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An extended Laha-Lukacs characterization result revisited *

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Dedicated to Professor C.R. Rao on his 80th birthday

Abstract

Recently Fosam and Shanbhag (1997) gave an extended version of the Laha-Lukacs characterization result based on a regression property, subsuming the Letac-Mora characterizations of the natural exponential families of distributions with variances as cubic functions of means. In the present note, we provide a new approach based on functional equations to arrive at the Fosam-Shanbhag result.

Key words: Laha-Lukacs result; Morris and Letac-Mora characterizations; Exponential families; Power series distributions; Polya-Eggenbeger distribution; Inverse Gaussian distribution; Regression property; Bhattacharyya matrices; Functional equation; Wet period of a dam.

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1 Introduction

Laha and Lukacs (1960) proved, under some mild conditions, that if $X_1, X_2, ..., X_n$, where $n \geq 2$, are independent identically distributed non-degenerate square-integrable random variables, then a quadratic expression of X_i's has its regression on $\sum_{j=1}^{n} X_j$ to be a quadratic function of $\sum_{j=1}^{n} X_j$ almost surely if and only if for some real $\alpha \neq 0$ and β, $\alpha X_1 + \beta$

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is either normal or Poisson or gamma or binomial or negative binomial or Meixner (hyperbolic cosine). Indeed, the main theme of the Laha-Lukacs result is contained in, essentially, the following specialized version of it:

Let $X_1, ..., X_n$, where $n \geq 2$, be independent identically distributed (iid) non-degenerate random variables. Then, with a, b, c as real,

$$E\{X_1^2 - aX_1X_2 - bX_1 | \sum_{i=1}^{n} X_i\} = c \quad a.s. \tag{1}$$

holds if and only if for some real $\alpha \neq 0$ and β, $\alpha X_1 + \beta$ is either normal or Poisson or gamma or binomial or negative binomial or Meixner, depending upon what a, b and c are. (A more precise statement of the result in this latter case will be met later as Theorem 3.1 in the present paper.) This specialized version of the Laha-Lukacs result gives, amongst other things, as a corollary, the Morris (1982) characterization of the class of natural exponential families where the members have their variances as quadratic functions of the corresponding means.

Rao and Shanbhag (1994) have given an elementary technique, based mainly on moments, to identify the class of distributions for which (1) holds. Considering $X_1, ..., X_n$ as iid non-degenerate random variables, where $n \geq 3$, and using an extended version of the Rao-Shanbhag argument, Fosam and Shanbhag (1997) have identified the class of distributions for which

$$E\{X_1^2 - aX_1X_2 - bX_1 - dX_1X_2X_3 | \sum_{i=1}^{n} X_i\} = c \quad a.s. \tag{2}$$

with a,b, c, d as real holds. The Fosam-Shanbhag result, in turn, yields as corollary, the Letac-Mora (1990) characterization of natural exponential families where variances are cubic functions of the means; Rao and Shanbhag (1994) and Fosam and Shanbhag (1997) cite some further literature that is linked with this result.

The purpose of the present short note is to provide an approach based on a functional equation of the type met in storage theory, in conjunction with that based on moments, to arrive at the Fosam-Shanbhag result. (The reader is advised to familiarise himself with the notation in Fosam and Shanbhag (1997) in order to understand the contents of the present note.)

2 Some auxiliary results

Lemma 2.1 *Let T be an open interval containing zero as one of its points and $\phi : T \longrightarrow R$ be a twice differentiable function (everywhere) with $\phi(0) = 0$ and $\phi''(t) > 0$ for all $t \in T$, where ϕ'' is the second derivative of ϕ. Then*

$$\phi''(t) = \alpha + \beta\phi'(t) + \gamma(\phi'(t))^2, \quad t \in T, \tag{3}$$

with α, β and γ as real numbers, ϕ' as the first derivative of ϕ, and ϕ'' as defined above, if and only if one of the following holds:

(i) $\beta = \gamma = 0$ *and* $\phi(t) = \mu t + \alpha\frac{t^2}{2}$, $t \in T$, *with* $\alpha > 0$ *and* μ *as a real number.*

(ii) $\beta \neq 0$,$\gamma = 0$ *and* $\phi(t) = -\lambda + \lambda e^{\beta t} - \frac{\alpha}{\beta}t$, $t \in T$, *with* λ *as a positive real number.*

(iii) $\gamma > 0$, $4\alpha\gamma = \beta^2$, *and* $\phi(t) = -\gamma^{-1}\log(1-\lambda t) - \frac{\beta}{2\gamma}t$, $t \in T$, *with* λ *as a non-zero real number,* $T \subset (-\infty, \lambda^{-1})$ *if* $\lambda > 0$ *and* $T \in (\lambda^{-1}, \infty)$ *if* $\lambda < 0$.

(iv) $\gamma < 0$ *and there exists a positive real number* λ *such that* $4\alpha\gamma = \beta^2 - \lambda^2$, *and* $\phi(t) = (-\gamma)^{-1}\log(1-p+pe^{\lambda t}) - \frac{\beta-\lambda}{2\gamma}t$, $t \in T$, *with* $p \in (0,1)$.

(v) $\gamma > 0$ *and there exists a non-zero real number* λ *such that* $4\alpha\gamma = \beta^2 - \lambda^2$, *and* $\phi(t) = \gamma^{-1}\{\log(1-p) - \log(1-pe^{\lambda t})\} - \frac{\beta-\lambda}{2\gamma}t$, $t \in T$, *with* $p \in (0,1)$, $T \subset (-\infty, \frac{-logp}{\lambda})$ *if* $\lambda > 0$ *and* $T \subset (\frac{-logp}{\lambda}, \infty)$ *if* $\lambda < 0$.

(vi) $\gamma > 0$ *and there exists a positive real number* λ *such that* $4\alpha\gamma = \beta^2 + \lambda^2$, *and* $\phi(t) = \gamma^{-1}\{\log(\cos\eta) - \log(\cos(\eta + \frac{\lambda}{2}t))\} - \frac{\beta}{2\gamma}t$, $t \in T$, *with* η *as real number lying in* $(-\frac{\pi}{2}, \frac{\pi}{2})$, $T \subset ((-\frac{\pi}{2} - \eta)/\lambda, (\frac{\pi}{2} - \eta)/\lambda)$.

Lemma 2.1 given above follows via a straightforward argument, and hence we shall not deal with its proof here.

Remark 2.2 In each of the cases (i)-(vi) of Lemma 2.1, $\exp\{\phi(t)\}$ has a Taylor series expansion about the origin in a neighbourhood of the origin. If we denote the coefficient of $\frac{t^m}{m!}$ by μ_m for each $m = 0, 1, ...$, then we have that $\{\mu_m\}$ is such that

$$\sum_{r=0}^{m}\binom{m}{r}\{\mu_{r+2}\mu_{m-r} - (\gamma+1)\mu_{r+1}\mu_{m-r+1} - \beta\mu_{r+1}\mu_{m-r} - \alpha\mu_r\mu_{m-r}\} = 0 \quad m = 0, 1, ... \quad (4)$$

implying that $\{\mu_m\}$ is determined given $(\alpha, \beta, \gamma, \mu_1)$ such that $\alpha + \beta\mu_1 + \gamma\mu_1^2 > 0$. Indeed, it is now clear that given $(\alpha, \beta, \gamma, \mu_1)$ such that $\alpha + \beta\mu_1 + \gamma\mu_1^2 > 0$, the sequence $\{\mu_m\}$ satisfying (4) is indeed the sequence $\{\mu_m\}$ realtive to the Taylor series expansion of $\exp\{\phi(t)\}$, in a neighbourhood of the origin. This is the moment sequence relative to a distribution that is determined uniquely by the corresponding moment sequence unless $\gamma < 0$ with γ^{-1} as a non-integer real number; in the case with $\gamma < 0$ and γ^{-1} as a non-integer real number, we have $\exp\{\phi(t+t_0) - \phi(t_0)\}$, $t \in I - t_0$, for some neighbourhood I of the origin and $t_0 \in I$, as the restriction to I of the Laplace transform of a signed measure that is not a non-negative measure, and hence the corresponding $\{\mu_m\}$ cannot be a moment sequence. In each case where the distribution is determined by the corresponding moment sequence, the restriction to T of the respective cumulant generating function (exists and) is defined by ϕ.

Lemma 2.3 *Let S be an open interval containing zero as one of its points and $\psi : S \longrightarrow R$ be a twice differentiable function with $\psi(0) = 0$ and (in obvious notation) $\psi'(s)$, $\psi''(s) > 0$ for all $s \in S$, Further, let β and γ be real numbers and δ be a non-zero real number. Then*

$$\psi''(s) = \beta\psi'(s) + \gamma(\psi'(s))^2 + \delta(\psi'(s))^3, \quad s \in S, \tag{5}$$

if and only if ϕ defined by

$$\phi(t) = t - \psi^{-1}(t), \quad t \in T, \tag{6}$$

where T is the range of ψ, is such that one of (i)-(vi) of Lemma 2.1 with (α, β, γ) replaced by $(\beta + \gamma + \delta, -(2\beta + \gamma), \beta)$, and $\phi'(t) < 1$ for all $t \in T$ is met. (In the present case, one could replace "$4\alpha\gamma = \beta^2$", "$4\alpha\gamma = \beta^2 - \lambda^2$" and "$4\alpha\gamma = \beta^2 + \lambda^2$" appearing in the lemma by "$4\beta\delta = \gamma^2$", "$4\beta\delta = \gamma^2 - \lambda^2$" and "$4\beta\delta = \gamma^2 + \lambda^2$" respectively.)

Proof: Because of the inverse function theorem (Apostol (1977), p. 372), (6) implies that ϕ is differentiable with derivative

$$\phi'(t) = 1 - (\psi'(\psi^{-1}(t)))^{-1}, \quad t \in T, \tag{7}$$

which, in turn, implies that ϕ is twice differentiable with $\phi''(t) > 0$ for all $t \in T$; also, trivially, (6) gives that $\phi(0) = 0$. (7) implies that

$$\psi'(s) = (1 - \phi'(\psi(s)))^{-1}, \quad s \in S,$$

and

$$\begin{aligned} \frac{\psi''(s)}{\psi'(s)} &= \frac{d}{d\psi(s)}(1 - \phi'(\psi(s)))^{-1} \\ &= \phi''(\psi(s))(1 - \phi'(\psi(s)))^{-2}, \quad s \in S. \end{aligned}$$

Consequently, it follows that (5) holds if and only if

$$\phi''(\psi(s)) = \beta(1 - \phi'(\psi(s)))^2 + \gamma(1 - \phi'(\psi(s))) + \delta, \quad s \in S,$$

which, in turn, is equivalent to the assertion that (3) with (α, β, γ) replaced by $(\beta + \gamma + \delta, -(2\beta + \gamma), \beta)$ holds. The result claimed, i.e. Lemma 2.3, is hence obvious in view of Lemma 2.1.

Remark 2.4 (6) can also be rewritten as

$$\psi(s) = s + \phi(\psi(s)), \quad s \in S;$$

a functional equation of this form has appeared in storage theory (see, for example, Kendall (1957) and Prabhu (1965; p. 237); see also Prabhu (1980) for further relevant material). In each of the cases (i)-(vi) in Lemma 2.3 , ψ possesses the property that it has a Taylor expansion about each point (in S) in some neighbourhood of the point; if we denote the coefficients of $\frac{s^m}{m!}$ in the Taylor series expansion about the origin, of $\exp\{\psi(s)\}$, in a neighbourhood of the origin, by μ_m for $m = 0, 1, ...$, we have, in view of (5), that

$$\sum_{r_1=0}^{m}\sum_{r_2=0}^{m-r_1}\frac{m!}{(m-r_1-r_2)!r_1!r_2!}\{\mu_{r_1+2}\mu_{r_2}\mu_{m-r_1-r_2} - \delta\mu_{r_1+1}\mu_{r_2+1}\mu_{m-r_1-r_2+1}$$

$$-(1+\gamma)\mu_{r_1+1}\mu_{r_2+1}\mu_{m-r_1-r_2} - \beta\mu_{r_1+1}\mu_{r_2}\mu_{m-r_1-r_2}\} = 0, \quad m = 0, 1, ...$$

With easy calculations, it can also be seen that the following assertions (in the notation of Fosam and Shanbhag (1997)) hold here:

(i)$'$ If $\beta = \gamma = 0$ and $\delta = 1$, $\{2^{-m}\mu_m\}$ is the moment sequence of an $IGD(1,.)$.

(ii)$'$ If $\beta = 0$, $\gamma > 0$ and $\delta = 1$, $\{\gamma^{-2m}\mu_m\}$ is the moment sequence of a distribution that is absolutely continuous with respect to Lebesgue measure with density, f, satisfying for some $s_0 > 0$, $f(x) \propto e^{-s_0 x}k(x)$, $x \in R$, where k is the density of the Kendall-Ressel distribution with parameter γ^{-1}.

(iii)$'$ If $\gamma > 0$, $\delta = 1$ and $0 < \beta \leq \frac{1}{4}\gamma^2$, then depending upon whether or not $\beta = \frac{1}{4}\gamma^2$, $\{\beta^{-m}\mu_m\}$ is the moment sequence (respectively) of a GPED $(\frac{2}{\gamma}, 1, 0, v)$ with $v < e^{-1}$ or a GPED$(h, 1, c^*, v)$ with $c^* = 2(\frac{\gamma}{(\gamma^2-4\beta)^{1/2}} - 1)^{-1}$, $h = (2 + c^*)/\gamma$ and $v < (1 + c^*)^{-(1+c^*)/c^*}$.

(iv)$'$ If $\gamma \geq 0$, $\delta = 1$ and $\beta > \frac{1}{4}\gamma^2$, then depending upon whether or not $\gamma = 0$, $\{\beta^{-m}\mu_m\}$ is the moment sequence of an AGPED $(\frac{2a^*}{\gamma}, a^* - i, 2i, v)$ with $a^* = \frac{\gamma}{(4\beta-\gamma^2)^{1/2}}$ and $v < e^{-a^* sin^{-1}((1+a^{*2})^{-1/2})}(1 + a^{*2})^{-1/2}$.

(v)$'$ If $\beta \leq 0$, $\delta = -1$, $\gamma > 0$ with $\gamma^2 > 4\beta$, or $\beta > 0$, $\delta = 1$, $\gamma < 0$ with $4\beta > \gamma^2$, $\{\mu_m\}$ is not a moment sequence. (ψ in this case can be extended to a domain with left extremity equal to $-\infty$ or right extremity equal to ∞, such that the extension has a Taylor series expansion in some neighbourhood about each point; if we denote the extension by ψ^*, then, it follows that we can find, unless ψ is not the restriction to S of a cumulant generating function, an s_0 and sufficiently small positive v such that, for s lying in some neighbourhood of the origin,

$$\exp\{\psi^*(s + s_0)\} \propto \sum_{k=0}^{\infty}\exp\{k\beta s\}J_k(2a^*/\gamma, a^* - i, 2i, -v),$$

where a^* is as defined in **(iv)$'$** but with $|\gamma|$ in place of γ, in the case with γ negative,

$$\exp\{\psi^*(s + s_0)\} \propto \sum_{k=0}^{\infty}\exp\{-k\gamma^2 s\}J_k(-\gamma^{-1}, 1, 0, -v)\exp\{\gamma s\},$$

in the case with $\delta - 1$ and $\beta = 0$, and

$$\exp\{\psi^*(s+s_0)\} \propto \sum_{k=0}^{\infty} \exp\{-k\gamma_2(\gamma_2-\gamma_1)s\} J_k(\frac{1}{\gamma_1-\gamma_2}, 1, \frac{\gamma_1}{\gamma_2-\gamma_1}, -v) \exp\{s\gamma_2\},$$

where $\gamma_1 = \frac{1}{2}(\gamma - (\gamma - 4|\beta|)^{1/2})$ and $\gamma_2 = \frac{1}{2}(\gamma + (\gamma - 4|\beta|)^{1/2})$, in the case of $\delta = -1$ and $\beta < 0$. In each of the cases, $\{J_k\}$ is not a non-negative sequence and hence we have a contradiction. (Note that as in Fosam and Shanbhag (1997), we take

$$J_k(A,t,c,u) = \begin{cases} 1 & \text{if} \quad k=0 \\ Au & \text{if} \quad k=1 \\ \frac{A(A+kt+c)\ldots(A+kt+(k-1)c)}{k!} u^k & \text{if} \quad k=2,3,\ldots\ldots) \end{cases}$$

Remark 2.5 *In the case of $\beta = 0$, $\delta = -1$, $\gamma > 0$ with $\gamma^2 > 4\beta$, we can choose the extension ψ^* such that there exists a point $s_0 \le 0$ and sufficiently small positive v such that (unless ψ is not the restriction to S of a cumulant generating function)*

$$\exp\{\psi^*(s+s_0)\} \propto \sum_{k=0}^{\infty} exp\{k\gamma_1(\gamma_2-\gamma_1)s\} J_k(\frac{1}{\gamma_2-\gamma_1}, 1, -(\frac{\gamma_2}{\gamma_2-\gamma_1}), v) \exp\{s\gamma_1\},$$

where γ_1 and γ_2 are as in $(\mathbf{v})'$ of Remark 2.4, leading us to a contradiction once more.

Remark 2.6 There do exist cases of (ϕ, ψ) with

$$\psi(s) = s + \phi(\psi(s)), \quad s \in S,$$

in which ψ, with obvious notational alterations, satisfies (3) and ϕ'' is a cubic function of ϕ'. The following example provides us with an illustration of this:

Example Let $m > 0$,

$$\psi(s) = -ms + s^2, \quad s \in (-\infty, \frac{m}{2})$$

and

$$\phi(t) = -\frac{m}{2} + 2 + (\frac{m^2}{4} + t)^{1/2}, \quad t \in (-\frac{m^2}{4}, \infty).$$

Note that we have here $\psi(s) = s + \phi(\psi(s)), \quad s \in (-\infty, \frac{m}{2})$.

3 The Fosam-Shanbhag extended version of the Letac-Mora result

In the present section, we prove the Fosam-Shanbhag extension of the Letac-Mora result, via a new argument. This result is given jointly by the following Theorems 3.1 and 3.2; the notation used in Theorem 3.2 is that introduced by Fosam and Shanbhag (1997).

Theorem 3.1 *Let n be an integer ≥ 2 and $X_1, ..., X_n$ be iid non-degenerate random variables. Then, with a, b, and c as real numbers, we have (1) to be valid if and only if one of the following is valid:*

(i) $a = 1$, $b = 0$ *and X_1 is normal with variance c (implying that $c > 0$).*

(ii) $a = 1$, $b \neq 0$ *and $(1/b)(X_1 + (c/b))$ is Poisson.*

(iii) $a > 1$, $4c(a-1) = b^2$, *and $(X_1 + b/[2(a-1)])$ or $-(X_1 + b/[2(a-1)])$ is Gamma with index $(a-1)^{-1}$.*

(iv) $a < 1$ *and there exists a number $\delta > 0$ such that $4c(a-1) = b^2 - \delta^2$, and $\delta^{-1}(X_1 + ((b-\delta)/[2(a-1)]))$ has a binomial $((1-a)^{-1}, .)$ distribution (implying that $(1-a)^{-1}$ is a positive integer).*

(v) $a > 1$ *and there exists a number $\delta \neq 0$ such that $4c(a-1) = b^2 - \delta^2$ and $\delta^{-1}(X_1 + ((b-\delta)/[2(a-1)]))$ has a negative binomial $((a-1)^{-1}, .)$ distribution.*

(vi) $a > 1$ *and there exists a number $\delta > 0$ such that $4c(a-1) = b^2 + \delta^2$ and $2\delta^{-1}(X_1 + (b/[2(a-1)]))$ has a Meixner distribution with index $(a-1)^{-1}$, i.e. has a distribution that is absolutely continuous with respect to Lebesgue measure with density of the form*

$$f(x) = (\cos\alpha)^{\rho}\frac{2^{\rho-2}}{\pi\Gamma(\rho)}e^{\alpha x}\Gamma(\frac{\rho}{2} + \frac{ix}{2})\Gamma(\frac{\rho}{2} - \frac{ix}{2}), \quad -\infty < x < \infty$$

with $\rho = (a-1)^{-1}$ and α real and lying in $(-\pi/2, \pi/2)$. (The moment generating function corresponding to the distribution in question is defined for $t \in ((-\pi/2) - \alpha, (\pi/2) - \alpha)$, by $(\cos\alpha)^{\rho}(\cos(\alpha + t))^{-\rho}$.)

Proof: The "if" part follows via the characteristic function argument appearing on page 231 in Rao and Shanbhag (1994), or on noting that for each of the distributions, the respective moment generating function M satisfies, on its domain of definition,

$$\{(M''(t) - bM'(t))M(t) - a(M'(t))^2 - c(M(t))^2\}(M(t))^{n-2} = 0, \tag{8}$$

where M' and M'' are the first two derivatives of M, implying that (1) holds. (Note that the moment generating function satisfies the differential equation referred to if and only if the corresponding cumulant generating function satisfies the differential equation (3) with $\gamma = a - 1$, $\beta = b$ and $\alpha = c$.).

To prove the "only if" part, observe first that (1) implies by induction that X_1 has moments of all order, since it implies that $E(X_1^2) < \infty$ and that, for each integer $k \geq 0$

$$E(X_1^2|\sum_{j=1}^{n} X_j|^k) \leq E\{(|a||X_1||X_2| + |b||X_1| + |c|)(\sum_{j=1}^{n}|X_j|)^k\}, \tag{9}$$

where the right hand side of the inequality is finite (implying that $E(|X_1|^{2+k} < \infty)$ if $E(|X_1|^{1+k}) < \infty$). Appeal to (1) once more to see then that the moment sequence $\{\mu_m : m = 0, 1, 2, ...\}$ relative to X_1 satisfies

$$\sum_{r=0}^{m} \binom{m}{r} \{\mu_{r+2}\mu_{m-r} - a\mu_{r+1}\mu_{m-r+1} - b\mu_{r+1}\mu_{m-r} - c\mu_r\mu_{m-r}\} = 0 \quad m = 0, 1, ..., \tag{10}$$

implying that $\{\mu_m\}$ is determined given (a, b, c, μ_1) such that $(a-1)\mu_1^2 + b\mu_1 + c > 0$. In view of what is revealed in Remark 2.2, we can now claim that the result sought holds; note that the situation of $a < 1$ with $(1-a)^{-1}$ as a non-integer does not occur since $\{\mu_m\}$ here is a moment sequence, and , in each of the other situations, the moment sequence $\{\mu_m\}$ determines the distribution uniquely as that in the assertion.

Theorem 3.2 *Let n be an integer ≥ 3 and $X_1, X_2, ..., X_n$ be iid non-degenerate random variables. Also, let a, b, c, d with $d \neq 0$ be real numbers and ξ be the largest real number such that $cd^2 + b\xi d + (a-1)\xi^2 + \xi^3 = 0$, α_1, α_2 be real numbers such that $\alpha_1 = bd + 2\xi(a-1) + 3\xi^2$ and $\alpha_2 = a - 1 + 3\xi$ and $Y_1 = dX_1 - \xi$. Then, (2) is met if and only if one the following conditions holds:*

(i) $\alpha_1 = \alpha_2 = 0$ *and* $\frac{1}{2}Y_1$ *has an IGD(1,.).*

(ii) $\alpha_1 = 0$, $\alpha_2 > 0$ *and the distribution of* $(1/\alpha_2^2)Y_1$ *is such that it is absolutely continuous with respect to Lebesgue measure with density, f, satisfying, for some $\lambda > 0$,*

$$f(x) \propto e^{-\lambda x} k(x), \quad x \in R,$$

where k is the density of the Kendall-Ressel distribution with parameter $1/\alpha_2$ (i.e. $u = 1/\alpha_2$ in the Fosam-Shanbhag notation).

(iii) $\alpha_2 > 0$, $0 < \alpha_1 \leq \frac{1}{4}\alpha_2^2$, *and depending upon whether or not* $\alpha_1 = \frac{1}{4}\alpha_2^2$, $\alpha_1^{-1}Y_1$ *has (respectively) a GPED$(2/\alpha_2, 1, 0, v)$ with $v < e^{-1}$ or a GPED$(h, 1, c^*, v)$ with $c^* = 2((\alpha_2/(\alpha_2^2 - 4\alpha_1)^{1/2}) - 1)^{-1}$, $h = (2 + c^*)/\alpha_2$ and $v < (1 + c^*)^{-(1+c^*)/c^*}$.*

(iv) $\alpha_2 \geq 0$, $\alpha_1 > \frac{1}{4}\alpha_2^2$, *and depending upon whether or not $\alpha_2 = 0$, $\alpha_1^{-1}Y_1$ has an AGPED$(\alpha_1^{-1/2}, -i, 2i, v)$ with $v < 1$ or an AGPED$(2a^*/\alpha_2, a^* - i, 2i, v)$ with $a^* = \alpha_2/(4\alpha_1 - \alpha_2^2)^{1/2}$ and $v < e^{-a^* \sin^{-1}((1+a^{*2})^{-1/2})}(1 + a^{*2})^{-1/2}$.*

Proof: There is no loss of generality in assuming that $b = \alpha_1$, $a - 1 = \alpha_2$, $c = 0$ and $d = 1$ with $\xi = 0$ (implying that $\alpha_1 \geq 0$ and $\alpha_2^2 \leq 4\alpha_1$.) The theorem then follows essentially via the argument used to prove the pervious theorem provided we take

$$\{(M''(t) - \alpha_1 M'(t))(M(t))^2 - (1 + \alpha_2)(M'(t))^2 M(t) - (M'(t))^3\}(M(t))^{n-3} = 0, \tag{11}$$

in place of (8), "$|X_1||X_2||X_3| + |1+\alpha_2||X_1||X_2| + |\alpha_1||X_1|$" in place of "$|a||X_1||X_2| + |b||X_1| + |c|$" in (9). "Remark 2.4" in place of"Remark 2.2", and the following system of equations in place of (10)

$$\sum_{r_1=0}^{m}\sum_{r_2=0}^{m-r_1}\frac{m!}{(m-r_1-r_2)!r_1!r_2!}\{\mu_{r_1+2}\mu_{r_2}\mu_{m-r_1-r_2} - \mu_{r_1+1}\mu_{r_2+1}\mu_{m-r_1-r_2+1}$$

$$-(1+\alpha_2)\mu_{r_1+1}\mu_{r_2+1}\mu_{m-r_1-r_2} - \alpha_1\mu_{r_1+1}\mu_{r_2}\mu_{m-r_1-r_2}\} = 0, \quad m = 0, 1, ... \tag{12}$$

(The statement under brackets, following (8) holds here, provided "(3) with $\gamma = a-1$, $\beta = b$ and $\alpha = c$" is replaced by (5) "with $\beta = \alpha_1$, $\gamma = \alpha_2$ and $\delta = 1$".)
Note that (12) implies that given $(\alpha_1, \alpha_2, \mu_1)$ such that $\mu_1^3 + \alpha_2\mu_1^2 + \alpha_1\mu_1 > 0$, the moment sequence $\{\mu_m\}$ is determined and the cases of $\mu_1 < 0$ and $\alpha_2 < 0$ do note arise here. (For $\mu_1 < 0$, $\mu_m^* = (-1)^m\mu_m$, $m = 0, 1, ...$ satisfy the system of equations relative to $\{\mu_m\}$ in Remark 2.2 with $\delta = -1$, $\beta = -\alpha_1$ and $\gamma = \alpha_2$.)

Remark 3.3 The proofs of Theorems 3.1 and 3.2 simplify slightly if it is assumed a priori that X_1 has a moment generating function. Also, since the distributions listed in the assertions in Theorems 3.1 and 3.2 are determined by the corresponding moment sequences, it is clear that the "only if" parts of the assertions follow from the respective "if" parts, provided the cases in which the moment sequences with desired properties do not exist, are dealt with appealing to the relevant observations in Remarks 2.2 and 2.4. Our findings in section 2 in their existing form, though throw further light on the behaviour of the distributions involved in the Letac-Mora result, and on its extension due to Fosam and Shanbhag.

The following is a corollary to Theorems 3.1 and 3.2:

Corollary 3.4 *Let $\{F_\theta : \theta \in \Theta\}$ be a family of non-degenerate d.f.'s on R that are absolutely continuous with respect to some σ-finite measure ν such that for each θ*

$$F_\theta(x) \propto \int_{(-\infty,x]} \exp\{g(\theta)y\}\nu(dy), \quad x \in R,$$

with $g : \Theta \to R$ satisfying the condition that the set of values of g is dense in an open interval. (The condition on g obviously places a restriction on Θ implicity, and the family that we have here is a version of an exponential family.) For each $\theta \in \Theta$, denote by μ_θ and σ_θ^2 the mean and the variance corresponding to F_θ respectively. Then

$$\sigma_\theta^2 = a_0 + a_1\mu_\theta + a_2\mu_\theta^2 + a_3\mu_\theta^3 \quad \text{for all } \theta \tag{13}$$

with a_0, a_1, a_2 and a_3 as real and independent of θ if and only if given a point $\theta_0 \in \Theta$, (2) with $n = 3$, $a = a_2 + 1$, $b = a_1$, $c = a_0$, $d = a_3$ and X_1 as a random variable with d.f. F_{θ_0}, holds.

Proof: Suppose for each $\theta \in \Theta$, we denote by $X_\theta^{(1)}$,$X_\theta^{(2)}$ and $X_\theta^{(3)}$ three independent random variables distributed with d.f. F_θ. Then, it easily follows that (13) holds if and only if for any fixed $\theta_0 \in \Theta$ we have (in obvious notation)

$$E_{\theta_0}\{(X_{\theta_0}^{(1)})^2 - (a_2+1)X_{\theta_0}^{(1)}X_{\theta_0}^{(2)} - a_1X_{\theta_0}^{(1)} - a_3X_{\theta_0}^{(1)}X_{\theta_0}^{(2)}X_{\theta_0}^{(3)}|X_{\theta_0}^{(1)} + X_{\theta_0}^{(2)} + X_{\theta_0}^{(3)}\} = a_0 \quad a.s. \tag{14}$$

The assertion now follows trivially.

The family $\{F_\theta\}$ of Corollary 3.4 is clearly such that given any $\theta_0 \in \Theta$, for each $\theta \in \Theta$, the moment generating function M_θ of F_θ is given by

$$M_\theta(.) = \frac{M_{\theta_0}(g(\theta) - g(\theta_0) + .)}{M_{\theta_0}(g(\theta) - g(\theta_0))}$$

with an appropriate domain of definition and M_{θ_0} as the moment generating function of F_{θ_0}. In other words, we have the family in question to be such that for each θ

$$F_\theta(x) \propto \int_{(-\infty,x]} \exp\{(g(\theta) - g(\theta_0)y)\}F_{\theta_0}(dy), \quad x \in R.$$

Assuming $\{F_\theta\}$ to be a natural exponential family, Letac and Mora (1990) have identified the cases under which (13) holds; Corollary 3.4 given above is hence essentially their result.

Remark 3.5 There exist typos in Theorem 1 and Remark 9 of Fosam and Shanbhag (1997): in (vi) of the theorem in question "$b - \delta$" should have appeared as "b", and in the remark referred to "(13)"should have appeared as "(17)".

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