On the Complexity of Many Faces in Arrangements of Circles*

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Abstract

We obtain improved bounds on the complexity of m distinct faces in an arrangement of n circles and in an arrangement of n unit circles. The bounds are worst-case tight for unit circles, and, for general circles, they nearly coincide with the best known bounds for the number of incidences between m points and n circles.

1 Introduction

Problem statement and motivation. The *arrangement* $\mathcal{A}(\Gamma)$ of a finite collection Γ of curves or surfaces in $\mathbb{R}^d$ is the decomposition of the space into relatively open connected cells of dimensions $0, \ldots, d$ induced by Γ, where each cell is a maximal connected set of points lying in the intersection of a fixed subset of Γ and avoiding all other elements of Γ. The *combinatorial complexity* (or *complexity* for short) of a cell ϕ in $\mathcal{A}(\Gamma)$, denoted as $|\phi|$, is the number of faces of $\mathcal{A}(\Gamma)$ of all dimensions that lie on the boundary of ϕ. Besides being interesting in their own right, due to the rich geometric, combinatorial, algebraic, and topological structure that they possess, arrangements also lie at the heart of numerous geometric problems arising in a wide range of applications, including robotics, computer graphics, and molecular modeling. The study of arrangements of lines and hyperplanes has a long, rich history, but most of the work until the 1980s dealt with the combinatorial structure of the entire arrangement or of a single cell in the arrangement (which, in this case, is a convex polyhedron); see [14] for a summary of early work. Motivated by problems in computational and combinatorial geometry, various substructures of, and algorithmic issues involving arrangements of hyperplanes, and, more generally, of hypersurfaces, have received considerable attention, mostly during the last two decades; see [2] for a recent survey.

In this paper we study the so-called *many-faces* problem for arrangements of circles in the plane. More precisely, given a set C of n circles in $\mathbb{R}^2$ and a set P of m points, none lying on any circle, we wish to obtain an upper bound for the maximum possible combined combinatorial complexity of the cells of $\mathcal{A}(C)$ that contain at least one point of P, as a function of n and m. The study of the complexity of many faces, and the accompanying algorithmic problem of computing many faces, in planar arrangements (as studied, e.g., in [3, 13]) has several motivations: (i) It arises in a variety of problems involving 3-dimensional arrangements [6, 15]. (ii) It is closely related to the problem of bounding the number of *incidences* between points and curves. Informally, in both cases we have points and curves; in the case of incidences, the points lie on the curves and an incidence is a pair (p, γ), where point p lies on curve γ. In the case of many faces, the points lie "in between" the curves, and we are essentially interested in "extended incidences," involving pairs (p, γ), where point p can reach curve γ without crossing any other curve (i.e., γ appears on the boundary of the face containing p). The incidence problem for points and curves has attracted considerable attention in combinatorial and computational geometry. The problem of many faces is typically much harder than the (already quite hard) corresponding incidence problem.

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(iii) The many-faces problem is the "loosest" (i.e., least restricted) of all problems that study substructures in arrangements. It poses the biggest challenge because there is less structure to exploit. Tackling this problem has led to the derivation of various tools, like the Combination Lemma [20], which are interesting in their own right, and have many algorithmic applications (see, e.g., [1] for a recent such application).

Previous results. An early paper by Canham [8] initiated the study of the many-faces problem for line arrangements. After a number of intermediate results, the tight bounds on the complexity of many faces in line and pseudoline arrangements were obtained by Clarkson *et al.* [11], using the random-sampling technique. This paper and a series of subsequent ones, proved near-optimal or nontrivial bounds on the complexity of many faces in arrangements of line segments, circles, and curves in the plane, and in arrangements of hyperplanes in higher dimensions; see [2] and the references therein. The currently best known bound on the complexity of m distinct faces in an arrangement of n circles in the plane is $O(m^{3/5}n^{4/5}4^{\alpha(n)/5} + n)$. If all circles are congruent, then the bound is $O(m^{2/3}n^{2/3}\alpha^{1/3}(n)+n)$; here $\alpha(n)$ is the extremely slowly growing inverse of the Ackermann's function [20]. These bounds were obtained in [11].

As mentioned above, the many-faces problem is closely related to the incidence problem, which, given a set Γ of curves and a set P of points in the plane, asks for bounding the number of pairs $(p, \gamma) \in P \times \Gamma$ such that $p \in \gamma$. For example, the tight bounds on the maximum number of incidences between points and lines (or segments, or pseudolines) are asymptotically the same as the maximum complexity of m distinct faces in an arrangement of n lines, viz., $\Theta(m^{2/3}n^{2/3} + m + n)$ [11]. (We note, though, that the best known bound for the complexity of many faces in an arrangement of line segments is slightly weaker [5].) The same has been true for arrangements of circles (except for the tiny $4^{\alpha(n)/5}$ factor in the leading term), until recently, when Aronov and Sharir [7] obtained an improved bound on the number of incidences between points and circles, showing that this number is $O(m^{2/3}n^{2/3} + m^{6/11+3\varepsilon}n^{9/11-\varepsilon} + m + n)$, for any $\varepsilon > 0$. They raised the question whether a similar bound can be obtained for the complexity of many faces in circle arrangements, which, after the cases of lines, segments, and pseudolines, is the next natural problem instance to be tackled.

Our results. In this paper we (almost) answer the question raised by Aronov and Sharir affirmatively. Let C be a set of n circles in the plane and P a set of m points. We will use $K(P, C)$ to denote the combined combinatorial complexity of the faces of $\mathcal{A}(C)$ that contain at least one point of P. Set $K(m, n) = \max K(P, C)$, where the maximum is taken over all families of n circles and all families of m points. We prove that $K(m, n)$ is

$$O\left((mn\log n)^{2/3} + m^{6/11+\varepsilon}n^{9/11} + (m+n)\log n\right),$$

where $\varepsilon > 0$ is an arbitrarily small constant.

Let $K'(m, n)$ denote the maximum value of $K(P, C)$ with the added assumption that all pairs of circles intersect. In this case, we obtain the following improved bound on $K'(m, n)$:

$$O\left((mn\log n)^{2/3} + m^{1/2}n^{5/6}\log^{3/4} n + (m+n)\log n\right).$$

If not all pairs of circles intersect, we obtain a bound that depends on X, the number of intersecting pairs of circles. Let $K(m, n, X) = \max K(P, C)$, where the maximum is taken over all families P of m points and C of n circles with X intersecting pairs. We prove that

$$K(m, n, X) = O\Big(m^{2/3}X^{1/3}\log^{2/3} n + m^{6/11+\varepsilon}X^{4/11}n^{1/11} + (m+n)\log n\Big),$$

where $\varepsilon > 0$ is an arbitrarily small constant. This bound is nearly the same as the new bound for incidences of [7], apart from small, logarithmic factors.

Our general technique is similar to the one used in [7], i.e., we first prove a weaker bound, which is almost optimal for large values of m, by cutting the circles of C into "pseudo-segments." Next, to handle small values of m, we use a partitioning scheme in the "dual space," decompose the problem into many subproblems, bound the complexity for each subproblem using the weaker bound, and estimate the increase in complexity as we merge the subproblems. However, several new ideas are needed to carry out each of these steps. For instance, merging subproblems is trivial for incidences—just add up the bounds of each subproblem. In contrast, it is a rather difficult and intricate step for face complexities.

Finally, for the case where all circles in C are congruent (the case of "unit circles"), we show that the complexity of m distinct faces in an arrangement of n congruent circles with X intersecting pairs, is $O(m^{2/3}X^{1/3} + n)$. This bound is asymptotically tight in the worst case, in contrast with the same asymptotic upper bound for the case of incidences, which is far away from the known, near-linear lower bound. Note that the improvement here is rather marginal—we only remove the factor $\alpha(n)^{1/3}$, appearing in the previous bound of [11].

The paper is organized as follows. We first prove in Section 2 the bound for congruent circles. Section 3 proves the bounds for general circles. We conclude in Section 4 by discussing a few generalizations and by stating some open problems.

2 The Case of Unit Circles

In this section we prove an optimal bound on the maximum complexity of many faces in an arrangement of unit circles in the plane. We first state the so-called *crossing lemma* (see, e.g., [19]), which we will use to prove the main result, as in [7]. A simple graph is said to be *drawn* in the plane if its vertices are mapped to distinct points in the plane, and each of its edges is mapped to a Jordan arc connecting the points corresponding to the end vertices of the edge. We further require that no curve passes through any other vertex and that each pair of curves meet a finite number of times. A *crossing* between two curves is a point at which their relative interiors intersect transversally. An *edge-crossing* in (the drawing of) the graph is a pair of crossing edges.

Lemma 2.1. *Any plane drawing of a simple graph G with e edges and n vertices must have $\Omega(e^3/n^2)$ edge-crossings, provided that $e \geq 4n$. Equivalently, if G can be drawn in the plane with X edge-crossings, then $e = O(n^{2/3}X^{1/3} + n)$.*

We now state and prove the main result of this section.

Theorem 2.2. *The combined combinatorial complexity of m distinct faces in an arrangement of n unit circles in the plane, X pairs of which intersect, is $O(m^{2/3}X^{1/3} + n)$. This bound is tight in the worst case.*

Proof. Let C be a collection of n unit circles in the plane and P a collection of m points marking (lying in the interior of) distinct faces in $\mathcal{A}(C)$. We aim to bound the total complexity $K = K(P, C)$ of the marked faces. Note that $m = O(X + n)$, as the total number of faces in the arrangement is at most $2X + n + 1$, since the rightmost vertex of every bounded face is either one of the at most $2X$ arrangement vertices or one of the n rightmost points of the circles, and each point can be used only once in this manner. In the remainder of the proof we assume, without loss of generality, that the union of the circles of C is connected, so $X = \Omega(n)$ and $m = O(X)$. The analysis can easily be extended to the case in which the union is disconnected.

The analysis begins in a manner similar to that for the case of a line arrangement, as presented in [12]. We fix a point q_e in the interior of every edge e of a marked face. For any circle $\gamma \in C$, we distinguish between faces touching γ "from the inside" and those that touch γ "from the outside." We construct two separate (multi)graphs G_- and G_+ to encode the two types of occurrences of a circle along a face boundary.

More precisely, suppose that $\gamma \in C$ encloses two distinct faces f_1, f_2 in its interior, and appears along their boundaries in two respective edges, e_1, e_2, of $\mathcal{A}(C)$, so that at least one of the two arcs of γ delimited by q_{e_1} and q_{e_2} contains no other edge of a marked face enclosed by γ; call this arc $[q_{e_1}, q_{e_2}]$. We connect the corresponding marking points p_1, p_2 by an edge in G_-, drawn as follows: p_1 is connected by a Jordan arc (see below) to q_{e_1}, then q_{e_1} is connected to q_{e_2} by the arc $[q_{e_1}, q_{e_2}]$ of γ, and finally q_{e_2} is connected by another Jordan arc to p_2. The Jordan arcs connecting the marking point p of a face f to its edges are chosen so that their relative interiors lie completely in the interior of f, and they do not cross one another. G_+ is constructed analogously, encoding the edges where faces touch a circle on the outside. See Figure 1 for an illustration.

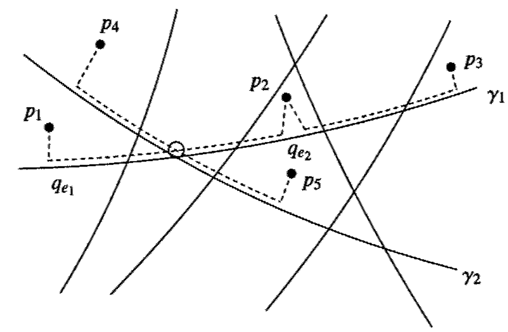

Figure 1: Three edges of the graph G_-. The edge (p_1, p_2) connecting p_1 and p_2 along the inner side of the circle γ_1 and the edge (p_4, p_5) connecting p_4 to p_5 along the inner side of γ_2 cross at an intersection point of γ_1 and γ_2.

A *face-circle incidence* is a pair (f, γ) where f is a marked face and γ is a circle appearing along ∂f. Let $I = I(P, C)$ be the total number of such incidences—note that it may in general be strictly smaller than $K = K(P, C)$, as it does not count multiple appearances of the same circle along a face boundary. Nevertheless, we have $K \leq 2I$, since the complexity of a single face in an arrangement of circles is at most $2\ell - 2$ if ℓ distinct circles appear along its boundary [20, Theorem 5.7]. To summarize, $I \leq K \leq 2I$, so we will somewhat freely switch between I and K (whenever it is safe to do so).

Let γ be a circle in C. We denote by $\sigma_- = (s_1^-, s_2^-, \ldots, s_\ell^-)$ the circular sequence of marked faces that lie in the interior of γ and such that γ appears on their boundary, in the order that these boundaries appear along γ, say in clockwise direction, with the additional provision that a maximal "run" of repeated appearances of the boundary of the same face along γ with no intervening appearances of other marked faces (enclosed by γ) is compressed in σ_- to a single symbol. We assume that γ appears on the boundary of at least three distinct marked faces enclosed by it (the remaining circles contribute at most $2n$ to the overall "inner" face-circle incidence count, and can thus be ignored). We denote by σ_+ the analogous sequence of faces that lie in the exterior of γ, provided that at least three distinct marked faces appear there.

The combined length of the sequences σ_-, σ_+, over all circles γ, is exactly $|G_-|+|G_+| \le K$, since, by construction, each graph edge represents the "connection" between the occurrences of two different consecutive faces along γ and thus is also equivalently encoded by a pair of consecutive elements in the corresponding sequence σ_-, σ_+. On the other hand, the number of *distinct* symbols appearing in each sequence, summed over all sequences, is exactly $I \ge K/2$, so it is sufficient to obtain an upper bound for $|G_-| + |G_+|$.

The analysis of Clarkson *et al.* [11] implies that the multiplicity of any edge of G_- is at most two. Actually, a stronger property holds: It is impossible for two distinct faces to touch three distinct unit circles on their interior sides. Hence, Lemma 2.1 implies that $|G_-|$ is $O(m^{2/3}X_-^{1/3} + m)$, where X_- is the number of edge-crossings in G_-. Since by construction an edge-crossing in G_- is also a crossing of a pair of circles in C and no two edge-crossings can use the same intersection point of the same pair of circles, it follows that $X_- \le 2X$ and $|G_-| = O(m^{2/3}X^{1/3} + m) = O(m^{2/3}X^{1/3})$ (the latter estimate follows from the fact that $m = O(X)$).

Handling the graph G_+ is somewhat more involved. It is shown in [11] that G_+ can be manipulated as follows. We first disregard the faces of the arrangement that lie outside all circles of C, if any of them are marked, because they can contribute at most $6n - 12$ (for $n \ge 3$) to K [18]. Each remaining marked face is enclosed by at least one circle of C and thus has diameter at most 2. We overlay the arrangement of the circles of C with the unit grid. Each circle meets the grid lines at most 8 times, so the total number of circle arcs of the form $[q_{e_1}, q_{e_2}]$ that are part of the drawing of G_+ and are met by the grid lines is at most $8n$—we remove the edges corresponding to these arcs from G_+. It can now be shown (adapting the analysis given in [11]) that in what remains of G_+ the edge multiplicities are all bounded by a constant, so we can apply an analysis similar to that above to conclude that $|G_+|$, and thus also the overall face-circle incidence count, are $O(m^{2/3}X^{1/3} + m + n) = O(m^{2/3}X^{1/3} + n)$ (the latter estimate follows, as above, from the fact that $m = O(X)$). This completes the proof of the bound asserted in Theorem 2.2.

To see that the bound is tight in the worst case, take an arrangement of n lines which has m faces whose combined complexity is $\Theta(m^{2/3}n^{2/3}+n)$ (see [19] for details). We can then "bend" the lines slightly into large but congruent circles without changing the combinatorial structure of any face. This shows that the bound is worst-case tight when $X = \Theta(n^2)$. For smaller values of X, put $k = \lceil n^2/X \rceil$, and take k copies of the preceding construction, placed far away from each other, each involving n/k circles and m/k faces, of combined complexity (within a single copy)

$$\Theta\left(\left(\frac{m}{k}\right)^{2/3}\left(\frac{n}{k}\right)^{2/3} + \frac{n}{k}\right).$$

Together, we have n congruent circles and m faces in their arrangement. The number of intersecting pairs is at most $k \cdot (n/k)^2 = X$, and the overall complexity of all the marked faces is

$$\begin{aligned} k \cdot \Theta\left(\left(\frac{m}{k}\right)^{2/3}\left(\frac{n}{k}\right)^{2/3} + \frac{n}{k}\right) &= \Theta\left(\frac{m^{2/3}n^{2/3}}{k^{1/3}} + n\right) \\ &= \Theta(m^{2/3}X^{1/3} + n). \end{aligned}$$

□

3 The Case of General Circles

Let C be a set of n circles in the plane with X intersecting pairs, and let P be a set of m points, not lying on any circle. In this section, we obtain a bound on $K(P, C)$. The proof proceeds in three stages. We first cut the circles in C into pseudo-segments. Next, we use the known results on the complexity of many faces in arrangements of pseudolines and the so-called Combination Lemma, to bound $K(P, C)$. This bound is near-optimal for large values of m, but is weak for smaller values. Finally, we use a different technique (based on decomposing the problem into smaller subproblems via cuttings in dual space) to improve the bound for small values of m.

3.1 Cutting circles into pseudo-segments

A planar collection Γ of bounded Jordan arcs (resp., unbounded Jordan curves, each separating the plane) called a family of *pseudo-segments* (resp. *pseudolines*) if every pair of them intersects in at most one point, where they cross each other. Tamaki and Tokuyama [21] had shown that a set of "pseudo-parabolic" arcs can be decomposed into $O(n^{5/3})$ pseudo-segments. The following improved bound is due to Aronov and Sharir [7].

Lemma 3.1. *Any n circles in the plane, with X intersecting pairs, can be cut into $O(n^{1/2-\varepsilon}X^{1/2+\varepsilon} + n)$ x-monotone pseudo-segments, for any arbitrarily small constant $\varepsilon > 0$.*

If every two circles intersect each other, an improved bound has recently been obtained by Agarwal *et al.* [4].

Lemma 3.2. *Any n pairwise-intersecting circles can be cut into $O(n^{4/3})$ x-monotone pseudo-segments.*

A collection Γ of x-monotone pseudo-segments is called *extendible* if each arc can be extended to an unbounded x-monotone curve (the graph of a totally defined continuous function), so that the resulting collection is a family of pseudolines. In general (even for circular arcs), pseudo-segments need not be extendible; see Figure 2 (i) for an illustration. Nevertheless, Chan [9] showed that if we store a collection Γ of x-monotone pseudo-segments in a segment tree and split each of them into up to $O(\log n)$ pieces, according to the way it is stored in the tree, the resulting collection of subarcs forms a family of extendible pseudo-segments; see Figure 2 (ii). Combining this result with Lemmas 3.1 and 3.2, we obtain the following.

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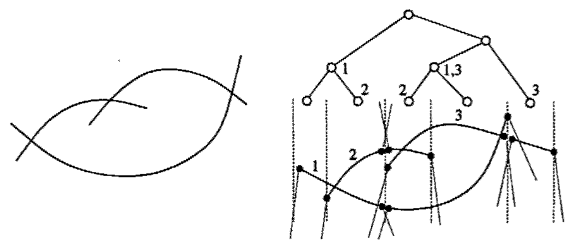

(i) (ii)

Figure 2: (i) Three pseudo-segments that are not extendible. (ii) Decomposing them into extendible pseudo-segments in a segment-tree fashion; dotted lines denote (roughly) the endpoints of the atomic intervals of the tree, at which the extension of the pseudo-segments to pseudolines takes place.

Corollary 3.3. *Any set C of n circles in the plane, with X intersecting pairs, can be cut into $O(n^{1/2-\varepsilon}X^{1/2+\varepsilon}+n\log n)$ extendible x-monotone pseudo-segments, for any arbitrarily small constant $\varepsilon > 0$. If every pair of circles in C intersect, then the bound can be improved to $O(n^{4/3}\log n)$.*

3.2 A weaker bound

Using Lemma 3.1, we cut the circles in C into a set of x-monotone pseudo-segments and then use Chan's procedure to cut them further into a set Γ of N extendible pseudo-segments. By Corollary 3.3, $N = O(n^{1/2-\varepsilon}X^{1/2+\varepsilon} + n\log n)$. Obviously, $K(P,C) \le K(P,\Gamma)$.

Chan's procedure constructs a segment tree $\mathcal{T}$ on the initial collection of pseudo-segments. For a node $v \in \mathcal{T}$, let σ_v be the vertical strip associated with v. By slightly displacing the points of P, and/or by slightly rotating the coordinate frame, we may assume that no point of P lies on a strip boundary. Let Γ_v be the set of pseudo-segments (clipped within σ_v) stored at v. The endpoints of all arcs in Γ_v lie on the boundary of σ_v; see Figure 2 (ii). Set $P_v = P \cap \sigma_v$, $n_v = |\Gamma_v|$, and $m_v = |P_v|$. Let X_v be the number of intersecting pairs in Γ_v.

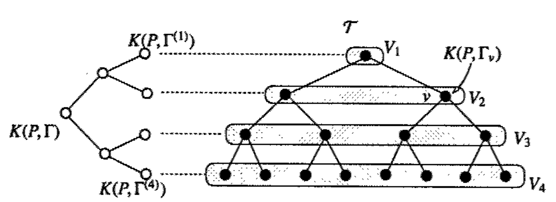

Figure 3: Using the segment tree $\mathcal{T}$ to generate subproblems from the initial (P,Γ); the rotated tree on the left shows how the various subproblems are merged together.

Consider the arrangement $\mathcal{A}(\Gamma_v)$. A face of $\mathcal{A}(\Gamma_v)$ that is marked by any point of $P \setminus P_v$ must be the (unique) unbounded face of $\mathcal{A}(\Gamma_v)$, and the complexity of this face is $O(\lambda_3(n_v)) = O(n_v\alpha(n_v))$ [20]. Therefore, $K(P,\Gamma_v) = O(n_v\alpha(n_v)) + K(P_v,\Gamma_v)$. From the point of view of P_v, we can regard Γ_v as a family of pseudolines, viewing the boundaries of σ_v as "vertical lines at infinity." (We may assume that no point of P lies in the unbounded face, since this face has already been taken into account.) By adapting the results and the proof technique in [12], similar to the way in which they have been exploited in the preceding section, we have $K(P_v,\Gamma_v) = O(m_v^{2/3}X_v^{1/3} + n_v)$, which implies that

$$K(P,\Gamma_v) = O\left(m_v^{2/3}X_v^{1/3} + n_v\alpha(n_v)\right).$$

Let V_h denote the set of all nodes of $\mathcal{T}$ at a fixed level h; see Figure 3. Since the strips spanned by these nodes are pairwise (openly) disjoint, we have $\sum_{v\in V_h} m_v = m$. Put $\Gamma^{(h)} = \bigcup_{v\in V_h}\Gamma_v$, $N_h = |\Gamma^{(h)}| = \sum_{v\in V_h} n_v$, and $X_h = \sum_{v\in V_h} X_v$. The disjointness of the strips σ_v, for $v \in V_h$, implies that

$$K(P,\Gamma^{(h)}) \le \sum_{v\in V_h} K(P,\Gamma_v).$$

Indeed, since each arc of $\Gamma^{(h)}$ lies within one strip σ_v, for $v \in V_h$, it follows that $\mathcal{A}(\Gamma^{(h)}) \cap \sigma_v = \mathcal{A}(\Gamma_v)$ for any $v \in V_h$, and every edge of $\mathcal{A}(\Gamma^{(h)})$ lies within a single strip. Let f be a marked face in $\mathcal{A}(\Gamma^{(h)})$, and let e be an edge of f. If e lies in a strip σ_v, then $e \in \mathcal{A}(\Gamma_v)$. Moreover $\Gamma_v \subseteq \Gamma^{(h)}$, therefore e lies in a face of $\mathcal{A}(\Gamma_v)$ marked by a point of P (which needs not lie in the same strip as e). Consequently, e is counted by $K(P,\Gamma_v)$, thereby implying the above equality.

Using Hölder's inequality, we have:

$$\begin{aligned} K(P,\Gamma^{(h)}) &= \sum_{v\in V_h} O(m_v^{2/3}X_v^{1/3} + n_v\alpha(n_v)) \\ &= O\left((\sum_{v\in V_h} m_v)^{2/3}(\sum_{v\in V_h} X_v)^{1/3} + N_h\alpha(n)\right) \\ &= O(m^{2/3}X_h^{1/3} + N_h\alpha(n)). \end{aligned}$$

We now overlay the marked faces in $\mathcal{A}(\Gamma^{(1)}), \mathcal{A}(\Gamma^{(2)}), \ldots, \mathcal{A}(\Gamma^{(\xi)})$, where $\xi = O(\log n)$ is the number of levels in $\mathcal{T}$, and estimate the complexity of the marked faces in their overlay, using the *combination lemma* of Edelsbrunner *et al.* [13], which provides an upper bound on the complexity of marked faces in an overlay of two arrangements of line segments. A close inspection of the proof of [13] shows that it also holds for arrangements of *extendible* pseudo-segments. We state the lemma for extendible pseudo-segments without a proof (which is easily derivable from the analysis in [13]).

Lemma 3.4 (Combination Lemma [13]). *The complexity of the faces marked by m points in an overlay of two arrangements of a total of N (extendible) pseudo-segments is at most $O(m+N)$ plus the sum of the complexities of the faces containing the marking points in each of the two arrangements.*

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In view of the above lemma,

$$K(P,\Gamma) = K\Big(P, \bigcup_{h=1}^{\xi} \Gamma^{(h)}\Big)$$

$$\leq K\Big(P, \bigcup_{h=1}^{\xi/2} \Gamma^{(h)}\Big) + K\Big(P, \bigcup_{h=\xi/2+1}^{\xi} \Gamma^{(h)}\Big) + O(m+N).$$

Since $\xi = O(\log n)$, the depth of the merge tree depicted on the left-hand side of Figure 3 is $O(\log\log n)$. Note that, by definition, $\sum_h N_h = N$. Moreover, we have $\sum_h X_h \leq 2X$, which follows from the fact that any intersection point between two circles is counted at most once in the left-hand side. Hence, we obtain the following estimate.

$$K(P,C) \leq \sum_{h=1}^{\xi} K(P,\Gamma^{(h)}) + O(m\log n + N\log\log n)$$

$$= \sum_{h=1}^{\xi} O(m^{2/3}X_h^{1/3} + N_h\alpha(n))$$

$$+ O(m\log n + N\log\log n)$$

$$= O(m^{2/3}X^{1/3}\log^{2/3} n + m\log n + N\log\log n).$$

By substituting the value of N in the above equation, by bounding X by n^2, and by letting the factor n^ε subsume the factor $\log\log n$), we conclude the following.

Theorem 3.5. *The maximum complexity of m distinct faces in an arrangement of n circles in the plane is*

$$K(m,n) = O((mn\log n)^{2/3} + n^{3/2+\varepsilon} + m\log n),$$

for any $\varepsilon > 0$.

If every pair of circles intersect, then, by Corollary 3.3, $N = O(n^{4/3}\log n)$. We thus obtain the following:

Theorem 3.6. *The maximum complexity of m distinct faces in an arrangement of n pairwise-intersecting circles in the plane is*

$$O((mn\log n)^{2/3} + n^{4/3}\log n\log\log n + m\log n). \qquad (1)$$

3.3 An improved bound

In Theorem 3.5, the term $n^{3/2+\varepsilon}$ becomes dominant when m is smaller than roughly $n^{5/4}$. In order to obtain an improved bound for small values of m, we (i) choose a parameter r, depending on the values of n and m, (ii) partition C into $O(r^3)$ subsets, each of size at most n/r^3, so that the points of P lie in at most m/r distinct faces of the arrangement of each subset, excluding faces in the common exterior or in the common interior of the circles in the subset, (iii) use Theorem 3.5 to bound the complexity of the faces in question in each subarrangement, and (iv) analyze the cost of overlaying all the subarrangements. Although this technique is similar in spirit to an analogous approach used in [7] for the case of incidences, it is considerably more involved when analyzing the complexity of many faces.

Cuttings. Although the following discussion holds in any dimension d, we only need it for $d = 3$, so, for simplicity, we confine the discussion to the three-dimensional case.

Let H be a set of m planes in $\mathbb{R}^3$, and let S be a set of n points in $\mathbb{R}^3$. For a simplex Δ, we use $H_\Delta \subset H$ to denote the set of planes that cross (meet the interior of) Δ, and S_Δ to denote $S \cap \Delta$. Set $m_\Delta = |H_\Delta|$ and $n_\Delta = |S_\Delta|$. Let k_Δ be the number of vertices of $\mathcal{A}(H)$ that lie inside Δ.

Let $1 \leq r \leq m$ be a parameter and Δ a simplex. A simplicial subdivision Ξ of Δ is called a $(1/r)$*-cutting* of H (with respect to Δ) if at most m/r planes of H cross any simplex of Ξ. We will use Chazelle's hierarchical cuttings [10] to compute a $(1/r)$-cutting Ξ of H, but with the additional twist that each simplex of the cutting contains at most n/r^3 points of S. We sketch the procedure for computing such a cutting.

We choose a sufficiently large constant r_0 and set $\nu = \lceil \log_{r_0} r \rceil$. We compute a sequence of cuttings $\Xi_0, \Xi_1, \ldots \Xi_\nu = \Xi$, where Ξ_i is a $(1/r_0^i)$-cutting of H and each simplex of Ξ_i contains at most n/r_0^{3i} points of S. Ξ_0 is simply Δ itself. Suppose we have computed $\Xi_0, \ldots, \Xi_{i-1}$. Let τ be a simplex of Ξ_{i-1}. If $m_\tau \leq m/r_0^i$, we add τ to Ξ_i. Otherwise, we compute a $(1/r_0)$-cutting Ξ_i^τ of H_τ (within τ) of size $c(r_0^2 + (k_\tau/m_\tau^3)r_0^3)$, for some absolute constant $c > 1$), as proposed by Chazelle [10], and add the simplices of Ξ_i^τ to Ξ_i. Finally, if a simplex of Ξ_i contains more than n/r_0^{3i} points of S, we partition it further into subsimplices, so that each resulting simplex contains at most n/r_0^{3i} points. The last step adds a total of at most r_0^{3i} simplices.

It is obvious that Ξ_ν is a $(1/r)$-cutting of H. As for the size of Ξ_i, we have, using the fact that $\sum_\tau k_\tau \leq m^3$,

$$|\Xi_i| \leq r_0^{3i} + |\Xi_{i-1}| + \sum_{\substack{\tau\in\Xi_{i-1} \\ m_\tau > m/r_0^i}} c\left(r_0^2 + \frac{k_\tau}{m_\tau^3} r_0^3\right)$$

$$\leq (1 + cr_0^2)|\Xi_{i-1}| + cr_0^{3i}\left(1 + r_0^3 \sum_\tau \frac{k_\tau}{m^3}\right)$$

$$\leq (1 + cr_0^2)^i|\Xi_0| + c(1 + r_0^3)\sum_{j=1}^{i} r_0^{3j}$$

$$\leq c' r_0^{3i} \qquad (2)$$

for an appropriate constant c' (which also depends on r_0). Since $|\Xi_0| = 1$ and $\nu = \lceil \log_{r_0} r \rceil$, a simple calculation shows that the size of the final cutting, Ξ, is $O(r^3)$.[1]

Problem decomposition. Returning to the problem of bounding $K(P,C)$, we use the above cutting to decompose

[1]If we do not require each simplex of Ξ_ν to contain at most n/r^3 points, then the size of the cutting will be $O(r^{2+\varepsilon} + (k_\Delta/m^3)r^3)$, as proved by Chazelle.

the problem of estimating $K(P, C)$ into subproblems, each involving appropriate subsets of P and C. We use the standard lifting transformation, as in [7], to map circles to points, and points to planes, in $\mathbb{R}^3$: A circle γ of radius ρ and center (a, b) in the plane is mapped to the point $\gamma^* = (a, b, a^2 + b^2 - \rho^2) \in \mathbb{R}^3$, and a point $p(\xi, \eta)$ in the plane is mapped to the plane $p^* : z = 2\xi x + 2\eta y - (\xi^2 + \eta^2)$ in $\mathbb{R}^3$. As is easily verified, a point p lies on (resp., inside, outside) a circle γ if and only if the dual plane p^* contains (resp., passes above, below) the dual point γ^*. Let P^* denote the set of planes dual to the points of P and let C^* denote the set of points dual to the circles of C. No three planes of P^* pass through a common line, as all planes of P^* are tangent to the paraboloid $\Pi : z = x^2 + y^2$.

Applying the above cutting procedure to P^*, C^*, and to a sufficiently large simplex that contains C^* and all vertices of $\mathcal{A}(P^*)$, with a value of r that will be fixed later, we obtain a $(1/r)$-cutting Ξ so that, for every simplex $\Delta \in \Xi$, we have $|P^*_\Delta| \le m/r$ and $|C^*_\Delta| \le n/r^3$; put $m_\Delta = |P^*_\Delta|$ and $n_\Delta = |C^*_\Delta|$. For a simplex $\Delta \in \Xi$, let C_Δ be the subset of circles in C that are dual to the points of C^*_Δ, and let P_Δ denote the set of points of P dual to the planes of P^*_Δ. We have $\sum_\Delta n_\Delta = n$. We define similar quantities for the intermediate cuttings Ξ_i.

For a point p not lying on any circle in C, let f_p denote the face of $\mathcal{A}(C)$ that contains p. For a subset $P' \subseteq P$ and a subset $C' \subset C$, we define $I(P', C')$ to be the number of pairs $(p, \gamma) \in P' \times C'$ such that an arc of γ appears on ∂f_p. Note that f_p is defined as a face of $\mathcal{A}(C)$ and not as a face of $\mathcal{A}(C')$; it is in fact a subset of the face of $\mathcal{A}(C')$ that contains p. Clearly, $I(P, C)$ is the same as defined in Section 2. As in the case of unit circles, we have $I(P, C) \le K(P, C) \le 2I(P, C)$. (Note, though, that this inequality need not hold when passing to subsets $P' \subset P$, $C' \subset C$.) It therefore suffices to bound $I(P, C)$.

Bounding $I(P, C)$. We will follow the notation introduced earlier for computing a $(1/r)$-cutting. By definition, the quantities $I(P', C')$ are additive, in the following sense:[2]

$$I(P, C) = \sum_{\Delta \in \Xi} (I(P_\Delta, C_\Delta) + I(P \setminus P_\Delta, C_\Delta)). \qquad (3)$$

Instead of bounding the right-hand side directly, we use a recursive approach, using the fact that Ξ was constructed hierarchically by computing a sequence of cuttings $\Xi_0, \Xi_1, \ldots, \Xi_\nu = \Xi$. We first prove the following lemma.

Lemma 3.7. *Let Δ be a simplex in one of the cuttings Ξ_i. Let $P' \subset P$ be a subset of the marking points. Then $I(P' \setminus P_\Delta, C_\Delta) = O(|P'| + n_\Delta)$.*

[2] In contrast, the quantities $K(P', C')$ are in general not additive, and they have to be combined by using a combination lemma, as in the preceding subsection; see [20] for details.

Proof. For any point $p \in P' \setminus P_\Delta$, the dual plane p^* does not intersect the simplex Δ. If p^* lies below (resp. above) Δ, and therefore below (resp. above) all points of C^*_Δ, then p lies in the common exterior (resp. common interior) of the circles in C_Δ. Let E denote the set of edges in these faces of $\mathcal{A}(C_\Delta)$; we have $|E| = O(n_\Delta)$ [20]. (If the common interior is nonempty, both the common interior and the common exterior are connected, so they constitute two faces of $\mathcal{A}(C_\Delta)$. Otherwise, the common exterior need not be connected, but all its faces together have at most $6n_\Delta - 12$ edges [18].) We construct a planar bipartite graph G whose nodes are the points of $P' \setminus P_\Delta$ on one side, and the edges of E on the other side. We connect a point p to an edge e by an arc of G if e appears on ∂f_p in the full arrangement $\mathcal{A}(C)$, drawing the arc within f_p, so that no two arcs intersect. By construction, G is a simple embedded planar graph, so the number of its edges is $O(|P'| + n_\Delta)$, which is clearly an upper bound for $I(P' \setminus P_\Delta, C_\Delta)$ (the number of edges of G may be larger because a circle may appear along the boundary of a face f_p along more than one arc of E). □

Therefore, applying (3) to Ξ_1, we have

$$\begin{aligned}
I(P, C) &\le \sum_{\Delta \in \Xi_1} (I(P_\Delta, C_\Delta) + I(P \setminus P_\Delta, C_\Delta)) \\
&\le \sum_{\Delta \in \Xi_1} I(P_\Delta, C_\Delta) + \sum_{\Delta \in \Xi_1} a(m + n_\Delta) \\
&\qquad\qquad \text{(by Lemma 3.7)} \\
&\le \sum_{\Delta \in \Xi_1} \Big(\sum_{\tau \in \Xi_2^\Delta} I(P_\tau, C_\tau) + I(P_\Delta \setminus P_\tau, C_\tau) \Big) \\
&\quad + a(n + c' r_0^3 m),
\end{aligned}$$

where a is the constant provided in Lemma 3.7, and c' is the constant defined in (2). Setting $a' = ac' r_0^3$ and using Lemma 3.7 again to bound $I(P_\Delta \setminus P_\tau, C_\tau)$, we obtain

$$\begin{aligned}
I(P, C) &\le \sum_{\tau \in \Xi_2} \left(I(P_\tau, C_\tau) + a(n_\tau + \frac{m}{r_0}) \right) + an + a'm \\
&\le \sum_{\tau \in \Xi_2} I(P_\tau, C_\tau) + 2an + a'm(1 + r_0^2),
\end{aligned}$$

because $\sum_\tau n_\tau = n$ and $|\Xi_2| \le c' r_0^6$. Continuing in this manner and recalling that for any simplex $\tau \in \Xi_{j-1}$, $m_\tau \le m/r_0^{j-1}$ and that $|\Xi_j| \le c r_0^{3j}$, $\sum_{\tau \in \Xi_j} n_\tau = n$, we obtain

$$\begin{aligned}
I(P, C) &\le \sum_{\tau \in \Xi_i} I(P_\tau, C_\tau) + ian + a'm \sum_{j=0}^{i-1} r_0^{2j} \\
&= \sum_{\tau \in \Xi_\nu} I(P_\tau, C_\tau) + O(n\nu + m r_0^{2\nu}) \\
&= \sum_{\tau \in \Xi} I(P_\tau, C_\tau) + O(n \log r + m r^2), \qquad (4)
\end{aligned}$$

because $\Xi_\nu = \Xi$ and $\nu = \lceil \log_{r_0} r \rceil$. Next, we bound $I(P_\tau, C_\tau)$.

Lemma 3.8. *For any* $\tau \in \Xi$,

$$I(P_\tau, C_\tau) \le 2m/r + 2n_\tau + K(P_\tau, C_\tau).$$

Proof. Let F be the set of faces of $\mathcal{A}(C_\tau)$ that contain a point of P_τ. Let f be a face in F that contains $m_f > 0$ points of P_τ, say $p_1, \dots, p_{m_f}$. The corresponding regions f_{p_j}, for $j = 1, \dots, m_f$, are pairwise-disjoint connected regions within f (because each face of $\mathcal{A}(C)$ is assumed to contain at most one point of P). Suppose ∂f has ξ_f connected components. For each connected component, we choose a point q_j, $1 \le j \le \xi_f$ that lies in the complement of f bounded by that component. We decompose each connected component of ∂f into maximal connected portions, so that each portion overlaps with the boundary of a single face f_{p_i} of $\mathcal{A}(C)$; such a portion might appear on ∂f_{p_i} in many disconnected pieces; see Figure 4. Let $\gamma_1, \dots, \gamma_{h_f}$ denote the resulting partition of ∂f. Then the points of P_τ lying in f contribute at most $h_f + |f|$ to $I(P_\tau, C_\tau)$, where $|f|$ is the number of edges in ∂f. Hence,

$$I(P_\tau, C_\tau) \le \sum_{f \in F} (h_f + |f|) = K(P_\tau, C_\tau) + \sum_{f \in F} h_f.$$

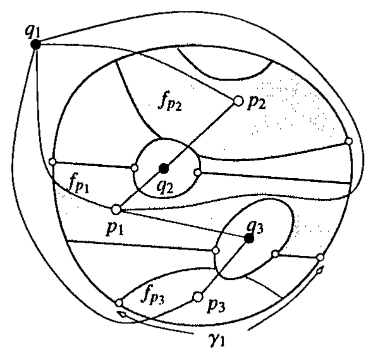

Figure 4: Construction of the bipartite graph to bound $I(P_\tau, C_\tau)$ within a single face of $\mathcal{A}(C_\tau)$; small white circles denote the partition of ∂f into $\gamma_1, \gamma_2, \dots$.

In order to bound h_f, we construct a planar bipartite graph whose vertices are the points p_j, for $j = 1, \dots, m_f$, on one side, and the points q_j, for $j = 1, \dots, \xi_f$, on the other side. For each γ_l, if γ_l is a portion of the jth connected component of ∂f and overlaps with f_{p_i}, we connect p_i to q_j by an edge; we draw the edge as an arc passing through γ_l; see Figure 4. This can easily be done so that these edge drawings are pairwise disjoint (except at their endpoints). Hence, the resulting graph is planar and has no faces of degree two (although there may be multiple edges between a pair of vertices). hence, the number, h_f, of edges in the graph is at most $2(m_f + \xi_f) - 4$.

The points of P_τ are partitioned among the faces of F, so $\sum_{f \in F} m_f = m_\tau \le m/r$. Moreover, $\sum_{f \in F} (\xi_f - 1) \le |C_\tau| = n_\tau$. Indeed, $\xi_f - 1$ is the total number of "islands" (inner boundary components) inside the face f, and a circle cannot belong to more than one island. This completes the proof of the lemma. □

Substituting the bounds from Lemma 3.8 and Theorem 3.5 in (4) and using that fact that $m_\tau \le m/r$, $n_\tau \le n/r^3$, and $|\Xi| = O(r^3)$, we obtain

$$\begin{aligned} I(P, C) &= \sum_{\tau \in \Xi} O\Big((m_\tau n_\tau \log n)^{2/3} + n_\tau^{3/2+\varepsilon} + m_\tau \log n \\ &\qquad + mr^2 + n \log r\Big) \\ &= O\Big((mn \log n)^{2/3} r^{1/3} + n^{3/2+\varepsilon}/r^{3/2+3\varepsilon} \\ &\qquad + mr^2 \log n + n \log r\Big). \end{aligned}$$

Choose $r = \lceil n^{5/11}/m^{4/11} \rceil$. For this value of r, $mr^2 > (n/r)^{3/2}$ only when $m < n^{1/3}$, in which case $K(m, n) = O(n)$. Using this and substituting the value of r, and including the bounds obtained when r does not fall into the required range, we have

$$I(P, C) = O\Big((mn \log n)^{2/3} + m^{6/11+\varepsilon} n^{9/11} + (m + n) \log n\Big),$$

for any $\varepsilon > 0$. Putting everything together, we obtain the following main result of the paper.

Theorem 3.9. *The maximum combined complexity of m distinct faces in an arrangement of n arbitrary circles in the plane is*

$$O\Big((mn \log n)^{2/3} + m^{6/11+\varepsilon} n^{9/11} + (m + n) \log n\Big),$$

for any arbitrarily small constant $\varepsilon > 0$.

We can extend Theorem 3.9 to obtain an upper bound for $K(m, n, X)$, which takes into account the number X of intersecting pairs of circles in C. Here is a sketch of the analysis: Put $s = \lceil n^2/X \rceil$, and construct a $(1/s)$-cutting of $\mathcal{A}(C)$ that consists of $O(s + s^2 X/n^2) = O(s)$ cells, each crossed at most n/s circles (see, e.g., [17]). We apply Theorem 3.9 to bound the complexity of the marked faces within each cell, add up the resulting complexity bounds, and also add the complexity of the zones of the cell boundaries to account for faces not confined to a single cell (as in [11]). This leads to the following result.

Theorem 3.10. *The maximum complexity of m distinct faces in an arrangement of n arbitrary circles in the plane with X intersecting pairs is* $O(m^{2/3} X^{1/3} \log^{2/3} n + m^{6/11+\varepsilon} X^{4/11} n^{1/11} + (m + n) \log n)$, *for an arbitrarily small constant* $\varepsilon > 0$.

The case of pairwise intersecting circles can be handled in a similar manner, using Theorem 3.6 to substitute the value of $K(P_\tau, C_\tau)$ in (4). Omitting the straightforward details, we obtain:

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Theorem 3.11. *The maximum complexity of m distinct faces in an arrangement of n arbitrary pairwise-intersecting circles in the plane is*

$$O\left((mn\log n)^{2/3} + m^{1/2}n^{5/6}\log^{3/4} n + (m+n)\log n\right).$$

4 Conclusion

In this paper we extended the analysis of [7] to obtain almost the same asymptotic bounds for the complexity of many faces in circle arrangements. In addition to the machinery developed and used in the preceding paper, our analysis involves the following ingredients:

(i) We have shown that cutting the circles of C into a collection of pseudo-segments facilitates the application of the crossing lemma to derive the first bound on $K(m,n)$— a considerably more involved step than in the case of incidences.

(ii) The use of the face-circle incidence count $I(m,n)$, instead of the face complexity $K(m,n)$ was instrumental for combining bounds for subproblems into a global bound. We believe that this idea can be useful in attacking other similar problems.

This paper raises many open problems. We mention two of the more obvious ones:

- Can one improve the upper bound, given in Theorem 3.9, on the complexity of many faces in an arrangement of arbitrary circles? (Of course, one should first aim at improving this bound for incidences!)
- Can the ideas developed here be applied to obtain an alternative derivation of the bound of [5] for the complexity of many faces in an arrangement of line segments?

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