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De Physica Belli

An Introduction to Lanchestrial Attrition Mechanics Part Three

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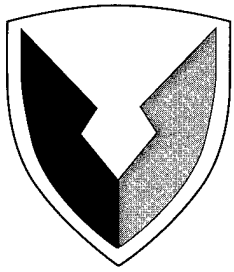

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13. ABSTRACT (*Maximum 200 words*)

Physics and War are inextricable. While it is often recognized that war is often the generator of progress in physics, the role of physics in war is less often recognized. This work is dedicated to examination of the physics inherent in some of the processes of war. The framework for this examination is the Lanchester model of attrition. Part I of this work is primarily concerned with the basic foundation of this model . Part II covers the basic physics inherent to attrition rate coefficients. This volume, which comprises Part III covers heterogeneous Lanchester theory and aggregation: This includes basic definition of the mathematics and symbology associated with the expansion from two forces, each comprised of only one type of element, to two forces, each comprised of several different types of elements either organizationally, or in terms of their weaponry. This diversity naturally leads to the problem of fire allocation, and hence to the formal theory of aggregation which permits the heterogeneous Lanchester problem to be reduced to the homogeneous Lanchester problem and provides the connection between Lanchester theory and high resolution combat simulation. Part IV (to be published) will cover special and advanced topics.

14. SUBJECT TERMS Attrition Theory, Lanchester, Lanchester Attrition Theory, Quadratic Lanchester Equations, Mixed Lanchester Equations, Attrition Differential Equations, Quantified Judgement Model, Epstein Model, Osipov's 3/2 Law, Constant Rate Attrition, Exponential Attrition, Helmbold's Attrition, Time Dependent Attrition, Range Dependent Attrition, Stochastic Duels, Stochastic Attrition, Stochastic Lanchester, Linear Attrition, Attrition Coefficients, Attrition Rates, One-on-one, Force-on-Force

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De Physica Belli

An Introduction to Lanchestrian Attrition Mechanics

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Part III:

Heterogeneous Lanchester Attrition Theory - Fire Allocation and Theory of Aggregation

Foreword

Warfare is as old as recorded human history. War has been especially prevalent in the last 500 years with the increasing conflict between large nation states. A great amount of analysis and thought has been given to the "Art of War". Nine principles of War have been defined: Objective, Offensive, Mass, Economy of Force, Maneuver, Unity of Command, Security, Surprise, and Simplicity. Despite these accepted principles, the science of war has remained elusive. Since World War II, investigators have searched for a theory on the physics of war--"De Physica Belli". Efforts have been more successful with the prominent rise of Operations Research as an analysis tool to assist combat operations. Dr. Bruce W. Fowler uses these modem analytical tools to seek the answer to the following question in this report--"Is there any scientific basis to describe the physics of war?" This report provides the answer to this question. His approach to a physics of war is the application of Lanchestrian attrition mechanics which first appeared in theory in the early 1900's.

In Part III, Dr. Fowler deals with the heterogeneous Lanchester attrition theory, which inherently deals with forces comprised of more that one type of unit each. The scope of this coverage is fundamental and in keeping with the two previous parts of this work, educational in nature. He introduces the formulation of the attrition differential equations as matrix differential equations, and describes the basics of eigenvalues and eigenvectors. On this basis, Dr. Fowler then covers the basic problem associated with mixed unit forces of fire allocation in the context both of mathematical optimization and human nature. Next, he introduces the formal theory of aggregation and establishes its role in modeling and simulation. Finally, Dr. Fowler reviews the formulation of stochastic Lanchester theory and relates this to other models.

"De Physica Belli" is intended to be a general reference and introduction to attrition theory suitable for the combat soldier, the student-soldier, or the military analyst. The manuscript succeeds in that respect and provides a good overall summary of the state-of-practice in attrition theory through 1990. However, given the great advances in modeling, simulation and computational power since 1990, it would not be surprising to see future updates to this work. The mathematical tools of complexity theory, fractal dimensions, fuzzy logic, information theory and the power of scientific visualization of data in interactive computer simulations may offer new and exciting insights into the physics of war. These new developments will most certainly provide opportunities to conduct experiments in the science of warfare that go beyond the limitations inherent in the analysis of historical data.

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(continued)

Chapter 1

Introduction to Part III

With this Part III of the work, we have now completed our consideration of both the basic fundamentals of classical Lanchester attrition theory, and the fundamental scientific underpinnings of the conjugate theory of the attrition rate coefficients, Parts I and II, respectively. This does not mean that we will be ignoring these results. Rather, we now move on to consideration of more advanced topics that build on and reinforce what has been considered previously.

In Part I, we were primarily concerned with the mathematical properties and underlying assumptions of the classical archetypes of the Lanchester attrition differential equations, the Linear and Quadratic equations, and the extended variations including the generalized Osipov differential equation. In Part II, we were primarily concerned with the conjugate theory of attrition rate coefficients that provides the connection between the neatness of the rate based attrition theory that draws its evolution from Lanchester and Osipov, and the fundamental physical and psychological principles that describe the phenomenology of our human oriented but physical world.

All of this consideration has essentially had a *lumpen* aspect that seems to harken back to the simplistic view of basic Newtonian mechanics. Forces, which we know either from direct experience with the Armed Forces, or study of history or doctrinal manuals, have structure both of an organizational and a physical sense. Armored battalions occupy space and are not just comprised of tanks firing their weapons against the same target in a consistently prescribed fashion. Yet most of our consideration in Part I considered such units as the equivalent of being comprised of identical elements which had fallen into some strange sort of black hole which compressed their spatial and temporal behavior and character into a sort of two-space and time geometrical point mass, while permitting their attritional process behavior to interact beyond the horizon of the black hole.

In Part II, we considered the physical and psychological basis of the conjugate theory of attrition rate coefficients, and nothing we obviously considered there was restricted to this geometric point compression. Indeed, the overt relaxation of the range constancy restriction implicit in the assumption structure of Part I has set the stage for consideration of further relaxation of the geometric point consideration of forces and their attritional interactions.

Chapter 1 Introduction to Part III

This then is both the prescription and the intent of Part III of the work. We shall now consider a succession of formalisms which relax the compression of force structure into a geometric point. Our intent here is twofold. First, we want to describe bases for consideration of battlefield attritional processes in a more generalized, more obviously realistic, in the sense of sensual observation of the world and our mental models of the same, form than the geometric point representation that we have described thus far. Second, we do not want to lose connection with the vast body of consideration, formalism, and insight that is embodied in that geometric point picture.

This leads to an outline that is somewhat reminiscent of some stylized dance such as may be performed about a May Pole. We shall move away from the centrality of the geometric point picture by generalization. of the mathematical formalism and then move back to the center whenever possible. In this manner, we shall weave a complex picture variation of greater generality in the treatment of space, time, and organization as it applies to attritional processes.

The fundamental concept of this complex picture is one that we have briefly dealt with previously, either overtly or covertly, either inherently or explicitly. This concept is *Aggregation*. Our goal throughout in establishing the connection between more generalized formalisms and the geometric point picture is to rest this connection on as firm a foundation of aggregation as possible. In some cases, we shall be successful in varying degrees, while in others, aggregation must proceed in a manner that can often be characterized as *ad hoc* at best.

In keeping with the custom established in previous Parts, I wish now to dedicate this Part III to individuals who have contributed to my investigations. Sadly, I cannot acknowledge all of those who have done so and so I present apologies to those many folk who have triggered my mind and who have contributed to my understanding and insight. In particular, I must thank Dr. James Taylor of the Naval Post Graduate School who triggered my interest in the subject of spatially distributed Lanchester theory during a lecture he presented during the Modeling, Simulation, and Gaming of Warfare course of Dr. Griff Callahan (COL, USA Ret.) at the Georgia Institute of Technology, I believe about 1982, and Dr. Ben Wise, whose affiliation I am now uncertain of, who through the kind offices of Griff Callahan, triggered my interest in the area of State Solutions for Heterogeneous Force Attrition Differential Equations in about 1989. Of course, I must acknowledge the information brokering and mentoring role of Griff, without whom none of this work would have occurred.

Chapter 35

Heterogeneous Force Attrition Differential Equations

35.1 Introduction

In this chapter, we begin our consideration of heterogeneous force attrition differential equations (ADEs). In this case, we examine relaxation of the organization aggregation of the force structure of units although we do retain the geometric point aggregation of their spatial extent. This is an obvious generalization of the methodology and formalism established in Part I. Indeed, a simplified variation of this formalism was considered by Osipov in his initial work.[1]

Osipov considered the addition of artillery and machine guns to an infantry force weaponed with rifles. He gives a pair of attrition solutions,

$$\begin{aligned}
\left[A_i + \frac{\zeta}{\alpha}A_{AR} + \frac{\varepsilon}{\alpha}A_{MG}\right](t) &= \left[A_i + \frac{\zeta}{\alpha}A_{AR} + \frac{\varepsilon}{\alpha}A_{MG}\right]\Big|_{t=0} \cosh\left(\sqrt{\alpha\beta}t\right) \\
&\quad -\sqrt{\frac{\beta}{\alpha}}\left[B_i + \frac{\delta}{\beta}B_{AR} + \frac{\xi}{\beta}B_{MG}\right]\Big|_{t=0} \sinh\left(\sqrt{\alpha\beta}t\right), \\
\left[B_i + \frac{\delta}{\beta}B_{AR} + \frac{\xi}{\beta}B_{MG}\right](t) &= \left[B_i + \frac{\delta}{\beta}B_{AR} + \frac{\xi}{\beta}B_{MG}\right]\Big|_{t=0} \cosh\left(\sqrt{\alpha\beta}t\right) \\
&\quad -\sqrt{\frac{\alpha}{\beta}}\left[A_i + \frac{\zeta}{\alpha}A_{AR} + \frac{\varepsilon}{\alpha}A_{MG}\right]\Big|_{t=0} \sinh\left(\sqrt{\alpha\beta}t\right).
\end{aligned} \tag{1}$$

We may note several things about these two equations. First, the roles of α and β are reversed from our normal convention. Second, it is assumed that although artillery (AR) and machine guns (MG) add to the attrition of the overall force, they are attrited at a rate proportional to their lethality. This is the first instance that I know of where force heterogeneity is taken into

account.

35.2 Homogeneous Lanchester Review

Before embarking on discussion of heterogeneity on the basis of force structure, we shall pause briefly to review the basic form of homogeneous Lanchester ADE archetypes. The basic archetype is the Quadratic Lanchester ADE which has the pair form,

$$\begin{aligned} \frac{dA}{dt} &= -\alpha B, \\ \frac{dB}{dt} &= -\beta A. \end{aligned} \tag{2}$$

The Linear Lanchester ADE has the form,

$$\begin{aligned} \frac{dA}{dt} &= -\alpha' AB, \\ \frac{dB}{dt} &= -\beta' BA. \end{aligned} \tag{3}$$

While the Osipov 3/2 ADE has the form,

$$\begin{aligned} \frac{dA}{dt} &= -\alpha" \sqrt{A} B, \\ \frac{dB}{dt} &= -\beta" \sqrt{B} A. \end{aligned} \tag{4}$$

The state and implicit solutions, and assumptions, of these ADEs are described in Part I.

In their original interpretation, these ADEs represented different battlefield conditions (constant versus variable density,) weapons mode of operation (point verses area attack), and doctrinal restrictions (area versus line fire.) We have seen in Part II that these archetypes represent limiting or asymptotic cases. Thus, we may write a general Lanchester ADE of the form

$$\begin{aligned} \frac{dA}{dt} &= -\alpha() B, \\ \frac{dB}{dt} &= -\beta() A. \end{aligned} \tag{5}$$

The Attrition Rate Coefficients (ARCs) α and β are indicated as being functions of some variables, but we have not specified these variables. In general, we know from Part II that these variables include force strengths and ranges.

35.3 Heterogeneous Lanchester ADEs I

We start our consideration of Heterogeneous Lanchester ADEs by considering a two forces, Red and Blue, which are comprised of different types of elements. In common usage, these elements may be characterized on the basis of different weapon systems, or different organizational elements. Care must be taken in both cases to conform to the restrictions and assumptions associated with geometric point aggregation, but especially for the latter. Assume that each force consists of N_A, N_B sets of elements and that we have aggregated these sets of elements together individually. Thus, the Red force consists of N_B force strengths, $A_i(t)$, and similarly for the blue force.

Further assume that we may define or calculate an ARC for each Red force strength to attrit each Blue force strength, and reciprocally for the Blue force strengths attriting the Red force strengths. All of these ARCs are not necessarily nonzero.

In this case, we may write two sets of ADEs for the two total forces. The ADEs for the Blue force are

$$\frac{dA_i}{dt} = -\sum_{j=1}^{N_B} \alpha_{i,j}() B_j, \tag{6}$$

while the ADEs for the Red force are

$$\frac{dB_j}{dt} = -\sum_{i=1}^{N_A} \beta_{j,i}() A_i. \tag{7}$$

The quantities $\alpha_{i,j}()$ and $\beta_{j,i}()$ are the ARCs for the i^{th} Red force being attrited by the j^{th} Blue force, and reciprocally for the Blue forces being attrited by the Red forces. There are thus a total of $N_A + N_B$ ADEs in this characterization.

In general, solution of systems of ADEs such as this are possible only if the ADEs are linear in a mathematical sense. We shall consider this restriction in detail later in this and other chapters. This means that in general we may find analytical solutions under only very restricted conditions. Of course, we still have the option of solving these equations numerically. As with the homogeneous ADEs, these equations are generally highly stable, so long as all of the force strengths are not too small (or negative), so that open integral approximations such as

$$\begin{aligned} A_i(k+1) &\simeq A_i(k) - \sum_{j=1}^{N_B} \alpha_{i,j}() B_j(k) \Delta t, \\ B_j(k+1) &\simeq B_j(k) - \sum_{i=1}^{N_A} \beta_{j,i}() A_i(t) \Delta t, \end{aligned} \tag{8}$$

where Δt is the time increment indexed by $k = 0..$ This class of approximation can be imple-

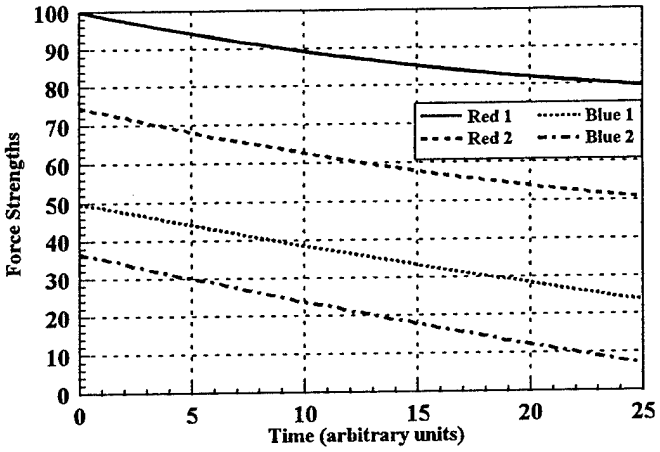

Figure 1: Two by Two Pure Quadratic Heterogeneous Attrition

mented using our usual spreadsheet simulation.[1]

To illustrate this, we present two sample calculations. The first is a 2 X 2 engagement (that is, 2 Red forces and 2 Blue forces) that are purely quadratic in attrition interaction. All Red forces fire on all Blue forces, and *visa versa*. The results of these calculations, performed using a simple spreadsheet simulation, are shown in Figure 1.

To contrast with this, we also present calculations where the first force on each side is Quadratic but the second is Linear in their attrition interactions. To promote comparison, the linear attrition ARCs have been scaled from the corresponding Quadratic ARCs of the preceding example by dividing by the initial force strengths. These calculations are shown in Figure 2.

[1]This Part witnesses a departure from our normal mode of simulation. Much of the formalism here and many of the techniques cannot be easily nor readily simulated using a spreadsheet. Recourse to actual coding becomes necessary with frequency. We shall try to be explicit in noting these cases.

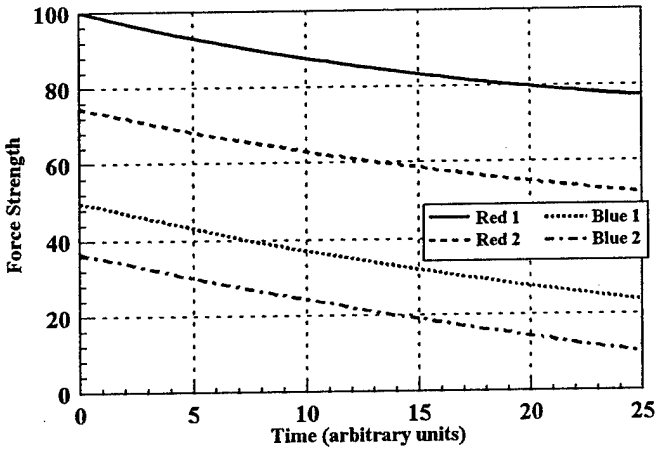

Figure 2: Two by Two Mixed Quadratic-Linear Hetherogeneous Attrition

35.4 Heterogeneous Lanchester ADEs II

It is sometimes useful to separate the two sets of force strengths into subsets based on the nature of the attrition interaction. Suppose that the Red force consists of two subsets of N_R^Q forces that attrit by point attack (thus, Quadratic), and N_R^L forces that attrit by area attack. The Blue force is similarly divided. In this case, the ADEs have the forms

$$\begin{aligned}
\frac{dA_i^Q}{dt} &= -\sum_{j=1}^{N_B^Q} \alpha_{i,j}^Q B_j^Q - \sum_{j=1}^{N_B^L} \alpha_{i,j}^L A_i^Q B_j^L, \\
\frac{dA_i^L}{dt} &= -\sum_{j=1}^{N_B^Q} \alpha_{i,j}^Q B_j^Q - \sum_{j=1}^{N_B^L} \alpha_{i,j}^L A_i^L B_j^L, \\
\frac{dB_j^Q}{dt} &= -\sum_{i=1}^{N_A^Q} \beta_{j,i}^Q A_i^Q - \sum_{i=1}^{N_A^L} \beta_{j,i}^L B_j^Q A_i^L,
\end{aligned} \tag{9}$$

$$\frac{dB_j^L}{dt} = -\sum_{i=1}^{N_A^Q} \beta_{j,i}^Q A_i^Q - \sum_{i=1}^{N_A^L} \beta_{j,i}^L B_j^L A_i^L.$$

We considered a much simplified version of this earlier in Part I.

This divided notation helps numerical calculations by explicitly showing the forms of the ADEs, but it also emphasizes the potential problems that we shall have in actually solving these sets of ADEs.

35.5 Mathematical Linearization

We have already indicated that straightforward solution of heterogeneous ADEs is largely a matter of having all of the differential equations linear in the mathematical sense. This presents some problems since we know that some attrition interactions will be Linear (in a Lanchester sense), and a few may be even more complex. In Chapter 33, however, we examined approximations for the exact ARCs that first removed the dependence on the ADEs on the target force strengths and then on range. Can we try the same approximation in the heterogeneous problem?

To examine this, we posit a set of calculations using the Quadratic, Linear, and 3/2 ADEs described above. Basically, we want to replace target force strengths in the latter two type of ADEs with the initial target force strengths. By doing this, approximate ADEs result which are Quadratic in a Lanchester sense and linear in a mathematical sense. That is, we replace Linear Lanchester ADEs of the form

$$\begin{aligned} \frac{dA}{dt} &= -\alpha AB, \\ \frac{dB}{dt} &= -\beta BA, \end{aligned} \tag{10}$$

with approximate Quadratic Lanchester ADEs of the form,

$$\begin{aligned} \frac{dA}{dt} &\simeq -\alpha A_0 B = -\alpha' B, \\ \frac{dB}{dt} &\simeq -\beta B_0 A = -\beta' A. \end{aligned} \tag{11}$$

A similar approximation is made for Osipov 3/2 attrition order ADEs.

To examine this, we generated a set of attrition rates for each type of equation that was scaled by the Quadratic ARCs divided by the target initial force strength raised to the power of two minus the attrition order. Thus, e.g.,

$$\alpha_{Linear} = \frac{\alpha_{Quadratic}}{A_0}. \tag{12}$$

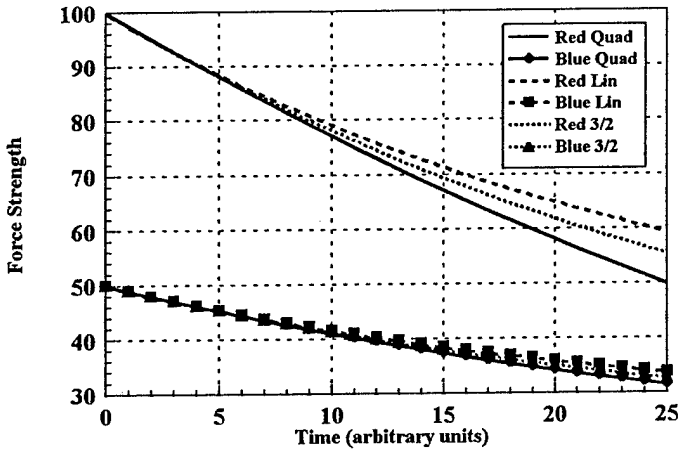

Figure 3: Comparison of Quadratic, Linear, and Osipov 3/2 Exact Calculations

Example calculations of the three "exact" ADEs are shown in Figure 3. We call these exact since they were solved numerically using a spreadsheet simulation.

Next, we made comparison calculations for the Linear Lanchester and Osipov 3/2 ADEs approximated as Quadratic Lanchester ADEs. These are shown in Figures 4 and 5.

We note that (visually), this approximation seems to be reasonable to force strengths of about 80% of initial. This is wonderful! It offers us the hope that we may approximate all Lanchester ADEs as Quadratic, and thus linear from a mathematical sense. The question is, can we live with having calculations truncated at 20% loses?

To examine this, we return to our historical data bases. In this case, we calculated the remaining force strength fraction for each side in each battle in our databases, and form an unbinned, Kolomogorov-Shmirnov type of cumulative distribution from each database. These are shown in Figure 6. From this, we may see that in 80-90% of the battles in each database, the remaining force strength fraction is greater then or equal to 80%. Thus, we are on reasonably comfortable ground if we "Quadraticize" all of our heterogeneous ADEs so that they are mathematically linear. Even for those cases where we know that we shall have less than 80% force strength fraction remaining, we may safely approximate attrition by dividing the calculation into successive

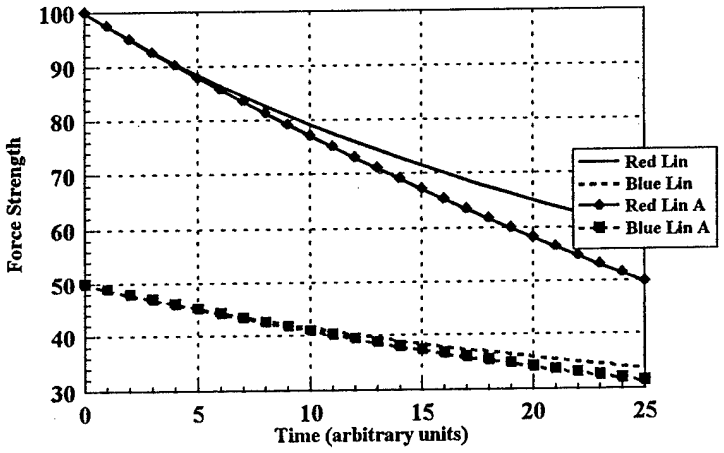

Figure 4: Comparison of Linear Lanchester and approximate Linear as Quadratic Lanchester Calculations

approximations of this extent.

35.6 Heterogeneous Lanchester ADEs III

Before concluding this brief chapter introducing the notation of heterogeneous Lanchester ADEs, it is useful to introduce one more set of notation. In this case, we reduce the two sets of Force Strength ADEs into one set. In this case, the force strengths are represented by a vector $\overrightarrow{F}$ with components F_i. By inspection, $F_i \equiv A_i, i = 1..N_A$, and $F_{i+N_A} \equiv B_i, i = 1..N_B$. The vector ADE has the form

$$\frac{d}{dt}\overrightarrow{F} = \overleftrightarrow{\Upsilon} \bullet \overrightarrow{F}, \tag{13}$$

where the ARC array is of the form

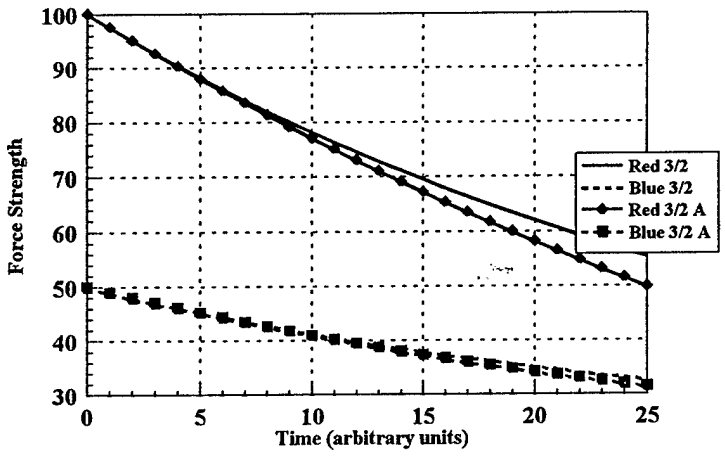

Figure 5: Comparison of Osipov 3/2 and approximate Osipov 3/2 as Quadratic Lanchester Calculations

$$\overleftrightarrow{\Upsilon} \equiv \begin{bmatrix} \overleftrightarrow{0} & -\overleftrightarrow{\alpha} \\ -\overleftrightarrow{\beta} & \overleftrightarrow{0} \end{bmatrix}, \tag{14}$$

where: $\overleftrightarrow{\alpha}$ = Blue ARC array on Red, and $\overleftrightarrow{\beta}$ = Red ARC array on Blue. We note that $\overleftrightarrow{\alpha}$ has N_A rows and N_B columns while $\overleftrightarrow{\beta}$ has N_B rows and N_A columns. Thus, the upper left hand zero subarray has N_A rows and columns, and the lower right hand zero subarray has N_B rows and columns. Equation 13 can also be written in component-summation notation as

$$\frac{dF_i}{dt} = \sum_{j=1}^{N_A+N_B} \Upsilon_{i,j} F_j. \tag{15}$$

This notation has the advantage of clearly reducing the perceived form of the ADEs to a single set. We shall use this notation fairly extensively in considering solutions and aggregation in a subsequent chapter. Note that the minus sign has been imbedded into the array itself.

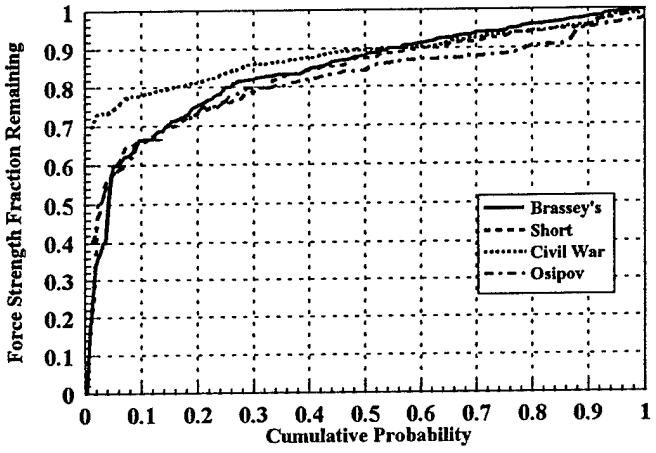

Figure 6: Cumulative Force Fraction Remaining Unbinned Distributions

35.7 References

[1] Osipov, M., "The Influence of the Numerical Strength of Engaged Forces on their Casualties", Voennie Shornik, 3-7, 1915, Robert L. Helmbold, and Allan S. Rehm, trans., U. S. Army Concepts Analysis Agency, Bethesda, MD., Research Paper CAA-RP-91-2, September 1992.

Chapter 36

Fire Allocation

36.1 Introduction

In this chapter, we take up a subject that is new to our view, fire allocation. This is not a subject that occurs in considering geometric point Lanchester theory since there are only two forces: one Red; and one Blue; both of which normally fire only on each other (neglecting fratricide, a subject that we shall take up in Part IV.) Thus, in homogeneous Lanchester theory, the problem is how to average different weapons or unit types. (This is a subject that we shall examine in the next chapter.) In heterogeneous Lanchester theory, the problem is how force strength is aligned against the target population.

The conjugate theory of attrition rate coefficients (ARCs) that we described in Part II provided a physical and psychological basis for the calculation of the mean time for a given weapon system to kill (acquire and engage) a given target system. From that mean time to kill, the rate at which weapon systems kill target systems can be calculated directly. This is the fundamental success of the conjugate theory. Unfortunately what that conjugate theory does not tell us is how weapon systems are paired with target systems. That is the subject of this chapter.

36.2 Fire Allocation

As in the preceding chapter, we assume that there are two forces: one Red; one Blue; that are engaged in combat. We designate these subforces as A_i and B_j, respectively. Each force is comprised of some number of subforces: N_A Red subforces; and N_B Blue subforces. We define two ARC matrices: $\alpha_{i,j}$; and $\beta_{j,i}$; respectively, $i = 1..N_A$, $j = 1..N_B$. If we combine these subforce strengths in a force strength vector

$$\overrightarrow{F} \equiv \begin{pmatrix} (A) \\ (B) \end{pmatrix} = A_i \hat{e}_i + B_j \hat{e}_j \tag{1}$$

in matrix notation as an $(N_A + N_B)$ X 1 matrix, then a combined ARC matrix is definable as

$$\overleftrightarrow{\Upsilon} \equiv \begin{pmatrix} \overleftrightarrow{0} & -\overleftrightarrow{\alpha} \\ -\overleftrightarrow{\beta} & \overleftrightarrow{0} \end{pmatrix} = \xi_{k,l}\hat{e}_k\hat{e}_l, \tag{2}$$

which is $(N_A + N_B)$ X $(N_A + N_B)$. The individual ARCs in this matrix are presumed to be calculated using the formalism described in Part II, and linearized using techniques described there and in the preceding chapter. The basis unit vectors $\hat{e}_i$ are assumed to have a Kronecker delta function orthogonality. We defined an evolution or rate equation for this notation as

$$\frac{d}{dt}\overrightarrow{F} = \overleftrightarrow{\Upsilon} \bullet \overrightarrow{F}. \tag{3}$$

We may now reveal that this picture is a lie, or at least only partly accurate. Since the $\xi_{k,l}$ are presumed calculated as ARCs on a pure basis, this picture is in error because it overcounts the attrition. Quite simply, the elements of each subforce can only engage a number of targets (point or area) equal in number to that subforce strength at any instant of time. Thus, since this picture applies all of a subforce to each ARC, the effect is to allow each subforce element to simultaneously engage as many targets as there are ARCs in the corresponding row of the ARC matrix.

To correct this overcounting, we introduce the concept of Fire Allocation which is simply the assignment of the totality of each subforce's elements among the potential targets. Since the subforces are aggregated, the most common notation for designating this allocation is to assign fractions of the total subforce strength against other subforces. These fractions are designated as $f_{k,l}$ where k indicates the target subforce and l indicates the engaging subforce. Since these are fractions, they must obey constitutive relations,

$$\sum_{k=1}^{N_A+N_B} f_{k,l} = 1. \tag{4}$$

These relations assure that only as many targets are engaged as there are subforce elements.

With these fractions and constitutive relations, the evolution or rate equation becomes

$$\frac{d}{dt}\overrightarrow{F} = \overleftrightarrow{\Gamma} \bullet \overrightarrow{F}, \tag{5}$$

where:

$$\overleftrightarrow{\Gamma} \equiv f_{k,l}\xi_{k,l}\hat{e}_k\hat{e}_l. \tag{6}$$

The remainder of this chapter will be primarily concerned with the different approaches to calculating the subforce fractions. These approaches basically fall into two categories:

- approaches based on optimization; and

- approaches based on physical and psychological processes.

The former category tends to be defined by optimization techniques that are the province of classical Operations Research. The latter tends to be defined by accessible states and their densities.

36.3 Example Format

As is our custom, we will be presenting examples wherever possible. Since we are dealing with optimization for several of these approaches, it is difficult to present visualizations for problems more complicated that 2 X 2. Thus, the examples that we shall present will be within a common format. Specifically, we shall consider a combat where the two forces are each comprised of two subforces. For clarity, we shall explicitly split the calculations into two separate sets of ADEs,

$$\begin{aligned} \frac{dA_i}{dt} &= -\alpha_{i,j} B_j, \\ \frac{dB_j}{dt} &= -\beta_{j,i} A_i. \end{aligned} \tag{7}$$

The ARC matrices will always be the same, specifically,

$$\alpha_{i,j} = \begin{pmatrix} 0.01 & 0.02 \\ 0.02 & 0.01 \end{pmatrix}, \tag{8}$$

and

$$\beta_{j,i} = \begin{pmatrix} 0.005 & 0.01 \\ 0.01 & 0.005 \end{pmatrix}. \tag{9}$$

We note in passing that these two matrices are symmetric in the sense that the ratios of elements are constant. This will have special significance in optimization approaches to fire allocation.

The initial force strengths are $(A_1(0), A_2(0), B_1(0), B_2(0)) = (100, 75, 50, 37)$. Note that in a homogeneous draw sense, the Blue on Red ARCs are too small by half.

Because of the constitutive relations, we have to consider only two fractions for each force. (Since for each subforce, the sum of fractions must be one. If we designate the fractions of the two Blue subforces allocated to Red subforce 1 by f_1 and f_2, and the fractions of the two Red subforces allocated to Blue subforce 1 by g_1 and g_2, then the combined ARC matrix has the form

$$\overleftrightarrow{\Gamma} = \begin{pmatrix} 0 & 0 & f_1\alpha_{1,1} & f_2\alpha_{1,2} \\ 0 & 0 & (1-f_1)\,\alpha_{2,1} & (1-f_2)\,\alpha_{2,2} \\ g_1\beta_{1,1} & g_2\beta_{1,2} & 0 & 0 \\ (1-g_1)\,\beta_{2,1} & (1-g_2)\,\beta_{2,2} & 0 & 0 \end{pmatrix}. \tag{10}$$

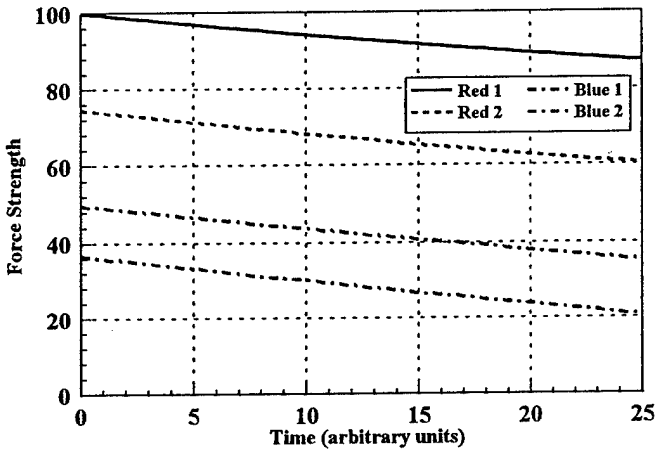

Figure 1: Nominal Fire Allocation - All Fractions = 0.5

As an introductory example, we consider the nominal case where the subforce fire allocations have a value of 0.5. This is shown in Figure 1.

36.4 Maximum Kills

The most common fire allocation approach found in the literature is based on maximizing kills. Over any period of time, the loss to Red is given by the integral equations,

$$\Delta A_i = A_i(0) - A_i(t) = \sum_{j=1}^{N_B} \int_0^t f_{i,j}\alpha_{i,j}B_j(t')\,dt', i = 1..N_A. \tag{11}$$

If t is small, then we may approximate the integrals as one point open integrations, so that these equations reduce to

$$\Delta A_i \simeq \sum_{j=1}^{N_B} f_{i,j}\alpha_{i,j}B_j(0)\,t, i = 1..N_A. \tag{12}$$

If we assess Red loss as simply the sum of these subforce losses, and recognize that time t is essentially arbitrary (if we stay away from any subforce strength becoming zero,) then we may note that the quantity

$$L_A = \sum_{i=1}^{N_A} \sum_{j=1}^{N_B} f_{i,j} \alpha_{i,j} B_j (0) \tag{13}$$

essentially represents the kills per time by the Blue force subject to the constitutive relations. This is the quantity that we want to maximize by the proper selection of values for $f_{i,j}$.

For the example we want to present, which is 2 X 2, the integral equations take on the form

$$\begin{aligned} \Delta A_1 &= A_1 (0) - A_1 (t) = \int_0^t \left[f_1 \alpha_{1,1} B_1 (t') + f_2 \alpha_{1,2} B_2 (t') \right] dt', \\ \Delta A_2 &= A_2 (0) - A_2 (t) = \int_0^t \left[(1 - f_1) \alpha_{2,1} B_1 (t') + (1 - f_2) \alpha_{2,2} B_2 (t') \right] dt', \end{aligned} \tag{14}$$

which approximate as

$$\begin{aligned} \Delta A_1 &\simeq \left[f_1 \alpha_{1,1} B_1 (0) + f_2 \alpha_{1,2} B_2 (0) \right] t, \\ \Delta A_2 &\simeq \left[(1 - f_1) \alpha_{2,1} B_1 (0) + (1 - f_2) \alpha_{2,2} B_2 (0) \right] t. \end{aligned} \tag{15}$$

The Red loss then becomes

$$\begin{aligned} L_A = &\left[f_1 \alpha_{1,1} B_1 (0) + f_2 \alpha_{1,2} B_2 (0) \right] + \\ &\left[(1 - f_1) \alpha_{2,1} B_1 (0) + (1 - f_2) \alpha_{2,2} B_2 (0) \right], \end{aligned} \tag{16}$$

which can be graphed as a surface plot by varying f_1 and f_2 from 0 to 1. This is shown in Figure 2. From this plot, it may easily be seen that $f_1 = 0$ and $f_2 = 1$ maximize Red kills by the Blue force. A plot of the force strength trajectories for these fractions is shown in Figure 3.

36.5 Minimum Loss

The minimum loss approach to Fire Allocation is an extension of the maximum loss approach. In this approach, the assumption is made that the enemy has made some sort of fire allocation, usually maximum kills. If we continue to examine the approach from the standpoint of the Blue force, then the starting point is the Blue losses, given by the integral equations

$$\Delta B_j = B_j (0) - B_j (t) = \sum_{i=1}^{N_A} \int_0^t g_{j,i} \beta_{j,i} A_i (t') \, dt', j = 1..N_B, \tag{17}$$

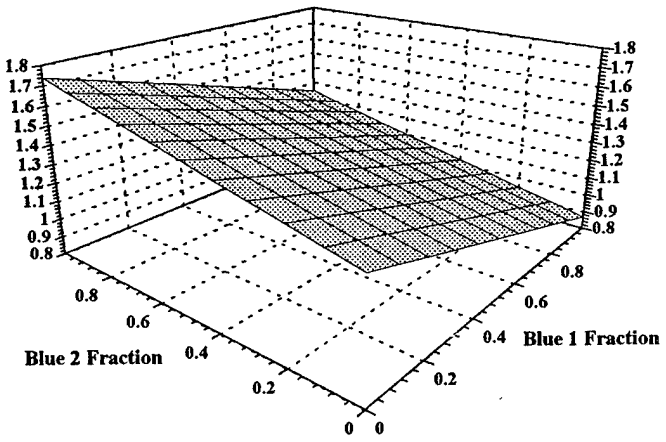

Figure 2: Maximum Kills 2 X 2 Example

which we write using the Red subforce equations as

$$\begin{aligned} \Delta B_j &= \sum_{i=1}^{N_A} \int_0^t g_{j,i}\beta_{j,i} dt' \left[A_i(0) - \sum_{k=1}^{N_B} \int_0^{t'} f_{i,k}\alpha_{i,k} B_k(t") \, dt" \right] \\ &= \sum_{i=1}^{N_A} \int_0^t g_{j,i}\beta_{j,i} A_i(0) \, dt' - \sum_{i=1}^{N_A} \int_0^t g_{j,i}\beta_{j,i} dt' \sum_{k=1}^{N_B} \int_0^{t'} f_{i,k}\alpha_{i,k} B_k(t") \, dt". \end{aligned} \tag{18}$$

Since the first integral on the rhs is a given and thereby constant (we have assumed some values for the $g_{j,i}$), the only quantity that contributes to the maximization is the second integral. If we apply the same integral approximation as before, then we may write a loss function to be minimized as

$$L'_B \approx -\sum_{i=1}^{N_B} \sum_{i=1}^{N_A} g_{j,i}\beta_{j,i} \sum_{k=1}^{N_B} f_{i,k}\alpha_{i,k} B_k(0). \tag{19}$$

Since this expression is negative only because of the leading minus sign, it is equally useful to maximize the negative of the expression.

Minimum Loss

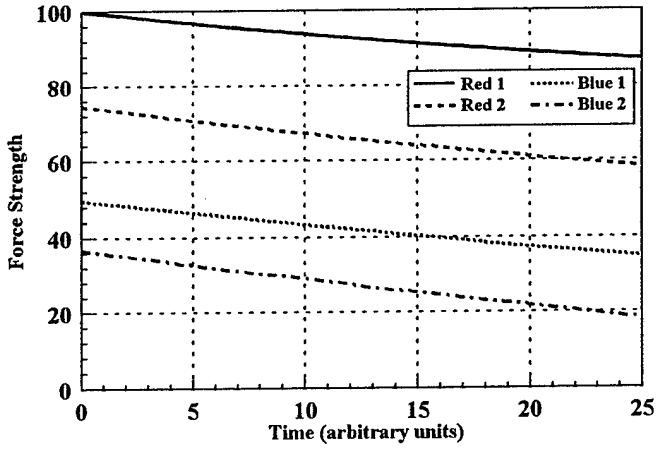

Figure 3: Heterogeneous Force Strength Trajectories - Maximum Kills Fire Allocation

In the context of our 2 X 2 example, if we assume that the Red force has allocated fire to maximize Blue losses, then the Red fire allocation matrix is

$$\overleftrightarrow{g} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{20}$$

The Blue loss is then

$$\begin{aligned} L'_B \approx & -\beta_{1,2}\left[(1-f_1)\,\alpha_{2,1}B_1(0) + (1-f_2)\,\alpha_{2,2}B_2(0)\right] \\ & -\beta_{2,1}\left[f_1\alpha_{1,1}B_1(0) + f_2\alpha_{1,2}B_2(0)\right] \end{aligned} \tag{21}$$

This surface is plotted in figure 4. It may be seen that the fire allocation fractions are $f_1 = 0$ and $f_2 = 1$, which is the same as we realized for the Maximum Kills example. Thus the force trajectories for this example are the same as shown in figure 3.

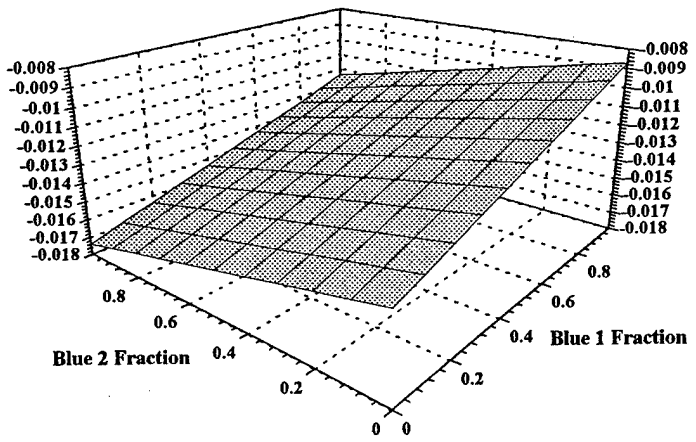

Figure 4: Minimum Losses 2 X 2 Example

36.6 Segue

This is not an exhaustive exposition of fire allocation schemes based on optimization. The astute student will note that the complicated mechanics is not really needed in many cases. If the loss functions are linear in the allocation fractions, then the optimized solution is usually to set to 1 the fraction in each column of the ARC matrix which corresponds to the maximum value ARC with the rest of the fractions set to zero. This is essentially a pure or integer optimization.

There seem to be a number of similar schemes, based on optimization of some loss function and implemented by some classical Operations Research technique such as Linear Programming, SIMPLEX, or even game theory. An interesting example of the latter is the work of Sternberg.[1]

There are two difficulties arising from the use of optimization schemes such as these. First, since there are other targets present on the battlefield, the calculation of the ARCs must be modified to account for the acquisition of targets of types that the fire allocation scheme effectively dictates not be engaged. This effectively forces target recognition/identification into the target acquisition process and adds a process of target rejection to decide not to pass these targets to the engagement process. Both of these increase the mean time to kill and thereby decrease the value

of the ARC, an effect that the fire allocation scheme is intended to counter by optimizing the killing process (e.g.) Despite this, the modified process is modelable and can thus be accounted for.

The second difficulty is more troublesome. Fundamentally, the question arises "Is this allocation scheme realistic?" If a target is acquired, is perceived to be both a danger and is engagable, then it seems reasonable that circumstances will arise not only naturally, but frequently, where those targets will be engaged despite doctrinal dictates arising from an optimal fire allocation scheme. The next two fire allocation schemes address this concern.

36.7 Natural Density

The first fire allocation scheme that tries to reflect human psychological proclivities is the "Natural" Density scheme. In this scheme, the assumption is made that the detectability of enemy targets is essentially the same despite variations in targets, and that friendly false targets can be instantly identified and rejected.[1] Thus, the fraction of fire allocated against an enemy subforce is just the fraction of target elements of that subforce in the total enemy force. Put mathematically, for the Blue force, this is simply

$$f_{i,j} = \frac{A_i}{\sum_{k=1}^{N_A} A_k}, \tag{22}$$

and the ADEs are thus just

$$\frac{dA_i}{dt} = -\sum_{j=1}^{N_B} \frac{A_i \alpha_{i,j} B_j}{\sum_{k=1}^{N_A} A_k}. \tag{23}$$

Note that by assuming this type of fire allocation, we have forced the ADEs to be Lanchester linear which violates our goal of keeping the ADEs Lanchester quadratic.

This difficulty does not prevent us from making numerical calculations and can be done simply using a spreadsheet simulation in our usual manner. The force strength trajectories are shown in figure 5.

To recover from this, we want to examine the application of the initial force strength approximation that we have so successfully applied before. In this case, the evolution equations become

$$\frac{dA_i}{dt} \simeq -\sum_{j=1}^{N_B} \frac{A_i(0)\, \alpha_{i,j} B_j}{\sum_{k=1}^{N_A} A_k(0)}. \tag{24}$$

The force strength trajectories are shown in figure 6. As we would expect, the losses in the example are of order 20%, so there is little difference. Thus, the approximation seems to have the same validity we have seen previously, and we thus recover the quadratic Lanchester behavior.

[1] We address this assumption explicitly in consideration of fratricide and amicide in Part IV.

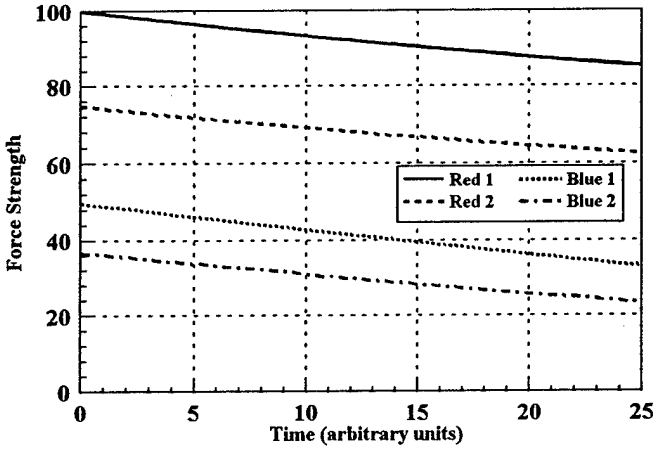

Figure 5: Heterogeneous Force Strength Trajectories - Natural Density Fire Allocation

36.8 Attrition Processes

The preceding section addressed fire allocation as a natural process with the so-called Natural Density approach. As we noted there, this approach assumes that fire is fractionally allocated as the fraction of available targets. This technique ignores the acquisition and engagement differences of targets and the problem of rejecting friendly ("false") targets. While we defer consideration of the latter effect for a later chapter in Part IV, we may further consider the variation in targets.

One approach that improves on the situation is to use the probability of detection as a weighting function. The idea here is that the product of probability of detection times subforce strength is approximately the number of targets that are detectable. Fire is then allocated amongst these detectable targets. If $p_d[i,j]$ is the probability of detection of the i^{th} target type by the j^{th} firer type, then for the Blue subforces, fire allocation fractions are

$$f_{i,j} = \frac{p_d[i,j]\,A_i}{\sum_{k=1}^{N_A} p_d[k,j]\,A_k}, \tag{25}$$

which can be quadraticized using the initial force strength approximation. Since we have to

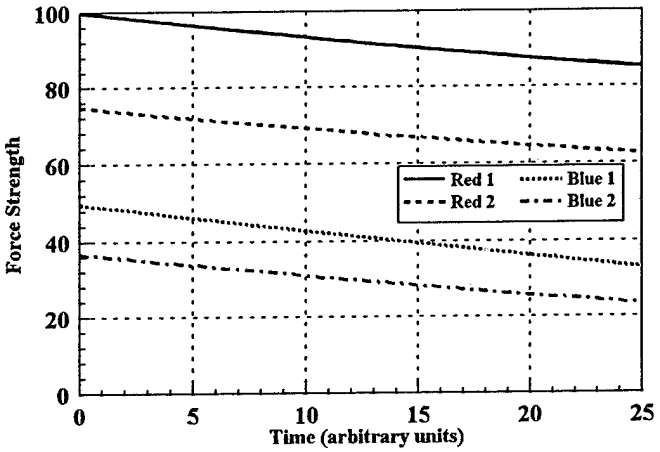

Figure 6: Heterogeneous Force Strength Trajectories - Natural Density Fire Allocation with Initial Force Strength Approximation

calculate ARCs anyway, and probability of detection is usually a factor in the calculation of the ARCs, this fire allocation scheme only adds marginal calculational burden.

Another approach is to consider not only the density of targets, but the amount of time it takes to acquire (or kill) a target. Let $\tau_{i,j}$ be the time for a firer of the j^{th} type to acquire (or kill) a target of the i^{th} type. If all of the targets had to be acquired (or killed) by the j^{th} subforce, ignoring both attrition to the subforce and the presence of the other subforces, then the total time to acquire (or kill) all of the targets is

$$T_j = \sum_{i=1}^{N_A} \tau_{i,j} A_i. \tag{26}$$

The idea here is that if there are B_j elements in the j^{th} subforce, then the fraction of these elements engaged in acquiring (or killing) elements of the i^{th} Red subforce is

$$f_{i,j} = \frac{\tau_{i,j} A_i}{T_j}. \tag{27}$$

At this point, it is useful to discuss the two schemes separately. For fire allocation on the basis of acquisition, we recall from Part II that the rate of acquiring targets of a given type is proportional to the number of targets of that type present. Thus,

$$\tau_{i,j} = \frac{1}{\kappa_{i,j} A_j}, \tag{28}$$

where $\kappa_{i,j}$ is the rate of acquiring one target of the i^{th} type by a searching element of the j^{th} subforce. This reduces the total time to acquire all targets to

$$T_j = \sum_{i=1}^{N_A} \frac{1}{\kappa_{i,j}}, \tag{29}$$

which is independent of the subforce strengths. Similarly, the fire allocation fractions reduce to

$$f_{i,j} = \frac{\frac{1}{\kappa_{i,j}}}{T_j}, \tag{30}$$

which is also independent of subforce strength.

The argument against this fire allocation approach is that it does not really account for how the subforce elements spend their time. They are not exclusively engaged in acquisition, but rather in killing (attrition). Thus an approach that is hailed as improving on the acquisition time scheme is the engagement or attrition time scheme. This scheme looks exactly like the scheme above except that the attrition times

$$\tau_{i,j} = \frac{1}{\alpha_{i,j}}, \tag{31}$$

are used, making the allocation fractions

$$f_{i,j} = \frac{\frac{A_i}{\alpha_{i,j}}}{\sum_{k=1}^{N_A} \frac{A_k}{\alpha_{k,j}}}. \tag{32}$$

This scheme is readily amenable to the initial subforce strength approximation to restore quadraciticity. An example of this is shown in figure 7.

Before proceeding, two matters need to be examined, which are related. The first of these is the situation where either due to weapon's alethality, or the amount of time to acquire that type of target, the time to kill is effectively infinite, or equivalently, the ARC is zero. This has the effect of making the normalizing summation in equation 32 effectively infinite (since the ARC is close to but not quite zero), and thereby, only the fraction corresponding to that ARC is nonzero (effectively).[2] This has the rather startling effect of allocating all fire against the targets that fire

[2]We are limiting this discussion to the case where only one ARC in a column of an ARC matrix is effectively zero. The extension to multiple zeros is straightforward and left as an exercise for the interested student.

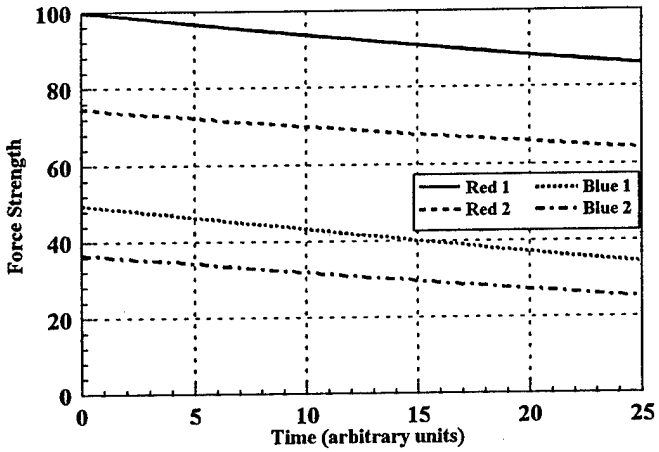

Figure 7: Heterogeneous Force Strength Trajectories - Attrition Time Fire Allocation with Initial Force Strength Approximation

is least effective against, in contrast to the effect of the optimization schemes (e.g., minimum loss and maximum kills).

Of course, the practical answer to these situations tactically is that the firing element will know it has poor lethality (or poor detection performance) against these targets and therefore reject them in most cases. (Infantry armed with rifles don't directly attack tanks.) We may do the same, leaving zero ARCs zero and excluding them from the scheme.

The second matter that we want to consider is what this type of fire allocation scheme does to the ADEs. If we substitute equation 32 into the ADEs, the result (for Red attrition) is

$$\begin{aligned} \frac{dA_i}{dt} &= -\sum_{j=1}^{N_B} \frac{\frac{A_i}{\alpha_{i,j}}}{\sum_{k=1}^{N_A} \frac{A_k}{\alpha_{k,j}}} \alpha_{i,j} B_j \\ &= -A_i \sum_{j=1}^{N_B} \frac{B_j}{\sum_{k=1}^{N_A} \frac{A_k}{\alpha_{k,j}}}, \end{aligned} \tag{33}$$

which is Lanchester linear (but we can alter to Lanchester quadratic, mathematically linear by

the initial force strength approximation.) What is more important is that the denominator is effectively only a function of j in this approximation. If we define

$$\varsigma_j \equiv \frac{1}{\sum_{k=1}^{N_A} \frac{A_k(0)}{\alpha_{k,j}}}, \tag{34}$$

as a fire allocated ARC, then the ADEs become

$$\frac{dA_i}{dt} \simeq -A_i(0) \sum_{j=1}^{N_B} \varsigma_j B_j. \tag{35}$$

(We will assume that we have excluded all values $\alpha_{i,j} \sim 0$ from the summation.) This has the effect not only of leveling the ARCs across the targets for each firing subforce, but effectively clouds the basis of the role of the conjugate theory of Part II.. Before leaping to the conclusion that all of our effort has been wasted, some further consideration is appropriate.

First, it is useful to apply the initial force strength approximation slightly differently to equation 33. In this case, we selectively apply the approximation only to the terms remaining in the summation, so that the ADEs become

$$\frac{dA_i}{dt} \simeq -A_i \sum_{j=1}^{N_B} \varsigma_j B_j(0), \tag{36}$$

which, since the individual ADEs now only depend on the attritted subforce strength (in the sense of the approximation,) they have solutions of the form

$$A_i(t) \simeq A_i(0)\, e^{-\sum_{j=1}^{N_B} \varsigma_j B_j(0) t}, \tag{37}$$

which at least indicates a handy approximate solution of a complexity that is still readily amenable to spreadsheet simulation, indicates a form of aggregation, and indicates exponential solution forms. The latter two presage the next chapter on aggregation.

Second, it is useful to examine the allocation scheme in the context of the forms of modern ARCs. A common form for both serial and parallel attrition processes is

$$\alpha_{i,j} = \frac{\eta_{i,j} A_i \upsilon_{i,j}}{\eta_{i,j} A_i + \upsilon_{i,j}}, \tag{38}$$

which we may substitute into the denominator summation of equation 33 as

$$\begin{aligned} \sum_{k=1}^{N_A} \frac{A_k}{\alpha_{k,j}} &= \sum_{k=1}^{N_A} \frac{\eta_{k,j} A_k + \upsilon_{k,j}}{\eta_{k,j} \upsilon_{k,j}} \\ &= \sum_{k=1}^{N_A} \frac{A_k}{\upsilon_{k,j}} + \sum_{k=1}^{N_A} \frac{1}{\eta_{k,j}}. \end{aligned} \tag{39}$$

We see from this that the effect of this fire allocation scheme is indeed to spread attrition across the attritability of the available targets, but the means by which this spread occurs is the componentry of the ARCs as described in Part II. What this type of fire allocation does is to make comparison of the conjugate theory with either historical data or detailed simulation extremely difficult - which we already knew, but now have complicated mathematical confirmation.

Of course, we do not know that this, or any other fire allocation scheme is accurate. What we can conclude, in the absence of other data, is that we have different techniques of modeling fire allocation that can span the range from total concentration of fire against one target type to complete averaging across all target types. This explains why considerable effort is spent in tuning simulations to have reasonable fire allocations.

36.9 Conclusion

We have reviewed several simple fire allocation schemes. More elaborate schemes are possible and are in use in different simulations. Of particular interest are combination schemes which combine the forms and merits of the optimization and "natural" schemes. These schemes produce allocations intermediary between the (often) single target allocations of the optimization schemes and the bland allocations of the "natural" schemes. An important consideration that we have only alluded to here is the necessity of considering target rejection in calculating ARCs when target rejection is an important (i.e., time consuming) subprocess of the overall attrition process. This can be accomplished using the same general modeling techniques detailed in Part II, so we shall not dwell on this now.

Additionally, our consideration thus far has been primarily concerned with just calculating force strength trajectories. Of course, attrition simulations have other applications. One particular application that deserves mention here is force structuring; that is, of which and how many weapons systems is a force comprised? If we know the structure of a potential enemy force, know all of the ARCs and have assumed some fire allocation scheme, then we may readily treat the initial subforce strengths as variables. Given some measures of merit for attrition performance of the force as a whole, the heterogeneous Lanchester methodology provides a basis for examining force structuring.

Finally, we must comment that the subject of fire allocation, for a variety of application purposes, is a subject of on-going research.[2] This dynamic nature of the subject area has been the primary motivation for the admittedly general approach that we have taken here. As we have seen from the figure herein, there can be little difference among different schemes in terms of force strength trajectories. (And there can be enormous differences!) The moral of this story is that the decision of what type of fire allocation scheme to use is not a matter of mathematical certainty, nor of compelling behavior. Rather the selection must be made within the context of the investigation to be performed.

36.10 References

[1] Sternberg, Stanley Ronald, "Development of Optimal Allocation Strategies in Heterogeneous Lanchester-type Processes", Dissertation, University of Michigan, 1971.

[2] Jaiswal, N. K., Meena Kumari, and B. S. Nagabhushana, "Optimal Force Mix in Heterogeneous Combat", Naval Research Logistics 42 September 1995, pp. 873-887.

Chapter 37

Basic Eigenmath

37.1 Introduction

With this chapter, we begin our approach to the problem of formal aggregation. In this chapter, we review the basic mathematics of eigenvalue and eigenvectors. This is a subject of academic study for physicists and engineers, primarily in the area of Mechanics, albeit this study is concentrated on matrices which have a high degree of symmetry. It is also an academic study of Operations Researchers in such areas as Saaty's Analytical Hierarchy Process where there is concern with matrices which are still symmetric but of a vastly different sort and possessing calculational problems akin to those we shall encounter in Lanchester Theory.[1] If not practiced, this learning may have rusted somewhat, and may be totally new to the military student although if they have served in Germany may have knowledge of the language. Resultingly, we are going to bore you with a chapter of mathematical manipulations. It is not complete nor rigorous and the student may wish to consult a real text on eigenmath at this point. A large number of texts in common use are available for this purpose.[2][1][3] If the reader has had such academic exposure, and still has the text (because most of us nerds keep ALL of our textbooks!), then you have a ready reference for most of what we will say in this chapter. If not, you have two choices: either take my word for things and suffer with the presentation; or go find a book at the library or college bookstore.

Basically, I find that the whole idea of eigenvalues and vectors is easiest understood by considering the problem we have posed in the preceding chapters in terms of differential equations. If we have a set of N first order differential equations, in principle, we can reduce this set to one N^{th} order differential equation. We know this single differential equation must have N general solutions and the specific solution (neglecting inhomogeneous terms in the differential equations) must consist of some linear combination of these N solutions. Now, we know that the same

[1] Physicists and Mechanical Engineers study matrix systems which are either symmetric or complex conjugate symmetric across the diagonal - that is Hermetian. The Analytical Heirarchy Process matrices tend to be inverse symmetric across the diagonal.

applies to the original N first order differential equations. The N specific solutions of these first order differential equations are also linear combinations of these N general solutions. Now let us turn this who situation inside out and ask the question - what linear combination of these N specific solutions is each individual general solution? Put another way, what linear combinations of first order differential equations give rise to a solution which is exactly one of the general solutions of the N^{th} order differential equation? This sounds horribly nasty and possibly even a very perverse thought, but this is exactly what the eigenmath problem is - turning the whole system inside out to get to the solutions.

Before we proceed, I need to make a few more caveats on this whole exposition. As I have said before this will not be a rigorous discussion. Search as you may, you won't even find any statement of Theorems, to say nothing of proofs. I am going to make the assumption that all of the composition of the attrition problem has been done correctly. Since we are going to be dealing with differential equations that have to be mathematically linear, I have to assume that all of the attrition contributions of attrition order not exactly equal to two have been quadraticized in a Lanchester sense. Initially, I am going to also require that the differential equations be homogeneous. This means that consideration of reinforcement is going to be left till later in the chapter. I am also going to defer any consideration of fratricide till the next part of the work. Finally, I am going to require that the ARC coefficient matrix is constant, being neither range nor time dependent, and come back to the matter later.

37.2 We've Seen This Already

Actually, we have already seen the basic ideas of this math, deriving simply and directly from the differential equation mathematics of Lanchester Theory. As we recall, the basic homogeneous Lanchester ADEs are:

$$\frac{dA}{dt} = -\alpha B; \tag{1}$$

and

$$\frac{dB}{dt} = -\beta A. \tag{2}$$

If we differentiate the first equation with respect to time,

$$\frac{d^2A}{dt^2} = -\alpha \frac{dB}{dt}, \tag{3}$$

and substitute the second, we get

$$\frac{d^2A}{dt^2} = \alpha\beta A. \tag{4}$$

If we now postulate that the solution is exponential,

$$A \propto e^{\lambda t}, \tag{5}$$

and substitute this into equation 4, and perform the differentiation, but retain the functional notation, then

$$\lambda^2 A = \alpha\beta A. \tag{6}$$

Since this equation contains A on both sides, we can remove the force strength from both sides, reducing the equation to

$$\lambda^2 = \alpha\beta. \tag{7}$$

This is a simple quadratic equation (no direct relation of Lanchester Quadratic) which has the solution,

$$\lambda = \pm\sqrt{\alpha\beta} \equiv \pm\gamma, \tag{8}$$

and is the definition of γ. This immediately gives the Red force strength solution as

$$A(t) = C_1 e^{\lambda t} + C_2 e^{-\lambda t}, \tag{9}$$

where C_1 and C_2 are constants which are determined from the initial conditions. We have seen this before in Part I. The new information is that the two λ are *eigenvalues*!

37.3 Eigenmath of Matrices

To proceed, we now consider square matrices, that is, matrices with as many rows as columns. As a general case, we denote these as

$$\overleftrightarrow{M} \equiv \begin{bmatrix} m_{1,1} & .. & m_{1,n} \\ .. & .. & .. \\ m_{n,1} & .. & m_{n,n} \end{bmatrix} \tag{10}$$

where the convention we use is that the first subscript indicates row number and the second subscript indicates column number. We shall be concerned with equations of the form

$$\overleftrightarrow{M} \bullet \overrightarrow{e} = \lambda \bullet \overrightarrow{e}. \tag{11}$$

The quantities (there are n of these) $\overrightarrow{e}$ and λ are called *eigenvectors*, and *eigenvalues*. We may also write matrices of eigenvalues and eigenvectors as

$$\overleftrightarrow{\lambda} \equiv \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_. & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}, \tag{12}$$

and

$$\overleftrightarrow{e} \equiv \begin{bmatrix} e_{1,1} & e_{1,..} & e_{1,n} \\ e..,1 & e_{..,..} & e_{..,n} \\ e_{n,1} & e_{n,..} & e_{n,n} \end{bmatrix}. \tag{13}$$

Then we may write equation 11 as

$$\overleftrightarrow{M} \bullet \overleftrightarrow{e} = \overleftrightarrow{\lambda} \bullet \overleftrightarrow{e}. \tag{14}$$

Alternately, we may also write this as n equations,

$$\overleftrightarrow{M} \bullet \overrightarrow{e_i} = \lambda_i \overrightarrow{e_i}, i = 1..n, \tag{15}$$

where:

$$\overrightarrow{e_i} \equiv \begin{bmatrix} e_{i,1} \\ e_{i,..} \\ e_{i,n} \end{bmatrix}. \tag{16}$$

We may take advantage of the fact that the eigenvectors are defined on both sides to rewrite equation 14 as

$$\left(\overleftrightarrow{M} - \overleftrightarrow{\lambda}\right) \bullet \overleftrightarrow{e} = \overleftrightarrow{0}, \tag{17}$$

where: $\overleftrightarrow{0}$ is a null matrix (all entries are zero.) Since all of the entries of the result of this matrix multiplication are zero, then we may require that the determinant of the quantity in parenthesis to be zero,

$$\left|\overleftrightarrow{M} - \overleftrightarrow{\lambda}\right| = 0. \tag{18}$$

This determinant has the matrix form,

$$\left|\begin{bmatrix} m_{1,1} - \lambda_1 & .. & m_{1,n} \\ .. & .. & .. \\ m_{n,1} & .. & m_{n,n} - \lambda_n \end{bmatrix}\right|. \tag{19}$$

It also has a polynomial form, (of order n,) called the *Characteristic Polynomial*,[2], which is

$$\sum_{i=0}^{n} c_i \lambda^i = 0. \tag{20}$$

[2]For the language buffs, the German name for the Characteristic Values (roots) of the Characteristic Polynomil is *Eigenwerte* from which we get eigenvalues.

The roots of this polynomial are the eigenvalues. From these individual values, the eigenvectors can be calculated using equation 15.

The eigenvalues can be calculated fairly directly using a variety of techniques since this is a scalar calculation. The eigenvector calculation is vector (matrix actually) so it is somewhat more difficult. The most elegant approach is to use what is know as a *Unitary Transformation.* In this method, the transform is a matrix $\overleftrightarrow{U}$ which has the property that

$$\overleftrightarrow{U} \bullet \overleftrightarrow{U}^{-1} = \overleftrightarrow{I}, \tag{21}$$

where $\overleftrightarrow{I}$ is the identity matrix and $\overleftrightarrow{U}^{-1}$ is the inverse of $\overleftrightarrow{U}$. To show the application of this method, let us start by assuming that a transformation can be found

$$\overleftrightarrow{M'} = \overleftrightarrow{U} \bullet \overleftrightarrow{M} \bullet \overleftrightarrow{U}^{-1}, \tag{22}$$

such that $\overleftrightarrow{M'}$ is diagonal [3] (that is all of the off diagonal elements are zero, but the diagonal elements are neither necessarily zero nor identical.) That is,

$$\overleftrightarrow{M'} = \begin{bmatrix} m'_1 & 0 & 0 \\ 0 & m'_{..} & 0 \\ 0 & 0 & m_n \end{bmatrix}. \tag{23}$$

This matrix has an eigenvalue equation,

$$\overleftrightarrow{M'} \bullet \overrightarrow{e'_i} = m'_i \overrightarrow{e'_i}, \tag{24}$$

where m'_i is the i^{th} diagonal value on the $\overleftrightarrow{M'}$ matrix, and $\overrightarrow{e'_i}$ is the i^{th} eigenvector of $\overleftrightarrow{M'}$. It is a simple matter to show that all of the components of each $\overrightarrow{e'_i}$ are zero except for the i^{th}, which is uniquely one. We may now rewrite equation 24 using equation 22 as

$$\overleftrightarrow{U} \bullet \overleftrightarrow{M} \bullet \overleftrightarrow{U}^{-1} \bullet \overrightarrow{e'_i} = m'_i \overrightarrow{e'_i}. \tag{25}$$

If we now multiply this equation on both sides by $\overleftrightarrow{U}^{-1}$ from the left, then we get

$$\overleftrightarrow{U}^{-1} \bullet \overleftrightarrow{U} \bullet \overleftrightarrow{M} \bullet \overleftrightarrow{U}^{-1} \bullet \overrightarrow{e'_i} = \overleftrightarrow{U}^{-1} \bullet m'_i \overrightarrow{e'_i}. \tag{26}$$

If we now take advantage of equation 21, and the fact that m'_i is a scalar, then we may reduce this equation to

$$\overleftrightarrow{M} \bullet \overleftrightarrow{U}^{-1} \bullet \overrightarrow{e'_i} = m'_i \overleftrightarrow{U}^{-1} \bullet \overrightarrow{e'_i}. \tag{27}$$

[3] In practice, it is difficult to find a Unitary Transform that reduces the ARC matrix to diagonal form. We can find a Unitary Transform which reduces it to upper triangular fairly easily, however. This is sufficient to determine both eigenvectors and eigenvalues. This does not compromise the math either.

If we now define

$$\overrightarrow{e_i} \equiv \overleftrightarrow{U}^{-1} \bullet \overrightarrow{e_i'}, \tag{28}$$

then equation 27 reduces to

$$\overleftrightarrow{M} \bullet \overrightarrow{e_i} = m_i' \overrightarrow{e_i}, \tag{29}$$

which we recognize as identical to equation 15.

It follows then that if this Unitary Transformation can be found, then

$$\lambda_i = m_i', \tag{30}$$

since the $\overrightarrow{e_i'}$ have only one nonzero element,

$$\overleftrightarrow{e} = \overleftrightarrow{U}, \tag{31}$$

and,

$$\overleftrightarrow{\lambda} = \overleftrightarrow{M'}. \tag{32}$$

Calculationally, the problem of finding eigenvalues and eigenvectors is thus reduced to finding this Unitary Transformation.

Before proceeding, we want to make one more association. We note that since $\overleftrightarrow{M'}$ contains the eigenvalues, and we may rewrite equation 22 as

$$\begin{aligned} \overleftrightarrow{M} &= \overleftrightarrow{U}^{-1} \bullet \overleftrightarrow{M'} \bullet \overleftrightarrow{U}, \\ &= \overleftrightarrow{U}^{-1} \bullet \overleftrightarrow{\lambda} \bullet \overleftrightarrow{U}. \end{aligned} \tag{33}$$

From this and equation 31, we may write

$$\overleftrightarrow{M} = \overleftrightarrow{e}^{-1} \bullet \overleftrightarrow{\lambda} \bullet \overleftrightarrow{e}, \tag{34}$$

and infer that the original matrix may be reconstructed from the eigenvalue and eigenvector matrices!

37.4 Back to Differential Equations

At this point, we want to return to consideration of differential equations. In particular, we want to consider a set of n first order differential equations that are linear in a mathematical sense. That is,

$$\frac{d}{dt}F_i(t) = C_{i,j}F_j(t), \tag{35}$$

where the F_i are the functions which we seek as solutions, and the $C_{i,j}$ are (constant) coefficients which define the functional relationship of the functions to their derivatives. In matrix notation, these differential equations may be written as one matrix differential equation,

$$\frac{d}{dt}\overrightarrow{F} = \overleftrightarrow{C} \bullet \overrightarrow{F}, \tag{36}$$

where:

$$\overrightarrow{F} \equiv \begin{bmatrix} F_1 \\ F_{..} \\ F_n \end{bmatrix}, \tag{37}$$

and

$$\overleftrightarrow{C} \equiv \begin{bmatrix} C_{1,1} & C_{1,..} & C_{1,n} \\ C_{..,1} & C_{..,..} & C_{..,n} \\ C_{n,1} & C_{n,..} & C_{n,n} \end{bmatrix}. \tag{38}$$

Because $\overleftrightarrow{C}$ is an ARC matrix, it is not symmetric and we must make a distinction between the eigenvectors calculated from the left or right sides of the matrix. In our case, we want the left eigenvectors (the eigenvalues are the same!) since they correspond to the solutions that we are interested in. Thus, the matrix differential equation that we want to calculate with is really

$$\frac{d}{dt}\overrightarrow{F}^T = \overrightarrow{F}^T \bullet \overleftrightarrow{C}^T, \tag{39}$$

where the T superscript indicates the transpose. Thus $\overrightarrow{F}^T$ is a row matrix. When we make this transposition, we can now extract the right eigenvectors which are the ones that most calculation algorithms and software compute. In other words, we are making this seeming meaningless transposition so that the calculations will come out right.

If we now make the same assumption that we did earlier, namely that the solutions are exponential,

$$F_i(t) \propto e^{\lambda_i t}, \tag{40}$$

then we may calculate the time derivatives as

$$\frac{d}{dt}\overrightarrow{F}^T = \begin{bmatrix} \lambda_1 F_1 & \lambda_{..} F_{..} & \lambda_n F_n \end{bmatrix}. \tag{41}$$

The rhs side of this equation may readily be rewritten as

$$\begin{bmatrix} \lambda_1 F_1 & \lambda_{..} F_{..} & \lambda_n F_n \end{bmatrix} = \begin{bmatrix} F_1 & F_{..} & F_n \end{bmatrix} \bullet \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_{..} & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}. \tag{42}$$

From our exposition of the previous section, we further recognize the rhs of this equation as

$$\begin{bmatrix} F_1 & F_{..} & F_n \end{bmatrix} \bullet \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_{..} & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} = \overrightarrow{F}^T \bullet \overleftrightarrow{\lambda}, \tag{43}$$

which allows us to rewrite equation 36 as

$$\overrightarrow{F}^T \bullet \overleftrightarrow{C} = \overrightarrow{F}^T \bullet \overleftrightarrow{\lambda}, \tag{44}$$

which is just an eigenvalue equation.

37.5 Solution Forms

We are now in a position to form solutions of our set of first order linear (in a mathematical sense) differential equations. Assuming we may find the proper Unitary Transformation, we may use our earlier definitions of the eigenvalue matrix as

$$\overleftrightarrow{\lambda} = \overleftrightarrow{U} \bullet \overleftrightarrow{C}^T \bullet \overleftrightarrow{U}^{-1}, \tag{45}$$

to simplify equation 36 by multiplying from the right by $\overleftrightarrow{U}^{-1}$, and using the identity property of the Unitary Transformation, equation 21, to write

$$\frac{d}{dt} \overrightarrow{F}^T \bullet \overleftrightarrow{U}^{-1} = \overrightarrow{F}^T \bullet \overleftrightarrow{U}^{-1} \bullet \overleftrightarrow{U} \bullet \overleftrightarrow{C}^T \bullet \overleftrightarrow{U}^{-1}. \tag{46}$$

If we now make use of equation 45 and define the eigensolutions as

$$\overrightarrow{f}^T \equiv \overrightarrow{F}^T \bullet \overleftrightarrow{U}^{-1}, \tag{47}$$

Since the Unitary Transformation is constant (since $\overleftrightarrow{C}$ is constant!), then we may exchange the order of transformation and differentiation to yield

$$\frac{d}{dt} \overrightarrow{f}^T = \overrightarrow{f}^T \bullet \overleftrightarrow{\lambda}, \tag{48}$$

(since the eigenvalues are the same regardless of left or right!) which, if we rewrite in index notation,

$$\begin{aligned} \frac{d}{dt} f_i &= \lambda_i \delta_{i,j} f_j, \\ &= \lambda_i f_i, \end{aligned} \tag{49}$$

has the obvious solutions,

$$f_i(t) = f_i(0)\, e^{\lambda_i t}. \tag{50}$$

It now remains to accommodate initial conditions and get back to the solutions of the original functions. It is useful in doing this to introduce two sets of notation. First, if we rewrite the rhs of equation 50 in opposite order, and define the matrix

$$\overleftrightarrow{T}(t) \equiv \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_n t} \end{bmatrix}, \tag{51}$$

then we may rewrite the eigensolutions in matrix form as

$$\vec{f}^{\,T}(t) = \vec{f}^{\,T}(0) \bullet \overleftrightarrow{T}(t). \tag{52}$$

We may use equation 47 to further rewrite this as

$$\vec{f}^{\,T}(t) = \vec{F}^{\,T}(0) \bullet \overleftrightarrow{U}^{-1} \bullet \overleftrightarrow{T}(t). \tag{53}$$

Since the transformation of equation 47 is invertible by the properties of the Unitary Transformation,

$$\vec{F}^{\,T} \equiv \vec{f}^{\,T} \bullet \overleftrightarrow{U}, \tag{54}$$

it follows immediately that we may write the solutions as

$$\begin{aligned} \vec{F}^{\,T}(t) &\equiv \vec{F}^{\,T}(0) \bullet \overleftrightarrow{U}^{-1} \bullet \overleftrightarrow{T}(t) \bullet \overleftrightarrow{U}, \\ &= \vec{F}^{\,T}(0) \bullet \overleftrightarrow{T'}(t), \end{aligned} \tag{55}$$

where:

$$\overleftrightarrow{T'}(t) \equiv \overleftrightarrow{U}^{-1} \bullet \overleftrightarrow{T}(t) \bullet \overleftrightarrow{U}. \tag{56}$$

If we want to return to our familiar column matrix notation, we have only to take the transpose of equation 55 which is

$$\vec{F}(t) = \overleftrightarrow{T'}^{\,T}(t) \bullet \vec{F}(t). \tag{57}$$

To develop the other notation, we first refer back to equation 47 which we may rewrite as

$$\overrightarrow{f} \equiv \overleftrightarrow{U}^{-1T} \bullet \overrightarrow{F}, \tag{58}$$

and its inverse as

$$\overrightarrow{F} \equiv \overleftrightarrow{U}^{T} \bullet \overrightarrow{f}. \tag{59}$$

This gives us a new definition of the eigenvector matrix,

$$\overleftrightarrow{e} \equiv \overleftrightarrow{U}^{-1T}, \tag{60}$$

which we may contrast with equation 31. If the ARC matrix, $\overleftrightarrow{C}$, were symmetric (or Hermitian), then this definition would be the same as before since then $\overleftrightarrow{U}^{-1T} = \overleftrightarrow{U}$, alternately, if we had properly extracted the left eigenvectors from the untransposed differential equations rather than the right eigenvectors of the transposed differential equations, we should not need this new definition! Instead, we must update this definition because of the asymmetry of the ARC matrix to get back to some semblance of our original notation. This equation indicates that the eigenvectors are the rows of the transpose of the inverse transformation, which are just the columns of the inverse transform, so we may write the components of the i^{th} eigenvector $\overrightarrow{e_i}$ as

$$e_{i|j} = u^{-1}_{,j,i}, \tag{61}$$

and similarly identify the inverse eigenvectors as the rows of the transpose of the transform or, the columns of the transform, thus,

$$e^{-1}_{i|j} = u_{j,i} \tag{62}$$

Next, we rewrite equation 55 in index notation (with implied summation of repeated indices) as

$$F_i(t) = T'_{l,i}(t)\, F_l(0), \tag{63}$$

and similarly rewrite equation 56

$$T'_{l,i}(t) = u^{-1}_{l,j} T_{j,k}(t)\, u_{k,l}. \tag{64}$$

Since the eigensolution matrix is diagonal, it can be written as

$$T_{j,k}(t) = e^{\lambda_j t} \delta_{j,k}, \tag{65}$$

and used to simplify equation 64 as

$$\begin{aligned} T'_{l,i}(t) &= u^{-1}_{l,j} e^{\lambda_j t} \delta_{j,k} u_{k,i}, \\ &= u^{-1}_{l,j} e^{\lambda_j t} u_{j,i}. \end{aligned} \tag{66}$$

By using equations 61 and 62, this may be rewritten as

$$T'_{l,i}(t) = e^{-1}_{j|l} e^{\lambda_j t} e_{i|j}, \tag{67}$$

which reduces the solutions to

$$F_i(t) = e^{-1}_{i|j} e^{\lambda_j t} e_{j|l} F_l(0). \tag{68}$$

Similarly, we may rewrite the eigensolutions, equation 53, as

$$f_i(t) = e^{\lambda_i t} e_{i|j} F_j(0). \tag{69}$$

This equation will be of considerable importance in the formal theory of aggregation.

37.6 The Basic Example

The simplest case that we may consider is the homogeneous Lanchester problem. Written in matrix notation, the attrition differential equations are:

$$\frac{d}{dt}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 & -\alpha \\ -\beta & 0 \end{bmatrix} \bullet \begin{bmatrix} A \\ B \end{bmatrix}. \tag{70}$$

If we take the transpose of this, calculate the eigenvalues and the right eigenvectors, and then transpose these again, we find the eigenvalues are

$$\begin{aligned} \lambda &= \pm\sqrt{\alpha\beta}, \\ &\equiv \pm\gamma, \end{aligned} \tag{71}$$

and the eigenvector matrix is

$$\overrightarrow{e} = \begin{bmatrix} \sqrt{\beta} & \sqrt{\alpha} \\ \sqrt{\beta} & -\sqrt{\alpha} \end{bmatrix}, \tag{72}$$

which has an inverse,

$$\overrightarrow{e}^{-1} = \frac{1}{2}\begin{bmatrix} \frac{1}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \\ \frac{1}{\sqrt{\alpha}} & -\frac{1}{\sqrt{\alpha}} \end{bmatrix}. \tag{73}$$

The eigensolutions are

$$\begin{aligned} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} &= \begin{bmatrix} \sqrt{\beta} & \sqrt{\alpha} \\ \sqrt{\beta} & -\sqrt{\alpha} \end{bmatrix} \bullet \begin{bmatrix} A \\ B \end{bmatrix}, \\ &= \begin{bmatrix} \sqrt{\beta}A + \sqrt{\alpha}B \\ \sqrt{\beta}A - \sqrt{\alpha}B \end{bmatrix}. \end{aligned} \tag{74}$$

The attrition differential equations become

$$\frac{d}{dt}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -\gamma & 0 \\ 0 & \gamma \end{bmatrix} \bullet \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{75}$$

which have solutions

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} e^{-\gamma t} & 0 \\ 0 & e^{\gamma t} \end{bmatrix} \bullet \begin{bmatrix} f_1(0) \\ f_2(0) \end{bmatrix}. \tag{76}$$

We may combine equations 74 and 76 to form the explicit time eigensolutions,

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} e^{-\gamma t} & 0 \\ 0 & e^{\gamma t} \end{bmatrix} \bullet \begin{bmatrix} \sqrt{\beta}A(0) + \sqrt{\alpha}B(0) \\ \sqrt{\beta}A(0) - \sqrt{\alpha}B(0) \end{bmatrix}, \tag{77}$$

and then use the inverse eigenvectors, equation 73, to write

$$\begin{bmatrix} A(t) \\ B(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{1}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \\ \frac{1}{\sqrt{\alpha}} & -\frac{1}{\sqrt{\alpha}} \end{bmatrix} \bullet \begin{bmatrix} e^{-\gamma t} & 0 \\ 0 & e^{\gamma t} \end{bmatrix} \bullet \begin{bmatrix} \sqrt{\beta}A(0) + \sqrt{\alpha}B(0) \\ \sqrt{\beta}A(0) - \sqrt{\alpha}B(0) \end{bmatrix}. \tag{78}$$

If we carry out the matrix multiplications, then the result will be

$$\begin{aligned} \begin{bmatrix} A(t) \\ B(t) \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} \frac{e^{-\gamma t}}{\sqrt{\beta}} & \frac{e^{\gamma t}}{\sqrt{\beta}} \\ \frac{e^{-\gamma t}}{\sqrt{\alpha}} & -\frac{e^{\gamma t}}{\sqrt{\alpha}} \end{bmatrix} \bullet \begin{bmatrix} \sqrt{\beta}A(0) + \sqrt{\alpha}B(0) \\ \sqrt{\beta}A(0) - \sqrt{\alpha}B(0) \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} \frac{e^{-\gamma t}}{\sqrt{\beta}}\left(\sqrt{\beta}A(0) + \sqrt{\alpha}B(0)\right) + \frac{e^{\gamma t}}{\sqrt{\beta}}\left(\sqrt{\beta}A(0) - \sqrt{\alpha}B(0)\right) \\ \frac{e^{-\gamma t}}{\sqrt{\alpha}}\left(\sqrt{\beta}A(0) + \sqrt{\alpha}B(0)\right) - \frac{e^{\gamma t}}{\sqrt{\alpha}}\left(\sqrt{\beta}A(0) - \sqrt{\alpha}B(0)\right) \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} A(0)\left(e^{-\gamma t} + e^{\gamma t}\right) + B(0)\sqrt{\frac{\alpha}{\beta}}\left(e^{-\gamma t} - e^{\gamma t}\right) \\ A(0)\sqrt{\frac{\beta}{\alpha}}\left(e^{-\gamma t} - e^{\gamma t}\right) + B(0)\left(e^{-\gamma t} - e^{\gamma t}\right) \end{bmatrix} \\ &= \begin{bmatrix} A(0)\cosh(\gamma t) - \sqrt{\frac{\alpha}{\beta}}B(0)\sinh(\gamma t) \\ B(0)\cosh(\gamma t) - \sqrt{\frac{\beta}{\alpha}}A(0)\sinh(\gamma t) \end{bmatrix}, \end{aligned} \tag{79}$$

which is what we expected.

37.7 Reinforcements

We may now turn our attention to the question of solutions of the heterogeneous attrition differential equations with reinforcement. In this case, we rewrite the matrix differential equation as

$$\frac{d}{dt}\vec{F} = \overleftrightarrow{C} \bullet \vec{F} + \vec{F_r}(t). \tag{80}$$

We note immediately that this matrix differential equation is also non-homogeneous in the mathematical sense. The quantity $\overrightarrow{F_r}(t)$ is the reinforcement vector or column matrix. If we define the eigenreinforcements as

$$f_R(t) = \overleftrightarrow{e} \bullet \overrightarrow{F_r}(t), \tag{81}$$

then this matrix differential equation can be reduced to

$$\frac{d}{dt}\overrightarrow{f} = \overleftrightarrow{C'} \bullet \overrightarrow{f} + \overrightarrow{f_r}(t), \tag{82}$$

where $\overleftrightarrow{C'}$ is the eigenvalue matrix. Since we are using the eigenvector solutions developed above, we do not need to concern ourselves at this time about right and left.

Since $\overleftrightarrow{C'}$ is diagonal, we may simply decompose equation 82 into individual differential equations,

$$\frac{d}{dt}f_i = \lambda_i f_i + f_r(t)_i. \tag{83}$$

We may now posit solutions of the form

$$f_i(t) = [g_i(t) + f_i(0)]\, e^{\lambda_i t}, \tag{84}$$

which has a derivative

$$\frac{d}{dt}f_i(t) = \frac{d}{dt}g_i(t)\, e^{\lambda_i t} + \lambda_i f_i(t). \tag{85}$$

If we compare equations 83 and 85, and eliminate common terms, then we find that

$$\frac{d}{dt}g_i(t)\, e^{\lambda_i t} = f_r(t)_i, \tag{86}$$

which may be solved immediately as

$$g_i(t) = \int_0^t e^{-\lambda_i t'} f_r(t')_i\, dt'. \tag{87}$$

This allows us to write the particular eigensolutions as

$$\begin{aligned} f_i(t) &= \left[\int_0^t e^{-\lambda_i t'} f_r(t')_i\, dt' + f_i(0)\right] e^{\lambda_i t} \\ &= f_i(0)\, e^{\lambda_i t} + e^{\lambda_i t}\int_0^t e^{-\lambda_i t'} f_r(t')_i\, dt'. \end{aligned} \tag{88}$$

If we now return to the vector/matrix notation, then we may write the eigensolutions as

$$\overrightarrow{f} = \overleftrightarrow{T}(t) \bullet \overrightarrow{f} + \overleftrightarrow{T}(t) \bullet \int_0^t \overleftrightarrow{T}^{-1}(t') \bullet \overrightarrow{f_R}(t')\, dt', \tag{89}$$

where we understand the integration to be over every matrix element product and recognize that $\overleftrightarrow{T}^{-1}(t) = \overleftrightarrow{T}(-t)$ because of its diagonal nature. We may now apply the definition of both the eigensolutions and the particular solutions to rewrite this in final form as

$$\overrightarrow{F}(t) = \overleftrightarrow{e}^{-1} \bullet \overleftrightarrow{T}(t) \bullet \overleftrightarrow{e} \bullet \overrightarrow{F}(0) + \overleftrightarrow{e}^{-1} \bullet \overleftrightarrow{T}(t) \bullet \int_0^t \overleftrightarrow{T}^{-1}(t') \bullet \overleftrightarrow{e} \bullet \overrightarrow{F_R}(t')\, dt'. \tag{90}$$

As we noted earlier for the homogeneous problem, the reinforcement solutions do not posses state solutions until after all reinforcements have been applied because the solutions are not stationary. The situation in the heterogeneous case is similar but in keeping with the greater number of force strengths, more complicated.

37.8 Solution Restrictions

Before concluding this chapter, a few words on solutions are in order. The reader will have already noted that the ARC matrix is inherently nonpositive. That is, all its entries are either negative or zero. Despite the desire to extract the minus signs, they are necessary to assure the proper eigensolutions. Since we have neglected fratricide, the diagonal entries are zero. This has a special effect since it indicates that the sum of the eigenvalues is zero. We shall treat this in more detail in the next chapter.

It is important that the ARC matrix be dense in a combat sense. In particular, it is necessary that every force strength component in the combat attrit at least one other force strength component, and be attrited by at least one other force strength component. If this is not the case, then the solutions do not usually exist.

It is also desirable that the ARC matrix not be divisible or partitionable as we have described previously. This does not effect the solution generation process, but will be important in the next chapter.

Finally, we come to the question of time or range dependent ARCs. If all of the nonzero ARCs vary only slightly from constants, then approximate solutions can be formed using series expansions in a manner akin to that used in the section above for reinforcements. Let the matrix attrition differential equation be

$$\frac{d}{dt}\overrightarrow{F} = \overleftrightarrow{C}(t) \bullet \overrightarrow{F}. \tag{91}$$

Now consider the expansion of the ARC matrix

$$\overleftrightarrow{C}(t) \simeq \overleftrightarrow{C}(t^*) + \frac{d}{dt}\overleftrightarrow{C}(t^*)(t - t^*), \tag{92}$$

where the derivative of the matrix indicates the derivative of every element of the matrix. We want to select the mean value $t^\star$ which represents a time which minimizes the deviation in some sense. One convenient way is to minimize the square deviation of the ARCs,

$$trace\left(\int_0^{t_e} \frac{d}{dt}\overleftrightarrow{C}\,(t^\star) \bullet \frac{d}{dt}\overleftrightarrow{C}^T\,(t^\star)\,(t'-t^\star)^2\,dt'\right) = \min imum, \tag{93}$$

where trace indicates the sum of the diagonal elements. Since the derivatives are constants, and the time integration simplifies to the same for all elements, it is possible then to select $t^\star$ so that is minimizes

$$trace\left(\frac{d}{dt}\overleftrightarrow{C}\,(t^\star) \bullet \frac{d}{dt}\overleftrightarrow{C}^T\,(t^\star)\right). \tag{94}$$

Given this, then we may rewrite equation 91 as

$$\begin{aligned} \frac{d}{dt}\overrightarrow{F} &\simeq \left[\overleftrightarrow{C}\,(t^\star) + \frac{d}{dt}\overleftrightarrow{C}\,(t^\star)\,(t-t^\star)\right] \bullet \overrightarrow{F} \\ &\simeq \overleftrightarrow{C}\,(t^\star) \bullet \overrightarrow{F} + (t-t^\star)\frac{d}{dt}\overleftrightarrow{C}\,(t^\star) \bullet \overrightarrow{F}. \end{aligned} \tag{95}$$

If we form eigensolutions from the mean value ARC matrix $\overleftrightarrow{C}\,(t^\star)$, then we may reduce the equation to the form,

$$\frac{d}{dt}\overrightarrow{f} \simeq \overleftrightarrow{C'}\,(t^\star) \bullet \overrightarrow{f} + (t-t^\star)\,\overleftrightarrow{e} \bullet \frac{d}{dt}\overleftrightarrow{C}\,(t^\star) \bullet \overleftrightarrow{e}^{-1} \bullet \overrightarrow{f}, \tag{96}$$

which may be written in component form as[4]

$$\frac{d}{dt}f_i \simeq \lambda_i f_i + (t-t^\star)\,e_{i,j}\left[\frac{d}{dt}C\,(t^\star)\right]_{j,k} e^{-1}_{k,l} f_l. \tag{97}$$

If we again assume a solution form,

$$f_i\,(t) = [f_i\,(0) + h_i\,(t)]\,e^{\lambda_i t}, \tag{98}$$

which has a derivative

$$\frac{d}{dt}f_i\,(t) = \lambda_i f_i\,(t) + e^{\lambda_i t}\frac{d}{dt}h_i\,(t), \tag{99}$$

and compare this equation with the approximate attrition differential equation, equation 97, then we see that

[4]We adopt the convention in this section that repeated indices are summed quantities unless otherwise noted.

$$e^{\lambda_i t}\frac{d}{dt}h_i(t) \simeq (t-t^\star)\, e_{i,j}\left[\frac{d}{dt}C(t^\star)\right]_{j,k} e^{-1}_{k,l} f_l(t). \tag{100}$$

This equation may be integrated directly,

$$h_i(t) \simeq \int_0^t e^{-\lambda_i t'}(t'-t^\star)\, e_{i,j}\left[\frac{d}{dt}C(t^\star)\right]_{j,k} e^{-1}_{k,l} f_l(t')\, dt', \tag{101}$$

and resubstituted to form the solution

$$f_i(t) \simeq \left[f_i(0) + \int_0^t e^{-\lambda_i t'}(t'-t^\star)\, e_{i,j}\left[\frac{d}{dt}C(t^\star)\right]_{j,k} e^{-1}_{k,l} f_l(t')\, dt'\right] e^{\lambda_i t}. \tag{102}$$

We see immediately that this is really an integral equation since it contains the solutions on both sides of the equation and under the integral sign. This presents as knotty a problem as the original eigenvector problem unless we introduce some approximation. A usual one is to approximate the eigensolution under the integral as the constant ARC matrix eigensolution. That is,

$$f_i(t) \simeq \left[f_i(0) + \int_0^t e^{-\lambda_i t'}(t'-t^\star)\, e_{i,j}\left[\frac{d}{dt}C(t^\star)\right]_{j,k} e^{-1}_{k,l} f_l(0)\, e^{\lambda_i t} dt'\right] e^{\lambda_i t}. \tag{103}$$

These integral are straightforward and it is a simple matter to reduce this equation back to force strengths, albeit the algebra is a bit messy in notation. We leave this as an exercise for the student to familiarize with these techniques.

As we have said before, these solutions can easily be computed numerically using spreadsheet simulation or with the help of a symbolic algebra or array manipulation program. Regardless of what method is used, care needs to be taken to avoid continued calculation past the point where any one force strength component approaches zero. We have previously noted that the attrition relationship changes when the density gets very low, so the minimum action should be to recast and recalculate the eigensolution at the time when this occurs.

37.9 References

[1] Kreyszig, Erwin, **Advanced Engineering Mathematics**, John Wiley & Sons, Inc., New York 1962.

[2] Hildebrand, Francis B., **Methods of Applied Mathematics**, Prentice-Hall, Inc., Englewood Cliffs, NJ 1965.

[3] Wylie, C. R., Jr., **Advanced Engineering Mathematics**, McGraw-Hill Book Company, New York 1966.

References

[4] Carnahan, Brice, H. A. Luther, and James O. Wilkes, **Applied Numerical Methods**, John Wiley & Sons, Inc., New York 1969.

[5] Jennings, Alan, **Matrix Computation for Engineers and Scientists**, John WIley & Sons, New York 1977.

Chapter 38

Formal Aggregation

38.1 Introduction

With this chapter, we take up the formal theory of aggregation. My daughter's school dictionary [1] (mine wasn't handy and hers was!) defines aggregate as: *"a group of things gathered into a total or whole"*, and aggregation as: *"a number of separate things brought together into a single group."* Both of these are nouns. As we would expect, the dictionary doesn't define a verb (to aggregate), but we might extrapolate a definition from the above:

"to gather or bring together a group of separate things into a single group".

Aggregation is at once a matter of interest and practice both in the normal course of physics and in attrition mechanics. In practical terms, it is the combination of separate things into a whole for the purpose of simplifying the model of those things. In classical mechanics, a collection of atoms in a solid may be aggregated as either a point mass (three degrees of freedom,) or as an extended point mass (six degrees of freedom.) Since each atom is comprised of several individual particles, an exact treatment would treat the solid as having six degrees of freedom per particle. This is clearly a large number (of approximately the order of Avogadro's Number - 10^{23}) which would be exceeding cumbersome (if practicable) to do bookkeeping on much less do computations with.

On the opposite hand, quantum theory indicates that each of these particles is individually different even if they do obey common laws and have common general behavior. Just as a common assumption of statistical physics is that some collection of particles is indistinguishable, and thus alike, we commonly make the assumption in attrition mechanics that some collection of things is effectively indistinguishable for the purposes of modeling. (Indeed, we make this approximation in any type of modeling. Thus any modeling must incorporate some degree of aggregation - assumption of indistinguishability or identicality.) As a general rule then, aggregation is necessary for modeling and the degree of aggregation is dictated by the observable aspects of the system that must be represented.

38.2 Proper Aggregation

There are two basic concepts of aggregation that we need to capture before proceeding to discuss the theory of aggregation as it applies to us here. These are Proper and Formal Aggregation. We discuss Proper Aggregation here and defer discussion of Formal Aggregation to later in the chapter.

Proper Aggregation is nothing more than a name for the process that we have alluded to above. That an aggregation is proper means little more than that the collection of separate things into a single thing has been done in keeping with the observable characteristics of the resulting one. There are two sides to this, albeit they are opposite sides of the same coin. One is recognizing what the observable characteristics of the aggregate are. The other is recognizing that these observable characteristics are consistent with the observable characteristics desired of the model. The second deals with the requirements for the model; the first with what the model can represent.

In practice, the degree of aggregation of a model seldom matches the requirements. If the model has greater characteristics than required, the model may be said to be underaggregated in terms of these requirements. If the model has fewer characteristics, it may be said to be overaggregated; and if they match, it may be said to be exactly aggregated.

Of course, exact aggregation is essentially an unattainable goal. In general, we want a model to be underaggregated and we want the degree of underaggregation to be minimal.

We must note that there is an inherently subjective aspect to aggregation. Recognition of this subjectivity is the genesis of Proper Aggregation as a consideration. As a case in point, we want to model the combat performance of some tanks. If we ignore for the moment the individuality of their positions and the individual performance characteristics of the tanks themselves and their crews, then we may choose to represent these tanks as being identical. This may or may not be proper.

Let us suppose that the group of tanks consists of equal numbers of M-1A1's and M-1A2's. A primary difference between these two versions of the model is their primary armament. The A1 has a 105mm main gun while the A2 has a 120mm gun. (They also have different size basic loads because of the constant storage volume in the tank.) These two guns (and their fire control systems) have somewhat different lethality properties. If these differences are small compared to either the variations within the performance of each subset, or (and more important) the degree of resolution sought from use of the model, then the two sets may properly be aggregated. If this is not the case, the two subsets cannot be properly be aggregated although they may still be formally aggregated.

Often, we are confronted with little choice on the subject of aggregation. Model makers and data collectors are not always either the same person or in close communication with each other. In this case, the degree of aggregation is often decided by the data collector. We have already noted this situation in Part I, in Chapters IX, XII, and XIV.

38.3 Formal Aggregation

In the past, several attempts have been made to develop algorithms for the aggregation and disaggregation of units. This effort has intensified with the advent of geographically distributed but electronically connected simulation interconnectivity (e.g., Distributed Interactive Simulation (DIS) on the Distributed Simulation Internet (DSI)). Often these connected simulations have different resolutions so that connecting them with each other and the basic environmental support software of the system necessitates some cross resolution mechanism.

Formal aggregation takes a somewhat different approach to the problem. It takes as a fundamental that the aggregation process must not only conserve some observables of the things being aggregated, but it must conserve the mathematical symmetry properties of those observables. Put in terms that we are used to dealing with in our discussions herein on Lanchester attrition theory, this says that if we aggregate (as we are about to do,) from a heterogeneous Lanchester picture to a homogeneous Lanchester picture, the mathematical process of that aggregation must conserve the symmetry properties of homogeneous Lanchester.[2]

In mathematical terms, we define a (linear) formal aggregation as

$$\begin{aligned} \sqrt{\beta A} &\equiv \sum_{i=1}^{n_a} b_i A_i, \\ \sqrt{\overline{\alpha} B} &\equiv \sum_{i=1}^{n_b} a_i B_i, \end{aligned} \tag{1}$$

and require that the aggregated force strengths satisfy the homogeneous Lanchester ADEs,

$$\begin{aligned} \frac{d}{dt}\overline{A} &= -\overline{\alpha}\overline{B}, \\ \frac{d}{dt}\overline{B} &= -\overline{\beta A}, \end{aligned} \tag{2}$$

where: $\overline{A}$ and $\overline{B}$ are aggregated force strengths, and $\overline{\alpha}$ and $\overline{\beta}$ are aggregate ARCs.

We distinguish two classes of Formal Aggregation: Intensive; and Extensive. Intensive aggregation is completely self-consistent and does not depend on any external decision or determination in the aggregation process. It is thus a consistent aggregation in that it contains no arbitrary component. In particular, intensive aggregation is independent of the initial force strengths. Extensive aggregation, on the other hand, does depend on an external decision in the aggregation process. That is, extensive aggregation can depend on the force strength components. This externality induces a degree of arbitrariness into the aggregation process which compromises its consistency.

38.3.1 Intensive Aggregation

As before in Chapter XXXV, we adopt the force strength matrix notation of the attrition differential equations,

$$\frac{d}{dt}\overrightarrow{F} = \overleftrightarrow{\Upsilon} \bullet \overrightarrow{F}, \tag{3}$$

where $\overrightarrow{F}$ is the force strength vector (column matrix), and $\overleftrightarrow{\Upsilon}$ is the Attrition Rate Coefficient tensor (matrix) which by definition is nonpositive, and neglecting fratricide, is traceless. Thus, we know from the start that the sum of the eigenvalues of this matrix is zero. The eigenvalues of the system are λ_i, $i = 1..n_a + n_b$, where n_a, n_b are the number of Red, Blue force strength components. The eigenvectors of the system are represented in the tensor (matrix) $\overleftrightarrow{e}$.[1]

The eigensolutions of this system are

$$\overrightarrow{f} \propto \overleftrightarrow{e} \bullet \overrightarrow{F}, \tag{4}$$

which since the eigensolutions have simple solutions,

$$f_i(t) = e^{\lambda_i t} f_i(0), \tag{5}$$

this gives Force Strength explicit solutions of the form,

$$\overrightarrow{F}(t) = \overleftrightarrow{e}^{-1} \bullet \overleftrightarrow{T}(t) \bullet \overleftrightarrow{e} \bullet \overrightarrow{F}(0), \tag{6}$$

where:

$$\overleftrightarrow{T}(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_{..} t} & 0 \\ 0 & 0 & e^{\lambda_n t} \end{bmatrix}, \tag{7}$$

is the propagator and $\overleftrightarrow{e}^{-1}$ is the inverse of the eigenvector tensor.

To develop the aggregation methodology, we must first examine the spectrum of the eigenvalues. As we have already indicated, the sum of these eigenvalues is zero. This leads us to speculate that the neatest way for the eigenvalues to uniquely have a zero sum is if the eigenvalues occur in pairs that are equal in magnitude but opposite in sign. Of course, if the number of eigenvalues, $n = n_a + n_b$, is odd, then one of the eigenvalues would have to be zero.

If we examine the eigenvalues of a dense ARC tensor, that is, one for which all force strength components are attrited by at least one other force strength element, and which attrit at least one other force strength component,[2] we indeed find that within the accuracy of the calculation, this

[1] Harkening back to the last chapter, these are the left eigenvectors of this equation. Since most eigenvector algorithms calculate right eigenvectors, it is usually necessary to calculate the right eigenvectors of the transposed equation.

[2] Actually, we also want to require that the ARC matrix not be reducible. See discussion later in this chapter under Extensive Aggregation.

is exactly the case, and it is a wonderful result![3][3] Why is this wonderful? Consider this in the context of equation 5. For a pair of matched eigenvalues, the solutions are

$$\begin{aligned} f_+(t) &= e^{\lambda t} f_+(0), \\ f_-(t) &= e^{-\lambda t} f_-(0). \end{aligned} \tag{8}$$

If we multiply these together,

$$\begin{aligned} f_+(t) f_-(t) &= e^{\lambda t} f_+(0) e^{-\lambda t} f_-(0) \\ &= f_+(0) f_-(0), \end{aligned} \tag{9}$$

then we have a result which is independent of time! This is the counterpart of the state solution in the homogeneous Lanchester problem, but where the state solution is used in homogeneous Lanchester theory to effect a solution, in heterogeneous Lanchester theory it is the basis for aggregation.

Of course, if the number of eigenvalues, n, is odd, then one eigenvalue is zero, and the particular solution corresponding to that eigenvalue,

$$f_0(t) = f_0(0), \tag{10}$$

is also a constant.

Since the general form of the eigensolutions is

$$f_i(t) = e_j^{(i)} F_j(t), \tag{11}$$

where $\overrightarrow{e}^{(i)}$ is the i^{th} eigenvector (note the change of notation here - it is intended to reduce confusion. Also, we are using the repeated index summation convention.), and the general form of the force strength solutions is

$$F_i(t) = e_j^{(i)-1} f_j(t), \tag{12}$$

we may rewrite equation 9 as

$$\begin{aligned} f_+(t) f_-(t) &= e_i^{(+)} F_i(t) e_j^{(-)} F_j(t) \\ &= e_i^{(+)} F_i(0) e_j^{(-)} F_j(0), \end{aligned} \tag{13}$$

which explicitly shows the state solution form.

[3]I cannot claim all of this result. Dr. Ben Wise, via the offices of Dr. Leslie G. Callahan (COL USA Ret.), piqued my interest in state solutions (time invariant relationships) for the heterogeneous Lanchester problem. Investigations following from that prod reesulted in the results given here.

As it stands, this is not sufficient to provide a basis for aggregation, but it is a beginning. Let us further consider that if among these pairs of eigenvalues and eigenvectors, there is one pair with some special properties. Specifically, these properties are that the magnitudes of the eigenvector components are equal,

$$\left|e_i^{(+)}\right| = \left|e_i^{(-)}\right|, i = 1..n, an \tag{14}$$

and that either all of the first n_a components of the two eigenvectors are of the same sign and the remaining n_b components are of opposite sign, or *visa versa*. That is,

$$\begin{aligned} e_i^{(+)} &= e_i^{(-)}, i = 1..n_a, \\ e_i^{(+)} &= -e_i^{(-)}, i = n_a + 1..n_a + n_b, \end{aligned} \tag{15}$$

or the opposite. In practice, we find this is the case although we cannot present a rigorous proof. Oh well, you can't say we don't offer challenges to the mathematicians.

Having said all this, it is useful before we proceed to make another shift of notation. Let us split the eigenvector components into two sets of components, defined by

$$\begin{aligned} b_i &\equiv e_i^{(+)}, i = 1..n_a, \\ a_j &\equiv -e_{j+n_a}^{(+)}, j = 1..n_b. \end{aligned} \tag{16}$$

This change of notation allows us to write the two eigensolutions as

$$\begin{aligned} f_+(t) &= e^{\lambda t}\left[\sum_{i=1}^{n_a} b_i A_i(0) - \sum_{j=1}^{n_b} a_j B_j(0)\right], \\ f_-(t) &= e^{-\lambda t}\left[\sum_{i=1}^{n_a} b_i A_i(0) + \sum_{j=1}^{n_b} a_j B_j(0)\right], \end{aligned} \tag{17}$$

where we have explicitly reintroduced the original Red, Blue force strength components. We also know that these eigensolutions can be written as

$$\begin{aligned} f_+(t) &= \left[\sum_{i=1}^{n_a} b_i A_i(t) - \sum_{j=1}^{n_b} a_j B_j(t)\right], \\ f_-(t) &= \left[\sum_{i=1}^{n_a} b_i A_i(t) + \sum_{j=1}^{n_b} a_j B_j(t)\right], \end{aligned} \tag{18}$$

so that if we combine equations 17 and 18, and perform a bit of adding and subtracting, we may form two new equations,

$$\begin{aligned} \sum_{i=1}^{n_a} b_i A_i(t) &= \cosh(\lambda t) \sum_{i=1}^{n_a} b_i A_i(0) - \sinh(\lambda t) \sum_{j=1}^{n_b} a_j B_j(0), \\ \sum_{j=1}^{n_b} a_j B_j(t) &= \cosh(\lambda t) \sum_{j=1}^{n_b} a_j B_j(0) - \sinh(\lambda t) \sum_{i=1}^{n_a} b_i A_i(0). \end{aligned} \tag{19}$$

If we now select two new mean "attrition rate coefficients" $\overline{\alpha}$ and $\overline{\beta}$, subject only to the restriction that

$$\overline{\alpha}\overline{\beta} = \lambda^2, \tag{20}$$

then we may define aggregate force strengths $\overline{A}$ and $\overline{B}$ by

$$\begin{aligned} \sqrt{\overline{\beta}}\overline{A}(t) &\equiv \sum_{i=1}^{n_a} b_i A_i(0), \\ \sqrt{\overline{\alpha}}\overline{B}(t) &\equiv \sum_{j=1}^{n_b} a_j B_j(0). \end{aligned} \tag{21}$$

If we now combine equations 19 and 21, and do a bit of algebra, we may find that

$$\begin{aligned} \overline{A}(t) &= \cosh(\lambda t)\overline{A}(0) - \sinh(\lambda t)\sqrt{\frac{\overline{\alpha}}{\overline{\beta}}}\overline{B}(0), \\ \overline{B}(t) &= \cosh(\lambda t)\overline{B}(0) - \sinh(\lambda t)\sqrt{\frac{\overline{\beta}}{\overline{\alpha}}}\overline{A}(0), \end{aligned} \tag{22}$$

which is nothing more than the homogeneous Lanchester quadratic attrition solution. This lucky (?) happenstance of the behavior of these two eigenvectors provides the basis for aggregation of the heterogeneous Lanchester formalism to the homogeneous Lanchester formalism in a formal sense. That is, the aggregation preserves the mathematical symmetry. In simple terms, the mathematical form of the end result of the aggregation (from heterogeneous Lanchester to homogeneous Lanchester) has the same mathematical properties as homogeneous Lanchester.

38.3.2 The Red Queen Problem

In the real world of crunching number and doing analyses, we have left a question unanswered. Namely, how do we pick the mean attrition rate coefficients, $\overline{\alpha}$ and $\overline{\beta}$? The answer is that we don't, at least directly. Rather, we take advantage of the fact that there is additional information in the original force strength components. In general, these components represent preexisting aggregations (we shall treat the problem of that aggregation in this formalism in a later chapter.) These components are:

- An aggregate of a single type of weapon system (e.g., tanks or missiles,) or
- An aggregate of a single military unit (e.g., a tank company or an infantry battalion.)

It is natural from a comparison standpoint to want to compare to a base force strength component. If one of the components is (e.g.,) tanks, then it is natural to compare the other force strength components to that force strength component. (This is the basic idea of Weapon Effectiveness Values (WEV's) and Unit Effectiveness Values (UEV's). [4] These values are often based on either test range or training exercise data which does not always reflect actual combat effectiveness. We do want to emphasize, and this is implicit in the development here, that the aggregation is situationally dependent! There is no one aggregation.) If we select a force strength component from each side, say A_{i^*}, and B_{j^*}, as the base comparison force strength components, then we may define the aggregation as

$$\begin{aligned} \overline{A} &\equiv A_{i^*} + \frac{1}{b_{i^*}} \sum_{\substack{i=1 \\ i \neq i^*}}^{n_a} b_i A_i(0), \\ \overline{B} &\equiv B_{j^*} + \frac{1}{a_{j^*}} \sum_{\substack{j=1 \\ j \neq j^*}}^{n_b} a_j B_j(0). \end{aligned} \tag{23}$$

With this definition, the aggregate solutions, equation 22, become

$$\begin{aligned} \overline{A}(t) &= \cosh(\lambda t)\,\overline{A}(0) - \sinh(\lambda t)\,\frac{a_{j^*}}{b_{i^*}}\overline{B}(0), \\ \overline{B}(t) &= \cosh(\lambda t)\,\overline{B}(0) - \sinh(\lambda t)\,\frac{b_{i^*}}{a_{j^*}}\overline{A}(0), \end{aligned} \tag{24}$$

and we may define the mean attrition rate coefficients from what we know from homogeneous Lanchester theory,

$$\begin{aligned} \overline{\alpha} &= \frac{a_{j^*}}{b_{i^*}}\lambda, \\ \overline{\beta} &= \frac{b_{i^*}}{a_{j^*}}\lambda. \end{aligned} \tag{25}$$

This provides a base of comparison for the aggregation (allows comparison of one weapon type to another, or one military unit to another) without compromising the symmetry of the aggregation. It does, however, introduce an aspect of arbitrariness to the aggregation since there is no mandate from the methodology for the force strength components to be selected.

38.3.3 Extensive Aggregation

As we have indicated, intensive aggregation works only for those situations where the ARC tensor is dense. At this point, we must extend our definition of (or at least restrictions on) denseness. Where before, we required only that all force strength components attrit and are attrited, this is a requirement only on the existence of solutions. As we indicated in an earlier chapter, an ARC tensor of the form,

$$\overleftrightarrow{\Upsilon} = \begin{bmatrix} 0 & 0 & -\alpha_1 & 0 \\ 0 & 0 & 0 & -\alpha_2 \\ -\beta_1 & 0 & 0 & 0 \\ 0 & -\beta_2 & 0 & 0 \end{bmatrix}, \tag{26}$$

while it has an eigensolution, that solution is separate in two parts since this ARC matrix is separable.

To again show this, if we transform the force strength vector,

$$\overrightarrow{F} = \begin{bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{bmatrix} \rightarrow \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{bmatrix} = \overrightarrow{F'}, \tag{27}$$

then the ARC tensor transforms to

$$\overleftrightarrow{\Upsilon} \rightarrow \overleftrightarrow{\Upsilon'} = \begin{bmatrix} 0 & -\alpha_1 & 0 & 0 \\ -\beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha_2 \\ 0 & 0 & -\beta_2 & 0 \end{bmatrix}, \tag{28}$$

which is obviously reducible.

Returning now to our problem, how do we aggregate (albeit approximately) heterogeneous force strength components which are not densely connected? Two choices present themselves to our fervid minds. The first is to make the ARC tensor dense by inserting small ARCs (as compared to the other ARCs). This gives a new ARC tensor of the form,

$$\overleftrightarrow{\Upsilon''} = \begin{bmatrix} 0 & 0 & -\alpha_1 & -\epsilon_1 \\ 0 & 0 & -\epsilon_2 & -\alpha_2 \\ -\beta_1 & -\epsilon_3 & 0 & 0 \\ -\epsilon_4 & -\beta_2 & 0 & 0 \end{bmatrix} \tag{29}$$

which we may aggregate using the method described above. The problem with this method is that it tends to drastically reduce the value of one of the subcombats, so that in addition to being arbitrarily dependent on the choice of the added ARCs, it may give a misleading picture of the value of some of the force strength components. There is a simple (but mathematically

complex) way around this problem, but we will defer this to a later time since what we are really interested in is extensive aggregation.

The second choice is to do something *ad hoc*. Let us assume that we have a situation where $n_a = n_b$, so that we have $\frac{n}{2}$ separate combats that we want to aggregate. If we write the Red force strength component solutions (which we again assume to be Lanchester homogeneous quadratic,) as

$$\frac{A_i(t)}{\sqrt{\alpha_i}} = \frac{A_i(0)}{\sqrt{\alpha_i}} \cosh(\gamma_i t) - \frac{B_i(0)}{\sqrt{\beta_i}} \sinh(\gamma_i t), \tag{30}$$

and similarly for the Blue components. We may postulate an aggregation based on the inverse square roots of the ARCS as weighting or aggregation factors,

$$\begin{aligned} \frac{\overline{A}(t)}{\sqrt{\overline{\alpha}}} &= \sum_{i=1}^{n_a} \frac{A_i(t)}{\sqrt{\alpha_i}} \\ &= \sum_{i=1}^{n_a} \frac{A_i(0)}{\sqrt{\alpha_i}} \cosh(\gamma_i t) - \sum_{i=1}^{n_a} \frac{B_i(0)}{\sqrt{\beta_i}} \sinh(\gamma_i t). \end{aligned} \tag{31}$$

If we write an aggregate eigensolution of the form,

$$\begin{aligned} \frac{\overline{A}(t)}{\sqrt{\overline{\alpha}}} + \frac{\overline{B}(t)}{\sqrt{\overline{\beta}}} &= \left[\frac{\overline{A}(t)}{\sqrt{\overline{\alpha}}} + \frac{\overline{B}(t)}{\sqrt{\overline{\beta}}}\right] e^{-\overline{\gamma}t} \\ &= \sum_{i=1}^{n_a} \left[\frac{A_i(0)}{\sqrt{\alpha_i}} + \frac{B_i(0)}{\sqrt{\beta_i}}\right] e^{-\gamma_i t}, \end{aligned} \tag{32}$$

this offers a definition of $\overline{\gamma}$ if we choose a suitable definition of t,

$$e^{-\overline{\gamma}t} = \frac{\sum_{i=1}^{n_a} \left[\frac{A_i(0)}{\sqrt{\alpha_i}} + \frac{B_i(0)}{\sqrt{\beta_i}}\right] e^{-\gamma_i t}}{\sum_{i=1}^{n_a} \left[\frac{A_i(0)}{\sqrt{\alpha_i}} + \frac{B_i(0)}{\sqrt{\beta_i}}\right]}. \tag{33}$$

One such selection would be a value of $t = 1$ which represents a unitary aggregate attrition.

It now remains to actually aggregate the force strengths. To do this, we proceed in the same (arbitrary) manner as for Intensive Aggregation, selecting a base (or comparison) force strength component,

$$\overline{A}(0) = A_{i^*}(0) + \sum_{\substack{i=1 \\ i \neq i^*}}^{n_a} \sqrt{\frac{\alpha_{i^*}}{\alpha_i}} A_i(0), \tag{34}$$

and similarly for Blue. This gives us an aggregate $\overline{\delta}$ of the form

$$\overline{\delta} = \sqrt{\frac{\alpha_{i*}}{\beta_{j*}}}, \tag{35}$$

which we may use to calculate aggregate ARCs as

$$\begin{aligned} \overline{\alpha} &= \overline{\gamma}\overline{\delta}, \\ \overline{\beta} &= \frac{\overline{\gamma}}{\overline{\delta}}. \end{aligned}$$

38.4 Examples

At this point, it is useful to present a few sample calculations. As is usual, I have made the data up and adjusted it so that the force strengths don't go negative too soon. The calculations were performed in two steps. I first calculated the eigenmath using MAPLE although any symbolic algebra program would suffice, and then actually generated the numbers in a spreadsheet. Since this eigenmath only depends on the values of the ARCs, this leaves considerable leeway.

Both of the examples are for 2 x 2 combats. That is, each side has two force strength components. The initial force strengths are

$$\vec{F}(0) = \begin{bmatrix} 75 \\ 35 \\ 75 \\ 50 \end{bmatrix}. \tag{36}$$

(If you go to repeat these calculations, note that I have done some judicious rounding, partly because of laziness and partly to correct for some of the inaccuracy in the calculations.)

38.4.1 Intensive Aggregation Example

In this case, we use an ARC tensor of the form

$$\overleftrightarrow{\Upsilon} = \begin{bmatrix} 0 & 0 & -.1 & -.2 \\ 0 & 0 & -.3 & -.35 \\ -.15 & -.25 & 0 & 0 \\ -.4 & -.3 & 0 & 0 \end{bmatrix} \tag{37}$$

which has eigenvalues of ± 0.52 and ± 0.07. The eigenvector tensor is

$$\overleftrightarrow{e} = \begin{bmatrix} .59 & .56 & -.43 & -.60 \\ -.72 & .35 & -.46 & .30 \\ -.72 & .35 & .46 & -.30 \\ .59 & 56 & .43 & .60 \end{bmatrix}, \tag{38}$$

which has been arranged in order of descending value of eigenvalue. (I want to emphasize again that I have rounded judiciously here. The symbolic algebra program gave me considerably more decimal places not all of which were right.) Of course, I also got the inverse of the above, but it isn't really valuable for the discussion. It should be obvious from just looking at the eigenvector tensor that the first and last eigenvectors (corresponding to the larger magnitude eigenvalue pair) are the aggregation eigenvectors.

Next, we may generate a variety of calculations from these data. First, we may calculate the force strengths by brute, numerical force in the manner that we have become used to. These are shown in figure 1. We may also calculate the eigensolutions (the $f_i(t)$) since we have the eigenvalues and the eigenvectors. These are shown in figure 2. Once we have these, it is a simple matter to calculate the force strength solutions using the inverse eigenvectors. We show these in figures 3 and 4 for the Red and Blue force strengths respectively. For comparison we have included the corresponding curves from figure 1. Finally, we can go ahead and perform the aggregation. The calculations shown are from a fundamental formal aggregation (initial force strengths and mean ARCS) applied to the analytical solution, and a pointwise aggregation of the individual force strengths as shown in earlier figures.

The agreement of these curves is noteworthy; the differences being largely due to differences in calculation and rounding. The reader may want to pay particular note to the agreement between the *ab initio* and pointwise aggregations in figure 5.

38.4.2 Extensive Aggregation Example

We now turn to an example of extensive aggregation. Where the previous example demonstrated the aggregation of a 2 x 2 dense or connected combat, this example demonstrates the aggregation of a 2 x 2 unconnected combat. As we have earlier indicated, there are approximate ways of performing the aggregation, both with drawbacks.

The ARC tensor for this problem is

$$\overleftrightarrow{\Upsilon} = \begin{bmatrix} 0 & 0 & -.1 & 0 \\ 0 & 0 & 0 & -.35 \\ -.15 & 0 & 0 & 0 \\ 0 & -.3 & 0 & 0 \end{bmatrix}. \tag{39}$$

As we noted earlier, this is a reducible ARC tensor which has eigenvalues ± 0.32 and ± 0.12. The eigenvector tensor is

$$\overleftrightarrow{e} = \begin{bmatrix} 0 & 0.68 & 0 & -0.73 \\ 0.78 & 0 & -0.64 & 0 \\ 0.78 & 0 & 0.64 & 0 \\ 0 & 0.68 & 0 & 0.73 \end{bmatrix}. \tag{40}$$

As before, we calculate the force strength trajectories for this example. These are presented in figure 6. These are the result of two independent pairs of eigenvectors as indicated by equation

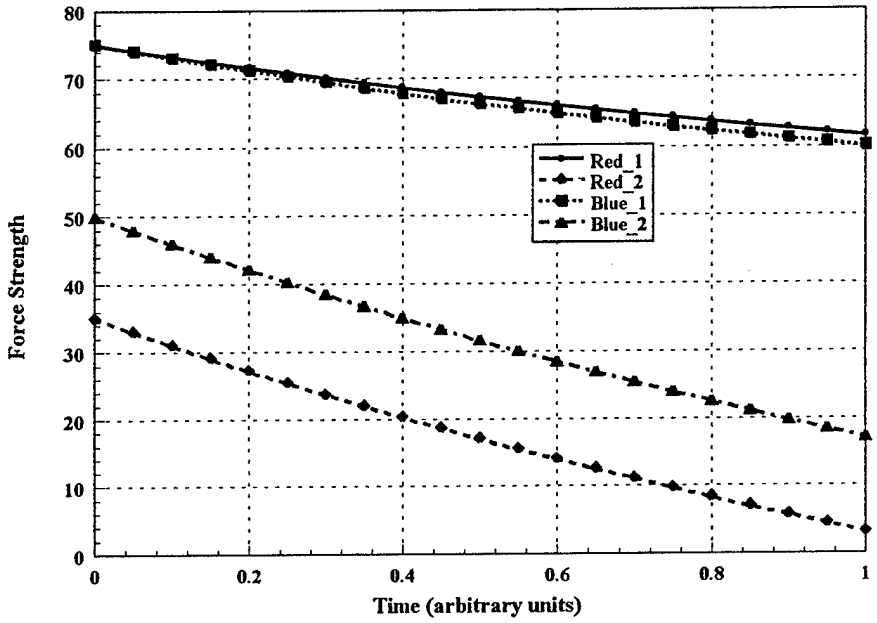

Figure 1: Red and Blue Force Strengths calculated numerically.

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38.4.2.1 Artificial Denseness

We may approximately aggregate these in one of two ways. First, we may insert small connecting ARCs in the ARC matrix,

$$\overleftrightarrow{\Upsilon} = \begin{bmatrix} 0 & 0 & -.1 & -.02 \\ 0 & 0 & -.03 & -.35 \\ -.15 & -.025 & 0 & 0 \\ -.04 & -.3 & 0 & 0 \end{bmatrix}, \tag{41}$$

which results in connected eigenvectors. This is equivalent to making the problem artificially dense, thereby reducing the problem to one of Intensive Aggregation which we may perform exactly. This permits formation of an aggregate solution, but suffers inaccuracy due to the arbitrary coupling of the force strength components via the inserted small ARCs. This is demonstrated by comparison of the aggregation of the force strength components propagated by the single pair of eigenvalues and comparable time by time aggregation of the individual force strength

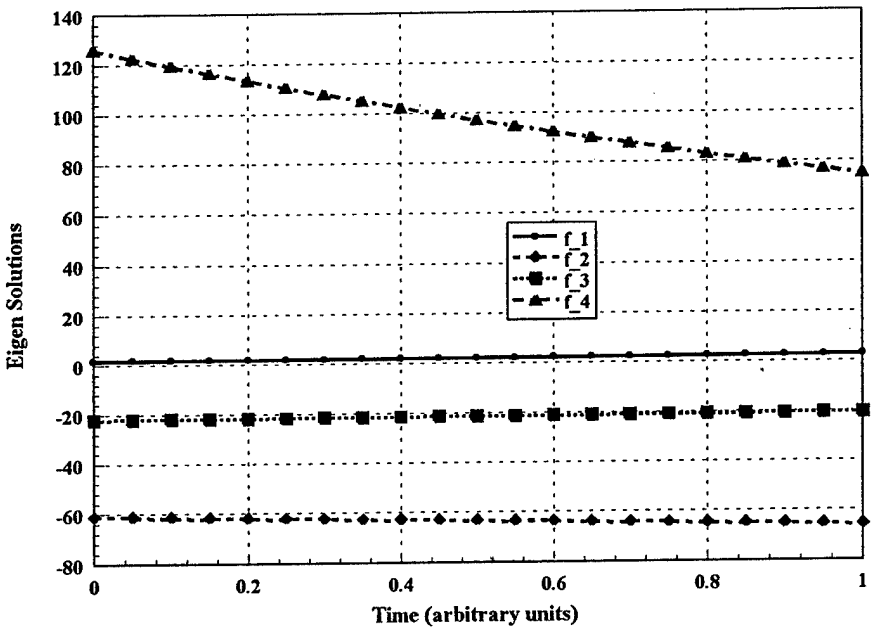

Figure 2: Eigensolutions for Intensive Aggregation Example.

trajectories. This is shown in figure 7. These curves should compare closely if the aggregation were useful.

38.4.2.2 Ad Hoc External Aggregation

Alternately, we may externally aggregate using the *ad hoc* method previously described. If we perform these calculations and perform the same time by time comparison, the result is shown in figure 8. Close comparison shows this to be a better aggregation.

38.5 Aggregation and Disaggregation

We have demonstrated two aggregation methodologies in this chapter. The first, Intensive Aggregation, which does not implicitly depend on the initial force strength components, is exact when the combat is dense. That is, when all force strength components each attrit at least one other force strength component and are attrited by at least one other force strength component. The second methodology, Extensive Aggregation, which does depend on the initial force strength

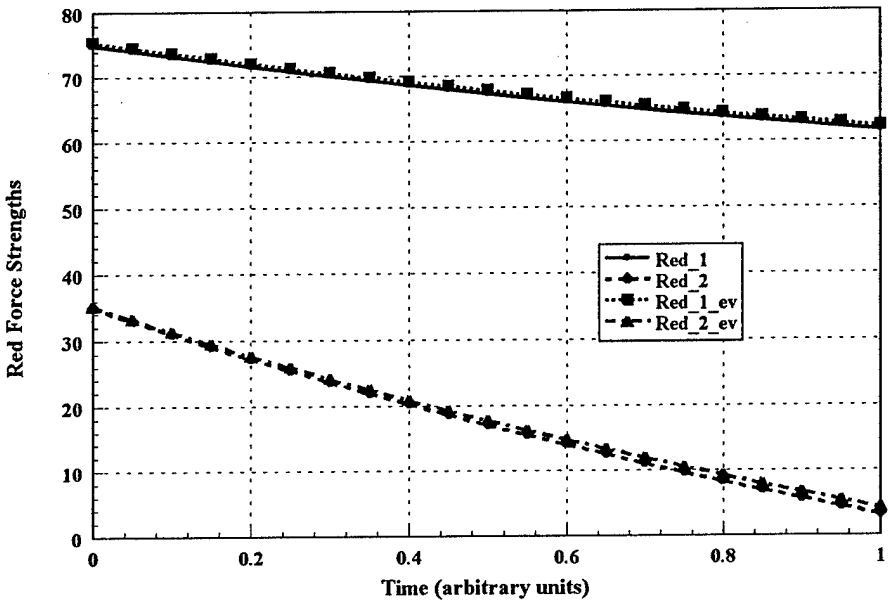

Figure 3: Comparison of numerically and eigensolution calculated Red force strengths - Intensive Aggregation Example

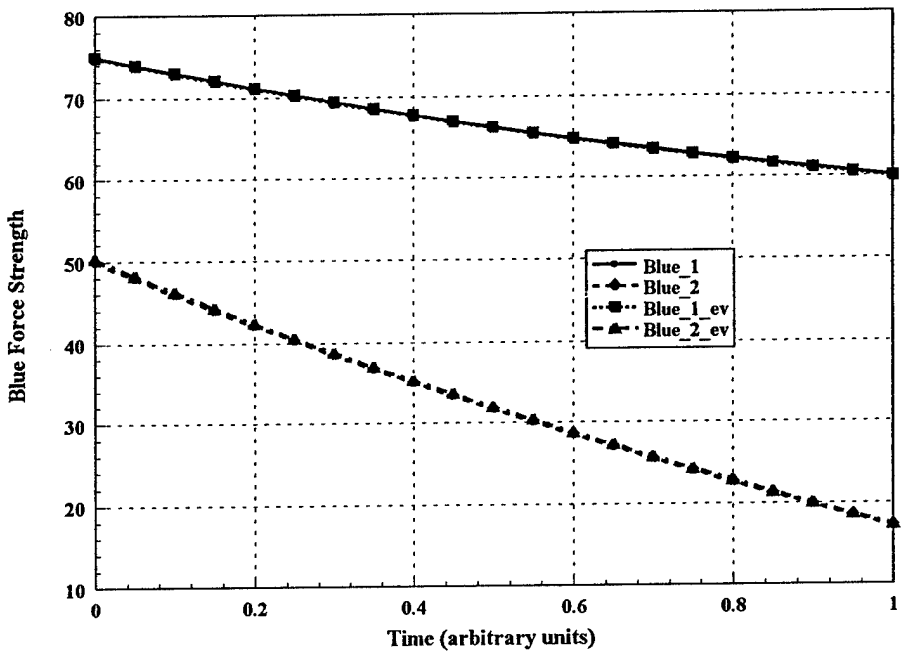

Figure 4: Comparison of numerically and eigensolution calculated Blue force strengths - Intensive Aggregation Example

components, can provide a useful approximate aggregation for those problems where combat is not dense by using the square roots of the ARCs as weight factors. In principle, these two methodologies provide a basis for aggregation of the heterogeneous Lanchester problem to the homogeneous Lanchester problem within the limits of the quadraticization of the attrition differential equations. This is sufficient to permit a collapse of resolution scale from a unit or weapon system level of aggregation to a higher level of aggregation. In principle therefore, one may start with a very complicated set of combats at the engagement level and aggregate up to battle or campaign level.

Care must be taken in performing this aggregation. Unless the combats are close to identical, it may be necessary to distinguish among variations in the ARCs. This may require the generation of a time series of aggregate ARCs. Since this process is straightforward but situationally dependent, we shall only allude to the process here.

Finally, some mention must be made of the process of disaggregation. It should be evident from the discussion above that aggregation discards information about the combat. This

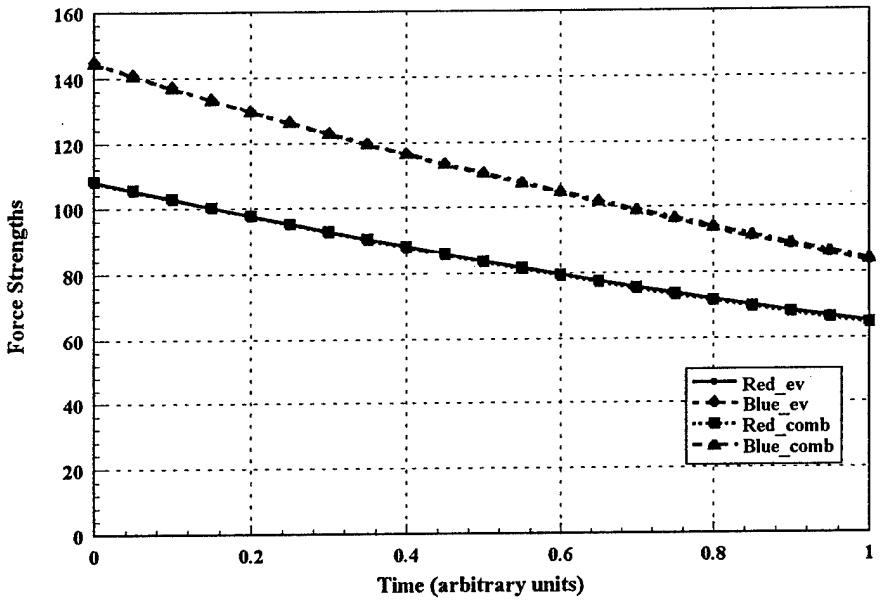

Figure 5: Comparison of Force Aggregate time solutions - Intensive Aggregation Example.

is blatantly obvious in the methodology of Intensive Aggregation where we retain only 2 of n eigensolutions in the aggregation. Although less obvious, the same is true of Extensive Aggregation. If we discard this information, then it is not available for disaggregation. Exact disaggregation is possible only if, for Intensive Aggregation, we retain all of the eigensolutions, not just the two selected ones. Exact disaggregation is not possible if, for the Intensive Aggregation methodology, we discard these other $n - 2$ eigensolutions, or at all for Extensive Aggregation.

It would be comforting to suggest that we might disaggregate by assuming proportional losses of force strength components. Sadly, this is not the case. Proportionality calculations of this type are fraught with error. (Indeed, we again note our caveat that Intensive Aggregation is valid only so long as all force strength components are positive.) This is a sad story in our modern world of interfacing multi-resolution simulations where on-the-fly aggregation/disaggregation is a desired capability. At best, we must recognize that disaggregation can only be accomplished in a arbitrary and *ad hoc* manner.

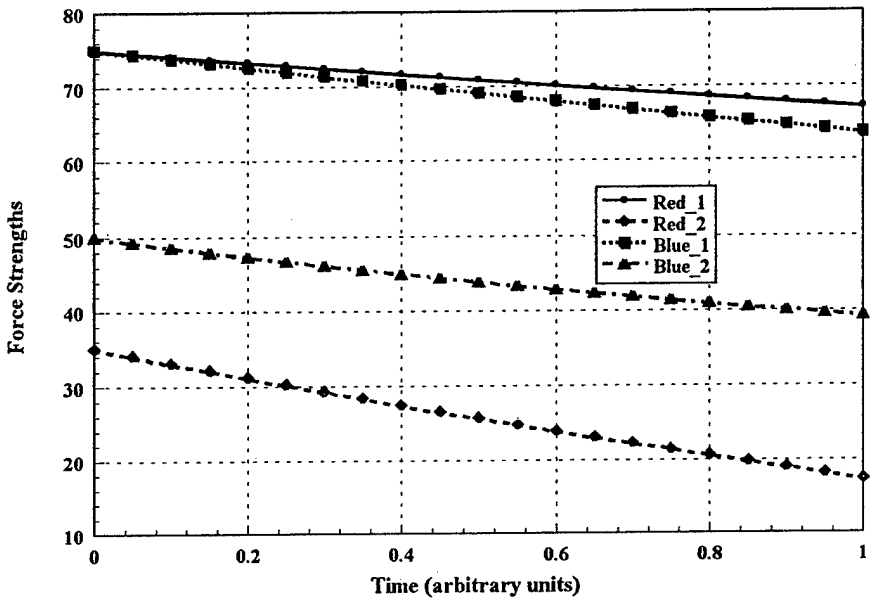

Figure 6: Red and Blue Force Strengths calculated numerically - Extensive Aggregation Example.

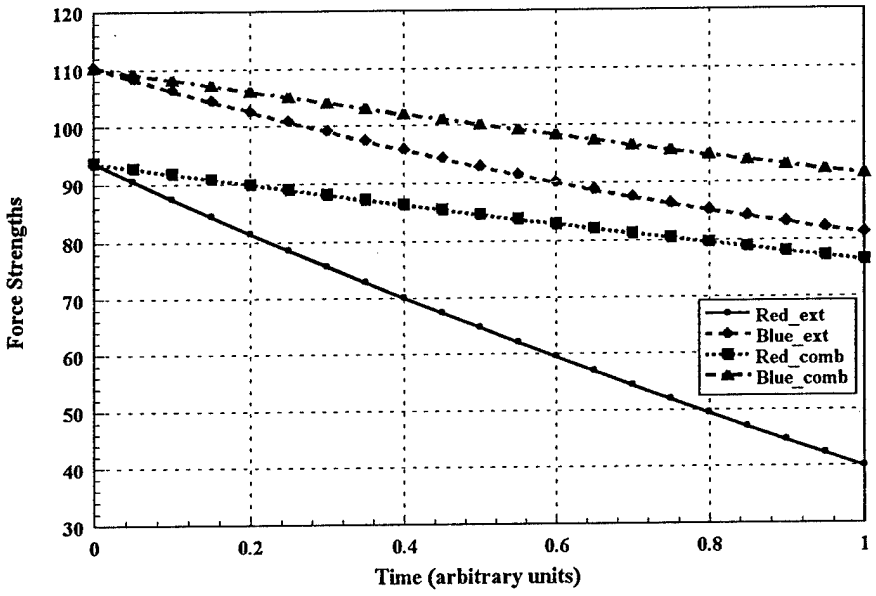

Figure 7: Extensive Aggregation by Artificial Denseness - Comparison of initial aggregation with time by time aggregation.

38.6 References

[1] Guralnik, David B., chief ed., **New World Dictionary for Young Readers**, Simon and Schuster, New York 1979.

[2] Fowler, Bruce W., "The Mathematical Theory of Attritional Aggregation", prepared for the (cancelled) Callaway Gardens Conference on Modeling, Simulation, and Gaming of Warfare, 1994.

[3] Fowler, Bruce W., "A *Perestroika* Program for Modeling and Simulation Tools and Techniques", in Leslie G. Callahan, Jr., and Ross A, Gagliano, eds., **Proceedings of the 1991 Callaway Workshop on Modeling, Simulation, and Gaming for Restructuring the U. S. Armed Forces**, Callaway Gardens, Pine Mountain, GA 1-4 December 1991, pp. 95-122.

[4] "Weapon Equivalence Studies", Chapter 30 in Frank E. Grubbs, chief contributor, **Engineer-**

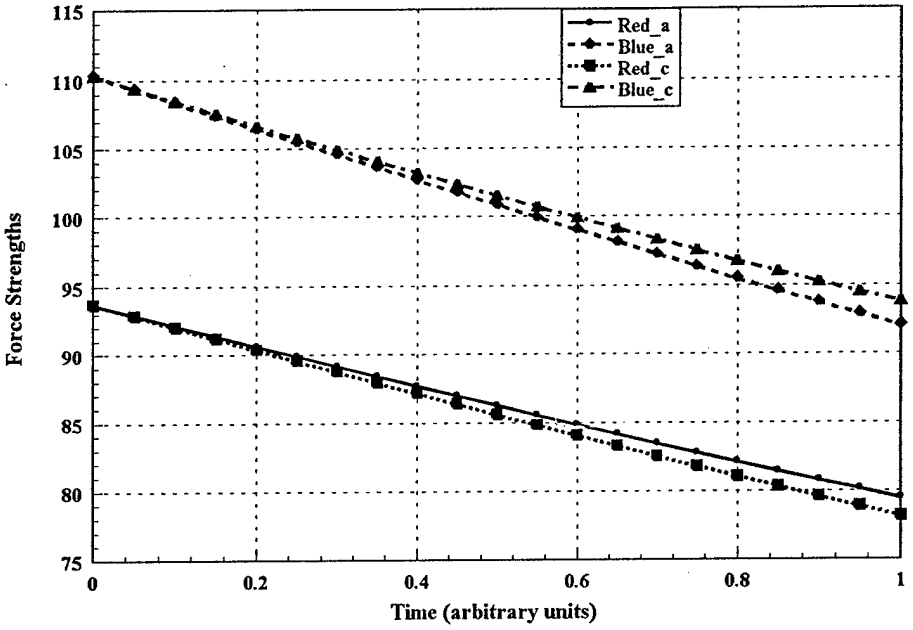

Figure 8: External Aggregation by ad hoc method - comparison of initial aggregation with aggregate ARCs and time by time step aggregation.

ing Design Handbook: Army Weapon Systems Analysis, Part Two, DARCOM-P 706-102, October 1972.

Chapter 39

Aggregation for Simulation

39.1 Introduction

In the previous chapter, we developed two methodologies for aggregating the heterogeneous Lanchester problem into a homogeneous Lanchester problem. These two methodologies were:

- Intensive Aggregation; and
- Extensive Aggregation.

Intensive Aggregation is an exact method of aggregation which is limited to dense combat. That is, each of the force strength components must both attrit at least one other force strength component and be attrited by another force strength component. This type of aggregation does not depend on the initial force strengths, but only on the Attrition Rate Coefficient (ARC) matrix. Extensive aggregation is an approximate method for aggregating non-dense combat and it does depend on the initial force strengths and the ARCs.

Both of these aggregations may be performed either across or by weapons type. This is depicted in figures 1 and 2 which are the same as shown in Chapter XV. In figure 1, aggregation of three units is depicted. Two of these units are infantry units, while the third is an armored unit. Alternately, these may be viewed as two collections (aggregates) of infantry weapons (e.g., assault rifles, etc.) and one collection of armored weapons (main tank gun). The aggregation here is to a composite which has the properties of all three types of weapons (units). (This division into three units is an artifact in terms of the methodology presented in the previous chapter.)

In figure 2, aggregation of three units is also depicted. Again, two of these units are infantry units, while the third is an armored unit. The aggregation here is to three units (aggregates), two of which have infantry properties and a third which has armor properties.

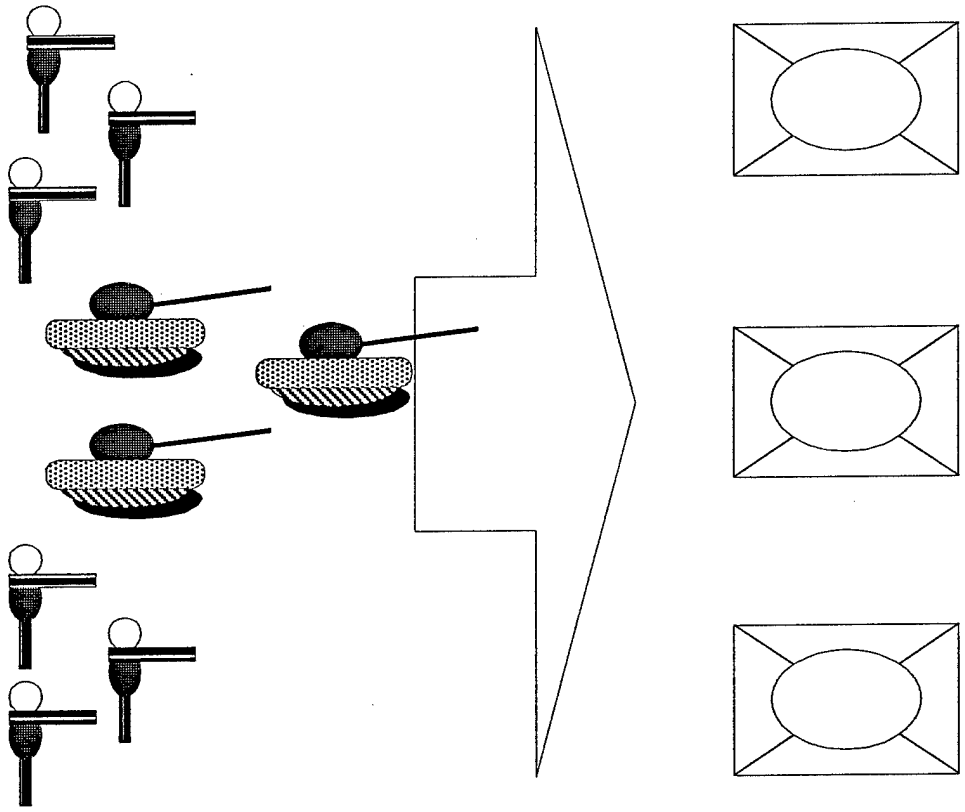

Figure 1: Aggregation across weapons types

The question arises, how do we arrive at the initial aggregation?

39.2 Process Models

In Chapter XV, we described the fundamental role of the process in the formulation of models and simulations. At that juncture, we identified two ways in which models and simulations could be constructed. These two ways are depicted in figure 3.

The left hand branch depicts the way in which process models are used to develop (and utilize) simulations at the platform or weapons' system level. The right hand branch depicts the way that Lanchester-type models and simulations are developed and utilized. In platform level simulations (nominally stochastic), the processes are randomly sampled to determine when processes conclude (assuming they are time dependent processes - this serves to define events), and to determine the outcome of random events (do shots hit and damage targets?)

Since these simulations are stochastic, they are executed several times for the same set of

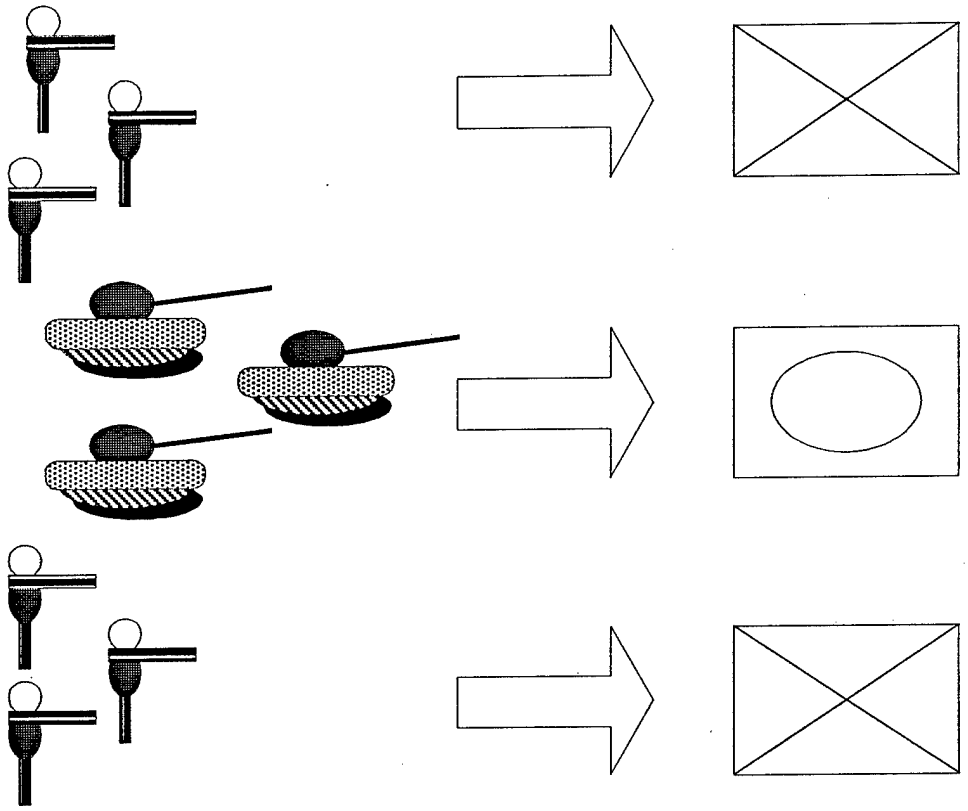

Figure 2: Aggregation by weapons types

input data, and the results are statistically analyzed. This provides information on attritional losses (and more, but that's what we are interested in here.) These results reflect a form of aggregation.

In Lanchester-type models, aggregation is performed initially, usually in the form of proper aggregation of the platforms into units and/or weapons' types, and in the formulation of ARCs. The attrition differential equations (e.g.,) are then coded into a simulation which is executed once (assuming the formulation is not stochastic Lanchester in form.) The results of this execution is then compared to the results coming from the platform level, stochastic simulation.

The question we want to address here is, how do we bootstrap these stochastic, platform level simulation results into Lanchester aggregations. The answer to this question is of interest for several reasons. It allows more direct comparison of the results of the two formalisms, and it provides a hierarchial framework of models and simulations. In general, this latter is important because of the limitations of computer size and available time to execute studies and analyses. (Of which, the latter is today the more important.)

In general, the time to execute a combat simulation is of the order of N^2 where N is the

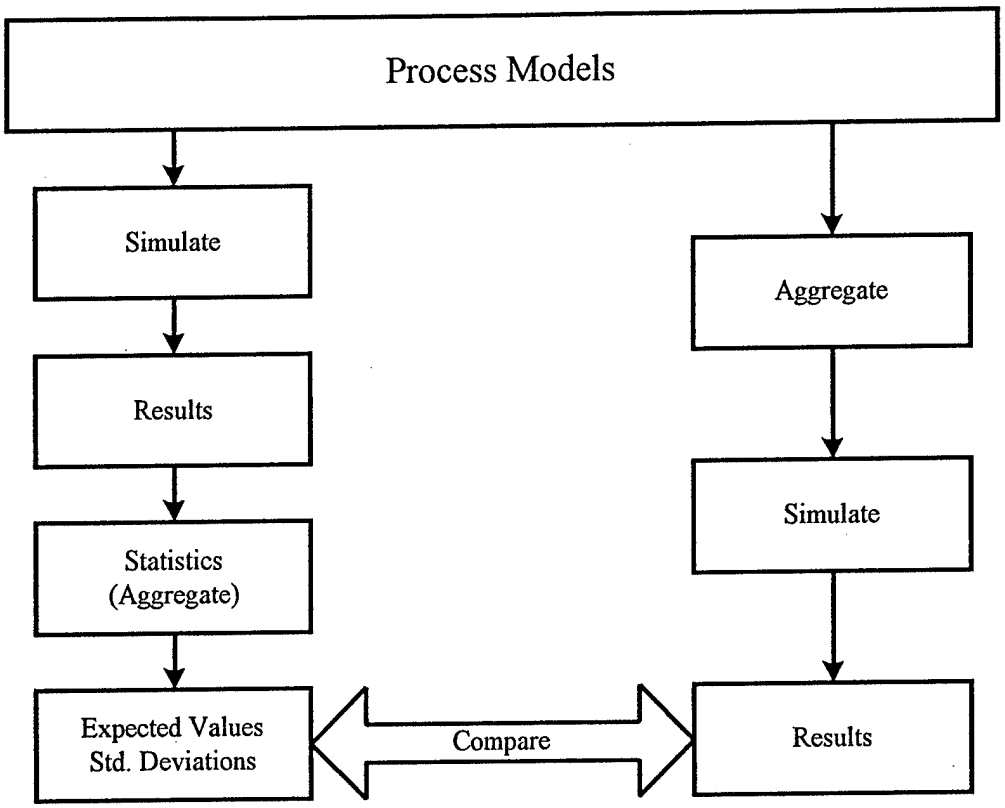

Figure 3: Alternate Routes for Process Models

number of individual degrees of freedom in the simulation and data. For stochastic, platform level simulations, N is related to the number of platforms while for Lanchester-type simulations, N is the number of force strength components. As we successively consider higher levels of combat (engagements → battles → campaigns → wars,) the number of platforms increases by a factor of approximately 3, so that if the number of degrees of freedom stays constant, each successive higher levels of combat increases its execution time by a factor of approximately 10. Since we generally want to support more executions that this in a study, the most reasonable way to reduce the time for execution is to successively reduce the number of degrees of freedom. This requires aggregation.

39.3 Platform Simulation Results

The attrition output of stochastic, platform level simulations is commonly expressed in the form of Killer-Victim Scoreboards (KVS.) These KVS are matrices whose columns represent attriting

systems/platforms and whose rows represent attrited systems/platforms. The entries are the losses. For a 3 (Red) by 3 (Blue), the one-sided KVS is:

Killer-Victim Scoreboard

Victims\Killers	Killer 1	Killer 2	Killer 3
Victim A	A loss to 1	A loss to 2	A loss to 3
Victim B	B loss to 1	B loss to 2	B loss to 3
Victim C	C loss to 1	C loss to 2	C loss to 3

If we combine the KVS for the two sides (again ignoring fratricide until a later chapter,) then for a 2 x 2 combat (two Red, two Blue,) then they can be formed into a differential loss matrix,

Loss Matrix

Victims\Killers	Red 1	Red 2	Blue 1	Blue 2
Red 1	0	0	$\Delta A_1 \mid_{\text{Blue 1}}$	$\Delta A_1 \mid_{\text{Blue 2}}$
Red 2	0	0	$\Delta A_2 \mid_{\text{Blue 1}}$	$\Delta A_2 \mid_{\text{Blue 2}}$
Blue 1	$\Delta B_1 \mid_{\text{Red 1}}$	$\Delta B_1 \mid_{\text{Red 2}}$	0	0
Blue 2	$\Delta B_2 \mid_{\text{Red 1}}$	$\Delta B_2 \mid_{\text{Red 2}}$	0	0

In addition, if these results reflect several executions, then there can also be a standard deviation (and correlation) matrix(s). Further, since these data can be collected at different times, a series of them can be generated representing different snapshots of the combat.

39.4 Connectivity

If we adopt the notation that we have used previously where we combine the force strength components into a single force strength vector (matrix),

$$\overrightarrow{F} \equiv \begin{bmatrix} A_1 \\ A_{\cdot\cdot} \\ A_{n_a} \\ B_1 \\ B_{\cdot\cdot} \\ B_{n_b} \end{bmatrix}, \tag{1}$$

then we may represent the entries of the Loss Matrix as $\Delta F_{i|j}$.

39.5 Back to Matrix Attrition

Recalling the form of the matrix attrition differential equation,

$$\frac{d}{dt}\overrightarrow{F}(t) = \overleftrightarrow{\Upsilon} \bullet \overrightarrow{F}(t), \tag{2}$$

which may be written in component (index) notation as

$$\frac{d}{dt}F_i(t) = \sum_{j=1}^{n_a+n_b} \gamma_{i,j} F_j(t), i = 1..n_a + n_b \tag{3}$$

where we recall that the ARC matrix is nonpositive by definition. If we integrate equation 3 with respect to time, then the left hand side is

$$\begin{aligned} \int_o^t \frac{d}{dt'} F_i(t')\, dt' &= F_i(t) - F_i(0), \\ &= -\Delta F_i(t), \end{aligned} \tag{4}$$

and the right hand side is

$$\int_o^t \sum_{j=1}^{n_a+n_b} \gamma_{i,j} F_j(t')\, dt' = \sum_{j=1}^{n_a+n_b} \gamma_{i,j} \int_o^t F_j(t')\, dt'. \tag{5}$$

From the eigenmath, we know that the eigensolutions have the form,

$$\overrightarrow{f}(t) = \overleftrightarrow{e} \bullet \overrightarrow{F}(t), \tag{6}$$

with specific solutions,

$$\begin{aligned} f_i(t) &= f_i(0)\, e^{\lambda_i t}, \\ &= \sum_{j=1}^{n_a+n_b} e_j^{(i)} F_j(o)\, e^{\lambda_i t}, \end{aligned} \tag{7}$$

where we have designated the i^{th} eigenvector in component notation as $e_j^{(i)}$. The inverse eigensolutions are given by

$$\begin{aligned} F_i(t) &= \sum_{j=1}^{n_a+n_b} e_j^{(i)-1} f_j(t), \\ &= \sum_{j=1}^{n_a+n_b} e_j^{(i)-1} \sum_{k=1}^{n_a+n_b} e_k^{(j)} F_k(0)\, e^{\lambda_j t}. \end{aligned} \tag{8}$$

This allows us to combine equations 5 and 8 to yield

$$\begin{aligned}\int_o^t F_j(t')\,dt' &= \int_o^t \sum_{j=1}^{n_a+n_b} e_j^{(i)-1} \sum_{k=1}^{n_a+n_b} e_k^{(j)} F_k(0)\, e^{\lambda_j t'} dt', \\ &= \sum_{j=1}^{n_a+n_b} e_j^{(i)-1} \int_o^t e^{\lambda_j t'} dt' \sum_{k=1}^{n_a+n_b} e_k^{(j)} F_k(0), \\ &= \sum_{j=1}^{n_a+n_b} e_j^{(i)-1} \frac{e^{\lambda_j t}-1}{\lambda_j} \sum_{k=1}^{n_a+n_b} e_k^{(j)} F_k(0).\end{aligned} \tag{9}$$

With this in place, we are now in a position to explicitly define the entries in the loss matrix. This requires that we identify the losses to force component i caused by fire from force component j to have an attrition differential equation,

$$\frac{d}{dt} F_{i|j}(t) = \gamma_{i,j} F_j(t). \tag{10}$$

If we integrate this equation, we obtain, courtesy of the above equations,

$$\begin{aligned}\int_o^t \frac{d}{dt'} F_{i|j}(t')\,dt' &= -\Delta F_{i|j}(t) \\ &= \gamma_{i,j} \sum_{j=1}^{n_a+n_b} e_j^{(i)-1} \frac{e^{\lambda_j t}-1}{\lambda_j} \sum_{k=1}^{n_a+n_b} e_k^{(j)} F_k(0),\end{aligned} \tag{11}$$

where we note there is no summation on j! If we write this in matrix notation, it becomes

$$\overrightarrow{\Delta F_{|}} = -\overleftrightarrow{\Upsilon} \diamond \overleftrightarrow{e}^{-1} \bullet \overleftrightarrow{\Lambda}(t) \bullet \overleftrightarrow{e} \bullet \overrightarrow{F}(0), \tag{12}$$

where we note the addition of a symbol $\diamond$ which indicates a special multiplication as defined in equation 11. The matrix product to the right of this symbol reduces to a column matrix. This symbol indicates that each row entry of this column matrix is to be multiplied by the corresponding column entry of the preceding matrix, *but the usual summation does not occur.* Thus, since $\overleftrightarrow{\Upsilon}$ in an N x N matrix, and the resulting matrix is an N x 1 matrix, the resulting matrix of this special multiplication is also N x N. This effectively constitutes the mathematical basis for extracting Lanchester information from stochastic, platform level simulation KVS.

Note the minus sign follows from the implicit nonpositivity of $\overleftrightarrow{\Upsilon}$.

39.6 Algorithm

I say effectively because the situation is not as completely defined as we should like. These equations, 12, depend on explicit knowledge of $\overleftrightarrow{\Upsilon}$! We do not have that knowledge and indeed need it to calculate both eigenvalues and eigenvectors. How do we get that knowledge?

Algorithm

The answer is to revert to a simple numerical trick. We must iterate to a solution. To do this, we must first guess what $\overleftrightarrow{\Upsilon}$ looks like. To do this, we make use of equation 10 and an approximation. For clarity, we shall now label the entries of the loss matrix, the composite KVS's, which are our original given as $L_{i|j}$, and the iterations of the calculations of $\overleftrightarrow{\Upsilon}$, and the eigenmath with superscripts. If we integrate equation 10,

$$\begin{aligned} \int_0^t \frac{d}{dt'} F_{i|j}(t')\, dt' &= -\Delta F_{i|j}(t) \\ &= \gamma_{i,j} \int_0^t F_j(t')\, dt', \end{aligned} \tag{13}$$

then we may recognize the differential losses as the entries of the loss matrix, and define our zeroth order guess of the $\overleftrightarrow{\Upsilon}$ matrix as

$$\gamma_{i,j}^{[0]} \equiv -\frac{L_{i|j}}{\int_0^t F_j(t')\, dt'}, \tag{14}$$

where the minus sign follows from the nonpositivity of the ARC matrix. It is now time to introduce the approximation. Since we do not as yet know the form of the solution for the force strength trajectories, we must approximately integrate the integral in the denominator of equation 14. If we define the total losses to force component i as

$$L_i \equiv \sum_{j=1}^{N} L_{i|j}, \tag{15}$$

then we may apply the trapezoid rule [1]

$$\int_0^t F_j(t')\, dt' \simeq \frac{t}{2}\left[F_j(t) + F_j(0)\right]. \tag{16}$$

Since

$$F_j(t) = F_j(0) - L_j, \tag{17}$$

equation 16 reduces to

$$\begin{aligned} \int_0^t F_j(t')\, dt' &\simeq \frac{t}{2}\left[2F_j(0) - L_j\right] \\ &= tF_j(0)\left[1 - \frac{L_j}{2F_j(0)}\right]. \end{aligned} \tag{18}$$

If $\frac{L_j}{2F_j(0)} \ll 1$, (the small loss limit,) then equation 18 can safely be reduced to

$$\int_0^t F_j(t')\,dt' \simeq tF_j(0). \tag{19}$$

This reduces equation 14 to

$$\gamma_{i,j}^{[0]} \simeq -\frac{L_{i|j}}{tF_j(0)}. \tag{20}$$

This introduces a potential ambiguity. As we indicated before, it is possible to execute the stochastic, platform level simulation to produce a number of KVS (loss matrices) at different times during the simulated combat. Presumably these are generated and averaged (to get the KVS) at the same combat clock times. Indeed we must assume this. These loss matrices may be viewed as a set $\left\{L_{i|j}(k\Delta t), k = 0..K\right\}$ where Δt is the combat clock time increment between KVS samples. The ambiguity arises since we do not know explicitly when the combat begins (i.e., when time is zero from a Lanchestrian sense.) In this case, we must proceed by calculating the differential loss matrices

$$\Delta L_{i|j}(k\Delta t) \equiv L_{i|j}(k\Delta t) - L_{i|j}((k-1)\Delta t), k = 1..K, \tag{21}$$

and proceed to perform K separate analyses, generating Lanchester ARCs for each, and treating the previous attrited force strength components as initial values. We may then further analyze these calculations using the techniques described in Chapter XXI to determine either the "true" combat start time (in a Lanchestrian sense) or be satisfied with time dependent ARCs.

The question is, what do we do with the time factor? The answer to this will become evident as we continue.

We may now use this calculated $\overleftrightarrow{\Upsilon}^{[0]}$ to compute eigenvalues and eigenvectors. These give computed differential losses given by a modified form of equation 12

$$\overrightarrow{\Delta F}_{|}^{[0]} = -\overleftrightarrow{\Upsilon}^{[0]} \diamond \overleftrightarrow{e}^{[0]-1} \bullet \overleftrightarrow{\Lambda}^{[0]}(t) \bullet \overleftrightarrow{e}^{0} \bullet \overrightarrow{F}(0), \tag{22}$$

which we may use to generate an error matrix,

$$\epsilon_{i,j} \equiv L_{i|j} - \Delta F_{i,j}^{[0]}. \tag{23}$$

If all of the entries in this error matrix are sufficiently small (and this must be determined in context depending on the size of the losses and the accuracy desired,) then we may conclude the calculation. Otherwise, we must generate a new $\overleftrightarrow{\Upsilon}^{[1]}$ and iterate again. This process is repeated until the errors are sufficiently small to be satisfactory (after all, we are doing this with a computer.) In this case, the new ARC matrix is defined from

$$\gamma_{i,j}^{[1]} \simeq -\frac{L_{i|j}}{\sum_{k=1}^{N} e_k^{[0](j)-1} \frac{e^{\lambda_k t}-1}{\lambda_k} \sum_{l=1}^{N} e_l^{[0](k)} F_l(0)}. \tag{24}$$

This equation also resolves the time ambiguity we identified earlier and safely allows us to select the time that we use somewhat arbitrarily for these calculations. We may fairly safely assume a unitary value for that time knowing that it simply scales the ARCs linearly. Thus we may perform the iterative calculations with a unitary time and then divide all of the ARCs by the actual time afterwards.

39.7 Examples

It is useful at this point to present a couple of examples of calculations, first to demonstrate the mechanism and then to show the scope of the results achieved.

39.7.1 1x1 Example

We begin by using our old friend (all right, at least admit acquaintance at this point in the relationship,) the homogeneous Lanchester problem. That is, we have data from a very simple simulation that has only two types of platforms. Nonetheless, the simulation may be much more differentially richer than a Lanchester calculation because of more explicit consideration of terrain, weather, process variations, etc. The KVS in this case are simple scalars, and the loss matrix is

$$\overrightarrow{L} = \begin{bmatrix} 0 & 10 \\ 5 & 0 \end{bmatrix}, \tag{25}$$

and the initial forces are

$$\overrightarrow{F}(0) = \begin{bmatrix} 100 \\ 75 \end{bmatrix}. \tag{26}$$

We may now define an ARC matrix from equation 20 (since the losses are fairly small,) as

$$\overleftrightarrow{\Upsilon}^{[0]} = \begin{bmatrix} 0 & -0.133 \\ -0.05 & 0 \end{bmatrix}, \tag{27}$$

which has eigenvalues of $\lambda = \pm 0.08164965809$ and eigenvectors of

$$\overleftrightarrow{e}^{[0]} = \begin{bmatrix} 0.5865023124 & 0.9565563235 \\ 0.5227083735 & -0.8525115579 \end{bmatrix} \tag{28}$$

which we note are not quite symmetric. This is a result partly of the iteration and partly of the calculation method. In practice, it may be necessary to perform some judicious averaging.

Application of these eigenvalues and vectors gives a differential loss matrix of

$$\overleftrightarrow{\Delta F}^{[0]} = \begin{bmatrix} 0.33101405 & 9.9860870625 \\ 5.005557425 & 0.249826053 \end{bmatrix}, \tag{29}$$

which gives us a maximum error of approximately 0.33, which is also approximately (time four-thirds, the per centum error.) This is amazingly good convergence and clearly indicates that the process may be continued to the desired degree of accuracy - assuming the computer will support it. We shall not proceed further with this example but proceed to one of greater complexity.

39.7.2 2x2 Example

We now consider a more complicated example, that of two Red force strength components in combat with two Blue force components. To perform this calculation, it was necessary to develop a stand-alone piece of code to calculate the eigenvalues and vectors. There are numerous methods available to perform these calculations. The only care that must be taken is to select an algorithm which does not assume the matrix to be processed is Hermitian or otherwise symmetric (or even nonnegative!) Accordingly, we shall not dwell on this aspect of the calculation beyond this advisory of some care in selecting the appropriate algorithm and its coded implementation. Such implementations are often available either as libraries or from an archive and thus do not require coding by the analyst, merely some checking.

The Loss Matrix is given by

$$\overrightarrow{L} = \begin{bmatrix} 0.00000 & 0.00000 & 7.29355 & 0.37381 \\ 0.00000 & 0.00000 & 2.18807 & 2.80358 \\ 4.84435 & 0.22952 & 0.00000 & 0.00000 \\ 1.45330 & 1.72143 & 0.00000 & 0.00000 \end{bmatrix}, \tag{30}$$

which we note is fully dense and therefore should give us the type of result we desire, and the initial force strengths are

$$\overrightarrow{F}(0) = \begin{bmatrix} 100.0 \\ 25.0 \\ 75.0 \\ 20.0 \end{bmatrix}. \tag{31}$$

The margin of accuracy assigned to the calculation was a maximum error component of 10^{-6}, which gives an ARC matrix of

$$\overleftrightarrow{\Upsilon} = \begin{bmatrix} 0.0000000 & 0.0000000 & -0.1007026 & -0.0203477 \\ 0.0000000 & 0.0000000 & -0.0302109 & -0.1526082 \\ -0.0503998 & -0.0102231 & 0.0000000 & 0.0000000 \\ -0.0151199 & -0.0766748 & 0.0000000 & 0.0000000 \end{bmatrix}. \tag{32}$$

This ARC matrix has eigenvalues of $\lambda = \pm 0.1151867, \pm 0.0642273$, and eigenvectors of

$$\overleftrightarrow{e} = \begin{bmatrix} -0.2525987 & -0.5200254 & 0.3572268 & 0.7335915 \\ 0.2525987 & 0.5200254 & 0.3572268 & 0.7335915 \\ 0.5487775 & -0.1802358 & -0.7756556 & 0.2543949 \\ -0.5487775 & 0.1802358 & -0.7756556 & 0.2543949 \end{bmatrix}, \tag{33}$$

and inverse eigenvectors of

$$\overleftrightarrow{e}^{-1} = \begin{bmatrix} -0.2723373 & 0.2723373 & 0.7857610 & -0.7857610 \\ -0.8292056 & 0.8292056 & -0.3816779 & 0.3816779 \\ 0.1927552 & 0.1927552 & -0.5558429 & -0.5558429 \\ 0.5877149 & 0.5877149 & 0.2706710 & 0.2706710 \end{bmatrix} \quad (34)$$

From these we may see that the first two eigenvectors (and thus first two eigenvalues) are the aggregation eigenvectors.

Because we had a Loss Matrix which was fully dense (all Blue/Red components attriting and being attrited by all Red/Blue components,) this calculation has demonstrated the aggregation of output from stochastic, platform level simulations into a Lanchester context. Of course, this problem was specified. All we really cared was that the Loss Matrix was not reducible. This could as easily have been satisfied by other loss configurations.

It is now possible to calculate the Lanchester force strength component trajectories which correspond to this simulation's results. These are shown in figure 4. The time corresponding to the KVS (Loss Matrix) in these scaled units is $t = 5$. This permits direct integration and comparison between the stochastic, platform level simulation and a simulation based on heterogeneous Lanchester.

It is also possible to aggregate these results using the techniques described in Chapter XXXVIII. Of course, this discards exactly half of the force strength information (since we are effectively throwing away two components,) but it does allow the use of these KVS results in a higher level simulation based on homogeneous Lanchester. These trajectories are shown in figure 5.

39.8 Conclusion

This concludes our exposition of applying the formal aggregation methodologies in the aggregation of results from higher resolution simulations such as stochastic, platform level simulations. These methods permit us to calculate both the heterogeneous Lanchester equivalent ARCs and to aggregate to heterogeneous Lanchester (if the combat is dense.) As such, they provide a valuable tool in not only comparing the results of simulations of similar level but different resolutions, but also in maintaining a hierarchy of simulations.

39.9 References

[1] Carnahan, Brice, H. A. Luther, James O. Wilkes, **Applied Numerical Methods**, Chapter 2, "Numerical Integration", John Wiley & Sons, Inc., New York, 1969, pp. 69-75.

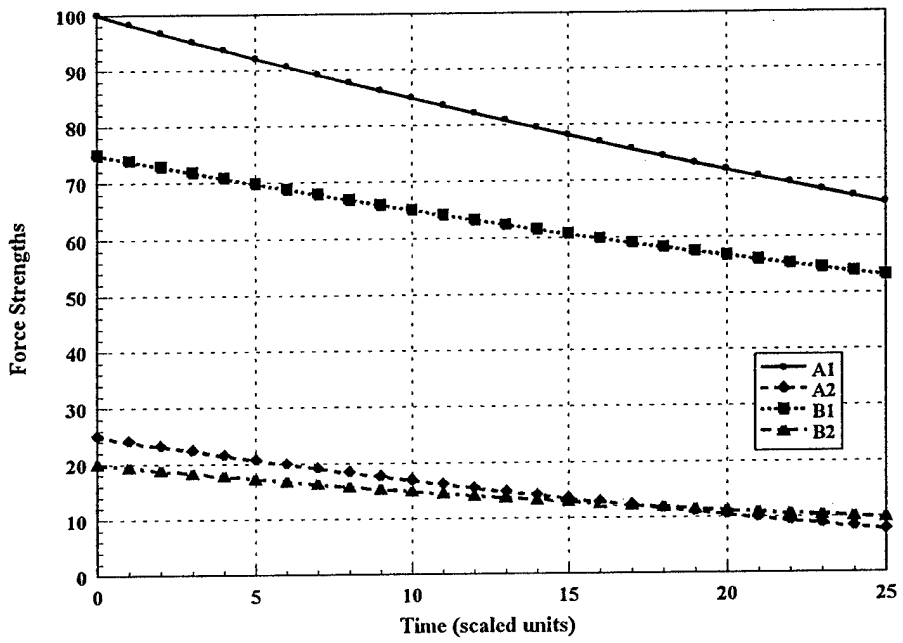

Figure 4: Component Force Strength Trajectories

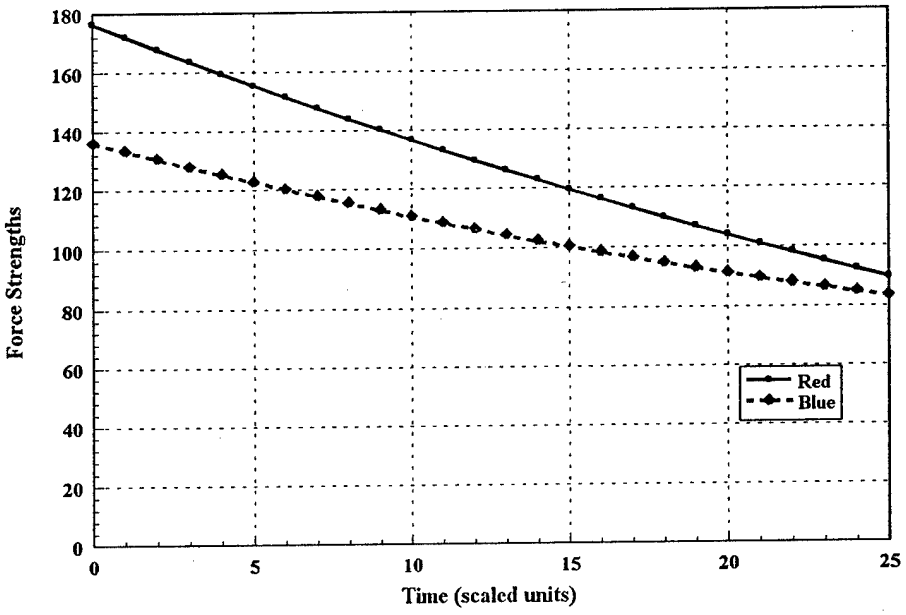

Figure 5: Aggregate Force Strength Trajectories

Chapter 40

Heterogeneous Stochastic Lanchester and Other Aggregations

40.1 Introduction

With this chapter, we come to the conclusion of Part III of the work. There are yet a few remaining topics to be treated, so the composition of this chapter will be a bit varied. In particular, we are going to deal with three topics:

- Stochastic Heterogeneous Lanchester;
- The Anti-Potential Potential Aggregation methodology; and
- the ATCAL methodology.

40.2 Stochastic Heterogeneous Lanchester Theory

Thus far, our consideration of heterogeneous Lanchester theory has been limited to the deterministic theory. From this, we have developed the theory of formal aggregation. Before concluding our consideration of heterogeneous Lanchester theory, it is fitting that we spend a few moments considering a stochastic formalism.

40.2.1 Review of Stochastic Homogeneous Lanchester Theory

We have earlier covered (again briefly,) stochastic homogeneous Lanchester theory in Chapter

XX in Part I of this work. At this time, we review the quadratic homogeneous stochastic attrition differential equations. For the expected force strengths, these are

$$\frac{d}{dt}\langle A\rangle = -\alpha\langle B\rangle, \tag{1}$$

and

$$\frac{d}{dt}\langle B\rangle = -\beta\langle A\rangle, \tag{2}$$

where $\langle X\rangle$ indicates the expected value of X. The variance equations are

$$\frac{d}{dt}\sigma_{AA}^2 = -2\alpha\sigma_{AB}^2 + \alpha\langle B\rangle, \tag{3}$$

and

$$\frac{d}{dt}\sigma_{BB}^2 = -2\beta\sigma_{AB}^2 + \beta\langle A\rangle, \tag{4}$$

and the covariance equation is

$$\frac{d}{dt}\sigma_{AB}^2 = -\beta\sigma_{AA}^2 - \alpha\sigma_{BB}^2, \tag{5}$$

where: $\sigma_{XY}^2 \equiv \langle XY\rangle - \langle X\rangle\langle Y\rangle$. The explicit solutions of these equations is given in Chapter XX.

In that chapter, we also introduced the Fokker-Planck-Kolmogorov (FPK) equation for the joint probability distribution function, [1]

$$\frac{\partial}{\partial t}P(\{x_i\},t) + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}a_i P - \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j}b_{ij}P = 0, \tag{6}$$

where there are assumed to be n random variables x_i, and the functions a_i and b_{ij} are known as the drift and diffusion coefficients. It is possible to show that the approximate FPK equation for homogeneous quadratic Lanchester attrition as

$$\frac{\partial}{\partial t}P(A,B,t) \simeq \left[\frac{\partial}{\partial A}\alpha B + \frac{\partial}{\partial B}\beta A\right]P + \frac{1}{2}\left[\frac{\partial^2}{\partial A^2}\alpha B + \frac{\partial^2}{\partial B^2}\beta A\right]P. \tag{7}$$

If we accept that $P(A,B,t) \to 0$ sufficient fast as $t \to \infty$, then this equation becomes an exact representation of equations 1-5. Of course, this set of equations is based on the assumption that the attrition rate coefficients (ARCs) are constants. Consideration of the case where the ARCs are themselves random variables (as we might expect) is deferred until Part IV.

40.2.2 Extension to Heterogeneity

If we again adopt our favorite heterogeneous Lanchester notation where the force strength components (now considered to be random variables) are represented by the components of the vector $\overrightarrow{F}$ and the ARC matrix (still constant and nonpositive, and now non-random,) is represented by $\overleftrightarrow{\Upsilon}$, then the FPK equation for heterogeneous stochastic Lanchester attrition can be readily extrapolated from equation 7 as

$$\frac{\partial}{\partial t}P\left(\{F_i\},t\right) = -\sum_{i,j=1}^{n}\frac{\partial}{\partial F_i}\gamma_{i,j}F_jP\left(\{F_i\},t\right) - \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial F_i^2}\gamma_{i,j}F_jP\left(\{F_i\},t\right). \tag{8}$$

Since we are only interested in deriving the ADEs at this point, an explicit solution of this equation is unnecessary. We again require the pdf to go to zero as t goes to infinity.

We may derive the ADEs for the expected force strength components in a direct manner. If we form the average,

$$\frac{d}{dt}\left\langle F_k\right\rangle = \int d\Gamma F_k\frac{\partial}{\partial t}P\left(\{F_i\},t\right), \tag{9}$$

where $d\Gamma \equiv \prod_{i=1}^{n} dF_i$ is the state space differential, then we may use the FPK equation to write

$$\begin{aligned}\frac{d}{dt}\left\langle F_k\right\rangle &= \int d\Gamma F_k\left[-\sum_{i,j=1}^{n}\frac{\partial}{\partial F_i}\gamma_{i,j}F_jP\left(\{F_i\},t\right) - \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial F_i^2}\gamma_{i,j}F_jP\left(\{F_i\},t\right)\right] \\ &= -\sum_{i,j=1}^{n}\int d\Gamma F_k\frac{\partial}{\partial F_i}\gamma_{i,j}F_jP\left(\{F_i\},t\right) - \frac{1}{2}\sum_{i,j=1}^{n}\int d\Gamma F_k\frac{\partial^2}{\partial F_i^2}\gamma_{i,j}F_jP\left(\{F_i\},t\right).\end{aligned} \tag{10}$$

So that formation of the ADEs is only a matter of calculating the expectation integrals. The algorithm that we shall use is to successively integrate by parts. To see this, we first examine an integral of the form given in the first right hand side sum of equation 10,

$$I_k = \int d\Gamma F_k\frac{\partial}{\partial F_i}\gamma_{i,j}F_jP. \tag{11}$$

Integration by parts with respect to F_i gives

$$I_k = \int d\Gamma_i'\left[F_k\gamma_{i,j}F_jP\mid_{\lim} - \int dF_i\frac{\partial F_k}{\partial F_i}\gamma_{i,j}F_jP\right], \tag{12}$$

where $d\Gamma_i'$ is missing the dF_i differential. If we assume the first right hand side term is zero (I am not going to show this, you just have to take my word for it,) then the integral reduces to

$$I_k = -\int d\Gamma\frac{\partial F_k}{\partial F_i}\gamma_{i,j}F_jP \tag{13}$$

$$
\begin{aligned}
&= -\int d\Gamma \delta_{i,k}\gamma_{i,j}F_j P \\
&= -\gamma_{k,j}\left\langle F_j\right\rangle .
\end{aligned}
$$

In a similar manner, the integrals of the second summation are all identically zero (derivative of a constant is zero!), so that equation 10 reduces to

$$\frac{d}{dt}\left\langle F_k\right\rangle = \sum_{j=1}^{n}\gamma_{k,j}\left\langle F_j\right\rangle , \tag{14}$$

which are identical to the deterministic equations.

The ADEs for the variances are calculated in a similar manner, except that we start by calculating the second moment equations,

$$\frac{d}{dt}\left\langle F_k F_l\right\rangle = \int d\Gamma F_k F_l \frac{\partial}{\partial t}P. \tag{15}$$

If we proceed as before, this gives

$$\frac{d}{dt}\left\langle F_k F_l\right\rangle = \sum_{j=1}^{n}\left[\gamma_{k,j}\left\langle F_l F_j\right\rangle + \gamma_{l,j}\left\langle F_k F_j\right\rangle\right] - \sum_{j=1}^{n}\delta_{k,l}\gamma_{l,j}\left\langle F_j\right\rangle . \tag{16}$$

It is a simple matter then to form the variance ADEs from the definition of the variances,

$$\frac{d}{dt}\Sigma^2_{kl} = \sum_{j=1}^{n}\left[\gamma_{k,j}\Sigma^2_{l,j} + \gamma_{l,j}\Sigma^2_{k,j}\right] - \sum_{j=1}^{n}\delta_{k,l}\gamma_{l,j}\left\langle F_j\right\rangle , \tag{17}$$

where: $\Sigma^2_{k,l} \equiv \left\langle F_k F_l\right\rangle - \left\langle F_k\right\rangle\left\langle F_j\right\rangle$. Because we are going to be referring to our eigensolutions, we introduce a notation here where the capital sigma indicates the force strength component variances and the lower case sigmas indicate the eigensolution variances.

At this point, it is appropriate to note that based on the form of the variances, we would expect transformation between the eigenvariances and the force strength component variances of

$$\sigma^2_{i,j} = \sum_{k,l=1}^{n} e_{i,k}e_{j,l}\Sigma^2_{kl}. \tag{18}$$

40.2.3 Back to Eigensolutions

At this point, it is convenient to return to consideration of the eigensolutions of the deterministic ADEs. The reader may recall that we defined the eigensolutions as

$$
\begin{aligned}
\overrightarrow{f} &= \overleftrightarrow{e}\bullet\overrightarrow{F} \\
f_i &= \sum_{j=1}^{n} e_{i,j}F_j,
\end{aligned} \tag{19}
$$

and that the transformation of the ARC matrix was

$$\begin{aligned} \overleftrightarrow{e} \bullet \overleftrightarrow{\Upsilon} \bullet \overleftrightarrow{e}^{-1} &= \overleftrightarrow{\lambda}, \\ \sum_{k,l=1}^{n} e_{i,k}\gamma_{k,l}e^{-1}_{l,j} &= \lambda_i \delta_{i,j}. \end{aligned} \tag{20}$$

We now want to make use of these equations to transform the FPK equation into the coordinate system of the eigensolutions. To do this, we must first convert the two derivative terms.

We start with the first order term, changing the variables of differentiation to eigenfunctions,

$$\frac{\partial}{\partial F_i}\gamma_{i,j}F_j = \sum_{k=1}^{n} \frac{\partial f_k}{\partial F_i}\frac{\partial}{\partial f_k}\gamma_{i,j}F_j, \tag{21}$$

and substitute eigenvector expansions for the first and last terms in the right hand side sum,

$$\begin{aligned} \frac{\partial}{\partial F_i}\gamma_{i,j}F_j &= \sum_{k=1}^{n} \frac{\partial \sum_{l=1}^{n} e_{k,l}F_l}{\partial F_i}\frac{\partial}{\partial f_k}\gamma_{i,j}\sum_{m=1}^{n} e^{-1}_{j,m}f_m \\ &= \sum_{k,l,m=1}^{n} \frac{\partial F_l}{\partial F_i}\frac{\partial}{\partial f_k}e_{k,l}\gamma_{i,j}e^{-1}_{j,m}f_m. \end{aligned} \tag{22}$$

Since by definition, $\frac{\partial F_l}{\partial F_i} = \delta_{i,l}$, this equation may be rewritten as

$$\begin{aligned} \frac{\partial}{\partial F_i}\gamma_{i,j}F_j &= \sum_{k,l,m=1}^{n} \delta_{i,l}\frac{\partial}{\partial f_k}e_{k,l}\gamma_{i,j}e^{-1}_{j,m}f_m \\ &= \sum_{k,m=1}^{n} \frac{\partial}{\partial f_k}e_{k,l}\gamma_{l,j}e^{-1}_{j,m}f_m, \end{aligned} \tag{23}$$

which we may reduce further using equation 20 to

$$\begin{aligned} \frac{\partial}{\partial F_i}\gamma_{i,j}F_j &= \sum_{k,m=1}^{n} \frac{\partial}{\partial f_k}\lambda_k\delta_{k,m}f_m \\ &= \sum_{k=1}^{n} \frac{\partial}{\partial f_k}\lambda_k f_k. \end{aligned} \tag{24}$$

The second order term is somewhat more complicated, reducing after somewhat more algebra (but via the same approach,) to

$$\frac{\partial^2}{\partial F_i^2}\gamma_{i,j}F_j = \sum_{l,k=1}^{n} \frac{\partial}{\partial f_l}\frac{\partial}{\partial f_j}e_{l,i}e_{j,i}e^{-1}_{i,k}\lambda_k f_k. \tag{25}$$

With these in hand, we may write the FPK equation (where P is now a function of the eigensolutions,) as

$$\frac{\partial}{\partial t}P = -\sum_{k=1}^{n}\frac{\partial}{\partial f_k}\lambda_k f_k P + \frac{1}{2}\sum_{i,j,l,k=1}^{n}\frac{\partial}{\partial f_l}\frac{\partial}{\partial f_j}e_{l,i}e_{j,i}e_{i,k}^{-1}\lambda_k f_k P. \tag{26}$$

This is a disappointing result since it indicates a highly complicated form for the diffusion components. It is this very complexity that points up an important distinction. Even though there are eigensolutions for the deterministic (expected value) ADEs, these eigensolutions correspond to the expected values, not to the variances. Therefore, we may draw a distinction between randomness that is natural to the force strength components and randomness that is natural to the eigensolutions. To explore the consequences of this, we shall address the solutions of each in turn.

40.2.3.1 Eigensolution Randomness

If the randomness (stochastic quality) of the attrition was natural to the coordinate frame of the eigensolutions (which we may argue would be the case of randomness due only to interaction among the force strength components rather than inherent to them,) then the FPK equation would have the form

$$\frac{\partial}{\partial t}P = -\sum_{k=1}^{n}\frac{\partial}{\partial f_k}\lambda_k f_k P + \frac{1}{2}\sum_{k=1}^{n}\frac{\partial^2}{\partial f_k^2}\lambda_k f_k P. \tag{27}$$

This FPK gives rise to ADEs of the form:

$$\frac{d}{dt}\langle f_j\rangle = \lambda_j\langle f_j\rangle, \tag{28}$$

for the expectation values; and

$$\frac{d}{dt}\sigma_{j,k}^2 = (\lambda_j+\lambda_k)\,\sigma_{j,k}^2 + \delta_{j,k}\lambda_j\langle f_j\rangle. \tag{29}$$

This is a wonderful result! Since it is common to assume that the variances have zero value at the initiation of combat, all of the covariances are zero. Only the variances are nonzero and they have the form,

$$\begin{aligned}\sigma_{j,j}^2(t) &= \langle f_j(0)\rangle\left[e^{2\lambda_j t}-e^{\lambda_j t}\right], \\ &= \langle f_j(t)\rangle\left[e^{\lambda_j t}-1\right].\end{aligned} \tag{30}$$

Two things are noteworthy about this: first, the variances in this representation do indeed scale as we would have expected, and they are indeed intensively formally aggregable in the sense that we have developed this methodology.

40.2.3.2 Force Component Randomness

If the randomness of the attrition is natural to the force component coordinate frame, then equation 26 is the appropriate FPK equation. In this case, we may again form ADEs and we find that the expected value ADEs are the same as equation 28 so that we at least have the joy of simple solution. The variance ADEs however, have a somewhat more complicated form,

$$\frac{d}{dt}\sigma_{j,k}^2 = (\lambda_j + \lambda_k)\,\sigma_{j,k}^2 + \frac{1}{2}(1+\delta_{j,k})\sum_{l,m=1}^{n} e_{j,l}e_{k,l}e_{l,m}^{-1}\lambda_m \langle f_m \rangle . \tag{31}$$

In this case, the covariances are not zero. If we again assume the variances are zero initially, these equations may be solved as

$$\sigma_{j,k}^2(t) = \frac{1}{2}(1+\delta_{j,k})\sum_{l,m=1}^{n} e_{j,l}e_{k,l}e_{l,m}^{-1}\frac{\lambda_m}{\lambda_j+\lambda_k-\lambda_m}\langle f_m(0)\rangle\left[e^{(\lambda_j+\lambda_k)t} - e^{\lambda_m t}\right]. \tag{32}$$

While these variances may be transformed into the force strength component coordinate system using the transformation described above, we note (sadly) that they cannot be aggregated in an intensive form because they exhibit time dependence of all of the eigenvalues.

40.3 AntiPotential Potential

The AntiPotential Potential (APP) is an aggregation methodology used in a variety of combat simulations to "value" weapon systems. [2] In this methodology, each weapons type (which corresponds directly to the force strength components) has a "value". In matrix notation, the Blue values are

$$\beta^B \overrightarrow{V^B} = \overleftrightarrow{\alpha}^T \bullet \overrightarrow{V^R}, \tag{33}$$

and the Red values are

$$\beta^R \overrightarrow{V^R} = \overleftrightarrow{\beta}^T \bullet \overrightarrow{V^B}, \tag{34}$$

where: β^B and β^R are Blue, Red proportionality constants; and $\overrightarrow{V^B}$ and $\overrightarrow{V^R}$ are "value" vectors. If we combine these two equations, eliminating the respective "value" vectors, we may easily find,

$$\begin{aligned} \beta^B\beta^R\overrightarrow{V^B} &= \overleftrightarrow{\alpha}^T \bullet \overleftrightarrow{\beta}^T \bullet \overrightarrow{V^B}, \\ \beta^B\beta^R\overrightarrow{V^R} &= \overleftrightarrow{\beta}^T \bullet \overleftrightarrow{\alpha}^T \bullet \overrightarrow{V^R}, \end{aligned} \tag{35}$$

which we may clearly see are clearly eigenvector equations.

Anderson notes a theorem, due to Frobenius, which states that if a matrix is "irreducible" and nonnegative, then there exists at least one eigenvalue which is positive, and the maximum value eigenvalue has corresponding eigenvector components which are all nonnegative. This provides a basis for aggregation and the eigenvector components provide a weighting of the force strength components in a manner analogous to what we have already seen in Intensive Formal Aggregation.

If we start with the basic heterogeneous matrix ADEs,

$$\begin{aligned} \frac{d}{dt}\overrightarrow{A} &= -\overleftrightarrow{\alpha} \bullet \overrightarrow{B}, \\ \frac{d}{dt}\overrightarrow{B} &= -\overleftrightarrow{\beta} \bullet \overrightarrow{A}, \end{aligned} \tag{36}$$

differentiate these with respect to time, and back substitute equations 36, then the resulting second order matrix heterogeneous ADEs are

$$\begin{aligned} \frac{d^2}{dt^2}\overrightarrow{A} &= \overleftrightarrow{\alpha} \bullet \overleftrightarrow{\beta} \bullet \overrightarrow{A}, \\ \frac{d^2}{dt^2}\overrightarrow{B} &= \overleftrightarrow{\beta} \bullet \overleftrightarrow{\alpha} \bullet \overrightarrow{B}. \end{aligned} \tag{37}$$

If we now define the left eigenvector matrices of $\overleftrightarrow{\alpha} \bullet \overleftrightarrow{\beta}$ and $\overleftrightarrow{\beta} \bullet \overleftrightarrow{\alpha}$ as $\overleftrightarrow{e}$ and $\overleftrightarrow{e'}$, respectively, and further denote the eigensolutions as

$$\begin{aligned} \overrightarrow{a} &= \overleftrightarrow{e} \bullet \overrightarrow{A}, \\ \overrightarrow{b} &= \overleftrightarrow{e'} \bullet \overrightarrow{B}, \end{aligned} \tag{38}$$

then the matrix heterogeneous ADEs, equations 36, become

$$\begin{aligned} \frac{d^2}{dt^2}\overrightarrow{a} &= \overleftrightarrow{\lambda^2} \bullet \overrightarrow{a}, \\ \frac{d^2}{dt^2}\overrightarrow{b} &= \overleftrightarrow{\lambda'^2} \bullet \overrightarrow{b}, \end{aligned} \tag{39}$$

where $\overleftrightarrow{\lambda^2}$ and $\overleftrightarrow{\lambda'^2}$ are diagonal matrices whose components are the squares of the eigenvalues.[1]

The eigensolution components are simply

[1] Actually, the components are the eigenvalues of the calculation which are the squares of the eigenvalues of the second order ADEs.

$$\begin{aligned} a_i(t) &= a_{+i}e^{\lambda_i t} + a_{-i}e^{-\lambda_i t}, i = 1..n_a, \\ b_i(t) &= b_{+i}e^{\lambda'_{ji} t} + b_{-i}e^{-\lambda'_j t}, j = 1..n_b, \end{aligned} \tag{40}$$

since the ADEs are second order. If the two maximum eigenvalues are the same, then we may designate these two solutions with the subscript m, and note that these solutions then have the form of homogeneous Lanchester solutions

$$\begin{aligned} a_m(t) &= \sum_{i=1}^{n_a} e_{mi} A_i(0) \cosh(\lambda_m t) - \frac{1}{\sqrt{\lambda_{mm}}} \sum_{i=1}^{n_a}\sum_{j=1}^{n_b} e_{mi}\alpha_{i,j} B_i(0) \sinh(\lambda_m t), \\ b_m(t) &= \sum_{i=1}^{n_b} e'_{mi} B_i(0) \cosh(\lambda_m t) - \frac{1}{\sqrt{\lambda_{mm}}} \sum_{i=1}^{n_b}\sum_{j=1}^{n_a} e'_{mi}\beta_{i,j} A_i(0) \sinh(\lambda_m t), \end{aligned} \tag{41}$$

which satisfy the homogeneous Lanchester ADEs

$$\begin{aligned} \frac{d}{dt}a_m &= -\overline{\alpha} b_m, \\ \frac{d}{dt}b_m &= -\overline{\beta} a_m, \end{aligned} \tag{42}$$

where:

$$\begin{aligned} \overline{\alpha} &= \sum_{i=1}^{n_a}\sum_{j=1}^{n_b} e_{mi}\alpha_{i,j} e'^{-1}_{mj}, \\ \overline{\beta} &= \sum_{i=1}^{n_b}\sum_{j=1}^{n_a} e'_{mi}\alpha_{i,j} e^{-1}_{mj}. \end{aligned} \tag{43}$$

If also follows that

$$\begin{aligned} \overrightarrow{V^R} &= \overrightarrow{e_m}, \\ \overrightarrow{V^B} &= \overrightarrow{e'_m}, \end{aligned} \tag{44}$$

and

$$\beta^R \beta^B = \lambda_m^2. \tag{45}$$

40.4 ATCAL Methodology

The ATCAL (an acronym for ATtrition CALibration) is an aggregation methodology which incorporates (among other parameters) fire allocation. It is a methodology for taking the results of high resolution (platform level) simulations and aggregating them for use in lower resolution simulations. It is noteworthy in our context here that the methodology is essentially not Lanchestrian although it does bear similarities.[3] Since this is an exceedingly complex methodology, we shall not provide a complete description here but will limit ourselves to an overview of certain aspects of the methodology to make associations with Lanchester theory. We do note that the methodology is iterative because of the non-linearity of some of the equations.

Like the example chapter on stochastic, platform level simulation connectivity to intensive aggregation, the ATCAL methodology is a bootstrap in that it uses the Killer Victim Scoreboards and also Rounds Fired Scoreboards from higher resolution simulations. The methodology distinguishes among platforms, weapons, and target types which are designated by the indices i, j, k. A key quantity is the exponential mean number of systems (platforms, weapons, and targets) during the combat. This is a key, non-Lanchestrian assumption, that the number of systems of a given type has a trajectory during the combat which is exponential. That is, if the number of systems (we suppress type indication for simplicity at this point,) is $S(t)$ then this number has trajectory,

$$S(t) \simeq S(0)\, e^{-\kappa t}, \tag{46}$$

where: $S(0)$ is the number of systems at the beginning of the combat (which we recall is taken from a high resolution simulation execution,) and κ is an "attrition" rate.

We may make a simple comparison of this approximation with homogeneous Lanchester theory. If we designate the number of enemy (firing) systems by $T(t)$, then the homogeneous Lanchester solution for $S(t)$ is just

$$S(t) = S(0)\sinh(\gamma t) - \delta T(0)\sinh(\gamma t), \tag{47}$$

which we are familiar with from Part I of this work. In the short time or small attrition limit, this equation may be expanded to first order in time as

$$S(t) \simeq S(0) - \alpha T(0)\, t, \tag{48}$$

which we may rewrite as

$$S(t) \simeq S(0)\left[1 - \frac{\alpha T(0)\, t}{S(0)}\right]. \tag{49}$$

Since the quantity in brackets is just an exponential to the same order of expansion, we may approximately replace this with

$$S(t) \simeq S(0)\, e^{-\frac{\alpha T(0) t}{S(0)}}, \tag{50}$$

which is exponential as is the ATCAL approximation.

The exponential average, which is used to account for large changes in system strength [2], is defined as

$$\overline{S} = \frac{\int_0^t S(t')\, dt'}{t}. \tag{51}$$

If we use equation 46 then this equation can be exactly integrated as

$$\overline{S} = \frac{S(0)\left[1 - e^{-\kappa t}\right]}{\kappa t}, \tag{52}$$

and if we make use of the change in system strength,

$$\Delta S = S(0)\left[1 - e^{-\kappa t}\right], \tag{53}$$

then this may be solved for the argument of the exponential,

$$\kappa t = -\ln\left(1 - \frac{\Delta S}{S(0)}\right), \tag{54}$$

and the two equations substituted into equation 52 to yield

$$\overline{S} = -\frac{\Delta S}{\ln\left(1 - \frac{\Delta S}{S(0)}\right)}, \tag{55}$$

which can be calculated entirely from the output of a high resolution simulation.

A key concept in the ATCAL methodology is the idea that during the duration of a combat, targets are available for engagement only during some fraction of this time. This is represented by quantities A_{ik} which are the Availability of target type k to engagement by a weapon/platform of type i. If there are a mean number of weapons/platforms of type i of $\overline{S_i}$, then the fraction of time that none of these targets are available (assuming they are completely uncorrelated,) is $(1 - A_{ik})^{\overline{S_i}}$. It is convenient to define the fraction of time that at least one target of type k is available for engagement by weapons/platforms of type i as

$$f_{ik} \equiv 1 - (1 - A_{ik})^{\overline{S_i}}. \tag{56}$$

A second key concept of the ATCAL Methodology is an absolute prioritization of target type for each weapon/platform type. That is, if a weapon/platform of type i is presented with the simultaneous opportunity to engage two targets of different type and one type target has a higher

[2] If the system is essentially completely attrited, then a linear average of system strength would approach $\frac{S(0)}{2}$ which is potentially misleading.

priority, that target is engaged completely preferentially to the other target. This allows a rigid system of determining the fraction of time that each target type is available for engagement. If, for convenience of notation, we assume that the targets are numerically ordered in decreasing priority, then we may define actual fractions of time targets are available for engagement as

$$\begin{aligned} f_{i1}^{\star} &= f_{i1}, \\ f_{i2}^{\star} &= f_{i2}\left(1-f_{i1}\right), \\ f_{i2}^{3} &= f_{i3}\left(1-f_{i2}\right)\left(1-f_{i1}\right), \end{aligned} \tag{57}$$

and so forth. In general terms, which do not assume ordering, these fractions may be written as

$$f_{ik}^{\star} = f_{ik}\prod\left(1-f_{ik'}\right), \tag{58}$$

where the product ranges over all targets of higher priority than target type k.

If the rate of fire by weapon/platform of type i is r_i, then the firing at targets of type k by weapons/platforms of type i is simply

$$F_{ik} = \overline{S_i} r_i f_{ik}^{\star}, \tag{59}$$

from which the point fire attrition equation is defined as

$$\begin{aligned} \left(\Delta S_k\right)_{ij} &= F_{ik} P_{ijk} \\ &= \overline{S_i} r_i f_{ik} \prod\left(1-f_{ik'}\right) P_{ijk}, \end{aligned} \tag{60}$$

where: P_{ijk} is a kills per round figure which is calculated by dividing the Killer Victim Scoreboard value by the Rounds Fired Scoreboard value. It thus represents a "probability of kill per shot" value.

If we designate the product of the rate of fire, the kills per round figure, and the actual fraction of available engagement time,

$$\kappa_{ijk}\tau \equiv r_i P_{ijk} f_{ik}^{\star}, \tag{61}$$

to be the product of an "Attrition Rate Coefficient" κ_{ijk} times the duration of the combat τ, then we may rewrite equation 60 as

$$\left(\Delta S_k\right)_{ij} = \overline{S_i}\kappa_{ijk}\tau, \tag{62}$$

then we have a simple attrition equation. If we now substitute equation 51 into equation 62, then we get

$$\left(\Delta S_k\right)_{ij} = \int_0^{\tau} S_i\left(t'\right) dt' \kappa_{ijk}, \tag{63}$$

which we may differentiate with respect to time to get

$$\frac{d}{dt}S_{k|_{ij}} = -\kappa_{ijk}S_i. \tag{64}$$

If we now sum this equation,

$$\begin{aligned} \frac{d}{dt}S_k &= -\sum_{i,j}\frac{d}{dt}S_{k|_{ij}} \\ &= -\sum_{i,j}\kappa_{ijk}S_i, \end{aligned} \tag{65}$$

which we recognize as nothing more than a heterogeneous "Lanchester" ADE written in component form. This association is meant to be descriptive only and is meant to indicate that the ATCAL methodology is Lanchestrian only within the limits of the approximations and assumptions outlined above.

For area fire, the ATCAL attrition equation is given as

$$\Delta S = S(0)\left[1 - e^{-\frac{A_L R}{A_S}}\right], \tag{66}$$

where: A_L = the area of lethality of a single round, R = the number of rounds fired, and A_S = the area occupied by the targets $S(0)$. In a Lanchester context, we may make an assumption that the latter area is a constant although this is not necessary, but is helpful for comparison purposes. An implicit assumption of the ATCAL methodology is that all rounds fall within this area. We recognize the exponential character of this equation as arising from the possibility of the lethal area of successive rounds overlapping (see Part I). Recognizing this, we may differentiate this equation with respect to time,

$$\begin{aligned} \frac{d}{dt}S(t) &= -\frac{d}{dt}\Delta S \\ &= -\frac{d}{dt}\left[\frac{A_L R}{A_S}\right]S(t), \end{aligned} \tag{67}$$

since

$$S(t) = S(0)\,e^{-\frac{A_L R}{A_S}}. \tag{68}$$

We may now designate the Rounds fired as

$$R \simeq \sum_i r'_i \tau \overline{S_i}, \tag{69}$$

which is just the product of the rate of area fire (rounds per time) per firer, the duration of the engagement, and the exponential mean of the number of firers. Further, we designate the area fire "attrition rate" as

$$\kappa_i' \equiv \frac{A_L r_i'}{A_S}, \tag{70}$$

which is comparable to what we have done earlier. The combination of equations 69 and 70 reduce equation 67 to

$$\frac{d}{dt}S(t) = -\frac{d}{dt}\left[\sum_i \kappa_i' \tau \overline{S_i}\right] S(t). \tag{71}$$

If we now again use equation 51, this equation reduces to

$$\frac{d}{dt}S(t) = -\frac{d}{dt}\left[\sum_i \kappa_i' \int_0^\tau S_i(t')\,dt'\right] S(t). \tag{72}$$

We may now differentiate this equation to obtain

$$\frac{d}{dt}S(t) = -\sum_i \kappa_i' S_i(t)\, S(t), \tag{73}$$

which is just a heterogeneous linear Lanchester ADE (in component notation). We emphasize that this comparison has been made simply for that purpose and does not constitute any claim that ATCAL is Lanchestrian.

40.5 Conclusion

This chapter concludes Part III of the work. This concludes our discussion of heterogeneous Lanchester attrition theory. It also concludes our discussion of basic Lanchester attrition theory and sets the stage for Part IV which is a collection of chapters dealing with advanced Lanchester theory in some sense.

40.6 References

[1] Sobczyk, Kazimierz, **Stochastic Differential Equations**, Kluwer Academic Publishers, Boston, MA 1990.

[2] Anderson, Lowell Bruce, and Frederic A. Miercort, "An Introduction to AntiPotential Potential Weapon Scores", Appendix A in "COMBAT: A Computer Program to Investigate Aimed Fire Attrition Equations, Allocations of Fire, and the Calculation of Weapons Scores", Institute for Defense Analysis, Alexandria, VA IDA Paper P-2248 September 1989.

[3] Analysis Support Division, "ATCAL: An Attrition Model Using Calibrated Parameters", U. S. Army Concepts Analysis Agency, Bethesda, MD, Technical Paper CAA-TP-83-3, August 1983.