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ACOUSTIC SCATTERING FROM LARGE ASPECT RATIO ELASTIC TARGETS

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FOREWORD

This paper applies the spheroidal T-matrix approach of Dr. Roger H Hackman described in his paper "The Transition Matrix for Acoustic and Elastic Wave Scattering in Prolate Spheroidal Coordinates", [Journal of the Acoustic Society of America, 75(1), PP. 35-45] to describe the scattering from large aspect ratio targets. Due to the limits of double precision arithmetic in the computation of the spheroidal wave functions, these methods are limited to frequencies below kL/2 = 30.

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Approved by

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Systems Department

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I. INTRODUCTION

Prolate spheroidal coordinates are one of eleven coordinate systems in which the scalar Helmholtz Equation in three dimensions is separable. Separability is the reason the scattering from a rigid (sound soft) prolate spheroid may be expressed as a partial wave series of spheroidal wave functions analogous to the partial wave series of a rigid sphere. However, the vector Helmholtz Equation is not separable in the prolate spheroidal coordinate system. This is the main reason for the lack of exact solutions for acoustic scattering from an elastic prolate spheroid. The lack of separability of the vector wave equation in spheroidal coordinates is also the reason there does not exist an expansion of the Green's Dyadic for the vector wave equation in terms of spheroidal functions.

In the early 1980's, Roger Hackman [1-3] at Coastal Systems Station used Betti's Identity to derive a T-matrix description of the scattering from large aspect ratio elastic targets based on the spheroidal wave function, rather than the popular spherical T-matrix description. This approach is numerically more stable than the spherical T-matrix approach for large aspect ratio targets. The author had the pleasure of working with Dr. Hackman during his tenure at COASTSYSTA, and worked on the scattering from large aspect ratio targets in a waveguide [4-8].

This article focuses on the use of the spheroidal T-matrix description of scattering from elastic targets to describe the low frequency scattering from these targets. The purpose for this work is to build an object oriented computer program for computing the low frequency scattering from large aspect ratio targets. This computer code is to be integrated into the sonar simulation PC Shallow Water Acoustic Tool-set (PC SWAT) to describe a low frequency imaging sonar in the presence of elastic returns from the target.

In addition to the Introduction and Reference sections of this report, the outline of this paper includes the following sections. Section II describes the spheroidal coordinate system and its relationship to linear acoustics. Section III outlines the computation of the spheroidal basis functions. Section IV describes the computation of the spheroidal T-matrix. Section V contains a collection of sample calculations and a comparison of the author's results with those of Hackman. Most of the details of the computations are contained in the appendices. Appendix A describes the computation of the eigenvalues and expansion coefficients for the spheroidal wave equation. Appendix B contains the orthogonality and completeness relationships of the expansion coefficients. Appendix C contains the expansion of the scalar Green's Function and plane wave in terms of spheroidal wave functions. Appendix D describes the transformation between spherical and spheroidal wave functions. Appendix E describes the computation of the radial spheroidal functions. Appendix F contains a tabulation of the connection components and its first order derivatives used in the computation of the second and third order covariant derivatives of the scalar basis functions. Appendix G contains a tabulation of the components of the vector basis functions and their covariant derivatives. Appendix H contains a tabulation of the parity of the basis functions. Appendix I describes the computation of the surface integrals and T-matrix in the case of a prolate spheroid. Appendix J describes the computation of the surface integrals and T-matrix, in the case of a finite cylinder with hemi-spherical end caps.

II. SPHEROIDAL COORDINATES

Prolate spheroidal coordinates are a generalization of the familiar spherical coordinates. The transformation from spheroidal to Cartesian coordinates is given by the following equations.

$$\begin{aligned} x &= f\sqrt{(\xi^2-1)(1-\eta^2)}\cos(\varphi) \\ y &= f\sqrt{(\xi^2-1)(1-\eta^2)}\sin(\varphi) \\ z &= f\xi\eta \end{aligned} \tag{1}$$

Here, $\xi \geq 1$ is the radial coordinate, $-1 \leq \eta \leq +1$ is the angular coordinate, $0 \leq \varphi \leq 2\pi$ is the azimuthal angle coordinate, and f is the semi-focal distance.

Adopt a right-handed coordinate system whose coordinates y^μ are given by the spheroidal coordinates (ξ, φ, η). The metric tensor in this coordinate system is a diagonal metric whose diagonal members are given by the following expressions in terms of the coordinate basis $\{\partial_\xi, \partial_\varphi, \partial_\eta\}$ of the tangent space:

$$\begin{aligned} g_{\xi\xi} &= f^2(\xi^2-\eta^2)/(\xi^2-1) \\ g_{\eta\eta} &= f^2(\xi^2-\eta^2)/(1-\eta^2) \\ g_{\varphi\varphi} &= f^2(\xi^2-1)(1-\eta^2) \end{aligned} \tag{2}$$

Define the ortho-normal triad $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \{\hat{\xi}, \hat{\varphi}, \hat{\eta}\}$ by the following relations:

$$\begin{aligned} \hat{e}_i &= e_i^\mu \partial_\mu \\ e_i^\mu &= \delta_i^\mu / h_\mu \\ h_\mu &= \sqrt{g_{\mu\mu}} \end{aligned} \tag{3}$$

The covariant derivative of a tensor of type $(1,1)$ in terms of the coordinate basis is of the following form:

$$\nabla_\mu T^\alpha{}_\beta = \partial_\mu T^\alpha{}_\beta + \Gamma^\alpha{}_{\sigma\mu} T^\sigma{}_\beta - \Gamma^\sigma{}_{\beta\mu} T^\alpha{}_\sigma \tag{4}$$

Here, $\Gamma^{\alpha}{}_{\mu\nu}$ are the components of the torsion-free affine connection defined by the following expression in terms of the metric tensor (An affine transform changes the relationship between the coordinate system a program uses to draw and the coordinate system used to display).

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma}(\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu}) \tag{5}$$

This connection has the property that the covariant derivative of the metric tensor vanishes.

The stress tensor of an elastic solid is defined by the following expression in terms of the metric tensor and the covariant derivative of the displacement vector *u*.

$$T_{\alpha\beta} = \lambda(\nabla_{\sigma} u^{\sigma}) g_{\alpha\beta} + \mu(\nabla_{\alpha} u_{\beta} + \nabla_{\beta} u_{\alpha}) \tag{6}$$

Here, λ and μ are the Lame constants of an isotropic elastic solid, which are related to the longitudinal and shear wave speeds by the following relations, where ρ is the density of the solid.

$$\begin{aligned} \lambda &= \rho(v_L^2 - 2v_T^2) \\ \mu &= \rho v_T^2 \end{aligned} \tag{7}$$

The equation of motion for an isotropic elastic solid is given by the following expression, where $T(u)_{\alpha\beta}$ is the stress tensor for the displacement vector u_{α}.

$$\rho \partial_t^2 u_{\mu} = \nabla_{\sigma} T^{\sigma}{}_{\mu} \tag{8}$$

A generalization of Green's Theorem gives rise to Betti's Identity, which states that given a pair (*u*,*v*) of regular solutions of Equation 8, the following surface integral over the boundary of the enclosed volume (*V*) vanishes.

$$\oiint_{\partial V} ds\{\vec{u} \bullet \vec{t}(v) - \vec{t}(u) \bullet \vec{v}\} = 0 \tag{9}$$

Here, the terms

$$t(u)_{\mu} = T(u)_{\mu\sigma} n^{\sigma} \tag{10}$$

are the components of the traction of the displacement *u*.

III. SPHEROIDAL FUNCTIONS

The scalar Helmholtz Equation in spheroidal coordinates is of the following form:

$$(\nabla_\sigma \nabla^\sigma + k^2)\psi = \frac{1}{\sqrt{g}}(\partial_\mu g^{\mu\nu}\sqrt{g}\partial_\nu + k^2\sqrt{g})\psi =$$

$$\frac{1}{f^2(\xi^2-\eta^2)}\{\partial_\xi(\xi^2-1)\partial_\xi\psi + \partial_\eta(1-\eta^2)\partial_\eta\psi$$

$$+\frac{(\xi^2-\eta^2)}{(\xi^2-1)(1-\eta^2)}\partial_\varphi^2\psi + c^2(\xi^2-\eta^2)\psi\} = 0 \tag{11}$$

$$g = Det(g_{\mu\nu})$$

Here, c is the dimensionless product of the wavenumber k and the semi-focal distance f. This wave equation is separable into the product of a solution of the radial equation:

$$\partial_\xi(\xi^2-1)\partial_\xi je_{ml} - (\lambda_{ml}(c) - c^2\xi^2 + \frac{m^2}{(\xi^2-1)})je_{ml} = 0 \tag{12}$$

the angular equation:

$$\partial_\eta(1-\eta^2)\partial_\eta S_{ml} + (\lambda_{ml}(c) - c^2\eta^2 - \frac{m^2}{(1-\eta^2)})S_{ml} = 0 \tag{13}$$

and the azimuthal equation. The spheroidal coordinates have the rather interesting property that the radial and angular equations are isomorphic under the interchange of the coordinates $\xi \leftrightarrow \eta$. This isomorphism implies a duality between the solutions of the radial and angular functions, that is, the radial functions may be regarded as the analytic continuation of the angular functions to the domain $+1 < \eta < +\infty$. In particular, the radial functions:

$$je_{ml}(c,\xi) = \frac{1}{\kappa_{ml}^{(1)}} S_{ml}^{(1)}(c,\xi)$$

$$ne_{ml}(c,\xi) = \frac{1}{\kappa_{ml}^{(2)}} S_{ml}^{(2)}(c,\xi) \tag{14}$$

are proportional to the corresponding angular functions. This transformation is an isometry of the metric tensor, that is, the metric tensor is invariant under this transformation. The classic method of solving the angular wave equation is to expand the angular functions of the first kind in terms of associated Legendre polynomials:

$$S_{ml}^{(1)}(c,\eta) = \sum_{k=0,1}^{\infty} d_k^{ml}(c) P_{m+k}^m(\eta) \tag{15}$$

Here, the sum is over either even or odd indices, depending upon whether the difference (*l-m*) is even or odd. The series expansion given in Equation 15 is a solution of the angular wave equation provided the expansion coefficients satisfies the following recurrence relations:

$$\alpha_k^m d_{k+2}^{ml} + (\beta_k^m - \lambda_{ml}) d_k^{ml} + \gamma_k^m d_{k-2}^{ml} = 0$$

$$\alpha_k^m = \frac{(2m+k+2)(2m+k+1)c^2}{(2m+2k+3)(2m+2k+5)}$$

$$\beta_k^m = (m+k)(m+k+1) + c^2 \frac{2(m+k)(m+k+1) - 2m^2 - 1}{(2m+2k-1)(2m+2k+3)} \tag{16}$$

$$\gamma_k^m = \frac{k(k-1)c^2}{(2m+2k-3)(2m+2k-1)}$$

The eigenvalue $\lambda_{ml}(c)$ is chosen such that the limit

$$\lim_{k\to\infty}\left(\frac{d_{k+2}^{ml}(c)}{d_k^{ml}(c)}\right) = 0 \tag{17}$$

vanishes. The conventional method of solving for the eigenvalue is to solve a transcendental equation defined in terms of a continued fraction of the above eigenvalue and coefficients as described by Flammer in Reference 9. A numerically more robust method of solving for the eigenvalues is to transform the above recurrence relation into the problem of solving for the eigenvalues of an infinite dimensional tri-diagonal matrix. This method is due to Hodge [10], and is described in References 10 through 12. This is the method adopted by the author since it is readily generalized to the case of complex wavenumber. The coefficients $d_k^{ml}(c)$ are related to the eigenvectors of this infinite dimensional matrix. Reference 13 contains a simple iterative method for solving for the eigenvalues and eigenvectors of arbitrarily large tri-diagonal matrices.

In the case of the radial wave equation, the fact that the radial functions are proportional to the analytic continuation of the angular functions can be used to calculate the radial functions for radial coordinates near unity ($+1 < \xi < 1.5$) as described in Reference 11. The computation of the irregular radial functions requires the computation of the angular functions of the second, which is derived by taking the following limit of the series expansion in terms of associated Legendre polynomials of the second kind.

$$S_{ml}^{(2)}(c,\eta) = \lim_{\rho\to 0} \sum_{k=-\infty}^{+\infty} d_{k+\rho}^{ml}(c) Q_{m+k+\rho}^{m}(\eta) \tag{18}$$

The associated Legendre polynomials of the second kind exhibit a simple pole at integral orders less than –m. At the same time, the coefficients $d_k^{ml}(c)$ exhibit a simple zero, and the product remains finite.

$$\lim_{\rho\to 0} d_{-k+\rho}^{ml} Q_{m-k+\rho}^{m} = d_{k|\rho}^{ml} P_{-m-1+k}^{m}, \quad k > +2m \tag{19}$$

This produces a representation of the angular function of the second kind that is a series of the associated Legendre functions of the first and second kind of the following form:

$$S_{ml}^{(2)}(c,\eta) = \sum_{k=-2m+\delta}^{+\infty} d_k^{ml}(c) Q_{m+k}^{m}(\eta) + \sum_{k=2m+2-\delta}^{+\infty} d_{k|\rho}^{ml}(c) P_{-m-1+k}^{m}(\eta) \tag{20}$$

The coefficients $d_{k|\rho}^{ml}(c)$ are defined by the limit:

$$d_{k|\rho}^{ml}(c) = \lim_{\rho\to 0} \frac{d_{-k+\rho}^{ml}(c)}{\rho} \tag{21}$$

A recurrence relation for these coefficients is given in Reference 11.

A popular representation of the regular radial function in terms of spherical Bessel Functions is of the following form described in References 9 through 12.

$$je_{ml}(c,\xi) = \frac{1}{N_{ml}^{r}} \left(\frac{(\xi^2-1)^{1/2}}{\xi}\right)^{m} \sum_{k=0,1}^{\infty} (+i)^{k+m-l} \frac{(2m+k)!}{k!} d_k^{ml} j_{m+k}(c\xi) \tag{22}$$

The above series expansion is absolutely convergent. The corresponding series expansion for the irregular radial function is obtained by replacing the regular spherical Bessel Function with the corresponding irregular spherical Bessel Function. However, in the case of the irregular radial function the series expansion is no longer absolutely convergent, and it is at best an asymptotic expansion.

The author has studied the convergence properties of about a half dozen different representations of the irregular radial functions based on integral representations of the following form:

$$ne_{ml}(c,\xi) = \int d\eta K(\xi,\eta) S^{(1)}_{ml}(c,\eta) \tag{23}$$

Flammer [9] provides a general discussion for the construction of integral representations of the above variety. None of these representations has satisfactory behavior at high frequencies. The high frequency convergence property of the irregular radial functions and the angular functions of the second kind is due to the fact that the expansion coefficients $d_k^{ml}(c)$ represent an alternating series of large terms leading to numerical round off due to the finite precision of the calculations.

IV. SPHEROIDAL T-MATRIX

In the case of the spherical T-matrix description, the expansion of the scalar Green's Function and Green's Dyadic in terms of spherical wave functions is often used in the derivation of the spherical T-matrix description from the scalar and vector Helmholtz Equation. The vector Helmholtz Equation in spheroidal coordinates is non-separable, and there exists no closed form solution for the corresponding Green's Dyadic in terms of spheroidal wave functions. To overcome this difficulty, Dr Hackman made use of Betti's Identity, which is a generalization of Green's Theorem, to generate a system of linear equations for the scattered field in terms of a collection of basis functions of the scalar and vector Helmholtz Equation.

The author introduces the following basis for the regular and outgoing basis functions for the scalar Helmholtz Equation in terms of spheroidal functions:

$$\operatorname{Re}\psi_{\sigma ml} = je_{ml}(c,\xi)S_{\sigma ml}(c,\eta,\varphi)$$

$$\psi_{\sigma ml} = he_{ml}(c,\xi)S_{\sigma ml}(c,\eta,\varphi) \tag{24}$$

The functions $S_{\sigma ml}(c,\eta,\varphi)$ are the following basis functions of the scalar spheroidal harmonics:

$$S_{\sigma ml}(c,\eta,\varphi) = \sqrt{\frac{\varepsilon(m)}{2\pi\Lambda_{ml}(c)}}S_{ml}^{(1)}(c,\eta)\begin{cases}\cos(m\varphi), \sigma = 0\\ \sin(m\varphi), \sigma = 1\end{cases}$$

$$\varepsilon(m) = \begin{cases}1, m = 0\\ 2, m \neq 0\end{cases} \tag{25}$$

Define the following basis functions of the vector Helmholtz Equation:

$$V_{1,\sigma ml} = \frac{1}{\sqrt{\lambda_{ml}}}\nabla\times(\vec{a}\psi_{\sigma ml}^{T})$$

$$V_{2,\sigma ml} = \frac{1}{c_T}\nabla\times V_{1,\sigma ml} \tag{26}$$

$$V_{3,\sigma ml} = \frac{1}{c_L}\nabla\,\psi_{\sigma ml}^{L}$$

Here, $\lambda_{ml}(c)$ is the eigenvalue of the spheroidal wave equation, and $c_T = k_T f$ and $c_L = k_L f$ are the dimensionless wavenumber for the transverse and longitudinal degrees of freedom. The functions

$$\begin{aligned}\psi^T_{\sigma ml} &= he_{ml}(c_T,\xi)S_{\sigma ml}(c_T,\eta,\varphi)\\ \psi^L_{\sigma ml} &= he_{ml}(c_L,\xi)S_{\sigma ml}(c_L,\eta,\varphi)\end{aligned} \tag{27}$$

are the outgoing basis functions for the scalar Helmholtz Equation for the transverse and longitudinal modes. The vector $\vec{a}$ is the conformal killing vector for dilatations whose components are as follows:

$$\begin{aligned}\vec{a} &= (x,y,z),\\ (a_\xi,a_\eta,a_\varphi) &= (f^2\xi, f^2\eta, 0)\end{aligned} \tag{28}$$

This vector satisfies the following condition:

$$\nabla_\mu a_\nu = g_{\mu\nu} \tag{29}$$

The curl in three dimensions is defined as follows:

$$\nabla \times V = e_\mu \varepsilon^{\mu\alpha\beta}\nabla_\alpha V_\beta \tag{30}$$

in terms of the Levi-Civita Tensor

$$\varepsilon^{\alpha\beta\gamma} = \frac{1}{\sqrt{Det(g_{\mu\nu})}}\delta^{\alpha\beta\gamma}_{123} \tag{31}$$

and the covariant derivative.

Define the vector spheroidal harmonics for the elastic solid as follows:

$$\begin{aligned}A_{1,\sigma ml} &= \frac{1}{\sqrt{\lambda_{ml}}}(\hat{\varphi}\sqrt{1-\eta^2}\partial_\eta - \hat{\eta}\frac{1}{\sqrt{1-\eta^2}}\partial_\varphi)S_{\sigma ml}(c_T,\eta,\varphi)\\ A_{2,\sigma ml} &= \frac{1}{\sqrt{\lambda_{ml}}}(\hat{\eta}\sqrt{1-\eta^2}\partial_\eta + \hat{\varphi}\frac{1}{\sqrt{1-\eta^2}}\partial_\varphi)S_{\sigma ml}(c_T,\eta,\varphi)\\ A_{3,\sigma ml} &= \hat{\xi}S_{\sigma ml}(c_L,\eta,\varphi)\end{aligned} \tag{32}$$

In the limit $f \to 0$ the vector spheroidal harmonics approach the vector spherical harmonics. They obey the following orthogonality conditions:

$$\oint d\varphi d\eta A_{1,\sigma ml} \bullet A_{1,\sigma' m' l'} = \delta_\sigma^{\sigma'} \delta_m^{m'} \Omega_{l,l'}^m$$
$$\oint d\varphi d\eta A_{2,\sigma ml} \bullet A_{2,\sigma' m' l'} = \delta_\sigma^{\sigma'} \delta_m^{m'} \Omega_{l,l'}^m$$
$$\oint d\varphi d\eta A_{3,\sigma ml} \bullet A_{3,\sigma' m' l'} = \delta_\sigma^{\sigma'} \delta_m^{m'} \delta_l^{l'}$$
$$\oint d\varphi d\eta A_{1,\sigma ml} \bullet A_{3,\sigma' m' l'} = 0 \qquad (33)$$
$$\oint d\varphi d\eta A_{2,\sigma ml} \bullet A_{3,\sigma' m' l'} = 0$$
$$\oint d\varphi d\eta A_{1,\sigma ml} \bullet A_{2,\sigma' m' l'} = 0$$

$$\Omega_{l,l'}^m = \delta_l^{l'} - \frac{c_T^2}{\sqrt{\lambda_{ml}\lambda_{ml'}\Lambda_{ml}\Lambda_{ml'}}} \int_{-1}^{+1} d\eta \eta^2 S_{ml}(c_T,\eta) S_{ml'}(c_T,\eta)$$

The vector basis functions $V_{\tau,\sigma ml}$ have the following asymptotic behavior in the limit $\xi \to +\infty$:

$$V_{1,\sigma ml} \to he_{ml}(c_T,\xi) A_{1,\sigma ml}$$
$$V_{2,\sigma ml} \to \frac{1}{c_T} \partial_\xi he_{ml}(c_T,\xi) A_{2,\sigma ml} \qquad (34)$$
$$V_{3,\sigma ml} \to \frac{1}{c_L} \partial_\xi he_{ml}(c_L,\xi) A_{3,\sigma ml}$$

The above asymptotic behavior of the vector basis functions coupled with Betti's Identity leads to the following important orthogonality condition for the vector basis functions. Let S be a smooth, closed surface, S_∞ be the sphere at infinity, V be the volume whose boundaries consists of the union of these two surfaces, and (u,v) be a pair of regular solutions of the vector Helmholtz Equation in this volume. Then Betti's Identity implies the following surface integrals over the closed surface S and the sphere at infinity are equal:

$$\iint_S dA\{t(u) \bullet v - u \bullet t(v)\} = \iint_{S_\infty} dA\{t(u) \bullet v - u \bullet t(v)\} \qquad (35)$$

By replacing the functions u and v with the vector basis functions and using the asymptotic expansions of the vector basis functions and Equation 33 to evaluate the surface integral over the sphere at infinity, we obtain the following orthogonality condition for the vector basis functions on the closed surface S:

$$\iint_S dA\{t(\operatorname{Re} V_{\tau,\sigma ml}) \bullet \operatorname{Re} V_{\tau',\sigma' m' l'} - \operatorname{Re} V_{\tau,\sigma ml} \bullet t(\operatorname{Re} V_{\tau',\sigma' m' l'})\} = 0$$

$$\iint_S dA\{t(V_{\tau,\sigma ml}) \bullet V_{\tau',\sigma' m' l'} - V_{\tau,\sigma ml} \bullet t(V_{\tau',\sigma' m' l'})\} = 0$$

$$\iint_S dA\{t(\operatorname{Re} V_{\tau,\sigma ml}) \bullet V_{\tau',\sigma' m' l'} - \operatorname{Re} V_{\tau,\sigma ml} \bullet t(V_{\tau',\sigma' m' l'})\} = \tag{36}$$

$$\begin{cases} -i\mu / k_T (-i)^{l-l'} \delta_\sigma^{\sigma'} \delta_m^{m'} \Omega_{l,l'}^m, \tau = \tau' \neq 3 \\ -i(\lambda + 2\mu) / k_L (-i)^{l-l'} \delta_\sigma^{\sigma'} \delta_m^{m'} \delta_l^{l'}, \tau = \tau' = 3 \\ 0, \tau \neq \tau' \end{cases}$$

The above orthogonality condition is the basis for the spheroidal T-matrix approach, which uses the above orthogonality condition to re-express the boundary conditions of the incident, scattered, and interior fields into a system of linear equations based upon expansions of the incident, scattered and surface fields in terms of these vector basis functions.

Consider the case of acoustic scattering from a solid elastic target immersed in water. The boundary conditions between the exterior and interior fields are given by the following conditions of continuity of normal displacement, continuity of the normal traction, and vanishing of the tangential interior traction.

$$\begin{aligned} \vec{u}_+ \bullet \vec{n} &= \vec{u}_- \bullet \vec{n} \\ \vec{t}_+ \bullet \vec{n} &= \vec{t}_- \bullet \vec{n} \\ \vec{t}_- \times \vec{n} &= 0 \end{aligned} \tag{37}$$

The single vector:

$$V_{\sigma ml}^W = \frac{1}{c} \nabla he_{ml}(c,\xi) S_{\sigma ml}(c,\eta,\varphi) \tag{38}$$

is the vector basis function for the fluid, which may be regarded as the limit $\mu \to 0$ of the vector basis for an elastic medium. The following vector is the corresponding traction vector of this basis:

$$\begin{aligned} &\vec{t}(V_{\sigma ml}^W) = \lambda^W \vec{n}(\nabla \bullet V_{\sigma ml}^W) = \\ &-\frac{\lambda^W k^2}{c} he_{ml}(c,\xi) S_{\sigma ml}(c,\eta,\varphi) \vec{n} \end{aligned} \tag{39}$$

Make the following expansions of the incident and scattered field in the fluid in terms of this basis:

$$u_{inc} = \sum_{\sigma ml} a_{\sigma ml} \operatorname{Re} V^{W}_{\sigma ml}$$
$$u_{Scatt} = \sum_{\sigma ml} f_{\sigma ml} V^{W}_{\sigma ml} \tag{40}$$

Applying Betti's Identity to the total field:

$$u_{Total} = u_{inc} + u_{scatt} = \sum_{\sigma ml} a_{\sigma ml} \operatorname{Re} V^{W}_{\sigma ml} + f_{\sigma ml} V^{W}_{\sigma ml} \tag{41}$$

and the regular and irregular basis functions on the surface S of the scatterer, one arrives at the following linear equations between the expansion coefficients of the incident and scattered fields and the surface fields

$$a_{\sigma ml} = -i \frac{k}{\lambda^W} \int_S dA\{t_+ \bullet V^{W}_{\sigma ml} - u_+ \bullet t(V^{W}_{\sigma ml})\}$$
$$f_{\sigma ml} = +i \frac{k}{\lambda^W} \int_S dA\{t_+ \bullet \operatorname{Re} V^{W}_{\sigma ml} - u_+ \bullet t(\operatorname{Re} V^{W}_{\sigma ml})\} \tag{42}$$

Using the fact that traction in the fluid is proportional to the normal vector to the surface and the continuity of the normal displacement, one can rewrite these equations in the following form:

$$a_{\sigma ml} = -i \frac{k}{\lambda^W} \int_S dA\{(t_+ \bullet n)(n \bullet V^{W}_{\sigma ml}) - (u_- \bullet n) n \bullet t(V^{W}_{\sigma ml})\}$$
$$f_{\sigma ml} = +i \frac{k}{\lambda^W} \int_S dA\{(t_+ \bullet n) n \bullet \operatorname{Re} V^{W}_{\sigma ml} - (u_- \bullet n) n \bullet t(\operatorname{Re} V^{W}_{\sigma ml})\} \tag{43}$$

Next, one may apply Betti's Identity to the regular vector basis functions of the solid and the surface displacement and traction of the interior fields to arrive at the following equations.

$$\int_S dA\{t_- \bullet \operatorname{Re} V_{\tau,\sigma ml} - u_- \bullet t(\operatorname{Re} V_{\tau,\sigma ml})\} = 0 \tag{44}$$

Use the boundary conditions at the surface to replace this integral with the following:

$$\int_S dA\{(t_+ \bullet n)(n \bullet \operatorname{Re} V_{\tau,\sigma ml}) - u_- \bullet t(\operatorname{Re} V_{\tau,\sigma ml})\} = 0 \tag{45}$$

Now make the following expansions in terms of the scalar harmonics in the exterior and the vector harmonics in the interior:

$$t_{+} \bullet n = \sum_{\sigma ml} \alpha_{\sigma ml} S_{\sigma ml}(c, \eta, \varphi)$$
$$u_{-} = \sum_{\tau\sigma ml} \beta_{\tau,\sigma ml} A_{\tau,\sigma ml} \tag{46}$$

Substituting the above expansions into Equations 43 and 45 one arrives at the following linear system of equations:

$$a = -(Q\beta - M\alpha)$$

$$f = +\mathrm{Re}\, Q\beta - \mathrm{Re}\, M\alpha$$

$$0 = P\alpha - R\beta$$

$$Q_{n,\tau' n'} = -i\frac{k}{\lambda^{W}} \int_{S} dA\{(n \bullet t(V_{n}^{W})(n \bullet A_{\tau',n'})\}$$

$$M_{n,n'} = -i\frac{k}{\lambda^{W}} \int_{S} dA\{(n \bullet V_{n}^{W})S_{n'}\}$$

$$P_{\tau n,n'} = \int_{S} dA\{(n \bullet \mathrm{Re}\, V_{\tau,n})S_{n'}\}$$

$$R_{\tau n,\tau' n'} = \int_{S} dA\{t(\mathrm{Re}\, V_{\tau n}) \bullet A_{\tau' n'}\} \tag{47}$$

The indices σml have been grouped into the single index n in the above equations. The solution of the above systems of equations is given by the following T-matrix, which relates the incident and scattered fields in the fluid.

$$T = -(\mathrm{Re}\, QR^{-1}P - \mathrm{Re}\, M)(QR^{-1}P - M)^{-1} \tag{48}$$

The above T-matrix is Method B described in Reference 3. This is the method used to perform the calculations in the next section.

V. SAMPLE CALCULATIONS

Figures 1 and 2 depict a comparison of the results of the present author and Dr. Hackman for the scattering from a 4:1 solid aluminum prolate spheroid at end-on incidence (0 degrees) and 30 degrees incidence.

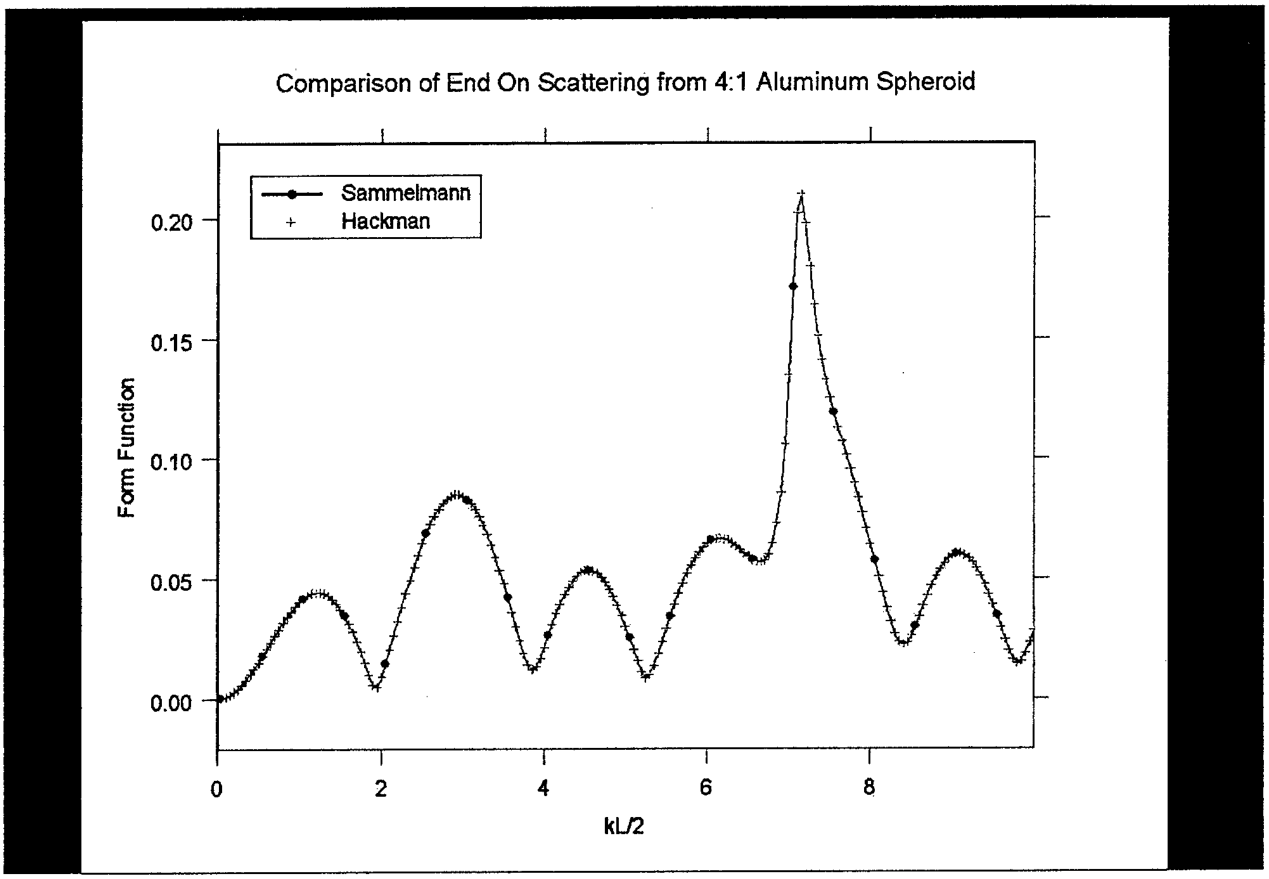

FIGURE 1. FORM FUNCTION AT END-ON INCIDENCE

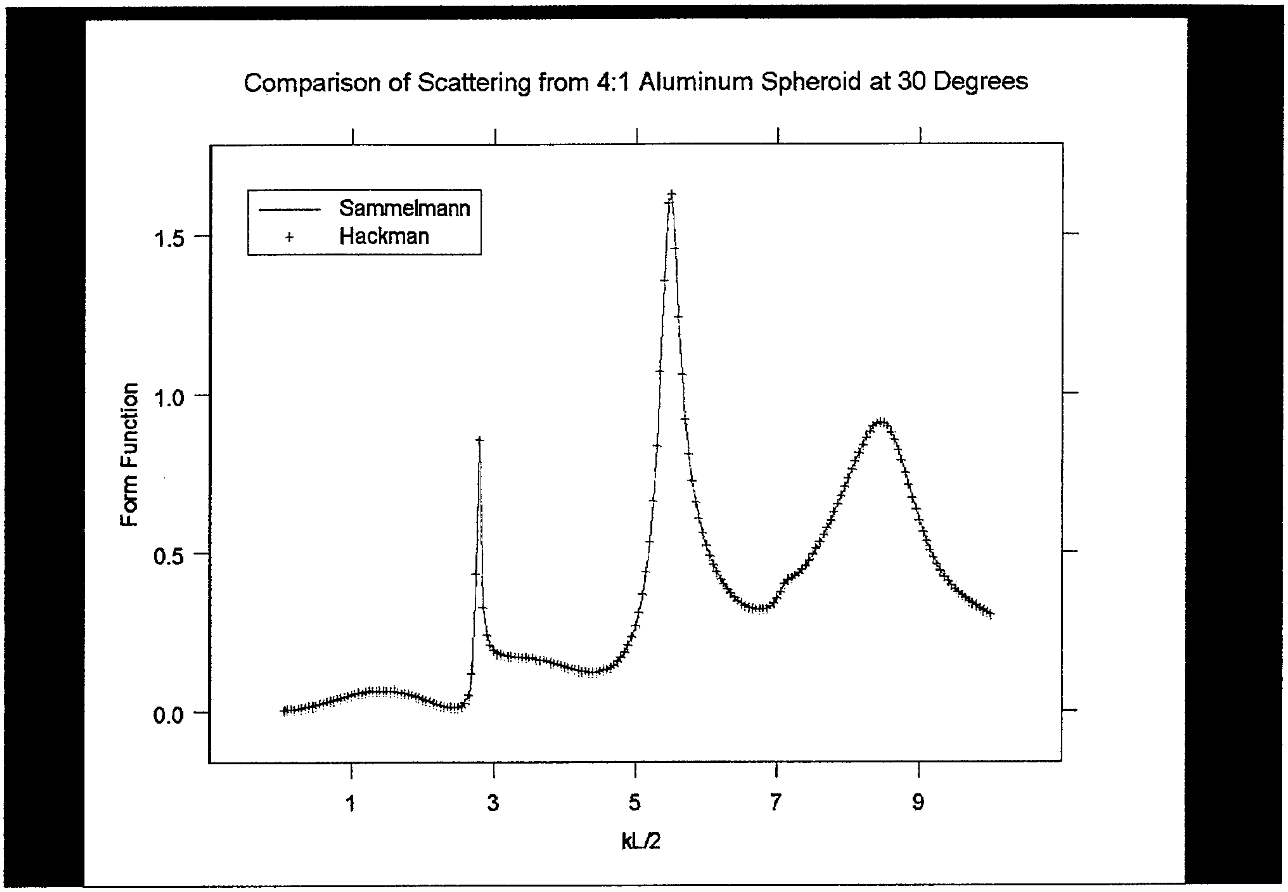

FIGURE 2. FORM FUNCTION AT 30 DEGREES INCIDENCE

Calculations made by the present author and Dr. Hackman agree within machine precision in the above two cases.

Figure 3 depicts a polar plot of the form function at kL/2=5.5, which corresponds to the frequency of the large flexural wave depicted in Figure 2. The directivity pattern of the flexural resonance at kL/2=5.5 has the characteristic shape of a quadrupole.

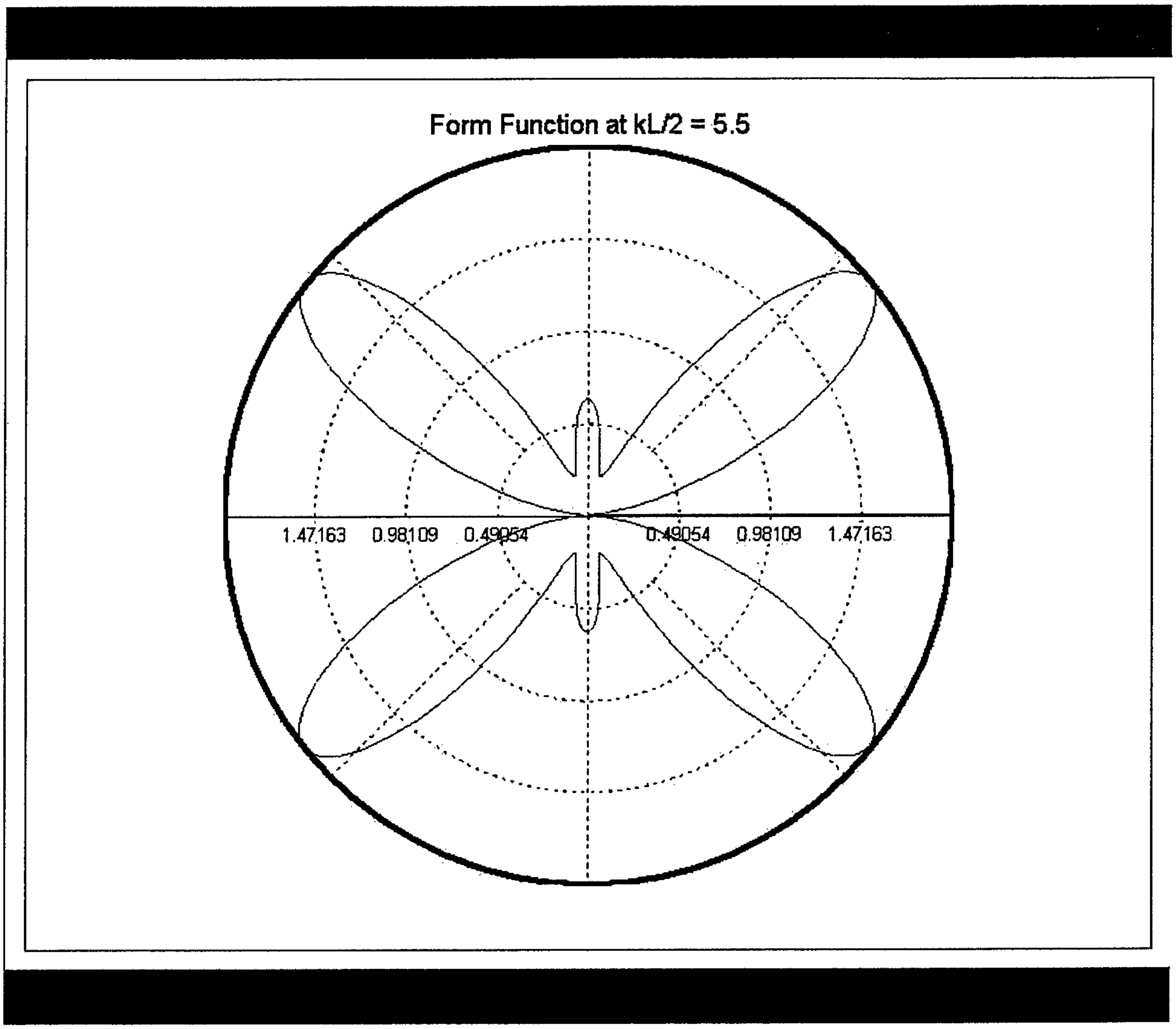

FIGURE 3. FORM FUNCTION AT KL/2 = 5.5

Figure 4 depicts a polar plot of the bistatic form function at kL/2=5.5 at end-on incidence. The forward scattering is strongly peaked at the radiation lobes of the flexural wave.

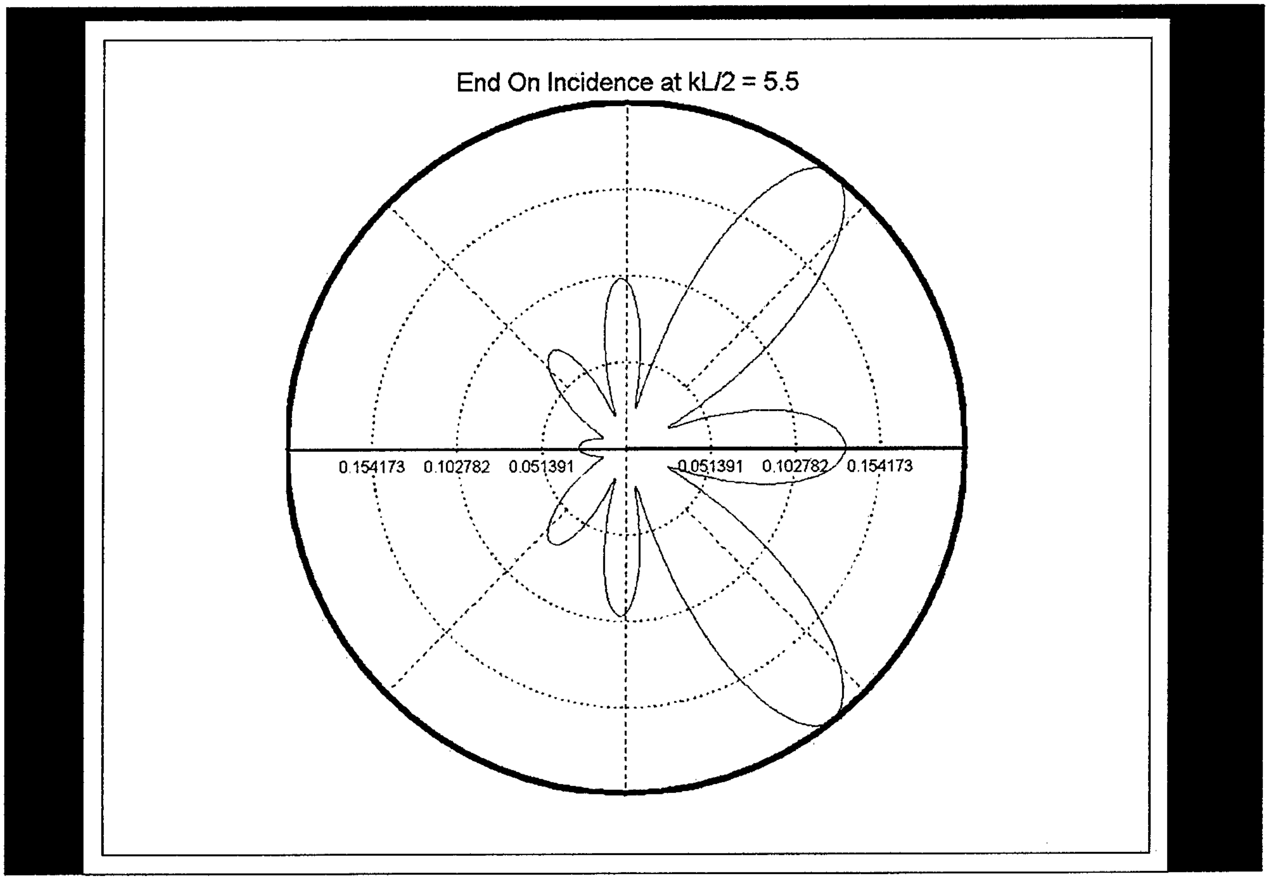

FIGURE 4. BISTATIC FORM FUNCTION AT KL/2 = 5.5 FOR END-ON INCIDENCE

Figure 5 illustrates a two-dimensional rendering of the form function of a 4:1 aluminum spheroid as a function of frequency and aspect angle. The first three flexural resonances of the spheroid are clearly legible.

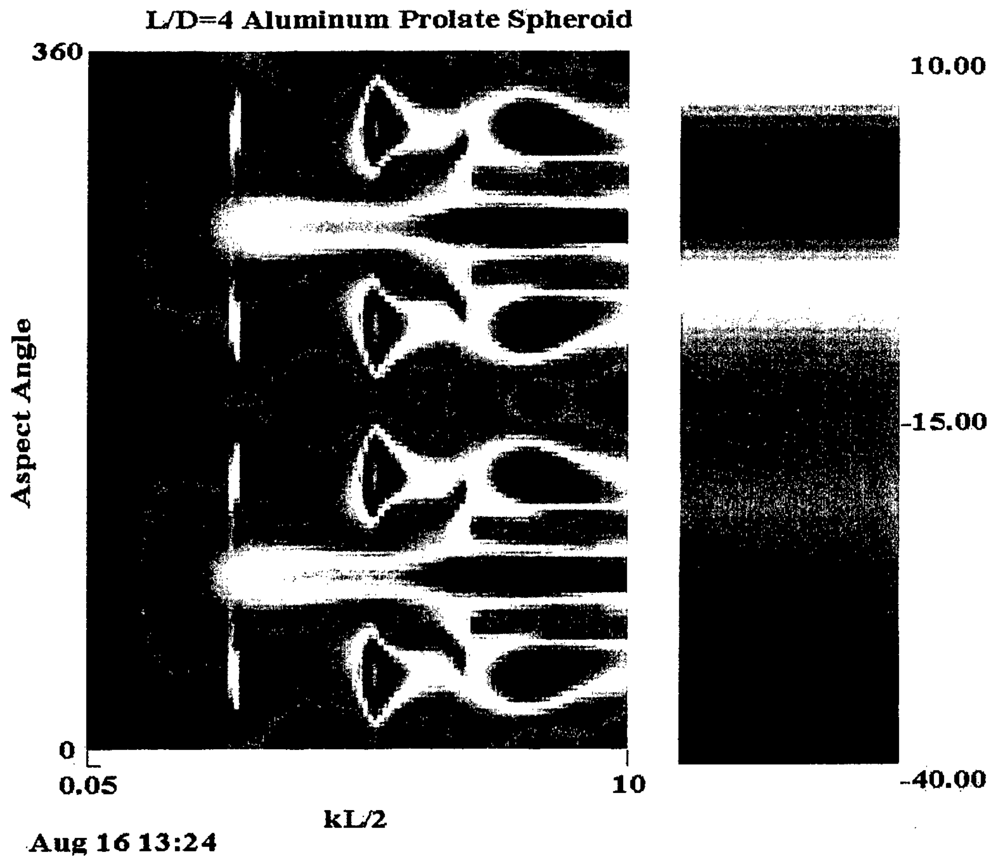

FIGURE 5. TWO-DIMENSIONAL PLOT OF THE FORM FUNCTION AS A FUNCTION OF FREQUENCY VERSUS ASPECT ANGLE

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APPENDIX A

COMPUTATION OF EIGENVALUES AND EIGENVECTORS

APPENDIX A - COMPUTATION OF EIGENVALUES AND EIGENVECTORS

This appendix describes the computation of the eigenvalues and coefficients d_k^{ml} of the spheroidal wave equation.

The expansion coefficients d_k^{ml} satisfy the following recurrence relation:

$$\alpha_k^m d_{k+2}^{ml} + (\beta_k^m - \lambda_{ml}) d_k^{ml} + \gamma_k^m d_{k-2}^{ml} = 0$$

$$\alpha_k^m = \frac{(2m+k+2)(2m+k+1)c^2}{(2m+2k+3)(2m+2k+5)}$$

$$\beta_k^m = (m+k)(m+k+1)$$
$$+c^2 \frac{2(m+k)(m+k+1) - 2m^2 - 1}{(2m+2k-1)(2m+2k+3)}$$

$$\gamma_k^m = \frac{k(k-1)c^2}{(2m+2k-3)(2m+2k-1)}$$

Define the ratio

$$N_k = -\alpha_{k-2}^m \frac{d_k^{ml}}{d_{k-2}^{ml}}$$

These ratios obey the following forward and backward recurrence relations:

$$N_{k+2} = \beta_k^m - \lambda - \frac{\alpha_{k-2}^m \gamma_k^m}{N_k}$$

$$N_k = \frac{\alpha_{k-2}^m \gamma_k^m}{\beta_k^m - \lambda - N_{k+2}}$$

$$N_{2+\delta} = \beta_\delta^m - \lambda$$

$$\delta \equiv (l-m) \bmod 2$$

The eigenvalue λ is chosen such that the following limit vanishes:

$$\lim_{k\to\infty} N_k = 0$$

The equation for the eigenvalues is obtained by using the forward recurrence relation to compute t he value of the above ratio at $k = 2 + l - m$. Next, one chooses a sufficiently large value of $k = K$ such that the ratio $N_{K+2} = 0$ vanishes and computes the value of the ratio of $k = 2 + l - m$ from the backward recurrence relation. The eigenvalue is adjusted to make the difference between these two estimates of the ratio at $k = 2 + l - m$ vanish.

Once the eigenvalue has been obtained, the values of the above ratio for $k \le 2 + l - m$ are computed from the forward recurrence relation. For indices $k \ge 2 + l - m$, the above ratio is calculated from the backward recurrence relation. The rationale for splitting the above calculation into a combination of forward and backward recurrence is to avoid loss of precision by recurring in the direction of increasing magnitude of the ratio. The expansion coefficients d_k^{ml} are obtained from the recurrence relation:

$$d_\delta^{ml} = 1$$

$$d_{k+2}^{ml} = -\frac{N_{k+2}}{\alpha_k^m} d_k^{ml}$$

The above prescription defines the expansion coefficients d_k^{ml} up to an arbitrary normalization condition. One adopts the normalization condition of References 1 and 2 defined below:

$$S_{ml}^{(1)}(0) = P_l^m(0), \text{l-m even}$$

$$S_{ml}^{(1)}{}'(0) = P_l^m{}'(0), \text{l-m odd}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(2n+2m+2\delta)!}{2^{2n+\delta}(n)!(n+m+\delta)!} d_{2n+\delta}^{ml} = (-1)^{(l-m-\delta)/2} \frac{(l+m+\delta)!}{2^{l-m}(\frac{l-m-\delta}{2})!(\frac{l+m+\delta}{2})!}$$

$$\delta \equiv (l-m) \bmod 2$$

An alternative method of computing the eigenvalues and eigenvectors of the spheroidal wave equations utilizes the fact that the recurrence relation for the expansion coefficients can be represented in the following matrix form as an eigenvalue problem:

$$\begin{pmatrix} \beta_\delta & \alpha_\delta & 0 & 0 \\ \gamma_{\delta+2} & \beta_{\delta+2} & \alpha_{\delta+2} & 0 \\ 0 & \gamma_{\delta+4} & \beta_{\delta+4} & \alpha_{\delta+4} \\ 0 & 0 & \gamma_{\delta+6} & \beta_{\delta+6} \end{pmatrix} \begin{pmatrix} d_\delta^{ml} \\ d_{\delta+2}^{ml} \\ d_{\delta+4}^{ml} \\ d_{\delta+6}^{ml} \end{pmatrix} = \lambda \begin{pmatrix} d_\delta^{ml} \\ d_{\delta+2}^{ml} \\ d_{\delta+4}^{ml} \\ d_{\delta+6}^{ml} \end{pmatrix}$$

The matrix can be transformed into a symmetric, tri-diagonal matrix by defining the following expansion coefficients:

$$\Pi_0 \equiv 1, \ \mathrm{n} = 0$$

$$\Pi_n \equiv \sqrt{\frac{\alpha_{\delta+2(n-1)}}{\gamma_{\delta+2n}}\Pi_{n-1}} = \sqrt{\Pi_{j=0}^{n-1} \frac{\alpha_{\delta+2j}}{\gamma_{\delta+2(j+1)}}}, \ \mathrm{n} > 0$$

$$e_0 \equiv d_\delta^{ml}, \ \mathrm{n} = 0$$

$$e_n \equiv \Pi_n d_{2n+\delta}^{ml}, \ \mathrm{n} > 0$$

These coefficients obey the recurrence relation:

$$a_n \equiv \sqrt{\alpha_{\delta+2n}^m \gamma_{\delta+2n+2}^m}$$

$$b_n \equiv \beta_{\delta+2n}^m$$

$$a_0 e_1 + (b_0 - \lambda) e_0 = 0, \mathrm{n} = 0$$

$$a_n e_{n+1} + (b_n - \lambda) e_n + a_{n-1} b_{n-1} = 0, \mathrm{n} > 0$$

The eigenvalues and the eigenvectors are solutions of the eigenvalue problem for the following symmetric tri-diagonal matrix:

$$\begin{pmatrix} b_0 & a_0 & 0 & 0 & \\ a_0 & b_1 & a_1 & 0 & \ddots \\ 0 & a_1 & b_2 & a_2 & \\ 0 & 0 & a_2 & b_3 & \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = \lambda \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

This method of solving for the eigenvectors is numerically more robust than solving the transcendental equation for the eigenvalues presented previously. In addition, this method is readily generalized to the case the wavenumber *(c)* is complex. *Note: the user is required to solve the above eigenvalue problem for even and odd parity eigenvalues separately.*

The value of the expansion coefficients d_k^{ml} for $-2m+\delta \le k \le -2+\delta$ can be calculated from the backward recurrence relation for the ratio of adjacent coefficients.

$$N_k = \frac{\alpha_{k-2}^m \gamma_k^m}{\beta_k^m - \lambda - N_{k+2}}$$

$$N_{2+\delta} = \beta_\delta^m - \lambda$$

$$\delta \equiv (l-m) \bmod 2$$

Alternatively one can use the following forward recurrence relation for the coefficients d_{-k}^{ml}":

$$r_{-k} = -\alpha_{-k}^m / (\beta_{-k}^m - \lambda + \gamma_{-k}^m r_{-k-2})$$

$$r_{-k} = \begin{cases} \dfrac{d_{-k}^{ml}}{d_{-k+2}^{ml}}, k < 2m+2-\delta \\ 0, k \ge 2m+2-\delta \end{cases}$$

Since the term $r_{-2m-2+\delta} = 0$ vanishes, the above recurrence relation expresses the ratio as a finite continued fraction. The expansion coefficients d_{-k}^{ml} are given by the following expression:

$$(\Pi_{j=0}^n r_{-2n-2+\delta}) d_\delta^{ml} = d_{-2n-2+\delta}^{ml}, n < m$$

In the case $k \ge -2m-2+\delta$, the coefficients $d_{k+\rho}^{ml}$ have a simple zero in the limit $\rho \to 0$. In this case, one is interested in the coefficients $d_{k|\rho}^{ml}$ for $k \ge -2m-2+\delta$ defined by the relation.

$$d_{k|\rho}^{ml} = \lim_{\rho \to 0} \frac{d_{-k+\rho}^{ml}}{\rho}, \mathrm{k} \ge 2m+2-\delta$$

One may use the following forward recurrence relation for the coefficients $d_{k|\rho}^{ml}$:

$$\hat{r}_{-k} = -\hat{\alpha}^m_{-k} / (\beta^m_{-k} - \lambda + \gamma^m_{-k} r_{-k-2}), k \geq 2m+2-\delta$$

$$\hat{r}_{-k} \equiv \begin{cases} \dfrac{d^{ml}_{2m+2-\delta|\rho}}{d^{ml}_{-2m+\delta}}, k = 2m+2-\delta \\ \dfrac{d^{ml}_{k|\rho}}{d^{ml}_{k-2|\rho}}, k > 2m+2-\delta \end{cases}$$

$$\hat{\alpha}^m_{-k} = \begin{cases} \alpha^m_{-k}, k \neq 2m+2-\delta \\ -\dfrac{c^2}{(2m-1)(2m+1)}, k = 2m+2-\delta, \delta = 0 \\ +\dfrac{c^2}{(2m-3)(2m-1)}, k = 2m+2-\delta, \delta = 1 \end{cases}$$

Here the coefficients $d^{ml}_{k|\rho}$ are given by the following expression in terms of the ratios $\hat{r}_k$:

$$(\prod^n_{j=0} \hat{r}_{-2(m+n)-2+\delta}) d^{ml}_{-2m+\delta} = d^{ml}_{+2(m+n)+2-\delta|\rho}$$

<u>REFERENCES</u>:

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APPENDIX B

EIGENVECTOR ORTHOGONALITY RELATIONS

APPENDIX B - EIGENVECTOR ORTHOGONALITY RELATIONS

This appendix contains the orthogonality relations satisfied by the expansion coefficients d_k^{ml}.

The expansion coefficients d_k^{ml} satisfy the following relationships:

$$\sum_{k=0,1}^{\infty} \frac{2}{(2m+2k+1)} \frac{(2m+k)!}{k!} d_k^{ml} d_k^{ml'} = \Lambda_{ml} \delta_l^{l'}$$

$$\sum_{l=m}^{\infty} d_k^{ml} d_{k'}^{ml} / \Lambda_{ml} = \frac{(2m+2k+1)}{2} \frac{k!}{(2m+k)!} \delta_k^{k'}$$

$$\Lambda_{ml} = \int_{-1}^{+1} d\eta S_{ml}^{(1)}(c,\eta) S_{ml}^{(1)}(c,\eta)$$

The above relationships represent the orthogonality and completeness of the expansion coefficients viewed as eigenvectors of an infinite dimensional tri-diagonal matrix.

APPENDIX C

GREEN'S FUNCTION EXPANSION

APPENDIX C - GREEN'S FUNCTION EXPANSION

This appendix contains the expansion of the scalar Green's Function and the plane wave in terms of spheroidal wave functions:

$$G(r,r') = \frac{\exp[+ik\,|r-r'|]}{4\pi\,|r-r'|} = ik\sum_{\sigma ml} he_{ml}(c,\xi_{>})\,je_{ml}(c,\xi_{<})S_{\sigma ml}(\eta,\varphi)S_{\sigma ml}(\eta',\varphi')$$

$$\exp[+i\vec{k}\bullet\vec{r}] = \exp[+ikr\cos(\gamma)] = 4\pi\sum_{\sigma ml}(+i)^{l}\,je_{ml}(c,\xi)S_{\sigma ml}(\eta,\varphi)S_{\sigma ml}(\eta',\varphi')$$

$$\cos(\gamma) = \cos(\vartheta)\cos(\vartheta') + \sin(\vartheta)\sin(\vartheta')\cos(\varphi-\varphi')$$

APPENDIX D

TRANSFORMATION FROM SPHERICAL TO SPHEROIDAL

APPENDIX D - TRANSFORMATION FROM SPHERICAL TO SPHEROIDAL

This appendix contains the transformation between spherical and spheroidal wave functions.

The transformation between spherical and spheroidal wave functions is given by the following transformation, where the matrix $B_{\sigma ml,\sigma' m' l'}$ is a unitary matrix:

$$he_{ml} S_{\sigma ml} = \sum_{\sigma' m' l'} B_{\sigma ml,\sigma' m' l'} h_{l'} Y_{\sigma' m' l'}$$

$$h_l Y_{\sigma ml} = \sum_{\sigma' m' l'} B^{-1}{}_{\sigma ml,\sigma' m' l'} he_{m' l'} S_{\sigma' m' l'}$$

$$B_{\sigma ml,\sigma' m' l'} = \delta_\sigma^{\sigma'} \delta_m^{m'} \frac{(+i)^{l'-l}}{\sqrt{\Lambda_{ml}}} \sqrt{\frac{2}{(2l'+1)} \frac{(l'+m)!}{(l'-m)!}} d_{l'-m}^{ml}$$

$$B^{-1}{}_{\sigma ml,\sigma' m' l'} = \delta_\sigma^{\sigma'} \delta_m^{m'} \frac{(+i)^{l'-l}}{\sqrt{\Lambda_{ml'}}} \sqrt{\frac{2}{(2l+1)} \frac{(l+m)!}{(l-m)!}} d_{l-m}^{ml'}$$

$$B^{-1}{}_{\sigma ml,\sigma' m' l'} = B^{\dagger}{}_{\sigma ml,\sigma' m' l'}$$

APPENDIX E

COMPUTATION OF RADIAL FUNCTIONS

APPENDIX E - COMPUTATION OF RADIAL FUNCTIONS

This appendix describes various representations of the radial functions. These representations are based on integral representations of the following form described by Flammer in Reference 1.

$$R_{ml}(c,\xi)=\int_a^b d\eta K(\xi,\eta)S_{ml}^{(1)}(c,\eta)$$

The following Theorem and Lemma are from Reference 1:

THEOREM:

Let L_ξ and L_η be the following differential operators:

$$L_\xi=\partial_\xi(\xi^2-1)\partial_\xi+c^2\xi^2-\frac{m^2}{(\xi^2-1)}$$

$$L_\eta=\partial_\eta(1-\eta^2)\partial_\eta-c^2\eta^2-\frac{m^2}{(1-\eta^2)}$$

Let $K(\xi,\eta)$ be a solution of the following partial differential equation in the complex domain D:

$$(L_\xi-L_\eta)K(\xi,\eta)=0$$

Let $[a,b]$ be the contour in the complex domain D such that the following integral vanishes:

$$\int_a^b d\eta\{(L_\eta K(\xi,\eta))S_{ml}^{(1)}(c,\eta)-K(\xi,\eta)L_\eta S_{ml}^{(1)}(c,\eta)\}=$$

$$(1-\eta^2)\{(\partial_\eta K(\xi,\eta))S_{ml}^{(1)}(c,\eta)-K(\xi,\eta)\partial_\eta S_{ml}^{(1)}(c,\eta)\}_a^b=0$$

Then the following contour integral is a solution of the radial equation in the complex domain D:

$$R_{ml}(c,\xi)=\int_a^b d\eta K(\xi,\eta)S_{ml}^{(1)}(c,\eta)$$

$$(L_\xi+\lambda_{ml})R_{ml}=0$$

LEMMA:

Suppose:

$$\psi(\xi,\eta,\varphi) = K(\xi,\eta)\exp[+im\varphi]$$

is a solution of the Helmholtz Wave Equation.

$$(\nabla^2 + k^2)\psi = 0$$

Then the function $K(\xi,\eta)$ is a solution of the partial differential equation:

$$(L_\xi - L_\eta)K(\xi,\eta) = 0$$

Some of the examples of kernels satisfying the above partial differential equation listed in Reference 1 are listed below:

$$(\xi^2-1)^{m/2}(1-\eta^2)^{m/2}\exp[+ic\xi\eta]$$

$$J_m(c\sqrt{(\xi^2-1)(1-\eta^2)})$$

$$\xi\eta J_m(c\sqrt{(\xi^2-1)(1-\eta^2)})$$

$$J_m(c\sqrt{(\xi^2-1)(1-\eta^2)}\sin(\alpha))\exp[\pm ic\xi\eta\cos(\alpha)]$$

$$j_m(c(\xi\pm\eta))(\xi^2-1)^{m/2}(1-\eta^2)^{m/2}(c(\xi\pm\eta))^{-m}$$

$$h^{(1)}{}_m(c(\xi\pm\eta))(\xi^2-1)^{m/2}(1-\eta^2)^{m/2}(c(\xi\pm\eta))^{-m}$$

The classic expansion [1-2] of the radial functions in terms of spherical Bessel Functions follows:

$$je_{ml}(c,\xi)=\frac{1}{\Lambda_{ml}^{(r)}}\frac{(\xi^2-1)^{m/2}}{\xi^m}\sum_{n=0}^{\infty}(+i)^{2n+\delta+m-l}\frac{(2m+2n+\delta)!}{(2n+\delta)!}d_{2n+\delta}^{ml}j_{2n+m+\delta}(c\xi)$$

$$ne_{ml}(c,\xi)=\frac{1}{\Lambda_{ml}^{(r)}}\frac{(\xi^2-1)^{m/2}}{\xi^m}\sum_{n=0}^{\infty}(+i)^{2n+\delta+m-l}\frac{(2m+2n+\delta)!}{(2n+\delta)!}d_{2n+\delta}^{ml}n_{2n+m+\delta}(c\xi)$$

$$\delta=(l-m)\bmod 2$$

$$\Lambda_{ml}^{(r)}=\sum_{n=0}^{\infty}\frac{(2n+2m+\delta)!}{(2n+\delta)!}d_{2n+\delta}^{ml}$$

The expansion for the regular function je_{ml} is absolutely convergent, whereas the expansion for the irregular function ne_{ml} is an asymptotic expansion unsuitable for radial coordinates near unity.

The expansions of the radial functions in terms of the analytic continuation of the angular functions are given by the following expressions:

$$je_{ml}(c,\xi)=\frac{1}{\kappa_{ml}^{(1)}}S_{ml}^{(1)}(c,\xi)$$

$$ne_{ml}(c,\xi)=\frac{1}{\kappa_{ml}^{(2)}}S_{ml}^{(2)}(c,\xi)$$

$$\kappa_{ml}^{(1)}=+\frac{(2m+2\delta+1)(l+m+\delta)!}{2^{l+m}c^{m+\delta}d_{\delta}^{ml}m!(\frac{l-m-\delta}{2})!(\frac{l+m+\delta}{2})!}\sum_{n=0}^{\infty}\frac{(2m+2n+\delta)!}{(2n+\delta)!}d_{2n+\delta}^{ml}$$

$$\kappa_{ml}^{(2)}=+\frac{2^{l-m}(2m)!(\frac{l-m}{2})!(\frac{l+m}{2})!d_{-2m}^{ml}}{(2m-1)m!(l+m)!c^{m-1}}\sum_{n=0}^{\infty}\frac{(2m+2n)!}{(2n)!}d_{2n}^{ml},\ \text{even (l-m)}$$

$$\kappa_{ml}^{(2)}=-\frac{2^{l-m}(2m)!(\frac{l-m-1}{2})!(\frac{l+m+1}{2})!d_{-2m+1}^{ml}}{(2m-1)(2m-3)m!(l+m+1)!c^{m-2}}\sum_{n=0}^{\infty}\frac{(2m+2n+\delta)!}{(2n+\delta)!}d_{2n+\delta}^{ml},\ \text{odd (l-m)}$$

The above representation of the irregular radial function ne_{ml} in terms of associated Legendre polynomials is useful for calculating the radial function for arguments in the range $+1<\xi\leq 1.5$. In the above expressions, one uses the analytic continuation of the associated Legendre polynomial to the complex plane:

$$P_l^m(z) = (z^2-1)^{m/2}\left(\frac{d}{dz}\right)^m P_l(z)$$

$$Q_l^m(z) = (z^2-1)^{m/2}\left(\frac{d}{dz}\right)^m Q_l(z)$$

The following expansion of the regular radial function in terms of spherical Bessel Functions is sometimes useful for arguments near unity:

$$je_{ml} = \frac{1}{S_{ml}^{(1)}(0)}\sum_k (+i)^{k+m-l} d_k^{ml} P_{m+k}^m(0) j_{m+k}(c\sqrt{\xi^2-1}) \quad \text{(even l-m)}$$

$$je_{ml} = \frac{1}{S'^{(1)}_{ml}(0)}\frac{\xi}{\sqrt{\xi^2-1}}\sum_k (+i)^{k+m-l} d_k^{ml} P'^{m}_{m+k}(0) j_{m+k}(c\sqrt{\xi^2-1}) \quad \text{(odd l-m)}$$

There is an error in Reference 1 in the case *l-m* is odd in the above expansion.

The radial functions have the following expansions in terms of the product of spherical Bessel Functions:

$\xi=\cosh(\mu/2)$

Even l-m

$$he_{ml} = (\xi^2-1)^{m/2}\sum_{n=0}^{\infty}(2n+2m+1)D_n^{ml} h_{m+n}^{(1)}(ce^{+\mu/2}) j_{m+n}(ce^{-\mu/2})$$

$$D_n^{ml} = \frac{(+i)^{l-m}2^m(n+2m)!}{d_0^{ml}c^m m!n!}\sum_{k=0}^{n} d_{2k}^{ml}\frac{(-n)_k(n+2m+1)_k(m+1/2)_k}{k!}\frac{\Gamma(m+3/2)}{\Gamma(m+3/2+2k)}$$

$$\times {}_3F_2(-n+k, n+k+2m+1, k+1/2; m+k+1, m+2k+3/2 : +1)$$

$$= \frac{(+i)^{l-m}(2m+1)2^{m-1}(n+2m)!}{d_0^{ml}c^m m!n!}\sum_{k=0}^{n} d_{2k}^{ml}\frac{(-n)_k(n+2m+1)_k(m+1/2)_k}{k!}\frac{\Gamma(m+n-k+1/2)}{\Gamma(m+n+k+3/2)}$$

Odd l-m

$$he_{ml} = \xi(\xi^2 - 1)^{m/2} \sum_{n=0}^{\infty} (2n + 2m + 3) h^{(1)}_{m+n+1}(ce^{+\mu/2}) j_{m+n+1}(ce^{-\mu/2}) D^{ml}_n$$

$$D^{ml}_n = \frac{(+i)^{l-m-1}(2m+3)2^m(2m+n+2)!}{d^{ml}_1 c^{m+1} m! n!} \sum_{k=0}^{n} d^{ml}_{2k+1} \frac{(-n)_k (m+3/2)_k (n+2m+3)_k}{k!} \frac{\Gamma(m+3/2)}{\Gamma(m+5/2+2k)}$$

$$\times {}_3F_2(-n+k, k+3/2, n+2m+k+3; m+k+2, m+2k+5/2 : +1)$$

The Pochammer and generalized Hypergeometric function are defined below:

$$(x)_k = \Gamma(x+k)/\Gamma(x) = \prod_{n=0}^{k-1}(x+n)$$

$${}_3F_2(a_1, a_2, a_3; b_1, b_2 : z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n}{n!(b_1)_n (b_2)_n} z^n$$

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1. Flammer, C., (1957), *Spheroidal Wave Functions*, Stanford University Press, Stanford CA.

2. Hodge, D.B., (1970), Eigenvalues and Eigenfunctions of the Spheroidal Wave Equation, Journal of Mathematics and Physics, V. 11 pp. 2308-2312

APPENDIX F

CONNECTION IN SPHEROIDAL COORDINATES

APPENDIX F - CONNECTION IN SPHEROIDAL COORDINATES

This appendix contains a tabulation of the non-zero terms of the linear connection, its derivatives, and the function $P^{\mu}_{\alpha\beta\gamma} = \partial_{\alpha}\Gamma^{\mu}_{\beta\gamma} + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\beta\gamma}$ in prolate spheroidal coordinates. These functions are used to compute the second and third order covariant derivatives of the scalar basis functions used in the construction of the vector basis and the stress tensor:

$$\Gamma^{\xi}_{\xi\xi} = -\frac{\xi(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}$$

$$\Gamma^{\eta}_{\eta\eta} = +\frac{\eta(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)}$$

$$\Gamma^{\xi}_{\xi\eta} = \Gamma^{\xi}_{\eta\xi} = -\frac{\eta}{(\xi^2-\eta^2)}$$

$$\Gamma^{\eta}_{\eta\xi} = \Gamma^{\eta}_{\xi\eta} = +\frac{\xi}{(\xi^2-\eta^2)}$$

$$\Gamma^{\xi}_{\eta\eta} = -\frac{\xi(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)}$$

$$\Gamma^{\eta}_{\xi\xi} = +\frac{\eta(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}$$

$$\Gamma^{\varphi}_{\varphi\eta} = \Gamma^{\varphi}_{\eta\varphi} = -\frac{\eta}{(1-\eta^2)}$$

$$\Gamma^{\varphi}_{\varphi\xi} = \Gamma^{\varphi}_{\xi\varphi} = +\frac{\xi}{(\xi^2-1)}$$

$$\Gamma^{\xi}_{\varphi\varphi} = -\frac{\xi(\xi^2-1)(1-\eta^2)}{(\xi^2-\eta^2)}$$

$$\Gamma^{\eta}_{\varphi\varphi} = +\frac{\eta(1-\eta^2)(\xi^2-1)}{(\xi^2-\eta^2)}$$

$$\partial_{\xi}\Gamma^{\xi}_{\xi\xi} = -\frac{(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}\{1-\frac{2\xi^2}{(\xi^2-1)}-\frac{2\xi^2}{(\xi^2-\eta^2)}\}$$

$$\partial_{\eta}\Gamma^{\xi}_{\xi\xi} = \frac{2\xi\eta}{(\xi^2-\eta^2)^2}$$

$$\partial_\xi \Gamma^\xi_{\xi\eta} = \partial_\xi \Gamma^\xi_{\eta\xi} = +\frac{2\xi\eta}{(\xi^2-\eta^2)^2}$$

$$\partial_\eta \Gamma^\xi_{\xi\eta} = \partial_\eta \Gamma^\xi_{\eta\xi} = -\frac{(\xi^2+\eta^2)}{(\xi^2-\eta^2)^2}$$

$$\partial_\xi \Gamma^\xi_{\eta\eta} = +\frac{1}{(1-\eta^2)(\xi^2-\eta^2)}\{1-3\xi^2+\frac{2\xi^2(\xi^2-1)}{(\xi^2-\eta^2)}\}$$

$$\partial_\eta \Gamma^\xi_{\eta\eta} = -\frac{2\xi\eta(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)}\{\frac{1}{(1-\eta^2)}+\frac{1}{(\xi^2-\eta^2)}\}$$

$$\partial_\xi \Gamma^\xi_{\varphi\varphi} = +\frac{(1-\eta^2)}{(\xi^2-\eta^2)}\{1-3\xi^2+\frac{2\xi^2(\xi^2-1)}{(\xi^2-\eta^2)}\}$$

$$\partial_\eta \Gamma^\xi_{\varphi\varphi} = +\frac{2\xi\eta(\xi^2-1)^2}{(\xi^2-\eta^2)^2}$$

$$\partial_\xi \Gamma^\varphi_{\varphi\xi} = \partial_\xi \Gamma^\varphi_{\xi\varphi} = -\frac{(\xi^2+1)}{(\xi^2-1)^2}$$

$$\partial_\eta \Gamma^\varphi_{\varphi\xi} = \partial_\eta \Gamma^\varphi_{\xi\varphi} = 0$$

$$\partial_\xi \Gamma^\varphi_{\varphi\eta} = \partial_\xi \Gamma^\varphi_{\eta\varphi} = 0$$

$$\partial_\eta \Gamma^\varphi_{\varphi\eta} = \partial_\eta \Gamma^\varphi_{\eta\varphi} = -\frac{(1+\eta^2)}{(1-\eta^2)^2}$$

$$\partial_\xi \Gamma^\eta_{\eta\eta} = +\frac{2\xi\eta}{(\xi^2-\eta^2)^2}$$

$$\partial_\eta \Gamma^\eta_{\eta\eta} = +\frac{(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)}\{1+\frac{2\eta^2}{(1-\eta^2)}+\frac{2\eta^2}{(\xi^2-\eta^2)}\}$$

$$\partial_\xi \Gamma^\eta_{\eta\xi} = \partial_\xi \Gamma^\eta_{\xi\eta} = -\frac{(\xi^2+\eta^2)}{(\xi^2-\eta^2)^2}$$

$$\partial_\eta \Gamma^\eta_{\eta\xi} = \partial_\eta \Gamma^\eta_{\xi\eta} = +\frac{2\xi\eta}{(\xi^2-\eta^2)^2}$$

$$\partial_\xi \Gamma^\eta_{\xi\xi} = -\frac{2\xi\eta(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}\{\frac{1}{(\xi^2-1)} + \frac{1}{(\xi^2-\eta^2)}\}$$

$$\partial_\eta \Gamma^\eta_{\xi\xi} = +\frac{1}{(\xi^2-1)(\xi^2-\eta^2)}\{1-3\eta^2 + \frac{2\eta^2(1-\eta^2)}{(\xi^2-\eta^2)}\}$$

$$\partial_\xi \Gamma^\eta_{\varphi\varphi} = +\frac{2\xi\eta(1-\eta^2)^2}{(\xi^2-\eta^2)^2}$$

$$\partial_\eta \Gamma^\eta_{\varphi\varphi} = +\frac{(\xi^2-1)}{(\xi^2-\eta^2)}\{1-3\eta^2 + \frac{2\eta^2(1-\eta^2)}{(\xi^2-\eta^2)}\}$$

$$P^\xi_{\xi\xi\xi} = +\frac{3\xi^2(1-\eta^2)}{(\xi^2-1)^2(\xi^2-\eta^2)}$$

$$P^\eta_{\xi\xi\xi} = +\frac{\xi\eta}{(\xi^2-1)(\xi^2-\eta^2)}$$

$$P^\xi_{\eta\xi\xi} = -\frac{3\xi\eta(1-\eta^2)}{(\xi^2-1)^2(\xi^2-\eta^2)}$$

$$P^\eta_{\eta\xi\xi} = -\frac{\eta^2}{(\xi^2-1)(\xi^2-\eta^2)}$$

$$P^\xi_{\xi\xi\eta} = P^\xi_{\xi\eta\xi} = +\frac{\xi\eta}{(\xi^2-1)(\xi^2-\eta^2)}$$

$$P^\eta_{\xi\xi\eta} = P^\eta_{\xi\eta\xi} = -\frac{\xi^2}{(1-\eta^2)(\xi^2-\eta^2)}$$

$$P^\xi_{\eta\xi\eta} = P^\xi_{\eta\eta\xi} = -\frac{\eta^2}{(\xi^2-1)(\xi^2-\eta^2)}$$

$$P^\eta_{\eta\xi\eta} = P^\eta_{\eta\eta\xi} = +\frac{\xi\eta}{(1-\eta^2)(\xi^2-\eta^2)}$$

$$P^\xi_{\xi\eta\eta} = -\frac{\xi^2}{(1-\eta^2)(\xi^2-\eta^2)}$$

$$P^\eta_{\xi\eta\eta} = -\frac{3\xi\eta(\xi^2-1)}{(1-\eta^2)^2(\xi^2-\eta^2)}$$

$$P^{\xi}_{\eta\eta\eta} = +\frac{\xi\eta}{(1-\eta^2)(\xi^2-\eta^2)}$$

$$P^{\eta}_{\eta\eta\eta} = +\frac{3\eta^2(\xi^2-1)}{(1-\eta^2)^2(\xi^2-\eta^2)}$$

$$P^{\xi}_{\xi\varphi\varphi} = -\frac{\xi^2(1-\eta^2)}{(\xi^2-\eta^2)}$$

$$P^{\xi}_{\eta\varphi\varphi} = +\frac{\xi\eta(1-\eta^2)}{(\xi^2-\eta^2)}$$

$$P^{\eta}_{\xi\varphi\varphi} = +\frac{\xi\eta(\xi^2-1)}{(\xi^2-\eta^2)}$$

$$P^{\eta}_{\eta\varphi\varphi} = -\frac{\eta^2(\xi^2-1)}{(\xi^2-\eta^2)}$$

$$P^{\xi}_{\varphi\xi\varphi} = P^{\xi}_{\varphi\varphi\xi} = -\frac{1}{(\xi^2-1)^2}$$

$$P^{\xi}_{\varphi\eta\varphi} = P^{\xi}_{\varphi\varphi\eta} = -\frac{\xi\eta}{(1-\eta^2)(\xi^2-1)}$$

$$P^{\eta}_{\varphi\xi\varphi} = P^{\eta}_{\varphi\varphi\xi} = -\frac{\xi\eta}{(1-\eta^2)(\xi^2-1)}$$

$$P^{\eta}_{\varphi\eta\varphi} = P^{\eta}_{\varphi\varphi\eta} = -\frac{1}{(1-\eta^2)^2}$$

$$P^{\varphi}_{\xi\xi\varphi} = P^{\varphi}_{\xi\varphi\xi} = -\frac{\xi^2(1-\eta^2)}{(\xi^2-\eta^2)}$$

$$P^{\varphi}_{\xi\eta\varphi} = P^{\varphi}_{\xi\varphi\eta} = +\frac{\xi\eta(\xi^2-1)}{(\xi^2-\eta^2)}$$

$$P^{\varphi}_{\eta\xi\varphi} = P^{\varphi}_{\eta\varphi\xi} = +\frac{\xi\eta(1-\eta^2)}{(\xi^2-\eta^2)}$$

$$P^{\varphi}_{\eta\eta\varphi} = P^{\varphi}_{\eta\varphi\eta} = -\frac{\eta^2(\xi^2-1)}{(\xi^2-\eta^2)}$$

$$P^{\varphi}_{\varphi\xi\xi} = -\frac{1}{(\xi^2 - 1)^2}$$

$$P^{\varphi}_{\varphi\xi\eta} = P^{\varphi}_{\varphi\eta\xi} = -\frac{\xi\eta}{(1-\eta^2)(\xi^2 - 1)}$$

$$P^{\varphi}_{\varphi\eta\eta} = -\frac{1}{(1-\eta^2)^2}$$

$$P^{\varphi}_{\varphi\varphi\varphi} = -1$$

APPENDIX G

VECTOR BASIS FUNCTIONS

APPENDIX G - VECTOR BASIS FUNCTIONS

This appendix contains the components of the vector basis functions and their covariant derivatives for an elastic medium in spheroidal coordinates. One may use the index n to signify the group of indices σml. Here, ψ_n^T and ψ_n^L are the scalar basis functions for the transverse and longitudinal modes. The vector basis functions are defined by the following relationships.

$$a_\xi = f^2\xi$$
$$a_\eta = f^2\eta$$
$$a_\varphi = 0$$

$$g = Det(g_{\mu\nu}) = f^6(\xi^2-\eta^2)^2$$

$$\sqrt{\lambda}(V_n^1)_\mu = -\frac{g_{\mu\nu}\delta_{\xi\varphi\eta}^{\nu\alpha\beta}a_\alpha\nabla_\beta\psi_n^T}{\sqrt{g}}$$

$$k_T\sqrt{\lambda}(V_n^2)_\mu = k_T^2 a_\mu\psi_n^T + 2\nabla_\mu\psi_n^T + a^\sigma\nabla_\sigma\nabla_\mu\psi_n^T$$

$$k_L(V_n^3)_\mu = \nabla_\mu\psi_n^L$$

$$\sqrt{\lambda_n}\nabla_\mu(V_n^1)_\nu = +\frac{g_{\mu\sigma}g_{\nu\alpha}\delta_{\xi\varphi\eta}^{\sigma\alpha\beta}\nabla_\beta\psi_n^T - g_{\nu\sigma}\delta_{\xi\varphi\eta}^{\sigma\alpha\beta}a_\alpha\nabla_\beta\nabla_\mu\psi_n^T}{\sqrt{g}}$$

$$k_T\sqrt{\lambda_n}\nabla_\mu(V_n^2)_\nu = k_T^2 g_{\mu\nu}\psi_n^T + k_T^2 a_\nu\nabla_\mu\psi_n^T + 3\nabla_\mu\nabla_\nu\psi_n^T + a^\sigma\nabla_\sigma\nabla_\mu\nabla_\nu\psi_n^T$$

$$\nabla_\mu(V_n^3)_\nu = \frac{1}{k_L}\nabla_\mu\nabla_\nu\psi_n^L$$

First, one tabulates the components of the second and third order covariant derivatives of the scalar basis functions. These functions are used to construct the vector basis and its covariant derivative. Since the Riemann Curvature Tensor vanishes, the covariant derivatives commute in a coordinate basis. Thus, the higher order covariant derivatives of the scalar basis functions are symmetric tensors. The radial, angular, and azimuthal wave equations have been used to eliminate second and higher order derivatives with respect to a given variable:

$$\nabla_\xi \nabla_\xi \psi = -\frac{\eta(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}\partial_\eta \psi - \frac{\xi}{(\xi^2-1)}[2-\frac{(1-\eta^2)}{(\xi^2-\eta^2)}]\partial_\xi \psi + \frac{1}{(\xi^2-1)}[\lambda - c^2\xi^2 + \frac{m^2}{(\xi^2-1)}]\psi$$

$$\nabla_\eta \nabla_\eta \psi = +\frac{\xi(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)}\partial_\xi \psi + \frac{\eta}{(1-\eta^2)}[2-\frac{(\xi^2-1)}{(\xi^2-\eta^2)}]\partial_\eta \psi - \frac{1}{(1-\eta^2)}[\lambda - c^2\eta^2 - \frac{m^2}{(1-\eta^2)}]\psi$$

$$\nabla_\xi \nabla_\eta \psi = \partial_\xi \partial_\eta \psi + \frac{1}{(\xi^2-\eta^2)}[\eta\partial_\xi \psi - \xi\partial_\eta \psi]$$

$$\nabla_\xi \nabla_\phi \psi = \partial_\xi \partial_\varphi \psi - \frac{\xi}{(\xi^2-1)}\partial_\varphi \psi$$

$$\nabla_\eta \nabla_\phi \psi = \partial_\eta \partial_\varphi \psi + \frac{\eta}{(1-\eta^2)}\partial_\varphi \psi$$

$$\nabla_\varphi \nabla_\varphi \psi = -m^2\psi + \frac{(1-\eta^2)(\xi^2-1)}{(\xi^2-\eta^2)}[\xi\partial_\xi \psi - \eta\partial_\eta \psi]$$

$$\begin{aligned}
\nabla_\xi \nabla_\xi \nabla_\xi \psi = & -\frac{3\eta(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}\partial_\xi \partial_\eta \psi \\
& + \frac{2\xi\eta(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}[\frac{2}{(\xi^2-\eta^2)} + \frac{1}{(\xi^2-1)} - \frac{(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}]\partial_\eta \psi \\
& + \frac{3(1-\eta^2)}{(\xi^2-1)^2(\xi^2-\eta^2)^2}[\xi^2+\eta^2+\xi^2\eta^2-3\xi^4]\partial_\xi \psi \\
& + \frac{1}{(\xi^2-1)}[\lambda+6-c^2\xi^2+\frac{(m^2+8)}{(\xi^2-1)}]\partial_\xi \psi \\
& - \frac{\xi}{(\xi^2-1)}\{2c^2 + \frac{4}{(\xi^2-1)}[\lambda - c^2\xi^2] + \frac{6m^2}{(\xi^2-1)^2} - \frac{3(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}[\lambda - c^2\xi^2 + \frac{m^2}{(\xi^2-1)}]\}\psi
\end{aligned}$$

$$\nabla_\eta \nabla_\eta \nabla_\eta \psi = +\frac{3\xi(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)}\partial_\xi \partial_\eta \psi$$

$$+\frac{2\xi\eta(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)}[\frac{2}{(\xi^2-\eta^2)}+\frac{1}{(1-\eta^2)}-\frac{(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)}]\partial_\xi \psi$$

$$-\frac{3(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)^2}[\xi^2+\eta^2+\xi^2\eta^2-3\eta^4]\partial_\eta \psi$$

$$-\frac{1}{(1-\eta^2)}\{\lambda+6-c^2\eta^2-\frac{(m^2+8)}{(1-\eta^2)}\}\partial_\eta \psi$$

$$+\frac{\eta}{(1-\eta^2)}\{2c^2-\frac{4}{(1-\eta^2)}[\lambda-c^2\eta^2]+\frac{6m^2}{(1-\eta^2)^2}+\frac{3(\xi^2-1)}{(1-\eta^2)(\xi^2-1)}[\lambda-c^2\eta^2-\frac{m^2}{(1-\eta^2)}]\}\psi$$

$$\nabla_\varphi \nabla_\varphi \nabla_\varphi \psi = +\frac{3(1-\eta^2)(\xi^2-1)}{(\xi^2-\eta^2)}[\xi\partial_\xi \partial_\varphi \psi - \eta\partial_\eta \partial_\varphi \psi]-(m^2+2)\partial_\varphi \psi$$

$$\nabla_\xi \nabla_\xi \nabla_\eta \psi = -\xi[\frac{2}{(\xi^2-1)}+\frac{2}{(\xi^2-\eta^2)}-\frac{(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}]\partial_\xi \partial_\eta \psi$$

$$+\frac{\eta}{(\xi^2-1)(\xi^2-\eta^2)}[3\lambda-2c^2\xi^2-c^2\eta^2+\frac{2m^2}{(\xi^2-1)}-\frac{m^2}{(1-\eta^2)}]\psi$$

$$+\frac{2\xi\eta}{(\xi^2-1)(\xi^2-\eta^2)^2}[3-4\xi^2+\eta^2]\partial_\xi \psi$$

$$+\frac{1}{(\xi^2-1)}[\lambda-c^2\xi^2+\frac{m^2}{(\xi^2-1)}]\partial_\eta \psi$$

$$+\frac{1}{(\xi^2-1)(\xi^2-\eta^2)^2}[3\eta^4-3\eta^2+\xi^2\eta^2-3\xi^2+2\xi^4]\partial_\eta \psi$$

$$\nabla_\xi \nabla_\eta \nabla_\eta \psi = \eta[\frac{2}{(1-\eta^2)}+\frac{2}{(\xi^2-\eta^2)}-\frac{(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)}]\partial_\xi \partial_\eta \psi$$

$$+\frac{\xi}{(1-\eta^2)(\xi^2-\eta^2)}\{3\lambda-2c^2\eta^2-c^2\xi^2-\frac{2m^2}{(1-\eta^2)}+\frac{m^2}{(\xi^2-1)}\}\psi$$

$$-\frac{2\xi\eta}{(1-\eta^2)(\xi^2-\eta^2)^2}[3-4\eta^2+\xi^2]\partial_\eta \psi$$

$$-\frac{1}{(1-\eta^2)}[\lambda-c^2\eta^2-\frac{m^2}{(1-\eta^2)}]\partial_\xi \psi$$

$$-\frac{1}{(1-\eta^2)(\xi^2-\eta^2)^2}[3\xi^4-3\xi^2+\xi^2\eta^2-3\eta^2+2\eta^4]\partial_\xi \psi$$

$$\nabla_\xi \nabla_\eta \nabla_\varphi \psi = \partial_\xi \partial_\eta \partial_\varphi \psi + [\frac{\eta}{(1-\eta^2)} + \frac{\eta}{(\xi^2-\eta^2)}]\partial_\xi \partial_\varphi \psi - [\frac{\xi}{(\xi^2-1)} + \frac{\xi}{(\xi^2-\eta^2)}]\partial_\eta \partial_\varphi \psi$$
$$-[\frac{\xi\eta}{(1-\eta^2)(\xi^2-\eta^2)} + \frac{\xi\eta}{(1-\eta^2)(\xi^2-1)} + \frac{\xi\eta}{(\xi^2-1)(\xi^2-\eta^2)}]\partial_\varphi \psi$$

$$\nabla_\xi \nabla_\xi \nabla_\varphi \psi = +\frac{(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}[\xi \partial_\xi \partial_\varphi \psi - \eta \partial_\eta \partial_\varphi \psi]$$
$$+\frac{1}{(\xi^2-1)}[\lambda + 2 - c^2\xi^2 + \frac{m^2+2}{(\xi^2-1)}]\partial_\varphi \psi - \frac{4}{(\xi^2-1)}\xi \partial_\xi \partial_\varphi \psi$$

$$\nabla_\eta \nabla_\eta \nabla_\varphi \psi = +\frac{(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)}[\xi \partial_\xi \partial_\varphi \psi - \eta \partial_\eta \partial_\varphi \psi]$$
$$-\frac{1}{(1-\eta^2)}[\lambda + 2 - c^2\eta^2 - \frac{m^2+2}{(1-\eta^2)}]\partial_\varphi \psi + \frac{4}{(1-\eta^2)}\eta \partial_\eta \partial_\varphi \psi$$

$$\nabla_\xi \nabla_\varphi \nabla_\varphi \psi = -\frac{\eta(1-\eta^2)(\xi^2-1)}{(\xi^2-\eta^2)}\partial_\xi \partial_\eta \psi$$
$$+\frac{2\xi\eta(1-\eta^2)(\xi^2-1)}{(\xi^2-\eta^2)^2}\partial_\eta \psi - \frac{2\xi^2(1-\eta^2)(\xi^2-1)}{(\xi^2-\eta^2)^2}\partial_\xi \psi - [m^2 + \frac{(1-\eta^2)(\xi^2+1)}{(\xi^2-\eta^2)}]\partial_\xi \psi$$
$$+\frac{\xi(1-\eta^2)}{(\xi^2-\eta^2)}[\lambda - c^2\xi^2 + \frac{m^2}{(\xi^2-1)}]\psi + \frac{2\xi m^2}{(\xi^2-1)}\psi$$

$$\nabla_\eta \nabla_\varphi \nabla_\varphi \psi = +\frac{\xi(1-\eta^2)(\xi^2-1)}{(\xi^2-\eta^2)}\partial_\xi \partial_\eta \psi$$
$$+\frac{2\xi\eta(1-\eta^2)(\xi^2-1)}{(\xi^2-\eta^2)^2}\partial_\xi \psi - \frac{2\eta^2(1-\eta^2)(\xi^2-1)}{(\xi^2-\eta^2)^2}\partial_\eta \psi - \{m^2 + \frac{(1+\eta^2)(\xi^2-1)}{(\xi^2-\eta^2)}\}\partial_\eta \psi$$
$$+\frac{\eta(\xi^2-1)}{(\xi^2-\eta^2)}[\lambda - c^2\eta^2 - \frac{m^2}{(1-\eta^2)}]\psi - \frac{2\eta m^2}{(1-\eta^2)}\psi$$

The component expression for the above vectors in terms of the scalar basis functions are as follows:

$$\sqrt{\lambda_n}(V_n^1)_\xi = +f\frac{\eta}{(\xi^2-1)}\partial_\varphi \psi_n^T$$

$$\sqrt{\lambda_n}(V_n^1)_\eta = -f\frac{\xi}{(1-\eta^2)}\partial_\varphi \psi_n^T$$

$$\sqrt{\lambda_n}(V_n^1)_\varphi = +f\frac{(1-\eta^2)(\xi^2-1)}{(\xi^2-\eta^2)}\{\xi\partial_\eta \psi_n^T - \eta\partial_\xi \psi_n^T\}$$

$$k_T\sqrt{\lambda_n}(V_n^2)_\xi = c_T^2\xi\psi_n^T + 2\nabla_\xi \psi_n^T + \frac{\xi(\xi^2 \mp 1)}{(\xi^2 \mp \eta^2)}\nabla_\xi \nabla_\xi \psi_n^T + \frac{\eta(1-\eta^2)}{(\xi^2 \mp \eta^2)}\nabla_\xi \nabla_\eta \psi_n^T$$

$$k_T\sqrt{\lambda_n}(V_n^2)_\eta = c_T^2\eta\psi_n^T + 2\nabla_\eta \psi_n^T + \frac{\xi(\xi^2 \mp 1)}{(\xi^2 \mp \eta^2)}\nabla_\xi \nabla_\eta \psi_n^T + \frac{\eta(1-\eta^2)}{(\xi^2 \mp \eta^2)}\nabla_\eta \nabla_\eta \psi_n^T$$

$$k_T\sqrt{\lambda_n}(V_n^2)_\varphi = 2\nabla_\varphi \psi_n^T + \frac{\xi(\xi^2 \mp 1)}{(\xi^2 \mp \eta^2)}\nabla_\xi \nabla_\varphi \psi_n^T + \frac{\eta(1-\eta^2)}{(\xi^2 \mp \eta^2)}\nabla_\eta \nabla_\varphi \psi_n^T$$

$$(V_n^3)_\xi = \frac{f}{c_L}\partial_\xi \psi_n^L$$

$$(V_n^3)_\eta = \frac{f}{c_L}\partial_\eta \psi_n^L$$

$$(V_n^3)_\varphi = \frac{f}{c_L}\partial_\varphi \psi_n^L$$

The components of the covariant derivatives of the vector basis functions are as follows:

$$\frac{1}{f}\sqrt{\lambda}\nabla_{\xi}(V_n^1)_{\xi} = +\eta(1-\eta^2)\nabla_{\xi}\nabla_{\varphi}\psi_n^T$$

$$\frac{1}{f}\sqrt{\lambda}\nabla_{\eta}(V_n^1)_{\eta} = -\xi(\xi^2-1)\nabla_{\eta}\nabla_{\varphi}\psi_n^T$$

$$\frac{1}{f}\sqrt{\lambda}\nabla_{\xi}(V_n^1)_{\eta} = -\frac{\xi}{(1-\eta^2)}\nabla_{\xi}\nabla_{\varphi}\psi_n^T - \frac{(\xi^2-\eta^2)}{(\xi^2-1)(1-\eta^2)}\nabla_{\varphi}\psi_n^T$$

$$\frac{1}{f}\sqrt{\lambda}\nabla_{\eta}(V_n^1)_{\xi} = +\frac{\eta}{(\xi^2-1)}\nabla_{\eta}\nabla_{\varphi}\psi_n^T + \frac{(\xi^2-\eta^2)}{(\xi^2-1)(1-\eta^2)}\nabla_{\varphi}\psi_n^T$$

$$\frac{1}{f}\sqrt{\lambda}\nabla_{\xi}(V_n^1)_{\varphi} = +(1-\eta^2)\nabla_{\eta}\psi_n^T - \frac{(\xi^2-1)(1-\eta^2)}{(\xi^2-\eta^2)}[\eta\nabla_{\xi}\nabla_{\xi}\psi_n^T - \xi\nabla_{\xi}\nabla_{\eta}\psi_n^T]$$

$$\frac{1}{f}\sqrt{\lambda}\nabla_{\eta}(V_n^1)_{\varphi} = -(\xi^2-1)\nabla_{\xi}\psi_n^T - \frac{(\xi^2-1)(1-\eta^2)}{(\xi^2-\eta^2)}[\eta\nabla_{\xi}\nabla_{\eta}\psi_n^T - \xi\nabla_{\eta}\nabla_{\eta}\psi_n^T]$$

$$\frac{1}{f}\sqrt{\lambda}\nabla_{\varphi}(V_n^1)_{\xi} = -(1-\eta^2)\nabla_{\eta}\psi_n^T + \frac{\eta}{(\xi^2-1)}\nabla_{\varphi}\nabla_{\varphi}\psi_n^T$$

$$\frac{1}{f}\sqrt{\lambda}\nabla_{\varphi}(V_n^1)_{\eta} = +(\xi^2-1)\nabla_{\xi}\psi_n^T - \frac{\xi}{(1-\eta^2)}\nabla_{\varphi}\nabla_{\varphi}\psi_n^T$$

$$\frac{1}{f}\sqrt{\lambda}\nabla_{\varphi}(V_n^1)_{\varphi} = -\frac{(\xi^2-1)(1-\eta^2)}{(\xi^2-\eta^2)}[\eta\nabla_{\xi}\nabla_{\varphi}\psi_n^T - \xi\nabla_{\eta}\nabla_{\varphi}\psi_n^T]$$

$$k_T\sqrt{\lambda_n}\nabla_\xi(V_n^2)_\xi = c_T^2\frac{(\xi^2-\eta^2)}{(\xi^2-1)}\psi_n^T + c_T^2\xi\nabla_\xi\psi_n^T + 3\nabla_\xi\nabla_\xi\psi_n^T + \frac{\xi(\xi^2-1)}{(\xi^2-\eta^2)}\nabla_\xi\nabla_\xi\nabla_\xi\psi_n^T + \frac{\eta(1-\eta^2)}{(\xi^2-\eta^2)}\nabla_\xi\nabla_\xi\nabla_\eta\psi_n^T$$

$$k_T\sqrt{\lambda_n}\nabla_\eta(V_n^2)_\eta = c_T^2\frac{(\xi^2-\eta^2)}{(1-\eta^2)}\psi_n^T + c_T^2\eta\nabla_\eta\psi_n^T + 3\nabla_\eta\nabla_\eta\psi_n^T + \frac{\xi(\xi^2-1)}{(\xi^2-\eta^2)}\nabla_\xi\nabla_\eta\nabla_\eta\psi_n^T + \frac{\eta(1-\eta^2)}{(\xi^2-\eta^2)}\nabla_\eta\nabla_\eta\nabla_\eta\psi_n^T$$

$$k_T\sqrt{\lambda_n}\nabla_\varphi(V_n^2)_\varphi = c_T^2(\xi^2-1)(1-\eta^2)\psi_n^T + 3\nabla_\varphi\nabla_\varphi\psi_n^T + \frac{\xi(\xi^2-1)}{(\xi^2-\eta^2)}\nabla_\xi\nabla_\varphi\nabla_\varphi\psi_n^T + \frac{\eta(1-\eta^2)}{(\xi^2-\eta^2)}\nabla_\varphi\nabla_\varphi\nabla_\eta\psi_n^T$$

$$k_T\sqrt{\lambda_n}\nabla_\xi(V_n^2)_\eta = c_T^2\eta\nabla_\xi\psi_n^T + 3\nabla_\xi\nabla_\eta\psi_n^T + \frac{\xi(\xi^2-1)}{(\xi^2-\eta^2)}\nabla_\xi\nabla_\xi\nabla_\eta\psi_n^T + \frac{\eta(1-\eta^2)}{(\xi^2-\eta^2)}\nabla_\xi\nabla_\eta\nabla_\eta\psi_n^T$$

$$k_T\sqrt{\lambda_n}\nabla_\eta(V_n^2)_\xi = c_T^2\xi\nabla_\eta\psi_n^T + 3\nabla_\xi\nabla_\eta\psi_n^T + \frac{\xi(\xi^2-1)}{(\xi^2-\eta^2)}\nabla_\xi\nabla_\xi\nabla_\eta\psi_n^T + \frac{\eta(1-\eta^2)}{(\xi^2-\eta^2)}\nabla_\xi\nabla_\eta\nabla_\eta\psi_n^T$$

$$k_T\sqrt{\lambda_n}\nabla_\xi(V_n^2)_\varphi = 3\nabla_\xi\nabla_\varphi\psi_n^T + \frac{\xi(\xi^2-1)}{(\xi^2-\eta^2)}\nabla_\xi\nabla_\xi\nabla_\varphi\psi_n^T + \frac{\eta(1-\eta^2)}{(\xi^2-\eta^2)}\nabla_\xi\nabla_\eta\nabla_\varphi\psi_n^T$$

$$k_T\sqrt{\lambda_n}\nabla_\eta(V_n^2)_\varphi = 3\nabla_\eta\nabla_\varphi\psi_n^T + \frac{\xi(\xi^2-1)}{(\xi^2-\eta^2)}\nabla_\xi\nabla_\eta\nabla_\varphi\psi_n^T + \frac{\eta(1-\eta^2)}{(\xi^2-\eta^2)}\nabla_\eta\nabla_\eta\nabla_\varphi\psi_n^T$$

$$k_T\sqrt{\lambda_n}\nabla_\varphi(V_n^2)_\xi = c_T^2\xi\nabla_\varphi\psi_n^T + 3\nabla_\xi\nabla_\varphi\psi_n^T + \frac{\xi(\xi^2-1)}{(\xi^2-\eta^2)}\nabla_\xi\nabla_\xi\nabla_\varphi\psi_n^T + \frac{\eta(1-\eta^2)}{(\xi^2-\eta^2)}\nabla_\xi\nabla_\eta\nabla_\varphi\psi_n^T$$

$$k_T\sqrt{\lambda_n}\nabla_\varphi(V_n^2)_\eta = c_T^2\eta\nabla_\varphi\psi_n^T + 3\nabla_\eta\nabla_\varphi\psi_n^T + \frac{\xi(\xi^2-1)}{(\xi^2-\eta^2)}\nabla_\xi\nabla_\eta\nabla_\varphi\psi_n^T + \frac{\eta(1-\eta^2)}{(\xi^2-\eta^2)}\nabla_\eta\nabla_\eta\nabla_\varphi\psi_n^T$$

$$\nabla_{\xi}(V_n^3)_{\xi} = \frac{f}{c_L}\nabla_{\xi}\nabla_{\xi}\psi_n^L$$

$$\nabla_{\eta}(V_n^3)_{\eta} = \frac{f}{c_L}\nabla_{\eta}\nabla_{\eta}\psi_n^L$$

$$\nabla_{\varphi}(V_n^3)_{\varphi} = \frac{f}{c_L}\nabla_{\varphi}\nabla_{\varphi}\psi_n^L$$

$$\nabla_{\xi}(V_n^3)_{\eta} = \frac{f}{c_L}\nabla_{\xi}\nabla_{\eta}\psi_n^L$$

$$\nabla_{\eta}(V_n^3)_{\xi} = \frac{f}{c_L}\nabla_{\xi}\nabla_{\eta}\psi_n^L$$

$$\nabla_{\xi}(V_n^3)_{\varphi} = \frac{f}{c_L}\nabla_{\xi}\nabla_{\varphi}\psi_n^L$$

$$\nabla_{\varphi}(V_n^3)_{\xi} = \frac{f}{c_L}\nabla_{\xi}\nabla_{\varphi}\psi_n^L$$

$$\nabla_{\eta}(V_n^3)_{\varphi} = \frac{f}{c_L}\nabla_{\eta}\nabla_{\varphi}\psi_n^L$$

$$\nabla_{\varphi}(V_n^3)_{\eta} = \frac{f}{c_L}\nabla_{\eta}\nabla_{\varphi}\psi_n^L$$

APPENDIX H

PARITY OF BASIS FUNCTIONS

APPENDIX H - PARITY OF BASIS FUNCTIONS

This appendix tabulates the parity of the various functions and their derivatives with respect to the transformation $\eta \to -\eta$, and the dependence of the functions on the first order derivative with respect to the azimuthal angle. These symmetries are important for determining the non-zero matrix elements obtained from the surface integrals for a symmetric target such as a prolate spheroid or finite cylinder.

Table H-1 contains the polar and azimuthal parity of the vector harmonics. Table H-2 contains the parity of the components of the vector basis functions, and Table H-3 contains the parity of the stress tensor formed from the covariant derivative of the vector basis functions. The interpretation of the polar parity column is as follows: +1 signifies the function transforms as $F(-\eta) = +(-1)^{l-m} F(\eta)$, -1 signifies the function transforms as $F(-\eta) = -(-1)^{l-m} F(\eta)$, and 0 signifies the function is zero. The interpretation of the azimuthal parity column is as follows: Y signifies the function is proportional to an odd number of derivatives of the function with respect to the azimuthal order, and N signifies it is proportional to an even number of derivatives.

The significance of the azimuthal parity is that it denotes the coupling of different parities in the azimuthal integration. The significance of the polar parity term is that it dictates the values of order l for which the integrals over the coordinate $-1 < \eta < +1$ are non-zero.

TABLE H-1. PARITY OF VECTOR HARMONICS

VECTOR HARMONIC	COMPONENT	POLAR PARITY	AZIMUTHAL PARITY
First Vector			
1	ξ	0	0
1	η	+1	Y
1	φ	-1	N
Second Vector			
2	ξ	0	0
2	η	-1	N
2	φ	+1	Y
Third Vector			
3	ξ	+1	N
3	η	0	0
3	φ	0	0

TABLE H-2. PARITY OF VECTOR BASIS FUNCTIONS

VECTOR BASIS FUNCTION	COMPONENT	POLAR PARITY	AZIMUTHAL PARITY
First Vector			
1	ξ	-1	Y
1	η	+1	Y
1	φ	-1	N
Second Vector			
2	ξ	+1	N
2	η	-1	N
2	φ	+1	Y
Third Vector			
3	ξ	+1	N
3	η	-1	N
3	φ	+	Y

TABLE H-3. PARITY OF STRESS TENSOR FORMED FROM VECTOR BASIS FUNCTIONS

VECTOR BASIS FUNCTION	COMPONENT OF STRESS TENSOR	POLAR PARITY	AZIMUTHAL PARITY
First Basis Function			
1	(ξ,ξ)	-1	Y
1	$(\xi,\eta)(\eta,\xi)$	+1	Y
1	$(\xi,\varphi)(\varphi,\xi)$	-1	N
1	(η,η)	-1	Y
1	$(\eta,\varphi)(\varphi,\eta)$	+1	N
1	(φ,φ)	-1	Y
Second Basis Function			
2	(ξ,ξ)	+1	N
2	$(\xi,\eta)(\eta,\xi)$	-1	N
2	$(\xi,\varphi)(\varphi,\xi)$	+1	Y
2	(η,η)	+1	N
2	$(\eta,\varphi)(\varphi,\eta)$	-1	Y
2	(φ,φ)	+1	N
Third Basis Function			
3	(ξ,ξ)	+1	N
3	$(\xi,\eta)(\eta,\xi)$	-1	N
3	$(\xi,\varphi)(\varphi,\xi)$	+1	Y
3	(η,η)	+1	N
3	$(\eta,\varphi)(\varphi,\eta)$	-1	Y
3	(φ,φ)	+1	N

APPENDIX I

COMPUTATION OF T-MATRIX FOR PROLATE SPHEROID

APPENDIX I - COMPUTATION OF T-MATRIX FOR PROLATE SPHEROID

This appendix describes the method for performing the surface integrals necessary to compute the spheroidal T-matrix for a prolate spheroid. Since the spheroid is both symmetric under the transformation $\eta \rightarrow -\eta$ and $\varphi \rightarrow \varphi + \alpha$, the T-matrix is diagonal in parity σ and azimuthal order *m*. In addition, only terms for which the difference *(l-l')* is even have non-zero elements for the T-matrix. These symmetries allow us to reduce the two-dimensional surface integrals to a one-dimensional integral over the variable $0 > \eta > 1$. Due to the above symmetries, the T-matrix is of the following form:

$$T_{\sigma ml,\sigma' m' l'} = \delta_{\sigma}^{\sigma'} \delta_{m}^{m'} T_{l,l'}^{m}$$

Define the following basis functions by replacing the trigonometric functions in the basis functions by the exponential functions $\exp[\pm im\varphi]$, respectively.

$$S^{W}{}_{\pm ml} \exp[\pm im\varphi] = \frac{1}{\sqrt{\varepsilon(m)}} (S^{W}{}_{eml} \pm i S^{W}{}_{oml})$$

$$A^{W}_{\pm ml} \exp[\pm im\varphi] = \frac{1}{\sqrt{\varepsilon(m)}} (A^{W}_{eml} \pm i A^{W}_{oml})$$

$$V^{W}_{\pm ml} \exp[\pm im\varphi] = \frac{1}{\sqrt{\varepsilon(m)}} (V^{W}_{eml} \pm i V^{W}_{oml})$$

$$A^{\tau}_{\pm ml} \exp[\pm im\varphi] = \frac{1}{\sqrt{\varepsilon(m)}} (A^{\tau}_{eml} \pm i A^{\tau}_{oml})$$

$$V^{\tau}_{\pm ml} \exp[\pm im\varphi] = \frac{1}{\sqrt{\varepsilon(m)}} (V^{\tau}_{eml} \pm i V^{\tau}_{oml})$$

$$t^{W}_{\pm ml} \exp[\pm im\varphi] = \frac{1}{\sqrt{\varepsilon(m)}} (t(V^{W}_{eml}) \pm it(V^{W}_{oml}))$$

$$t^{\tau}_{\pm ml} \exp[\pm im\varphi] = \frac{1}{\sqrt{\varepsilon(m)}} (t(V^{\tau}_{eml}) \pm it(V^{\tau}_{oml}))$$

The functions S^{W}, A^{W}, V^{W}, and t^{W}, are the scalar harmonic, vector harmonic, vector basis function, and traction vector in the external fluid, respectively. The functions A^{τ}, V^{τ}, and t^{τ} are the vector harmonics, vector basis functions, and traction vectors in the interior, respectively. Depending upon whether a given function $F_{\pm ml}$ contains an even or odd number of derivatives with respect to the angle φ the function will be proportional to either "1" or $\pm i$.

Define the following notation:

$$F_{\pm ml} = \tilde{F}_{ml} \begin{cases} +1, & \text{even azimuthal parity} \\ \pm i, & \text{odd azimuthal parity} \end{cases}$$

Define the following functions based on their azimuthal parity:

$$\tilde{S}^{W}_{ml} = S^{W}_{\pm ml}$$

$$(\tilde{A}^1_{ml})_\eta = (A^1_{\pm ml})_\eta / \pm i$$
$$(\tilde{A}^1_{ml})_\varphi = (A^1_{\pm ml})_\varphi$$
$$(\tilde{A}^2_{ml})_\eta = (A^2_{\pm ml})_\eta$$
$$(\tilde{A}^2_{ml})_\varphi = (A^2_{\pm ml})_\varphi / \pm i$$
$$(\tilde{A}^3_{ml})_\xi = (A^3_{\pm ml})_\xi$$

$$(\tilde{V}^1_{ml})_\xi = (V^1_{\pm ml})_\xi / \pm i$$
$$(\tilde{V}^1_{ml})_\eta = (V^1_{\pm ml})_\eta / \pm i$$
$$(\tilde{V}^1_{ml})_\varphi = (V^1_{\pm ml})_\varphi$$
$$(\tilde{V}^2_{ml})_\xi = (V^2_{\pm ml})_\xi$$
$$(\tilde{V}^2_{ml})_\eta = (V^2_{\pm ml})_\eta$$
$$(\tilde{V}^2_{ml})_\varphi = (V^2_{\pm ml})_\varphi / \pm i$$
$$(\tilde{V}^3_{ml})_\xi = (V^3_{\pm ml})_\xi$$
$$(\tilde{V}^3_{ml})_\eta = (V^3_{\pm ml})_\eta$$
$$(\tilde{V}^3_{ml})_\varphi = (V^3_{\pm ml})_\varphi / \pm i$$

$$(\tilde{t}^1_{ml})_\xi = (t^1_{\pm ml})_\xi / \pm i$$
$$(\tilde{t}^1_{ml})_\eta = (t^1_{\pm ml})_\eta / \pm i$$
$$(\tilde{t}^1_{ml})_\varphi = (t^1_{\pm ml})_\varphi$$
$$(\tilde{t}^2_{ml})_\xi = (t^2_{\pm ml})_\xi$$
$$(\tilde{t}^2_{ml})_\eta = (t^2_{\pm ml})_\eta$$
$$(\tilde{t}^2_{ml})_\varphi = (t^2_{\pm ml})_\varphi / \pm i$$
$$(\tilde{t}^3_{ml})_\xi = (t^3_{\pm ml})_\xi$$
$$(\tilde{t}^3_{ml})_\eta = (t^3_{\pm ml})_\eta$$
$$(\tilde{t}^3_{ml})_\varphi = (t^3_{\pm ml})_\varphi / \pm i$$

The computation of the T-matrix requires the evaluation of the following surface integrals over the surface of the target:

$$Q_{n,\tau'n'} = -i\frac{k}{\lambda^W}\int_S dA\{(n \bullet t(V_n^W)(n \bullet A_{\tau',n'})\}$$

$$M_{n,n'} = -i\frac{k}{\lambda^W}\int_S dA\{(n \bullet V_n^W)S^W{}_{n'}\}$$

$$P_{\tau n,n'} = \int_S dA\{(n \bullet \mathrm{Re}\, V_{\tau,n})S^W{}_{n'}\}$$

$$R_{\tau n,\tau' n'} = \int_S dA\{t(\mathrm{Re}\, V_{\tau n}) \bullet A_{\tau' n'}\}$$

In the above integrals one adopts the convention that the left hand term in the integral has $+m$ and the right hand term has $-m$ for the azimuthal angle dependence of the integrand. The above integrals are diagonal in m, and they can be replaced by the following one-dimensional integrals:

$$h_\xi = f\sqrt{(\xi^2-\eta^2)/(\xi^2-1)}$$
$$h_\eta = f\sqrt{(\xi^2-\eta^2)/(1-\eta^2)}$$
$$h_\varphi = f\sqrt{(\xi^2-1)(1-\eta^2)}$$

$$\Delta_n = h_\eta h_\varphi n_{\hat{\xi}} - h_\xi h_\varphi n_{\hat{\eta}} \frac{\partial \xi(\eta)}{\partial \eta}$$

$$Q^{m}{}_{l,\tau'l'} = -2\pi i \frac{k}{\lambda^{W}} \int_{-1}^{+1} d\eta \Delta_{n} (t^{W}_{+ml})_{\xi} (A^{\tau'}_{-ml'})_{\xi}$$

$$M^{m}{}_{l,l'} = -2\pi i \frac{k}{\lambda^{W}} \int_{-1}^{+1} d\eta \Delta_{n} (V^{W}_{+ml})_{\xi} (S^{W}_{-ml'})$$

$$P^{m}{}_{\tau l,l'} = 2\pi \int_{-1}^{+1} d\eta \Delta_{n} (\mathrm{Re} V^{\tau}_{+ml})_{\xi} (S^{W}_{-ml'})$$

$$R^{m}{}_{\tau l,\tau' l'} = 2\pi \int_{-1}^{+1} d\eta \Delta_{n} (\mathrm{Re}\, t^{\tau}_{+ml}) \bullet (A^{\tau'}_{-ml'})$$

Using the polar parity and the azimuthal parity properties of the integrands, the non-zero components of the above integrals in the case *l-l'* is even are given by the following integrals:

$$Q^{m}{}_{l,3l'} = -4\pi i \frac{k}{\lambda^{W}} \int_{0}^{+1} d\eta \Delta_{n} (\tilde{t}^{W}_{ml})_{\hat{\xi}} (\tilde{A}^{3}_{ml'})_{\hat{\xi}}$$

$$M^{m}{}_{l,l'} = -4\pi i \frac{k}{\lambda^{W}} \int_{0}^{+1} d\eta \Delta_{n} (\tilde{V}^{W}_{ml})_{\hat{\xi}} (\tilde{S}^{W}_{ml'})$$

$$P^{m}{}_{2l,l'} = 4\pi \int_{0}^{+1} d\eta \Delta_{n} (\mathrm{Re}\tilde{V}^{2}_{ml})_{\hat{\xi}} (\tilde{S}^{W}_{ml'})$$

$$P^{m}{}_{3l,l'} = 4\pi \int_{0}^{+1} d\eta \Delta_{n} (\mathrm{Re}\tilde{V}^{3}_{ml})_{\hat{\xi}} (\tilde{S}^{W}_{ml'})$$

$$R^{m}{}_{1l,1l'} = 4\pi \int_{0}^{+1} d\eta \Delta_{n} [+(\mathrm{Re}\tilde{t}^{1}_{ml})_{\hat{\eta}} (\tilde{A}^{1}_{ml'})_{\hat{\eta}} + (\mathrm{Re}\tilde{t}^{1}_{ml})_{\hat{\varphi}} (\tilde{A}^{1}_{ml'})_{\hat{\varphi}}]$$

$$R^{m}{}_{2l,2l'} = 4\pi \int_{0}^{+1} d\eta \Delta_{n} [+(\mathrm{Re}\tilde{t}^{2}_{ml})_{\hat{\eta}} (\tilde{A}^{2}_{ml'})_{\hat{\eta}} + (\mathrm{Re}\tilde{t}^{2}_{ml})_{\hat{\varphi}} (\tilde{A}^{2}_{ml'})_{\hat{\varphi}}]$$

$$R^{m}{}_{3l,3l'} = 4\pi \int_{0}^{+1} d\eta \Delta_{n} (\mathrm{Re}\tilde{t}^{3}_{ml})_{\hat{\xi}} (\tilde{A}^{3}_{ml'})_{\hat{\xi}}$$

$$R^{m}{}_{2l,3l'} = 4\pi \int_{0}^{+1} d\eta \Delta_{n} (\mathrm{Re}\tilde{t}^{2}_{ml})_{\hat{\xi}} (\tilde{A}^{3}_{ml'})_{\hat{\xi}}$$

$$R^{m}{}_{3l,2l'} = 4\pi \int_{0}^{+1} d\eta \Delta_{n} [+(\mathrm{Re}\tilde{t}^{3}_{ml})_{\hat{\eta}} (\tilde{A}^{2}_{ml'})_{\hat{\eta}} + (\mathrm{Re}\tilde{t}^{3}_{ml})_{\hat{\varphi}} (\tilde{A}^{2}_{ml'})_{\hat{\varphi}}]$$

The non-zero components of the integrals in the case *l-l'* is odd are given by the following integrals:

$$P^m{}_{1l,l'} = +4\pi i \int_0^{+1} d\eta \Delta_n (\mathrm{Re}\tilde{V}^1_{ml})_{\hat{\xi}} (\tilde{S}^W_{ml'})$$

$$R^m{}_{1l,2l'} = +4\pi i \int_0^{+1} d\eta \Delta_n [+(\mathrm{Re}\tilde{t}^1_{ml})_{\hat{\eta}} (\tilde{A}^2_{ml'})_{\hat{\eta}} - (\mathrm{Re}\tilde{t}^1_{ml})_{\hat{\varphi}} (\tilde{A}^2_{ml'})_{\hat{\varphi}}]$$

$$R^m{}_{2l,1l'} = -4\pi i \int_0^{+1} d\eta \Delta_n [+(\mathrm{Re}\tilde{t}^2_{ml})_{\hat{\eta}} (\tilde{A}^1_{ml'})_{\hat{\eta}} - (\mathrm{Re}\tilde{t}^2_{ml})_{\hat{\varphi}} (\tilde{A}^1_{ml'})_{\hat{\varphi}}]$$

$$R^m{}_{3l,1l'} = -4\pi i \int_0^{+1} d\eta \Delta_n [+(\mathrm{Re}\tilde{t}^3_{ml})_{\hat{\eta}} (\tilde{A}^1_{ml'})_{\hat{\eta}} - (\mathrm{Re}\tilde{t}^3_{ml})_{\hat{\varphi}} (\tilde{A}^1_{ml'})_{\hat{\varphi}}]$$

$$R^m{}_{1l,3l'} = +i4\pi \int_0^{+1} d\eta \Delta_n (\mathrm{Re}\tilde{t}^1_{ml})_{\hat{\xi}} (\tilde{A}^3_{ml'})_{\hat{\xi}}$$

In the case (*l-l'*) is even, the components of the *R* and *P* matrices are real. In the case (*l-l'*) is odd the *R* and *P* matrices are imaginary. The *R* and *P* matrices can be transformed into real matrices by making the following similarity transformation on the vector indices of the *R, P,* and *Q* matrices:

$$S_{\tau\tau'} = -i\delta^{\tau'}_{\tau}\delta^1_{\tau} + \delta^{\tau'}_{\tau}\delta^2_{\tau} + \delta^{\tau'}_{\tau}\delta^3_{\tau}$$

$$S^{-1}{}_{\tau\tau'} = S^{\dagger}{}_{\tau\tau'} = +i\delta^{\tau'}_{\tau}\delta^1_{\tau} + \delta^{\tau'}_{\tau}\delta^2_{\tau} + \delta^{\tau'}_{\tau}\delta^3_{\tau}$$

$$R^m_{\tau l,\tau' l'} = \sum_{\tau'',\tau'''} S_{\tau',\tau''} R^m_{\tau'' l,\tau''' l'} S^{\dagger}{}_{\tau''',\tau'}$$

$$P^m_{\tau l,l'} = \sum_{\tau''} S_{\tau',\tau''} P^m_{\tau'' l,l'}$$

$$Q^m_{l,\tau' l'} = \sum_{\tau''} Q^m_{l,\tau'' l'} S^{\dagger}{}_{\tau'',\tau'}$$

In the case (*l-l'*) is even, the components of the *Q, R,* and *P* matrices are left invariant by this transformation, whereas in the case (*l-l'*) is odd they are transformed to the following real matrices:

$$P^m{}_{1l,l'} = +4\pi \int_0^{+1} d\eta \Delta_n (\mathrm{Re}\tilde{V}^1_{ml})_{\hat{\xi}} (\tilde{S}^W_{ml'})$$

$$R^m{}_{1l,2l'} = +4\pi \int_0^{+1} d\eta \Delta_n [+(\mathrm{Re}\tilde{t}^1_{ml})_{\hat{\eta}} (\tilde{A}^2_{ml'})_{\hat{\eta}} - (\mathrm{Re}\tilde{t}^1_{ml})_{\hat{\varphi}} (\tilde{A}^2_{ml'})_{\hat{\varphi}}]$$

$$R^m{}_{2l,1l'} = +4\pi \int_0^{+1} d\eta \Delta_n [+(\mathrm{Re}\tilde{t}^2_{ml})_{\hat{\eta}} (\tilde{A}^1_{ml'})_{\hat{\eta}} - (\mathrm{Re}\tilde{t}^2_{ml})_{\hat{\varphi}} (\tilde{A}^1_{ml'})_{\hat{\varphi}}]$$

$$R^m{}_{3l,1l'} = +4\pi \int_0^{+1} d\eta \Delta_n [+(\mathrm{Re}\tilde{t}^3_{ml})_{\hat{\eta}} (\tilde{A}^1_{ml'})_{\hat{\eta}} - (\mathrm{Re}\tilde{t}^3_{ml})_{\hat{\varphi}} (\tilde{A}^1_{ml'})_{\hat{\varphi}}]$$

$$R^m{}_{1l,3l'} = +4\pi \int_0^{+1} d\eta \Delta_n (\mathrm{Re}\tilde{t}^1_{ml})_{\hat{\xi}} (\tilde{A}^3_{ml'})_{\hat{\xi}}$$

To convert the above integrals into matrices one adopts the following convention for the index of the pair τl of vector and order indices in the case m=0, where negative indices are ignored:

$$n(\tau = 1, l) = 3(l - m) - 2$$
$$n(\tau = 2, l) = 3(l - m) - 1$$
$$n(\tau = 3, l) = 3(l - m)$$

In the case $m \neq 0$, the following index notation for the pair τl of vector and order indices is adopted:

$$n(\tau = 1, l) = 3(l - m) + 2$$
$$n(\tau = 2, l) = 3(l - m) + 1$$
$$n(\tau = 3, l) = 3(l - m)$$

Here one uses the C-style index notation, where the first element in a vector starts at the index 0. The above index convention is used to project out the zero components of the vector basis functions in the case l=m=0. For example, the square R-matrix has dimension $(3(l_{\max} - m) + 1)$ in the case m=0, and $(3(l_{\max} - m) + 3)$ otherwise, where $l_{\max}$ is the maximum value for the order l.

The T-matrix is defined by the following equation in terms of the above matrices:

$$T = -(\mathrm{Re}\, QR^{-1}P - \mathrm{Re}\, M)(QR^{-1}P - M)^{-1}$$

APPENDIX J

COMPUTATION OF T-MATRIX FOR FINITE CYLINDER

APPENDIX J - COMPUTATION OF T-MATRIX FOR FINITE CYLINDER

This appendix describes the computation of the spheroidal T-matrix for a finite cylinder with hemi-spherical end caps.

First, the author begins with a discussion of the geometry of the surface of the cylinder. Let L denote the length of the cylinder and D its diameter, where the aspect ratio $\chi = (L/D)$ is greater than unity. The semi-focal distance of the spheroidal coordinate system is defined by the following equation in terms of the length and aspect ratio of the cylinder:

$$f = (L/2)\frac{\sqrt{\chi^2 - 1}}{\chi}$$

Define the quantity $L_{cyl} = L - D$ as the length of the cylindrical portion of the target excluding the end caps.

In the case $-L_{cyl}/2 < z < +L_{cyl}/2$, the normal to the surface is given by the following equation:

$$(n_{cyl})_{\hat{\xi}} = +\xi\sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}}$$

$$(n_{cyl})_{\hat{\eta}} = -\eta\sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}}$$

$$(n_{cyl})_{\hat{\varphi}} = 0$$

The spheroidal coordinates of the point on the surface in the $\varphi = 0$ plane are given by the following equations:

$$\tilde{\rho} = \rho / f = D / f / 2$$

$$\tilde{z} = z / f$$

$$\tilde{r} = r / f = \sqrt{\tilde{\rho}^2 + \tilde{z}^2}$$

$$\xi = \sqrt{\frac{1 + \tilde{r}^2 + \sqrt{(1 + \tilde{r}^2)^2 - 4\tilde{z}^2}}{2}}$$

$$\eta = \tilde{z} / \xi$$

To describe the surface of the target at its end caps one uses the polar angle $-\pi / 2 < \vartheta < +\pi / 2$ to parameterize the surface at its end caps. The coordinates of a point on the end caps in the $\varphi = 0$ plane are given by the following relationships:

$$\tilde{\rho} = (D / 2f)\sin(\vartheta)$$

$$\tilde{z} = z / f = \pm((D / 2f)\cos(\vartheta) + L_{cyl} / 2)$$

$$\tilde{r}^2 = \tilde{\rho}^2 + \tilde{z}^2$$

$$\xi = \sqrt{\frac{1 + \tilde{r}^2 + \sqrt{(1 + \tilde{r}^2)^2 - 4\tilde{z}^2}}{2}}$$

$$\eta = \tilde{z} / \xi$$

The normal to the surface at its end caps in the $\varphi = 0$ plane is given by the following equations:

$$x = f\sqrt{(\xi^2-1)(1-\eta^2)}$$
$$z = f\xi\eta$$

$$n_x = \sin(\vartheta)$$
$$n_z = \cos(\vartheta)$$

$$n_\xi = n_x\partial_\xi x + n_z\partial_\xi z = f[+\xi\sqrt{\frac{1-\eta^2}{\xi^2-1}}\sin(\vartheta) + \eta\cos(\vartheta)]$$
$$n_\eta = n_x\partial_\eta x + n_z\partial_\eta z = f[-\eta\sqrt{\frac{\xi^2-1}{1-\eta^2}}\sin(\vartheta) + \xi\cos(\vartheta)]$$
$$n_\varphi = 0$$

$$n_{\hat\xi} = [+\xi\sqrt{1-\eta^2}\sin(\vartheta) + \eta\sqrt{\xi^2-1}\cos(\vartheta)]/\sqrt{\xi^2-\eta^2}$$
$$n_{\hat\eta} = [-\eta\sqrt{\xi^2-1}\sin(\vartheta) + \xi\sqrt{1-\eta^2}\cos(\vartheta)]/\sqrt{\xi^2-\eta^2}$$
$$n_{\hat\varphi} = 0$$

The component $n_{\hat\xi}$ of the normal has even parity under the transformation $\eta \to -\eta$, whereas the component $n_{\hat\eta}$ has odd parity. In the case of the normal for the end caps, the angle ϑ becomes $\vartheta \to \vartheta + \pi$ under the transformation $\eta \to -\eta$.

Since the finite cylinder is both symmetric under the transformation $\eta \to -\eta$ and $\varphi \to \varphi + \alpha$, the T-matrix is diagonal in parity σ and azimuthal order *m*. In addition, only terms for which the difference (*l-l'*) is even have non-zero elements for the T-matrix. These symmetries allow us to reduce the two-dimensional surface integrals to a one-dimensional integral over the variable $0 > \eta > 1$. Due to the above symmetries, the T-matrix is of the following form:

$$T_{\sigma ml,\sigma' m' l'} = \delta_\sigma^{\sigma'}\delta_m^{m'}T_{l,l'}^m$$

Define the following basis functions by replacing the trigonometric functions in the basis functions by the exponential functions $\exp[\pm im\varphi]$, respectively:

$$S^W{}_{\pm ml}\exp[\pm im\varphi]=\frac{1}{\sqrt{\varepsilon(m)}}(S^W{}_{eml}\pm iS^W{}_{oml})$$

$$A^W_{\pm ml}\exp[\pm im\varphi]=\frac{1}{\sqrt{\varepsilon(m)}}(A^W_{eml}\pm iA^W_{oml})$$

$$V^W_{\pm ml}\exp[\pm im\varphi]=\frac{1}{\sqrt{\varepsilon(m)}}(V^W_{eml}\pm iV^W_{oml})$$

$$A^\tau_{\pm ml}\exp[\pm im\varphi]=\frac{1}{\sqrt{\varepsilon(m)}}(A^\tau_{eml}\pm iA^\tau_{oml})$$

$$V^\tau_{\pm ml}\exp[\pm im\varphi]=\frac{1}{\sqrt{\varepsilon(m)}}(V^\tau_{eml}\pm iV^\tau_{oml})$$

$$t^W_{\pm ml}\exp[\pm im\varphi]=\frac{1}{\sqrt{\varepsilon(m)}}(t(V^W_{eml})\pm it(V^W_{oml}))$$

$$t^\tau_{\pm ml}\exp[\pm im\varphi]=\frac{1}{\sqrt{\varepsilon(m)}}(t(V^\tau_{eml})\pm it(V^\tau_{oml}))$$

The functions S^W, A^W, V^W, and t^W, are the scalar harmonic, vector harmonic, vector basis function, and traction vector in the external fluid, respectively. The functions A^τ, V^τ, and t^τ are the vector harmonics, vector basis functions, and traction vectors in the interior, respectively. Depending upon whether a given function $F_{\pm ml}$ contains an even or odd number of derivatives with respect to the angle φ, the function will be proportional to either "1" or $\pm i$.

Define the following notation:

$$F_{\pm ml}=\tilde{F}_{ml}\begin{cases}+1, & \text{even azimuthal parity}\\ \pm i, & \text{odd azimuthal parity}\end{cases}$$

Define the following functions based on their azimuthal parity:

$$\tilde{S}^W_{ml}=S^W_{\pm ml}$$

$$(\tilde{A}^1_{ml})_\eta=(A^1_{\pm ml})_\eta/\pm i$$
$$(\tilde{A}^1_{ml})_\varphi=(A^1_{\pm ml})_\varphi$$
$$(\tilde{A}^2_{ml})_\eta=(A^2_{\pm ml})_\eta$$
$$(\tilde{A}^2_{ml})_\varphi=(A^2_{\pm ml})_\varphi/\pm i$$
$$(\tilde{A}^3_{ml})_\xi=(A^3_{\pm ml})_\xi$$

$$(\tilde{V}^1_{ml})_\xi = (V^1_{\pm ml})_\xi / \pm i$$
$$(\tilde{V}^1_{ml})_\eta = (V^1_{\pm ml})_\eta / \pm i$$
$$(\tilde{V}^1_{ml})_\varphi = (V^1_{\pm ml})_\varphi$$
$$(\tilde{V}^2_{ml})_\xi = (V^2_{\pm ml})_\xi$$
$$(\tilde{V}^2_{ml})_\eta = (V^2_{\pm ml})_\eta$$
$$(\tilde{V}^2_{ml})_\varphi = (V^2_{\pm ml})_\varphi / \pm i$$
$$(\tilde{V}^3_{ml})_\xi = (V^3_{\pm ml})_\xi$$
$$(\tilde{V}^3_{ml})_\eta = (V^3_{\pm ml})_\eta$$
$$(\tilde{V}^3_{ml})_\varphi = (V^3_{\pm ml})_\varphi / \pm i$$

$$(\tilde{t}^1_{ml})_\xi = (t^1_{\pm ml})_\xi / \pm i$$
$$(\tilde{t}^1_{ml})_\eta = (t^1_{\pm ml})_\eta / \pm i$$
$$(\tilde{t}^1_{ml})_\varphi = (t^1_{\pm ml})_\varphi$$
$$(\tilde{t}^2_{ml})_\xi = (t^2_{\pm ml})_\xi$$
$$(\tilde{t}^2_{ml})_\eta = (t^2_{\pm ml})_\eta$$
$$(\tilde{t}^2_{ml})_\varphi = (t^2_{\pm ml})_\varphi / \pm i$$
$$(\tilde{t}^3_{ml})_\xi = (t^3_{\pm ml})_\xi$$
$$(\tilde{t}^3_{ml})_\eta = (t^3_{\pm ml})_\eta$$
$$(\tilde{t}^3_{ml})_\varphi = (t^3_{\pm ml})_\varphi / \pm i$$

The computation of the T-matrix requires the evaluation of the following surface integrals over the surface of the target:

$$Q_{n,\tau'n'} = -i\frac{k}{\lambda^W}\int_S dA\{(n\bullet t(V_n^W)(n\bullet A_{\tau',n'})\}$$

$$M_{n,n'} = -i\frac{k}{\lambda^W}\int_S dA\{(n\bullet V_n^W)S^W{}_{n'}\}$$

$$P_{\tau n,n'} = \int_S dA\{(n\bullet \mathrm{Re}V_{\tau,n})S^W{}_{n'}\}$$

$$R_{\tau n,\tau'n'} = \int_S dA\{t(\mathrm{Re}V_{\tau n})\bullet A_{\tau'n'}\}$$

In the above integrals one adopts the convention that the left hand term in the integral has $+m$ and the right hand term has $-m$ for the azimuthal angle dependence of the integrand. The above integrals are diagonal in m, and they can be replaced by the following one-dimensional integrals:

$$h_\xi = f\sqrt{(\xi^2-\eta^2)/(\xi^2-1)}$$
$$h_\eta = f\sqrt{(\xi^2-\eta^2)/(1-\eta^2)}$$
$$h_\varphi = f\sqrt{(\xi^2-1)(1-\eta^2)}$$

$$\Delta_n = h_\eta h_\varphi n_{\hat\xi} - h_\xi h_\varphi n_{\hat\eta}\frac{\partial\xi(\eta)}{\partial\eta}$$

$$Q^m{}_{l,\tau'l'} = -2\pi i\frac{k}{\lambda^W}\int_{-1}^{+1} d\eta\Delta_n[(t^W_{+ml})_{\hat\xi}n_{\hat\xi} + (t^W_{+ml})_{\hat\eta}n_{\hat\eta}][(A^{\tau'}_{-ml'})_{\hat\xi}n_{\hat\xi} + (A^{\tau'}_{-ml'})_{\hat\eta}n_{\hat\eta}]$$

$$M^m{}_{l,l'} = -2\pi i\frac{k}{\lambda^W}\int_{-1}^{+1} d\eta\Delta_n[(V^W_{+ml})_{\hat\xi}n_{\hat\xi} + (V^W_{+ml})_{\hat\eta}n_{\hat\eta}](S^W_{-ml'})$$

$$P^m{}_{\tau l,l'} = 2\pi\int_{-1}^{+1} d\eta\Delta_n[(\mathrm{Re}V^\tau_{+ml})_{\hat\xi}n_{\hat\xi} + (\mathrm{Re}V^\tau_{+ml})_{\hat\eta}n_{\hat\eta}](S^W_{-ml'})$$

$$R^m{}_{\tau l,\tau'l'} = 2\pi\int_{-1}^{+1} d\eta\Delta_n(\mathrm{Re}\,t^\tau_{+ml})\bullet(A^{\tau'}_{-ml'})$$

Using the polar parity and the azimuthal parity properties of the integrands, the non-zero components of the above integrals in the case l-l' is even are given by the following integrals:

$$Q^{m}{}_{l,2l'} = -4\pi i \frac{k}{\lambda^{W}} \int_{0}^{+1} d\eta \Delta_{n} [+(\tilde{t}^{W}_{ml})_{\hat{\xi}} n_{\hat{\xi}} + (\tilde{t}^{W}_{ml})_{\hat{\eta}} n_{\hat{\eta}}][+(\tilde{A}^{2}_{ml'})_{\hat{\eta}} n_{\hat{\eta}}]$$

$$Q^{m}{}_{l,3l'} = -4\pi i \frac{k}{\lambda^{W}} \int_{0}^{+1} d\eta \Delta_{n} [+(\tilde{t}^{W}_{ml})_{\hat{\xi}} n_{\hat{\xi}} + (\tilde{t}^{W}_{ml})_{\hat{\eta}} n_{\hat{\eta}}][+(\tilde{A}^{3}_{ml'})_{\hat{\xi}} n_{\hat{\xi}}]$$

$$M^{m}{}_{l,l'} = -4\pi i \frac{k}{\lambda^{W}} \int_{0}^{+1} d\eta \Delta_{n} [+(\tilde{V}^{W}_{ml})_{\hat{\xi}} n_{\hat{\xi}} + (\tilde{V}^{W}_{ml})_{\hat{\eta}} n_{\hat{\eta}}](\tilde{S}^{W}_{ml'})$$

$$P^{m}{}_{2l,l'} = 4\pi \int_{0}^{+1} d\eta \Delta_{n} [+(\mathrm{Re}\,\tilde{V}^{2}_{ml})_{\hat{\xi}} n_{\hat{\xi}} + (\mathrm{Re}\,\tilde{V}^{2}_{ml})_{\hat{\eta}} n_{\hat{\eta}}](\tilde{S}^{W}_{ml'})$$

$$P^{m}{}_{3l,l'} = 4\pi \int_{0}^{+1} d\eta \Delta_{n} [+(\mathrm{Re}\,\tilde{V}^{3}_{ml})_{\hat{\xi}} n_{\hat{\xi}} + (\mathrm{Re}\,\tilde{V}^{3}_{ml})_{\hat{\eta}} n_{\hat{\eta}}](\tilde{S}^{W}_{ml'})$$

$$R^{m}{}_{1l,1l'} = 4\pi \int_{0}^{+1} d\eta \Delta_{n} [+(\mathrm{Re}\,\tilde{t}^{1}_{ml})_{\hat{\eta}} (\tilde{A}^{1}_{ml'})_{\hat{\eta}} + (\mathrm{Re}\,\tilde{t}^{1}_{ml})_{\hat{\varphi}} (\tilde{A}^{1}_{ml'})_{\hat{\varphi}}]$$

$$R^{m}{}_{2l,2l'} = 4\pi \int_{0}^{+1} d\eta \Delta_{n} [+(\mathrm{Re}\,\tilde{t}^{2}_{ml})_{\hat{\eta}} (\tilde{A}^{2}_{ml'})_{\hat{\eta}} + (\mathrm{Re}\,\tilde{t}^{2}_{ml})_{\hat{\varphi}} (\tilde{A}^{2}_{ml'})_{\hat{\varphi}}]$$

$$R^{m}{}_{3l,3l'} = 4\pi \int_{0}^{+1} d\eta \Delta_{n} (\mathrm{Re}\,\tilde{t}^{3}_{ml})_{\hat{\xi}} (\tilde{A}^{3}_{ml'})_{\hat{\xi}}$$

$$R^{m}{}_{2l,3l'} = 4\pi \int_{0}^{+1} d\eta \Delta_{n} (\mathrm{Re}\,t^{2}_{+ml})_{\hat{\xi}} (A^{3}_{+ml'})_{\hat{\xi}}$$

$$R^{m}{}_{3l,2l'} = 4\pi \int_{0}^{+1} d\eta \Delta_{n} [+(\mathrm{Re}\,\tilde{t}^{3}_{ml})_{\hat{\eta}} (\tilde{A}^{2}_{ml'})_{\hat{\eta}} + (\mathrm{Re}\,\tilde{t}^{3}_{ml})_{\hat{\varphi}} (\tilde{A}^{2}_{ml'})_{\hat{\varphi}}]$$

The non-zero components of the integrals in the case *l-l'* is odd are given by the following integrals:

$$Q^{m}{}_{l,1l'} = -4\pi i(-i)\frac{k}{\lambda^{W}}\int_{0}^{+1} d\eta\Delta_{n}[+(\tilde{t}^{W}_{ml})_{\hat{\xi}}n_{\hat{\xi}} + (\tilde{t}^{W}_{ml})_{\hat{\eta}}n_{\hat{\eta}}][(\tilde{A}^{1}_{ml'})_{\hat{\eta}}n_{\hat{\eta}}]$$

$$P^{m}{}_{1l,l'} = +4\pi i\int_{0}^{+1} d\eta\Delta_{n}[+(\mathrm{Re}\tilde{V}^{1}_{ml})_{\hat{\xi}}n_{\hat{\xi}} + (\mathrm{Re}\tilde{V}^{1}_{ml})_{\hat{\eta}}n_{\hat{\eta}}](\tilde{S}^{W}_{ml'})$$

$$R^{m}{}_{1l,2l'} = +4\pi i\int_{0}^{+1} d\eta\Delta_{n}[+(\mathrm{Re}\tilde{t}^{1}_{ml})_{\hat{\eta}}(\tilde{A}^{2}_{ml'})_{\hat{\eta}} - (\mathrm{Re}\tilde{t}^{1}_{ml})_{\hat{\varphi}}(\tilde{A}^{2}_{ml'})_{\hat{\varphi}}]$$

$$R^{m}{}_{2l,1l'} = -4\pi i\int_{0}^{+1} d\eta\Delta_{n}[+(\mathrm{Re}\tilde{t}^{2}_{ml})_{\hat{\eta}}(\tilde{A}^{1}_{ml'})_{\hat{\eta}} - (\mathrm{Re}\tilde{t}^{2}_{ml})_{\hat{\varphi}}(\tilde{A}^{1}_{ml'})_{\hat{\varphi}}]$$

$$R^{m}{}_{3l,1l'} = -4\pi i\int_{0}^{+1} d\eta\Delta_{n}[+(\mathrm{Re}\tilde{t}^{3}_{ml})_{\hat{\eta}}(\tilde{A}^{1}_{ml'})_{\hat{\eta}} - (\mathrm{Re}\tilde{t}^{3}_{ml})_{\hat{\varphi}}(\tilde{A}^{1}_{ml'})_{\hat{\varphi}}]$$

$$R^{m}{}_{1l,3l'} = +4\pi i\int_{0}^{+1} d\eta\Delta_{n}(\mathrm{Re}\tilde{t}^{1}_{ml})_{\hat{\xi}}(\tilde{A}^{3}_{ml'})_{\hat{\xi}}$$

In the case *(l-l')* is even, the components of the *R* and *P* matrices are real. In the case *(l-l')* is odd the *R* and *P* matrices are imaginary. The *R* and *P* matrices can be transformed into real matrices by making the following similarity transformation on the vector indices of the *R, P,* and *Q* matrices:

$$S_{\tau\tau'} = -i\delta^{\tau'}_{\tau}\delta^{1}_{\tau} + \delta^{\tau'}_{\tau}\delta^{2}_{\tau} + \delta^{\tau'}_{\tau}\delta^{3}_{\tau}$$

$$S^{-1}{}_{\tau\tau'} = S^{\dagger}{}_{\tau\tau'} = +i\delta^{\tau'}_{\tau}\delta^{1}_{\tau} + \delta^{\tau'}_{\tau}\delta^{2}_{\tau} + \delta^{\tau'}_{\tau}\delta^{3}_{\tau}$$

$$R^{m}_{\tau l,\tau' l'} = \sum_{\tau'',\tau'''} S_{\tau',\tau''}R^{m}_{\tau'' l,\tau''' l'}S^{\dagger}{}_{\tau''',\tau'}$$

$$P^{m}_{\tau l,l'} = \sum_{\tau''} S_{\tau',\tau''}P^{m}_{\tau'' l,l'}$$

$$Q^{m}_{l,\tau' l'} = \sum_{\tau''} Q^{m}_{l,\tau'' l'}S^{\dagger}{}_{\tau'',\tau'}$$

In the case (*l-l'*) is even, the components of the *Q, R,* and *P* matrices are left invariant by this transformation, whereas in the case (*l-l'*) is odd they are transformed to the following matrices:

$$Q^{m}{}_{l,1l'} = -4\pi i \frac{k}{\lambda^{W}} \int_{0}^{+1} d\eta \Delta_{n} [+(\tilde{t}^{W}_{ml})_{\hat{\xi}} n_{\hat{\xi}} + (\tilde{t}^{W}_{ml})_{\hat{\eta}} n_{\hat{\eta}}][(\tilde{A}^{1}_{ml'})_{\hat{\eta}} n_{\hat{\eta}}]$$

$$P^{m}{}_{1l,l'} = +4\pi \int_{0}^{+1} d\eta \Delta_{n} [+(\mathrm{Re}\tilde{V}^{1}_{ml})_{\hat{\xi}} n_{\hat{\xi}} + (\mathrm{Re}\tilde{V}^{1}_{ml})_{\hat{\eta}} n_{\hat{\eta}}](\tilde{S}^{W}_{ml'})$$

$$R^{m}{}_{1l,2l'} = +4\pi \int_{0}^{+1} d\eta \Delta_{n} [+(\mathrm{Re}\tilde{t}^{1}_{ml})_{\hat{\eta}} (\tilde{A}^{2}_{ml'})_{\hat{\eta}} - (\mathrm{Re}\tilde{t}^{1}_{ml})_{\hat{\varphi}} (\tilde{A}^{2}_{ml'})_{\hat{\varphi}}]$$

$$R^{m}{}_{2l,1l'} = +4\pi \int_{0}^{+1} d\eta \Delta_{n} [+(\mathrm{Re}\tilde{t}^{2}_{ml})_{\hat{\eta}} (\tilde{A}^{1}_{ml'})_{\hat{\eta}} - (\mathrm{Re}\tilde{t}^{2}_{ml})_{\hat{\varphi}} (\tilde{A}^{1}_{ml'})_{\hat{\varphi}}]$$

$$R^{m}{}_{3l,1l'} = +4\pi \int_{0}^{+1} d\eta \Delta_{n} [+(\mathrm{Re}\tilde{t}^{3}_{ml})_{\hat{\eta}} (\tilde{A}^{1}_{ml'})_{\hat{\eta}} - (\mathrm{Re}\tilde{t}^{3}_{ml})_{\hat{\varphi}} (\tilde{A}^{1}_{ml'})_{\hat{\varphi}}]$$

$$R^{m}{}_{1l,3l'} = +4\pi \int_{0}^{+1} d\eta \Delta_{n} (\mathrm{Re}\tilde{t}^{1}_{ml})_{\hat{\xi}} (\tilde{A}^{3}_{ml'})_{\hat{\xi}}$$

To convert the above integrals into matrices one adopts the following convention for the index of the pair τl of vector and order indices in the case m=0, where negative indices are ignored:

$$n(\tau = 1, l) = 3(l - m) - 2$$
$$n(\tau = 2, l) = 3(l - m) - 1$$
$$n(\tau = 3, l) = 3(l - m)$$

In the case $m \neq 0$, one adopts the following index notation for the pair τl of vector and order indices.

$$n(\tau = 1, l) = 3(l - m) + 2$$
$$n(\tau = 2, l) = 3(l - m) + 1$$
$$n(\tau = 3, l) = 3(l - m)$$

Here one uses the C-style index notation, where the first element in a vector starts at the index 0. The above index convention is used to project out the zero components of the vector basis functions in the case *l=m=0*. For example, the square R-matrix has dimension $(3(l_{\max} - m) + 1)$ in the case m=0, and $(3(l_{\max} - m) + 3)$ otherwise, where $l_{\max}$ is the maximum value for the order l.

The T-matrix is defined by the following equation in terms of the above matrices.

$$T = -(\operatorname{Re} QR^{-1}P - \operatorname{Re} M)(QR^{-1}P - M)^{-1}$$

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