AFWAL-TM-81-76-FIMM

The Flow of Marking Particles for a Two Dimensional Source/Sink and a Two Dimensional Potential Vortex

March 1981

George L. Seibert

Approved for public release; distribution unlimited

**FLIGHT DYNAMICS LABORATORY
AIR FORCE WRIGHT AERONAUTICAL LABORATORIES
WRIGHT-PATTERSON AIR FORCE BASE, OHIO 45433**

AFWAL-TM-81-76-FIMM

The Flow of Marking Particles for a Two Dimensional Source/Sink and a Two Dimensional Potential Vortex

March 1981

George L. Seibert

Trisonic Experimental Group
Aerodynamics and Airframe Branch

Foreword

This Technical Memorandum was prepared by Dr. George L. Seibert of the Trisonic Experimental Group, Aerodynamics and Airframe Branch, Aeromechanics Division, Flight Dynamics Laboratory, Air Force Wright Aeronautical Laboratories, Wright-Patterson Air Force Base, Ohio. The work was performed December 1980 through February 1981 under Project Number 24041048.

Lowell C Keel

LOWELL C. KEEL, MAJ, USAF
Chief, Aerodynamics and
Airframe Branch
Aeromechanics Division

Table of Contents

Abstract

The flow of marker particles in a two dimensional source/sink and a two dimensional potential vortex are theoretically calculated. The flow is considered to be defined by a complex potential and is limited to steady, inviscid, incompressible, irrotational flow. The particles are assumed to be spherical and not to interact with each other. The gas and dust streamlines are determined for two cases; one, the particles follow the gas streamlines, analogous to no dust in the gas, and secondly, when the particles are free to move independently.

Introduction

Following the work of Saffman,[2] Catalano[1] took the equations of motion of a dusty gas given by Saffman as

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^{\,2}} + K_N\ (v_i - u_i) \tag{1}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2}$$

$$M\left(\frac{\partial u_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = K\ (u_i - v_i) \tag{3}$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x_j}\ (NU_i) = 0 \tag{4}$$

and developed the equations of motion in a general sense for a first order perturbation of a dusty gas. The concentration of dust is considered to be small, i.e. to first order. The resulting equation for the velocity of the dusty gas is written as

$$\bar{V} = \text{grad}\ \left(\phi - \frac{\tau}{2}\ (\text{grad}\ \phi)^2\right) \tag{5}$$

where ϕ is the potential of the flow and τ is the time relaxation parameter defined as

$$\tau = \frac{M}{K} \tag{6}$$

M is the mass of the dust particle and K is the Stokes coefficient of resistance where Stokes flow requires that the particle Reynolds number be ≤ 1 based on particle size and the relative velocity between the dust and gas. τ has the physical significance that as $\tau \to 0$, the gas and dust particle motion becomes more identical.

Based on this development and the previously stated assumptions, the motion of a dusty gas may be calculated for any flow where a potential function may be defined. In most applicable fluid flow situations, this potential is mathematically complicated and the solution requires numerical integration to determine the streamlines of the flow. Catalano[1] calculated the case for a dusty gas jet impinging against a two dimensional wall normal to the flow. This note will calculate the flow pattern for 2 simple types of flow; namely a source/sink flow and a potential vortex.

Case Studies

<u>Case 1.</u> Source/Sink Flow

For the case of a source/sink flow the complex function defining this flow is

$$F = \frac{c}{2\pi} \ln z \tag{6}$$

where $z = re^{i\theta}$

Then

$$F = \frac{c}{2\pi} [\ln r + i\theta], \quad (r = \sqrt{x^2 + y^2})$$

with the equipotential lines

$$\phi = \text{Re } F(z) = \frac{c}{2\pi} \ln r = \text{const} \tag{7}$$

which are concentric circles, and the streamlines

$$\psi = \text{Im } F(z) = \frac{c\theta}{2\pi} = \text{const}, \quad (\theta = \arg z) \tag{8}$$

are straight lines through the origin. z=0 is a singular point of the flow where the solution is not valid, the fluid "disappearing" in the case of a sink (c negative real) and "emanating" in the case of a source (c positive real).

For the case of a dusty gas we consider the dust velocity equation given by Michael[4] and derived formally by Catalano[1] as

$$\bar{V} = \text{grad } (\phi - \frac{1}{2} \tau (\text{grad } \phi)^2) \quad (5)$$

Then

$$\text{grad } \phi = \frac{c}{2\pi} [\frac{x}{r^2} \hat{e}_1 + \frac{y}{r^2} \hat{e}_2]$$

and

$$(\text{grad } \phi)^2 = \frac{c^2}{4\pi^2} [\frac{x^2}{r^4} + \frac{y^2}{r^4}]$$

then

$$\phi - \frac{1}{2}\tau\ (\text{grad}\ \phi)^2 = \frac{c}{2\pi}\ (\ell n\ r - \frac{\tau c}{4\pi r^2})$$

and

$$\text{grad}\ (\phi_o - \frac{1}{2}\tau\ (\text{grad}\ \phi)^2) =$$

$$\frac{c}{2\pi}\ [\frac{x}{r^2} + \frac{\tau c}{2\pi}\frac{x}{r^4}]\ \hat{e}_1 + \frac{c}{2\pi}\ [\frac{y}{r^2} + \frac{\tau c^2}{4\pi^2}\frac{y}{r^4}]\ \hat{e}_2$$

then

$$V_x = \frac{c}{2\pi}\frac{x}{r^2}\ [1 + \frac{\tau c}{2\pi r^2}] \tag{9}$$

$$V_y = \frac{c}{2\pi}\frac{y}{r^2}\ [1 + \frac{\tau c}{2\pi r^2}] \tag{10}$$

The equation of the streamline is defined as:

$$\frac{dy}{dx} = \frac{V_y}{V_x}$$

and substituting for V_y and V_x gives

$$\frac{dy}{dx} = \frac{\frac{cy}{2\pi r^2}\ [1 + \frac{\tau c}{2\pi r^2}]}{\frac{cx}{2\pi r^2}\ [1 + \frac{\tau c}{2\pi r^2}]}$$

then

$$\frac{dy}{dx} = \frac{y}{x}$$

and the equation of the streamline is just

$$y = Ax.$$

The streamline of the dusty gas is independent of τ and, as expected, identical to the original streamline of a gas with no dust. From the case with no dust where the streamfunction is given by

$$\psi = \frac{c\theta}{2\pi} = \text{const}$$

we determine the constant A to be

$$A = \tan \frac{2\pi\psi}{c}$$

and the equation for both the gas and dust streamline becomes

$$y = (\tan \frac{2\pi\psi}{c})\ X \tag{11}$$

This equation is shown in figure 1. The magnitude of the dust velocity along the streamlines is

$$|V| = \frac{c}{2\pi r} \ [1 + \frac{\tau c}{2\pi r^2}] \tag{12}$$

As $\tau \to 0$, this reduces to the velocity of the dust free gas.

From Marble[3] who defined the "velocity range" of a particle as

$\lambda = \tau U_o$ where U_o is the local characteristic velocity of the system,

we write

$$U_o \equiv U_{gas} = \frac{c}{2\pi r}$$

then

$$\lambda = \frac{\tau c}{2\pi r}$$

and equation 12 may be written as

$$|V| = U_{gas} \ [1 + \frac{\lambda}{r}] \tag{13}$$

For the Stokes flow assumption to be valid $\frac{\lambda}{r} << 1$ in which case the particle motion depends on the local gas flow conditions. When $\frac{\lambda}{r} \sim 1$, the particle has a "memory of events" that took place prior to its transition into the Stokes flow regime, i.e. conditions where significant flow changes took place. For the source/sink flow example to be considered here this region is close to the origin and the critical radius is shown in Figure 1.

For the case where $\frac{\lambda}{r} >> 1$, the particle is not influenced by the local gas flow conditions (analogous to a bullet fired from a rifle). Equation 13 for the particle velocity is not valid in this case; however, in any system where the velocity range, λ, if the particle is much greater than the characteristic dimension, r, of the system, the particle will pass through the system virtually unaffected, regardless of the form of the drag on the particle.

For c positive real (source flow) with $r>0$, the dust velocity exceeds the gas velocity for any finite r with the velocity $\rightarrow 0$ as $r \rightarrow \infty$ where the dust velocity exactly "equals the flow velocity".

For the particular case of a 1 μm dust particle in air with $T=520^{\circ}R$, the $Re \leq 1$ limitation allows a relative velocity of $\sim$15m/sec between the dust and gas before the Stokes' flow assumption becomes invalid.

Using a value of $C = \pm 10\ m^2/sec$ for the source/sink flow the initial values of $\frac{\lambda}{r}$ were calculated. For $r>0$ ($\neq 0$) the deviation between the gas and dust velocity decreases until at r=1.25 cm ($\frac{\lambda}{r} = .12$) the Stokes flow limitation is reached. This illustrates that the dust in the flow from the source achieved its velocity from other than Stokes flow effects. The difference between the gas and dust velocity is $\sim$12% at this point decreasing to less than a 1% difference at r=5 cm ($\frac{\lambda}{r} \sim 7.6 \times 10^{-3}$).

For a sink of the same strength the dust velocity lags the gas velocity by the same ratio as it exceeded the gas velocity from the source. A change in the strength c will shift the applicable Stokes flow region closer to or farther from the origin.

Depending on the strength $\pm$ c any LV measurement errors in a source/sink type flow would be limited to regions where $\frac{\lambda}{r} > 1$, outside the Stokes flow regime.

Case 2. Potential Vortex

A potential vortex is, in general, irrotational except for a singularity at the origin where the rotation is infinite, therefore the original limitation that the flow be irrotational is not violated.

The complex potential function for this flow may be written as

$$F(z) = \frac{-i\Gamma}{2\pi} \ln z \tag{14}$$

where Γ is the circulation around the vortex.

Then

$$F(z) = \frac{i\Gamma}{2\pi} [\ln z + i\theta]$$

and the equipotential lines

$$\phi = \mathrm{Re}F(z) = \frac{\Gamma\theta}{2\pi} \tag{15}$$

are straight lines through the origin.

The streamlines of the flow are given by

$$\psi = \mathrm{Imag}\ F(z) = \frac{-\Gamma}{2\pi} \ln r \tag{16}$$

which are concentric circles around the origin.

Recalling the dust velocity equation;

$$\bar{V} = \mathrm{grad}\ (\phi - \frac{\tau}{2} (\mathrm{grad}\ \phi)^2) \qquad (5)$$

and writing the grad in cylindrical polar coordinates gives;

$$\mathrm{grad}\ \Phi = \frac{\partial}{\partial r} (\frac{\Gamma\theta}{2\pi})\ \hat{e}_1 + \frac{1}{r}\frac{\partial}{\partial\theta} (\frac{\Gamma\theta}{2\pi})\ \hat{e}_2 = \frac{\Gamma}{2\pi r}\ \hat{e}_2$$

$$(\mathrm{grad}\ \Phi)^2 = \frac{\Gamma^2}{4\pi^2 r^2}$$

then

$$\bar{V}_i = \mathrm{grad}\ (\frac{\Gamma\theta}{2\pi} - \frac{\tau}{2} (\frac{\Gamma^2}{4\pi^2 r^2})\ )$$

where τ is defined as before.

Then

$$V_i = \frac{\partial}{\partial r}\left[\left(\frac{\Gamma\theta}{2\pi} - \frac{\tau}{2}\left(\frac{\Gamma 2}{4\pi^2 r^2}\right)\right]\hat{e}_1 + \frac{1}{r}\frac{\partial}{\partial\theta}\left[\left(\frac{\Gamma\theta}{2\pi} - \frac{\tau}{2}\left(\frac{\Gamma 2}{4\pi^2 r^2}\right)\right]\hat{e}_2$$

$$V_i = \frac{\tau\Gamma^2}{4\pi^2 r^3}\hat{e}_1 + \frac{\Gamma}{2\pi r}\hat{e}_2$$

or

$$V_r = \frac{\tau\Gamma^2}{4\pi r^2} \tag{17}$$

$$V_\theta = \frac{\Gamma}{2\pi r} \tag{18}$$

and the streamlines of the flow are

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{V_r}{V_\theta}$$

and

$$\frac{dr}{d\theta} = \frac{\tau\Gamma}{2\pi r}$$

or

$$r = \sqrt{\frac{\tau\Gamma\theta}{\pi}} + C$$

Returning to equation of streamline with no dust, (Eq 16)

$$\psi = -\frac{\Gamma}{2\pi}\ln r$$

where

$$r = \frac{r}{L}$$

with L a unit length gives

$$r = L\,e^{-\frac{2\pi\psi}{\Gamma}} = \text{const.}$$

for given streamline and circulation Γ.

Therefore, the equation of the dust streamline becomes

$$r = \sqrt{\frac{\tau \Gamma \theta}{\pi}} + L\, e^{-\frac{2\pi\psi}{\Gamma}} \qquad (19)$$

with r having the dimensions consistent with the circulation and unit length L. The dimensions on Γ are as $\frac{L^2}{t}$. When $\tau \to 0$, the equation reduces to the streamline for a dust free gas.

Equation 19 is plotted in Figure 2 for the 1 μm diameter dust particle considered in the source/sink flow with ψ and Γ chosen to simplify the calculation. The potential increases by the value Γ each time we travel around the vortex, therefore we will consider the range of θ from 0 to 2π only.

The magnitude of the dust velocity is given by

$$|V| = \frac{\Gamma}{2\pi r} \sqrt{1 + \frac{\Gamma^2 \tau^2}{4\pi^2 r^4}}$$

reducing to the gas speed as $\tau \to 0$.

Analogous to the development for the source/sink flow, we may cast the dust speed equation for the potential vortex flow as

$$|V| = U_{gas} \left(1 + \left(\frac{\lambda}{r}\right)^2\right)^{\frac{1}{2}} \qquad (20)$$

For $\frac{\lambda}{r} >> 1$ (close to the origin of the vortex) the particle speed will be much greater than the gas speed, and the Stokes flow solution will not be valid. For the 1 μm particle in air with $\Gamma = 10m^2/sec$ and ψ chosen for convenience, $\frac{\lambda}{r}$ may be calculated defining the region where the Stokes flow solution is valid. The Stokes limit for this case is reached when $\frac{\lambda}{r} = .37$ ($\frac{\lambda}{r} \sim 1$) and is shown in Figure 2. Practically speaking, the dust speed quickly adjusts to the gas speed and the Stokes flow solution governs the flow pattern for $r > r_s$ where for $\frac{\lambda}{r} \sim .04$ ($\frac{\lambda}{r} << 1$) the relative speed difference is less than 1%. The region where $\frac{\lambda}{r} \sim 1$ governs a region of the vortex flow where the particle motions are influenced by their previous history, in this case the vortex "start" since Γ is an instantaneous, constant value at any θ. Figure 2 shows this effect as the particle radius is increased by the component of particle velocity in the r direction as θ increases from zero. This motion would not be the case for flow where the particle was translating uniformly with the gas and the flow suddenly turned; the particle would continue in a straight line reacting to the flow as a function of $\frac{\lambda}{r}$. Thus with this solution, we can not simulate a flow with entrained dust particles traveling with zero velocity relative to the gas and then turning into a vortex. This solution would be valid for a vortex generating source flow under the constraints of the original assumptions.

Conclusions

The solution for the dust streamlines for two related, potential, fluid flows is presented in closed form, considering a first order solution to the equations of motion with dust concentration small (i.e. no collisions between particles) and the Stokes flow approximation valid. It is seen that the relative velocity difference between the particles and gas is dependent on their history prior to the Stokes flow evaluation. This is a constraint of the solution since both flows are required to exist at the start of the solution imposing a prior history. The parameter $\frac{\lambda}{r}$ defines the interaction of the particles and the gas, characterizing the entire flow regime, with λ defining the relaxation parameter and local characteristic velocity and r a characteristic length for the flow.

LV measurements of similar flows would give accurate velocity measurements where the Stokes solution was valid becoming increasingly suspect when $\frac{\lambda}{r} \geq 1$. For the potential vortex, the particles spiral radially outward but in the Stokes flow regime they match the local gas velocity at any point in the flow. If particles were not continually added at the vortex center, however, the "core" region would not be measurable with an LV system.

References

1. Catalano, G., "The Flow of Marking Particles in a Two Dimensional Stagnation Region", AFFDL-TM-78-21, December 1977.

2. Saffman, P. G., "On the Stability of Laminar Flow of a Dusty Gas", 1962, J. Fluid Mechanics, 13, 120.

3. Marble, F. E., "Dynamics of a Gas-Containing Small Solid Particles", Fifth AGARD Colloquium on Combustion and Propulsion, 1962.

4. Michael, D. H., "The Steady Motion of a Sphere in a Dusty Gas", J. Fluid Mechanics, 1968.

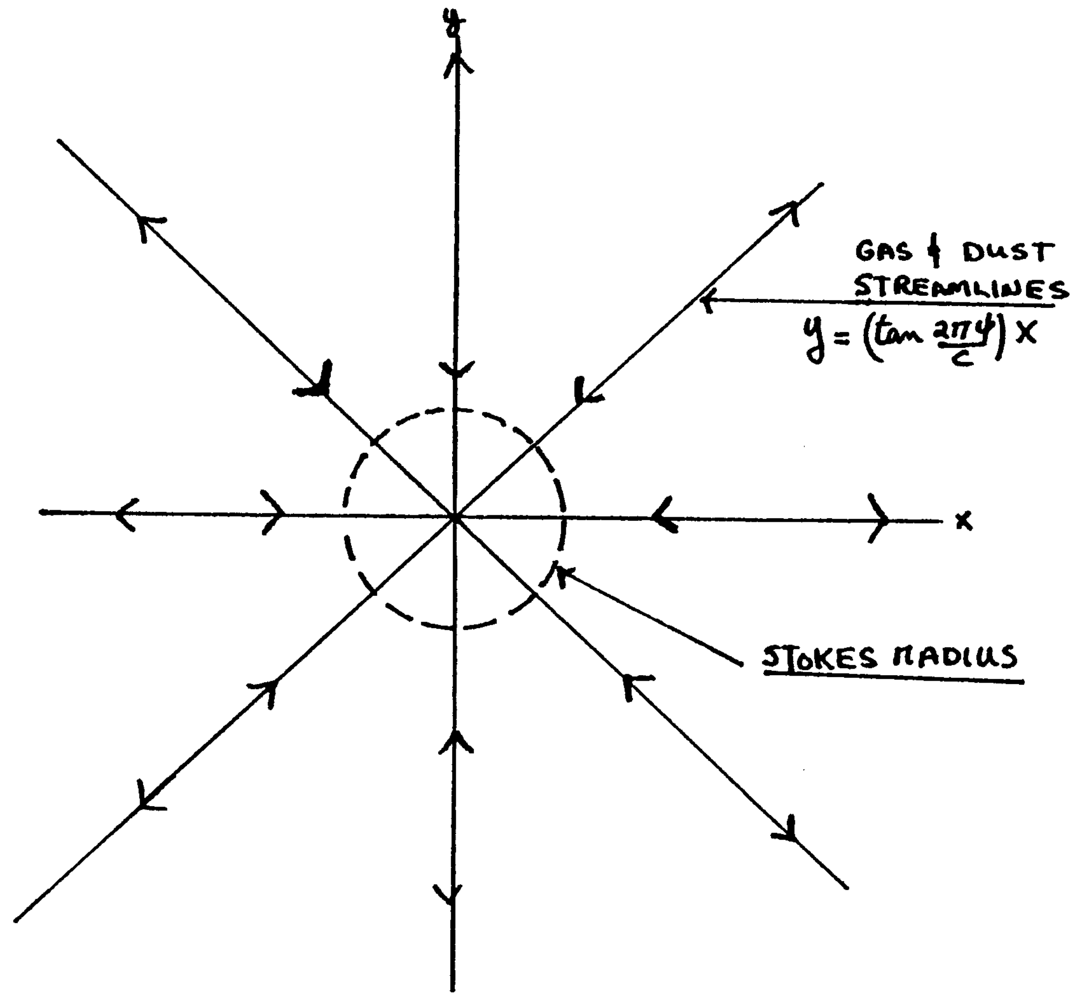

Figure 1. - GAS & DUST STREAMLINES;
SOURCE/SINK

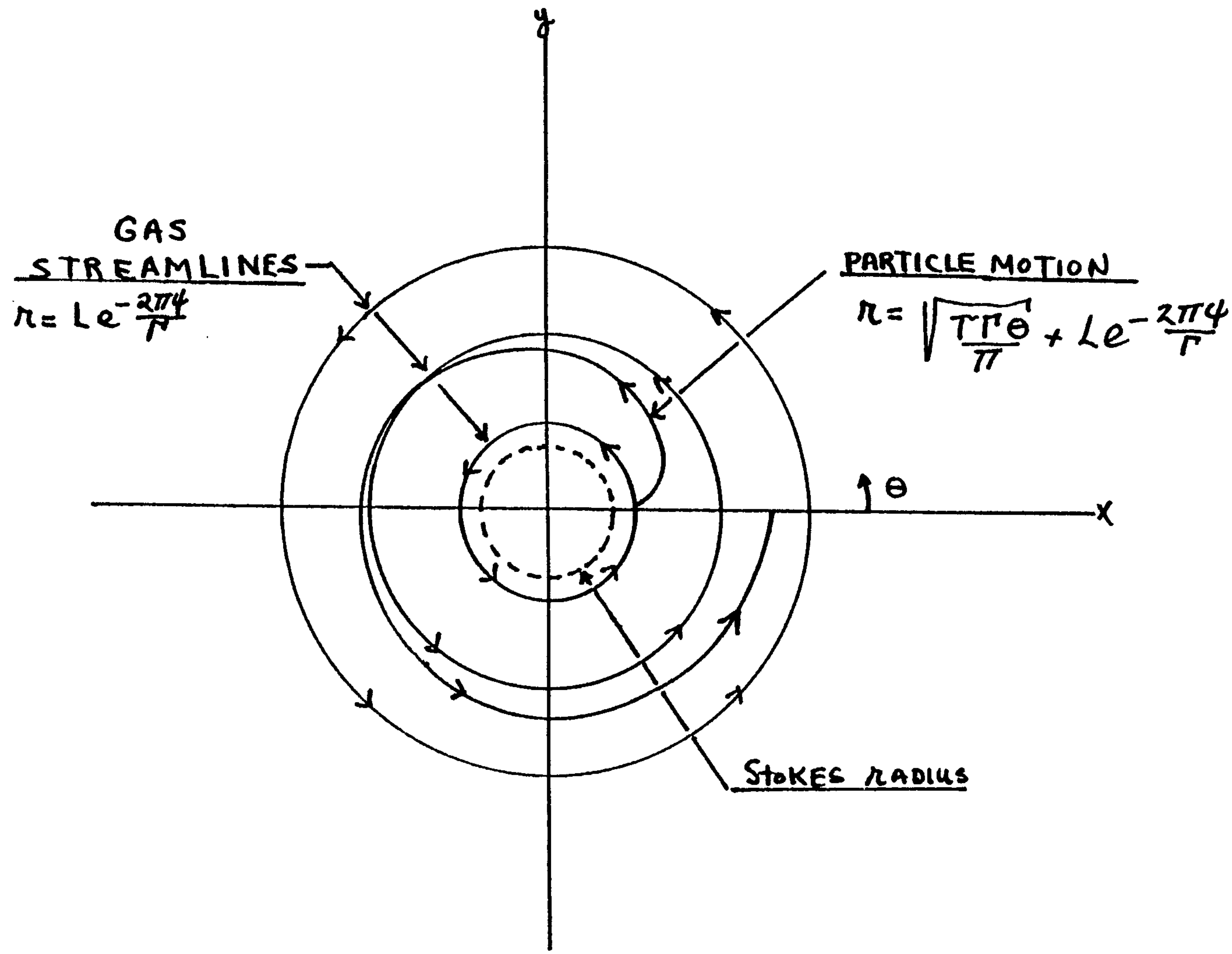

Figure 2 - GAS & DUST STREAMLINES; POTENTIAL VORTEX