AN INVESTIGATION OF THE PENETRATION OF PRESSURE

BELOW THE SURFACE OF A COMPRESSIBLE FLUID

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Assume that pressure is exerted at a certain point
O on the free surface of a compressible fluid. This is
then propagated along this surface in the form of a shock
wave (Fig. 1). We will assume that $R(t)$ describes the
propagation of the wave front and the distribution of
pressure on the surface and is a known but arbitrary
function of time. The pressure distribution in the depths
of the fluid is then found with the aid of a quadrature
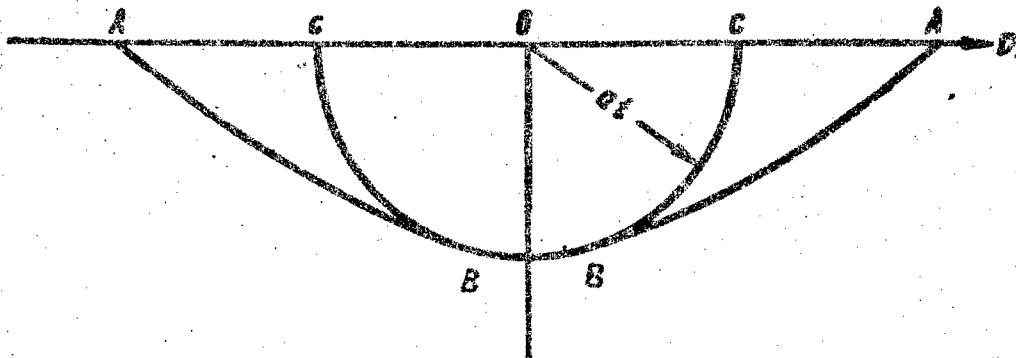/1/.

Fig. 1

1. Pressure distribution along a head wave.

In cases where the velocity of the front is greater
than the velocity of sound in the fluid, the disturbed
region will be bounded by the characteristic arc (the

head wave) AB and a circle of radius at (Fig. 1). Let us select an axis Or in the plane of the free surface and an axis Oz perpendicular to Or.

In the region ACB the pressure distribution takes the form:

$$p(r_0,\ z,\ t)=\frac{1}{\pi}\int\limits_{\psi_1}^{\psi}\int\limits_{r_1^*}^{r_2^*}\frac{(r,\ t')\,zr\,dr\,d\psi}{(z^2+r^2+r_0^2-2rr_0\cos\psi)^{3/2}}+\frac{1}{\pi a}\int\limits_0^{\psi_1}\int\limits_{r_2^*}^{r_2^*}\frac{\frac{\partial p_1(r,\ t')}{\partial t}\,zr\,dr\,d\psi}{z^2+r^2+r_0^2-2rr_0\cos\psi}+$$

$$+\frac{1}{\pi a}\int\limits_0^{\psi_1}\frac{p_1(r_2^*,\ t_*')\,zr_2^*\frac{dr_2^*}{dt}}{z^2+r_0^2+r_2^{*2}-2r_0r_2^*\cos\psi}\,d\psi-\frac{1}{\pi a}\int\limits_0^{\psi_1}\frac{p_1(r_1^0,\ t_*')\,zr_1^*\frac{dr_1^*}{dt}}{z^2+r_0^2+r_1^{*2}-2r_0r_1^*\cos\psi}\,d\psi. \quad (1,1)$$

where $r_{1,2}^*$ are the roots of the equation

$$r_{1,2}^*=R\left(t-\frac{\sqrt{r_{1,2}^{*2}+r_0^2-2r_{1,2}^*r_0\cos\psi+z^2}}{a}\right), \quad (1,2)$$

$$t_*'=f(r_1^*);\quad t_*''=f(r_2^*),$$

$f(x)$ is the inverse function of R(t), ψ_1 is the value of ψ for which $r_1^*(\psi_1)=r_2^*(\psi_1)=j_0^*$. Then ψ_1 and f_0^* are found from the equations

$$r_0^*=R\left(t-\frac{\sqrt{r_0^{*2}+r_0^2-2r_0r_0^*\cos\psi_1+z^2}}{a}\right),$$

$$1+\frac{1}{a}R'\left(t-\frac{\sqrt{r_0^{*2}+r_0^2-2r_0^*r_0\cos\psi_1+z^2}}{a}\right)\frac{r_0^*-r_0\cos\psi_1}{\sqrt{r_0^2+r_0^{*2}-2r_0r_0^*\cos\psi_1+z^2}}=0. \quad (1,3)$$

The head wave may be expressed parametrically as the envelope of the elementary disturbances arising at a

point on the free surface at the instant of time t' at which the wave front reaches it.

$$[r_0 - R(t')]^2 + z^2 = a^2(t-t')^2,$$
$$R'(t')[R(t') - r_0] = a^2(t'-t). \qquad (1,4)$$

The Eqs. (1.3) and (1.4) constitute a system of equations for the determination of r_0^*, ψ_1, t' and z as functions of r_0 and t. It is obvious that in this case it is possible to set $\psi_1 = 0$ and

$$t' = t - \frac{\sqrt{r_0^2 + r_0^{*2} - 2r_0^* r_0 + z^2}}{a}.$$

Then the Eqs. (1.4) will follow from (1.3) and for the determination of r_0^*, and z there will remain the two Eqs. (1.3). Thus we have ascertained that $\psi_1 = 0$ on AB. Substituting $\psi_1 = 0$ in Eqs. (1.3) we obtain equations for r_0^* and the equation of the line AB. Let us denote r_0^* on the head wave as $\bar{r}_0^*$. It is clear that this quantity may be found from the equations

$$\bar{r}_0^* = R\left[t - \frac{\sqrt{(\bar{r}_0^* - r_0)^2 + z^2}}{a}\right],$$

which
$$1 + \frac{1}{a} R'\left[t - \frac{\sqrt{(\bar{r}_0^* - r_0)^2 + z^2}}{a}\right] \frac{r_0^* - r_0}{\sqrt{(\bar{r}_2^* - r_0)^2 + z^2}} = 0, \qquad (1,5)$$

which serve at the same time for the determination of the equation for AB: $z = z(r_0, t)$. In particular, if $R' = \text{const} = V_0$, then

$$\bar{r}^* = \frac{r_0 - \frac{at}{M}}{1 - \frac{1}{M^2}},$$

where $M \equiv V_0/a$. After substituting this value of r_0^* in the first of Eqs. (1.5), we get the equation of the head wave front:

$$z = \frac{V_0 t - r_0}{\sqrt{M^2 - 1}}.$$

Let us now find from Eq. (1.2) an expression for $dr_{1,2}^*/dt$, which appears under the integral sign in Formula (1.1):

$$\frac{dr_{1,2}^*}{dt} = \frac{R'\left(t - \frac{\sqrt{r_{1,2}^{*2} + r_0^2 - 2r_0 r_{1,2}^* \cos\psi + z^2}}{a}\right)}{1 + \frac{1}{a} R'\left(t - \frac{\sqrt{r_{1,2}^{*2} + r_0^2 - 2r_0 r_{1,2}^* \cos\psi + z^2}}{a}\right)\frac{r_{1,2}^* - r_0 \cos\psi}{\sqrt{r_{1,2}^{*2} + r_0^2 - 2r_0 r_{1,2}^* \cos\psi + z^2}}}$$

$$(1.6)$$

From Eqs. (1.6) and (1.3) it is clear that at the upper limit ($\psi = \psi_1$) $dr_{1,2}^*/dt$ becomes infinite.

All the integrals in (1.1), with the exception of the last two, vanish on the head wave AB. In the two exceptions the interval of the integration degenerates to a point while the integrand goes to infinity. From this it follows from among other things that in the case where the boundary condition is given in the form:

$$p_1(r, t) = p_A(R) f_1(r/R),$$

where R(t) is the radius of the front at the time t, then the following equations will hold:

$$p_1(r_2^*, t_a') = p_A(r_2^*),$$
$$p_1(r_1^*, t_a') = p_A(r_1^*),$$

that is, the pressure distribution $p(r_0, z, t)$ along the head wave AB does not depend upon the pressure distribution behind the front $f_1(r/R)$.
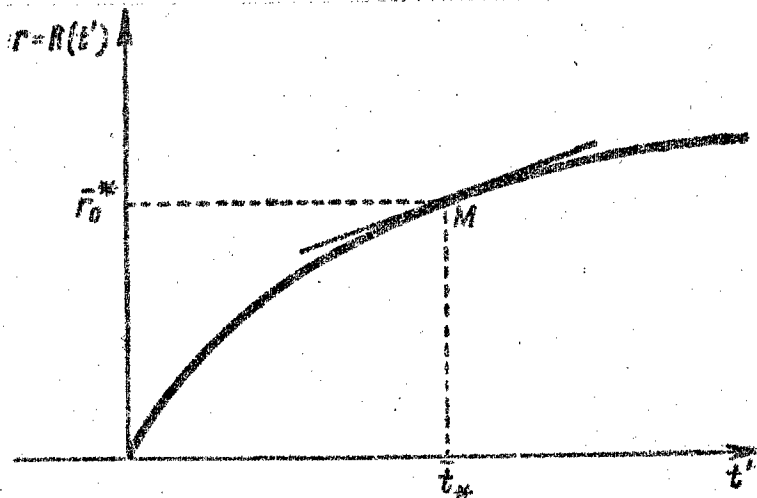

Fig. 2

To determine the pressure distribution over the head wave, in the general case, we must find the limit approached by the last two integrals in Formula (1.1) as we move to AB. We note that the entire continuous part of the integrand may be taken from under the integral sign after setting

$$\psi = 0, \quad r_1^* = r_2^* = \bar{r}_0^*; \quad t_0' = t_0' = \bar{t}_*.$$

in it. Thus the problem ultimately becomes one of finding the limit of the expression:

$$\frac{1}{\pi a} \frac{p_1(\bar{r}_0^*, \bar{t}_*)}{z^2 + (\bar{r}_0^* - r_0)^2} z\bar{r}_0 \int_0^{\psi_1} \frac{d}{dt}(r_2^* - r_2^*) \, d\psi_0 \qquad (1.7)$$

where $\vec{r}_0^*$ and $z = z(r_0, t)$ are determined by Eqs. (1.5). It is clear from (1.6) and (1.3) that only the function $R(t')$ itself and its first derivative $R'(t')$ enter into the integral in Formula (1.7). Aside from this, the values of the arguments of the functions R and R' in (1.7) approach $\overline{t}_*$ as $\psi_1 \to 0$, that is, for every r_0, z and t belonging to AB.

Let us consider the graph of $R(t')$ (Fig. 2). From Eq. (1.5) we find $\vec{r}_0^*$ and $\overline{t}_*$, for fixed values of r_0, z, and t on AB.

In the neighborhood of $r = \vec{r}_0^*$ and $t' = \overline{t}_*$ we substitute the equation of the tangent at the point M for the curve $r = R(t')$. This equation takes the form:

$$r - \overline{r}_0^* = R'(\overline{t}_*)(t' - t_*)$$

The equation for the determination of $r_{1,2}^*$ now takes the form:

$$r_{1,2}^* - \overline{r}_0^* = R'(\overline{t}_*)\left(t - \frac{\sqrt{r_{1,2}^{*2} + r_0^2 - 2 r_0 r_{1,2}^* \cos\psi + z^2}}{a} - \overline{t}_* \right).$$

We obtain an expression for $r_{1,2}^*$ by solving this equation:

$$r_{1,2}^* = \frac{M^2 r_0 \cos\psi - \overline{r}_1 \pm \sqrt{(M^2 r_0 \cos\psi - \overline{r}_1)^2 - [M^2(r_0^2 + z^2) - \overline{r}_1^2](M^2 - 1)}}{M^2 - 1}, \quad (1.8)$$

where

$$M = \frac{R'(\overline{t}_*)}{a}, \quad \overline{r}_1 = r_0^* + R'(\overline{t}_*)(t - \overline{t}_*).$$

We have the following expression for ψ_1:

$$\psi_1 = \arccos \frac{\bar{r}_1 + \sqrt{(M^2 - 1)[M^2(r_0^2 + z^2) - \bar{r}_1^2]}}{M^2 r_0}.$$

When differentiating (1.8) over time it is necessary to take into account the dependence of $\bar{t}_*$ on t. In performing this differentation it is not difficult to see that the integrand will consist of some continuous function, which we will discard, plus a fraction. The numerator of this fraction is also a continuous function, but the denominator is a radical which also enters into Eq. (1.8), and which vanishes when $\psi = \psi_1$. As has been previously noted, the continuous part of the integrand may be taken from under the integral sign on setting $\psi = 0$ in it. Thus the entire matter reduces to the calculation of the integral:

$$\int_0^{\psi_1} \frac{d\psi}{\sqrt{(M^2 r_0 \cos \psi - \bar{r}_1)^2 - [M^2(r_0^2 + z^2) - \bar{r}_1^2](M^2 - 1)}}.$$

Substituting in the last integral $\cos \psi \approx 1 - \psi^2/2$, as is possible because of the smallness of ψ, and calculating the resulting tabular integral, we finally have along AB:

$$p(r_0, z, t) = \frac{p_1(\bar{r}_0^*, \bar{t}_*)\sqrt{M^2 r_0 - \bar{r}_1}}{a\sqrt{r_0 M^2}\sqrt{M^2 - 1}} \left[\frac{d\bar{r}_1}{dt} - \frac{M \frac{dM}{dt}(\bar{r}_1 - r_0)}{M^2 - 1} \right]. \qquad (1,9)$$

where $\quad M = \dfrac{R'(\bar{t}_*)}{a}; \quad \bar{r}_1 = \bar{r}_0^* + R'(\bar{t}_*)(t - \bar{t}_*),$

and the pressure distribution along the head wave is given in terms of the elementary functions.

On the free surface $z = 0$, $\overline{r}_0^* = R(t)$, and $p(r_0, z, t,) = p_1(R, t)$. If $p_1 = $ const and $R'(t) = $ const. $= V_0$, we have for the pressure distribution along AB:

$$p(r_0, z, t) = p_1 \sqrt{\frac{M^2 r_0 - V_0 t}{r_0(M^2 - 1)}}. \tag{1,10}$$

The Expression (1.10) was derived by A. Ya. Sagomonyan by the method of characteristics.

2. An Investigation of the Solution in a Half Space

It is interesting to investigate various particular cases of the definition of the law giving the pressure distribution on the surface of the fluid. For simplicity we will assume that the pressure distribution to be uniform on the liquid surface behind the front, but to vary in time: $p_1(r, t) = p_A(R)$, where $R = R(t)$ is the equation of motion of the front over the surface.

The simplest form of pressure distribution along the axis Oz below the surface will be

$$p(0, z, t) = -\frac{\partial}{\partial t} \int_0^{r_0^*} \frac{p_1(r, t') r \, dr}{\sqrt{r^2 + z^2}}, \tag{2,1}$$

where

$$r_0^a = R\left(t - \frac{\sqrt{r_0^{*2} + z^2}}{a}\right); \qquad t' = t - \frac{\sqrt{r^2 + z^2}}{a}.$$

We will assume that $p_1(r,t) = p_A(R) = C/R^\alpha$, where C and α are certain constants, and $R(t) = Bt^\beta$ for the law of motion of the front. Then we obtain the pressure distribution along Oz in the form:

$$p(0,z,t) = -\frac{C}{aB^a}\left[\frac{z + r_0^a \frac{\partial r_0^a}{\partial z}}{\left(t - \frac{\sqrt{r_0^{*2} + z^2}}{a}\right)^{a\beta}\sqrt{r_0^{*2} + z^2}} - \frac{1}{\left(t - \frac{z}{a}\right)^{a\beta}}\right]. \qquad (2,2)$$

From the conservation equations it follows that $\beta = 2/(2 + \alpha)$ on the front at the point A. The greatest interest centers about the behavior of the solution when $z \to at$. When this happens it is clear that $r^*_0 \to 0$. Evaluating the indeterminate form in (2.2) it is easy to convince ourselves that, for example, with $\alpha = 3$, (the pressure at $z = at$) (at the front of the sound wave) becomes negatively infinite. Thus in those cases where $p_1(r,t)$ has a singularity close to the front of the sound wave as $t \to 0$, the linear theory ceases to apply. Calculations show that even when this singularity is absent, negative pressures are possible below the surface. But this appears to have a definite physical meaning. Calculation also shows that zones of strong rarefaction occur when the pressure on the free surface drops sharply with time, and that they are distributed between the free surface of the liquid and regions of higher pressure

close to the sound-wave front. This result is somewhat unexpected if it is kept in mind that when P_1 and $R' = V_0$ are constant in time, the pressure is essentially monotonic and positive; here, for example, when $V_0 = a$ the pressure changes according to the linear equation:

$$\frac{p(0, z, t)}{p_1} = 1 - \frac{z}{at}. \tag{2,3}$$

In order to illustrate the above considerations, we will investigate the following example. Let the pressure distribution on the surface behind the front be constant while the pressure on the front changes according to the law illustrated in Fig. 3.
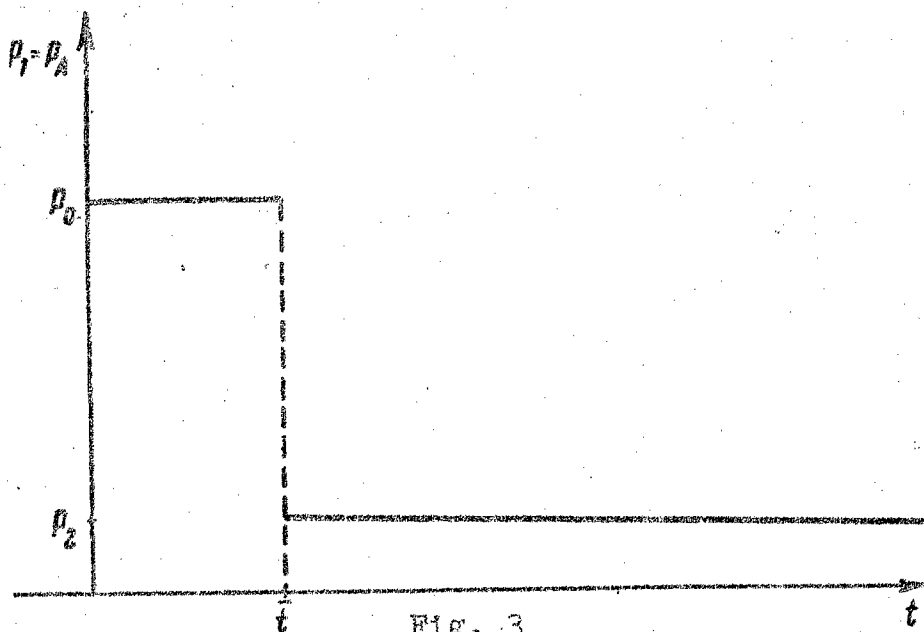
Fig. 3

Here $\bar{t}$ is some instant of time. Let the pressure p_0 correspond to a front velocity $V_0 = a$ and the pressure p_2 to $V_2 < a$.

In accordance with (2.1) and the law adopted for $p_1(r,t)$, there results:

$$p(0, z, t) = -a \frac{\partial}{\partial t} \int_{t - \frac{\sqrt{r_0^{*2} + z^2}}{a}}^{t - z/a} p_A[R(t')] dt'. \qquad (2,4)$$

Clearly for $t < \bar{t}$ we have t' that much smaller than t and all integration intervals in (2.4) lie to the left of $t' = \bar{t}$ in Fig. 3. For such values of t Eq. (2.3) holds.

We will investigate two different cases for $t < \bar{t}$.
1) $z > a(t - \bar{t})$; here $t' < \bar{t}$ also and Eq. (2.3) holds;
2) $z < a(t - \bar{t})$; here $t - z/a > t$, but for some interval on the z-axis we have $t - \sqrt{r_0^{*2} + z^2}/a < \bar{t}$.
Then (2.4) takes the form:

$$p(0, z, t) = -a \frac{\partial}{\partial z} \int_{t - \frac{\sqrt{r_0^{*2} + z^2}}{a}}^{\bar{t}} p_0 \, dt' - a \frac{\partial}{\partial z} \int_{\bar{t}}^{t - \frac{z}{a}} p_2 \, dt' = -p_0 \frac{z}{at} + p_2. \qquad (2,5)$$

At the point $z = a(t - \bar{t})$ there is a pressure discontinuity:

$$p(0, z, t) = -p_0 + p_0 \frac{\bar{t}}{t} + p_2,$$

and we take $\bar{t} \ll t$ and $p_2 \ll p_0$, then it is clear that $p(0,z,t) = 0$. Thus the step-function pressure distribution on the free surface produces a nonmonotonic pressure distribution differing from (2.3) below the surface, and

in certain cases produces zones of rarefaction.

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