# Thermophoresis of Axially Symmetric Bodies

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**Abstract.** Thermophoresis of axially symmetric bodies is investigated to first order in the Knudsen-number, $Kn$. The study is made in the limit where the typical length of the immersed body is small compared to the mean free path. It is shown that in this case, in contrast to what is the case for spherical bodies, the arising thermal force on the body is not in general antiparallel to the temperature gradient. It is also shown that the gas excerts a torque on the body, which in magnitude and direction depends on the body geometry. Equations of motion describing the body movement are derived. Asymptotic solutions are studied.

## INTRODUCTION

Thermophoresis as a phenomenon has been known for a long time, and several authors have approached the problem. For example, Einstein calculated the final velocity of a spherical particle in a heat conducting gas using elementary kinetic theory. A recent review of the phenomenon is given in an article by Sone, see [1]. Another review is given by Talbot, see [2]. The first systematic attempt to describe the thermophoresis phenomenon using kinetic theory is found in an article by Waldmann from 1959, see [4]. That work was made under the assumption that the mean free path of the gas is much larger than the dimension of the body. Further results are found in a variety of articles, but these mostly apply to larger bodies, using asymptotic methods, and often to spheres, cf [3].

In this work, thermophoresis of axially symmetric bodies is considered. The typical macroscopical length $L$ over which the temperature varies is assumed to be much larger than $\lambda$, the mean free path of the gas. To this end we define the Knudsen number according to $Kn = \lambda/L$. Thus $Kn \ll 1$, and we can use the first order Chapman-Enskog solution to the Boltzmann equation, see [5], to describe the heat conducting gas from the macroscopical viewpoint. The typical diameter $R$ of the body is assumed to bee much smaller than $\lambda$. $R$ is also assumed to be much larger than $d$, the diameter of a gas molecule. In calculating the momentum transfer from the gas to the body we therefore use the equations of free molecular flow ( [5]). We also assume that $Ma = v/c_s \ll 1$, where $v$ is the speed of a body surface element and where $c_s$ is the speed of sound.

## LOCAL MOMENTUM TRANSFER TO THE BODY

If the one-particle distribution function $f = f(\mathbf{x}, \mathbf{c}, t)$ describing the gas is known and the proper gas-body surface boundary conditions are specified it is in principle possible to calculate the net transfer of momentum from the gas molecules to a body immersed in the gas. Let us first of all consider the interaction between the gas and a surface element of the body, with area $dS$ and normal $\mathbf{n}$, that is at rest in the gas. The net momentum transfer from the gas to the body surface element can now be expressed by

$$dF_i = -\left( \int_{\mathbf{c}\cdot\mathbf{n}<0} mc_i\mathbf{c}\cdot\mathbf{n}f^{(i)}d^3c + \int_{\mathbf{c}\cdot\mathbf{n}>0} mc_i\mathbf{c}\cdot\mathbf{n}f^{(r)}d^3c \right) dS \tag{1}$$

Here $f^{(i)}$ describes the incident molecular stream and $f^{(r)}$ describes the reflected stream. $m$ is the mass of a gas molecule, and $\mathbf{c}$ is the velocity. The surface element is taken to be impermeable, and thus the stream of gas molecules incident to the surface element equals the stream of outgoing molecules. Therefore

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**Abstract**

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$$\int_{\mathbf{c}\cdot\mathbf{n}<0} \mathbf{c}\cdot\mathbf{n} f^{(i)} d^3c + \int_{\mathbf{c}\cdot\mathbf{n}>0} \mathbf{c}\cdot\mathbf{n} f^{(r)} d^3c = 0. \tag{2}$$

As the typical length of the body is small compared to the mean free path in the gas and as the body surface is taken to be convex, the incident stream of molecules is unaffected by the presence of the body, and thus $f^{(i)} = f$, where $f$ is the distribution function for the gas in the absence of the body. The reflected stream is determined from a Maxwell-type boundary condition and aquires thus upon reflection two separate parts: One part that is specularly reflected, and one part that is diffusely reflected. Here, the relative amount of momentum tangential to the surface element transferred to the body from the incident stream is measured by the momentum accommodation coefficient $\alpha_\tau$, see [5]. This boundary condition has the following form:

$$f^{(r)} = (1-\alpha_\tau)f' + \alpha_\tau (n^{(r)}/n^{(0)})f^{(0)}, \tag{3}$$

where $f' \equiv f(\mathbf{c}')$, in which $c_i' = c_i - 2n_i n_j c_j$, is thus the distribution function of a specularly reflected stream. The second term describes the diffusely reflected part of the outgoing stream. $f^{(0)}$ is a Maxwellian distribution function given by

$$f^{(0)} = n \left(\frac{2\pi k_B T}{m}\right)^{-3/2} e^{-\mathcal{C}^2}, \tag{4}$$

describing gas molecules which are in thermal equilibrium with the surface element. In this expression, $n_r$ is the number density of the reflected stream, and the dimensionless velocity $\mathcal{C}$ is defined by

$$\mathcal{C}_i \equiv c_i / \sqrt{\frac{2k_B T}{m}}.$$

$T$ is the local gas temperature and $k_B$ is the Boltzmann constant. It is assumed here that the temperature of the body is the same as the local temperature of the gas. The integral (1) can now be simplified. Introducing a dimensionless mass flux and a dimensionless momentum flux incident on the body according to

$$\mathcal{M} \equiv -\frac{1}{n}\left(\frac{2k_B T}{m}\right)^{3/2} \int_{\mathcal{C}_i n_i < 0} \mathcal{C}_i n_i f d^3\mathcal{C} \quad \text{and} \quad \mathcal{M}_i \equiv -\frac{1}{n}\left(\frac{2k_B T}{m}\right)^{3/2} \int_{\mathcal{C}_i n_i < 0} \mathcal{C}_i \mathcal{C}_j n_j f d^3\mathcal{C}, \tag{5}$$

the force on a surface element becomes, where $p = n k_B T$,

$$dF_i = p\left[4(1-\alpha_\tau)n_i n_j \mathcal{M}_j + \alpha_\tau\left(2\mathcal{M}_i - \sqrt{\pi}\mathcal{M} n_i\right)\right] dS. \tag{6}$$

In the present case, we shall consider a body which is small compared to the mean free path. This body will start to move under the influence of the force, and thus the surface element will experience a homogenous flow of the surrounding gas. This flow will excert an additional force on the surface element. Therefore, when calculating the force on the surface element according to (6), we consider a distribution function corresponding to a gas which is subject to a temperature gradient and a homogenous flow. It is assumed that $Kn \equiv \lambda/L$, where $\lambda$ is the mean free path and where $L$ is the macroscopic length scale, is small. We also assume that the resulting motion of the surface element is small compared to the speed of sound.

For a gas subject to a non-uniform temperature distribution the distribution function takes the form, see [5],

$$f = f_0 \left(1 + \phi\right), \tag{7}$$

where $\phi$ can be found through the Chapman-Enskog expansion to first order in $Kn$. The result for pure heat conduction is

$$\phi = -\frac{1}{nT}\sqrt{\frac{2k_B T}{m}} A(\mathcal{C}^2)\mathcal{C}_i T_{,i}. \tag{8}$$

The function $A$ is usually expressed in Sonine polynomials according to

$$A(\mathcal{C}^2) = -\sum_{n=1}^{\infty} a_n S_{3/2}^{(n)}(\mathcal{C}^2), \quad \text{with} \quad a_1 = -\frac{2m\kappa}{5k_B^2 T}. \tag{9}$$

Here $\kappa$ is the heat conductivity. If we introduce

$$\hat{A}(\mathcal{C}^2) \equiv A(\mathcal{C}^2)/a_1,$$

we may express $\phi$ in terms of the heat current $q_i = -\kappa T_{,i}$ according to

$$\phi = -\frac{4}{5p}\left(\frac{2k_BT}{m}\right)^{-1/2}\hat{A}(\mathcal{C}^2)\mathcal{C}_iq_i. \tag{10}$$

Now $\phi$ is integrated to give the moments (5) and one arrives at

$$\mathcal{M} = \frac{1}{2\sqrt{\pi}} \quad \text{and} \quad \mathcal{M}_i = -\frac{1}{4}n_i + \frac{\gamma}{10p}\left(\frac{2\pi k_BT}{m}\right)^{-1/2}(\delta_{ij}+n_in_j)\,q_j. \tag{11}$$

The formulas (11) thus apply to a body surface element at rest. Here, the integral $\gamma$ is given by

$$\gamma = \int_0^\infty \mathcal{C}^5\hat{A}(\mathcal{C}^2)e^{-\mathcal{C}^2}\,d\mathcal{C}. \tag{12}$$

If only the first term in the Sonine polynomial expansion of $\hat{A}$ is retained, $\gamma = 1$. The corresponding integral in $\mathcal{M}$,

$$\int_0^\infty \mathcal{C}^4\hat{A}(\mathcal{C}^2)e^{-\mathcal{C}^2}\,d\mathcal{C},$$

vanishes since the gas is at rest.

The temperature gradient will give rise to a force on the immersed body and thus set the body in motion. Following Waldmann, if we denote the velocity of a body surface element by $u_i$, this motion is taken into account by transforming the distribution function to a frame of reference in which the surface element of the body is momentarily at rest. When the transformation $c_i \to c_i - u_i$ is made, both the Maxwellian part and the heat conduction part of (7) is transformed. However, since $Ma$ is small, the cross term coupling translation and heat conduction will be of the order of $Kn\,Ma$, and will thus be ignored in the present context. The moments $\mathcal{M}$ and $\mathcal{M}_i$ get additional contributions of the order of $Ma$, and the result is

$$\mathcal{M} = \frac{1}{2\sqrt{\pi}} - \frac{\sqrt{\pi}}{2}\left(\frac{2\pi k_BT}{m}\right)^{-1/2}n_iu_i, \tag{13}$$

$$\mathcal{M}_i = \frac{1}{2}\left(\frac{2\pi k_BT}{m}\right)^{-1/2}(\delta_{ij}+n_in_j)\left(\frac{\gamma}{5p}q_j + u_j\right). \tag{14}$$

The resulting force on a body surface element (6) becomes

$$dF_i = -pn_idS + \left(\frac{2\pi k_BT}{m}\right)^{-1/2}\left\{\alpha_\tau\left(\frac{\gamma}{5}q_i + pu_i\right) + \left(\frac{\gamma}{5}(4-3\alpha_\tau)q_j + p\left[4-\left(3-\frac{\pi}{2}\right)\alpha_\tau\right]u_j\right)n_jn_i\right\}dS \tag{15}$$

This force will in general also produce a torque acting on the body through the surface element according to

$$dM_i = \epsilon_{ijk}x_jdF_k, \tag{16}$$

where $\mathbf{x}$ is the vector from the center of mass of the body to the surface element. As the force and the torque are applied to the body, a velocity and also an angular velocity will be produced. Therefore, the appropriate Gallileian transformation to be used in this context is

$$u_i = -(v_i + \epsilon_{ijk}\omega_jx_k),$$

where the minus sign stems from transforming to the frame of reference where the gas is at rest. The integration of the force and the torque over the total body surface involves calculating geometrical tensor integrals of the type

$$\int_S n_{i_1}n_{i_2}...n_{i_k}x_{j_1}x_{j_2}...x_{j_l}\,dS \equiv I^{(k,l)}_{i_1i_2...i_k|j_1j_2...j_l}. \tag{17}$$

These are all isotropic functions of the unit vector $\mathbf{N}$ defining the axis of symmetry, and are listed in the appendix.

# FORCE AND TORQUE ON THE BODY

Now we are in a position to write down the expressions for the force and the torque acting on the body. In what follows, the dimensionless dynamical coefficients $a_1, .., a_9$ are introduced for simplicity. They consist of products between linear functions of $\alpha_\tau$ and geometrical coefficients given by contractions of the geometrical tensor integrals encountered in the previous section(17). The coefficients $a_1, .., a_9$ are listed in the appendix.

The force becomes

$$F_i = \left(\frac{2\pi k_B T}{m}\right)^{-1/2} \left\{\frac{4\gamma S}{15}\left(\delta_{ij} + a_1 N_{<i}N_{j>}\right) q_j - pS\left(a_2\delta_{ij} + a_3 N_{<i}N_{j>}\right) v_j + 3pS^{3/2}a_4\epsilon_{ijk}N_j\omega_k\right\}. \quad (18)$$

The first term on the right hand side of (18) is a thermal contribution to the force proportional to the heat current. It contains one isotropic part, $a_1\delta_{ij}$ and one symmetric traceless part, $a_2 N_{<i}N_{j>}$. While the former part acts on a spherically symmetric body, the latter one, which vanishes in case of spherical body surface, projects momentum onto the direction of the axis of symmetry. The order of magnitude of this term is $\sim pSKn$. The second contribution to the total force is a friction force with geometrical features similar to the thermal force. The last term is a force acting on the body as a consequence of the body rotation. The two last terms both have the magnitude $\sim pSMa$.

The torque is given by

$$M_i = \left(\frac{2\pi k_B T}{m}\right)^{-1/2} \left\{\frac{3\gamma S^{3/2}}{5}a_5\epsilon_{ijk}N_j q_k - 3pS^{3/2}a_6\epsilon_{ijk}N_j v_k - \frac{pS^2}{2}\left(a_7\delta_{ij} + a_8 N_{<i}N_{j>}\right)\omega_j\right\}. \quad (19)$$

In the expression for the torque, the first term is a thermal torque that acts to align the axis of symmetry to the heat current. This contribution is of the order of $\sim pS^{3/2}Kn$. The second term is active whenever the axis of symmetry is not parallel or antiparallel to the velocity. The third term is a friction torque. The two latter contributions are of the order of $\sim pS^{3/2}Ma$.

# EQUATIONS OF MOTION

In order to formulate the equations of motion for the body immersed in the heat conducting gas, we introduce a body-fixed orthonormal frame of reference with the origin in the body's center of mass. The basis vectors are denoted by $\mathbf{e}^{(\alpha)}$ where the index $\alpha = 1, 2, 3$ numbers the basis vectors. One of these basis vectors is naturally chosen to coincide with the axis of symmetry $\mathbf{N}$. The equations of motion become

$$\frac{d}{dt}(m_B v_i) = \left(\frac{2\pi k_B T}{m}\right)^{-1/2}\left\{\frac{4\gamma S}{15}\left(\delta_{ij} + a_1 N_{<i}N_{j>}\right) q_j - pS\left(a_2\delta_{ij} + a_3 N_{<i}N_{j>}\right) v_j + 3pS^{3/2}a_4\epsilon_{ijk}N_j\omega_k\right\},$$

$$\frac{d}{dt}(I_{ij}\omega_j) = \left(\frac{2\pi k_B T}{m}\right)^{-1/2}\left\{\frac{3\gamma S^{3/2}}{5}a_5\epsilon_{ijk}N_j q_k - 3pS^{3/2}a_6\epsilon_{ijk}N_j v_k - \frac{pS^2}{2}\left(a_7\delta_{ij} + a_8 N_{<i}N_{j>}\right)\omega_j\right\},$$

$$\frac{d}{dt}(e_i^{(\alpha)}) = \epsilon_{ijk}\omega_j e_k^{(\alpha)},$$

$$(20)$$

where $m_B$ is the body mass and where $I_{ij}$ is the inertia tensor. This tensor has the form

$$I_{ij} = I_0\left(b_1\delta_{ij} + b_2 N_{<i}N_{j>}\right). \quad (21)$$

For values of $b_1, b_2, I_0$, see Appendix.

It is possible to construct two asymptotic solutions to these equations by observing that $\mathbf{F} = \mathbf{M} = \mathbf{0}$ if $\mathbf{v}$ is parallel to $\mathbf{q}$ and $\mathbf{N}$ is parallel or antiparallel to $\mathbf{q}$, together with $\boldsymbol{\omega} = \mathbf{0}$: The equation of motion is then reduced to an expression for the asymptotic velocity $\mathbf{V}$:

$$V_i = \frac{4}{15}\gamma\frac{3 + 2a_1}{3a_2 + 2a_3}q_i = \frac{\gamma}{5p}\frac{1 + \frac{1}{2}\left(4 - 3\alpha_\tau\right)c^{(2,0)}}{1 + \frac{\pi}{8}\alpha_\tau + \frac{1}{4}\left[8 - (6 - \pi)\alpha_\tau\right]c^{(2,0)}}q_i. \quad (22)$$

Here, $\gamma$ is given by (12). It depends on the intramolecular forces of the gas molecules; $\gamma = 1$ if only the first term of the Sonine polynomial expansion of the heat conducting part of the distributionfunction is retained.

The new terms are those containing the parameter $c^{(2,0)}$ that depends on the body geometry. $c^{(2,0)}$ measures deformations of the body surface from a sphere: For a sphere, $c^{(2,0)} = 0$. In this case the asymptotic velocity (22) coincides with Waldmann's result, [4]. If the the body is extremely oblate, or shaped like a coin, $c^{(2,0)} = 1$. If the body is extremely prolate, or shaped like needle, $c^{(2,0)} = -\frac{1}{2}$. -The order of magnitude of this velocity is $Kn \cdot c_s$.

Non-dimensional variables are now introduced according to $v_i = \frac{\gamma q}{5p} v_i^*$, $t = \frac{m_B}{pS} \left(\frac{2\pi k_B T}{m}\right)^{1/2} t^*$, and $\omega_i = \frac{pS}{m_B q} \left(\frac{2\pi k_B T}{m}\right)^{1/2} \omega_i^*$, where $q = |\mathbf{q}|$. The non-dimensional heat current is chosen to be $\hat{\mathbf{q}} = (1,0,0)$. A simulation of the non-dimensional equations of motion yielding the body orbit of a cone and a coin is plotted in figure 1.
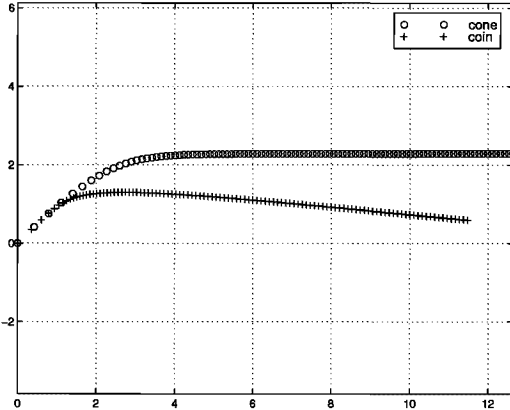


**FIGURE 1.** Orbit of a cone and a coin, with the heat current in the $x$-direction. Initially the axis of symmetry $\mathbf{N}$ of the two bodies makes the angle $45°$ with the heat current. The initial non-dimensional velocity of the two bodies is 5 and is directed along $\mathbf{N}$. The spacing between the points representing the body orbits corresponds to a fixed time interval. Note that the cone assumes a final velocity parallel to the heat current, whereas the final velocity of the coin also has a component in the plane orthogonal to the heat current.

Numerical simulations seem to suggest that the bodies reach asymptotic states after a distance of the order of 1 in non-dimensional units, that is, $l = \left(\frac{2\pi k_B T}{m}\right)^{1/2} \frac{\gamma m_B q}{5p^2 S} \sim \frac{R}{d} \lambda \cdot Kn$ in dimensional units. Here, $R$ is the dimension of the body, and $d$ is the diameter of a gas molecule. The difference in asymptotic velocity is due to a symmetry in the body shape of the coin that is lacking in the body shape of the cone. This is explained in the proceeding paragraph.

In order to investigate the stability of the two asymptotic states we linearize the equations of motion around the state where $\mathbf{N}$ is nearly parallel to $\mathbf{q}$. The eigenvalues of the resulting linear system is calculated for a special type of body, a 'double cone', shown in figure 2. This body consists of two cones with a common base. The radius of the base is denoted by $D$, and the total length by $R$. The base is situated a distance $s \cdot R$ from the left cusp, where the dimensionless parameter obeys $0 \leq s \leq 1$. When $s = 0$ the double cone degenerates to a single cone with its cusp pointing in the direction of $\mathbf{N}$. When $s = 1$ we recover another single cone, pointing in the direction of $-\mathbf{N}$.
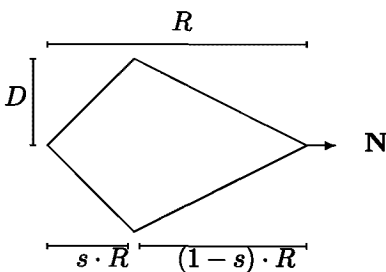


FIGURE 2. Test body: a 'double cone'

Plotting the eigenvalues against $s$ yields a picture of the preferred orientation of the body. It turns out that stabillity is determined by two identical eigenvalues (the other eigenvalues are either negative or zero for all

values of $s$). This eigenvalue is plotted against $s$ in figure 3 for a somewhat prolate body $(R/D = 3)$ and for an oblate body $(R/D = 1/3)$.
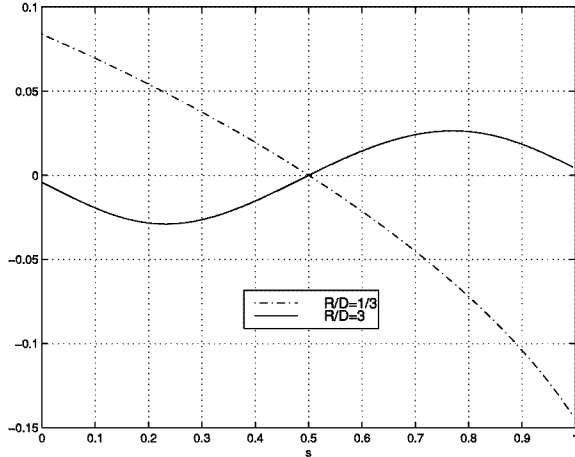


**FIGURE 3.** Critical eigenvalue plotted against $s$ for two different double cones

It is apparent that the stable state for the prolate body occures when $s < 0.5$ and for the oblate body when $s > 0.5$. This may indicate that the stabillity is determined by the orientation of the center of mass of the body and the center of mass of the surface of the body along the direction of the heat current.

## BODY INVARIANT UNDER REFLEXION IN A PLANE ORTHOGONAL TO THE AXIS OF SYMMETRY

Now we consider a body that is symmetric with respect to an equator, that is, a body that is invariant under the transformation $\mathbf{N} \rightarrow -\mathbf{N}$. Then the coefficients $a_4, a_5$ and $a_6$ vanish. It it easy to see that then the rotational motion decouples from the translation and becomes independent of the heat current. The remaining part of the angular momentum equation in (20) is then

$$\frac{d}{dt}\left(I_{ij}\omega_j\right) = -\frac{pS^2}{2}\left(\frac{2\pi k_B T}{m}\right)^{-1/2}\left(a_7\delta_{ij} + a_8 N_{<i}N_{j>}\right)\omega_j. \tag{23}$$

This equation states that as $t \rightarrow \infty$ all constant values of $\mathbf{N}$ is a solution. In the momentum equation (20) the coupling to $\boldsymbol{\omega}$ vanishes, but the dependence on $\mathbf{N}$ still remains through the tensor $N_{<i}N_{j>}$. The asymptotic solution to the momentum equation in this case becomes

$$\tilde{V}_i = \left[\delta_{ij} + k\left(\delta_{ij} - N_i N_j\right)\right]V_j, \tag{24}$$

where $V_j$ is the asymptotic velocity (22) found in the previous section. The tensor $\delta_{ij} - N_i N_j$ picks up the projection of $\mathbf{q}$ that lies in the plane perpendicular to $\mathbf{N}$. The parameter $k$ is introduced for simplicity and is given by (we recall that $c^{(2,0)}$ is a geometrical parameter encountered in (22))

$$k = \frac{6\pi\alpha_\tau^2 c^{(2,0)}}{2(8 + \pi\alpha_\tau) + [16 - (12 - 2\pi)\alpha_\tau - 3\pi\alpha_\tau^2]c^{(2,0)} - [32 - (48 - 4\pi)\alpha_\tau + (18 - 3\pi)\alpha_\tau^2]\left[c^{(2,0)}\right]^2}. \tag{25}$$

This means that a nonspherical body for general values of $\alpha_\tau$ will aquire a velocity component in the plane perpendicular to the heat current. This effect is apparent in the final behaviour exposed by the body orbit of the coin in figure 1. The maximum angle between the heat current and the body velocity, $\psi_M$, occures in case of purely diffuse reflexion for an optimal value of the final orientation of the axis of symmetry relative to the heat current, and is given by

$$\psi_M = \arccos\left\{2\frac{\sqrt{1+k}}{2+k}\right\} = \begin{cases} 16.4^0 \text{ (coin)} \\ 12.0^0 \text{ (needle)} \end{cases} \tag{26}$$

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These results thus predict that in an initially localized assembly of equatorially symmetric bodies with different orientations relative to the heat current will, in the plane perpendicular to $\mathbf{q}$, with time spread without bound. This is an interesting result, but perhaps not the whole story: Going to higher order contributions to the torque, a term of the order of $Kn^2$ will recouple the angular momentum of equatorially symmetric bodies to the temperature gradient. However, since in the present context $Kn \ll 1$, this coupling is clearly weak.

# APPENDIX

The geometric tensor integrals encountered when the expressions for the force and the torque are integrated over the body surface are listed below:

$$I_{i|}^{(1,0)} = 0$$

$$I_{|j}^{(0,1)} = S^{3/2} c^{(0,1)} N_j$$

$$I_{i_1 i_2|}^{(2,0)} = S \left[ \tfrac{1}{3} \delta_{i_1 i_2} + c^{(2,0)} N_{<i_1} N_{i_2>} \right]$$

$$I_{i|j}^{(1,1)} = S^{3/2} \left[ c_1^{(1,1)} \delta_{ij} + c_2^{(1,1)} N_{<i} N_{j>} \right],$$

$$I_{|j_1 j_2}^{(0,2)} = S^2 \left[ c_1^{(0,2)} \delta_{j_1 j_2} + c_2^{(0,2)} N_{<j_1} N_{j_2>} \right]$$

$$I_{i_1 i_2|j}^{(2,1)} = S^{3/2} \left[ c_1^{(2,1)} \delta_{i_1 i_2} N_j + c_2^{(2,1)} \left( \delta_{i_1 j} N_{i_2} + \delta_{i_2 j} N_{i_1} \right) + c_3^{(2,1)} N_{i_1} N_{i_2} N_{i_3} \right]$$

$$\begin{aligned}
I_{i_1 i_2|j_1 j_2}^{(2,2)} = S^2 \Big[ &c_1^{(2,2)} \delta_{i_1 i_2} \delta_{j_1 j_2} + c_2^{(2,2)} \left( \delta_{i_1 j_1} \delta_{i_2 j_2} + \delta_{i_1 i j_2} \delta_{i_2 j_1} \right) + c_3^{(2,2)} \delta_{i_1 i_2} N_{j_1} N_{j_2} + c_4^{(2,2)} \delta_{j_1 j_2} N_{i_1} N_{i_2} \\
&+ c_5^{(2,2)} \left( \delta_{i_1 j_1} N_{i_2} N_{j_2} + \delta_{i_1 j_2} N_{i_2} N_{j_1} + \delta_{i_2 j_1} N_{i_1} N_{j_2} + \delta_{i_2 j_2} N_{i_1} N_{j_1} \right) \\
&+ c_6^{(2,2)} N_{i_1} N_{i_2} N_{j_1} N_{j_2} \Big]
\end{aligned}$$

The coefficients $c_n^{(k,l)}$ in the expressions above are given by

$$
\begin{aligned}
c^{(0,1)} &= \mathcal{J}_2 \\
c^{(2,0)} &= \tfrac{1}{2} \left( 3\mathcal{J}_1 - 1 \right) \\
c_1^{(1,1)} &= \tfrac{1}{3} \mathcal{J}_2 \\
c_2^{(1,1)} &= \tfrac{1}{2} \left( 3\mathcal{J}_3 - \mathcal{J}_2 \right) \\
c_1^{(0,2)} &= \tfrac{1}{3} \mathcal{J}_5 \\
c_2^{(0,2)} &= \tfrac{1}{2} \left( 3\mathcal{J}_7 - \mathcal{J}_5 \right) \\
c_1^{(2,1)} &= \tfrac{1}{2} \left( \mathcal{J}_2 - \mathcal{J}_4 \right) \\
c_2^{(2,1)} &= \tfrac{1}{2} \left( \mathcal{J}_3 - \mathcal{J}_4 \right) \\
c_3^{(2,1)} &= \tfrac{1}{2} \left( -2\mathcal{J}_2 - 2\mathcal{J}_3 + 5\mathcal{J}_4 \right) \\
c_1^{(2,2)} &= \tfrac{1}{8} \left( -3\mathcal{J}_4 + 17\mathcal{J}_5 + 2\mathcal{J}_6 - 8\mathcal{J}_7 - 3\mathcal{J}_8 + \mathcal{J}_9 \right) \\
c_1^{(2,2)} &= \tfrac{1}{8} \left( 2\mathcal{J}_5 - 2\mathcal{J}_6 - 3\mathcal{J}_7 - 3\mathcal{J}_8 + 4\mathcal{J}_9 + \mathcal{J}_{10} \right) \\
c_2^{(2,2)} &= \tfrac{1}{8} \left( -\mathcal{J}_5 + 2\mathcal{J}_6 + \mathcal{J}_7 + \mathcal{J}_8 - 4\mathcal{J}_9 + \mathcal{J}_{10} \right) \\
c_3^{(2,2)} &= \tfrac{1}{8} \left( -3\mathcal{J}_5 + 2\mathcal{J}_6 + 7\mathcal{J}_7 + 3\mathcal{J}_8 - 4\mathcal{J}_9 - 5\mathcal{J}_{10} \right) \\
c_4^{(2,2)} &= \tfrac{1}{8} \left( -3\mathcal{J}_5 + 2\mathcal{J}_6 + 3\mathcal{J}_7 + 7\mathcal{J}_8 - 4\mathcal{J}_9 - 5\mathcal{J}_{10} \right) \\
c_5^{(2,2)} &= \tfrac{1}{8} \left( -\mathcal{J}_5 - 2\mathcal{J}_6 - \mathcal{J}_7 - \mathcal{J}_8 - 8\mathcal{J}_9 - 5\mathcal{J}_{10} \right) \\
c_6^{(2,2)} &= \tfrac{1}{8} \left( \mathcal{J}_5 + 2\mathcal{J}_6 - 5\mathcal{J}_7 - 5\mathcal{J}_8 - 5\mathcal{J}_9 + 35\mathcal{J}_{10} \right)
\end{aligned}
$$

In these coefficients the integrals $\mathcal{J}_1 - \mathcal{J}_{10}$ are given by

**Intgrals**

$$\mathcal{J}_1 = S^{-1} \int_S \left( \mathbf{N} \cdot \mathbf{n} \right)^2 dS$$
$$\mathcal{J}_2 = S^{-3/2} \int_S \mathbf{x} \cdot \mathbf{N} dS$$
$$\mathcal{J}_3 = S^{-3/2} \int_S \left( \mathbf{x} \cdot \mathbf{n} \right) \left( \mathbf{n} \cdot \mathbf{N} \right) dS$$
$$\mathcal{J}_4 = S^{-3/2} \int_S \left( \mathbf{x} \cdot \mathbf{N} \right) \left( \mathbf{n} \cdot \mathbf{N} \right)^2 dS$$
$$\mathcal{J}_5 = S^{-2} \int_S \mathbf{x}^2 dS$$
$$\mathcal{J}_6 = S^{-2} \int_S \left( \mathbf{x} \cdot \mathbf{n} \right)^2 dS$$
$$\mathcal{J}_7 = S^{-2} \int_S \left( \mathbf{x} \cdot \mathbf{N} \right)^2 dS$$
$$\mathcal{J}_8 = S^{-2} \int_S \mathbf{x}^2 \left( \mathbf{n} \cdot \mathbf{N} \right)^2 dS$$
$$\mathcal{J}_9 = S^{-2} \int_S \left( \mathbf{x} \cdot \mathbf{n} \right) \left( \mathbf{x} \cdot \mathbf{N} \right) \left( \mathbf{n} \cdot \mathbf{N} \right) dS$$
$$\mathcal{J}_{10} = S^{-2} \int_S (\mathbf{x} \cdot \mathbf{N})^2 \left( \mathbf{n} \cdot \mathbf{N} \right)^2 dS$$

In the following table, the values of the coefficients $a_1 - a_9$ and there resp. equatorial symmetry are listed:

| Coefficient | Value | Eql. sym. |
|---|---|---|
| $a_1$ | $\frac{1}{6} \left( 4 - 3\alpha_\tau \right) \left( 3\mathcal{J}_1 - 1 \right)$ | $+$ |
| $a_2$ | $\frac{1}{6} \left( 8 + \pi\alpha_\tau \right)$ | $+$ |
| $a_3$ | $\frac{1}{4} \left[ 8 - (6 - \pi) \alpha_\tau \right] \left( 3\mathcal{J}_1 - 1 \right)$ | $+$ |
| $a_4$ | $\frac{1}{12} \left\{ \left[ 8 + (\pi - 2) \alpha_\tau \right] \mathcal{J}_2 - \left[ 8 - (6 - \pi) \alpha_\tau \right] \mathcal{J}_3 \right\}$ | $-$ |
| $a_5$ | $\frac{1}{6} \left[ \left( 4 - \alpha_\tau \right) \mathcal{J}_2 - \left( 4 - 3\alpha_\tau \right) \mathcal{J}_3 \right]$ | $-$ |
| $a_6$ | $\frac{1}{12} \left\{ \left[ 8 + (\pi - 2) \alpha_\tau \right] \mathcal{J}_2 - \left[ 8 - (6 - \pi) \alpha_\tau \right] \mathcal{J}_3 \right\}$ | $-$ |
| $a_7$ | $\frac{1}{12} \left\{ \left[ 8 + (\pi - 2) \alpha_\tau \right] \mathcal{J}_5 - \left[ 8 - (6 - \pi) \alpha_\tau \right] \mathcal{J}_6 \right\}$ | $+$ |
| $a_8$ | $\frac{1}{8} \left\{ \left[ 16 - (10 - 2\pi) \alpha_\tau \right] \mathcal{J}_5 - \left[ 16 - (12 - 2\pi) \alpha_\tau \right] \mathcal{J}_6 - \left[ 24 - (12 - 3\pi) \alpha_\tau \right] \mathcal{J}_7 \right.$ $\left. - \left[ 24 - (18 - 3\pi) \alpha_\tau \right] \mathcal{J}_8 + \left[ 48 - (36 - 6\pi) \alpha_\tau \right] \mathcal{J}_9 \right\}$ | $+$ |

The tensor of inertia, $I_{ij}$, is conveniently expressed according to (21) with the choice

$$I_0 \equiv m_B S,$$

and the coefficients $b_1$ and $b_2$ become

$$b_1 = \frac{2}{3m_B S} \int_V \rho(\mathbf{x}) \mathbf{x}^2 d^3 x, \qquad b_2 = \frac{1}{2m_B S} \int_V \rho(\mathbf{x}) \left[ \mathbf{x}^2 - 3(\mathbf{N} \cdot \mathbf{x})^2 \right] d^3 x.$$

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