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13. ABSTRACT

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A NOTE ON THE ASYMPTOTIC BEHAVIOR OF THE LSE'S OF THE PARAMETERS FOR SUPERIMPOSED EXPONENTIAL SIGNALS IN PRESENCE OF STATIONARY NOISE

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ABSTRACT: Superimposed exponential signals play an important role in Statistical Signal Processing and Time series analysis. In this note, the asymptotic behavior of the least squares estimators of the parameters are obtained in presence of stationary noise for the undamped exponential model. It is well known that this model does not satisfy the sufficient conditions of Jennrich (1969), Wu (1981) or Kundu (1991) for the least squares estimators to be consistent even when the errors are independent and identically distributed random variables with mean zero and finite variance. This paper extends some of the earlier works of Hannan (1971, 1973), Walker (1971), Bai *et al.* (1991), Rao and Zhao (1993), Kundu (1995) and Kundu and Mitra (1995, 1998) in different ways. Some numerical experiments are performed to observe the small sample behavior of the least squares estimators.

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1. INTRODUCTION:

We consider the following model of multiple superimposed exponential signals in presence of stationary noise;

$$Y(t) = \sum_{k=1}^{M} \alpha_k^0 e^{i\omega_k^0 t} + X(t); \quad \text{for} \quad t = 1, \ldots, N. \tag{1}$$

Here $\alpha_1^0, \ldots, \alpha_M^0$ are unknown complex numbers known as amplitudes and all of them are different from zero, $i = \sqrt{-1}$. The $\omega_1^0, \ldots, \omega_M^0$ are unknown frequencies lying between 0 and 2π and they are distinct. $X(t)$'s are stationary processes with mean zero and they satisfy Assumption 1 as given at the end of this section. M is assumed to be known. Given a sample of size N, the problem is to estimate the unknown parameters α's and ω's and some times the error variance.

This is a very important and well discussed model in Statistical Signal processing. For example, in electromagnetic pulse (EMP) situations (Ricketts *et al.;* 1976 and Sircar; 1987), the EMP pickup can be characterized by a sum of complex exponentials whose parameters are to be determined. The parameters are a means of coding the various pulse wave forms and the signal approximation thus obtained can be readily employed to analyze responses in various subsystems under EMP environment. In system identification problems, the characterization of the impulse responses of a linear system by a sum of complex exponentials and then identifying or approximating the complex amplitudes and natural frequencies with high degree of accuracy has it's special importance in a wide variety of applications.

Quite a number of papers appeared dealing with the estimation of the parameters for this model particularly when the $X(t)$'s are independent and identically distributed (*i.i.d.*) random variables. See for example the review articles of Rao (1988), Prasad *et al.* (1995) and also the paper of Stoica (1993) for an extensive list of references. Few articles appeared in the last few years developing the theoretical properties of the least squares estimators, like Bai *et al.* (1991), Rao and Zhao (1993), Kundu (1995), Kundu and Mitra (1995, 1998). Pillai (1989) pointed out that although the assumptions of the *i.i.d.* errors are quite common, but in many situations the noise might be correlated. The problem has some theoretical interest also. No body, at least not known to the authors considered the theoretical properties of the LSE's under this general set up.

It is worth mentioning that the model (1) does not satisfy the standard sufficient conditions of Jennrich (1969), Wu (1981) or Kundu (1991) for the LSE's to be consistent even when the errors are *i.i.d.* random variables, see Kundu and Mitra (1998) for details. Therefore, it is not immediate how the LSE's will behave in this general set up. Hannan (1971) and Walker (1971) considered a similar kind of model in the context of non-linear time series regression. They considered the following model

$$Y(t) = \beta_1 + \beta_2 cos(t\omega) + \beta_3 sin(t\omega) + X(t) \tag{2}$$

where $X(t)$'s are i.i.d. random variables with mean zero and finite variance. They mentioned that it may be difficult to obtain the asymptotic properties of the LSE's directly. Instead of considering the LSE's they make the Fourier transform of the data and obtained the estimators of the frequencies by maximizing the periodogram. They called them as approximate least squares estimators (ALSE's). They obtained the consistency and the asymptotic normality property of the ALSE's. Hannan (1973) extended the results of Hannan (1971) and Walker (1971) to the case when $X(t)$'s are from a stationary time series with mean zero and finite variance and has a continuous spectrum. Hannan (1973)'s approach is quite similar to that of Hannan (1971) or Walker (1973).

Bai *et al.* (1991) first proved directly the consistency of the LSE's of the parameters under the assumptions that $X(t)$'s are *i.i.d.* normal random variables with mean zero and finite variance. Rao and Zhao (1993) obtained the asymptotic distribution of the LSE's under the same assumption as that of Bai *et al.* (1991). Kundu (1995) obtained the consistency of the LSE's when the errors are i.i.d random variables with mean zero and finite variance, but under some restrictions on the parameter space. Kundu and Mitra (1995) obtained the strong consistency of the LSE's under the *i.i.d.* assumptions of the error random variables and very recently Kundu and Mitra (1998) obtained the asymptotic distribution of the LSE's under the same assumption as that of Kundu and Mitra (1995). In this paper we try to generalize the result when the errors are from a stationary sequence. Our approach here is quite different from Hannan (1971, 1973), Walker (1971), Bai *et al.* (1991), Rao and Zhao (1993) and Kundu (1995). We make the following assumptions on $X(t)$.

Assumption 1: $X(t)$ is a complex valued stationary process and $X(t) = U(t) + iV(t)$. Here

$$U(t) = \sum_{j=-\infty}^{\infty} \alpha(j)\epsilon(t-j) \qquad V(t) = \sum_{j=-\infty}^{\infty} \beta(j)e(t-j) \tag{3}$$

where $\epsilon(t)$'s and $e(t)$'s are *i.i.d.* real valued random variables with mean zero and finite variance σ^2. Also $\epsilon(t)$'s and $e(t)$' are independent of each other and $\sum_{j=-\infty}^{\infty} |\alpha(j)| < \infty$, $\sum_{j=-\infty}^{\infty} |\beta(j)| < \infty$.

We denote $Var(X(t)) = \sigma^2 \left(\sum_{j=-\infty}^{\infty} |\alpha(j)|^2 + \sum_{j=-\infty}^{\infty} |\beta(j)|^2 \right) = \gamma^2 < \infty$. The major aim of this paper is to prove directly the consistency of the LSE's of the parameters of the model (1) under Assumption 1, of the error random variables. We also obtain that the least squares estimators are asymptotically normal with mean vector zero and certain variance covariance matrix. The explicit expression of the asymptotic variance covariance matrix is obtained, which it seems is not available in the literature. We also obtain the consistency and the asymptotic properties of the estimator of γ^2. This paper generalizes the works of Bai *et al.* (1991), Rao and Zhao (1993), Kundu (1995), Kundu and Mitra (1995, 1998). It is important to point out that Hannan (1971, 1973) or Walker (1971) did not consider the exact LSE's and moreover, Hannan (1973) did not consider the multiparameter situation or the estimation of the variance of error random variables. The exact expression of the asymptotic distribution of the multiparameter situation is not available in the literature.

Therefore, our results extends the results of Hannan (1971, 1973) and Walker (1971) to the complex parameter and also to the multiparameter situation. In this paper the almost sure convergence means with respect to the Lebesgue measure and it will be denoted by *a.s.*. The notation $a = O(N^b)$ means $\frac{a}{N^b}$ is bounded for all N. Also the real and imaginary part of a complex number, say a, will be denoted by a_R and a_I respectively. Therefore any complex number a can be written as $a = a_R + ia_I$. $N_p(\mathbf{a}, \mathbf{B})$ denotes the p variate normal distribution with mean vector $\mathbf{a}$ and the covariance matrix $\mathbf{B}$ and $N(a,b)$ denotes the univariate normal with mean a and variance b.

The rest of the paper is organized as follows. In Section 2, we prove the consistency and asymptotic normality of the LSE's of the parameters of the model (1) for the case $M = 1$. For general M, the result is established in Section 3. The asymptotic properties of the estimator of γ^2 is obtained in Section 4. Some experimental findings are discussed in Section 5 and finally we draw conclusions from our results in Section 6.

2. CONSISTENCY AND ASYMPTOTIC NORMALITY OF THE LSE'S:

Let's denote $\hat{\theta} = (\hat{\alpha}, \hat{\omega})$ be the LSE's of $\theta^0 = (\alpha^0, \omega^0)$, obtained by minimizing;

$$Q(\alpha, \omega) = \sum_{t=1}^{N} \left| Y(t) - \alpha e^{i\omega t} \right|^2 \tag{4}$$

Note that $\hat{\theta}$, $\hat{\alpha}$, $\hat{\omega}$ and $Q(\alpha, \omega)$ depend on N, but for brevity we don't make it explicit. To prove the necessary results we need the following lemma.

Lemma 1: Let $U(t)$ be a stationary sequence which satisfies Assumption 1, then

$$\lim_{N\to\infty} \sup_{\omega} \left| \frac{1}{N} \sum_{t=1}^{N} U(t) cos(t\omega) \right| = 0 \qquad a.s. \tag{5}$$

Proof: See Appendix.

Comments: The following result also can be proved along the same line.

$$\lim_{N\to\infty} \sup_{\omega} \left| \frac{1}{N^{L+1}} \sum_{t=1}^{N} t^L X(t) cos(t\omega) \right| = 0 \qquad a.s. \tag{6}$$

for $L = 0, 1, 2, \ldots$, and $X(t)$ satisfies Assumption 1. Note that the above results (5) and (6) are true if the cosine function is replaced by sine function.

Using Lemma 1 and the same techniques as of Theorem 1 of Kundu and Mitra (1995), the following result can be established:

Theorem 1: For $M = 1$, $\hat{\theta}$ is a strongly consistent estimator of θ^0.

Now we will establish the asymptotic normality property of the LSE's. The idea is quite similar to that of Kundu and Mitra (1998). The major difference here is that, the center limit theorem for *i.i.d.* random variables was used in Kundu and Mitra (1998) to prove the asymptotic normality , but here we need to use the central limit theorem for stochastic processes. The details will be explained below. Let's denote

$$Q'(\theta) = \left(\frac{\partial Q(\theta)}{\partial \alpha_R}, \frac{\partial Q(\theta)}{\partial \alpha_I}, \frac{\partial Q(\theta)}{\partial \omega} \right) \tag{7}$$

and $Q''(\theta)$ be the corresponding 3×3 matrix which contains the double derivative of $Q(\theta)$. Therefore, expanding $Q'(\hat{\theta})$ around θ^0 by multivariate Taylor series, we have

$$Q'(\hat{\theta}) - Q'(\theta^0) = (\hat{\theta} - \hat{\theta}^0) Q''(\bar{\theta}) \tag{8}$$

where $\bar{\theta}$ is a point line joining $\bar{\theta}$ and $\hat{\theta}^0$. Since $Q'(\hat{\theta}) = 0$, (8) implies,

$$(\hat{\theta} - \hat{\theta}^0) = -Q'(\theta^0)[Q''(\bar{\theta})]^{-1} \tag{9}$$

The main idea to prove that $(\hat{\theta} - \hat{\theta}^0)$ converges to a multivariate normal distribution will be as follows. Consider the following 3×3 diagonal matrix

$$\mathbf{D} = \operatorname{diag}\left\{ N^{-\frac{1}{2}}, N^{-\frac{1}{2}}, N^{-\frac{3}{2}} \right\} \tag{10}$$

Now write

$$(\hat{\theta} - \theta^0) = -Q'(\theta^0)\mathbf{D}\left[\mathbf{D}Q''(\bar{\theta})\mathbf{D}\right]^{-1} \tag{11}$$

It can be easily shown (see Kundu and Mitra; 1998) that

$$\lim_{N\to\infty} \left[\mathbf{D}Q''(\bar{\theta})\mathbf{D}\right] = \lim_{N\to\infty} \left[\mathbf{D}Q''(\theta^0)\mathbf{D}\right] = \Sigma \tag{12}$$

where

$$\Sigma = \begin{bmatrix} 2 & 0 & -\alpha_I^0 \\ 0 & 2 & \alpha_R^0 \\ -\alpha_I^0 & \alpha_R^0 & \frac{2}{3}|\alpha^0|^2 \end{bmatrix}. \tag{13}$$

and Σ^{-1} exists if $\alpha^0 \neq 0$ and it is as follows;

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{2} + \frac{3}{2}\frac{\alpha_I^{0^2}}{|\alpha^0|^2} & -\frac{3}{2}\frac{\alpha_R^0\alpha_I^0}{|\alpha^0|^2} & 3\frac{\alpha_I^0}{|\alpha^0|^2} \\ -\frac{3}{2}\frac{\alpha_R^0\alpha_I^0}{|\alpha^0|^2} & \frac{1}{2} + \frac{3}{2}\frac{\alpha_R^{0^2}}{|\alpha^0|^2} & -3\frac{\alpha_R^0}{|\alpha^0|^2} \\ 3\frac{\alpha_I^0}{|\alpha^0|^2} & -3\frac{\alpha_R^0}{|\alpha^0|^2} & \frac{6}{|\alpha^0|^2} \end{bmatrix}. \tag{14}$$

Using the Central limit theorem of stochastic processes (see Fuller 1976, page 251), it easily follows that $Q(\theta^0)\mathbf{D}$ tends to a multivarite (3-variate) normal distribution as given below;

$$Q'(\theta^0)\mathbf{D} \to N_3\left\{\mathbf{0}, \sigma^2 c\Sigma\right\}$$

here $c = \sum_{h=-\infty}^{\infty} \rho_U(h) e^{i\omega^0 h} + \sum_{h=-\infty}^{\infty} \rho_V(h) e^{i\omega^0 h}$, where $\rho_U(h)$ and $\rho_V(h)$ are the autocorrelation functions of the stationary process $\{U(t)\}$ and $\{V(t)\}$ respectively. Therefore, we have

$$(\hat{\theta} - \theta^0)\mathbf{D}^{-1} \rightarrow N_3(\mathbf{0}, \sigma^2 c \Sigma^{-1})$$

Now we can state the result as the following theorem:

Theorem 2: Under the conditions of Theorem 1,

$$\left\{N^{\frac{1}{2}}(\hat{\alpha}_R - \alpha_R^0), N^{\frac{1}{2}}(\hat{\alpha}_I - \alpha_I^0), N^{\frac{3}{2}}(\hat{\omega} - \omega^0)\right\} \rightarrow N_3(\mathbf{0}, \sigma^2 c \Sigma^{-1})$$

where c and Σ^2 are as defined before.

Remark: It may be mentioned that if we rewrite the model (1) as follows

$$Y(t) = \alpha^0 e^{i\omega^0 (t-\bar{t})} + X(t); \qquad for \quad t = 1, \ldots, N, \tag{15}$$

where $\bar{t} = \frac{N+1}{2}$, then it can be shown along the same line as above that

$$\left\{N^{\frac{1}{2}}(\hat{\alpha}_R - \alpha_R^0), N^{\frac{1}{2}}(\hat{\alpha}_I - \alpha_I^0), N^{\frac{3}{2}}(\hat{\omega} - \omega^0)\right\} \rightarrow N_3(\mathbf{0}, \sigma^2 c \Gamma^{-1})$$

and

$$\Gamma^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{6}{|\alpha^0|^2} \end{bmatrix}. \tag{16}$$

Here $\hat{\alpha}_R$, $\hat{\alpha}_I$ and $\hat{\omega}$ are LSE's of the corresponding parameters of the model (15). Although, the model (15) is more convenient to deal with, because of the independence structure of the LSE's, but in practice the data may not be available in that form.

3. MULTIPARAMETER CASE:

In this section we extend the result of Section 2 of the model (1) for general M. Let's use the following notations; $\theta^M = (\theta_1, \ldots, \theta_M), \theta_j = (\alpha_{jR}, \alpha_{jI}, \omega_j)$, where $\alpha_j = \alpha_{jR} + i\alpha_{jI}$ for $j = 1, \ldots, M$. Similarly, $\hat{\theta}^M$ and θ^{M0} are also defined. θ^{M0} and $\hat{\theta}^M$ denote the true parameter value and the LSE's of the true parameter value, respectively. We have the following results.

Theorem 3: If $X(t)$ satisfies Assumption 1, then $\hat{\theta}^M$ is a strongly consistent estimator of θ^{M0}.

Proof: The proof can be obtained similarly as Theorem 1, see also Kundu and Mitra (1995).

To obtain the asymptotic distribution, first let's define a $3M \times 3M$ diagonal matrix $\mathbf{V}$ as follows

$$\mathbf{V} = diag\{N^{\frac{1}{2}}, N^{\frac{1}{2}}, N^{\frac{3}{2}}, \ldots, N^{\frac{1}{2}}, N^{\frac{1}{2}}, N^{\frac{3}{2}}\}$$

Theorem 4: Under the assumptions of Theorem 1, $(\hat{\theta}^M - \theta^{M0})\mathbf{V}$ converges in distribution to a $3M$ variate normal distribution with mean vector $\mathbf{0}$ and the dispersion matrix $\sigma^2\Psi^{-1}$, where

$$\Psi^{-1} = \begin{bmatrix} c_1\Sigma_1^{-1} & 0 & \ldots & 0 \\ 0 & c_2\Sigma_2^{-1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & c_M\Sigma_M^{-1} \end{bmatrix}. \tag{17}$$

and

$$\Sigma_j^{-1} = \begin{bmatrix} \frac{1}{2} + \frac{3}{2}\frac{\alpha_{jI}^{0^2}}{|\alpha_j^0|^2} & -\frac{3}{2}\frac{\alpha_{jR}^0\alpha_{jI}^0}{|\alpha_j^0|^2} & 3\frac{\alpha_{jI}^0}{|\alpha_j^0|^2} \\ -\frac{3}{2}\frac{\alpha_{jR}^0\alpha_{jI}^0}{|\alpha_j^0|^2} & \frac{1}{2} + \frac{3}{2}\frac{\alpha_{jR}^{0^2}}{|\alpha_j^0|^2} & -3\frac{\alpha_{jR}^0}{|\alpha_j^0|^2} \\ 3\frac{\alpha_{jI}^0}{|\alpha_j^0|^2} & -3\frac{\alpha_{jR}^0}{|\alpha_j^0|^2} & \frac{6}{|\alpha_j^0|^2} \end{bmatrix}. \tag{18}$$

and $c_j = \sum_{h=-\infty}^{\infty} \rho_U(h)e^{i\omega_j^0 h} + \sum_{h=-\infty}^{\infty} \rho_V(h)e^{i\omega_j^0 h}$.

Proof: The proof can be obtained similarly as that of Theorem 2, so it is omitted.

4. CONSISTENCY AND ASYMPTOTIC NORMALITY OF $\hat{\gamma}^2$

In this section we obtain the consistency and the asymptotic properties of the LSE of $Var(X(t)) = \gamma^2$, given by $\hat{\gamma}^2 = \frac{1}{N}Q(\hat{\theta}^M)$. Other than Assumption 1, we need to make the following assumptions on $\{e(t)\}$'s and $\{\epsilon(t)\}$'s,

Assumption 2: Both $\{e(t)\}$'s and $\{\epsilon(t)\}$'s have finite six moments and $E(e(t)^4) = E(\epsilon(t)^4) = \eta\sigma^4$. First prove the following lemma;

Lemma 2: If $\hat{\omega}_j$ is the LSE of ω_j^0 for $j = 1, \ldots, M$, for the model (1), then as $N \to \infty$,

$$N(\hat{\omega}_j - \omega_j^0) \to 0 \qquad a.s.$$

for $j = 1, \ldots, M$. Expanding $Q(\hat{\theta}^M)$ around θ^{M0} by multivariate Taylor series and using (6) the result can be obtained. Note that

$$\hat{\gamma}^2 = \frac{1}{N}Q(\hat{\theta}^M) = T_1 + T_2 + T_3$$

where

$$
\begin{aligned}
T_1 &= \frac{1}{N}\sum_{t=1}^{N}\left(U(t)^2+V(t)^2\right) \\
T_2 &= \frac{1}{N}\sum_{t=1}^{N}\left[\left(\sum_{j=1}^{M}\alpha_{jR}^0 cos(\omega_j^0 t)-\hat{\alpha}_{jR}cos(\hat{\omega}_j t)-\alpha_{jI}^0 sin(\omega_j^0 t)+\hat{\alpha}_{jI}sin(\hat{\omega}_j t)\right)^2\right]+ \\
&\quad \frac{1}{N}\sum_{t=1}^{N}\left[\left(\sum_{j=1}^{M}\alpha_{jI}^0 cos(\omega_j^0 t)-\hat{\alpha}_{jI}cos(\hat{\omega}_j t)+\alpha_{jR}^0 sin(\omega_j^0 t)-\hat{\alpha}_{jR}sin(\hat{\omega}_j t)\right)^2\right] \\
T_3 &= \frac{2}{N}\sum_{t=1}^{N}\left[U(t)\left(\sum_{j=1}^{M}\alpha_{jR}^0 cos(\omega_j^0 t)-\hat{\alpha}_{jR}cos(\hat{\omega}_j t)-\alpha_{jI}^0 sin(\omega_j^0 t)+\hat{\alpha}_{jI}sin(\hat{\omega}_j t)\right)\right]+ \\
&\quad \frac{2}{N}\sum_{t=1}^{N}\left[V(t)\left(\sum_{j=1}^{M}\alpha_{jI}^0 cos(\omega_j^0 t)-\hat{\alpha}_{jI}cos(\hat{\omega}_j t)+\alpha_{jR}^0 sin(\omega_j^0 t)-\hat{\alpha}_{jR}sin(\hat{\omega}_j t)\right)\right]
\end{aligned}
\tag{19}
$$

Observe that T_1 converges to γ^2 *a.s.*, T_3 converges to zero *a.s.* by Theorem 3 and (6) and T_2 converges to zero *a.s.* can be established with the help of Lemma 2 and Theorem 3 along the same line of Rao and Zhao (1993). Thus we have the following theorem.

Theorem 5: If $\hat{\theta}^M$ is the LSE of θ^{M0} for the model (1), with $X(t)$ satisfies Assumptions 1 and 2, then $\hat{\gamma}^2$ is a strongly consistent estimator of γ^2

Now we obtain the asymptotic distribution $\hat{\gamma}^2$. First we need to consider the following lemmas.

Lemma 3: If $\{X(t)\}$ is the sequence of random variables as defined in Section 1 and $\hat{\omega}_j$ is the LSE of ω_j^0, for $j=1,\ldots,M$, of the model (1), then

$$\frac{1}{\sqrt{N}}\sum_{t=1}^{N}X(t)\left(cos(\omega_j^0 t)-cos(\hat{\omega}_j t)\right)\to 0 \qquad \text{in probability}$$

Proof: Using the mean value theorem on $(cos(\omega_j^0 t)-cos(\hat{\omega}_j t))$ and using Theorems 3, 4 and (6) it can be obtained.

Lemma 4: If $\hat{\omega}_j$ and $\hat{\alpha}_j$ are the LSE's of ω_j^0 and α_j^0 respectively of the model (1), then

$$\frac{1}{\sqrt{N}}\left[|\alpha_j^0|^2\sum_{t=1}^{N}cos^2(\omega_j^0 t)-\alpha_j^0\hat{\alpha}_j\sum_{t=1}^{N}cos(\omega_j^0 t)cos(\hat{\omega}_j t)\right]\to 0 \qquad \text{in probability}$$

$$\frac{1}{\sqrt{N}}\left[|\alpha_j^0|^2\sum_{t=1}^{N}cos^2(\hat{\omega}_j t)-\alpha_j^0\hat{\alpha}_j\sum_{t=1}^{N}cos(\omega_j^0 t)cos(\hat{\omega}_j t)\right]\to 0 \qquad \text{in probability}$$

Proof: Using Theorem 3 and Lemma 2 the results can be established.

The results in Lemmas 3 and 4 are true if the cosine functions are replaced by sine functions.

Lemma 5: If $\hat{\gamma}^2$ is the LSE of γ^2, then

$$\sqrt{N}\left\{\hat{\gamma}^2 - \frac{1}{N}|X(t)|^2\right\} \to 0 \qquad \text{in probability}$$

Proof: Consider

$$\sqrt{N}\hat{\gamma}^2 = \sqrt{N}T_1 + \sqrt{N}T_2 + \sqrt{N}T_3 = G_1 + G_2 + G_3 \qquad \text{(say)}$$

Expanding G_2 and using Lemma 4 it follows that G_2 converges to zero in probability. Using Lemma 3 and Theorem 3, it can be shown that G_3 also converges to zero. Therefore, the result follows.

We also have the following lemma.

Lemma 6: If $U(t)$ and $V(t)$ satisfy Assumptions 1 and 2, then

$$\sqrt{N}\left(\frac{1}{N}\sum_{t=1}^{N} U(t)^2 - \sigma^2 \sum_{j=-\infty}^{\infty} \alpha(j)^2\right) \to N\left(0, (\eta-3)\Gamma_1(0)^2 + 2\sum_{j=-\infty}^{\infty} \Gamma_1(h)^2\right),$$

similarly

$$\sqrt{N}\left(\frac{1}{N}\sum_{t=1}^{N} V(t)^2 - \sigma^2 \sum_{j=-\infty}^{\infty} \beta(j)^2\right) \to N\left(0, (\eta-3)\Gamma_2(0)^2 + 2\sum_{j=-\infty}^{\infty} \Gamma_2(h)^2\right)$$

and they are independent. Here $\Gamma_1(h)$ and $\Gamma_2(h)$ are the auto covariance functions of $U(t)$ and $V(t)$ respectively.

Proof: See Fuller (1976).

Therefore, we have the following result:

Theorem 6: Under the Assumptions 1 and 2

$$\sqrt{N}\left(\hat{\gamma}^2 - \sigma^2(\sum_{j=-\infty}^{\infty} \alpha(j)^2 + \sum_{j=-\infty}^{\infty} \beta(j)^2\right) \to$$

$$N\left(0, (\eta-3)(\Gamma_1(0)^2 + \Gamma_2(0)^2) + 2\sum_{j=-\infty}^{\infty} \Gamma_1(h)^2 + 2\sum_{j=-\infty}^{\infty} \Gamma_2(h)^2\right)$$

Proof: Using Lemmas 5 and 6, it follows immediately.

Note that when both $e(t)$'s and $\epsilon(t)$'s are normally distributed random variables, with $U(t) = \epsilon(t)$ and $V(t) = e(t)$ for all t, then the asymptotic distribution $\hat{\sigma}^2$ as obtained by Rao and Zhao (1993) can be obtained from Theorem 5, by substituting $\eta = 3$, $\Gamma_1(0) = \Gamma_2(0) = \sigma^2$ and $\Gamma_1(h) = \Gamma_2(h) = 0$ for all $h \neq 0$

5. NUMERICAL EXPERIMENTS AND DISCUSSIONS:

In this section we perform some Monte Carlo simulations to see how the asymptotic results work for small sample. We considered the following model:

$$Y(t) = 2.0e^{i2\pi f_1 t} + 3.0e^{i2\pi f_2 t} + X(t); \qquad t = 1, \ldots, 25. \tag{20}$$

where $f_1 = .15Hz$ and $f_2 = .17Hz$. $X(t) = U(t) + iV(t)$, $U(t) = \epsilon(t) + .5\epsilon(t-1)$ and $V(t) = e(t) + .5e(t-1)$. Note that $|f_1 - f_2| < \frac{1}{N}$ and this is quite important from the applications point of view (see Tufts and Kumaresan; 1982 or Breslar and Macovski; 1986). Here $\epsilon(t)$ are *i.i.d.* random variables with mean zero and finite variance, similarly $e(t)$'s are also normal random variables with mean zero and finite variance. $e(t)$'s and $\epsilon(t)$'s are independent. Numerical results are obtained for different SNR = 5dB, 10dB, 15dB and 20 dB. SNR is defined as

$$SNR = 10log_{10}\frac{\text{signal power}}{\text{noise power}}$$

where signal power $= |\alpha_1|^2 + |\alpha_2|^2$ and noise power $= E|X(t)|^2$.

All these computations are performed at the Pennsylvania State University , using SUN workstation and using the IMSL random deviate generators. For a particular data set we estimate the nonlinear as well as the linear parameters by the least squares method. We compute the average estimates and the average mean squared errors (MSE's)over 1000 replications. The results can be obtained directly from the authors on request. We present the main findings. From the results it is observed that SNR increases the MSE's decrease for both the linear and the nonlinear parameters which indicates the consistency of the least squares estimators of both the linear as well as nonlinear parameters. It is also clear from the results that the rate of convergence of the nonlinear parameters are more than that of the linear parameters. As SNR increases the MSE becomes closer to the asymptotic variance (ASVA), although in some cases, particularly for small SNR, it is observed that the MSE is more than the ASVA. For the nonlinear parameters MSE-ASVA converges to zero faster than that of the linear parameters.

6. CONCLUSIONS:

In this paper we consider the one parameter and multiparameter superimposed exponential signal model under the assumption of additive stationary noise. We obtain the asymptotic properties of the least squares estimators directly which generalizes some of the existing results. We also obtain the explicit expression of the covariance matrix for the multiparameter case, which it seems is not available in the literature. We prove the consistency and the asymptotic normality of the estimator of the error variance in this general set up. From the numerical studies it is observed that the finite sample inference can be drawn from the asymptotic result for reasonable sample sizes for both the linear and nonlinear parameters.

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APPENDIX:

Proof of Lemma 1:

$$\frac{1}{N}\sum_{t=1}^{N} U(t)cos(t\theta) = \frac{1}{N}\sum_{t=1}^{N}\sum_{j=-\infty}^{\infty} \alpha(j)\epsilon(t-j)cos(t\theta)$$

$$= \frac{1}{N}\sum_{j=-\infty}^{\infty} \alpha(j)cos(j\theta)\sum_{t=1}^{N}\epsilon(t-j)cos((t-j)\theta)$$

$$-\frac{1}{N}\sum_{j=-\infty}^{\infty} \alpha(j)sin(j\theta)\sum_{t=1}^{N}\epsilon(t-j)sin((t-j)\theta)$$

Therefore

$$\begin{aligned} \sup_{\theta}\left|\frac{1}{N}\sum_{t=1}^{N} U(t)cos(t\theta)\right| &\leq \sup_{\theta}\left|\frac{1}{N}\sum_{j=-\infty}^{\infty} \alpha(j)cos(j\theta)\sum_{t=1}^{N}\epsilon(t-j)cos((t-j)\theta)\right| \\ &+ \sup_{\theta}\left|\frac{1}{N}\sum_{j=-\infty}^{\infty} \alpha(j)sin(j\theta)\sum_{t=1}^{N}\epsilon(t-j)sin((t-j)\theta)\right| \qquad (21) \end{aligned}$$

We would like to prove that both the terms on the right hand side of (21) converges to zero as N tends to infinity. Now observe that

$$E\sup_{\theta}\left|\frac{1}{N}\sum_{j=-\infty}^{\infty}\alpha(j)cos(j\theta)\sum_{t=1}^{N}\epsilon(t-j)cos((t-j)\theta)\right| \leq$$
$$\frac{1}{N}\sum_{j=-\infty}^{\infty}|\alpha(j)|E\sup_{\theta}\left|\sum_{t=1}^{N}\epsilon(t-j)cos((t-j)\theta)\right| \leq$$
$$\frac{1}{2N}\sum_{j=-\infty}^{\infty}|\alpha(j)|\{E\sup_{\theta}\left|\epsilon(t-j)e^{i(t-j)\theta}\right| + E\sup_{\theta}\left|\epsilon(t-j)e^{-i(t-j)\theta}\right|\} \tag{22}$$

Also

$$\frac{1}{2N}\sum_{j=-\infty}^{\infty}|\alpha(j)|E\sup_{\theta}\left|\epsilon(t-j)e^{i(t-j)\theta}\right| \leq \frac{1}{2N}\sum_{j=-\infty}^{\infty}|\alpha(j)|\left\{E\sup_{\theta}\left|\epsilon(t-j)e^{i(t-j)\theta}\right|^2\right\}^{\frac{1}{2}}$$
$$\leq \frac{1}{2N}\sum_{j=-\infty}^{\infty}|\alpha(j)|\left\{N+\sum_{t=-N+1}^{N}E\left(\left|\sum_{m}\epsilon(m)\epsilon(m+t)\right|\right)\right\}^{\frac{1}{2}} \tag{23}$$

where the sum $\sum_{t=-N+1}^{N}$ omits the term $t=0$ and the sum $\sum_m$ is over $N-|t|$ terms (dependent on j). Since

$$\sum_{t=-N+1}^{N}E\left(\left|\sum_{m}\epsilon(m)\epsilon(m+t)\right|\right) \leq \sum_{t=-N+1}^{N}\left\{E\left|\sum_{m}\epsilon(m)\epsilon(m+t)\right|^2\right\}^{\frac{1}{2}} = O(N^{\frac{3}{2}})$$

uniformly in j, therefore (23) is $O(N^{-\frac{1}{4}})$ and so (22) is also $O(N^{-\frac{1}{4}})$. Observe that if we choose any subsequence $\{N^{\delta}\}$ of $\{N\}$, where $\delta > 4$, then we can make (22) to be summable over that subsequence, so let's choose in particular the subsequence $K = N^5$. Therefore,

$$E\sup_{\theta}\left|\frac{1}{K}\sum_{j=-\infty}^{\infty}\alpha(j)cos(j\theta)\sum_{t=1}^{K}\epsilon(t-j)cos((t-j)\theta)\right| = O(K^{-\frac{5}{4}})$$

Similarly the result is true if the cosine function is replaced by sine function. So we have

$$E\sup_{\theta}\left|\frac{1}{K}\sum_{t=1}^{K}U(t)cos(t\theta)\right| = O(K^{-\frac{5}{4}})$$

Therefore, by using the Chebyshev's inequality we obtain,

$$\sup_{\theta}\left|\frac{1}{K}\sum_{t=1}^{K}U(t)cos(t\theta)\right| \to 0 \qquad a.s.$$

when $K = N^5$. Now

$$\sup_{\theta} \sup_{N^5 < J \leq (N+1)^5} \left| \frac{1}{N^5} \sum_{t=1}^{N} U(t) cos(t\theta) - \frac{1}{N^5} \sum_{t=1}^{J} U(t) cos(t\theta) \right| \leq \frac{1}{N^5} \sum_{t=N^5+1}^{(N+1)^5} |U(t)| \qquad (24)$$

The mean squared errors of the right hand side of (24) is dominated by N^{-2}, therefore, the left hand side of (24) converges to zero *a.s.*, which proves the lemma.

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