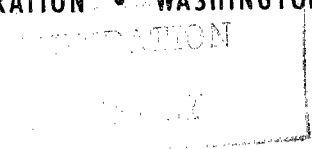**NASA CONTRACTOR REPORT**

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# UPPER AND LOWER BOUNDS FOR THE EIGENVALUES OF VIBRATING BEAMS WITH LINEARLY VARYING AXIAL LOAD

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# UPPER AND LOWER BOUNDS FOR THE EIGENVALUES OF VIBRATING

# BEAMS WITH LINEARLY VARYING AXIAL LOAD

By William M. Laird and Guy Fauconneau

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# ABSTRACT

Previous investigations have demonstrated the importance of the effect of linearly varying axial or in-plane loading on the vibration characteristics of beams and flat plates. It has already been established that the problem reduces to solving for the eigenvalues of a fourth order, variable coefficient differential equation that can not be solved in closed form. Beginning with a variational representation of the eigenvalue problem, methods are discussed by which both upper and lower bounds for the eigenvalues may be formed. The true eigenvalues may thus be estimated as being bracketed by the upper and lower bounds which are shown to approach each other. The bounds for the eigenvalues may also be estimated by an averaging procedure which may or may not compare favorably with the true values depending on the values of the loading parameters. Finally, numerical values for upper bounds, lower bounds, and average lumped end-load eigenvalues are computed on an IBM 7090 Computer.

## NOMENCLATURE

| | |
|---|---|
| A | Differential operator of loaded beam |
| c | Eigenvectors |
| $c_i$ | Constants |
| E | Modulus of Elasticity |
| f | Natural frequency of vibrating beam |
| I | Moment of inertia |
| $K_b$ | Class of admissible functions in elastic stability problems |
| $K_v$ | Class of admissible functions in vibration problems |
| L | Length of the beam |
| $P_1$, $P_2$ | Constant end loads |
| u | Function |
| v | Function |
| x | Axial coordinate |
| $\alpha$ | Distributed axial load parameter |
| $\alpha_c$ | Critical axial load |
| $\beta$ | Ratio of end load to total distributed load |
| $\gamma^4$ | Separation constant |
| $\phi$ | Function |
| $\lambda$ | Eigenvalue |
| $\tilde{\lambda}$ | Upper bound |
| $\xi$ | Nondimensional axial variable |
| $\zeta$ | Density per unit of length |
| $\psi$ | Mode shape, dependent deflection variable |

# I. INTRODUCTION

In recent years much attention has been given to the effect of linearly varying axial or in-plane loads on the vibrational characteristics of beams and plates. This topic is of particular interest in aerospace applications where inertia and friction drag forces manifest themselves as axial or in-plane loads. A detailed formulation of the problem is the subject of a prior NASA report by authors [1] and is the subject of considerable literature (see ref. 2 through 16).

## Formulation of the Problem

As described in references (1) and (2), the eigenvalue problem for both the beam and the rectangular plate may be resolved, under certain restrictions, to a solution of the ordinary differential equation

$$\frac{d^4\Psi}{d\xi^4} + \alpha \frac{d}{d\xi}\left\{ (\beta + \xi) \frac{d\Psi}{d\xi} \right\} - \lambda \Psi = 0 \tag{1}$$

and the boundary conditions

$$\frac{d^2\Psi}{d\xi^2} = 0 \quad , \quad \frac{d^3\Psi}{d\xi^3} + \alpha(\beta+\xi)\frac{d\Psi}{d\xi} = 0 \qquad \text{at a free end}$$

$$\Psi = 0 \quad , \quad \frac{d^2\Psi}{d\xi^2} = 0 \qquad \text{at a simply supported end} \tag{2}$$

$$\Psi = 0 \quad , \quad \frac{d\Psi}{d\xi} = 0 \qquad \text{at a clamped end.}$$

where

$$\xi = \frac{x}{L} \qquad \alpha = \frac{\omega L^3}{EI} \quad , \quad \beta = \frac{A}{\omega L} \qquad \text{and} \quad \lambda = \frac{\gamma^4 S L^4}{EI} \tag{3}$$

In view of the definition of the parameter $\beta$, it is clear that for a given compressive distributed load $\omega$, the following cases may occur:

1) $\beta > 0$ , the beam is entirely in compression

2) $0 > \beta > -1$ , the beam is partly in tension and partly in compression

3) $-1 \geqslant \beta$ , the beam is entirely in tension since the tensile and load $P_1$ is larger than the total distributed load L.

In the last case, the problem of elastic stability does not exist.

The determination of mode shapes and natural frequencies involves the solution of the differential eigenvalue problem defined by eqs. (1) and (2). Variational techniques [1] [2] finally resolve this to obtaining solutions to the variational principle

$$\lambda_1 = \min_{u \in K} \frac{\langle Au, u \rangle}{\langle u, u \rangle} \qquad * \tag{4}$$

where K is the class of functions constituting the domain of definition of the operator A, and, hence, satisfying both the prescribed and the natural boundary conditions, and $\langle u, v \rangle$ denotes the inner product between two functions u,v , where

$$\langle u, v \rangle = \int_0^1 u\, v\, d\xi \qquad \text{and} \qquad A = \frac{d^4}{d\xi^4} + \alpha \frac{d}{d\xi}\left\{ (\beta + \xi) \frac{d}{d\xi} \right\} \tag{5}$$

Equation (4) may be characterized by Courant's maximum-minimum characterization (ref. 18, Chap. III) given by

$$\lambda_j = \max_{\{u_i\}} \left\{ \min_{\langle \phi, u_i \rangle = 0} \frac{\int_0^1 \left[ \left( \frac{d^2\phi}{d\xi^2} \right)^2 - \alpha(\beta + \xi)\left( \frac{d\phi}{d\xi} \right)^2 \right] d\xi}{\int_0^1 \phi^2\, d\xi} \right\}_{i=1 \text{ to } j-1.} \tag{6}$$

where $\phi$ and $\mu_i$ belong to $K_v$, where $K_v$ is the class of admissible functions required to satisfy only the prescribed boundary conditions.

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* This functional is known as Rayleigh's quotient.

In resume, the situation is as follows: if $\beta$ is such that buckling may occur, there exists for the given falue of $\beta$ a critical value of this distributed axial load parameter, $\alpha_c$ , for which the beam is unstable and the potential energy is equal to zero. For any value of $\alpha$ less than $\alpha_c$ , the potential energy is positive, and the beam has discrete natural frequencies whose square are proportional to the eigenvalues of the operator A,

where

$$A = \frac{d^4}{d\xi^4} + \alpha \frac{d}{d\xi}\left\{(\beta + \xi)\frac{d}{d\xi}\right\}$$  (7)

These eigenvalues are assumed to be ordered in the non-decreasing sequences

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots$$

The eigenfunctions corresponding to distinct eigenvalues are mutually orthogonal, and correspond to the mode shapes of the beam. For a given value of $\beta$ , as $\alpha$ increases, the numerator of the Rayleigh quotient decreases and the eigenvalues decrease. Buckling occurs when $\alpha$ becomes equal to $\alpha_c$ , for which the first eigenvalue goes to zero.

In the next section, we review the methods used in this work to obtain approximate solutions.


## II.  BOUNDS FOR EIGENVALUES


There appear in the literature many methods for finding bounds for eigenvalues. Upper bounds are usually found without too many difficulties by the Rayleigh-Ritz method. Lower bounds present considerably more difficulties, and it can be said that no method having the generality, simplicity, and success of the Rayleigh-Ritz method exists for the computation of lower bounds. The most suitable method usually depends on the problem at hand.

In this section, we review  briefly  the methods used in this work in the calculations of approximations to eigenvalues. They are the Rayleigh-Ritz method, the method of Kato, and the method of intermediate problems of Weinstein and Aronszajn, with some modifications introduced by Bazley and Fox.

## A. The Rayleigh-Ritz Method

The Rayleigh-Ritz method for numerical computations of approximations to eigenvalues has been used extensively and with great success in the literature.[*] Consequently, it will only be outlined briefly here.

The basic idea of the method consists in determining the stationary values of the Rayleigh quotient, not over all admissible functions u, but only over the linear manifold spanned by an arbitrary set of n linearly independent functions $\{u_i\}$.[*] satisfying the boundary conditions of the operator A. The problem then consists in finding the functions u of the form

$$u = \sum_{i=1}^{n} c_i u_i \tag{8}$$

i.e., in finding the constants $c_i$, making the Rayleigh quotient stationary, and the stationary value of the quotient. Substitution of Equation (8) into Rayleigh's quotient yields

$$\frac{\langle u, Au \rangle}{\langle u, u \rangle} = \frac{\sum_{i,j=1}^{n} c_i c_j \langle u_i, A u_j \rangle}{\sum_{i,j=1}^{n} c_i c_j \langle u_i, u_j \rangle} \tag{9}$$

which is the ratio of two quadratic forms in the n real variables $c_1$, $c_2$, ...$c_n$. Its stationary values can be obtained by finding, for instance, the stationary values of the quadratic form in the numerator, subject to the auxiliary condition that the denominator be equal to one, and using the method of the Lagrange undetermined multiplier. The result is the general matrix eigenvalue problem

$$\left[ \langle u_i, A u_j \rangle \right] \left[ c_j \right] = \tilde{\lambda} \left[ \langle u_i, u_j \rangle \right] \left[ c_j \right] \tag{10}$$

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[*] See, for instance, references 17, 18, and 19.

[*] The functions $u_i$ are often called coordinate functions.

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Since the class of admissible functions was restricted to the finite dimensional manifold, it follows that the eigenvalues $\tilde{\lambda}_i$ are upper-bounds for the eigenvalues of A, i.e.,

$$\lambda_i \leq \tilde{\lambda}_i \quad , \quad i=1,2,\dots n \quad\quad (11)$$

Furthermore, it follows that as n increases, the upper bounds are improved, or at least, not worsened.

From a computational standpoint, it is advantageous to choose mutually orthogonal coordinate functions to avoid the solution of a general eigenvalue problem. Also, Equation(9)may be written as in Equation(6)with the functions $\{u_i\}$ required to satisfy only the prescribed boundary conditions. This point is discussed in detail in references 18 and 19. The coordinate functions utilized in this work satisfy both the prescribed and the natural boundary conditions, as will be seen later.

## B. The Method of Kato

The Rayleigh-Ritz method described above furnishes upper bounds for eigenvalues. The results, particularly for the first eigenvalue, are usually in agreement with the exact eigenvalues for the cases where the latter can be obtained. However, in general, the question regarding the closeness of these bounds to the true values remains unanswered, although in some instances, an estimate of the error is possible.* One way of determining how good the approximations are is to compute also lower bounds. If these turn out close to the upper bounds, the question is essentially answered. The method of Kato[22], which is an extension of Temple's method, furnishes lower bounds, provided rough estimates of the sought eigenvalues are known. This is outlined by Freidman [22, p. 212].

## C. The Method of Intermediate Problems

The methods described in the preceding two sections furnish upper and lower bounds for eigenvalues. In both methods, the quality of the results

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depends strongly on how well the trial functions approximate the eigenvectors of the operator. Hence, both methods may require considerable ingenuity in the selection of the trial functions. Furthermore, for different sets of trial functions, there is little prior knowledge of which set will give the best results. For these reasons, it is in order to consider also another method for the computation of the lower bounds. The method used here is the method of intermediate problem, which presents the advantage that the bounds can be improved.

Quite a few years back, Weinstein [24] introduced the method of intermediate problems, which gives improvable lower bounds by changing the boundary conditions of differential operators. Briefly, the method consists in relaxing the boundary conditions to obtain a solvable problem, the base problem, whose eigenvalues give rough lower bounds for the eigenvalues of the given problem. A sequence of intermediate problems linking the base problem to the given problem is then introduced. These are such that they can be solved in terms of the base problem, and that they give improved lower bounds. The details of the procedure are exposed in references 17 and 25.

In 1951, Aronszajn [26] pointed out that a base problem can be obtained by changing the differential operator, and indicated the method of construction of the intermediate problems. The solution of these intermediate problems requires the determination of the poles and the zeroes of a meromorphic function given in its partial fractions representations. From a computational standpoint, the determination of the zeroes present many difficulties which have been removed in a dissertation by Bazley [27], and in a series of recent papers by Bazley and Fox [28-33]. These authors have applied their method to the determination of the eigenvalues of Schrodinger's equation and Mathieu's equation with excellent results.

A more detailed resume of the Method of Kato and the Method of Intermediate Problems is given in Reference (2). Reference (2) also describes specific application to the simply supported beam and the beam with built-in ends. These procedures are not particularly difficult in principle, but the calculations involved are somewhat laborious.

## D. Lumped Constant End Load Approximation

An approximation to the response of beams with distributed axial load may be accomplished by replacing the distributed load and its reaction with equal and opposite average end loads. This results in an ordinary linear differential equation with constant coefficient which may be solved exactly in terms of trigonometric functions. A comparison of the eigenvalues calculated in this manner is made with the upper and lower bounds in the section on Results and Discussion.

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# III. RESULTS AND DISCUSSION

Following the methods described above, upper and lower bounds for the eigenvalues of the simply supported and clamped beam were calculated on an IBM 7090 Computer in the Computation and Data Processing Center of the University of Pittsburgh. The results are displayed in Tables I and II and Figures 1, 2, and 3. Upper bounds, lower bounds and lumped end-load eigenvalues are displayed for a wide range of loading parameters $\alpha$ and $\beta$.

## A. Simply Supported Beam

The bounds for the first five eigenvalues of the simply supported beam are presented in Table I. To facilitate the comparison between the Rayleigh-Ritz upper bounds and the lower bounds by the method of intermediate problems, the ratio of their difference to their average has been computed and is also presented in Table I. Since the eigenvalues of a simply supported beam are easy to obtain, it is interesting to compare the upper and lower bounds of the eigenvalues obtained by lumping half of the total distributed load as a constant load at each end. These results are also included in Table I.

Analysis of the results in Table I indicates that the Rayleigh-Ritz upper bounds and the lower bounds by the method of intermediate problems remain close over the whole range of axial loadings. This is particularly true for the first eigenvalue. Only when the beam is extremely close to buckling does the relative error increase greatly as a result of the smallness of the eigenvalues. For eigenvalues of order higher than one, the error is slightly higher, but, if necessary, it could be reduced by considering higher intermediate problems.

The lower bounds for the first eigenvalue by the method of Kato remain close to the upper bounds for moderate loading, but drop off considerably at the loading increases. Perhaps, this effect might be attributed to the fact that as the first eigenvalue approaches zero, the choice arbitrary trial variations becomes more and more critical. For higher eigenvalues, this selection is not as critical, and consequently, the lower bounds remain close to the upper bounds. However, in the cases where the beam can not become elastically unstable, the Kato lower bounds eventually decrease as the loading becomes very large, and no explanation for this behavior can be offered.

The eigenvalues of the beam with lumped constant end load are remarkably close to those of the beam with distributed load for compressive end thrusts, i.e., for $\beta > 0$. For negative $\beta$, the results are quite far apart. In particular, for $\beta = -.5$, the beam with distributed axial load may become elastically unstable, while the beam with lumped load can not buckle, because its net thrust is zero. Consequently, extreme care should be exercised in the lumping of the loads when they are of opposite signs.

The effect of the axial loads on the first frequency of the simply supported beam is shown in Figures 1 and 2. Figure 1 represents the ratio of the first frequency of the loaded beam to that of the unloaded beam as a function of the distributed load parameter $\alpha$ , as obtained by Kato's method and the Rayleigh-Ritz method. The lower bounds of the method of intermediate problems are not shown because their curve practically coincides with the Rayleigh-Ritz curve for the scale used. The curves correspond to $\beta = 0$ . Figure 2 also represents the ratio of the fundamental frequency of the loaded beam to that of the unloaded beam as a function of $\alpha$ for various values of $\beta$ . The curves were obtained by using the average of the upper bounds and lower bounds by the method of intermediate problems.

The values of the critical axial load $\alpha_c$ are given at the intersection of the frequency ratio curve with the horizontal axis. The buckling loads obtained from graphs having a larger scale than that of Figure 2 compare favorably with the exact results of Tyler and Rouleau [11]. For $\beta = 0$ , the graphs indicate that $\alpha_c \simeq 18.7 \, EI/L^3$ while Tyler and Rouleau's result is $\alpha_c = 18.763 \, EI/L^3$ For $\beta = 1.0$ , we obtain $\alpha_c \simeq 6.5 \, EI/L^3$ while the exact answer is $\alpha_c = 6.519 \, EI/L^3$ and for $\beta = -.50$ we have $\alpha_c \simeq 83 \, EI/L^3$ against the exact result of $82.8819 \, EI/L^3$ . The approximate values are certainly close enough for engineering application.

## B. Clamped Beam

The bounds for the first four eigenvalues of the clamped beam are presented in Table 2. The ratio of the difference between the upper bound and the corresponding lower bound by the method of intermediate problems to their average has also been computed. The eigenvalues of the clamped beam carrying a constant end load equal to half the total distributed load and the constant end load are also presented in Table II to indicate for what values of the loading parameters this lumping is acceptable.

Examination of the results indicate the following:

i) The lower bounds by the method of intermediate problems are very close to the Rayleigh-Ritz upper bounds for all eigenvalues and for the whole range of the loading parameters.

ii) The lower bounds by the method of Kato present the same features demonstrated in the simply supported beam calculations: whenever the loading is small, the bounds are fairly good but become worse as the loads increase.

iii) The eigenvalues of the beam with lumped end load are fairly close to the upper bounds for moderate loading, particularly for $\beta > 0$ . For negative values of $\beta$ , they can be quite remote from the upper bounds, particularly for $\beta$ for which the beam with distributed axial load may buckle while the beam with lumped end load can not.

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The effect of the axial loads on the first frequency of the clamped beam are shown in Figure 3, which represents the ratio of the first frequency of the loaded beam to that of the unloaded beam as a function of the axial load parameter $\alpha$ for various values of $\beta$ .

## IV. CONCLUDING REMARKS

Bounds for the eigenvalues of a simply supported and a clamped beam carrying linearly distributed axial loads have been presented. The main difficulty in problems of this nature arises from the fact that the governing differential equation has a varying coefficient which usually prevents one from obtaining exact solutions. Upper bounds were easily obtained by the Rayleigh-Ritz method. Lower bounds by the method of Kato were also easy to obtain. In both methods, the closeness of the results to the true eigenvalues depends on the quality of the coordinate functions. It appears that for moderate loading, the eigenfunctions of the unloaded beams were good coordinate functions, as our results indicate.

The lower bounds computed by the method of intermediate problems were very close to the upper bounds, both for the simply supported and the clamped beam. The modifications introduced by Bazley and Fox eliminate the computational difficulties which prevented extensive use of the method of intermediate problems.

For engineering applications, it appears that lumping the axial loads gives eigenvalues that are larger than the true eigenvalues, and that care must be exercixed whenever the distributed load and the constant end thrust are of opposite signs. In this case, the buckling loads predicted by the lumped end load problem can be quite remote from the actual critical loads.

The present research could be extended to the consideration of beams with other boundary conditions, closer determinations of the buckling loads, and the methods used here can be applied to other problems giving rise to differential equations with variable coefficients, such as in the problems of the determination of natural frequencies and buckling loads of beams of varying cross sections, plates with varying in-plane loads, and plates of non-uniform thickness, to mention a few. Information of this nature would be valuable to designers, particularly in the Aerospace industry.

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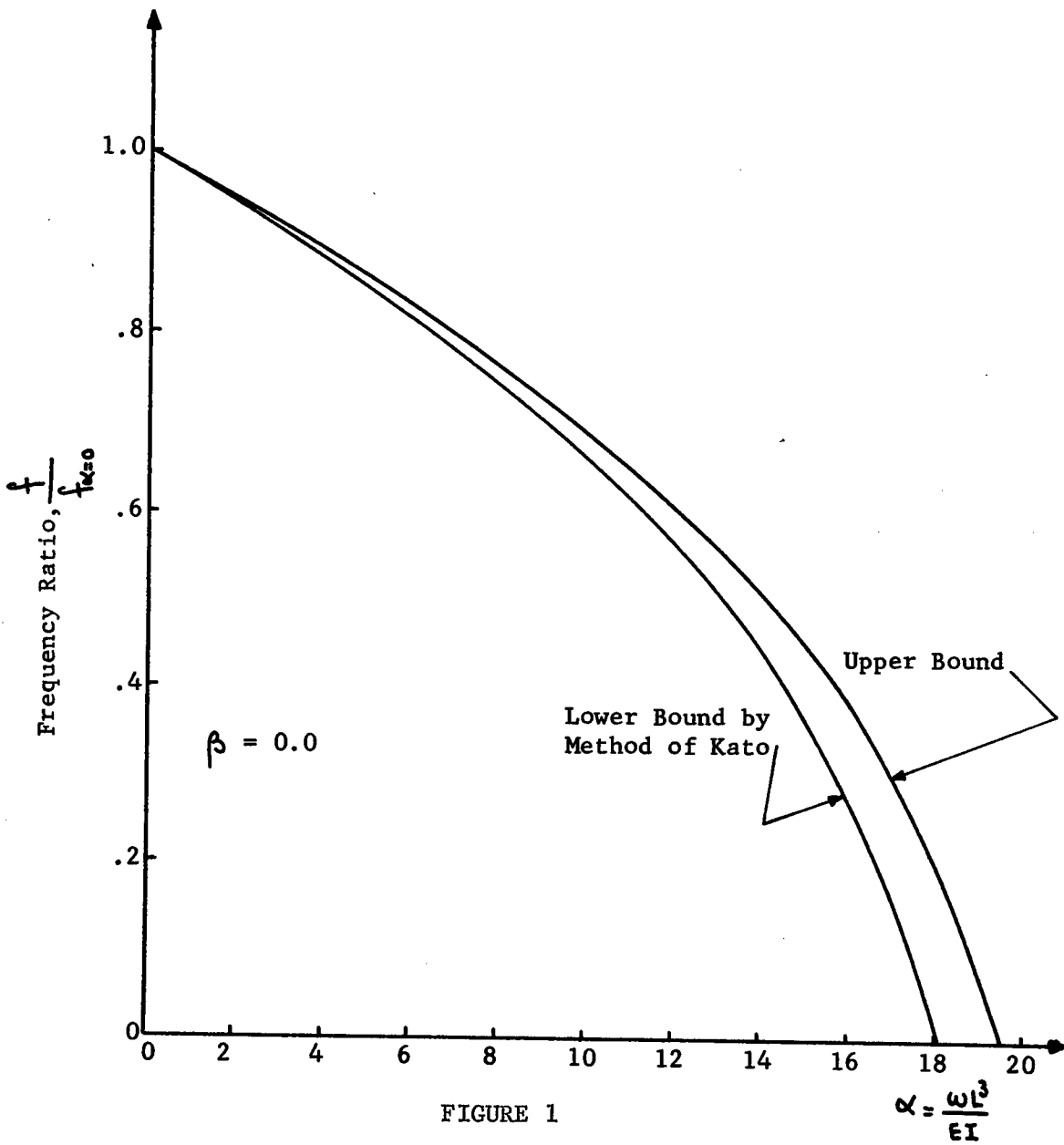
FIGURE 1

Effect of Distributed Axial Load on the First Frequency

of the Simply Supported Beam
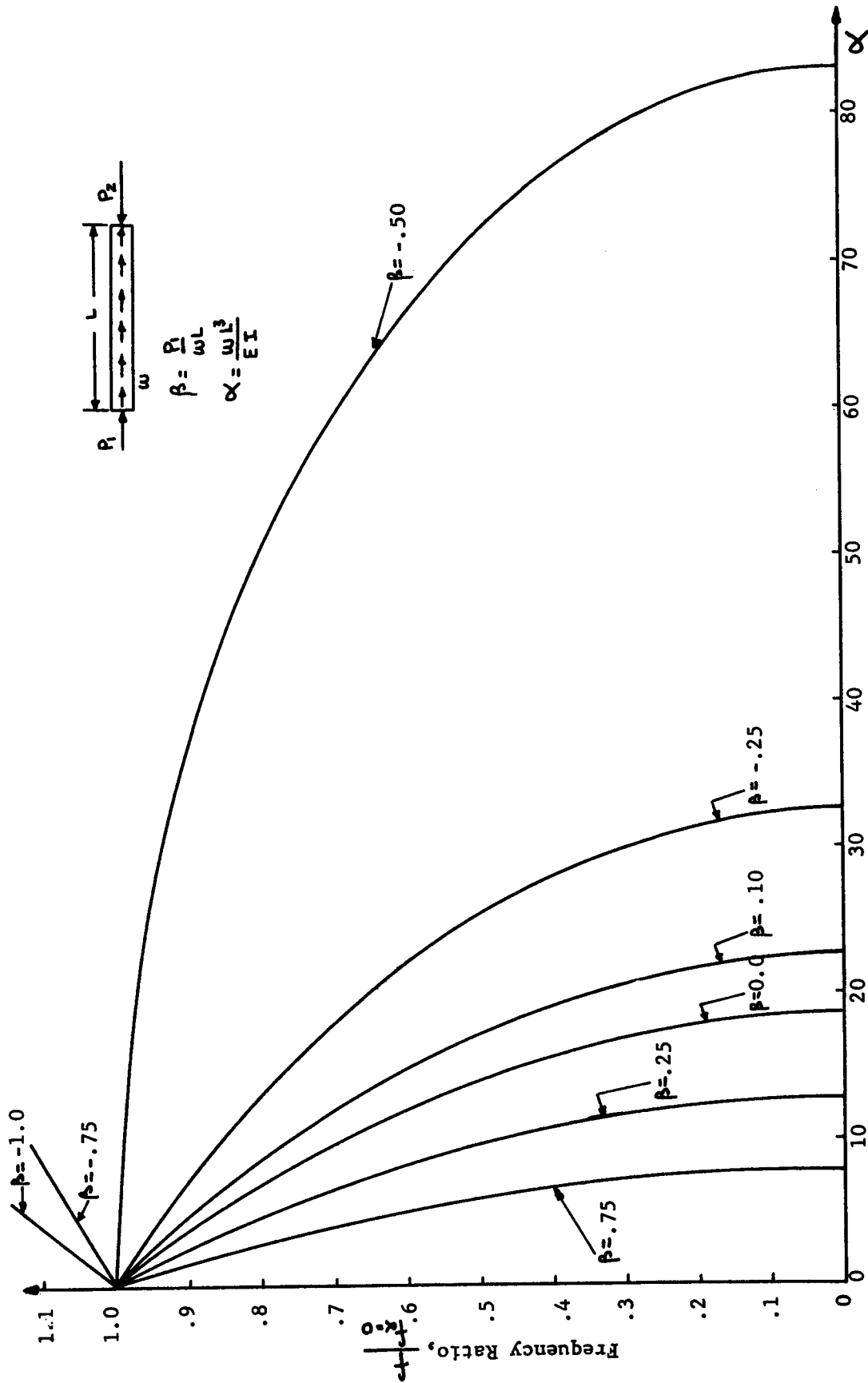
FIGURE 2

Effect of Axial Loads on the Fundamental Frequency of a Simply Supported Beam

15
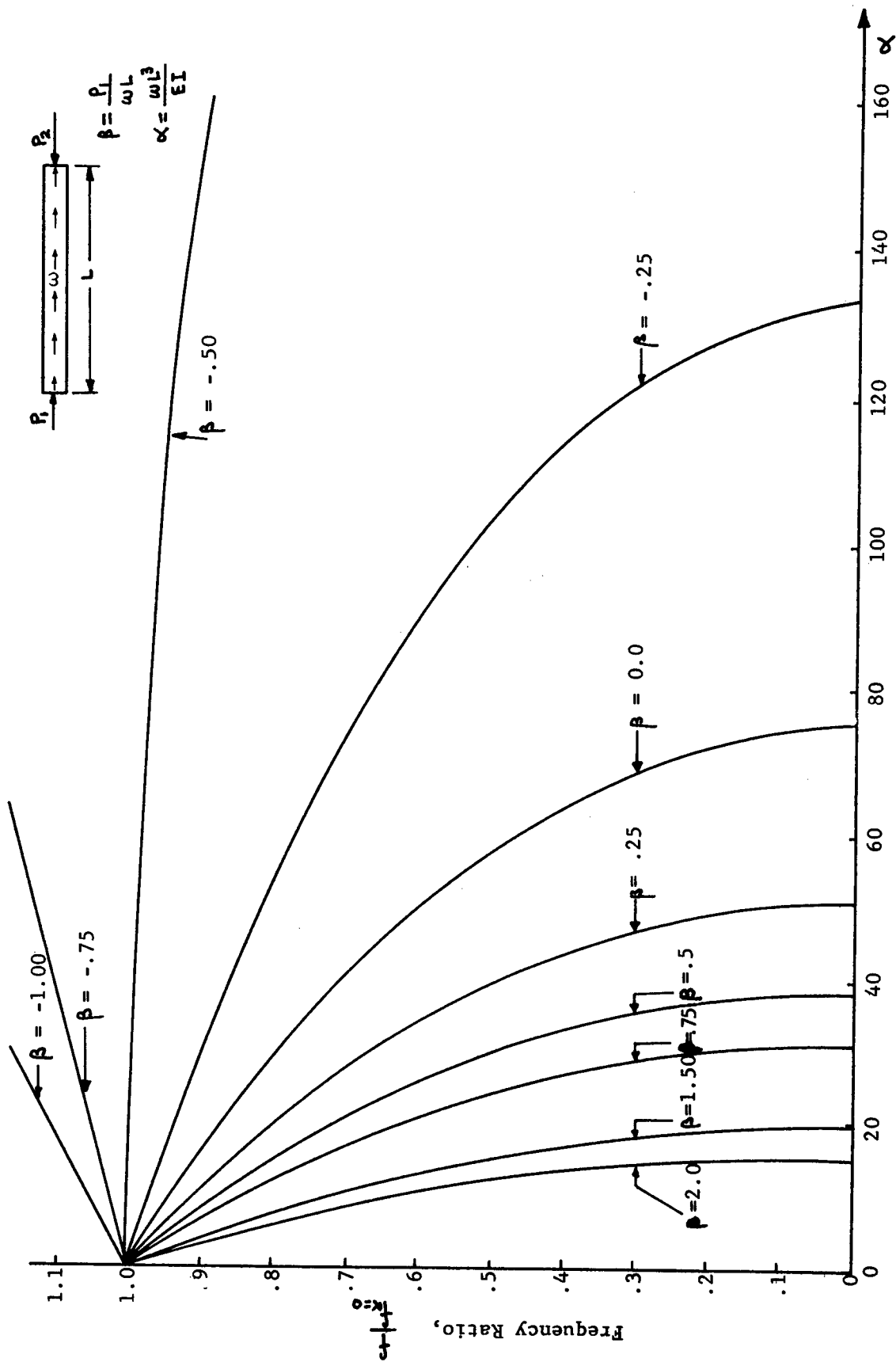
FIGURE 3

Effect of Axial Loads on the Fundamental Frequency of a Clamped Beam

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TABLE   I   -   BOUNDS  FOR  THE  EIGENVALUES  OF  THE  SIMPLY  SUPPORTED  BEAM          β = 0.0

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 0.00 | 1 | 97.40909 | 97.40909 | 97.40909 | 0.0 | 97.40909 |
| | 2 | 1 518.545 | 1 558.545 | 1 558.545 | 0.0 | 1 558.545 |
| | 3 | 7 890.136 | 7 890.136 | 7 890.136 | 0.0 | 7 890.136 |
| | 4 | 24 936.73 | 24 936.73 | 24 936.73 | 0.0 | 24 936.73 |
| | 5 | 60 880.68 | 60 880.68 | 60 880.68 | 0.0 | 60 880.68 |
| 2.00 | 1 | 87.48401 | 86.82118 | 87.47622 | 0.009 | 87.5395 |
| | 2 | 1 519.022 | 1 509.258 | 1 518.916 | 0.007 | 1 519.067 |
| | 3 | 7 801.265 | 7 759.954 | 7 798.981 | 0.029 | 7 801.309 |
| | 4 | 24 778.77 | 24 668.74 | 24 740.64 | 0.154 | 24 778.81 |
| | 5 | 60 633.89 | 60 403.19 | 60 441.32 | 0.318 | 60 633.94 |
| 4.00 | 1 | 77.44337 | 76.09688 | 77.42758 | 0.020 | 77.6699 |
| | 2 | 1 479.411 | 1 459.780 | 1 479.201 | 0.014 | 1 479.588 |
| | 3 | 7 712.310 | 7 629.280 | 7 707.814 | 0.058 | 7 712.483 |
| | 4 | 24 620.72 | 24 399.70 | 24 544.67 | 0.309 | 24 620.89 |
| | 5 | 60 387.02 | 59 923.86 | 60 001.91 | 0.640 | 60 387.20 |
| 6.00 | 1 | 67.27963 | 65.20801 | 67.25670 | 0.034 | 67.8003 |
| | 2 | 1 439.716 | 1 410.094 | 1 439.409 | 0.021 | 1 440.110 |
| | 3 | 7 623.273 | 7 498.086 | 7 616.639 | 0.087 | 7 623.656 |
| | 4 | 24 462.60 | 24 120.60 | 24 348.82 | 0.466 | 24 462.98 |
| | 5 | 60 140.07 | 59 442.65 | 59 562.45 | 0.965 | 60 140.46 |

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\qquad$ $\beta = 0.0$

| α | Order | Upper Bound by Rayleigh-Ritz Method | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 8.00 | 1 | 56.98461 | 54.11895 | 56.95145 | 0.058 | 57.9307 |
|  | 2 | 1 399.940 | 1 360.181 | 1 399.538 | 0.029 | 1 400.631 |
|  | 3 | 7 534.153 | 7 366.345 | 7 525.454 | 0.116 | 7 534.830 |
|  | 4 | 24 304.39 | 23 858.37 | 24 153.09 | 0.624 | 24 305.07 |
|  | 5 | 59 893.04 | 58 959.51 | 59 122.92 | 1.294 | 59 893.71 |
| 10.00 | 1 | 46.54933 | 42.78417 | 46.51453 | 0.075 | 48.0611 |
|  | 2 | 1 360.085 | 1 310.021 | 1 359.597 | 0.036 | 1 361.153 |
|  | 3 | 7 444.952 | 7 234.029 | 7 434.263 | 0.144 | 7 446.003 |
|  | 4 | 24 146.10 | 23 585.99 | 23 957.49 | 0.784 | 24 147.15 |
|  | 5 | 59 645.92 | 58 474.38 | 58 683.34 | 1.627 | 59 646.98 |
| 12.00 | 1 | 35.96395 | 31.14474 | 35.92827 | 0.099 | 38.1915 |
|  | 2 | 1 320.157 | 1 259.591 | 1 319.593 | 0.043 | 1 321.674 |
|  | 3 | 7 355.672 | 7 101.108 | 7 343.061 | 0.172 | 7 357.177 |
|  | 4 | 23 987.74 | 23 312.42 | 23 762.01 | 0.945 | 23 989.24 |
|  | 5 | 59 398.72 | 57 987.22 | 58 243.69 | 1.964 | 59 400.23 |
| 14.00 | 1 | 25.21769 | 19.12343 | 25.17796 | 0.158 | 28.3219 |
|  | 2 | 1 280.161 | 1 208.868 | 1 279.623 | 0.050 | 1 282.196 |
|  | 3 | 7 266.316 | 6 967.549 | 7 251.870 | 0.199 | 7 268.350 |
|  | 4 | 23 829.29 | 23 037.61 | 23 566.66 | 1.109 | 23 831.33 |
|  | 5 | 59 151.45 | 57 497.99 | 57 803.98 | 2.304 | 59 153.49 |

TABLE  I -  BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM          $\beta = 0.0$

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 16.00 | 1 | 14.29866 | 6.61749 | 14.25562 | 0.301 | 18.4523 |
| | 2 | 1 240.103 | 1 157.825 | 1 239.414 | 0.056 | 1 242.718 |
| | 3 | 7 175.883 | 6 833.320 | 7 160.670 | 0.226 | 7 179.524 |
| | 4 | 23 670.76 | 22 761.53 | 23 371.45 | 1.273 | 23 673.41 |
| | 5 | 58 904.09 | 57 006.62 | 57 364.20 | 2.649 | 58 906.75 |
| 18.00 | 1 | 3.193809 | ---- | 3.148092 | 1.442 | 8.5827 |
| | 2 | 1 199.990 | 1 106.434 | 1 199.233 | 0.063 | 1 203.239 |
| | 3 | 7 087.378 | 6 698.383 | 7 069.476 | 0.253 | 7 090.697 |
| | 4 | 23 512.15 | 22 484.12 | 23 176.36 | 1.438 | 23 515.50 |
| | 5 | 58 656.65 | 56 513.06 | 56 924.36 | 2.998 | 58 660.02 |
| 18.50 | 1 | .386901 | ---- | .341591 | 12.448 | 6.1152 |
| | 2 | 1 189.955 | 1 093.528 | 1 189.187 | 0.065 | 1 193.370 |
| | 3 | 7 064.990 | 6 664.534 | 7 046.678 | 0.260 | 7 068.491 |
| | 4 | 23 472.49 | 22 414.56 | 23 127.62 | 1.480 | 23 476.02 |
| | 5 | 58 594.78 | 56 389.33 | 56 814.39 | 3.085 | 58 598.33 |

TABLE  I -  BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM

β = 0.25

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 2.00 | 1 | 82.54863 | 81.91434 | 82.54596 | 0.003 | 82.5047 |
|  | 2 | 1 499.283 | 1 489.607 | 1 499.177 | 0.007 | 1 499.327 |
|  | 3 | 7 756.852 | 7 715.692 | 7 754.567 | 0.029 | 7 756.896 |
|  | 4 | 24 699.80 | 24 589.99 | 24 661.68 | 0.357 | 24 699.85 |
|  | 5 | 60 510.51 | 60 280.09 | 60 317.95 | 0.319 | 60 510.57 |
| 4.00 | 1 | 67.56889 | 66.34465 | 67.55165 | 0.026 | 67.8003 |
|  | 2 | 1 439.935 | 1 420.664 | 1 439.726 | 0.015 | 1 440.110 |
|  | 3 | 7 523.485 | 7 541.064 | 7 618.987 | 0.059 | 7 623.656 |
|  | 4 | 24 462.81 | 24 242.65 | 24 386.76 | 0.311 | 24 462.98 |
|  | 5 | 60 140.28 | 59 678.24 | 59 755.17 | 0.642 | 60 140.46 |
| 6.00 | 1 | 52.45787 | 50.68089 | 52.43477 | 0.044 | 52.9959 |
|  | 2 | 1 380.505 | 1 351.718 | 1 380.197 | 0.022 | 1 380.892 |
|  | 3 | 7 490.036 | 7 366.244 | 7 483.402 | 0.089 | 7 490.416 |
|  | 4 | 24 225.73 | 23 894.68 | 24 111.95 | 0.471 | 24 226.11 |
|  | 5 | 59 769.96 | 59 075.06 | 59 192.34 | 0.971 | 59 770.35 |
| 8.00 | 1 | 37.20190 | 34.89742 | 37.16991 | 0.086 | 38.1915 |
|  | 2 | 1 321.000 | 1 282.770 | 1 320.595 | 0.031 | 1 321.674 |
|  | 3 | 7 356.507 | 7 191.216 | 7 347.814 | 0.118 | 7 357.177 |
|  | 4 | 23 988.57 | 23 546.06 | 23 837.28 | 0.633 | 23 989.24 |
|  | 5 | 59 399.56 | 58 470.54 | 58 629.44 | 1.305 | 59 400.23 |

TABLE  I  -   BOUNDS  FOR  THE  EIGENVALUES  OF  THE  SIMPLY  SUPPORTED  BEAM          β = 0.25

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 10.00 | 1 | 21.78530 | 18.95922 | 21.74532 | 0.184 | 23.3871 |
| | 2 | 1 261.426 | 1 213.824 | 1 260.936 | 0.039 | 1 262.457 |
| | 3 | 7 222.901 | 7 015.972 | 7 212.221 | 0.148 | 7 223.937 |
| | 4 | 23 751.33 | 23 196.75 | 23 562.72 | 0.797 | 23 752.37 |
| | 5 | 59 029.08 | 57 864.62 | 58 066.49 | 1.644 | 59 030.12 |
| 12.00 | 1 | 6.190051 | 2.817389 | 6.1546701 | 0.573 | 8.5827 |
| | 2 | 1 201.793 | 1 144.882 | 1 201.233 | 0.047 | 1 203.239 |
| | 3 | 7 089.221 | 6 840.497 | 7 076.629 | 0.178 | 7 090.697 |
| | 4 | 23 514.01 | 22 846.74 | 23 288.30 | 0.965 | 23 515.01 |
| | 5 | 58 658.51 | 57 257.28 | 57 503.46 | 1.989 | 58 660.01 |
| 12.50 | 1 | 2.260952 | ---- | 2.224899 | 1.607 | 4.8815 |
| | 2 | 1 186.877 | 1 127.648 | 1 186.297 | 0.049 | 1 188.435 |
| | 3 | 7 055.789 | 6 796.590 | 7 042.739 | 0.185 | 7 057.387 |
| | 4 | 23 454.67 | 22 759.13 | 23 219.72 | 1.007 | 23 456.28 |
| | 5 | 58 565.86 | 57 105.22 | 57 362.69 | 2.076 | 58 567.49 |

TABLE  I  -  BOUNDS  FOR  THE  EIGENVALUES  OF  THE  SIMPLY  SUPPORTED  BEAM     β = 0.50

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bounds by Kato's Method | Lower Bounds by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 1.00 | 1 | 87.52560 | 87.20957 | 87.51748 | 0.009 | 87.5395 |
|  | 2 | 1 519.055 | 1 514.228 | 1 519.001 | 0.004 | 1 519.067 |
|  | 3 | 7 801.297 | 7 780.766 | 7 800.157 | 0.015 | 7 801.309 |
|  | 4 | 24 778.79 | 24 724.01 | 24 759.71 | 0.078 | 24 778.81 |
|  | 5 | 60 633.91 | 60 518.92 | 60 537.63 | 0.159 | 60 633.94 |
| 3.00 | 1 | 67.67012 | 66.80439 | 67.65601 | 0.021 | 67.8003 |
|  | 2 | 1 440.011 | 1 425.730 | 1 439.847 | 0.011 | 1 440.110 |
|  | 3 | 7 623.559 | 7 562.125 | 7 620.156 | 0.045 | 7 623.656 |
|  | 4 | 24 462.88 | 24 298.45 | 24 405.78 | 0.234 | 24 462.98 |
|  | 5 | 60 140.35 | 59 794.90 | 59 851.52 | 0.481 | 60 140.46 |
| 5.00 | 1 | 47.68328 | 46.38864 | 47.66803 | 0.032 | 48.0611 |
|  | 2 | 1 360.886 | 1 337.431 | 1 360.623 | 0.019 | 1 361.153 |
|  | 3 | 7 445.739 | 7 343.619 | 7 440.165 | 0.075 | 7 446.003 |
|  | 4 | 24 146.89 | 23 872.72 | 24 051.95 | 0.394 | 24 147.15 |
|  | 5 | 59 646.70 | 59 070.15 | 59 165.95 | 0.809 | 59 646.98 |
| 7.00 | 1 | 27.54646 | 25.95841 | 27.51930 | 0.099 | 28.3219 |
|  | 2 | 1 281.687 | 1 249.359 | 1 281.329 | 0.028 | 1 282.196 |
|  | 3 | 7 267.840 | 7 125.257 | 7 260.169 | 0.106 | 7 268.350 |
|  | 4 | 23 830.81 | 23 446.81 | 23 698.26 | 0.558 | 23 831.33 |
|  | 5 | 59 152.97 | 58 344.64 | 58 479.11 | 1.146 | 59 153.49 |

TABLE  I  -   BOUNDS  FOR  THE  EIGENVALUES  OF  THE  SIMPLY  SUPPORTED  BEAM                 $\beta = 0.50$

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 8.00 | 1 | 17.41498 | 15.73609 | 17.38809 | 0.155 | 18.4523 |
| | 2 | 1 242.063 | 1 205.418 | 1 241.659 | 0.033 | 1 242.718 |
| | 3 | 7 178.863 | 7 016.133 | 7 170.170 | 0.121 | 7 179.524 |
| | 4 | 23 672.75 | 23 233.79 | 23 521.46 | 0.641 | 23 573.41 |
| | 5 | 58 906.08 | 57 981.60 | 58 135.96 | 1.316 | 58 906.75 |
| 9.00 | 1 | 7.237294 | 5.507694 | 7.2031238 | 0.473 | 8.5827 |
| | 2 | 1 202.425 | 1 161.547 | 1 201.981 | 0.037 | 1 203.239 |
| | 3 | 7 089.866 | 6 907.049 | 7 080.177 | 0.137 | 7 090.698 |
| | 4 | 23 514.66 | 23 020.71 | 23 344.70 | 0.725 | 23 515.50 |
| | 5 | 58 659.17 | 57 918.37 | 57 792.81 | 1.488 | 58 660.02 |
| 9.60 | 1 | 1.107041 | - - - - | 1.070184 | 3.386 | 2.6609 |
| | 2 | 1 178.637 | 1 135.259 | 1 178.171 | 0.040 | 1 179.552 |
| | 3 | 7 036.460 | 6 841.618 | 7 026.174 | 0.146 | 7 037.401 |
| | 4 | 23 419.80 | 22 892.85 | 23 238.65 | 0.776 | 23 420.75 |
| | 5 | 58 511.01 | 57 400.33 | 57 586.90 | 1.592 | 58 511.97 |

TABLE  I  -   BOUNDS  FOR  THE  EIGENVALUES  OF  THE  SIMPLY  SUPPORTED  BEAM                    β = .75

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 1.00 | 1 | 85.05812 | 84.74914 | 85.05647 | 0.002 | 85.0721 |
|  | 2 | 1 509.186 | 1 504.381 | 1 509.134 | 0.003 | 1 509.197 |
|  | 3 | 7 779.090 | 7 758.597 | 7 777.946 | 0.015 | 7 779.102 |
|  | 4 | 24 739.31 | 24 684.58 | 24 720.24 | 0.077 | 24 739.33 |
|  | 5 | 60 572.23 | 60 457.30 | 60 475.94 | 0.159 | 60 572.25 |
| 2.00 | 1 | 72.67783 | 72.10252 | 72.66977 | 0.011 | 72.7351 |
|  | 2 | 1 459.805 | 1 450.306 | 1 459.697 | 0.007 | 1 459.849 |
|  | 3 | 7 668.026 | 7 627.168 | 7 665.747 | 0.030 | 7 668.069 |
|  | 4 | 24 541.89 | 24 432.50 | 24 503.77 | 0.155 | 24 541.94 |
|  | 5 | 60 263.77 | 60 033.91 | 60 071.21 | 0.320 | 60 263.83 |
| 3.00 | 1 | 60.26579 | 59.47118 | 60.25156 | 0.024 | 60.3981 |
|  | 2 | 1 410.403 | 1 396.327 | 1 410.242 | 0.011 | 1 410.501 |
|  | 3 | 7 556.940 | 7 495.851 | 7 553.539 | 0.005 | 7 557.036 |
|  | 4 | 24 344.45 | 24 180.50 | 24 287.33 | 0.235 | 24 344.55 |
|  | 5 | 59 995.29 | 59 610.47 | 59 666.46 | 0.483 | 59 955.40 |
| 4.00 | 1 | 47.81929 | 46.85748 | 47.80695 | 0.026 | 48.0611 |
|  | 2 | 1 360.982 | 1 342.451 | 1 360.769 | 0.016 | 1 361.153 |
|  | 3 | 7 445.834 | 7 364.650 | 7 441.338 | 0.060 | 7 446.003 |
|  | 4 | 24 146.98 | 23 928.56 | 24 070.93 | 0.315 | 24 147.16 |
|  | 5 | 59 646.80 | 59 187.00 | 59 261.69 | 0.065 | 59 646.98 |

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM          β = .75

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 5.00 | 1 | 35.33536 | 34.26425 | 35.31343 | 0.062 | 35.7241 |
|  | 2 | 1 311.542 | 1 288.684 | 1 311.276 | 0.020 | 1 311.805 |
|  | 3 | 7 334.708 | 7 233.570 | 7 329.137 | 0.076 | 7 334.970 |
|  | 4 | 23 949.50 | 23 676.70 | 23 859.56 | 0.376 | 23 949.76 |
|  | 5 | 59 338.28 | 58 763.48 | 58 856.91 | 0.815 | 59 338.55 |
| 6.00 | 1 | 22.81063 | 21.69949 6 | 22.78719 | 0.103 | 23.3871 |
|  | 2 | 1 262.085 | 1 235.035 | 1 261.775 | 0.025 | 1 262.457 |
|  | 3 | 7 223.563 | 7 102.614 | 7 216.936 | 0.092 | 7 223.937 |
|  | 4 | 23 751.99 | 23 424.91 | 23 638.22 | 0.480 | 23 752.37 |
|  | 5 | 59 029.74 | 58 339.92 | 58 452.12 | 0.983 | 59 030.12 |
| 7.00 | 1 | 10.24133 | 9.153818 | 10.21664 | 0.241 | 11.0501 |
|  | 2 | 1 212.514 | 1 181.513 | 1 212.262 | 0.029 | 1 213.109 |
|  | 3 | 7 112.400 | 6 971.787 | 7 104.729 | 0.108 | 7 112.904 |
|  | 4 | 23 554.47 | 2 317.319 | 23 421.92 | 0.564 | 23 554.98 |
|  | 5 | 58 721.18 | 57 916.33 | 58 047.31 | 1.154 | 58 721.70 |
| 7.50 | 1 | 3.938669 | 2.895383 | 3.907605 | 0.792 | 4.8815 |
|  | 2 | 1 187.873 | 1 154.802 | 1 187.496 | 0.032 | 1 188.435 |
|  | 3 | 7 056.811 | 6 906.424 | 7 048.634 | 0.116 | 7 057.388 |
|  | 4 | 23 455.70 | 23 047.36 | 23 313.78 | 0.607 | 23 456.28 |
|  | 5 | 58 566.89 | 57 704.51 | 57 844.89 | 1.240 | 58 567.49 |

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM

$\beta = .75$

| $\alpha$ Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|
| 1 | 0.7825819 | ---- | 0.7542064 | 3.693 | 1.7973 |
| 2 | 1 175.502 | 1 141.460 | 1 175.109 | 0.033 | 1 176.098 |
| 7.75  3 | 7 029.015 | 6 873.755 | 7 020.583 | 0.120 | 7 029.629 |
| 4 | 23 406.31 | 22 984.45 | 23 259.71 | 0.628 | 23 406.94 |
| 5 | 58 4 89.75 | 57 598.60 | 57 743.69 | 1.284 | 58 490.38 |

TABLE  I  -  BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM  β = 1.00

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 1.00 | 1 | 82.59065 | 82.28879 | 82.58765 | 0.004 | 82.6047 |
| | 2 | 1 499.316 | 1 494.533 | 1 499.263 | 0.004 | 1 499.327 |
| | 3 | 7 756.884 | 7 736.428 | 7 755.743 | 0.015 | 7 756.896 |
| | 4 | 24 699.84 | 24 645.15 | 24 680.75 | 0.077 | 24 699.85 |
| | 5 | 60 510.55 | 60 395.69 | 60 414.26 | 0.159 | 60 510.57 |
| 3.00 | 1 | 52.86138 | 52.14043 | 52.85093 | 0.020 | 52.9959 |
| | 2 | 1 380.795 | 1 366.927 | 1 380.634 | 0.012 | 1 380.892 |
| | 3 | 7 490.320 | 7 429.579 | 7 486.927 | 0.045 | 7 490.416 |
| | 4 | 24 226.01 | 24 062.55 | 24 158.90 | 0.236 | 24 226.11 |
| | 5 | 59 770.24 | 59 426.05 | 59 481.41 | 0.484 | 59 770.35 |
| 5.00 | 1 | 22.98678 | 22.15408 | 22.96599 | 0.090 | 23.3871 |
| | 2 | 1 262.199 | 1 239.950 | 1 261.929 | 0.021 | 1 262.457 |
| | 3 | 7 223.677 | 7 123.531 | 7 218.113 | 0.077 | 7 223.937 |
| | 4 | 23 752.11 | 23 480.69 | 23 657.17 | 0.401 | 23 752.37 |
| | 5 | 59 029.85 | 58 456.83 | 58 548.49 | 0.819 | 59 030.13 |
| 6.00 | 1 | 7.984866 | 7.241968 | 7.961275 | 0.296 | 8.5827 |
| | 2 | 1 202.877 | 1 176.730 | 1 202.567 | 0.026 | 1 203.239 |
| | 3 | 7 090.327 | 6 970.827 | 7 083.696 | 0.094 | 7 090.698 |
| | 4 | 23 515.12 | 23 190.04 | 23 401.35 | 0.485 | 23 515.5 |
| | 5 | 58 659.63 | 57 972.37 | 58 082.01 | 0.990 | 58 660.02 |

TABLE   I  -   BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM          β = 1.00

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| | 1 | 4.22 834 | 3.524225 | 4.197706 | 0.692 | 4.8816 |
| | 2 | 1 188.045 | 1 160.955 | 1 187.721 | 0.027 | 1 188.435 |
| 6.25 | 3 | 7 056.987 | 6 932.686 | 7 050.098 | 0.098 | 7 057.388 |
| | 4 | 23 455.87 | 23 117.41 | 23 337.41 | 0.506 | 23 456.28 |
| | 5 | 58 567.07 | 57 851.28 | 57 965.38 | 1.033 | 58 567.49 |
| | 1 | .4656114 | - - - - | .4387692 | 5.938 | 1.1804 |
| | 2 | 1 173.212 | 1 145.193 | 1 172.879 | 0.028 | 1 173.630 |
| 6.50 | 3 | 7 023.645 | 6 894.559 | 7 016.496 | 0.102 | 7 024.078 |
| | 4 | 23 396.62 | 23 044.79 | 23 273.46 | 0.528 | 23 397.06 |
| | 5 | 58 474.51 | 57 730.19 | 57 848.76 | 1.076 | 58 474.96 |

TABLE  I  -  BOUNDS  FOR  THE  EIGENVALUES  OF  THE  SIMPLY  SUPPORTED  BEAM                                      β = 1.50

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 1.00 | 1 | 77.65570 | 77.36830 | 77.65391 | 0.002 | 77.6699 |
|  | 2 | 1 479.577 | 1 474.838 | 1 479.628 | 0.003 | 1 479.588 |
|  | 3 | 7 712.471 | 7 692.090 | 7 711.329 | 0.015 | 7 712.483 |
|  | 4 | 24 620.88 | 24 566.31 | 24 601.80 | 0.078 | 24 620.89 |
|  | 5 | 60 387.17 | 60 272.46 | 60 290.89 | 0.160 | 60 387.20 |
| 2.00 | 1 | 57.871 | 57.38978 | 57.86322 | 0.014 | 57.9307 |
|  | 2 | 1 400.588 | 1 391.361 | 1 400.479 | 0.008 | 1 400.631 |
|  | 3 | 7 534.786 | 7 494.387 | 7 532.506 | 0.030 | 7 534.830 |
|  | 4 | 24 305.02 | 24 196.28 | 24 266.90 | 0.157 | 24 305.07 |
|  | 5 | 59 893.66 | 59 664.63 | 59 701.10 | 0.322 | 59 893.71 |
| 3.00 | 1 | 38.05234 | 37.48683 | 38.04164 | 0.028 | 38.1915 |
|  | 2 | 1 321.579 | 1 308.134 | 1 321.421 | 0.072 | 1 321.674 |
|  | 3 | 7 357.081 | 7 297.041 | 7 353.682 | 0.046 | 7 357.177 |
|  | 4 | 23 989.14 | 23 826.66 | 23 932.03 | 0.238 | 23 989.24 |
|  | 5 | 59 400.13 | 59 057.20 | 59 111.29 | 0.487 | 59 400.23 |
| 4.00 | 1 | 18.19300 | 17.67678 | 18.18 21 | 0.070 | 18.45523 |
|  | 2 | 1 242.554 | 1 225.180 | 1 242.339 | 0.017 | 1 242.718 |
|  | 3 | 7 179.357 | 7 100.068 | 7 174.870 | 0.063 | 7 179.524 |
|  | 4 | 23 673.24 | 23 457.46 | 23 597.19 | 0.322 | 23 673.41 |
|  | 5 | 58 906.58 | 58 450.16 | 58 521.47 | 0.656 | 58 906.75 |

TABLE  I  -  BOUNDS  FOR  THE  EIGENVALUES  OF  THE  SIMPLY  SUPPORTED  BEAM          β = 1.50

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 4.80 | 1 | 2.272578 | 1.910738 | 2.251292 | 0.941 | 2.6609 |
|  | 2 | 1 179.323 | 1 159.031 | 1 179.062 | 0.022 | 1 179.552 |
|  | 3 | 7 037.165 | 6 942.770 | 7 031.815 | 0.076 | 7 037.402 |
|  | 4 | 23 420.41 | 23 162.41 | 23 329.35 | 0.390 | 23 420.75 |
|  | 5 | 58 511.72 | 57 964.83 | 58 049.61 | 0.793 | 58 511.97 |
| 4.90 | 1 | .2892582 | ---- | .2603751 | 7.358 | .686968 |
|  | 2 | 1 171.419 | 1 150.776 | 1 171.164 | 0.022 | 1 171.657 |
|  | 3 | 7 019.390 | 6 923.126 | 7 013.928 | 0.078 | 7 019.636 |
|  | 4 | 23 388.92 | 23 125.55 | 23 295.88 | 0.399 | 23 389.17 |
|  | 5 | 58 462.36 | 57 904.18 | 57 990.62 | 0.810 | 58 462.62 |

TABLE I – BOUNDS FOR THE EIGENVLAUES OF THE SIMPLY SUPPORTED BEAM $\beta = 2.00$

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 1.00 | 1 | 72.72075 | 72.44813 | 72.71235 | 0.012 | 72.7351 |
|  | 2 | 1 459.838 | 1 455.143 | 1 459.786 | 0.004 | 1459.849 |
|  | 3 | 7 668.058 | 7 647.753 | 7 666.911 | 0.015 | 7 668.069 |
|  | 4 | 24 541.92 | 24 487.45 | 24 522.84 | 0.045 | 24 541.94 |
|  | 5 | 60 263.81 | 60 149.23 | 60 167.52 | 0.150 | 60 263.83 |
| 2.00 | 1 | 48.00062 | 47.58488 | 47.99471 | 0.012 | 48.0611 |
|  | 2 | 1 361.110 | 1 352.067 | 1 360.999 | 0.008 | 1 361.153 |
|  | 3 | 7 445.960 | 7 405.870 | 7 443.674 | 0.031 | 7 445.003 |
|  | 4 | 24 147.11 | 24 038.80 | 24 108.99 | 0.158 | 24 147.15 |
|  | 5 | 59 646.92 | 59 418.45 | 59 454.36 | 0.323 | 59 646.98 |
| 3.00 | 1 | 23.24296 | 22.84481 | 23.232020 | 0.047 | 23.3871 |
|  | 2 | 1 262.364 | 1 249.352 | 1 262.204 | 0.013 | 1 262.457 |
|  | 3 | 7 223.842 | 7 164.512 | 7 220.445 | 0.097 | 7 223.937 |
|  | 4 | 23 752.27 | 23 590.78 | 23 695.16 | 0.241 | 23 752.37 |
|  | 5 | 59 030.02 | 58 688.35 | 58 741.18 | 0.491 | 59 030.13 |
| 3.75 | 1 | 4.645885 | 4.391169 | 4.629351 | 0.357 | 4.88155 |
|  | 2 | 1 188.294 | 1 172.574 | 1 188.087 | 0.017 | 1 188.435 |
|  | 3 | 7 057.242 | 6 983.855 | 7 053.022 | 0.050 | 7 057.388 |
|  | 4 | 23 456.13 | 23 255.20 | 23 384.81 | 0.305 | 23 456.28 |
|  | 5 | 58 567.33 | 58 141.24 | 58 206.29 | 0.618 | 58 567.49 |

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM    $\beta = 2.00$

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| | 1 | 2.164188 | 1.938379 | 2.150358 | 0.641 | 2.41419 |
| | 2 | 1 178.418 | 1 162.355 | 1 178.210 | 0.018 | 1 178.565 |
| 3.85 | 3 | 7 035.028 | 6 959.792 | 7 030.706 | 0.061 | 7 035.181 |
| | 4 | 23 416.64 | 23 210.48 | 23 343.44 | 0.313 | 23 416.81 |
| | 5 | 58 505.64 | 58 068.32 | 58 134.96 | 9.636 | 58 505.81 |

TABLE   I  -  BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM          β = -0.10

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 1.00 | 1 | 93.44753 | 93.11508 | 93.44343 | 0.004 | 93.5612 |
|  | 2 | 1 542.742 | 1 537.864 | 1 542.688 | 0.004 | 1 542.754 |
|  | 3 | 7 854.593 | 7 833.972 | 7 853.450 | 0.015 | 7 854.605 |
|  | 4 | 24 873.54 | 24 818.62 | 24 854.46 | 0.077 | 24 873.56 |
|  | 5 | 60 781.96 | 60 666.80 | 60 685.67 | 0.159 | 60 781.98 |
| 4.00 | 1 | 81.39310 | 80.01163 | 81.37477 | 0.023 | 81.6177 |
|  | 2 | 1 495.202 | 1 475.431 | 1 494.993 | 0.014 | 1 495.380 |
|  | 3 | 7 747.841 | 7 664.568 | 7 743.345 | 0.058 | 7 748.013 |
|  | 4 | 24 683.89 | 24 462.53 | 24 607.84 | 0.309 | 24 684.06 |
|  | 5 | 60 485.71 | 60 022.11 | 60 100.61 | 0.639 | 60 485.89 |
| 8.00 | 1 | 64.89664 | 61.93527 | 64.87003 | 0.041 | 65.8264 |
|  | 2 | 1 431.516 | 1 391.182 | 1 431.118 | 0.028 | 1 432.214 |
|  | 3 | 7 605.211 | 7 436.416 | 7 596.508 | 0.115 | 7 605.891 |
|  | 4 | 24 430.72 | 23 983.31 | 24 279.42 | 0.621 | 24 431.40 |
|  | 5 | 60 090.42 | 59 155.1 | 59 320.31 | 1.290 | 60 091.11 |
| 12.00 | 1 | 47.86730 | 42.97865 | 47.83155 | 0.075 | 50.0350 |
|  | 2 | 1 367.508 | 1 305.609 | 1 366.943 | 0.041 | 1 369.049 |
|  | 3 | 7 462.254 | 7 205.422 | 7 449.643 | 0.169 | 7 463.769 |
|  | 4 | 24 117.23 | 23 498.74 | 23 951.47 | 0.938 | 24 178.74 |
|  | 5 | 59 694.81 | 58 279.23 | 58 539.68 | 1.954 | 59 696.32 |

TABLE   I  -   BOUNDS  FOR  THE  EIGENVALUES  OF  THE  SIMPLY  SUPPORTED  BEAM            β = -0.10

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 15.00 | 1 | 34.70939 | 28.00187 | 34.66177 | 0.137 | 38.1915 |
| | 2 | 1 319.305 | 1 240.427 | 1 318.638 | 0.051 | 1 321.675 |
| | 3 | 7 354.827 | 7 030.143 | 7 339.476 | 0.209 | 7 357.177 |
| | 4 | 23 986.98 | 23 131.58 | 23 705.87 | 1.178 | 23 989.24 |
| | 5 | 59 397.88 | 57 616.27 | 57 954.20 | 2.460 | 59 400.24 |
| 19.00 | 1 | 16.59538 | 6.639269 | 16.55101 | 0.268 | 22.4001 |
| | 2 | 1 254.801 | 1 151.969 | 1 254.016 | 0.063 | 1 258.509 |
| | 3 | 7 211.323 | 6 793.473 | 7 192.587 | 0.260 | 7 215.055 |
| | 4 | 23 732.84 | 22 636.78 | 23 378.85 | 1.503 | 23 736.58 |
| | 5 | 59 001.70 | 56 723.86 | 57 173.23 | 3.148 | 59 005.45 |
| 21.00 | 1 | 7.271955 | - - - - | 7.220387 | 0.712 | 14.5044 |
| | 2 | 1 222.460 | 1 106.989 | 1 221.621 | 0.069 | 1 226.927 |
| | 3 | 7 139.458 | 6 673.771 | 7 119.119 | 0.285 | 7 143.994 |
| | 4 | 23 605.69 | 22 387.01 | 23 215.54 | 1.667 | 23 610.25 |
| | 5 | 58 803.48 | 56 273.92 | 56 782.65 | 3.497 | 58 808.06 |
| 22.00 | 1 | 2.538900 | - - - - | 2.489753 | 1.955 | 10.5565 |
| | 2 | 1 206.270 | 1 084.293 | 1 205.413 | 0.071 | 1 211.136 |
| | 3 | 7 103.499 | 6 613.559 | 7 082.395 | 0.298 | 7 108.563 |
| | 4 | 23 542.09 | 22 261.51 | 23 133.93 | 1.749 | 23 547.08 |
| | 5 | 58 704.35 | 56 047.99 | 56 587.34 | 3.672 | 58 709.36 |

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta =-0.10$

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Consent End Load |
|---|---|---|---|---|---|---|
| 22.25 | 1 | 1.347959 | - - - - | 1.298447 | 3.741 | 9.56961 |
| | 2 | 1 202.220 | 1 078.596 | 1 201.356 | 0.072 | 1 207.187 |
| | 3 | 7 094.506 | 6 598.468 | 7 073.216 | 0.301 | 7 099.580 |
| | 4 | 23 526.18 | 22 230.07 | 23 113.49 | 1.770 | 23 531.29 |
| | 5 | 58 679.56 | 55 991.40 | 56 538.51 | 3.717 | 58 684.69 |
| 22.50 | 1 | 0.1538999 | - - - - | 0.1040043 | 38.682 | 8.58265 |
| | 2 | 1 198.17 | 1 072.890 | 1 197.298 | 0.073 | 1 203.329 |
| | 3 | 7 085.512 | 6 583.360 | 7 064.037 | 0.304 | 7 090.697 |
| | 4 | 23 510.28 | 22 198.61 | 23 093.09 | 1.790 | 23 515.50 |
| | 5 | 58 654.77 | 55 934.77 | 56 489.67 | 3.761 | 58 660.02 |

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM    β = -0.25

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 1.00 | 1 | 94.92801 | 94.59147 | 94.92394 | 0.004 | 94.9417 |
|  | 2 | 1 548.664 | 1 543.772 | 1 548.610 | 0.003 | 1 548.675 |
|  | 3 | 7 867.917 | 7 847.274 | 7 866.778 | 0.014 | 7 867.929 |
|  | 4 | 24 897.23 | 24 842.28 | 24 878.15 | 0.077 | 24 897.24 |
|  | 5 | 60 818.97 | 60 703.77 | 60 722.68 | 0.158 | 60 818.99 |
| 5.00 | 1 | 84.72365 | 82.85905 | 84.70111 | 0.027 | 85.0721 |
|  | 2 | 1 508.919 | 1 483.754 | 1 508.651 | 0.018 | 1 509.197 |
|  | 3 | 7 778.833 | 7 673.832 | 7 773.258 | 0.072 | 7 779.102 |
|  | 4 | 24 739.06 | 24 460.84 | 24 644.12 | 0.385 | 24 739.33 |
|  | 5 | 60 571.98 | 59 990.19 | 60 090.62 | 0.798 | 60 572.25 |
| 10.00 | 1 | 71.30376 | 67.34356 | 71.27029 | 0.047 | 72.7351 |
|  | 2 | 1 458.752 | 1 409.939 | 1 458.265 | 0.033 | 1 459.849 |
|  | 3 | 7 667.004 | 7 459.246 | 7 656.303 | 0.140 | 7 668.069 |
|  | 4 | 24 540.88 | 23 979.70 | 24 352.25 | 0.772 | 24 541.94 |
|  | 5 | 60 262.77 | 59 089.72 | 59 300.19 | 1.610 | 60 263.83 |
| 15.00 | 1 | 57.08734 | 50.16623 | 57.03932 | 0.083 | 60.3981 |
|  | 2 | 1 408.065 | 1 334.337 | 1 407.394 | 0.048 | 1 410.501 |
|  | 3 | 7 554.657 | 7 241.396 | 7 539.280 | 0.204 | 7 557.036 |
|  | 4 | 24 342 18 | 23 489.36 | 24 061.14 | 1.161 | 24 344.55 |
|  | 5 | 59 953.04 | 58 174.63 | 58 509.38 | 2.606 | 59 955.40 |

TABLE  I --  BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM          β = -0.25

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 20.00 | 1 | 42.00398 | 30.90309 | 41.95275 | 0.122 | 48.0611 |
| | 2 | 1 356.890 | 1 257.749 | 1 356.070 | 0.060 | 1 361.153 |
| | 3 | 7 441.805 | 7 021.661 | 7 422.214 | 0.264 | 7 446.003 |
| | 4 | 24 142.97 | 22 990.42 | 23 770.82 | 1.553 | 24 147.15 |
| | 5 | 59 642.79 | 57 245.57 | 57 718.18 | 3.280 | 59 646.98 |
| 25.00 | 1 | 25.97378 | 8.89325 | 25.91837 | 0.214 | 35.7241 |
| | 2 | 1 305.264 | 1 179.673 | 1 304.336 | 0.071 | 1 311.805 |
| | 3 | 7 328.459 | 6 802.433 | 7 305.086 | 0.319 | 7 334.970 |
| | 4 | 23 943.26 | 22 482.28 | 23 481.30 | 1.948 | 23 949.76 |
| | 5 | 59 332.04 | 56 301.99 | 56 926.59 | 4.138 | 59 338.55 |
| 30.00 | 1 | 8.906012 | ---- | 8.848833 | 0.644 | 23.3871 |
| | 2 | 1 253.232 | 1 100.010 | 1 252.223 | 0.081 | 1 262.457 |
| | 3 | 7 214.636 | 6 579.224 | 7 187.091 | 0.371 | 7 223.937 |
| | 4 | 23 743.05 | 21 964.30 | 23 192.62 | 2.345 | 23 752.37 |
| | 5 | 59 020.79 | 55 388.08 | 56 134.63 | 5.013 | 59 030.12 |
| 32.00 | 1 | 1.765746 | ---- | 1.710738 | 3.165 | 18.4522 |
| | 2 | 1 232.318 | 1 067.875 | 1 231.279 | 0.084 | 1 242.718 |
| | 3 | 7 168.977 | 6 489.106 | 7 141.022 | 0.391 | 7 179.524 |
| | 4 | 23 662.83 | 21 755.39 | 23 077.39 | 2.505 | 23 673.41 |
| | 5 | 58 896.14 | 54 954.51 | 55 817.73 | 5.367 | 58 906.75 |

TABLE  I  -  BOUNDS  FOR  THE  EIGENVALUES  OF  THE  SIMPLY  SUPPORTED  BEAM

$\beta = -0.25$

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 32.25 | 1 | 0.8600384 | ---- | 0.8096142 | 6.038 | 17.8354 |
| | 2 | 1 229.700 | 1 063.699 | 1 228.664 | 0.084 | 1 240.250 |
| | 3 | 7 163.264 | 6 478.028 | 7 135.161 | 0.393 | 7 173.972 |
| | 4 | 23 652.79 | 21 725.42 | 23 062.99 | 2.525 | 23 663.54 |
| | 5 | 58 880.55 | 54 906.15 | 55 778.11 | 5.412 | 58 891.33 |

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM    β = -0.50

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 1.00 | 1 | 97.39548 | 97.05219 | 97.38885 | 0.007 | 97.4091 |
|  | 2 | 1 558.533 | 1 553.621 | 1 558.481 | 0.003 | 1 558.545 |
|  | 3 | 7 890.123 | 7 869.443 | 7 888.977 | 0.015 | 7 890.135 |
|  | 4 | 24 936.71 | 24 881.71 | 24 917.63 | 0.077 | 24 936.72 |
|  | 5 | 60 880.66 | 60 765.39 | 60 784.37 | 0.158 | 60 880.68 |
| 5.00 | 1 | 97.06945 | 95.02756 | 97.05359 | 0.016 | 97.4091 |
|  | 2 | 1 558.264 | 1 532.551 | 1 558.000 | 0.017 | 1 558.545 |
|  | 3 | 7 889.865 | 7 783.922 | 7 884.291 | 0.071 | 7 890.135 |
|  | 4 | 24 936.45 | 24 656.89 | 24 841.50 | 0.381 | 24 936.73 |
|  | 5 | 60 880.40 | 60 296.88 | 60 399.04 | 0.794 | 60 880.68 |
| 10.00 | 1 | 96.05006 | 91.43544 | 96.01405 | 0.037 | 97.4091 |
|  | 2 | 1 557.425 | 1 506.462 | 1 556.936 | 0.031 | 1 558.545 |
|  | 3 | 7 889.057 | 7 677.593 | 7 878.347 | 0.136 | 7 890.134 |
|  | 4 | 24 935.66 | 24 368.81 | 24 747.01 | 0.759 | 24 936.73 |
|  | 5 | 60 879.61 | 59 699.54 | 59 917.04 | 1.594 | 60 880.68 |
| 20.00 | 1 | 91.96428 | 78.64063 | 91.91397 | 0.055 | 97.4091 |
|  | 2 | 1 554.071 | 1 446.067 | 1 553.239 | 0.054 | 1 558.545 |
|  | 3 | 7 885.830 | 7 451.941 | 7 866.169 | 0.250 | 7 890.135 |
|  | 4 | 24 932.48 | 23 756.87 | 24 560.22 | 1.504 | 24 936.73 |
|  | 5 | 60 876.46 | 58 451.08 | 58 951.90 | 3.212 | 60 880.68 |

TABLE  I -   BOUNDS  FOR  THE  EIGENVALUES  OF  THE  SIMPLY  SUPPORTED  BEAM                    β = -0.50

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 30.00 | 1 | 85.12538 | 56.42424 | 85.06259 | 0.074 | 97.4091 |
|  | 2 | 1 548.503 | 1 375.513 | 1 547.459 | 0.067 | 1 558.545 |
|  | 3 | 7 880.455 | 7 210.208 | 7 853.494 | 0.343 | 7 890.135 |
|  | 4 | 24 927.18 | 23 095.63 | 24 376.41 | 2.234 | 24 936.73 |
|  | 5 | 60 871.19 | 57 129.12 | 57 985.23 | 4.856 | 60 880.68 |
| 40.00 | 1 | 75.48948 | 21.68015 | 75.42380 | 0.087 | 97.4091 |
|  | 2 | 1 540.754 | 1 292.824 | 1 539.589 | 0.076 | 1 558.545 |
|  | 3 | 7 872.940 | 6 950.046 | 7 840.212 | 0.417 | 7 890.135 |
|  | 4 | 24 919.78 | 22 387.13 | 24 195.62 | 2.949 | 24 936.73 |
|  | 5 | 60 863.84 | 55 727.85 | 57 017.05 | 6.527 | 60 880.68 |
| 50.00 | 1 | 62.99520 | ---- | 62.93328 | 0.098 | 97.4091 |
|  | 2 | 1 530.869 | 1 195.846 | 1 529.658 | 0.079 | 1 558.545 |
|  | 3 | 7 863.296 | 6 669.137 | 7 826.211 | 0.473 | 7 890.135 |
|  | 4 | 24 910.26 | 21 611.65 | 24 017.91 | 3.648 | 24 936.73 |
|  | 5 | 60 854.38 | 54 239.88 | 56 047.35 | 8.224 | 60 880.68 |
| 60.00 | 1 | 47.56379 | ---- | 47.49907 | 0.136 | 97.4091 |
|  | 2 | 1 518.906 | 1 082.880 | 1 517.698 | 0.080 | 1 558.545 |
|  | 3 | 7 851.536 | 6 364.444 | 7 811.384 | 0.513 | 7 890.135 |
|  | 4 | 24 898.63 | 20 767.33 | 23 843.33 | 4.330 | 24 936.73 |
|  | 5 | 60 842.82 | 52 657.41 | 55 076.16 | 9.949 | 60 880.68 |

TABLE  I -  BOUNDS  FOR  THE  EIGENVALUES  OF  THE  SIMPLY  SUPPORTED  BEAM          $\beta = -0.50$

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 70.00 | 1 | 29.09930 | ---- | 29.03900 | 0.207 | 97.4091 |
| | 2 | 1 504.937 | 951.7511 | 1 503.762 | 0.078 | 1 558.545 |
| | 3 | 7 837.675 | 6 034.095 | 7 795.619 | 0.538 | 7 890.135 |
| | 4 | 24 884.90 | 19 863.03 | 23 671.88 | 4.996 | 24 936.73 |
| | 5 | 60 829.17 | 50 971.82 | 54 103.51 | 11.704 | 60 880.68 |
| 80.00 | 1 | 7.488954 | ---- | 7.427070 | 0.830 | 97.4091 |
| | 2 | 1 489.045 | 801.0849 | 1 487.923 | 0.075 | 1 558.545 |
| | 3 | 7 821.731 | 5 689.257 | 7 778.816 | 0.550 | 7 890.135 |
| | 4 | 24 869.08 | 18 858.86 | 23 503.62 | 5.646 | 24 936.73 |
| | 5 | 60 813.43 | 49 175.77 | 53 129.41 | 13.487 | 60 880.68 |
| 81.00 | 1 | 5.150067 | ---- | 5.085385 | 1.264 | 97.4091 |
| | 2 | 1 487.354 | 785.6676 | 1 486.242 | 0.075 | 1 558.545 |
| | 3 | 7 820.022 | 5 652.000 | 7 777.073 | 0.551 | 7 890.135 |
| | 4 | 24 8 7.38 | 18 753.30 | 23 486.96 | 5.710 | 24 936.73 |
| | 5 | 60 811.74 | 48 988.74 | 53 031.92 | 13.668 | 60 880.68 |
| 82.00 | 1 | 2.778307 | ---- | 2.713758 | 2.351 | 97.4091 |
| | 2 | 1 485.644 | 768.9310 | 1 484.539 | 0.074 | 1 558.545 |
| | 3 | 7 818.294 | 5 615.842 | 7 775.319 | 0.551 | 7 890.135 |
| | 4 | 24 86 .67 | 18 646.77 | 23 470.35 | 5.773 | 24 936.73 |
| | 5 | 60 810.03 | 48 801.56 | 52 934.43 | 13.848 | 60 880.68 |

TABLE   I  -   BOUNDS  FOR  THE  EIGENVALUES  OF  THE  SIMPLY  SUPPORTED  BEAM                    β = -0.75

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 1.00 | 1 | 99.86295 | 99.51297 | 99.86157 | 0.001 | 99.8765 |
|  | 2 | 1 568.403 | 1 563.469 | 1 568.352 | 0.003 | 1 568.415 |
|  | 3 | 7 912.330 | 7 891.613 | 7 911.188 | 0.014 | 7 912.342 |
|  | 4 | 24 976.19 | 24 921.13 | 24 957.11 | 0.076 | 24 976.20 |
|  | 5 | 60 942.34 | 60 827.01 | 60 846.06 | 0.158 | 60 942.36 |
| 5.00 | 1 | 109.4148 | 107.2052 | 109.3978 | 0.016 | 109.7461 |
|  | 2 | 1 607.610 | 1 581.359 | 1 607.345 | 0.016 | 1 607.893 |
|  | 3 | 8 000.896 | 7 894.022 | 7 995.323 | 0.070 | 8 001.168 |
|  | 4 | 25 133.84 | 24 852.95 | 25 038.90 | 0.378 | 25 134.11 |
|  | 5 | 61 1881.83 | 60 603.59 | 60 707.47 | 0.790 | 61 189.10 |
| 10.00 | 1 | 120.7894 | 115.5869 | 120.7548 | 0.029 | 122.0831 |
|  | 2 | 1 656.102 | 1 603.075 | 1 655.616 | 0.029 | 1 657.241 |
|  | 3 | 8 111.112 | 7 896.020 | 8 100.397 | 0.132 | 8 112.202 |
|  | 4 | 25 330.43 | 24 758.00 | 25 141.78 | 0.748 | 25 331.51 |
|  | 5 | 61 496.46 | 60 309.52 | 60 533.90 | 1.578 | 61 497.53 |
| 20.00 | 1 | 141.8121 | 126.6892 | 141.7554 | 0.040 | 146.7571 |
|  | 2 | 1 751.334 | 1 635.145 | 1 750.491 | 0.048 | 1 755.937 |
|  | 3 | 8 329.371 | 7 880.263 | 8 310.140 | 0.237 | 8 334.268 |
|  | 4 | 25 721.99 | 24 523.88 | 25 349.63 | 1.458 | 25 726.29 |
|  | 5 | 62 110.12 | 59 656.94 | 60 185.61 | 3.147 | 62 114.38 |

TABLE   I -   BOUNDS  FOR  THE  EIGENVALUES  OF  THE  SIMPLY  SUPPORTED  BEAM                β = -0.75

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 30.00 | 1 | 160.7671 | 129.6790 | 160.7000 | 0.042 | 171.4311 |
|  | 2 | 1 844.200 | 1 653.497 | 1 843.138 | 0.058 | 1 854.633 |
|  | 3 | 8 546.365 | 7 840.957 | 8 519.187 | 0.319 | 8 556.334 |
|  | 4 | 26 111.35 | 24 234.53 | 25 560.22 | 2.133 | 26 121.079 |
|  | 5 | 62 721.62 | 58 916.90 | 59 835.84 | 4.709 | 62 731.23 |
| 40.00 | 1 | 177.8850 | 123.8125 | 177.8083 | 0.043 | 196.1051 |
|  | 2 | 1 934.700 | 1 656.769 | 1 933.479 | 0.063 | 1 953.329 |
|  | 3 | 8 760.552 | 7 776.235 | 8 727.331 | 0.380 | 8 778.400 |
|  | 4 | 26 498.48 | 23 875.30 | 25 773.49 | 2.774 | 26 515.86 |
|  | 5 | 63 330.94 | 58 083.83 | 59 484.58 | 6.264 | 63 348.08 |
| 50.00 | 1 | 193.3510 | 108.4000 | 193.2635 | 0.045 | 220.7791 |
|  | 2 | 2 022.857 | 1 643.877 | 2 021.548 | 0.065 | 2 052.0256 |
|  | 3 | 8 972.409 | 7 684.302 | 8 934.409 | 0.424 | 9 000.466 |
|  | 4 | 26 883.35 | 23 454.02 | 25 989.35 | 3.382 | 26 910.64 |
|  | 5 | 63 938.06 | 57 149.33 | 59 131.86 | 7.811 | 63 964.93 |
| 60.00 | 1 | 207.3155 | 82.13972 | 207.2298 | 0.041 | 245.4531 |
|  | 2 | 2 108.712 | 1 611.067 | 2 107.367 | 0.064 | 2 150.726 |
|  | 3 | 9 181.915 | 7 572.918 | 9 140.270 | 0.455 | 9 222.532 |
|  | 4 | 27 265.95 | 22 945.47 | 26 207.73 | 3.958 | 27 305.43 |
|  | 5 | 64 542.95 | 56 109.45 | 58 777.69 | 9.350 | 64 581.78 |

**TABLE  I  -  BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM**          β = -0.75

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 70.00 | 1 | 219.9011 | 45.50295 | 219.8101 | 0.041 | 270.1271 |
| | 2 | 2 192.316 | 1 565.173 | 2 190.959 | 0.062 | 2 249.417 |
| | 3 | 9 389.065 | 7 423.49 | 9 344.764 | 0.473 | 9 444.598 |
| | 4 | 27 646.24 | 22 435.10 | 26 428.54 | 4.504 | 27 700.22 |
| | 5 | 65 145.60 | 54 952.82 | 58 422.11 | 10.882 | 65 198.63 |
| 80.00 | 1 | 231.2094 | ---- | 231.1131 | 0.042 | 294.8011 |
| | 2 | 2 273.728 | 1 497.529 | 2 272.369 | 0.060 | 2 348.114 |
| | 3 | 9 593.853 | 7 241.964 | 9 547.784 | 0.481 | 9 666.664 |
| | 4 | 28 024.21 | 21 680.87 | 26 651.69 | 5.021 | 28 094.99 |
| | 5 | 65 745.98 | 53 671.96 | 58 065.14 | 12.407 | 65 815.48 |
| 90.00 | 1 | 241.3245 | ---- | 241.2256 | 0.041 | 319.4752 |
| | 2 | 2 353.010 | 1 410.343 | 2 351.666 | 0.057 | 2 446.809 |
| | 3 | 9 796.286 | 7 026.768 | 9 749.202 | 0.482 | 9 888.731 |
| | 4 | 28 399.86 | 20 992.29 | 26 877.07 | 5.510 | 28 489.78 |
| | 5 | 66 344.09 | 52 257.63 | 57 706.82 | 13.925 | 66 432.33 |
| 100.00 | 1 | 250.3169 | ---- | 250.2154 | 0.041 | 344.1492 |
| | 2 | 2 430.225 | 1 303.072 | 2 428.900 | 0.055 | 2 545.505 |
| | 3 | 9 996.371 | 6 795.214 | 9 948.945 | 0.476 | 10 110.79 |
| | 4 | 28 773.16 | 20 018.08 | 27 104.57 | 5.972 | 28 884.57 |
| | 5 | 66 939.92 | 50 702.86 | 57 347.18 | 15.436 | 67 047.18 |

TABLE  I -  BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM     β = -0.75

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| | 1 | 280.2315 | ---- | 280.0769 | 0.057 | 467.5192 |
| | 2 | 2 787.495 | 439.3405 | 2 786.273 | 0.044 | 3 038.986 |
| 150.00 | 3 | 10 962 20 | 5 034.360 | 10 919.79 | 0.406 | 11 221.12 |
| | 4 | 30 604.40 | 13 548.06 | 28 269.28 | 7.933 | 30 858.49 |
| | 5 | 69 884.39 | 40 397.25 | 55 531.11 | 22.889 | 70 133.43 |

TABLE  I  -  BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM

β = -1.00

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 1.00 | 1 | 102.3304 | 101.9738 | 102.3264 | 0.004 | 102.3439 |
|  | 2 | 1 578.273 | 1 573.317 | 1 578.219 | 0.003 | 1 578.284 |
|  | 3 | 7 934.537 | 7 913.782 | 7 933.391 | 0.014 | 7 934.549 |
|  | 4 | 25 015.66 | 24 960.05 | 24 996.58 | 0.076 | 25 015.68 |
|  | 5 | 61 004.03 | 60 888.62 | 60 907.74 | 0.158 | 61 004.05 |
| 5.00 | 1 | 121.7597 | 119.3913 | 121.7338 | 0.021 | 122.0831 |
|  | 2 | 1 656.956 | 1 630.178 | 1 656.690 | 0.016 | 1 657.241 |
|  | 3 | 8 111.928 | 8 004.133 | 8 106.352 | 0.069 | 8 112.202 |
|  | 4 | 25 331.23 | 25 049.03 | 25 236.29 | 0.375 | 25 331.51 |
|  | 5 | 61 497.25 | 60 910.29 | 61 015.89 | 0.786 | 61 497.52 |
| 10.00 | 1 | 145.6227 | 139.7899 | 145.4817 | 0.028 | 145.7571 |
|  | 2 | 1 754.785 | 1 699.563 | 1 754.292 | 0.028 | 1 755.937 |
|  | 3 | 8 333.166 | 8 114.520 | 8 322.439 | 0.087 | 8 334.268 |
|  | 4 | 25 724.21 | 25 148.07 | 25 536.55 | 0.736 | 25 726.29 |
|  | 5 | 62 113.30 | 60 914.02 | 61 150.74 | 1.562 | 62 114.38 |
| 20.00 | 1 | 191.5759 | 174.8713 | 191.5207 | 0.029 | 196.1051 |
|  | 2 | 1 948.654 | 1 825.068 | 1 947.809 | 0.043 | 1 953.329 |
|  | 3 | 8 773.930 | 8 311.577 | 8 754.135 | 0.226 | 8 778.400 |
|  | 4 | 26 511.51 | 25 294.93 | 26 139.05 | 1.415 | 26 515.86 |
|  | 5 | 63 343.78 | 60 863.53 | 61 419.33 | 3.085 | 63 348.08 |

TABLE  I  -  BOUNDS  FOR  THE  EIGENVALUES  OF  THE  SIMPLY  SUPPORTED  BEAM                    β = -1.00

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 30.00 | 1 | 236.0319 | 203.0425 | 235.9576 | 0.031 | 245.4531 |
|  | 2 | 2 140.146 | 1 933.952 | 2 139.055 | 0.051 | 2 150.721 |
|  | 3 | 9 212.350 | 8 476.840 | 9 184.954 | 0.298 | 9 222.532 |
|  | 4 | 27 295.54 | 25 372.56 | 26 744.07 | 2.041 | 27 304.43 |
|  | 5 | 64 572.05 | 60 706.93 | 61 686.44 | 4.571 | 64 581.78 |
| 40.00 | 1 | 279.2148 | 224.5836 | 279.1330 | 0.029 | 294.8012 |
|  | 2 | 2 329.327 | 2 026.117 | 2 328.056 | 0.055 | 2 348.114 |
|  | 3 | 9 648.386 | 8 615.351 | 9 614.699 | 0.350 | 9 666.664 |
|  | 4 | 28 077.27 | 25 380.02 | 27 351.47 | 2.619 | 28 094.99 |
|  | 5 | 65 798.06 | 60 444.15 | 61 952.11 | 6.021 | 65 815.48 |
| 50.00 | 1 | 321.3589 | 239.9198 | 321.2640 | 0.030 | 344.1492 |
|  | 2 | 2 516.301 | 2 099.304 | 2 514.910 | 0.055 | 2 545.506 |
|  | 3 | 10 082.02 | 8 716.340 | 10 043.17 | 0.386 | 10 110.79 |
|  | 4 | 28 856.64 | 25 305.54 | 27 961.06 | 3.152 | 28 884.57 |
|  | 5 | 67 021.81 | 60 068.90 | 62 215.35 | 7.437 | 67 049.18 |
| 75.00 | 1 | 423.2159 | 252.5182 | 423.102 | 0.027 | 467.5192 |
|  | 2 | 2 974.865 | 2 216.332 | 2 973.316 | 0.052 | 3 038.986 |
|  | 3 | 11 155.69 | 8 839.134 | 11 108.01 | 0.428 | 11 221.13 |
|  | 4 | 30 794.66 | 24 732.32 | 29 493.62 | 4.316 | 30 858.49 |
|  | 5 | 70 070.98 | 58 596.06 | 62 870.95 | 10.832 | 70 133.43 |

TABLE   I  -  BOUNDS  FOR  THE  EIGENVALUES  OF  THE  SIMPLY  SUPPORTED  BEAM         β = -1.00

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 100.00 | 1 | 521.3549 | 230.0603 | 521.2197 | 0.026 | 590.8893 |
|  | 2 | 3 422.233 | 2 221.951 | 3 420.612 | 0.047 | 3 532.466 |
|  | 3 | 12 214.89 | 8 756.443 | 12 162.74 | 0.428 | 12 331.46 |
|  | 4 | 32 717.68 | 23 562.65 | 31 036.50 | 5.274 | 32 832.41 |
|  | 5 | 73 105.38 | 56 270.24 | 63 517.38 | 14.036 | 73 217.68 |
| 125.00 | 1 | 616.7877 | 179.7252 | 616.6295 | 0.026 | 714.2593 |
|  | 2 | 3 860.049 | 2 144.900 | 3 858.391 | 0.043 | 4 025.946 |
|  | 3 | 13 260.35 | 8 451.943 | 13 206.70 | 0.405 | 13 441.79 |
|  | 4 | 34 625.77 | 21 666.47 | 32 587.38 | 6.065 | 34 806.33 |
|  | 5 | 76 124.82 | 52 967.30 | 64 156.24 | 17.064 | 76 301.94 |
| 150.00 | 1 | 710.1394 | 99.52777 | 709.9560 | 0.026 | 837.6294 |
|  | 2 | 4 289.673 | 1 958.556 | 4 287.981 | 0.039 | 4 519.426 |
|  | 3 | 14 292.91 | 7 961.67 | 14 239.73 | 0.373 | 14 552.12 |
|  | 4 | 36 519.20 | 18 874.88 | 34 143.94 | 6.723 | 36 780.25 |
|  | 5 | 79 129.26 | 48 545.04 | 64 788.19 | 19.930 | 79 386.18 |

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM          β = 0.0

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 0.0 | 1 | 500.564 | 500.564 | 500.564 | 0.000 | 500.564 |
|  | 2 | 3 803.54 | 3 803.54 | 3 803.54 | 0.000 | 3 803.54 |
|  | 3 | 14 617.6 | 14 617.6 | 14 617.6 | 0.000 | 14 617.6 |
|  | 4 | 39 943.8 | 39 943.8 | 39 943.8 | 0.000 | 39 943.8 |
| 10.0 | 1 | 438.485 | 436.964 | 438.281 | 0.047 | 438.857 |
|  | 2 | 3 572.38 | 3 570.81 | 3 571.05 | 0.037 | 3 573.03 |
|  | 3 | 14 122.2 | 14 119.9 | 14 115.8 | 0.045 | 14 123.0 |
|  | 4 | 39 084.9 | 39 081.3 | 39 069.3 | 0.040 | 39 085.8 |
| 20.0 | 1 | 375.192 | 366.109 | 374.825 | 0.098 | 376.742 |
|  | 2 | 3 339.36 | 3 320.29 | 3 336.87 | 0.074 | 3 342.01 |
|  | 3 | 13 624.9 | 13 589.2 | 13 612.6 | 0.091 | 13 628.2 |
|  | 4 | 38 224.2 | 38 189.7 | 38 196.0 | 0.074 | 38 227.7 |
| 30.0 | 1 | 310.543 | 289.161 | 310.010 | 0.171 | 314.184 |
|  | 2 | 3 104.42 | 3 062.73 | 3 100.83 | 0.115 | 3 110.44 |
|  | 3 | 13 125.9 | 13 047.8 | 13 107.8 | 0.138 | 13 133.2 |
|  | 4 | 37 361.7 | 37 285.5 | 37 318.9 | 0.114 | 37 369.5 |
| 40.0 | 1 | 244.366 | 200.691 | 243.766 | 0.245 | 251.143 |
|  | 2 | 2 867.50 | 2 786.55 | 2 863.13 | 0.152 | 2 878.30 |
|  | 3 | 12 625.1 | 12 472.4 | 12 601.6 | 0.186 | 12 638.1 |
|  | 4 | 36 497.3 | 36 364.3 | 36 440.4 | 0.156 | 36 511.2 |

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM    β = 0.0

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 50.0 | 1 | 176.455 | 110.389 | 175.815 | 0.363 | 187.572 |
| | 2 | 2 628.54 | 2 505.85 | 2 623.39 | 0.195 | 2 645.57 |
| | 3 | 12 122.6 | 11 891.2 | 12 094.1 | 0.234 | 12 142.9 |
| | 4 | 35 631.2 | 35 422.2 | 35 562.0 | 0.194 | 35 652.8 |
| 60.0 | 1 | 106.557 | 5.130 | 105.877 | 0.639 | 123.421 |
| | 2 | 2 387.52 | 2 164.11 | 2 381.38 | 0.257 | 2 412.22 |
| | 3 | 11 618.4 | 11 249.9 | 11 584.8 | 0.288 | 11 647.6 |
| | 4 | 34 763.3 | 34 453.4 | 34 679.2 | 0.242 | 34 794.3 |
| 70.0 | 1 | 34.354 | ---- | 33.672 | 2.004 | 58.631 |
| | 2 | 2 144.43 | 1 801.43 | 2 137.92 | 0.304 | 2 178.22 |
| | 3 | 11 112.6 | 10 592.4 | 11 074.6 | 0.342 | 11 152.3 |
| | 4 | 33 893.8 | 33 459.1 | 33 799.5 | 0.278 | 33 935.9 |
| 72.0 | 1 | 19.603 | ---- | 18.915 | 3.567 | 45.590 |
| | 2 | 2 095.57 | 1 735.77 | 2 088.83 | 0.321 | 2 131.34 |
| | 3 | 11 011.2 | 10 464.7 | 10 971.8 | 0.358 | 11 053.2 |
| | 4 | 33 719.7 | 33 261.5 | 33 621.8 | 0.290 | 33 764.2 |
| 73.0 | 1 | 12.185 | ---- | 11.517 | 5.632 | 39.059 |
| | 2 | 2 071.11 | 1 702.82 | 2 064.43 | 0.322 | 2 107.88 |
| | 3 | 10 960.5 | 10 400.7 | 10 920.7 | 0.363 | 11 003.6 |
| | 4 | 33 632.5 | 33 097.8 | 33 535.9 | 0.287 | 33 678.3 |

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM          $\beta = 0.0$

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| | 1 | 4.740 | ---- | 4.101 | 14.457 | 32.521 |
| 74.0 | 2 | 2 046.62 | 1 669.78 | 2 039.78 | 0.334 | 2 084.43 |
| | 3 | 10 909.8 | 10 336.5 | 10 870.1 | 0.364 | 10 954.1 |
| | 4 | 33 545.5 | 32 997.1 | 33 445.2 | 0.299 | 33 592.5 |

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM       β = .25

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 5.0 | 1 | 454.229 | 454.011 | 454.138 | 0.020 | 454.321 |
|  | 2 | 3 630.54 | 3 629.90 | 3 629.87 | 0.018 | 3 630.71 |
|  | 3 | 14 246.5 | 14 245.5 | 14 243.2 | 0.022 | 14 246.7 |
|  | 4 | 39 300.1 | 39 298.6 | 39 291.9 | 0.021 | 39 300.3 |
| 10.0 | 1 | 407.474 | 403.672 | 407.270 | 0.050 | 407.853 |
|  | 2 | 3 456.93 | 3 449.07 | 3 455.61 | 0.038 | 3 457.59 |
|  | 3 | 13 874.8 | 13 859.6 | 13 868.4 | 0.046 | 13 875.6 |
|  | 4 | 38 655.9 | 38 641.5 | 38 641.2 | 0.038 | 38 656.8 |
| 15.0 | 1 | 360.266 | 352.036 | 359.978 | 0.080 | 361.146 |
|  | 2 | 3 282.68 | 3 265.48 | 3 280.76 | 0.058 | 3 284.16 |
|  | 3 | 13 502.7 | 13 469.4 | 13 493.3 | 0.069 | 13 504.5 |
|  | 4 | 38 011.2 | 37 979.6 | 37 989.0 | 0.058 | 38 013.2 |
| 20.0 | 1 | 312.568 | 297.357 | 312.215 | 0.113 | 314.184 |
|  | 2 | 3 107.77 | 3 078.04 | 3 105.32 | 0.079 | 3 110.44 |
|  | 3 | 13 130.0 | 13 072.4 | 13 117.7 | 0.094 | 13 133.2 |
|  | 4 | 37 366.1 | 37 310.7 | 37 335.5 | 0.082 | 37 369.5 |
| 25.0 | 1 | 264.339 | 241.495 | 263.913 | 0.161 | 266.951 |
|  | 2 | 2 932.19 | 2 881.39 | 2 929.19 | 0.102 | 2 936.39 |
|  | 3 | 12 756.1 | 12 669.1 | 12 741.7 | 0.119 | 12 761.9 |
|  | 4 | 36 720.5 | 36 635.3 | 36 684.8 | 0.097 | 36 725.8 |

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta$ = .25

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 30.0 | 1 | 215.531 | 181.248 | 215.058 | 0.220 | 219.427 |
|  | 2 | 2 755.92 | 2 684.91 | 2 752.53 | 0.123 | 2 762.01 |
|  | 3 | 12 383.3 | 12 259.8 | 12 365.3 | 0.145 | 12 390.5 |
|  | 4 | 36 074.4 | 35 947.8 | 36 030.6 | 0.121 | 35 082.0 |
| 35.0 | 1 | 166.090 | 117.249 | 165.609 | 0.289 | 171.591 |
|  | 2 | 2 578.95 | 2 485.09 | 2 575.09 | 0.149 | 2 587.29 |
|  | 3 | 12 009.3 | 11 824.7 | 11 988.7 | 0.171 | 12 019.1 |
|  | 4 | 35 427.8 | 35 258.3 | 35 376.8 | 0.144 | 35 438.2 |
| 40.0 | 1 | 115.952 | 45.459 | 115.458 | 0.426 | 123.422 |
|  | 2 | 2 401.27 | 2 266.53 | 2 397.11 | 0.173 | 2 412.22 |
|  | 3 | 11 634.8 | 11 399.8 | 11 611.7 | 0.198 | 11 647.6 |
|  | 4 | 34 780.9 | 34 546.6 | 34 724.5 | 0.162 | 34 794.4 |
| 45.0 | 1 | 65.043 | ---- | 64.531 | 0.789 | 74.892 |
|  | 2 | 2 222.86 | 2 033.24 | 2 218.17 | 0.211 | 2 236.78 |
|  | 3 | 11 259.9 | 10 969.9 | 11 234.2 | 0.228 | 11 276.1 |
|  | 4 | 34 133.4 | 33 841.8 | 34 070.1 | 0.185 | 34 150.5 |
| 49.0 | 1 | 23.705 | ---- | 23.192 | 2.186 | 35.791 |
|  | 2 | 2 079.62 | 1 823.99 | 2 074.83 | 0.230 | 2 096.16 |
|  | 3 | 10 959.7 | 10 601.7 | 10 932.1 | 0.251 | 10 978.9 |
|  | 4 | 33 615.2 | 33 227.0 | 33 546.8 | 0.203 | 33 635.4 |

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM          $\beta = .25$

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|----------|-------|------------------------------|------------------------------|--------------------------------------|----------------------|--------------------------|
| 50.0     | 1     | 13.279                       | ----                         | 12.767                               | 3.927                | 25.975                   |
|          | 2     | 2 043.73                     | 1 779.62                     | 2 038.75                             | 0.243                | 2 060.96                 |
|          | 3     | 10 884.7                     | 10 513.8                     | 10 856.5                             | 0.258                | 10 904.6                 |
|          | 4     | 33 485.6                     | 33 082.9                     | 33 415.6                             | 0.209                | 33 506.6                 |

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM                    β = .50

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 5.0 | 1 | 438.765 | 437.247 | 438.674 | 0.021 | 438.857 |
|  | 2 | 3 572.87 | 3 569.78 | 3 572.20 | 0.018 | 3 573.03 |
|  | 3 | 14 122.8 | 14 116.7 | 14 119.6 | 0.023 | 14 123.0 |
|  | 4 | 39 085.1 | 39 079.9 | 39 077.9 | 0.019 | 39 085.8 |
| 10.0 | 1 | 376.346 | 370.642 | 376.159 | 0.052 | 376.743 |
|  | 2 | 3 341.35 | 3 329.51 | 3 340.05 | 0.039 | 3 342.01 |
|  | 3 | 13 627.4 | 13 604.2 | 13 621.1 | 0.046 | 13 628.2 |
|  | 4 | 38 226.8 | 38 204.9 | 38 211.4 | 0.040 | 38 227.7 |
| 15.0 | 1 | 313.277 | 300.721 | 312.998 | 0.089 | 314.183 |
|  | 2 | 3 108.94 | 3 080.15 | 3 107.06 | 0.060 | 3 110.437 |
|  | 3 | 13 131.5 | 13 081.0 | 13 122.1 | 0.071 | 13 133.2 |
|  | 4 | 37 367.6 | 47 313.1 | 37 343.8 | 0.063 | 37 369.5 |
| 20.0 | 1 | 249.456 | 226.799 | 249.107 | 0.140 | 251.143 |
|  | 2 | 2 875.62 | 2 826.68 | 2 873.20 | 0.084 | 2 878.30 |
|  | 3 | 12 635.0 | 12 548.4 | 12 622.7 | 0.097 | 12 638.1 |
|  | 4 | 36 507.9 | 36 413.4 | 36 478.8 | 0.079 | 36 511.2 |
| 25.0 | 1 | 184.806 | 148.818 | 184.415 | 0.212 | 187.573 |
|  | 2 | 2 641.34 | 2 568.21 | 2 638.51 | 0.107 | 2 645.57 |
|  | 3 | 12 138.0 | 12 007.4 | 12 122.9 | 0.124 | 12 142.9 |
|  | 4 | 35 647.7 | 35 503.7 | 35 610.2 | 0.105 | 35 652.8 |

TABLE  II  -  BOUNDS  FOR  THE  EIGENVALUES  OF  THE  CLAMPED  BEAM          β = .50

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 30.0 | 1 | 119.229 | 66.432 | 118.810 | 0.351 | 123.422 |
|  | 2 | 2 406.08 | 2 292.07 | 2 402.80 | 0.136 | 2 412.22 |
|  | 3 | 11 640.6 | 11 458.7 | 11 622.7 | 0.153 | 11 647.6 |
|  | 4 | 34 787.0 | 34 555.9 | 34 745.1 | 0.120 | 34 794.4 |
| 35.0 | 1 | 52.603 | ---- | 52.188 | 0.791 | 58.631 |
|  | 2 | 2 169.82 | 2 000.18 | 2 166.18 | 0.167 | 2 178.22 |
|  | 3 | 11 142.7 | 10 874.9 | 11 122.3 | 0.182 | 11 152.3 |
|  | 4 | 33 925.9 | 33 570.4 | 33 875.6 | 0.148 | 33 935.9 |
| 37.9 | 1 | 13.422 | ---- | 13.003 | 3.165 | 20.733 |
|  | 2 | 2 032.32 | 1 806.94 | 2 028.44 | 0.190 | 2 042.19 |
|  | 3 | 10 853.7 | 10 546.2 | 10 831.8 | 0.201 | 10 864.9 |
|  | 4 | 33 426.2 | 33 016.6 | 33 372.7 | 0.160 | 33 437.9 |

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM          β = .75

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 5.0 | 1 | 423.275 | 421.095 | 423.173 | 0.024 | 423.368 |
|  | 2 | 3 515.16 | 3 510.73 | 3 514.51 | 0.018 | 3 515.33 |
|  | 3 | 13 999.1 | 13 990.4 | 13 995.9 | 0.023 | 13 999.3 |
|  | 4 | 38 871.1 | 38 862.9 | 38 864.1 | 0.018 | 38 871.3 |
| 10.0 | 1 | 345.128 | 336.790 | 344.945 | 0.053 | 345.522 |
|  | 2 | 3 225.64 | 3 208.86 | 3 224.36 | 0.040 | 3 226.29 |
|  | 3 | 13 379.9 | 13 346.6 | 13 373.6 | 0.047 | 13 380.7 |
|  | 4 | 37 797.8 | 37 766.3 | 37 783.4 | 0.038 | 37 798.6 |
| 15.0 | 1 | 266.015 | 247.405 | 265.745 | 0.101 | 266.951 |
|  | 2 | 2 934.90 | 2 894.78 | 2 933.03 | 0.063 | 2 936.39 |
|  | 3 | 12 760.2 | 12 688.5 | 12 750.8 | 0.073 | 12 761.9 |
|  | 4 | 36 724.0 | 36 655.3 | 36 700.9 | 0.063 | 36 725.8 |
| 20.0 | 1 | 185.806 | 153.039 | 185.486 | 0.172 | 187.573 |
|  | 2 | 2 642.88 | 2 575.69 | 2 640.56 | 0.087 | 2 645.57 |
|  | 3 | 12 139.9 | 12 018.2 | 12 127.6 | 0.100 | 12 142.9 |
|  | 4 | 35 649.6 | 35 516.0 | 35 620.7 | 0.081 | 35 652.8 |
| 25.0 | 1 | 104.342 | 51.289 | 103.995 | 0.333 | 107.287 |
|  | 2 | 2 349.54 | 2 237.66 | 2 346.74 | 0.119 | 2 353.78 |
|  | 3 | 11 519.0 | 11 337.2 | 11 504.04 | 0.129 | 11 523.8 |
|  | 4 | 34 574.8 | 34 344.0 | 34 537.7 | 0.107 | 34 579.7 |

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM   $\beta = .75$

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 30.0 | 1 | 21.426 | ---- | 21.064 | 1.703 | 25.9749 |
|  | 2 | 2 054.82 | 1 852.78 | 2 051.66 | 0.153 | 2 060.96 |
|  | 3 | 10 897.7 | 10 617.7 | 10 880.1 | 0.161 | 10 904.6 |
|  | 4 | 33 499.6 | 33 181.0 | 33 456.0 | 0.130 | 33 506.6 |
| 30.5 | 1 | 13.045 | ---- | 12.685 | 2.799 | 17.782 |
|  | 2 | 2 025.27 | 1 818.03 | 2 022.09 | 0.156 | 2 031.62 |
|  | 3 | 10 835.6 | 10 547.5 | 10 817.7 | 0.164 | 10 842.7 |
|  | 4 | 33 392.0 | 33 061.3 | 33 348.2 | 0.131 | 33 399.2 |

TABLE  II  –  BOUNDS  FOR  THE  EIGENVALUES  OF  THE  CLAMPED  BEAM                    $\beta$ = 1.00

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 5.0 | 1 | 407.759 | 404.794 | 407.663 | 0.023 | 407.853 |
|     | 2 | 3 457.42 | 3 451.38 | 3 456.75 | 0.019 | 3 457.59 |
|     | 3 | 13 875.4 | 13 863.4 | 13 872.2 | 0.023 | 13 875.6 |
|     | 4 | 38 656.6 | 38 645.4 | 38 648.7 | 0.020 | 38 656.8 |
| 10.0 | 1 | 313.783 | 302.702 | 313.598 | 0.059 | 314.184 |
|      | 2 | 3 109.78 | 3 084.36 | 3 108.52 | 0.040 | 3 110.44 |
|      | 3 | 13 132.5. | 13 087.2 | 13 126.2 | 0.048 | 13 133.2 |
|      | 4 | 37 368.7 | 37 325.8 | 37 354.4 | 0.038 | 37 369.5 |
| 15.0 | 1 | 218.460 | 192.281 | 218.202 | 0.117 | 219.427 |
|      | 2 | 2 760.52 | 2 700.07 | 2 758.67 | 0.066 | 2 762.01 |
|      | 3 | 12 388.9 | 12 292.5 | 12 379.5 | 0.075 | 12 390.5 |
|      | 4 | 36 080.3 | 35 975.2 | 36 058.3 | 0.061 | 36 082.0 |
| 20.0 | 1 | 121.567 | 74.571 | 121.262 | 0.250 | 123.422 |
|      | 2 | 2 409.53 | 2 295.25 | 2 407.26 | 0.094 | 2 412.22 |
|      | 3 | 11 644.6 | 11 482.5 | 11 632.4 | 0.104 | 11 647.6 |
|      | 4 | 34 791.4 | 34 611.8 | 34 761.8 | 0.085 | 34 794.4 |
| 22.0 | 1 | 82.309 | 22.824 | 82.001 | 0.374 | 84.628 |
|      | 2 | 2 268.63 | 2 135.06 | 2 266.17 | 0.108 | 2 271.90 |
|      | 3 | 11 346.8 | 11 154.9 | 11 333.5 | 0.116 | 11 350.4 |
|      | 4 | 34 275.6 | 34 051.7 | 34 243.6 | 0.093 | 34 279.3 |

TABLE  II  -  BOUNDS  FOR  THE  EIGENVALUES  OF  THE  CLAMPED  BEAM          $\beta$ = 1.00

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 25.0 | 1 | 22.822 | - - - - | 22.515 | 1.354 | 25.975 |
|  | 2 | 2 056.73 | 1 865.36 | 2 054.05 | 0.130 | 2 060.96 |
|  | 3 | 10 899.9 | 10 574.8 | 10 885.1 | 0.135 | 10 904.6 |
|  | 4 | 33 501.9 | 33 194.9 | 33 465.6 | 0.108 | 33 506.6 |
| 25.6 | 1 | 10.832 | - - - - | 10.502 | 3.086 | 14.173 |
|  | 2 | 2 014.26 | 1 814.86 | 2 011.50 | 0.136 | 2 081.71 |
|  | 3 | 10 810.6 | 10 472.3 | 10 795.1 | 0.143 | 10 815.4 |
|  | 4 | 33 347.2 | 33 026.9 | 33 310.1 | 0.111 | 33 352.1 |

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM

β = 1.50

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 5.0 | 1 | 376.647 | 371.774 | 376.549 | 0.026 | 376.743 |
| | 2 | 3 341.85 | 3 331.81 | 3 341.18 | 0.020 | 3 342.01 |
| | 3 | 13 628.0 | 13 607.9 | 13 624.8 | 0.023 | 13 628.2 |
| | 4 | 38 227.5 | 38 208.8 | 38 220.1 | 0.019 | 38 227.7 |
| 10.0 | 1 | 250.726 | 231.953 | 250.551 | 0.069 | 251.143 |
| | 2 | 2 877.65 | 2 836.71 | 2 876.39 | 0.044 | 2 878.30 |
| | 3 | 12 637.4 | 12 563.1 | 12 631.1 | 0.050 | 12 638.1 |
| | 4 | 36 510.5 | 36 430.0 | 36 494.9 | 0.043 | 36 511.2 |
| 15.0 | 1 | 122.384 | 80.317 | 122.166 | 0.174 | 123.422 |
| | 2 | 2 410.73 | 2 316.69 | 2 408.97 | 0.071 | 2 412.22 |
| | 3 | 11 646.1 | 11 490.9 | 11 636.8 | 0.079 | 11 647.6 |
| | 4 | 34 792.9 | 34 621.3 | 34 770.9 | 0.063 | 34 794.4 |
| 17.0 | 1 | 70.251 | 15.309 | 70.006 | 0.349 | 71.644 |
| | 2 | 2 223.15 | 2 090.54 | 2 221.21 | 0.087 | 2 225.07 |
| | 3 | 11 249.4 | 11 032.7 | 11 238.9 | 0.092 | 11 251.33 |
| | 4 | 34 105.7 | 33 859.8 | 34 081.1 | 0.072 | 34 107.6 |
| 18.0 | 1 | 43.993 | - - - - | 43.747 | 0.561 | 45.590 |
| | 2 | 2 129.18 | 1 984.01 | 2 127.15 | 0.095 | 2 131.34 |
| | 3 | 11 051.0 | 10 812.0 | 11 039.9 | 0.099 | 11 053.2 |
| | 4 | 33 752.1 | 33 489.6 | 33 736.1 | 0.076 | 33 764.2 |

TABLE  II  -  BOUNDS  FOR  THE  EIGENVALUES  OF  THE  CLAMPED  BEAM        $\beta$ = 1.50

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| | 1 | 17.599 | ---- | 17.348 | 1.432 | 19.422 |
| 19.0 | 2 | 2 035.09 | 1 850.68 | 2 032.97 | 0.104 | 2 037.49 |
| | 3 | 10 852.6 | 10 574.2 | 10 841.0 | 0.106 | 10 885.1 |
| | 4 | 33 418.4 | 33 070.8 | 33 390.8 | 0.082 | 33 420.8 |

TABLE  II  -  BOUNDS  FOR  THE  EIGENVALUES  OF  THE  CLAMPED  BEAM                 β = 2.00

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 5.0 | 1 | 345.425 | 343.790 | 345.341 | 0.024 | 345.521 |
|  | 2 | 3 226.13 | 3 221.23 | 3 225.48 | 0.020 | 3 226.29 |
|  | 3 | 13 380.6 | 13 372.4 | 13 377.3 | 0.024 | 13 380.7 |
|  | 4 | 37 798.5 | 37 785.5 | 37 790.6 | 0.021 | 37 798.6 |
| 10.0 | 1 | 187.138 | 158.651 | 186.972 | 0.088 | 187.573 |
|  | 2 | 2 644.93 | 2 577.60 | 2 643.70 | 0.046 | 2 645.57 |
|  | 3 | 12 142.3 | 12 032.7 | 12 136.0 | 0.052 | 12 142.9 |
|  | 4 | 35 652.3 | 35 532.4 | 35 637.5 | 0.041 | 35 652.8 |
| 12.0 | 1 | 122.762 | 80.012 | 122.576 | 0.151 | 123.422 |
|  | 2 | 2 411.29 | 2 305.74 | 2 409.85 | 0.059 | 2 412.22 |
|  | 3 | 11 646.7 | 11 494.8 | 11 639.2 | 0.064 | 11 674.6 |
|  | 4 | 34 793.6 | 34 625.7 | 34 775.8 | 0.051 | 34 794.4 |
| 14.0 | 1 | 57.682 | ---- | 57.474 | 0.360 | 58.6312 |
|  | 2 | 2 176.94 | 2 018.56 | 2 175.32 | 0.075 | 2 178.22 |
|  | 3 | 11 151.1 | 10 928.2 | 11 142.3 | 0.078 | 11 152.3 |
|  | 4 | 33 934.9 | 33 681.5 | 33 914.1 | 0.061 | 33 935.9 |
| 15.0 | 1 | 24.852 | ---- | 24.652 | 0.806 | 25.975 |
|  | 2 | 2 059.50 | 1 883.64 | 2 057.77 | 0.083 | 2 060.96 |
|  | 3 | 10 903.3 | 10 652.4 | 10 893.9 | 0.085 | 10 904.6 |
|  | 4 | 33 595.4 | 33 218.8 | 33 483.6 | 0.065 | 33 506.6 |

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM    $\beta = 2.00$

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 15.3 | 1 | 14.954 | ---- | 14.757 | 1.388 | 16.142 |
| | 2 | 2 024.23 | 1 843.09 | 2 022.46 | 0.087 | 2 025.75 |
| | 3 | 10 828.9 | 10 569.5 | 10 819.4 | 0.087 | 10 830.3 |
| | 4 | 33 376.6 | 33 079.7 | 33 353.8 | 0.068 | 33 377.8 |

TABLE  II  -  BOUNDS  FOR  THE  EIGENVALUES  OF  THE  CLAMPED  BEAM          β  =  -.25

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 10.0 | 1 | 469.394 | 467.968 | 469.193 | 0.043 | 469.760 |
|  | 2 | 3 687.69 | 3 684.62 | 3 686.35 | 0.036 | 3 688.35 |
|  | 3 | 14 369.5 | 14 364.0 | 14 363.1 | 0.044 | 14 370.3 |
|  | 4 | 39 513.9 | 39 508.5 | 39 498.7 | 0.038 | 39 514.8 |
| 20.0 | 1 | 437.367 | 431.682 | 436.975 | 0.090 | 438.857 |
|  | 2 | 3 570.41 | 3 558.22 | 3 567.84 | 0.072 | 3 573.03 |
|  | 3 | 14 119.7 | 14 097.8 | 14 107.3 | 0.088 | 14 123.0 |
|  | 4 | 39 082.3 | 39 060.8 | 39 051.9 | 0.078 | 39 085.8 |
| 40.0 | 1 | 370.529 | 347.856 | 369.834 | 0.188 | 376.743 |
|  | 2 | 3 331.41 | 3 283.41 | 3 326.61 | 0.144 | 3 342.01 |
|  | 3 | 13 615.1 | 13 529.4 | 13 591.1 | 0.176 | 13 628.2 |
|  | 4 | 38 213.6 | 38 129.0 | 38 156.6 | 0.149 | 38 227.7 |
| 60.0 | 1 | 299.567 | 244.237 | 298.689 | 0.293 | 314.184 |
|  | 2 | 3 086.33 | 2 949.91 | 3 079.92 | 0.207 | 2 110.44 |
|  | 3 | 13 103.8 | 12 914.7 | 13 069.2 | 0.263 | 13 133.2 |
|  | 4 | 37 337.9 | 37 149.8 | 37 257.5 | 0.215 | 37 369.5 |
| 80.0 | 1 | 223.878 | 116.025 | 222.877 | 0.448 | 251.143 |
|  | 2 | 2 835.04 | 2 625.53 | 2 826.75 | 0.292 | 2 878.30 |
|  | 3 | 12 585.8 | 12 256.2 | 12 541.8 | 0.349 | 12 638.1 |
|  | 4 | 36 455.1 | 36 081.7 | 36 344.4 | 0.303 | 36 511.2 |

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM   $\beta = -.25$

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|----------|-------|------------------------------|------------------------------|--------------------------------------|----------------------|--------------------------|
| 100.0    | 1     | 142.697                      | - - - -                      |                                      | - - -                | 187.573                  |
|          | 2     | 2 577.45                     | 2 255.24                     |                                      | 13.33                | 2 645.57                 |
|          | 3     | 12 061.2                     | 11 555.8                     |                                      | 4.25                 | 12 142.9                 |
|          | 4     | 35 565.4                     | 34 990.1                     |                                      | 1.63                 | 35 652.8                 |

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM    β = -.50

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 10.0 | 1 | 500.205 | 499.105 | 499.992 | 0.042 | 500.564 |
|      | 2 | 3 802.89 | 3 800.44 | 3 801.55 | 0.035 | 3 803.54 |
|      | 3 | 14 616.8 | 14 613.5 | 14 610.4 | 0.044 | 14 617.6 |
|      | 4 | 39 942.9 | 39 938.6 | 39 926.8 | 0.040 | 39 943.8 |
| 20.0 | 1 | 499.129 | 494.724 | 498.723 | 0.081 | 500.564 |
|      | 2 | 3 800.94 | 3 791.15 | 3 798.33 | 0.068 | 3 803.54 |
|      | 3 | 14 614.4 | 14 601.4 | 14 601.8 | 0.086 | 14 617.6 |
|      | 4 | 39 940.2 | 39 923.2 | 39 909.6 | 0.077 | 39 943.8 |
| 40.0 | 1 | 494.821 | 477.099 | 494.061 | 0.153 | 500.564 |
|      | 2 | 3 793.17 | 3 763.95 | 3 788.27 | 0.129 | 3 803.54 |
|      | 3 | 14 604.6 | 14 552.6 | 14 580.3 | 0.166 | 14 617.6 |
|      | 4 | 39 929.6 | 39 861.4 | 39 874.9 | 0.137 | 39 943.8 |
| 60.0 | 1 | 487.621 | 447.372 | 486.533 | 0.223 | 500.564 |
|      | 2 | 3 780.22 | 3 591.84 | 3 773.26 | 0.184 | 3 803.54 |
|      | 3 | 14 588.3 | 14 471.2 | 14 552.9 | 0.243 | 14 617.6 |
|      | 4 | 39 912.0 | 39 758.5 | 39 829.7 | 0.206 | 39 943.8 |
| 80.0 | 1 | 477.550 | 405.005 | 476.125 | 0.288 | 500.564 |
|      | 2 | 3 762.09 | 3 604.65 | 3 753.56 | 0.227 | 3 803.54 |
|      | 3 | 14 565.6 | 14 357.2 | 14 519.8 | 0.314 | 14 617.6 |
|      | 4 | 39 887.3 | 39 14.4 | 39 779.1 | 0.271 | 39 943.8 |

TABLE   II  -  BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM             $\beta$ = -.50

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 100.0 | 1 | 464.419 | 349.230 | 462.905 | 0.326 | 500.564 |
|  | 2 | 3 738.80 | 3 492.14 | 3 728.65 | 0.271 | 3 803.54 |
|  | 3 | 14 536.3 | 14 210.1 | 14 481.2 | 0.379 | 14 617.6 |
|  | 4 | 39 855.6 | 39 429.1 | 39 726.5 | 0.324 | 39 943.8 |
| 125.0 | 1 | 443.820 | 259.076 | 442.025 | 0.405 | 500.564 |
|  | 2 | 3 702.43 | 3 315.45 | 3 689.41 | 0.352 | 3 893.54 |
|  | 3 | 14 490.7 | 13 981.0 | 14 423.9 | 0.461 | 14 617.6 |
|  | 4 | 39 806.0 | 39 139.5 | 39 640.2 | 0.417 | 39 943.8 |

TABLE   II   -   BOUNDS   FOR   THE   EIGENVALUES   OF   THE   CLAMPED   BEAM            β = -.75

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 10.0 | 1 | 530.922 | 529.723 | 530.701 | 0.042 | 531.274 |
|  | 2 | 3 917.96 | 3 915.17 | 3 916.53 | 0.036 | 3 918.60 |
|  | 3 | 14 864.1 | 14 859.8 | 14 857.6 | 0.043 | 14 864.9 |
|  | 4 | 40 371.8 | 40 366.3 | 40 355.9 | 0.039 | 40 372.7 |
| 20.0 | 1 | 560.511 | 555.692 | 560.062 | 0.080 | 561.893 |
|  | 2 | 4 030.98 | 4 020.35 | 4 028.28 | 0.067 | 4 033.54 |
|  | 3 | 15 108.8 | 15 091.5 | 15 096.1 | 0.084 | 15 112.1 |
|  | 4 | 40 798.1 | 40 768.6 | 40 768.9 | 0.072 | 40 801.7 |
| 40.0 | 1 | 617.532 | 599.287 | 616.673 | 0.139 | 622.874 |
|  | 2 | 4 252.96 | 4 212.01 | 4 247.82 | 0.121 | 4 263.06 |
|  | 3 | 15 593.4 | 15 523.6 | 15 568.6 | 0.158 | 15 606.3 |
|  | 4 | 41 645.3 | 41 566.3 | 41 585.6 | 0.143 | 41 659.4 |
| 60.0 | 1 | 671.896 | 634.041 | 670.667 | 0.183 | 683.530 |
|  | 2 | 4 469.68 | 4 380.84 | 4 462.22 | 0.167 | 4 492.12 |
|  | 3 | 16 071.4 | 15 819.7 | 16 035.1 | 0.225 | 16 100.2 |
|  | 4 | 42 485.3 | 42 309.6 | 42 399.3 | 0.203 | 42 517.0 |
| 80.0 | 1 | 723.829 | 655.591 | 722.256 | 0.217 | 743.883 |
|  | 2 | 4 681.35 | 4 521.11 | 4 671.69 | 0.206 | 4 720.74 |
|  | 3 | 16 542.9 | 16 294.4 | 16 495.6 | 0.286 | 16 494.0 |
|  | 4 | 43 318.2 | 43 016.2 | 43 203.9 | 0.264 | 43 374.4 |

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM          $\beta = -.75$

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 100.0 | 1 | 773.522 | 671.565 | 770.979 | 0.329 | 803.955 |
| | 2 | 4 888.16 | 4 646.52 | 4 864.31 | 0.489 | 4 948.93 |
| | 3 | 17 008.0 | 16 648.5 | 16 936.9 | 0.419 | 17 087.5 |
| | 4 | 44 144.0 | 43 687.5 | 43 861.5 | 0.642 | 44 231.7 |
| 125.0 | 1 | 832.739 | 683.535 | 829.534 | 0.386 | 878.676 |
| | 2 | 5 140.13 | 4 790.86 | 5 110.31 | 0.582 | 5 233.58 |
| | 3 | 17 580.6 | 17 009.6 | 17 493.0 | 0.499 | 17 704.0 |
| | 4 | 45 166.2 | 44 443.2 | 44 827.8 | 0.752 | 45 303.1 |
| 150.0 | 1 | 888.985 | 696.494 | 885.218 | 0.425 | 953.016 |
| | 2 | 5 385.16 | 4 896.41 | 5 350.45 | 0.646 | 5 517.62 |
| | 3 | 18 143.7 | 17 377.3 | 18 040.9 | 0.568 | 18 230.2 |
| | 4 | 46 177.5 | 45 167.2 | 45 775.5 | 0.874 | 46 374.2 |

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM            $\beta = -1.00$

| $\alpha$ | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problems | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 10.0 | 1 | 561.548 | 559.521 | 561.338 | 0.037 | 561.893 |
|  | 2 | 4 032.90 | 4 028.51 | 4 031.55 | 0.033 | 4 033.54 |
|  | 3 | 15 111.2 | 15 103.9 | 15 104.9 | 0.042 | 15 112.1 |
|  | 4 | 40 800.7 | 40 792.7 | 40 784.9 | 0.039 | 40 801.7 |
| 20.0 | 1 | 621.541 | 613.689 | 621.114 | 0.069 | 622.874 |
|  | 2 | 4 260.54 | 4 243.43 | 4 257.90 | 0.062 | 4 263.06 |
|  | 3 | 15 603.1 | 15 577.1 | 15 590.7 | 0.079 | 15 606.3 |
|  | 4 | 41 655.9 | 41 623.8 | 41 608.6 | 0.114 | 41 659.4 |
| 40.0 | 1 | 738.893 | 709.280 | 738.076 | 0.111 | 743.883 |
|  | 2 | 4 710.92 | 4 642.44 | 4 705.95 | 0.105 | 4 720.74 |
|  | 3 | 16 581.3 | 16 482.1 | 16 557.3 | 0.145 | 16 594.0 |
|  | 4 | 43 360.5 | 43 233.6 | 43 308.4 | 0.120 | 43 374.4 |
| 60.0 | 1 | 853.195 | 793.191 | 851.688 | 0.177 | 863.763 |
|  | 2 | 5 155.16 | 5 001.03 | 5 147.73 | 0.144 | 5 176.70 |
|  | 3 | 17 552.5 | 17 339.1 | 17 513.6 | 0.222 | 17 580.8 |
|  | 4 | 45 057.6 | 44 775.5 | 44 901.7 | 0.347 | 45 088.8 |
| 80.0 | 1 | 964.891 | 857.790 | 963.176 | 0.178 | 982.652 |
|  | 2 | 5 593.76 | 5 332.86 | 5 584.65 | 0.163 | 5 631.07 |
|  | 3 | 18 516.9 | 18 153.4 | 18 471.8 | 0.244 | 18 566.5 |
|  | 4 | 46 747.3 | 46 251.4 | 46 584.8 | 0.348 | 46 802.6 |

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM       β = -1.00

| α | Order | Upper Bound by Rayleigh-Ritz | Lower Bound by Kato's Method | Lower Bound by Intermediate Problem | Gap/Average Per Cent | Lumped Constant End Load |
|---|---|---|---|---|---|---|
| 100.0 | 1 | 1 074.33 | 917.851 | 1 071.61 | 0.253 | 1 100.66 |
|  | 2 | 6 027.13 | 5 637.97 | 6 015.50 | 0.193 | 6 083.95 |
|  | 3 | 19 474.6 | 18 929.7 | 19 407.2 | 0.346 | 19 551.3 |
|  | 4 | 48 429.6 | 47 680.0 | 48 255.9 | 0.359 | 48 515.6 |
| 125.0 | 1 | 1 208.37 | 975.666 | 1 204.80 | 0.296 | 1 247.09 |
|  | 2 | 6 562.08 | 5 998.45 | 6 548.45 | 0.209 | 6 648.11 |
|  | 3 | 20 662.6 | 19 837.7 | 20 578.1 | 0.411 | 20 780.8 |
|  | 4 | 50 522.1 | 49 425.7 | 50 209.0 | 0.626 | 50 655.8 |
| 150.0 | 1 | 1 339.76 | 1 019.69 | 1 336.44 | 0.249 | 1 392.44 |
|  | 2 | 7 090.13 | 6 269.69 | 7 075.31 | 0.209 | 7 210.27 |
|  | 3 | 21 881.0 | 20 689.1 | 21 763.0 | 0.358 | 22 008.6 |
|  | 4 | 52 603.5 | 50 963.6 | 52 352.5 | 0.478 | 52 794.9 |
| 175.0 | 1 | 1 468.84 | 1 051.54 | 1 464.90 | 0.269 | 1 536.86 |
|  | 2 | 7 611.88 | 6 679.71 | 7 594.8 | 0.224 | 7 770.58 |
|  | 3 | 23 010.1 | 21 434.2 | 22 919.3 | 0.396 | 23 234.9 |
|  | 4 | 54 673.8 | 52 581.9 | 54 546.7 | 0.233 | 54 9328 |
| 200.0 | 1 | 1 595.89 | 1 072.52 | 1 592.01 | 0.243 | 1 680.44 |
|  | 2 | 8 127.84 | 6 940.16 | 8 110.06 | 0.219 | 8 329.17 |
|  | 3 | 24 170.5 | 22 240.3 | 24 084.8 | 0.355 | 24 459.6 |
|  | 4 | 56 733.4 | 54 163.9 | 56 511.6 | 0.392 | 57 069.4 |