FINAL REPORT
ONR GRANT# N00014-96-1-0584

GRANT TITLE: Space-Time Adaptive Processing of Electromagnetic Waves: A First-Principles Approach Based on Extended Physical Wavelets

PROJECT PERIOD: April 10, 1996 to April 9, 1998, Extended to September 30, 1998

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OBJECTIVES:
1. Formulate mathematically a class of electromagnetic wavelets, defined in spacetime, possessing good *directivity* and capable of being *focused*.

2. Apply such wavelets to radar imaging and target recognition.

NEW FINDINGS:
At the end of my book *A Friendly Guide to Wavelets,* I derived causal acoustic and electromagnetic wavelets which are exact solutions of the wave equation or Maxwell's equations with compact sources consisting formally of points located in complex spacetime. However, due to time constraints, only the case of nonzero imaginary time was there investigated, and it was found that the imaginary time represents a scale parameter giving the pulse duration. Since then I discovered that for nonzero imaginary *space* coordinates, these wavelets are closely related to the *complex-source pulsed beams* (CSPB) studied extensively in the engineering literature by E. Heyman, L. Felsen and others. These pulsed beams are emitted by a real disk whose radius and orientation are specified by the imaginary space vector giving the source location. Thus what appears mathematically as a *point source* located in *complex* spacetime is reinterpreted as an *extended source* in *real* spacetime. The advantage is that the formal description in terms of complex-point source is much simpler and economical, both conceptually and computationally.

I have proposed using such CSPB to formulate a very general radar scheme allowing an arbitrary number of transmitting antennas, an arbitrary number of receiving antennas, and an arbitrary number of targets, with each platform and target allowed an arbitrary motion including rotations and accelerations. (Of course the more larger the number of platforms and targets, the more impractical the resulting computations; but *in principle,* the model is physically correct for any number of

platforms and targets.) This idea is explained in my paper *Short-pulse radar via electromagnetic wavelets,* published in the Proceedings of the Third Ultrawideband, Short-Pulse Electromagnetics Conference, Plenum Press, 1997 (attached). The basic idea is to generalize the "ambiguity functional" formalism for radar I first proposed in the February 1996 issue of IEEE *Antennas and Propagation Magazine* from *point* transmitters, *point* receivers and *point* targets to *extended* transmitters, receivers and targets using CSPB combined with the *analytic-signal transform* as defined in Section 9.3 of my book.

This formulation is potentially useful because it is intuitively clear and physically correct (within the obvious limitations of the chosen model, e.g. dispersionless propagation through a homogeneous medium). The conceptual simlicity results from the fact that the theory is based directly in space-time, hence lends itself to intuitive interpretation. The usual time-frequency (narrowband) and time-scale (wideband) formalisms make unnecessary assumptions and are, moreover, less clear because they depend on a "dictionary" related to these assumptions, for example that the Doppler effect translates to a frequency shift (narrowband) or a scaling factor (wideband). In such a restricted context, it is not obvious how to deal with *new* situations not originally built into the model, such as accelerations, rotations, multistatic configurations, multiple reflections, etc. Such situations appear in the old models as distortions which can be corrected (if at all) only by using computationally intensive schemes.

In May 1998 I was invited to teach a 4-day course at the European Space Agency in Noordwijk, Holland on the applications of pulsed-beam wavelets ro radar. In preparing the lecture notes, I made further progress in understanding these wavelets, in particular deriving their far fields and *time-domain radiation patterns.* The usual definition of radiation patterns is geared to narrowband fields, hence cannot be applied in this case. When a proper time-domain definition is formulated, it is found that the pulsed-beam fields have *no sidelobes* and can be focused and collimated to an arbitrarily high degree, in principle. (The ESA notes are available on request.)

I have also found a method for filtering signals in scale rather than frequency. This is described in my paper *Wavelet filtering in the scale domain* (attached). Thus operations usually performed in the frequency domain (filters, convolutions, etc.) can also be performed in the wavelet domain: Take the wavelet transform of the signal, multiply it by a scale-dependent "transfer function," then apply the inverse wavelet transform. A one-to-one correspondence was established between certain "admissible" convolution operators and such scale transfer functions, based on Mellin convolutions and Mellin transforms.

One application of scale filtering would be to Doppler processing of wideband signals, where the Doppler effect is more properly represented by scaling rather than modulation.

RELATED ONGOING WORK

As a result of the progress made in the past two years, I am now engaged in a project investigating how pulsed-beam wavelets can be realized physically in hardware. This work is being done jointly with Ehud Heyman (one of the originators of CSPB) under contract with AFOSR. We are investigating the precise nature of a real source distribution in space (supported on the source disk) which will produce the same effect as the (unattainable) single complex source point. We have also computed the 4D (space-time) Fourier transform of the beams, which is needed to find the source power and total energy as well as simplify many of the computations. This work is in progress.

PUBLICATIONS SUPPORTED BY THE GRANT

• G. Kaiser, *Wavelet analysis as a wideband generalization of time-frequency analysis,* Invited paper, 136th Meeting of the Acoustical Society of America, JASA Vol. 104 #3, September 1998.

• G. Kaiser, *Electromagnetic pulsed-beam wavelets for radar,* Invited paper, 28th European Microwave Conference, Amsterdam, October 5-9, Miller Freeman, 1998.

• G. Kaiser, *The fast Haar transform: Gateway to wavelets,* IEEE Potentials Magazine, April/May, 1998.

• G. Kaiser, L. Rossi and D. Washburn, *Recovery of Kolmogorov statistics in thermal mixing in the troposphere: The hazards of real data,* in Airborne Laser Advanced Technology, SPIE Conference Proceedings #3381, Orlando, FL, April, 1998.

• G. Kaiser, *Short-pulse radar via electromagnetic wavelets,* in Proceedings of the Third Ultrawide Band, Short-Pulse Electromagnetics Conference, Plenum Press, 1997.

• G. Kaiser, *Scale filtering in the wavelet domain,* in Proceedings of the 30th Asilomar Conference on Signals, Systems, and Computers, Pacific Grove, CA, November, 1996.

• G. Kaiser, *Filtering in scale,* in Wavelet Applications in Signal and Image Processing, IEEE Conference Proceedings #2825, Denver, CO, August, 1996.

• G. Kaiser, *Wavelet filtering in the scale domain, invited paper,* in Wavelet Applications III, SPIE Conference Proceedings #2762, Orlando, FL, April, 1996.

• Wavelet filtering with the Mellin transform, *Applied Mathematics Letters* 9 #5, 69-74, 1996.

• E. Heyman and G. Kaiser, *Real source realization of complex-source pulsed beams,* in preparation.

SHORT COURSES GIVEN BY INVITATION DURING 4/10/96-9/30/98

• 5/16/96: Applications of Electromagnetic Wavelets to Radar and Sonar (3 hours), IEEE 1996 National Radar Conference, Ann Arbor, MI.

• 6/4-6/96: Mathematics and Physics of Wavelets (12 hours), Applied Technology Institute, College Park, MD.

• 7/21/96: Electromagnetic Wavelets in Radar and Sonar (6 hours), IEEE-AP-S International Symposium and URSI Radio Science Meeting, Baltimore, MD.

• 8/4/96: Wavelet Transforms: Theory and Applications (4 hours), SPIE International Symposium on Optical Science, Engineering, and Instrumentation, Denver, CO

• 12/17-19/96: Mathematics and Physics of Wavelets (12 hours), Applied Technology Institute, Brookline, MA.

• 2/10/97: Wavelet Fundamentals (6 hours), Internat. Soc. for Optical Engineering (SPIE) Photonics West, San Jose, CA

• 3/21/97: Radar via Physical Wavelets (6 hours), Applied Computational Electromagnetics Society (ACES), Monterey, CA

• 4/21/97: Remote Sensing via Physical Wavelets (6 hours), Internat. Soc. for Optical Engineering (SPIE) AeroSense, Orlando, FL

• 7/30/97: Introduction to Radar via Physical Wavelets (6 hours), Internat. Soc. for Optical Engineering (SPIE) Annual Meeting, San Diego, CA

• 12/16-18/97: Mathematics and Physics of Wavelets (12 hours), Applied Technology Institute, Brookline, MA.

• 5/11-14/98: A Detailed Introduction to Mathematical and Physical Wavelets (18 hours), ESTEC/ESA (European Space Agency), Noordwijk, The Netherlands.

SEMINARS AND TALKS (Recent samples):

• Nov 8, 1996: Seminar on scale filtering at Stanford University Mathematics Dept.

• Jan 5, 1997: Talk on scale filtering in the wavelet domain at AFOSR Electromagnetics workshop in San Antonio, TX.

• Jan 31, 1997: Seminar on wavelet scale filtering at Tufts University Mathematics Dept.

• Feb 18, 1997: Seminar on wavelet analysis of atmospheric intermittency at UMass Amherst ECE Dept.

• Nov 20, 1997: Talk given at Schlumberger on using Complex-Source pulsed beams in radar.

CONSULTATIVE AND ADVISORY FUNCTIONS TO LABS AND AGENCIES
•Dec 9, 1996: Seminar given on wavelet analysis of atmospheric intermittency at NOAA. I was invited by Bob Weber, who is heading a project to develop clear-air radar for the Air force. The detection and analysis of intermittent events (such as the passing of a flock of birds) is an important issue.

• Dec 11, 1996: Talk given on wavelet analysis of atmospheric intermittency at the Atmospheric Modeling and Laser Propagation Workshop, Phillips Laboratory (PL/LI), Kirtland AFB. This work later developed into a joint project with Lou Rossi and Don Washburn.

• April 10, 1997: Seminar on wavelet analysis of atmospheric intermittency at MIT Lincoln Laboratory (invited by the IEEE Geophysics Remote Sensing Society).

SHORT-PULSE RADAR VIA ELECTROMAGNETIC WAVELETS[*]

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INTRODUCTION

The new theory of physical wavelets makes it possible to perform radar and sonar analysis directly in the space-time domain, based on fundamental principles underlying the emission, reflection, and reception of electromagnetic and acoustic waves[1,2]. Being independent of the Fourier transform and even of the usual (affine) wavelet transform, this formalism is therefore equally well suited for ultrawideband or short-pulse radar as for narrowband or continuous-wave radar. However, Fourier analysis does have a natural place in this theory and can be used easily when spectral questions are of interest.

A transmitting antenna following an arbitrary (possibly accelerating or nonlinear) space-time trajectory $\alpha(t)$ emits a physical (acoustic or electromagnetic) wavelet which is propagated in space by the appropriate Green function. This defines an *emission operator* E_α which, acting on any time signal $\psi(t)$, gives the emitted space-time wave $(E_\alpha \psi)(\mathbf{r}, t)$. The *reception operator* R_α is dual to E_α, measuring any incident space-time wave $F(\mathbf{r}, t)$ along the given antenna trajectory $\alpha(t)$ to produce the received time signal $(R_\alpha F)(t)$. Reflection is modeled as reception followed by re-emission, i.e., by the operator $E_\alpha R_\alpha$ transforming any incident space-time wave to the reflected space-time wave. Let the receiving antenna follow another arbitrary space-time trajectory $\gamma(t)$ (possibly different from the trajectory $\alpha(t)$ of the transmitter), let the target follow a third arbitrary space-time trajectory $\beta_T(t)$, and let the transmitted and received signals be $\psi(t)$ and $\chi(t)$, respectively. The objective is to estimate the target trajectory $\beta_T(t)$ from a knowledge of $\alpha(t), \gamma(t), \psi(t)$ and $\chi(t)$. This is achieved by maximizing the modulus of the the *normalized ambiguity functional* $\tilde{\chi}_N(\beta)$, obtained by matching the *actual* return $\chi(t)$ with the *computed* return due to a *trial trajectory* $\beta(t)$. When the radar is monostatic and the target is assumed to move uniformly in the radial direction, then $\tilde{\chi}_N(\beta)$ reduces to the usual *wideband ambiguity function*, which is just the ordinary *time-scale (wavelet) transform* of $\chi(t)$ with $\psi(t)$ as the basic wavelet. In the narrowband approximation, it reduces further to the usual time-frequency ambiguity function, which is a windowed Fourier transform of the *video signal* of the return. This shows that our "physical wavelet analysis" is a generalization of the usual ("mathematical") wavelet analysis, which is in turn a generalization of time-frequency analysis. In particular, our analysis applies equally well to *bistatic wideband*

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radar, where the Doppler effect can no longer be represented simply by scaling and hence the usual affine wavelet analysis breaks down.

In Reference 2, the transmitting and receiving antennas and the target were all assumed to be *points*, so that the trajectories $\alpha(t), \beta(t)$, and $\gamma(t)$ fully describe their motions. Consequenctly, the emitted and reflected waves are omnidirectional and hence are not very useful in practice. In this paper we generalize the above model to include *extended* antennas and targets, follwing an idea introduced by Heyman and Felsen[3]. The associated radar beams are much more useful since they have a measure of directivity.

EXTENDED PHYSICAL WAVELETS AND AMBIGUITY FUNCTIONALS

Suppose we are given an antenna located at the space point $\mathbf{x}$, emitting the response to an impulse at time t. Ignoring polarization for simplicity, the resulting wave at the observation point $\mathbf{x}'$ at time t' can be represented by a solution of the scalar wave equation, which we write as

$$K(\mathbf{x}', t' \mid \mathbf{x}, t),$$

with a source distribution appropriate to the antenna. If the antenna is very small (essentially a point) and omnidirectional, then $K(\mathbf{x}', t' \mid \mathbf{x}, t)$ is well approximated by the retarded Green function

$$G(\mathbf{x}' - \mathbf{x}, t' - t) = \frac{\delta(t' - t - |\mathbf{x}' - \mathbf{x}|/c)}{4\pi |\mathbf{x}' - \mathbf{x}|},$$

where c is the speed of light. Following Heyman and Felsen[3], a simple model can be formulated for an *extended* antenna by allowing *complex* antenna space-time coordinates $\mathbf{x} \to \mathbf{z} = \mathbf{x} + i\mathbf{y}$ and $t \to u = t + is$ and taking $K(\mathbf{x}', t' \mid \mathbf{z}, u)$ to be an analytic extension of the above retarded Green function.* Heyman and Felsen showed that for $|\mathbf{y}| < cs$, $K(\mathbf{x}', t' \mid \mathbf{z}, u)$ can be interpreted as a *pulsed beam field* emitted by a circular disk of radius $|\mathbf{y}|$ in the direction of $\mathbf{y}$. Thus $\mathbf{y}$ is a convenient "handle" by which the radius and orientation of the antenna can be controlled, without having to construct a messy model for the antenna involving a continuous distribution of point sources. More complicated extended antennas and arrays can be modeled by a distribution (continuous or discrete) of such complex source points.

To keep the notation uncluttered, we combine the space and time coordinates into a single symbol:

$$x' \equiv (\mathbf{x}', t') \in \mathbf{R}^4, \qquad z \equiv (\mathbf{z}, u) = x + iy \in \mathbf{C}^4,$$

where

$$x = (\mathbf{x}, t) \quad \text{and} \quad y = (\mathbf{y}, s).$$

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* Note that our convention differs from that of Reference 3 in that our positive-frequency time-harmonic waves vary as $e^{i\omega t}$ rather than $e^{-i\omega t}$. For this reason, the analytic continuation of the retarded Green function (using the analytic signal of the delta function) is to the upper-half time plane ($s > 0$) rather than the lower-half time plane. This is consistent with the convention used in Reference 1.

The condition $|\mathbf{y}| < cs$ means that the imaginary space-time four-vector y belongs to the *future cone*, so that z actually belongs to the complex *future tube*[1,4]

$$z \in \mathcal{T}_+ \equiv \{x + iy \in \mathbf{C}^4 : x \in \mathbf{R}^4 \text{ and } y = (\mathbf{y}, s) \text{ with } |\mathbf{y}| < cs\},$$

which is a four-dimensional generalization of the upper-half complex time plane.

Suppose now that our antenna executes an arbitrary motion, including possible rotations and accelerations. Using the complex source coordinates, this can be parameterized as

$$z = \alpha(t) = x(t) + iy(t), \quad \text{where } x(t) = (\mathbf{x}(t), t) \in \mathbf{R}^4 \text{ and } y(t) = (\mathbf{y}(t), s). \tag{1}$$

It is reasonable (but mathematically unnecessary) to assume that the radius of the antenna remains constant during the motion, so that $|\mathbf{y}(t)| = |\mathbf{y}(0)| \equiv R < cs$, although the direction of $\mathbf{y}(t)$ may vary to allow tracking, scanning, etc. While executing this motion, the antenna is fed an input time signal $\psi(t)$. Then the output beam is

$$\Psi_\alpha(x') = \int_{-\infty}^{\infty} dt \, K(x' \,|\, \alpha(t)) \, \psi(t).$$

For reasons explained in Reference 2, we call $\Psi_\alpha(x')$ the *extended physical wavelet* generated by $\psi(t)$ along the antenna motion $\alpha(t)$. Given $\alpha(t)$, we define the *emission operator* E_α as the operator transforming the time signal $\psi(t)$ to the space-time wave $\Psi_\alpha(x')$, i.e.,

$$(E_\alpha \psi)(x') \equiv \int_{-\infty}^{\infty} dt \, K(x' \,|\, \alpha(t)) \, \psi(t). \tag{2}$$

Thus E_α takes a function of one variable (the input signal) to a function of four variables (the output beam). On the other hand, if the antenna is used as a receiver, it converts space-time waves into time signals. Again, assume that the complex antenna motion $\alpha(t)$ is given as in (1). Then the simplest model for the received signal due to an incident wave $F(x')$ is

$$(R_\alpha F)(t) = g_\alpha F(\alpha(t)), \tag{3}$$

where g_α is a "gain factor." Thus R_α simply measures the field along the complex trajectory $\alpha(t)$. More complicated receivers can be formulated which measure derivatives of F along $\alpha(t)$. (In the full electromagnetic formalism, for example, R_α could measure the induced current rather than the field.) Since $\alpha(t)$ is complex, the "evaluation" of the field $F(x')$ at $x' = \alpha(t)$ must be defined in (3). For this we use the *analytic-signal transform* of F, which extends F to complex space-time[1,4]:

$$F(x + iy) \equiv \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i} \, F(x + \tau y).$$

When $y = (\mathbf{0}, s)$ with $s > 0$, $F(x+iy)$ reduces to the usual Gabor analytic signal $F(\mathbf{x}, t+is)$ corresponding to $F(\mathbf{x}, t)$, with $\mathbf{x}$ regarded as an external parameter; this function is analytic in the upper-half complex time plane. It is further shown in Reference 1 that if $F(x')$ is any solution of the homogeneous wave equation (or Klein-Gordon equation[4]), then $F(x + iy)$ is analytic in the future tube $\mathcal{T}_+$ (i.e., $|\mathbf{y}| < cs$). The reception operator then evaluates the analytic-signal transform $F(x + iy)$ in its region of analyticity.

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With emission and reception modeled by (2) and (3), we are almost ready to formulate a general radar problem. The only missing element is a model for *reflection*. In the spirit of regarding a scattered electromagnetic wave as being emitted by the current induced on the scatterer by the incident wave, we propose the following model: Suppose we are given an oriented circular "target" disk executing a motion described by a complex space-time trajectory $\alpha(t) = x(t) + iy(t)$ as in (1). Again, we interpret the imaginary position vector $\mathbf{y}(t)$ as defining the radius and orientation of the disk. (To say that the disk is "oriented" means that its two sides are not equivalent; for example, one side could be reflective while the other side is not. Then every unit vector $\hat{\mathbf{y}} = \mathbf{y}/|\mathbf{y}|$ corresponds to a unique orientation of the disk. This is useful if, for example, we approximate a complicated target by patching together disks of various sizes and orientations, as in Section 10.2 of Reference 1; their non-reflecting sides should then be oriented towards the interior.) A given space-time wave $F(x')$ will now be assumed to be reflected from the disk as follows: First the disk acts as a *receiver*, then as a *transmitter*. Thus the reflected wave is

$$F_{\text{refl}}(x') = (E_\alpha R_\alpha F)(x') = g_\alpha \int_{-\infty}^{\infty} dt\ K(x'\,|\,\alpha(t))\,F(\alpha(t)).$$

Note that in the present context, the original "gain factor" g_α is re-interpreted as a *reflection coefficient*. When a complicated target is patched together from circular targets of various radii and orientations, the reflection coefficient becomes a function defined over the target surface as desired.

The ambiguity functional formalism developed in Reference 2 generalizes easily and naturally to the present setting of extended physical wavelets. Given the outgoing time signal ψ and the motions α, β, and γ of the transmitter, target, and receiver (all complex), our model for the time signal received at γ is

$$\begin{aligned}
\psi_\beta(t'') &= (R_\gamma E_\beta R_\beta E_\alpha \psi)(t'') \\
&= g_\gamma g_\beta \iint dt'\,dt\ K(\gamma(t'')\,|\,\beta(t'))\,K(\beta(t')\,|\,\alpha(t))\,\psi(t).
\end{aligned} \tag{4}$$

Of course, the received signal depends functionally on all three trajectories α, β, γ, as is evident from the right-hand side of (4). But to simplify the notation, we have suppressed the dependence on the *known* trajectories α and γ and displayed only the dependence on the target trajectory β. To estimate the *actual* target trajectory $\beta_T(t)$, we compute $\psi_\beta(t)$ for a *trial trajectory* $\beta(t)$ and match the result with the actual return $\chi(t)$ by taking the inner product of the two time signals. We denote the result by $\tilde{\chi}(\beta)$, which we call the *ambiguity functional* of the return:

$$\begin{aligned}
\tilde{\chi}(\beta) &\equiv \langle\, \chi, \psi_\beta \,\rangle \equiv \int_{-\infty}^{\infty} dt''\ \chi(t'')\,\psi_\beta(t'') \\
&= \iiint dt''\,dt'\,dt\ \chi(t'')\,K(\gamma(t'')\,|\,\beta(t'))\,K(\beta(t')\,|\,\alpha(t))\,\psi(t).
\end{aligned} \tag{5}$$

(We assume that $\psi(t)$ and $\chi(t)$ are real; if they are complex, then $\chi(t)$ should be replaced by its complex conjugate in (5).) Assuming that $\psi_\beta(t)$ and $\chi(t)$ have finite energies $\|\psi_\beta\|^2$ and $\|\chi\|^2$, the Schwarz inequality implies that

$$\begin{aligned}
|\tilde{\chi}(\beta)| &= |\langle\, \chi, \psi_\beta \,\rangle| \leq \|\chi\|\,\|\psi_\beta\| \\
|\tilde{\chi}(\beta)| &= \|\chi\|\,\|\psi_\beta\| \iff \chi(t) = C\psi_\beta(t).
\end{aligned} \tag{6}$$

Therefore, to estimate the true target trajectory $\beta_T(t)$, we need to maximize the *normalized* ambiguity functional

$$\tilde{\chi}_N(\beta) \equiv \frac{\tilde{\chi}(\beta)}{\|\psi_\beta\|} .$$

By (6),

$$|\tilde{\chi}_N(\beta)| \leq \|\chi\| \quad \text{and} \quad |\tilde{\chi}_N(\beta)| = \|\chi\| \iff \chi(t) = C\psi_\beta(t).$$

Equivalently, we can minimize the *error functional* defined by

$$\mathcal{E}(\beta) \equiv 1 - \frac{|\chi(\beta)|}{\|\chi\| \, \|\psi_\beta\|} ,$$

since the Schwarz inequality states that

$$0 \leq \mathcal{E}(\beta) \leq 1 \quad \text{and} \quad \mathcal{E}(\beta) = 0 \iff \chi(t) = C\psi_\beta(t).$$

Thus $|\tilde{\chi}_N(\beta)|$ and $\mathcal{E}(\beta)$ attain their maximum and minimum values, respectively, only when the trial return is indistinguishable from the actual return. Of course, this does not guarantee that the trial trajectory $\beta(t)$ coincides with the actual target trajectory $\beta_T(t)$, since the return does not, in general, uniquely determine the target trajectory. That is, *the functionals $\tilde{\chi}_N(\beta)$ and $\mathcal{E}(\beta)$ are generally not one-to-one*. The class of all trajectories β such that $\tilde{\chi}_N(\beta) = \tilde{\chi}_N(\beta_T)$ or, equivalently, $\mathcal{E}(\beta) = \mathcal{E}(\beta_T)$, represents the inherent *ambiguity* of the radar problem. A problem of obvious importance is to find outgoing signals $\psi(t)$ which minimize this ambiguity class.

We have assumed above that the return is due to a reflection from a *single* target. If N distinct targets are involved, then we can approximate the return as a superposition

$$\psi_{\beta_1,\beta_2,\ldots,\beta_N} \approx \psi_{\beta_1} + \cdots + \psi_{\beta_N} . \tag{7}$$

As noted, (7) is an approximation because it ignores *multiple reflections*. Although these can often be ignored, they can also cause resonances (ringing), hence must sometimes be taken into account. This can be easily done, in principle. For example, the signal received by the doubly-reflecting path $\alpha \rightarrow \beta_m \rightarrow \beta_n \rightarrow \gamma$ is

$$\psi_{\beta_m,\beta_n} = R_\gamma E_{\beta_n} R_{\beta_n} E_{\beta_m} R_{\beta_m} E_\alpha \psi ,$$

which can be immediately converted to a triple integral by using the definitions (2) and (3). Sums of contributions from various "trial" scattering paths may then be matched with the actual return, defining a generalized ambiguity functional

$$\tilde{\chi}(\beta_1, \beta_2, \ldots, \beta_N) \equiv \langle \chi, \psi_{\beta_1,\beta_2,\ldots,\beta_N} \rangle,$$

and the Schwarz inequality may be used as in the case of a single path to optimize the match.

This method is reminiscent of *Feynman diagrams*[5], where fundamental processes are represented by multiple integrals with corresponding intuitive diagrams. Because the physics is built into the formalism from the beginning through the Green functions, our

model can handle such complications in a conceptually straightforward (if computationally nontrivial) way. The resemblance to Feynman diagrams is no coincidence, and the present formalism may be modified to include quantum (photonic) aspects of radar simply by using *Feynman propagators* in place of the retarded Green functions.

REFERENCES

1. G. Kaiser, *A Friendly Guide to Wavelets*, Birkhäuser, Boston, 1994.

2. G. Kaiser, Physical wavelets and radar, *IEEE Antennas and Propagation Magazine*, February, 1996.

3. E. Heyman and L.B. Felsen, Complex-source pulsed-beam fields, *Journal of the Optical Society of America A* **6**, 806–817.

4. G. Kaiser, *Quantum Physics, Relativity, and Complex Spacetime: Towards a New Synthesis*, North-Holland, Amsterdam, 1990. Second edition to be published by Birkhäuser, Boston.

5. R.P. Feynman, *QED: The Strange Theory of Light and Matter*, Princeton University Press, 1985.

Wavelet Filtering in the Scale Domain

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ABSTRACT

For a given basic wavelet $\psi(t)$, two distinct correspondences (called C1 and C2) are established between *frequency filters*, defined in the frequency domain through multiplication by a transfer function $W(f)$, and *scale filters*, defined in the wavelet domain through multiplication by a *scale transfer function* $w(\sigma)$. $W(f)$ is obtained by performing a *scaling convolution* of $w(\sigma)$ with $\hat{\psi}(f)^*$ (for C1) or its spectral energy density $|\hat{\psi}(f)|^2$ (for C2). For a large class of transfer functions $W(f)$, this relation can be solved for $w(\sigma)$ by applying the Mellin transform. We call such frequency filters and their associated time-domain convolution operators C1- or C2-*admissible* with respect to ψ. In particular, the identity operator ($W(f) \equiv 1$) is C2-admissible if and only if the wavelet ψ is admissible in the conventional sense. The implementation of the correspondence C1 is computationally simpler than C2, but C2 can be generalized to time-dependent filters. Applications are proposed to the analysis of atmospheric turbulence data and wideband Doppler filtering.

Keywords: wavelets, filters, transfer functions, scale, Mellin transform, spectral smoothing, wideband Doppler filtering.

1. SCALE FILTERS AS TIME-DOMAIN CONVOLUTIONS

The wavelet transform $\tilde{\chi}(\sigma, \tau)$ of a real time signal $\chi(t)$ with respect to a real wavelet $\psi(t)$ can be defined as the inner product of $\chi(t)$ with the wavelet family

$$\psi_{\sigma,\tau}(t) \equiv \psi(\sigma t - \tau), \quad \sigma > 0.$$

That is,

$$\tilde{\chi}(\sigma, \tau) \equiv \langle \psi_{\sigma,\tau}, \chi \rangle = \int_{-\infty}^{\infty} dt\, \psi(\sigma t - \tau)\, \chi(t). \tag{1}$$

Since we consider only real time signals and filters (*i.e.*, filters with a real-valued impulse response), it will suffice to use *positive scales* only.

To be practical, the wavelet transform must have an inverse. The derivation of the inverse leads to the *reconstruction formula* giving $\chi(t)$ in terms of $\tilde{\chi}(\sigma, \tau)$, analogous to the inverse Fourier transform. However, one of the most useful applications of Fourier analysis is in *frequency processing, i.e.*, performing certain operations in the frequency domain before applying the inverse transform. Our aim in this paper is to derive analogous techniques for the wavelet transform and suggest some of their applications. The standard reconstruction formula will then emerge as a special case, when *no* processing is performed in the wavelet domain before transforming back.

Towards this end, note that the Fourier transform of $\psi_{\sigma,\tau}(t)$ is

$$\hat{\psi}_{\sigma,\tau}(f) \cong \int_{-\infty}^{\infty} dt\, e^{-2\pi i f t}\, \psi_{\sigma,\tau}(t) = \frac{1}{\sigma}\, e^{-2\pi i f \tau / \sigma}\, \hat{\psi}(f/\sigma). \tag{2}$$

Parseval's identity can therefore be used to rewrite the inner product $\langle \psi_{\sigma,\tau}, \chi \rangle$ in the frequency domain as

$$\tilde{\chi}(\sigma,\tau) = \langle \hat{\psi}_{\sigma,\tau}, \hat{\chi} \rangle = \frac{1}{\sigma} \int_{-\infty}^{\infty} df\, e^{2\pi i f \tau / \sigma}\, \hat{\psi}(f/\sigma)^*\, \hat{\chi}(f) = \frac{1}{\sigma} \int_{-\infty}^{\infty} df\, e^{2\pi i f \tau / \sigma}\, R_\psi(f/\sigma)\, \hat{\chi}(f), \tag{3}$$

where the asterisk denotes complex conjugation and we have defined

$$\boxed{R_\psi(f) \equiv \hat{\psi}(f)^*.}$$

Our construction of scale filters will be motivated by a solid understanding of the meaning of the wavelet parameter (σ, τ) used in our convention. The next two paragraphs are therefore devoted to the extraction of this meaning directly from the expressions (1) and (3) for the wavelet transform in the time- and frequency domains.

To understand σ, note (as will be seen later) that for the wavelet transform to have an inverse, we need $\hat{\psi}(0) = \int dt\, \psi(t) = 0$, which means that the wavelet must "wiggle" (hence the name). If $\psi(t)$ has compact support or rapid decay, then $\hat{\psi}(f)$ is smooth, so $\hat{\psi}(f)$ must also decay as $f \to 0$. In addition, we assume that $\psi(t)$ is reasonably smooth, so that $\hat{\psi}(f)$ has rapid decay at infinity. Thus $R_\psi(f)$ must decay both as $f \to 0$ and $f \to \infty$, and so it may be considered a *bandpass filter*. The reality of $\psi(t)$ implies $\hat{\psi}(-f) = \hat{\psi}(f)^*$, and we may therefore restrict our attention to the positive frequencies. There, let $R_\psi(f)$ be concentrated mainly on the frequency band $F_\psi = [\alpha, \beta]$ with geometric mean $f_\psi = \sqrt{\alpha\beta}$. Then $R_\psi(f/\sigma)$ is concentrated mainly on the band $\sigma F_\psi = [\sigma\alpha, \sigma\beta]$ with geometric mean $f = \sigma f_\psi$, and (3) shows that $\tilde{\chi}(\sigma,\tau)$ contains information about the spectrum $\hat{\chi}(f)$ mainly in this band. Since f_ψ is fixed for a given wavelet, we may *identify* the mean frequency σf_ψ with σ. In fact, by rescaling $(\sigma \to \sigma/f_\psi)$ we may assume, without loss, that $f_\psi = 1$, so that $\tilde{\chi}(\sigma,\tau)$ contains information about $\hat{\chi}(f)$ mainly near $f = \sigma$. Therefore *we may regard σ as a "macroscopic" (blurred) frequency variable*, representing the band σF_ψ of the signal. Note, however, that the bands σF_ψ have *proportional bandwidth*, which is a trademark of wavelet analysis. (When the wavelet transform is discretized, this means that high-frequency bands get sampled at proportionately high rates. This makes wavelet analysis more efficient than windowed Fourier analysis, where all bands have the same width and are therefore sampled at the same rate.)

To interpret τ, we return to the time domain. If $\psi(t)$ is concentrated mainly in a time interval T_ψ centered near $t = 0$, then (1) shows that $\tilde{\chi}(\sigma,\tau)$ contains information about $\chi(t)$ mainly in the time interval $(\tau + T_\psi)/\sigma$ centered near τ/σ. Therefore *τ is interpreted as a scaled "macroscopic" time variable,* i.e., $\tau \sim \sigma t - T_\psi$. Had we defined $\psi_{\sigma,\tau}(f) = \psi(\sigma(t - \tau))$, which is closer to the usual convention, then τ would have been an unscaled time variable. However, *the present convention has the advantage of automatically slowing down high-frequency features of the signal and speeding up low-frequency features, so that τ actually measures the phases of all frequency components uniformly across the frequency scales.* That is, with the above interpretation of σ, we have for a given increment $\Delta\tau$:

$$\Delta\tau \sim \sigma\Delta t \sim f\Delta t = \text{phase shift.}$$

2

The advantage of using scaled time becomes especially clear in the discrete case, where orthogonal and biorthogonal dyadic wavelet bases have the form

$$\Psi_{m,n}(t) = \psi(2^m t - n) = \psi_{2^m,n}(t),$$

showing that τ is sampled *uniformly* since it is already adapted to the scale. (Note that with this normalization, the energies are $\|\Psi_{m,n}\|^2 = 2^{-m}$.)

> Equations (1) and (3) lead to an interpretation of the wavelet parameters (σ, τ) as a pair of *macroscopic time-frequency variables* differing in two fundamental ways from the usual interpretation in "time-frequency analysis" using windowed Fourier transforms: (a) Equation (3) shows that the bandpass filter $R_\psi(f/\sigma)$ is "tuned" to $f = \sigma$ by *scaling*, whereas the corresponding operation for the windowed Fourier transform is *sliding* (modulation in the time domain). (b) The time variable τ is now *scale-adapted*, hence a given interval $\Delta\tau$ measures the same phase shift in all frequency components.

With this interpretation of the wavelet domain established, we are ready to proceed . The interpretation of σ as a frequency variable will be used to construct scale filters. Taking the inverse Fourier transform of (3) with respect to τ/σ gives

$$R_\psi(f/\sigma)\,\hat{\chi}(f) = \int_{-\infty}^{\infty} d\tau\, e^{-2\pi i f \tau/\sigma}\, \tilde{\chi}(\sigma,\tau) = \sigma \int_{-\infty}^{\infty} dt\, e^{-2\pi i f t}\, \tilde{\chi}(\sigma,\sigma t), \qquad (4)$$

which restates the above assertion that $\tilde{\chi}(\sigma,\tau)$, in its dependence on τ, contains localized spectral information about χ. Now integrate both sides over $\sigma > 0$ with an arbitrary weight function $w(\sigma)/\sigma$:

$$\int_0^\infty \frac{d\sigma}{\sigma}\, w(\sigma)\, R_\psi(f/\sigma)\,\hat{\chi}(f) = \iint_{\sigma>0} d\sigma dt\, e^{-2\pi i f t}\, w(\sigma)\, \tilde{\chi}(\sigma,\sigma t). \qquad (5)$$

Let us define $W_1(f)$ by

$$W_1(f) \equiv \int_0^\infty \frac{d\sigma}{\sigma}\, w(\sigma)\, R_\psi(f/\sigma), \qquad (6)$$

so that (5) reads

$$W_1(f)\,\hat{\chi}(f) = \iint_{\sigma>0} d\sigma dt\, e^{-2\pi i f t}\, w(\sigma)\, \tilde{\chi}(\sigma,\sigma t).$$

Applying the inverse Fourier transform to both sides gives our first main result:

$$\int_{-\infty}^{\infty} df\, e^{2\pi i f t}\, W_1(f)\,\hat{\chi}(f) = \int_0^\infty d\sigma\, w(\sigma)\, \tilde{\chi}(\sigma,\sigma t). \qquad (7)$$

The left side displays $W_1(f)$ as (the transfer function of) a *frequency filter* with impulse response

$$
\begin{aligned}
w_1(t) &\equiv \int_{-\infty}^{\infty} df\, e^{2\pi i f t}\, W_1(f) = \int_{-\infty}^{\infty} df\, e^{2\pi i f t} \int_0^\infty \frac{d\sigma}{\sigma}\, w(\sigma)\, \hat{\psi}(f/\sigma)^* \\
&= \int_0^\infty \frac{d\sigma}{\sigma}\, w(\sigma) \int_{-\infty}^{\infty} df\, e^{2\pi i f t}\, \hat{\psi}(f/\sigma)^* \qquad (8) \\
&= \int_0^\infty d\sigma\, w(\sigma)\, \psi(-\sigma t).
\end{aligned}
$$

3

Evidently, according to our conventions, $\psi(t)$ must be supported in the negative time axis $t \leq 0$ to make w_1 *causal* ($w_1(t) = 0$ for $t < 0$). This is also the condition for $\tilde{\chi}(\sigma, \tau)$ to be causal, *i.e.*, for it to depend on $\chi(t)$ only for $t \leq \tau/\sigma$.

Equation (7) shows that the convolution of w_1 with χ can be represented in the wavelet domain as a *scale filter* with the weight function $w(\sigma)$:

$$(w_1 * \chi)(t) \equiv \int_{-\infty}^{\infty} du \, w_1(t - u) \chi(u) = \int_0^{\infty} d\sigma \, w(\sigma) \, \tilde{\chi}(\sigma, \sigma t). \tag{9}$$

If the integral in (6) converges, it therefore defines a correspondence

$$C1 : w(\sigma) \to W_1(f) \tag{10}$$

from scale filters to frequency filters and their associated convolution operators. Note that since $R_\psi(-f)^* = R_\psi(f)$, we have

$$W_1(-f)^* = \int_0^{\infty} \frac{d\sigma}{\sigma} \, w(\sigma)^* \, R_\psi(f/\sigma). \tag{11}$$

Therefore $w_1(t)$ is real if $w(\sigma)$ is real:

$$w(\sigma)^* = w(\sigma) \;\Rightarrow\; W_1(-f)^* = W_1(f) \;\Rightarrow\; w_1(t)^* = w_1(t). \tag{12}$$

In Section 2 we examine how the correspondence C1 can be inverted to find $w(\sigma)$ for a given filter $W_1(f)$ and wavelet $\psi(t)$. But first we discuss an alternative correspondence which, although computationally more intensive, can be easily generalized to represent *time-dependent* filters.

Returning to (4), multiply both sides by $\sigma^{-1}\hat{\psi}(f/\sigma)$ and use (2) to obtain

$$\frac{1}{\sigma} \, |\hat{\psi}(f/\sigma)|^2 \, \hat{\chi}(f) = \int_{-\infty}^{\infty} d\tau \, \hat{\psi}_{\sigma,\tau}(f) \, \tilde{\chi}(\sigma, \tau). \tag{13}$$

Again we integrate over $\sigma > 0$ with a weight function $w(\sigma)$:

$$\int_0^{\infty} \frac{d\sigma}{\sigma} \, w(\sigma) \, |\hat{\psi}(f/\sigma)|^2 \hat{\chi}(f) = \iint_{\sigma>0} d\sigma d\tau \, \hat{\psi}_{\sigma,\tau}(f) \, w(\sigma) \, \tilde{\chi}(\sigma, \tau). \tag{14}$$

Let $S_\psi(f)$ be the spectral energy density of ψ, *i.e.*,

$$\boxed{S_\psi(f) \equiv |\hat{\psi}(f)|^2,} \tag{15}$$

and define the frequency filter $W_2(f)$ separately for positive and negative frequencies by

$$\boxed{\begin{aligned} W_2(f) &\equiv \int_0^{\infty} \frac{d\sigma}{\sigma} \, w(\sigma) \, S_\psi(f/\sigma), & f > 0 \\ W_2(-f) &\equiv W_2(f)^* = \int_0^{\infty} \frac{d\sigma}{\sigma} \, w(\sigma)^* \, S_\psi(f/\sigma), & f > 0. \end{aligned}} \tag{16}$$

4

Then (14) becomes

$$W_2(f)\,\hat{\chi}(f) = \iint_{\sigma > 0} d\sigma d\tau\, \hat{\psi}_{\sigma,\tau}(f)\, w(\sigma)\, \tilde{\chi}(\sigma,\tau), \quad f > 0. \tag{17}$$

The reason for the two-folded definition (16) is that the top line alone would imply $W_2(-f) = W_2(f)$, since $S_\psi(-f) = S_\psi(f)$. Instead, we want the *impulse response*

$$\mathrm{w}_2(t) \equiv \int_{-\infty}^{\infty} df\, e^{2\pi i f t}\, W_2(f)$$

to be real, which is guaranteed by the bottom line. (As mentioned earlier, we must have $\hat{\psi}(0) = 0$, hence $W_2(0) = 0$ and the definition (16) is complete.) Note that whereas we needed a real scale filter $w(\sigma)$ to make $\mathrm{w}_1(t)$ real, $\mathrm{w}_2(t)$ is real whether $w(\sigma)$ is real or not. This resembles the situation in the frequency domain, where transfer functions need not be real. In fact, we could extend our C2-formalism to include *negative scales* by defining $w(-\sigma) \equiv w(\sigma)^*$ in parallel with (16). The restriction to positive *scales* throughout this paper is closely tied to the practice of ignoring the (redundant) negative *frequencies* of real signals by using the corresponding "analytic signals." This will indeed be done next, for reasons to be seen later in connection with the Mellin transform.

We want to take the inverse Fourier transform of (17) to obtain the action of the filter $W_2(f)$ in the time domain. But since (17) is restricted to $f > 0$, we integrate over positive frequencies only. On the right side, we need therefore to introduce the function

$$\psi^+_{\sigma,\tau}(t) \equiv \int_0^{\infty} df\, e^{2\pi i f t}\, \hat{\psi}_{\sigma,\tau}(f),$$

known as the *analytic signal* of $\psi_{\sigma,\tau}(t)$. The reason for this name is that $\psi^+_{\sigma,\tau}(t)$ is necessrily complex and extends analytically to the upper half of the complex time plane. It is related to $\psi_{\sigma,\tau}(t)$ by

$$2\,\mathrm{Re}\,\psi^+_{\sigma,\tau}(t) = \int_0^{\infty} df\, \left[e^{2\pi i f t}\, \hat{\psi}_{\sigma,\tau}(f) + e^{-2\pi i f t}\, \hat{\psi}_{\sigma,\tau}(-f) \right] = \psi_{\sigma,\tau}(t).$$

Similarly, $2\,\mathrm{Im}\,\psi^+_{\sigma,\tau}(t)$ is the Hilbert transform of $\psi_{\sigma,\tau}(t)$. (See [1], Sections 3.3 and 9.3 for a discussion of analytic signals and their multidimensional generalization.) Like $\psi_{\sigma,\tau}(t)$, the entire family $\psi^+_{\sigma,\tau}(t)$ can be obtained from a single 'basic wavelet' $\psi^+(t)$. Namely, (2) implies that

$$\psi^+_{\sigma,\tau}(t) = \int_0^{\infty} df\, e^{2\pi i f(\sigma t - \tau)}\, \hat{\psi}(f) = \psi^+(\sigma t - \tau),$$

where $\psi^+(t)$ is the analytic signal of $\psi(t)$. Furthermore, the wavelet transform $\tilde{\chi}(\sigma,\tau)$ can be expressed in terms of inner products with $\psi^+_{\sigma,\tau}(t)$ by

$$\tilde{\chi}(\sigma,\tau) = \int_{-\infty}^{\infty} dt\, [2\,\mathrm{Re}\,\psi^+(\sigma t - \tau)^*]\, \chi(t),$$

$$= 2\,\mathrm{Re} \int_{-\infty}^{\infty} dt\, \psi^+(\sigma t - \tau)^*\, \chi(t)$$

$$= 2\,\mathrm{Re}\,\langle\, \psi^+_{\sigma,\tau}, \chi\,\rangle = 2\,\mathrm{Re}\,\langle\, \hat{\psi}^+_{\sigma,\tau}, \hat{\chi}\,\rangle = 2\,\mathrm{Re}\,\langle\, \hat{\psi}^+_{\sigma,\tau}, \hat{\chi}^+\,\rangle$$

$$= 2\,\mathrm{Re}\,\langle\, \psi^+_{\sigma,\tau}, \chi^+\,\rangle \equiv 2\,\mathrm{Re}\,\tilde{\chi}^+(\sigma,\tau),$$

5

where $\chi^+(\sigma, \tau)$ is the wavelet transform of χ^+ with respect to ψ^+.

Returning to (17) and taking the inverse Fourier transform over $0 < f < \infty$, we obtain our second main result:

$$\int_0^\infty df\, e^{2\pi i f t}\, W_2(f)\, \hat{\chi}(f) = \iint_{\sigma > 0} d\sigma d\tau\, \psi_{\sigma,\tau}^+(t)\, w(\sigma)\, \tilde{\chi}(\sigma, \tau). \tag{18}$$

The left side presents $W_2(f)$ as the transfer function of a positive-frequency filter, while the right side defines the action of $w(\sigma)$ as a scale filter operating in the wavelet domain. The full action, including the negative frequencies, can be obtained by taking real parts:

$$\int_{-\infty}^\infty df\, e^{2\pi i f t}\, W_2(f)\, \hat{\chi}(f) = \int_0^\infty df\, \left[e^{2\pi i f t}\, W_2(f)\hat{\chi}(f) + e^{-2\pi i f t}\, W_2(-f)\hat{\chi}(-f) \right]$$

$$= \int_0^\infty df\, \left[e^{2\pi i f t}\, W_2(f)\hat{\chi}(f) + e^{-2\pi i f t}\, W_2(f)^*\hat{\chi}(f)^* \right]$$

$$= 2\,\mathrm{Re} \iint_{\sigma > 0} d\sigma d\tau\, \psi_{\sigma,\tau}^+(t)\, w(\sigma)\, \tilde{\chi}(\sigma, \tau).$$

Since $\tilde{\chi}(\sigma, \tau)$ is real, we have

$$\int_{-\infty}^\infty df\, e^{2\pi i f t}\, W_2(f)\, \hat{\chi}(f) = \iint_{\sigma > 0} d\sigma d\tau\, 2\,\mathrm{Re}\left[\psi_{\sigma,\tau}^+(t)\, w(\sigma) \right] \tilde{\chi}(\sigma, \tau). \tag{19}$$

This is the "real-signal" representation of scale filters, compared with the analytic-signal representation (18). It is interesting to note that if $w_2(-t) \neq w_2(t)$ (e.g., if the filter is causal!), then $W_2(-f) \neq W_2(f)$; hence (16) implies that $w(\sigma)^* \neq w(\sigma)$, and (19) shows that the action of W_2 on real signals necessarily involves the Hilbert transform of $\psi(t)$, as it does also in the analytic-signal representation (18).

If the integrals (16) converge, then (18) and (19) establish a mapping

$$C2 : w(\sigma) \to W_2(f)$$

from scale filters to frequency filters and their associated convolution operators. An examination of (18) reveals the following precise correspondence between objects in the positive-frequency Fourier domain and the positive-scale wavelet domain:

Fourier 'basis functions'	$e^{2\pi i f t}$	$\longleftrightarrow$ $\psi_{\sigma,\tau}^+(t)$	wavelet 'basis functions'
frequency filter	$W_2(f)$	$\longleftrightarrow$ $w(\sigma)$	scale filter
Fourier transform	$\hat{\chi}(f)$	$\longleftrightarrow$ $\tilde{\chi}(\sigma, \tau)$	wavelet transform.

To complete our frequency – scale "dictionary," we must also investigate the inverse problem: Given a frequency filter $W(f)$, find equivalent scale filters $w_1(\sigma)$ and $w_2(\sigma)$ corresponding to $W(f)$ under C1 and C2, and the conditions under which they exists. The key is to observe that Equations (6) and (16)

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represent $W_1(f)$ and $W_2(f)$ as *scaling convolutions* of $w(\sigma)$ with $R_\psi(f)$ and $S_\psi(f)$, respectively. This means that instead of the difference $t - u$ in the usual convolution (9), we have the *ratio* f/σ in (6) and (16), and instead of the *translation-invariant* (Lebesgue) measure du in (9) we have the *scaling-invariant* measure $d\sigma/\sigma$. By analogy with (9), we write (6) and (16) as

$$
\begin{aligned}
W_1(f) &= \int_0^\infty \frac{d\sigma}{\sigma} \, w(\sigma) \, R_\psi(f/\sigma) \equiv (w \bullet R_\psi)(f) \\
W_2(f) &= \int_0^\infty \frac{d\sigma}{\sigma} \, w(\sigma) \, S_\psi(f/\sigma) \equiv (w \bullet S_\psi)(f),
\end{aligned}
\tag{20}
$$

where we now assume $f > 0$ in both equations and define $W_1(-f) \equiv W_1(f)^*$ and $W_2(-f) \equiv W_2(f)^*$. (Note that $W_1(0) = W_2(0) = 0$ since $R_\psi(0) = S_\psi(0) = 0$.)

Definition. *A given filter $W(f)$ is* **C1-admissible** *or* **C2-admissible** *with respect to a given wavelet ψ if and only if the integral equations*

$$
\text{C1:} \quad W(f) = (w_1 \bullet R_\psi)(f) = \int_0^\infty \frac{d\sigma}{\sigma} \, w_1(\sigma) \, R_\psi(f/\sigma)
$$

$$
\text{C2:} \quad W(f) = (w_2 \bullet S_\psi)(f) = \int_0^\infty \frac{d\sigma}{\sigma} \, w_2(\sigma) \, S_\psi(f/\sigma)
$$

can be solved (deconvolved) to give $w_1(\sigma)$ or $w_2(\sigma)$, respectively.

> *If a frequency filter W is C1- or C2-admissible with respect to ψ, then there exists a C1- or C2-equivalent scale filter w.*

We must therefore examine when and how the above scaling convolution equations can be solved. Just as ordinary convolutions are untangled by the Fourier transform, so are scaling convolutions untangled by the *Mellin transform* [2, 3]. The following definition will be convenient in our case: Given a function $X(\sigma)$ defined for $\sigma > 0$, its Mellin transform is a new function of the complex variable p, given by

$$
\check{X}(p) \equiv \int_0^\infty \frac{d\sigma}{\sigma} \, \sigma^{-p} \, X(\sigma).
\tag{21}
$$

Of course, this integral will not converge in general. If $X(\sigma)$ is reasonable (say $X(\sigma)$ is piecewise smooth for $0 < \sigma < \infty$, although more singular functions can be accomodated), then the convergence will depend on the behavior of $X(\sigma)$ as $\sigma \to 0^+$ and $\sigma \to \infty$. Suppose that for some pair of *characteristic exponents* α, β (which can be assumed to be real) we have

$$
X(\sigma) = \begin{cases} O(\sigma^\alpha) & \text{as } \sigma \to \infty \\ O(\sigma^\beta) & \text{as } \sigma \to 0^+. \end{cases}
\tag{22}
$$

Then the integral in (21) over the subinterval $1 \le \sigma < \infty$ will converge if and only if $\operatorname{Re} p > \alpha$, and the integral over $0 \le \sigma \le 1$ will converge if and only if $\operatorname{Re} p < \beta$. Hence the Mellin transform $\check{X}(p)$ exists in the vertical *strip*

$$
\mathcal{S}_X = \{p \in \mathbf{C} : \alpha < \operatorname{Re} p < \beta\} \equiv S(\alpha, \beta).
\tag{23}
$$

In particular, the integral converges *nowhere* if $\alpha > \beta$. It can be shown that when $\alpha < \beta$, $\check{X}(p)$ is actually *analytic* in $S(\alpha, \beta)$. In the limiting case $\alpha = \beta$, $S(\alpha, \beta)$ is empty but $\check{X}(p)$ can still exist as a *generalized* function on the vertical line $\operatorname{Re} p = \alpha$. See [4], and Section 2 of this paper. (When $\alpha < \beta$, the boundary values of $\check{X}(p)$ on $\operatorname{Re} p = \alpha$ and $\operatorname{Re} p = \beta$ are distributions.)

The Mellin transform can be related to the (inverse) Fourier transform by the substitutions

$$\sigma \to e^f, \ p \to -2\pi i t, \ X(\sigma) \to \hat{x}(f), \ \check{X}(p) \to x(t). \tag{24}$$

Consequently, the *inverse Mellin transform* can be represented by

$$X(\sigma) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} dp \ \sigma^p \ \check{X}(p), \tag{25}$$

where $\alpha < r < \beta$ so that the integration contour is inside the strip of analyticity. Since $\check{X}(p)$ is analytic in $S(\alpha, \beta)$, the value of the integral is independent of the particular choice of r within these bounds. (In the singular case $\alpha = \beta$, take $r = \alpha$.)

It is easy to see that under the substitutions (24), the scaling convolution $X \bullet Y$ becomes an ordinary convolution $\hat{x} * \hat{y}$. It follows that the Mellin transform maps $X \bullet Y$ to the *pointwise product* of the two Mellin transforms:

$$Z(f) \equiv \int_0^\infty \frac{d\sigma}{\sigma} \ X(\sigma) \, Y(f/\sigma) = (X \bullet Y)(f) \ \Rightarrow \ \check{Z}(p) = \check{X}(p) \, \check{Y}(p), \tag{26}$$

provided all the transforms exist. The inverse problem stated above (given $W(f)$, find $w(\sigma)$) corresponds here to solving for $X(\sigma)$ in terms of $Z(f)$ and $Y(f)$. Let us *define* the function

$$F(p) = \check{Z}(p) \big/ \check{Y}(p), \tag{27}$$

which should, according to (26), be the Mellin transform of the solution X. A sufficient (but not necessary) set of conditions for $X(\sigma)$ to exist is: (a) The analyticity strips $\mathcal{S}_Z$ and $\mathcal{S}_Y$ of $\check{Z}(p)$ and $\check{Y}(p)$ overlap, so that their nonempty intersection

$$\mathcal{S}_X \equiv \mathcal{S}_Z \cap \mathcal{S}_Y$$

is a potential analyticity strip for $\check{X}(p)$, (b) $\check{Y}(p) \neq 0$ in $\mathcal{S}_X$, so that $F(p)$ *is* analytic in $\mathcal{S}_X$, and (c) $F(p)$ decays sufficiently rapidly as $|p| \to \infty$ in $\mathcal{S}_X$ for the integral defining its inverse Mellin transform to converge.

Given a filter $W(f)$, the action of its C1-equivalent scale filter is computationally simpler than that of its C2-equivalent scale filter since the representation (7) involves only an integration over scales whereas (18) and (19) involve integration over time as well. Does C2-equivalence have any advantages over C1-equivalence? This question needs further study, both numerical and theoretical. One such advantage may be that C2-equivalence can be easily generalized to time-dependent filters, whereas C1-equivalence cannot. We need only replace $w(\sigma)$ by $w(\sigma, \tau)$ and define the operators $\mathcal{W}^+$ and $\mathcal{W}$, acting on real signals $\chi(t)$, by

$$(\mathcal{W}^+ \chi)(t) \equiv \iint_{\sigma > 0} d\sigma d\tau \ \psi_{\sigma,\tau}^+(t) \, w(\sigma, \tau) \, \check{\chi}(\sigma, \tau)$$

$$(\mathcal{W}\chi)(t) \equiv 2 \operatorname{Re} (\mathcal{W}^+ \chi)(t).$$

These generalize (18) and (19), respectively. They are no longer convolution operators since they are not time-translation invariant. Instead, they describe *time-varying filters*.

2. EXAMPLE: SPECTRAL POWER LAWS

Pure powers, *i.e.*, functions of the form

$$X(\sigma) = \sigma^q$$

with q a possibly complex exponent, play a special role in Mellin transform theory. Note that since $|\sigma^q| = \sigma^{\operatorname{Re} q}$, the characteristic exponents are now equal:

$$\alpha = \beta = \operatorname{Re} q \equiv r,$$

and the analyticity strip $S(\alpha, \beta)$ is empty. Actually, the Mellin transform $\check{X}(p)$ exists in the generalized sense of *distributions* [4], namely

$$\check{X}(p) = 2\pi i \delta(p - q).$$

This means that in (25) we must integrate along the vertical line $\operatorname{Re} p = r$ and

$$\frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} dp \; \sigma^p \, 2\pi i \delta(p - q) = \sigma^q = X(\sigma). \tag{28}$$

This is not surprising, since under the correspondence (24), σ^p is mapped to $e^{-2\pi i f t}$. Pure powers are, therefore, the Mellin version of complex exponentials. This means that although their Mellin transforms are singular, the computations become simple.

We demonstrate this now by choosing a scale filter which is a pure power and computing the corresponding frequency filters W_1 and W_2. Thus let

$$w(\sigma) = \sigma^q,$$

where q is an arbitrary complex exponent. Then for any fixed $f > 0$, the substitution $\sigma \to f/\sigma$ in (20) gives

$$\begin{aligned}
W_1(f) &\equiv \int_0^\infty \frac{d\sigma}{\sigma} \, \sigma^q \, R_\psi(f/\sigma) = \int_0^\infty \frac{d\sigma}{\sigma} \, (f/\sigma)^q \, R_\psi(\sigma) \\
&= f^q \int_0^\infty \frac{d\sigma}{\sigma} \, \sigma^{-q} \, R_\psi(\sigma) = f^q \, \check{R}_\psi(q),
\end{aligned} \tag{29}$$

with a similar computation for $W_2(f)$. Hence

$$w(\sigma) = \sigma^q \implies \begin{cases} W_1(f) = f^q \, \check{R}_\psi(q) \\ W_2(f) = f^q \, \check{S}_\psi(q). \end{cases} \tag{30}$$

We can invert (30) to go from the given frequency filter $W(f) = f^q$ to the equivalent scale filters:

$$W(f) = f^q \implies \begin{cases} w_1(\sigma) = \sigma^q / \check{R}_\psi(q) \\ w_2(\sigma) = \sigma^q / \check{S}_\psi(q). \end{cases} \tag{31}$$

The C1- and C2-admissibility conditions now become

$$C1 : \check{R}_\psi(q) \neq 0, \infty$$
$$C2 : \check{S}_\psi(q) \neq 0, \infty.$$

Theorem. *The frequency filter $W(f) = f^q$, with q complex, is C1- or C2-admissible if and only if the wavelet ψ satisfies*

$$C1: \quad 0 < |\check{R}_\psi(q)| \equiv \left| \int_0^\infty \frac{df}{f^{q+1}} \, \hat{\psi}(f)^* \right| < \infty$$
$$C2: \quad 0 < |\check{S}_\psi(q)| \equiv \left| \int_0^\infty \frac{df}{f^{q+1}} \, |\hat{\psi}(f)|^2 \right| < \infty. \tag{32}$$

When this is the case, the action of W can be represented in the wavelet domain by

$$\int_{-\infty}^\infty df \, e^{2\pi i f t} \, f^q \, \hat{\chi}(f) = \frac{1}{\check{R}_\psi(q)} \int_0^\infty d\sigma \, \sigma^q \, \tilde{\chi}(\sigma, \sigma t)$$
$$\int_0^\infty df \, e^{2\pi i f t} \, f^q \, \hat{\chi}(f) = \frac{1}{\check{S}_\psi(q)} \iint_{\sigma>0} d\sigma d\tau \, \sigma^q \, \psi_{\sigma,\tau}^+(t) \, \tilde{\chi}(\sigma, \tau), \tag{33}$$

respectively. For real q, we have

$$\int_{-\infty}^\infty df \, e^{2\pi i f t} \, f^q \, \hat{\chi}(f) = \frac{1}{\check{S}_\psi(q)} \iint_{\sigma>0} d\sigma d\tau \, \sigma^q \, \psi_{\sigma,\tau}(t) \, \tilde{\chi}(\sigma, \tau). \tag{34}$$

To connect with the usual admissibility condition and reconstruction formula, choose $q = 0$, so that $W(f) \equiv 1$ and the associated convolution is the *identity operator* ($w_2 * \chi = \chi$). Then (32) becomes

$$C1: \quad 0 < |\check{R}_\psi(0)| = \left| \int_0^\infty \frac{df}{f} \, \hat{\psi}(f)^* \right| < \infty$$
$$C2: \quad 0 < \check{S}_\psi(q) = \int_0^\infty \frac{df}{f} \, |\hat{\psi}(f)|^2 < \infty, \tag{35}$$

and (33) becomes

$$\chi(t) = \frac{1}{\check{R}_\psi(0)} \int_0^\infty d\sigma \, \tilde{\chi}(\sigma, \sigma t)$$
$$\chi^+(t) = \frac{1}{\check{S}_\psi(0)} \iint_{\sigma>0} d\sigma d\tau \, \psi_{\sigma,\tau}^+(t) \, \tilde{\chi}(\sigma, \tau). \tag{36}$$

Taking the real part of the second equation gives

$$\chi(t) = \frac{1}{\check{S}_\psi(0)} \iint_{\sigma>0} d\sigma d\tau \, \psi_{\sigma,\tau}(t) \, \tilde{\chi}(\sigma, \tau). \tag{37}$$

The second line in (35) is just the usual admissibility condition for the *wavelet*, and (37) is the usual reconstruction formula. Note that either of the conditions (35) imply the zero-moment condition $\hat{\psi}(0) = 0$ to which we refered earlier.

> *The wavelet ψ is admissible in the conventional sense if and only if the identity operator is C2-admissible as a convolution operator.*

3. SOME POSSIBLE APPLICATIONS

The general advantage of scale filtering over frequency filtering is that it can be performed *locally in time*. Its advantage over *time-frequency filtering* (using windowed Fourier transforms instead of wavelet transforms) is that the *spectral smoothing* implied in the scale convolutions $W = w \bullet R_\psi$ and $W = w \bullet S_\psi$ is performed with *proportional bandwidth*: long-duration or low-frequency components (represented by $w(\sigma)$ with $0 < \sigma < 1$) are smoothed over a narrow band (since $R_\psi(f/\sigma)$ and $S_\psi(f/\sigma)$ are compressed versions of $R_\psi(f)$ and $S_\psi(f)$), while short-duration or high-frequency components (represented by $w(\sigma)$ with $\sigma > 1$) are smoothed over a wide band (since $R_\psi(f/\sigma)$ and $S_\psi(f/\sigma)$ are stretched versions of $R_\psi(f)$ and $S_\psi(f)$). This is in the general spirit of wavelet analysis, as explained earlier.

The above examples suggest that systems obeying spectral power laws (even if only asymptotically) may have an especially simple representation. An example occurs in the study of electromagnetic waves propagating through turbulence, where the power spectrum follows Kolmogorov's $f^{-11/3}$ law between the "inner" and the "outer" scales. The Mellin transform has already found natural applications in this area [5], and it is hoped that the wavelet-based program proposed here can contribute by bringing *time* into the picture. In fact, there is solid empirical evidence that the analysis of atmospheric turbulence data in the scale domain is superior to the usual analysis in the frequency domain because it tends to smooth the spectrum *without smearing the scales*, so that the exponents in the power laws are clearly resolved [6].

Another potentially promising application is to *Doppler filtering* in wideband radar and sonar analysis. There, the usual representation of the Doppler effect as a uniform frequency shift breaks down. Instead, the Doppler effect must be represented by scaling in the time domain, the scale parameter $\sigma = (c - v)/(c + v)$ being directly related to the radial target velocity v. Therefore, the Doppler "frequency shift" is itself *proportional* to the frequency:

$$\Delta f = \sigma f - f = -\frac{2vf}{c + v} .$$

Only in the narrowband approximation can Δf be made uniform by replacing f with the carrier frequency in the above formula. The usual time-frequency analysis of the echo using the ambiguity function is generalized to a time-scale analysis using the so-called *wideband ambiguity function*. The latter is simply the wavelet transform of the echo, with the outgoing signal playing the role of basic wavelet [1,7]. Since the scale parameter σ (but *not* the frequency!) now has a direct physical significance related to the velocity of the reflector, *it is actually more natural to filter directly in scale rather than frequency* when analyzing the motion. For example, when trying to identify the echo $\chi(t)$ of a moving target embedded in a stationary background ("clutter"), one might simply multiply the wavelet transform $\tilde{\chi}(\sigma, \tau)$ by a function $w(\sigma)$ concentrated on the scale interval corresponding to the velocity range of interest, thus eliminating the clutter. Our results then show that this is equivalent to using a particular frequency filter, with a more convoluted transfer function. Possibly this may be used to speed up the computation of the original scale filtering operation with the help of the FFT.

4. ACKNOWLEDGEMENTS

5. REFERENCES

[1] G Kaiser, *A Friendly Guide to Wavelets,* Birkhäuser, Boston, 1994.

[2] J Bertrand, P Bertrand, and J-P Ovarlez, The Mellin Transform, Chapter 12 in *The Transforms and Applications Handbook,* CRC Press, Boca Raton, 1996.

[3] OI Marichev, *Handbook of Integral Transforms of Higher Transcendental Functions: Theory and Algorithmic Tables,* Ellis Horwood Ltd., Chichester, England, 1983.

[4] IM Gelfand and GE Shilov, *Generalized Functions,* Volume 1, Academic Press, New York, 1964.

[5] RJ Sasiela, *Electromagnetic Wave Propagation in Turbulence: Evaluation and Application of Mellin Transforms,* Springer-Verlag, New York, 1994.

[6] L Hudgins, CA Friehe, and ME Mayer, Wavelet transforms and atmospheric turbulence, *Physical Review Letters* **71**, 3279–3282 (1993).

[7] G Kaiser, Physical wavelets and radar: A variational approach to remote sensing, *IEEE Antennas and Propagation Magazine,* February, 1996.

[8] G Kaiser, Wavelet filtering with the Mellin transform, to appear in *Applied Mathematics Letters,* 1996.