $\omega = -j\frac{1}{\mu_m}$ $(m = N_I + 1, \cdots, 2N_I$

For $N_I + 1 \leq m \leq 2N_I$, the poles $\omega = -j\frac{1}{\mu_m}$ are of order 1.

$$
\begin{aligned}
& \mathrm{Res}\left[\frac{\Phi_D(j\omega)}{\omega}; -j\frac{1}{\mu_m}\right] \\
= & \left[\omega - \left(-j\frac{1}{\mu_m}\right)\right] \frac{\Phi_D(j\omega)}{\omega}\bigg|_{\omega=-j\frac{1}{\mu_m}} \\
= & \frac{1}{(-j)^{2L}} \left(\prod_{n=1}^{2N_I} \frac{1}{\mu_n}\right) \frac{1}{(\mu_{00}\mu_{01})^{L-N_I}} \frac{1}{\omega} \prod_{n=1}^{m-1} \frac{1}{\left(\omega + j\frac{1}{\mu_n}\right)} \\
& \prod_{n=m+1}^{2N_I} \frac{1}{\left(\omega + j\frac{1}{\mu_n}\right)} \frac{1}{\left(\omega + j\frac{1}{\mu_{00}}\right)^{L-N_I}} \frac{1}{\left(\omega + j\frac{1}{\mu_{01}}\right)^{L-N_I}}\Bigg|_{\omega=-j\frac{1}{\mu_m}} \\
= & \frac{1}{(-j)^{2L}} \left(\prod_{n=1}^{2N_I} \frac{1}{\mu_n}\right) \frac{1}{(\mu_{00}\mu_{01})^{L-N_I}} \frac{1}{-j\frac{1}{\mu_m}} \prod_{n=1}^{m-1} \frac{1}{\left(-j\frac{1}{\mu_m} + j\frac{1}{\mu_n}\right)} \\
& \prod_{n=m+1}^{2N_I} \frac{1}{\left(-j\frac{1}{\mu_m} + j\frac{1}{\mu_n}\right)} \frac{1}{\left(-j\frac{1}{\mu_m} + j\frac{1}{\mu_{00}}\right)^{L-N_I}} \frac{1}{\left(-j\frac{1}{\mu_m} + j\frac{1}{\mu_{01}}\right)^{L-N_I}}.
\end{aligned}
\tag{B.3}
$$

After some simplification, we have

$$
\begin{aligned}
\mathrm{Res}\left[\frac{\Phi_D(j\omega)}{\omega}; -j\frac{1}{\mu_m}\right] = & \mu_m^{2L-1} \left(\prod_{n=1}^{m-1} \frac{1}{\mu_n - \mu_m}\right) \left(\prod_{n=m+1}^{2N_I} \frac{1}{\mu_n - \mu_m}\right) \\
& \frac{1}{(\mu_{00} - \mu_m)^{L-N_I}} \frac{1}{(\mu_{01} - \mu_m)^{L-N_I}}.
\end{aligned}
\tag{B.4}
$$

B.2 Evaluate the residue at pole $\omega = -j\frac{1}{\mu_{01}}$

Since the pole $\omega = -j\frac{1}{\mu_{01}}$ is of order $(L - N_I)$,

$$\begin{aligned}
&\text{Res}\left[\frac{\Phi_D(j\omega)}{\omega}; -j\frac{1}{\mu_{01}}\right] \\
&= \frac{1}{(L-N_I-1)!}\frac{d^{L-N_I-1}}{d\omega^{L-N_I-1}}\left\{\left[\omega - \left(-j\frac{1}{\mu_{01}}\right)\right]^{L-N_I}\frac{\Phi_{D|s_t}(j\omega)}{\omega}\right\}\Bigg|_{\omega=-j\frac{1}{\mu_{01}}} \\
&= \frac{1}{(-j)^{2L}}\left(\prod_{n=1}^{2N_I}\frac{1}{\mu_n}\right)\frac{1}{(\mu_{00}\mu_{01})^{L-N_I}}\frac{1}{(L-N_I-1)!} \\
&\quad \frac{d^{L-N_I-1}}{d\omega^{L-N_I-1}}\left\{\frac{1}{\omega}\left[\prod_{n=1}^{2N_I}\frac{1}{\left(\omega+j\frac{1}{\mu_n}\right)}\right]\frac{1}{\left(\omega+j\frac{1}{\mu_{00}}\right)^{L-N_I}}\right\}\Bigg|_{\omega=-j\frac{1}{\mu_{01}}}.
\end{aligned} \tag{B.5}$$

Define $\mu_0 = \infty$, then

$$\frac{1}{\omega}\left[\prod_{n=1}^{2N_I}\frac{1}{\left(\omega+j\frac{1}{\mu_n}\right)}\right] = \prod_{n=0}^{2N_I}\frac{1}{\left(\omega+j\frac{1}{\mu_n}\right)} = \sum_{n=0}^{2N_I}\frac{A_n}{\left(\omega+j\frac{1}{\mu_n}\right)} \tag{B.6}$$

where

$$\begin{aligned}
A_n &= \prod_{i=0}^{n-1}\frac{1}{\left(\omega+j\frac{1}{\mu_i}\right)}\prod_{i=n+1}^{2N_I}\frac{1}{\left(\omega+j\frac{1}{\mu_i}\right)}\Bigg|_{\omega=-j\frac{1}{\mu_n}} \\
&= (-1)^{N_I}\prod_{i=0}^{n-1}\frac{\mu_i\mu_n}{(\mu_i-\mu_n)}\prod_{i=n+1}^{2N_I}\frac{\mu_i\mu_n}{(\mu_i-\mu_n)}.
\end{aligned} \tag{B.7}$$

For $n = 0$,

$$A_0 = (-1)^{N_I}\prod_{i=1}^{2N_I}\frac{\mu_0\mu_n}{(\mu_0-\mu_n)} = (-1)^{N_I}\prod_{i=1}^{2N_I}\mu_n \quad (\text{since } \mu_0 = \infty). \tag{B.8}$$

For $1 \le n \le 2N_I$,

$$\begin{aligned}
A_n &= (-1)^{N_I}\prod_{i=0}^{n-1}\frac{\mu_i\mu_n}{(\mu_i-\mu_n)}\prod_{i=n+1}^{2N_I}\frac{\mu_i\mu_n}{(\mu_i-\mu_n)} \\
&= (-1)^{N_I}\mu_n^{2N_I-1}\prod_{i=1}^{2N_I}\mu_n\prod_{i=1}^{n-1}\frac{1}{(\mu_i-\mu_n)}\prod_{i=n+1}^{2N_I}\frac{1}{(\mu_i-\mu_n)}.
\end{aligned} \tag{B.9}$$

Using (B.6) and the derivation expressions

$$\frac{d^n}{d\omega^n}\left[f(\omega)\, g(\omega)\right] = \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{d^{n-\ell} f(\omega)}{d\omega^{n-\ell}} \frac{d^\ell g(\omega)}{d\omega^\ell} \tag{B.10}$$

$$\frac{d^n}{d\omega^n} \frac{1}{(\omega+a)^m} = (-1)^n \frac{(m+n-1)!}{(m-1)!} \frac{1}{(\omega+a)^{m+n}}, \tag{B.11}$$

we get the derivation in (B.5) as

$$\begin{aligned}
&\frac{d^{L-N_I-1}}{d\omega^{L-N_I-1}} \left\{ \frac{1}{\omega} \left[\prod_{n=1}^{2N_I} \frac{1}{\left(\omega + j\frac{1}{\mu_n}\right)} \right] \frac{1}{\left(\omega + j\frac{1}{\mu_{00}}\right)^{L-N_I}} \right\} \\
= &\sum_{\ell=0}^{L-N_I-1} \binom{L-N_I-1}{\ell} \frac{d^{L-N_I-1-\ell}}{d\omega^{L-N_I-1-\ell}} \left[\sum_{n=0}^{2N_I} \frac{A_n}{\left(\omega + j\frac{1}{\mu_n}\right)} \right] \frac{d^\ell}{d\omega^\ell} \frac{1}{\left(\omega + j\frac{1}{\mu_{00}}\right)^{L-N_I}} \\
= &\sum_{\ell=0}^{L-N_I-1} \binom{L-N_I-1}{\ell} \left\{ (-1)^{L-N_I-1-\ell} \frac{(1+L-N_I-1-\ell-1)!}{(1-1)!} \right. \\
&\left. \sum_{n=0}^{2N_I} \frac{A_n}{\left(\omega + j\frac{1}{\mu_n}\right)^{1+L-N_I-1-\ell}} \right\} (-1)^\ell \frac{(L-N_I+\ell-1)!}{(L-N_I-1)!} \\
&\frac{1}{(\omega + j\frac{1}{\mu_{00}})^{L-N_I+\ell}}.
\end{aligned} \tag{B.12}$$

After some simple algebra,

$$\begin{aligned}
&\frac{d^{L-N_I-1}}{d\omega^{L-N_I-1}} \left\{ \frac{1}{\omega} \left[\prod_{n=1}^{2N_I} \frac{1}{\left(\omega + j\frac{1}{\mu_n}\right)} \right] \frac{1}{\left(\omega + j\frac{1}{\mu_{00}}\right)^{L-N_I}} \right\} \\
= &(-1)^{L-N_I-1} \sum_{\ell=0}^{L-N_I-1} \frac{(L-N_I+\ell-1)!}{\ell!} \left[\sum_{n=0}^{2N_I} \frac{A_n}{\left(\omega + j\frac{1}{\mu_n}\right)^{L-N_I-\ell}} \right] \\
&\frac{1}{(\omega + j\frac{1}{\mu_{00}})^{L-N_I+\ell}}.
\end{aligned} \tag{B.13}$$

Substitute (B.13) into (B.5),

$$\operatorname{Res}\left[\frac{\Phi_D(j\omega)}{\omega}; -j\frac{1}{\mu_{01}}\right]$$

$$= \frac{1}{(-j)^{2L}}\left(\prod_{n=1}^{2N_I}\frac{1}{\mu_n}\right)\frac{1}{(\mu_{00}\mu_{01})^{L-N_I}}\frac{1}{(L-N_I-1)!}(-1)^{L-N_I-1}$$

$$\sum_{\ell=0}^{L-N_I-1}\frac{(L-N_I+\ell-1)!}{\ell!}\left[\sum_{n=0}^{2N_I}\frac{A_n}{\left(-j\frac{1}{\mu_{01}}+j\frac{1}{\mu_n}\right)^{L-N_I-\ell}}\right] \tag{B.14}$$

$$\frac{1}{(-j\frac{1}{\mu_{01}}+j\frac{1}{\mu_{00}})^{L-N_I+\ell}} \tag{B.15}$$

which can be simplified to

$$\operatorname{Res}\left[\frac{\Phi_D(j\omega)}{\omega}; -j\frac{1}{\mu_{01}}\right]$$

$$= \frac{1}{(\mu_{00}\mu_{01})^{L-N_I}}\frac{1}{(L-N_I-1)!}(-1)^{L-N_I-1}\sum_{\ell=0}^{L-N_I-1}\frac{(L-N_I+\ell-1)!}{\ell!}$$

$$\left[\mu_{01}^{L-N_I-\ell}+\sum_{n=1}^{2N_I}\frac{(\mu_n\mu_{01})^{L-N_I-\ell}}{(\mu_n-\mu_{01})^{L-N_I-\ell}}\mu_n^{2N_I-1}\left(\prod_{i=1}^{n-1}\frac{1}{\mu_i-\mu_n}\right)\left(\prod_{i=n+1}^{2N_I}\frac{1}{\mu_i-\mu_n}\right)\right]$$

$$\frac{(\mu_{00}\mu_{01})^{L-N_I+\ell}}{(\mu_{00}-\mu_{01})^{L-N_I+\ell}}. \tag{B.16}$$

Appendix C

Express $F_{I-N_I-k}(y_m)$ as a Summation of $(y_m^2 - \eta^2)$ to Integer Power

In this appendix, we derive (3.50), which expresses the function $F_k(y_m)$ in (3.45) as a summation of $(y_m^2 - \eta^2)$ to integer power.

For simplicity, define

$$F_k = -(1+y_m)(\eta - y_m)^k + (1-y_m)(\eta+y_m)^k. \quad \text{(C.1)}$$

Then

$$F_0 = -2y_m \quad \text{(C.2)}$$
$$F_1 = 2(1-\eta)y_m. \quad \text{(C.3)}$$

By substituting in F_{k-1} and F_{k-2}, it could be easily proved that

$$F_k = 2\eta F_{k-1} + \left(y_m^2 - \eta^2\right) F_{k-2}. \quad \text{(C.4)}$$

To simplify the notations, define mathematical symbols $P = 2\eta$, $Q = y_m^2 - \eta^2$. Then (C.4) becomes

$$F_k = PF_{k-1} + QF_{k-2}. \quad \text{(C.5)}$$

We try to find the relation between F_k and F_0, F_1. First we express F_k as a function of F_{k-2} and F_{k-3}. From (C.5),

$$\begin{aligned} F_k &= PF_{k-1} + QF_{k-2} \\ &= P\left(PF_{k-2} + QF_{k-3}\right) + QF_{k-2} \\ &= \left(P^2 + Q\right) F_{k-2} + PQF_{k-3} \end{aligned} \quad \text{(C.6)}$$

Continue on in this way, we have

$$F_k = \left(P^3 + 2PQ\right) F_{k-3} + \left(P^2Q + Q^2\right) F_{k-4} \quad \text{(C.7)}$$
$$F_k = \left(P^4 + 2P^2Q + P^2Q + Q^2\right) F_{k-4} + \left(P^3Q + 2PQ^2\right) F_{k-5}. \quad \text{(C.8)}$$

Judging from (C.5) to (C.8), we guess that the relation between F_k and F_{k-I}, F_{k-I-1} (I is any positive integer) is:

- when I is odd,

$$F_k = \left[\sum_{t=0}^{O_I} \binom{I-t}{t} P^{I-2t} Q^t\right] F_{k-I} + \left[\sum_{t=0}^{O_I} \binom{I-1-t}{t} P^{I-2t-1} Q^{t+1}\right] F_{k-I-1}. \quad \text{(C.9)}$$

where $O_I = (I-1)/2$.

- when I is even,

$$F_k = \left[\sum_{t=0}^{O_I} \binom{I-t}{t} P^{I-2t} Q^t\right] F_{k-I} + \left[\sum_{t=0}^{O_I-1} \binom{I-1-t}{t} P^{I-2t-1} Q^{t+1}\right] F_{k-I-1}. \quad \text{(C.10)}$$

where $O_I = I/2$.

(C.5) to (C.8) have shown that (C.9) and (C.10) are valid for $I = 1, 2, 3, 4$. The validity of (C.9) and (C.10) for any positive integer I could be proved easily by the method of mathematical induction.

From (C.9) and (C.10), we can get the relation between F_k and F_1, F_0 as follow:

- When k is even

Let $I = k-1$ and substitute it into (C.9),

$$F_k = \left[\sum_{t=0}^{O_I} \binom{k-1-t}{t} P^{k-1-2t} Q^t\right] F_1 + \left[\sum_{t=0}^{O_I} \binom{k-1-1-t}{t} P^{k-1-2t-1} Q^{t+1}\right] F_0. \quad \text{(C.11)}$$

Define $T \triangleq k/2$. since $I = 2O_I + 1$,

$$O_I = \frac{I-1}{2} = \frac{k-1-1}{2} = \frac{k}{2} - 1 = T - 1. \quad \text{(C.12)}$$

Substituting (C.12) into (C.11), we get the relation between F_k and F_1, F_0 as

$$F_k = \left[\sum_{t=0}^{T-1} \binom{k-1-t}{t} P^{k-1-2t} Q^t\right] F_1 + \left[\sum_{t=0}^{T-1} \binom{k-1-1-t}{t} P^{k-1-2t-1} Q^{t+1}\right] F_0. \quad \text{(C.13)}$$

- When k is odd

Similarly, we can get

$$F_k = \left[\sum_{t=0}^{T} \binom{k-1-t}{t} P^{k-1-2t} Q^t\right] F_1 + \left[\sum_{t=0}^{T-1} \binom{k-2-t}{t} P^{k-2-2t} Q^{t+1}\right] F_0 \quad \text{(C.14)}$$

where $T = (k-1)/2$.

Finally, substituting $P = 2\eta$, $Q = y_m^2 - \eta^2, F_0 = -2y_m$, and $F_1 = 2(1-\eta)y_m$ in (C.13) and (C.14), after some manipulations, we have

$$F_k(y_m) = 2y_m \sum_{t=0}^{[k/2]} a_{k,t} \left(y_m^2 - \eta^2\right)^t \quad \text{(C.15)}$$

where $[k/2]$ denotes the largest integer that is less than $k/2$, and $a_{k,t}$ is calculated differently for when k is even or odd as follow:

- When k is even,

$$a_{k,t} = \begin{cases} (2\eta)^{k-1}(1-\eta) & t=0 \\ \left[\binom{k-1-t}{t}(1-\eta) - 2\eta\binom{k-1-t}{t-1}\right](2\eta)^{k-1-2t} & 1 \le t \le [k/2]-1 \\ -1 & t = [k/2] \end{cases} . \tag{C.16}$$

- When k is odd,

$$a_{k,t} = \begin{cases} (2\eta)^{k-1}(1-\eta) & t=0 \\ \left[\binom{k-1-t}{t}(1-\eta) - 2\eta\binom{k-1-t}{t-1}\right](2\eta)^{k-1-2t} & 1 \le t \le [k/2] \end{cases} . \tag{C.17}$$

If we assume $\binom{m}{n} = 0$ for $m < n$ or $n < 0$, we can express (C.16) and (C.17) as a single expression:

$$a_{k,t} = \left[\binom{k-1-t}{t}(1-\eta) - 2\eta\binom{k-1-t}{t-1}\right](2\eta)^{k-1-2t},$$

for $0 \le t \le [k/2]$.

Appendix D

Evaluation of the Sum of Integrals

In this appendix, we evaluate the sum of integrals $\int f_m(\mathbf{y}) p_{\mathbf{y}}(y)dy$. The basic steps are:

1. Combine the sum of N_I integrals (each integral is a N_I−fold integral) into one N_I−fold integral.
2. Change the integration limits of some variables from finite to semi-infinite.
3. For $N_I > 1$, separate the N_I−fold integration into N_I independent integrations.

D.1 Combine the sum of N_I integrals

By expressing the integrals into a slightly different form, we can convert the sum of N_I integrals into one integral. We first consider the integrals $\int f_m(\mathbf{y}) p_{\mathbf{y}}(y)dy$ for $1 < m < N_I$, then for $m = 1$ and $m = N_I$.

For $1 < m < N_I$, $\int f_m(\mathbf{y}) p_{\mathbf{y}}(y)dy$ is an N_I−fold integral. We first carry out the integration over y_m, then over y_{N_I}, next y_{N_I-1}, at last over y_1. Since $\infty > y_1 \geqslant y_2 \geqslant \cdots \geqslant y_{N_I} \geqslant \eta$, $\int f_m(\mathbf{y}) p_{\mathbf{y}}(y)dy$ can be expressed as

$$\begin{aligned} & \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y})d\mathbf{y} \\ = & \int_\eta^\infty \left\{ \int_\eta^{y_1} \cdots \left\{ \int_\eta^{y_{m-2}} \left\{ \int_\eta^{y_{m-1}} \cdots \left\{ \int_\eta^{y_{N_I-1}} \left\{ \int_{y_{m+1}}^{y_{m-1}} f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) dy_m \right\} dy_{N_I} \right\} \right. \right. \right. \right. \\ & \cdots dy_{m+1} \big\} \, dy_{m-1} \big\} \cdots dy_2 \big\} \, dy_1, \end{aligned} \tag{D.1}$$

with the inner integrations being carried out before the outer integrations.

From (3.43) and (3.46),

$$\begin{aligned} f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) = & \frac{1-y_m}{2y_m} \frac{(1-\eta^2)^{L-N_I}}{(y_m^2-\eta^2)^{L-N_I}} \left\{ \prod_{n=1}^{m-1} \frac{1-y_n^2}{y_m^2-y_n^2} \right\} \left\{ \prod_{n=m+1}^{N_I} \frac{1-y_n^2}{y_m^2-y_n^2} \right\} \\ & K_1 \left[\prod_{i=1}^{N_I} \exp\left[-\beta\left(y_i^2-\eta^2\right)\right] \left(y_i^2-\eta^2\right)^{L-N_I} \right] \\ & \left[\prod_{1\le i<j\le N_I-1} \left(y_i^2-y_j^2\right)^2 \right] y_1 y_2 \cdots y_{N_I}. \end{aligned} \tag{D.2}$$

Convert the variables from y to t as:

$$y_1 \rightarrow t_1, y_2 \rightarrow t_2 \cdots, y_{m-1} \rightarrow t_{m-1}, y_{m+1} \rightarrow t_m, \cdots, y_{N_I} \rightarrow t_{N_I-1}, y_m \rightarrow t_{N_I}. \tag{D.3}$$

Then (D.1) becomes

$$\begin{aligned}&\int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\ =& \int_\eta^\infty \left\{ \int_\eta^{t_1} \cdots \left\{ \int_\eta^{t_{m-2}} \left\{ \int_\eta^{t_{m-1}} \cdots \left\{ \int_\eta^{t_{N_I-2}} \left\{ \int_{t_m}^{t_{m-1}} f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) dt_{N_I} \right\} dt_{N_I-1} \right\} \right.\right.\right.\right. \\ & \cdots dt_m \Big\} dt_{m-1} \Big\} \cdots dt_2 \Big\} dt_1, \end{aligned} \tag{D.4}$$

and (D.2) becomes

$$\begin{aligned} f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) =& \frac{1-t_{N_I}}{2t_{N_I}} \frac{\left(1-\eta^2\right)^{L-N_I}}{\left(t_{N_I}^2-\eta^2\right)^{L-N_I}} \left\{ \prod_{n=1}^{N_I-1} \frac{1-t_n^2}{t_{N_I}^2-t_n^2} \right\} \\ & K_1 \left[\prod_{i=1}^{N_I} \exp\left[-\beta\left(t_n^2-\eta^2\right)\right] \left(t_i^2-\eta^2\right)^{L-N_I} \right] \\ & \left[\prod_{1\le i<j\le N_I-1} \left(t_i^2-t_j^2\right)^2 \right] t_1 t_2 \cdots t_{N_I}, \end{aligned} \tag{D.5}$$

which could be changed to (by re-arranging the terms)

$$\begin{aligned} & f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) \\ =& \frac{1}{2} K_1 \left(1-\eta^2\right)^{L-N_I} \left\{ \prod_{n=1}^{N_I-1} \left(1-t_n^2\right) \exp\left[-\beta\left(t_n^2-\eta^2\right)\right] \left(t_n^2-\eta^2\right)^{L-N_I} t_n \right\} \\ & \left[\prod_{1\le i<j\le N_I-1} \left(t_i^2-t_j^2\right)^2 \right] \left(1-t_{N_I}\right) \exp\left[-\beta\left(t_{N_I}^2-\eta^2\right)\right] \\ & \left[\prod_{n=1}^{N_I-1} \left(t_{N_I}^2-t_n^2\right) \right]. \end{aligned} \tag{D.6}$$

Substitute (D.6) in (D.4),

$$\begin{aligned} & \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\ =& \frac{1}{2} K_1 \left(1-\eta^2\right)^{L-N_I} \int_\eta^\infty \int_\eta^{t_1} \cdots \int_\eta^{t_{N_I-2}} \\ & \left\{ \prod_{n=1}^{N_I-1} \left(1-t_n^2\right) \exp\left[-\beta\left(t_n^2-\eta^2\right)\right] \left(t_n^2-\eta^2\right)^{L-N_I} t_n \right\} \left[\prod_{1\le i<j\le N_I-1} \left(t_i^2-t_j^2\right)^2 \right] \\ & \left\{ \int_{t_m}^{t_{m-1}} \left(1-t_{N_I}\right) \exp\left[-\beta\left(t_{N_I}^2-\eta^2\right)\right] \left[\prod_{n=1}^{N_I-1} \left(t_{N_I}^2-t_n^2\right) \right] dt_{N_I} \right\} \\ & dt_{N_I-1} \cdots dt_2 dt_1. \end{aligned} \tag{D.7}$$

The above equation is for $1 < m < N_I$. Similarly, for $N_I = 1$ and N_I, we have

$$\begin{aligned}
&\int f_1(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\
= \;& \frac{1}{2} K_1 \left(1-\eta^2\right)^{L-N_I} \int_\eta^\infty \int_\eta^{t_1} \cdots \int_\eta^{t_{N_I-2}} \\
& \left\{ \prod_{n=1}^{N_I-1} \left(1-t_n^2\right) \exp\left[-\beta\left(t_n^2-\eta^2\right)\right] \left(t_n^2-\eta^2\right)^{L-N_I} t_n \right\} \left[\prod_{1\le i<j\le N_I-1} \left(t_i^2-t_j^2\right)^2 \right] \\
& \left\{ \int_{t_1}^\infty \left(1-t_{N_I}\right) \exp\left[-\beta\left(t_{N_I}^2-\eta^2\right)\right] \left[\prod_{n=1}^{N_I-1} \left(t_{N_I}^2-t_n^2\right) \right] dt_{N_I} \right\} \\
& dt_{N_I-1} \cdots dt_2 dt_1
\end{aligned} \tag{D.8}$$

and

$$\begin{aligned}
&\int f_{N_I}(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\
= \;& \frac{1}{2} K_1 \left(1-\eta^2\right)^{L-N_I} \int_\eta^\infty \int_\eta^{t_1} \cdots \int_\eta^{t_{N_I-2}} \\
& \left\{ \prod_{n=1}^{N_I-1} \left(1-t_n^2\right) \exp\left[-\beta\left(t_n^2-\eta^2\right)\right] \left(t_n^2-\eta^2\right)^{L-N_I} t_n \right\} \left[\prod_{1\le i<j\le N_I-1} \left(t_i^2-t_j^2\right)^2 \right] \\
& \left\{ \int_\eta^{t_{N_I-1}} \left(1-t_{N_I}\right) \exp\left[-\beta\left(t_{N_I}^2-\eta^2\right)\right] \left[\prod_{n=1}^{N_I-1} \left(t_{N_I}^2-t_n^2\right) \right] dt_{N_I} \right\} \\
& dt_{N_I-1} \cdots dt_2 dt_1.
\end{aligned} \tag{D.9}$$

Notice that the integrands in (D.7), (D.8) and (D.9) are the same. The integration limits for $t_{N_I-1}, \cdots, t_2, t_1$ are the same as well. Therefore we can write the sum of the N_I integrals as

$$\begin{aligned}
&\sum_{m=1}^{N_I} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\
= \;& \frac{1}{2} K_1 \left(1-\eta^2\right)^{L-N_I} \int_\eta^\infty \int_\eta^{t_1} \cdots \int_\eta^{t_{N_I-2}} \\
& \left\{ \prod_{n=1}^{N_I-1} \left(1-t_n^2\right) \exp\left[-\beta\left(t_n^2-\eta^2\right)\right] \left(t_n^2-\eta^2\right)^{L-N_I} t_n \right\} \left[\prod_{1\le i<j\le N_I-1} \left(t_i^2-t_j^2\right)^2 \right] \\
& \left\{ \left(\int_{t_1}^\infty + \sum_{m=2}^{N_I-1} \int_{t_m}^{t_{m-1}} + \int_\eta^{t_{N_I-1}} \right) \left(1-t_{N_I}\right) \exp\left[-\beta\left(t_{N_I}^2-\eta^2\right)\right] \right. \\
& \left. \left[\prod_{n=1}^{N_I-1} \left(t_{N_I}^2-t_n^2\right) \right] dt_{N_I} \right\} dt_{N_I-1} \cdots dt_2 dt_1.
\end{aligned} \tag{D.10}$$

Since

$$\int_{t_1}^{\infty} + \sum_{m=2}^{N_I-1} \int_{t_m}^{t_{m-1}} + \int_{\eta}^{t_{N_I-1}} = \int_{\eta}^{\infty},$$

$$\begin{aligned}
&\sum_{m=1}^{N_I} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\
= \;& \frac{1}{2} K_1 \left(1-\eta^2\right)^{L-N_I} \int_{\eta}^{\infty} \int_{\eta}^{t_1} \cdots \int_{\eta}^{t_{N_I-2}} \\
& \left\{ \prod_{n=1}^{N_I-1} \left(1-t_n^2\right) \exp\left[-\beta\left(t_n^2-\eta^2\right)\right] \left(t_n^2-\eta^2\right)^{L-N_I} t_n \right\} \left[\prod_{1 \le i < j \le N_I-1} \left(t_i^2 - t_j^2\right)^2 \right] \\
& \left\{ \int_{\eta}^{\infty} \left(1-t_{N_I}\right) \exp\left[-\beta\left(t_{N_I}^2-\eta^2\right)\right] \left[\prod_{n=1}^{N_I-1} \left(t_{N_I}^2 - t_n^2\right) \right] dt_{N_I} \right\} \\
& dt_{N_I-1} \cdots dt_2 dt_1.
\end{aligned} \tag{D.11}$$

which consists only one N_I−fold integral.

To simplify the notations, we convert variables as: $(t_n^2 - \eta^2) \to z_n$ for $n = 1, 2, \cdots, N_I - 1$, and define

$$z_0 = 1 - \eta^2, \tag{D.12}$$

then (D.11) becomes

$$\begin{aligned}
&\sum_{m=1}^{N_I} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\
= \;& \frac{1}{2^{N_I}} K_1 z_0^{L-N_I} \int_0^{\infty} \int_0^{z_1} \cdots \int_0^{z_{N_I-2}} \\
& \left\{ \prod_{n=1}^{N_I-1} \left(z_0 - z_n\right) \exp\left(-\beta z_n\right) z_n^{L-N_I} \right\} \left[\prod_{1 \le i < j \le N_I-1} \left(z_i - z_j\right)^2 \right] \\
& \left\{ \int_{\eta}^{\infty} \left(1-t_{N_I}\right) \exp\left[-\beta\left(t_{N_I}^2-\eta^2\right)\right] \left[\prod_{n=1}^{N_I-1} \left(t_{N_I}^2 - \eta^2 - z_n\right) \right] dt_{N_I} \right\} \\
& dz_{N_I-1} \cdots dz_2 dz_1.
\end{aligned} \tag{D.13}$$

D.2 Change the integration limits of some variables

Consider the integrations in (D.13). The integration area for $z_{N_I-1}, \cdots, z_2, z_1$ is $\infty > z_1 \geqslant z_2 \geqslant z_3 \geqslant \cdots \geqslant z_{N_I-1} \geqslant 0$. There are other similar areas such as:

$$\begin{aligned}
\infty &> z_1 \geqslant z_3 \geqslant z_2 \geqslant \cdots \geqslant z_{N_I-1} \geqslant 0, \\
\infty &> z_2 \geqslant z_1 \geqslant z_3 \geqslant \cdots \geqslant z_{N_I-1} \geqslant 0,
\end{aligned}$$

etc. There are $(N_I - 1)!$ such areas in the region: $\infty > z_1, z_2, z_3, \cdots, z_{N_I-1} \geqslant 0$. Since the integrand is symmetric to $z_1, z_2, \cdots, z_{N_I-1}$, the integration over these $(N_I - 1)!$ areas must be equal. Therefore the integration over one such area is $1/(N_I-1)!$ of the integration over the region: $\infty > z_1, z_2, z_3, \cdots, z_{N_I-1} \geqslant 0$. Therefore

$$\begin{aligned}
&\sum_{m=1}^{N_I} \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\
= \; & \frac{1}{(N_I-1)!} \frac{1}{2^{N_I}} K_1 z_0^{L-N_I} \int_0^\infty \int_0^\infty \cdots \int_0^\infty \\
& \left\{ \prod_{n=1}^{N_I-1} (z_0 - z_n) \exp(-\beta z_n) z_n^{L-N_I} \right\} \left[\prod_{1 \le i < j \le N_I - 1} (z_i - z_j)^2 \right] \\
& \left\{ \int_\eta^\infty (1 - t_{N_I}) \exp\left[-\beta \left(t_{N_I}^2 - \eta^2\right)\right] \left[\prod_{n=1}^{N_I-1} \left(t_{N_I}^2 - \eta^2 - z_n\right) \right] dt_{N_I} \right\} \\
& dz_{N_I-1} \cdots dz_2 dz_1.
\end{aligned} \tag{D.14}$$

The above integration is treated differently for $N_I = 1$ and $N_I > 1$.

D.3 $N_I = 1$

From (3.48) we have

$$K_1 = \frac{2}{(L-1)!} \beta^L. \tag{D.15}$$

(D.14) simplifies to

$$\begin{aligned}
&\sum_{m=1}^{N_I} \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\
= \; & \frac{1}{2} \frac{2}{(L-1)!} \beta^L z_0^{L-1} \int_\eta^\infty (1 - t_1) \exp\left[-\beta \left(t_1^2 - \eta^2\right)\right] dt_1.
\end{aligned} \tag{D.16}$$

Since $\eta = \sqrt{\frac{1}{\gamma} + 1}$,

$$z_0 = 1 - \eta^2 = -\frac{1}{\gamma}. \tag{D.17}$$

Using the above equation and the result in Appendix E, we have

$$\sum_{m=1}^{N_I} \int f_m(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} = \frac{1}{(L-1)!} \beta^L \left(-\frac{1}{\gamma}\right)^{L-1} \left(B_0 - \frac{1}{2}\frac{1}{\beta}\right) \tag{D.18}$$

where

$$\begin{aligned}
B_0 &= \int_\eta^\infty (1 - t_1) \exp\left[-\beta \left(t_1^2 - \eta^2\right)\right] dt_1 \\
&= \sqrt{\frac{\pi}{\beta}} \exp\left(\beta \eta^2\right) Q\left(\sqrt{2\beta}\eta\right).
\end{aligned} \tag{D.19}$$

D.4 $N_I > 1$

For this case, we first try to separate the multiple integration over $z_{N_I-1}, \cdots, z_2, z_1$. According to [39, Chapter 3], for any bounded function $f(\cdot)$,

$$\begin{aligned} & \int_0^\infty \int_0^\infty \cdots \int_0^\infty \prod_{n=1}^{N_I-1} f(z_n) \left[\prod_{1\le i<j\le N_I-1} (z_i - z_j)^2 \right] dz_{N_I-1} \cdots dz_2 dz_1 \\ = & \ (N_I-1)! \int_0^\infty \int_0^\infty \cdots \int_0^\infty \prod_{n=1}^{N_I-1} f(z_n) \begin{vmatrix} 1 & z_2 & \cdots & z_{N_I-1}^{N_I-2} \\ z_1 & z_2^2 & \cdots & z_{N_I-1}^{N_I-1} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{N_I-2} & z_2^{N_I-1} & \cdots & z_{N_I-1}^{2N_I-4} \end{vmatrix} \\ & dz_{N_I-1} \cdots dz_2 dz_1 \end{aligned} \tag{D.20}$$

where $|\cdot|$ is the determinant of a matrix. The element on the ith row, j-th column of the matrix in (D.21) is z_j^{i+j-2}. Notice that all the elements on the j-th column of the matrix depend only on variable z_j.

Therefore (D.14) is equivalent to

$$\begin{aligned} & \sum_{m=1}^{N_I} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\ = & \ \frac{1}{2^{N_I}} K_1 z_0^{L-N_I} \int_0^\infty \int_0^\infty \cdots \int_0^\infty \\ & \left\{ \prod_{n=1}^{N_I-1} (z_0 - z_n) \exp(-\beta z_n) z_n^{L-N_I} \right\} \begin{vmatrix} 1 & z_2 & \cdots & z_{N_I-1}^{N_I-2} \\ z_1 & z_2^2 & \cdots & z_{N_I-1}^{N_I-1} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{N_I-2} & z_2^{N_I-1} & \cdots & z_{N_I-1}^{2N_I-4} \end{vmatrix} \\ & \left\{ \int_\eta^\infty (1 - t_{N_I}) \exp\left[-\beta \left(t_{N_I}^2 - \eta^2 \right) \right] \left[\prod_{n=1}^{N_I-1} \left(t_{N_I}^2 - \eta^2 - z_n \right) \right] dt_{N_I} \right\} \\ & dz_{N_I-1} \cdots dz_2 dz_1. \end{aligned} \tag{D.21}$$

Next we try to expand the products $\prod_{n=1}^{N_I-1} (z_0 - z_n)$ and $\prod_{n=1}^{N_I-1} \left(t_{N_I}^2 - \eta^2 - z_n \right)$. Obviously,

$$\prod_{n=1}^{1} (z_0 - z_n) = \sum_{p=0}^{1} (-1)^{1-p} z_0^p \sum_{\substack{m_1=0,1 \\ m_1=1-p}} z_1^{m_1} \tag{D.22}$$

$$\prod_{n=1}^{2} (z_0 - z_n) = \sum_{p=0}^{2} (-1)^{2-p} z_0^p \sum_{\substack{m_1,m_2=0,1 \\ m_1+m_2=2-p}} z_1^{m_1} z_2^{m_2}. \tag{D.23}$$

In (D.23), the second summation is taken over all the sets of (m_1, m_2) which satisfy the two restrictions: (1) each m_i $(i = 1, 2)$ must be 0 or 1; (2) The summation of all m_i's must be equal to $2 - p$.

Similarly, we have

$$\prod_{n=1}^{N_I-1} (z_0 - z_n) = \sum_{p=0}^{N_I-1} (-1)^{N_I-1-p} z_0^p \sum_{\substack{m_1+m_2+\cdots+m_{N_I-1}=N_I-1-p \\ m_i\in\{0,1\}}} z_1^{m_1} z_2^{m_2} \cdots z_{N_I-1}^{m_{N_I-1}} \tag{D.24}$$

$$\prod_{n=1}^{N_I-1} \left(t_{N_I}^2 - \eta^2 - z_n\right) = \sum_{q=0}^{N_I-1} (-1)^{N_I-1-q} \left(t_{N_I}^2 - \eta^2\right)^q \left[\sum_{\substack{n_1+n_2+\cdots+n_{N_I-1}=N_I-1-q \\ n_i\in\{0,1\}}} z_1^{n_1} z_2^{n_2} \cdots z_{N_I-1}^{n_{N_I-1}}\right]. \tag{D.25}$$

Substitute (D.24) and (D.25) in (D.21),

$$\begin{aligned}
&\sum_{m=1}^{N_I} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\
&= \frac{1}{2^{N_I}} K_1 z_0^{L-N_I} \sum_{p=0}^{N_I-1} (-1)^{N_I-1-p} z_0^p \sum_{q=0}^{N_I-1} (-1)^{N_I-1-q} \\
&\quad \sum_{\substack{m_1+m_2+\cdots+m_{N_I-1}=N_I-1-p \\ m_i\in\{0,1\}}} \; \sum_{\substack{n_1+n_2+\cdots+n_{N_I-1}=N_I-1-q \\ n_i\in\{0,1\}}} \\
&\quad \int_0^\infty \int_0^\infty \cdots \int_0^\infty z_1^{m_1+n_1} z_2^{m_2+n_2} \cdots z_{N_I-1}^{m_{N_I-1}+n_{N_I-1}} \\
&\quad \left\{\prod_{n=1}^{N_I-1} \exp\left(-\beta z_n\right) z_n^{L-N_I}\right\} \begin{vmatrix} 1 & z_2 & \cdots & z_{N_I-1}^{N_I-2} \\ z_1 & z_2^2 & \cdots & z_{N_I-1}^{N_I-1} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{N_I-2} & z_2^{N_I-1} & \cdots & z_{N_I-1}^{2N_I-4} \end{vmatrix} \\
&\quad \left\{\int_\eta^\infty \left(1 - t_{N_I}\right) \left(t_{N_I}^2 - \eta^2\right)^q \exp\left[-\beta\left(t_{N_I}^2 - \eta^2\right)\right] dt_{N_I}\right\} \\
&\quad dz_{N_I-1} \cdots dz_2 dz_1.
\end{aligned} \tag{D.26}$$

After combining $z_n^{m_n+n_n} \exp(-\beta z_n)\, z_n^{L-N_I}$ (for $n = 1, 2, \cdots, N_I - 1$) into the n-th column of matrix, (D.26) becomes

$$\sum_{m=1}^{N_I} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}$$

$$= \frac{1}{2^{N_I}} K_1 z_0^{L-N_I} \sum_{p=0}^{N_I-1} (-1)^{N_I-1-p} z_0^p \sum_{q=0}^{N_I-1} (-1)^{N_I-1-q}$$

$$\sum_{\substack{m_1+m_2+\cdots+m_{N_I-1}=N_I-1-p \\ m_i\in\{0,1\}}} \sum_{\substack{n_1+n_2+\cdots+n_{N_I-1}=N_I-1-q \\ n_i\in\{0,1\}}}$$

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \begin{vmatrix} U_{1,1} & U_{1,2} & \cdots & U_{1,N_I-1} \\ U_{2,1} & U_{2,2} & \cdots & U_{2,N_I-1} \\ \vdots & \vdots & \ddots & \vdots \\ U_{N_I-1,1} & U_{N_I-1,2} & \cdots & U_{N_I-1,N_I-1} \end{vmatrix}$$

$$\left\{ \int_\eta^\infty (1-t_{N_I}) \left(t_{N_I}^2 - \eta^2\right)^q \exp\left[-\beta\left(t_{N_I}^2-\eta^2\right)\right] dt_{N_I} \right\}$$

$$dz_{N_I-1} \cdots dz_2 dz_1. \tag{D.27}$$

where

$$U_{i,j} = \exp(-\beta z_j)\, z_j^{m_j+n_j+L-N_I+i+j-2}. \tag{D.28}$$

Since the elements on the j-th column of the matrix only depend on variable z_j, we could put the integration over z_j into the matrix and obtain

$$\sum_{m=1}^{N_I} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y}$$

$$= \frac{1}{2^{N_I}} K_1 z_0^{L-N_I} \sum_{p=0}^{N_I-1} (-1)^{N_I-1-p} z_0^p \sum_{q=0}^{N_I-1} (-1)^{N_I-1-q}$$

$$\sum_{\substack{m_1+m_2+\cdots+m_{N_I-1}=N_I-1-p \\ m_i\in\{0,1\}}} \sum_{\substack{n_1+n_2+\cdots+n_{N_I-1}=N_I-1-q \\ n_i\in\{0,1\}}} \det \mathbf{V}$$

$$\left\{ \int_\eta^\infty (1-t_{N_I}) \left(t_{N_I}^2 - \eta^2\right)^q \exp\left[-\beta\left(t_{N_I}^2-\eta^2\right)\right] dt_{N_I} \right\} \tag{D.29}$$

where

$$\det \mathbf{V} = \begin{vmatrix} V_{1,1} & V_{1,2} & \cdots & V_{1,N_I-1} \\ V_{2,1} & V_{2,2} & \cdots & V_{2,N_I-1} \\ \vdots & \vdots & \ddots & \vdots \\ V_{N_I-1,1} & V_{N_I-1,2} & \cdots & V_{N_I-1,N_I-1} \end{vmatrix} \tag{D.30}$$

and

$$V_{i,j} = \int_0^\infty \exp\left(-\beta z_j\right) z_j^{m_j+n_j+L-N_I+i+j-2} dz_j$$
$$= \frac{(m_j+n_j+L-N_I+i+j-2)!}{\beta^{m_j+n_j+L-N_I+i+j-2+1}}. \tag{D.31}$$

Substituting (D.31) in (D.30) and taking out the terms with β, we have

$$\det \mathbf{V} = \frac{1}{\beta^{\sum_{j=1}^{N_I-1}(m_j+n_j+L-N_I+j-2+1)+\sum_{i=1}^{N_I-1} i}} \begin{vmatrix} W_{1,1} & W_{1,2} & \cdots & W_{1,N_I-1} \\ W_{2,1} & W_{2,2} & \cdots & W_{2,N_I-1} \\ \vdots & \vdots & \ddots & \vdots \\ W_{N_I-1,1} & W_{N_I-1,2} & \cdots & W_{N_I-1,N_I-1} \end{vmatrix} \tag{D.32}$$

where

$$W_{i,j} = (m_j+n_j+L-N_I+i+j-2)!. \tag{D.33}$$

Since

$$\sum_{j=1}^{N_I-1}(m_j+n_j+L-N_I+j-2+1)+\sum_{i=1}^{N_I-1} i$$
$$= \sum_{j=1}^{N_I-1} m_j + \sum_{j=1}^{N_I-1} n_j + (L-N_I)(N_I-1) + \sum_{j=1}^{N_I-1} j - (N_I-1) + \sum_{i=1}^{N_I-1} i$$
$$= N_I - 1 - p + N_I - 1 - q + (L-N_I)(N_I-1)$$
$$+\frac{(N_I-1)N_I}{2} - (N_I-1) + \frac{(N_I-1)N_I}{2}$$
$$= (L+1)(N_I-1) - p - q, \tag{D.34}$$

$$\det \mathbf{V} = \frac{1}{\beta^{(L+1)(N_I-1)-p-q}} \det \mathbf{W}. \tag{D.35}$$

Next let's evaluate the final integration in (D.29). Separating the first term, then

$$\int_\eta^\infty (1-t_{N_I})\left(t_{N_I}^2-\eta^2\right)^q \exp\left[-\beta\left(t_{N_I}^2-\eta^2\right)\right] dt_{N_I}$$
$$= \int_\eta^\infty \left(t_{N_I}^2-\eta^2\right)^q \exp\left[-\beta\left(t_{N_I}^2-\eta^2\right)\right] dt_{N_I}$$
$$- \int_\eta^\infty t_{N_I}\left(t_{N_I}^2-\eta^2\right)^q \exp\left[-\beta\left(t_{N_I}^2-\eta^2\right)\right] dt_{N_I}$$
$$= \int_\eta^\infty \left(t_{N_I}^2-\eta^2\right)^q \exp\left[-\beta\left(t_{N_I}^2-\eta^2\right)\right] dt_{N_I}$$
$$-\frac{1}{2}\int_\eta^\infty \left(t_{N_I}^2-\eta^2\right)^q \exp\left[-\beta\left(t_{N_I}^2-\eta^2\right)\right] d\left(t_{N_I}^2-\eta^2\right)$$
$$= B_q - \frac{1}{2}\frac{q!}{\beta^{q+1}} \tag{D.36}$$

where

$$B_q = \int_{\eta}^{\infty} \left(t_{N_I}^2 - \eta^2\right)^q \exp\left[-\beta\left(t_{N_I}^2 - \eta^2\right)\right] dt_{N_I}, \tag{D.37}$$

which can be calculated recurrently as discussed in Appendix E.

Substituting (3.48), (D.35) and (D.36) in (D.29), we get

$$\begin{aligned}
&\sum_{m=1}^{N_I} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\
= &\frac{1}{\left[\prod_{i=1}^{N_I}(L-i)!\right]\left[\prod_{i=1}^{N_I}(N_I-i)!\right]} \beta^{LN_I} z_0^{L-N_I} \\
&\sum_{p=0}^{N_I-1} (-1)^{N_I-1-p} z_0^p \sum_{q=0}^{N_I-1} (-1)^{N_I-1-q} \left(B_q - \frac{1}{2}\frac{q!}{\beta^{q+1}}\right) \\
&\sum_{\substack{m_1+m_2+\cdots+m_{N_I-1}=N_I-1-p \\ m_i\in\{0,1\}}} \quad \sum_{\substack{n_1+n_2+\cdots+n_{N_I-1}=N_I-1-q \\ n_i\in\{0,1\}}} \\
&\frac{1}{\beta^{(L+1)(N_I-1)-p-q}} \det \mathbf{W}.
\end{aligned} \tag{D.38}$$

Define

$$\begin{aligned}
&H(p,q) \\
= &\frac{1}{\left[\prod_{i=1}^{N_I}(L-i)!\right]\left[\prod_{i=1}^{N_I}(N_I-i)!\right]} \\
&\sum_{\substack{m_1,m_2,\cdots,m_{N_I-1}=0,1 \\ m_1+m_2+\cdots+m_{N_I-1}=N_I-1-p}} \quad \sum_{\substack{n_1,n_2,\cdots,n_{N_I-1}=0,1 \\ n_1+n_2+\cdots+n_{N_I-1}=N_I-1-q}} \det \mathbf{W}.
\end{aligned} \tag{D.39}$$

Substituting in $z_0 = -\frac{1}{\gamma}$ and after some manipulations, we can simply the expression in (D.38) as

$$\begin{aligned}
&\sum_{m=1}^{N_I} \int f_m(\mathbf{y})\, p_{\mathbf{y}}(\mathbf{y}) d\mathbf{y} \\
= &\left(-\frac{1}{\gamma}\right)^{L-N_I} \beta^{L-N_I+1} \sum_{p=0}^{N_I-1}\sum_{q=0}^{N_I-1} (-1)^q H(p,q) \\
&\frac{1}{\gamma^p}\left(B_q - \frac{1}{2}\frac{q!}{\beta^{q+1}}\right)\beta^{p+q}
\end{aligned} \tag{D.40}$$

which is the expression we want to get.

If we assume $H(p,q) = \frac{1}{(L-1)!}$ for $N_I = 1$, then by setting $N_I = 1$, (D.40) is equal to (D.18). That means (D.40) could be used for any N_I.

Appendix E

Derivation of Series B_q

In this appendix, we derive the method to calculate the series B_q $(q = 0, 1, 2, \cdots)$, which is defined as

$$B_q = \int_{\eta}^{\infty} \exp\left(-\beta\left(t^2 - \eta^2\right)\right)\left(t^2 - \eta^2\right)^q dt. \tag{E.1}$$

In the following, we first derive the expressions for B_0 and B_1. Then we will show that for $q \geq 2$, B_q could be evaluated by B_{q-1} and B_{q-2}.

E.1 $q = 0$

$$B_0 = \int_{\eta}^{\infty} \exp\left(-\beta\left(t^2 - \eta^2\right)\right) dt = \exp\left(\beta\eta^2\right) \int_{\eta}^{\infty} \exp\left(-\beta t^2\right) dt. \tag{E.2}$$

Define $\frac{1}{2}z^2 = \beta t^2$, then $x = \frac{1}{\sqrt{2\beta}}z$, $z = \sqrt{2\beta}t$, and

$$\begin{aligned} B_0 &= \exp\left(\beta\eta^2\right) \int_{\eta}^{\infty} \exp\left(-\beta t^2\right) dt = \exp\left(\beta\eta^2\right) \int_{\sqrt{2\beta}\eta}^{\infty} \exp\left(-\frac{1}{2}z^2\right) \frac{1}{\sqrt{2\beta}} dz \\ &= \frac{1}{\sqrt{2\beta}} \exp\left(\beta\eta^2\right) \sqrt{2\pi} Q\left(\sqrt{2\beta}\eta\right) \\ &= \sqrt{\frac{\pi}{\beta}} \exp\left(\beta\eta^2\right) Q\left(\sqrt{2\beta}\eta\right). \end{aligned} \tag{E.3}$$

E.2 $q = 1$

$$\begin{aligned} B_1 &= \int_{\eta}^{\infty} \exp\left(-\beta\left(t^2 - \eta^2\right)\right)\left(t^2 - \eta^2\right) dt \\ &= \exp\left(\beta\eta^2\right)\left[\int_{\eta}^{\infty} \exp\left(-\beta t^2\right) t^2 dt - \eta^2 \int_{\eta}^{\infty} \exp\left(-\beta t^2\right) dt\right]. \end{aligned} \tag{E.4}$$

Define $\frac{1}{2}z^2 = \beta t^2$, then $t = \frac{1}{\sqrt{2\beta}}z$, $z = \sqrt{2\beta}t$, and

$$\begin{aligned}
B_1 &= \exp\left(\beta\eta^2\right)\left[\int_{\sqrt{2\beta}\eta}^{\infty} \exp\left(-\frac{1}{2}z^2\right)\frac{1}{2\beta}z^2\frac{1}{\sqrt{2\beta}}dz - \eta^2\int_{\sqrt{2\beta}\eta}^{\infty} \exp\left(-\frac{1}{2}z^2\right)\frac{1}{\sqrt{2\beta}}dz\right] \\
&= \frac{1}{\sqrt{2\beta}}\exp\left(\beta\eta^2\right)\left[\frac{1}{2\beta}\int_{\sqrt{2\beta}\eta}^{\infty} \exp\left(-\frac{1}{2}z^2\right)z^2dz - \eta^2\int_{\sqrt{2\beta}\eta}^{\infty} \exp\left(-\frac{1}{2}z^2\right)dz\right] \\
&= \frac{1}{\sqrt{2\beta}}\exp\left(\beta\eta^2\right)\left[\frac{1}{2\beta}\int_{\sqrt{2\beta}\eta}^{\infty} -z d\exp\left(-\frac{1}{2}z^2\right) - \eta^2\sqrt{2\pi}Q\left(\sqrt{2\beta}\eta\right)\right] \\
&= \frac{1}{\sqrt{2\beta}}\exp\left(\beta\eta^2\right) \\
&\quad \left\{\frac{1}{2\beta}\left[-z\exp\left(-\frac{1}{2}z^2\right)\Big|_{\sqrt{2\beta}\eta}^{\infty} + \int_{\sqrt{2\beta}\eta}^{\infty} \exp\left(-\frac{1}{2}z^2\right)dz\right] - \eta^2\sqrt{2\pi}Q\left(\sqrt{2\beta}\eta\right)\right\} \\
&= \frac{1}{\sqrt{2\beta}}\exp\left(\beta\eta^2\right)\left\{\frac{1}{2\beta}\left[\sqrt{2\beta}\eta\exp\left(-\beta\eta^2\right) + \sqrt{2\pi}Q\left(\sqrt{2\beta}\eta\right)\right] - \eta^2\sqrt{2\pi}Q\left(\sqrt{2\beta}\eta\right)\right\} \\
&= \frac{1}{\sqrt{2\beta}}\exp\left(\beta\eta^2\right)\left\{\frac{\eta}{\sqrt{2\beta}}\exp\left(-\beta\eta^2\right) + \sqrt{2\pi}\left[\frac{1}{2\beta} - \eta\right]Q\left(\sqrt{2\beta}\eta\right)\right\} \\
&= \frac{\eta}{2\beta} + \sqrt{\frac{\pi}{\beta}}\left[\frac{1}{2\beta} - \eta^2\right]\exp\left(\beta\eta^2\right)Q\left(\sqrt{2\beta}\eta\right). \qquad \text{(E.5)}
\end{aligned}$$

E.3 $q \geq 2$

In this case, we express B_q as a function of B_{q-1} and B_{q-2}. The details are as follows:

$$
\begin{aligned}
B_q &= \int_\eta^\infty \exp\left(-\beta\left(t^2-\eta^2\right)\right)\left(t^2-\eta^2\right)^{q-1}\left(t^2-\eta^2\right)dt \\
&= \int_\eta^\infty \exp\left(-\beta\left(t^2-\eta^2\right)\right)\left(t^2-\eta^2\right)^{q-1}t^2dt \\
&\quad -\eta^2\int_\eta^\infty \exp\left(-\beta\left(t^2-\eta^2\right)\right)\left(t^2-\eta^2\right)^{q-1}dt \\
&= \int_\eta^\infty \left(t^2-\eta^2\right)^{q-1}\frac{1}{-2\beta}td\exp\left(-\beta\left(t^2-\eta^2\right)\right)-\eta^2B_{q-1} \\
&= \left(t^2-\eta^2\right)^{q-1}\frac{1}{-2\beta}t\exp\left(-\beta\left(t^2-\eta^2\right)\right)\Big|_\eta^\infty \\
&\quad +\frac{1}{2\beta}\int_\eta^\infty \exp\left(-\beta\left(t^2-\eta^2\right)\right)\left[(q-1)\left(t^2-\eta^2\right)^{q-2}2t^2+\left(t^2-\eta^2\right)^{q-1}\right]dt-\eta^2B_{q-1} \\
&= \frac{1}{2\beta}\int_\eta^\infty \exp\left(-\beta\left(t^2-\eta^2\right)\right)\left[2(q-1)\left(t^2-\eta^2\right)^{q-2}\left(t^2-\eta^2+\eta^2\right)+\left(t^2-\eta^2\right)^{q-1}\right]dt \\
&\quad -\eta^2B_{q-1} \\
&= \frac{1}{2\beta}\int_\eta^\infty \exp\left(-\beta\left(t^2-\eta^2\right)\right)\left[(2q-1)\left(t^2-\eta^2\right)^{q-1}+2(q-1)\eta^2\left(t^2-\eta^2\right)^{q-2}\right]dt \\
&\quad -\eta^2B_{q-1} \\
&= \frac{1}{2\beta}\left[(2q-1)B_{q-1}+2(q-1)\eta^2B_{q-2}\right]-\eta^2B_{q-1} \\
&= \left[\frac{(2q-1)}{2\beta}-\eta^2\right]B_{q-1}+\frac{(q-1)}{\beta}\eta^2B_{q-2}. \qquad \text{(E.6)}
\end{aligned}
$$

Therefore B_q can be obtained from B_{q-1} and B_{q-2}.

Appendix F

Integration of I_0

In this appendix, we prove the following integration of zeroth order modified Bessel function:

$$\int_0^\infty te^{-\alpha t^2} I_0(2\sqrt{\beta}t)dt = \frac{1}{2\alpha}e^{\frac{\beta}{\alpha}} \tag{F.1}$$

Eq. (6.614.3) in [23] is

$$\int_0^\infty e^{-\alpha x} I_{2\nu}(2\sqrt{\beta x})dx = \frac{e^{\frac{1}{2}\frac{\beta}{\alpha}}}{\sqrt{\alpha\beta}} \frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)} M_{-\frac{1}{2},\nu}(\frac{\beta}{\alpha}) \tag{F.2}$$

Let $\nu = 0$,

$$\int_0^\infty e^{-\alpha x} I_0(2\sqrt{\beta x})dx = \frac{e^{\frac{1}{2}\frac{\beta}{\alpha}}}{\sqrt{\alpha\beta}} M_{-\frac{1}{2},0}(\frac{\beta}{\alpha})$$

Using equation of $M_{-\mu-\frac{1}{2},\mu}(x)$ in [40, P. 432],

$$\int_0^\infty e^{-\alpha x} I_0(2\sqrt{\beta x})dx = \frac{e^{\frac{1}{2}\frac{\beta}{\alpha}}}{\sqrt{\alpha\beta}} (\frac{\beta}{\alpha})^{\frac{1}{2}} e^{\frac{1}{2}\frac{\beta}{\alpha}} = \frac{1}{\alpha}e^{\frac{\beta}{\alpha}} \tag{F.3}$$

Let $x = t^2$,

$$\int_0^\infty e^{-\alpha x} I_0(2\sqrt{\beta x})dx = \int_0^\infty 2te^{-\alpha t^2} I_0(2\sqrt{\beta}t)dt = \frac{1}{\alpha}e^{\frac{\beta}{\alpha}} \tag{F.4}$$

It follows

$$\int_0^\infty te^{-\alpha t^2} I_0(2\sqrt{\beta}t)dt = \frac{1}{2\alpha}e^{\frac{\beta}{\alpha}} \tag{F.5}$$

Appendix G

Derivation of the Characteristic Function for MSDD

In this appendix, we derive the expression (5.32) for the characteristic function $\Phi_D(j\omega)$ of the MSDD test statistic D. To that end, using (5.24) and (5.30), we express the test statistic D in quadratic form:

$$D = \sum_{\ell=1}^{L} A_\ell^2 \left(|y_\ell(\mathbf{s}_k)|^2 - |y_\ell(\mathbf{s}_k')|^2 \right), \tag{G.1}$$

where

$$A_\ell = \sqrt{\frac{1}{\lambda_\ell (KP_s + \lambda_\ell)}} \tag{G.2}$$

for $\ell = 1, 2, \cdots, L$. Define

$$d_\ell = A_\ell^2 \left(|y_\ell(\mathbf{s}_k)|^2 - |y_\ell(\mathbf{s}_k')|^2 \right), \tag{G.3}$$

then

$$D = \sum_{\ell=1}^{L} d_\ell. \tag{G.4}$$

Also define vector $\mathbf{y}(\mathbf{s}_k) = [y_1(\mathbf{s}_k), y_2(\mathbf{s}_k), \cdots, y_L(\mathbf{s}_k)]^T$, then from (5.12) and (5.8), we have

$$\begin{aligned} \mathbf{y}(\mathbf{s}_k) &= \left(\sum_{i=0}^{K-1} \mathbf{x}_{k-i} s_{k-i}^* \right) \\ &= \mathbf{U}_z^H \left(\sum_{i=0}^{K-1} \mathbf{r}_{k-i} s_{k-i}^* \right). \end{aligned} \tag{G.5}$$

From the signal model in Section II, $E[\mathbf{y}(\mathbf{s}_k)] = E[\mathbf{y}(\mathbf{s}'_k)] = 0$. After some algebra, the covariance matrix of $\mathbf{y}(\mathbf{s}_k)$ can be evaluated as

$$\begin{aligned} E\left[\mathbf{y}(\mathbf{s}_k)\,\mathbf{y}(\mathbf{s}_k)^H\right] &= E\left[\mathbf{U}_z^H\left(\sum_{i=0}^{K-1}\mathbf{r}_{k-i}s_{k-i}^*\right)\left(\sum_{\ell=0}^{K-1}\mathbf{r}_{k-\ell}s_{k-\ell}^*\right)^H\mathbf{U}_z\right] \\ &= K^2P_s\mathbf{I}_L + K\Lambda_z. \end{aligned} \tag{G.6}$$

Similarly, we have the following results:

$$E\left[\mathbf{y}(\mathbf{s}'_k)\,\mathbf{y}(\mathbf{s}'_k)^H\right] = |v(\mathbf{s}_k,\mathbf{s}'_k)|^2 P_s\mathbf{I}_L + K\Lambda_z \tag{G.7}$$

$$E\left[\mathbf{y}(\mathbf{s}_k)\,\mathbf{y}(\mathbf{s}'_k)^H\right] = v^*(\mathbf{s}_k,\mathbf{s}'_k)\left(KP_s\mathbf{I}_L + \Lambda_z\right) \tag{G.8}$$

$$E\left[\mathbf{y}(\mathbf{s}'_k)\,\mathbf{y}(\mathbf{s}_k)^H\right] = v(\mathbf{s}_k,\mathbf{s}'_k)\left(KP_s\mathbf{I}_L + \Lambda_z\right), \tag{G.9}$$

where $v(\mathbf{s}_k,\mathbf{s}'_k) = \mathbf{s}_k'^H\mathbf{s}_k$.

To use the results in [7, Appendix B], we identify the following quantities using the notation in the reference: $X_\ell = y_\ell(\mathbf{s}_k)$, $Y_\ell = y_\ell(\mathbf{s}'_k)$. Then from (G.6) to (G.9), we have in the notation of the reference

$$\overline{X}_\ell = \overline{Y}_\ell = 0 \tag{G.10}$$

$$\mu_{xx,\ell} = \frac{1}{2}\left(K^2P_s + K\lambda_\ell\right) \tag{G.11}$$

$$\mu_{yy,\ell} = \frac{1}{2}\left(|v(\mathbf{s}_k,\mathbf{s}'_k)|^2 P_s + K\lambda_\ell\right) \tag{G.12}$$

$$\mu_{xy,\ell} = \frac{1}{2}v^*(\mathbf{s}_k,\mathbf{s}'_k)\left(KP_s + \lambda_\ell\right) \tag{G.13}$$

$$\mu_{yx,\ell} = \frac{1}{2}v(\mathbf{s}_k,\mathbf{s}'_k)\left(KP_s + \lambda_\ell\right). \tag{G.14}$$

Substitute the above equations into Eq. (B-6) and Eq. (B-5) in [7, Appendix B], together with $A = A_\ell^2$, $B = -A_\ell^2$, and $C_\ell = 0$. After some straightforward manipulations, we get the characteristic function of d_ℓ as

$$\phi_{d_\ell}(j\omega) = \frac{\theta_{1,\ell}\theta_{2,\ell}}{(\omega + j\theta_{1,\ell})(\omega - j\theta_{2,\ell})}. \tag{G.15}$$

where

$$\theta_{1,\ell} = \frac{1}{2\left[KP_s + \lambda_\ell\right]\lambda_\ell\zeta b_\ell^2}\left[\sqrt{\zeta^2P_s^2 + 4\left[KP_s + \lambda_\ell\right]\lambda_\ell\zeta} - \zeta P_s\right] \tag{G.16}$$

$$\theta_{2,\ell} = \frac{1}{2\left[KP_s + \lambda_\ell\right]\lambda_\ell\zeta b_\ell^2}\left[\sqrt{\zeta^2P_s^2 + 4\left[KP_s + \lambda_\ell\right]\lambda_\ell\zeta} + \zeta P_s\right], \tag{G.17}$$

and

$$\zeta = K^2 - |v(\mathbf{s}_k,\mathbf{s}'_k)|^2. \tag{G.18}$$

It follows that the characteristic function of D is

$$\Phi_D(j\omega) = \prod_{\ell=1}^{L} \phi_{d_\ell}(j\omega) = \prod_{\ell=1}^{L} \frac{1}{\left(1 - j\omega \frac{1}{\theta_{1,\ell}}\right)\left(1 + j\omega \frac{1}{\theta_{2,\ell}}\right)}. \tag{G.19}$$

For a system with a single interference source, the eigenvalues of the interference plus noise covariance matrix are $\lambda_\ell = \sigma^2$ for $\ell = 2, 3, \cdots, L$. It follows that $\theta_{1,\ell} = \theta_{1,2}$ and $\theta_{2,\ell} = \theta_{2,2}$ for $\ell = 2, 3, \cdots, L$. Hence the characteristic function can be expressed as

$$\Phi_D(j\omega) = \frac{1}{(1 - j\mu_1\omega)} \frac{1}{(1 - j\mu_2\omega)} \frac{1}{(1 - j\mu_3\omega)^{L-1}} \frac{1}{(1 - j\mu_4\omega)^{L-1}}, \tag{G.20}$$

where $\mu_1 = 1/\theta_{1,1}$, $\mu_2 = -1/\theta_{2,1}$, $\mu_3 = 1/\theta_{1,2}$, and $\mu_4 = -1/\theta_{2,2}$.

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