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Representation in Skilled Mental Arithmetic

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Two experiments were performed to investigate the nature of skilled arithmetic performance. In Experiment 1, college subjects were trained extensively on a set of simple multiplication (e.g., __ = 4 × 9) and division problems (e.g., 56 = __ × 8). They were then tested on each problem seen at practice, and on three altered versions of each practice problem; a change in operand order v (e.g., __ = 4 × 9 at practice, __ = 9 × 4 at test), a change in operation (e.g., __ 4 × 9 at practice, 36 = __ × 9 at test), and change in both operand order and operation (e.g., 4 × 9 at practice, 36 = __ × 4 at test). In Experiment 2, both multiplication and division problems were again presented at practice and test. In addition, half of the problems had the symbol "×" and half had the symbol "÷". On the immediate and delayed tests, subjects again solved four versions of each practice problem; the actual practice problem, a problem with the symbol reversed, a problem with the operation reversed, and a problem with both symbol and operation reversed. Results from both experiments showed: (1) improvement in reaction time with practice follows a power law for all tested problem types, (2) across practice, division is more difficult that multiplication, and problems with the symbol "÷" are more difficult than problems with the symbol "×", regardless of the actual arithmetic operation required, (3) transfer of learning is substantial across changes in symbol, and across a change in operand order for multiplication, but is at best minimal across all other changes that were tested, (4) there is good to excellent retention of RT improvements gained.

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REPRESENTATION IN SKILLED

MENTAL ARITHMETIC

by

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retention of RT improvements gained through practice. The above results should be relevant in evaluating present and future models of skilled arithmetic performance. The finding that improvement in RT with practice follows a power law provides an important constraint for arithmetic learning models. Further, the test results differentiate between two sharply contrasting hypotheses of arithmetic fact representation; an abstract representation hypothesis, according to which representation is totally independent of perceptual characteristics of the problems, and a specific representation hypothesis, according to which represtation is totally dependent on the perceptual characteristics of the problem.

CONTENTS

CHAPTER 1

INTRODUCTION

Fundamental to any mental calculation is the skill of simple arithmetic, that is, the ability to quickly and accurately determine the answers to problems like 4 x 7 = ?. In recent years, there has been increasing interest in the cognitive processes underlying this skill in both children and adults (e.g., Ashcraft, 1985, 1987; Campbell, 1985, 1987a b and c, 1991; Campbell & Graham, 1985; Fendrich, Healy, & Bourne, in press; Koshmider & Ashcraft, 1991; McCloskey, Hartley, & Sokol, 1991; Miller & Paraedes, 1990; Miller, Pertmutter, & Keating, 1984; Siegler, 1988; Zbordoff & Logan, 1990). A basic finding of this research is that, while children often use explicit, consciously mediated counting algorithms, especially while they are first acquiring the skill (e.g., children often solve 3 x 7 by adding 7 + 7 + 7), there is a transition toward retrieval of arithmetic facts directly from memory as skill improves (e.g., Siegler, 1988). By adulthood, performance on most single-digit operand problems appears to reflect direct retrieval from memory. Several phenomena in adult mental multiplication performance have provided insight into details of the processes that underlie retrieval of these arithmetic facts. For example, under speeded conditions, typically 70 to 90% of errors that college students make are table-related; that is, they are answers to problems that share an operand with the problem being solved (e.g., Campbell & Graham, 1985; Graham, 1987; Sokol, McCloskey, Cohen, and Aliminosa, 1991). Thus, 21 (the answer to 3 x 7) is a frequent error to 4 x 7. In contrast, table-unrelated error are relatively infrequent (e.g., 30, the answer to 5 x 6, is a relatively infrequent error to 4 x 7). A second robust error pattern is that most errors are close in magnitude to the correct answer (Campbell & Graham, 1985; Graham, 1987; Miller et al., 1984). For example, 32 is a relatively frequent error to 4 x 9, while 12 is relatively infrequent. These error patterns suggest that representations for multiple problems that share operands and/or have answers of similar magnitude become active whenever a problem is being solved, and that these representations interfere with one another during the retrieval process.

A host of studies exploring priming effects in mental multiplication provide converging evidence in support of this conclusion (e.g., Campbell, 1987 b, 1991; Koshmider & Ashcraft, 1991; Winkelman & Schmidt, 1974). For example, Campbell (1991) presented a two-digit number that is a valid multiplication answer (e.g., 24) as a prime for 200 ms, and then presented a multiplication problem to be solved. He found that RT is slower and error rates are higher when the prime is table-related to the problems than when it is table-unrelated.

The convergent theme in recent theoretical accounts of these interference effects (see Anderson, Spoehr, & Bennett, in press; Campbell & Oliphant, in press, McCloskey & Lindemann, in press) is that retrieval of arithemtic facts reflects a process interactive-activation and competition, similar to that proposed by Rumelhart and McClelland (1982). According to this account, when an arithmetic problem is being solved by the skilled performer, multiple problem representations become active to the extent that they are in some way similar (e.g., share an operand) with the problem, and these active representations then compete until one representation (usually the correct one) reaches a high enough level of activation to be selected as the answer.

Further support for this general account of skilled arithmetic performance comes from a recent investigation of practice and transfer by Campbell (1987a). He pretested college subjects on a set of simple multiplication problems, trained them for several sessions on a subset of these problems, and then tested them again on all problems. Campbell (1987 a) found that response time (RT) improved considerably (by several hundred milliseconds) across practice. Also, on the post training test, subjects performed worse (had significantly longer RTs and a higher error rate) on unpracticed problems than they had on these same problems at pretest. Further, most errors to unpracticed problems on the post training test were answers to problems seen at practice. The finding that practice does not transfer positively across problems within multiplication constitutes important converging evidence for the basic theoretical assumption that skilled arithmetic performance reflects retrieval of individual facts from memory, rather than the execution of more general procedures as is reflected in children's performance. Further, the finding that practice actually transfers negatively to new problems is consistent with the assumption that representations for multiple facts compete for activation during the retrieval process (i.e., the problems on which the subject's practiced would compete more strongly after practice, thus slowing RT for unpracticed problems).

In another practice/transfer study, Fendrich, Healy, and Bourne (in press) trained college students for three one-hour sessions on simple multiplication problems (e.g., 6 x 8) and then tested them on these same problems, on operand order reverses of these problems (e.g., 8 x 6), and on new problems that were not seen during practice. Like Campbell (1987a) they found that RT improved considerably across practice sessions. Also, they found that learning transfered positively, although not completely, to operand order reversed problems. The Fendrich et al. (in press) study establishes a useful qualifier to the Campbell (1987a) findings; while practice does not transfer positively across problems involving different operands, there is positive transfer to problems related by operand reversal.

Both the Campbell (1987a) and the Fendrich (in press) studies demonstrate the value of a practice/transfer experimental approach to exploring basic issues of representation in mental arithmetic. As others have pointed out (e.g., Campbell, 1991), research to date on mental arithmetic has focused on outlining the basic processes involved in arithmetic fact retrieval (see the various investigations into interference effects cited above). Now that these processes are relatively well understood, researchers might profit from shifting the emphasis toward gaining a better understanding the details of the representations on which these processes operate. The current studies employ a practice/transfer approach in an effort to make progress in this direction. In the first phase of both experiments, college students were given 3 sessions of practice on simple multiplication and division problems. Across Experiments 1 and 2, four types of problems were presented at practice; multiplication problems with symbol "x" (e.g., __ = 4 x 7), multiplication problems with symbol "÷" (e.g., _ ÷ 6 = 9), division problems with symbol "x" (e.g., 35 = _ x 5), and division problems with symbol "÷", (e.g., 48 ÷ _ = 6). The primary purposes of practice in these experiments was to provide data on skill acquisition and to permit evaluation of performance across various altered problems in a subsequent transfer phase of the experiment. In both the Campbell (1987a) and the Fendrich et al. (in press) studies, average RTs for multiplication problems demonstrated a more of less constant speed-up from session to session of practice. In the current studies, improvement in RT will be evaluated on a trial to trial basis, allowing more precise estimates of speed-up with practice. More specifically, the practice data will provide a test of the applicability of the power law of practice (Newell & Rosenbloom, 1981) in the domains of mental multiplication and division. The power law had previously been demonstrated to hold for mental addition (Crossman, 19early), but to date has not been applied to multiplication or division skill. The practice data will further be used to establish the relative difficulty of multiplication and division problems, and of problems with the symbol "x" and the symbol "÷".

In the second, test phase of the experiments, subjects were tested on each of the exact problems seen at practice, as well as on several altered versions of each of the practice problems. For example, in Experiment 1, the altered versions at test included a reversal of operands (e.g., __ = 4 x 7 at practice, __ = 7 x 4 at test), a change in operation (e.g., __ = 4 x 7 at practice, 28 = _ x 7 at test), and a change in both operand order and operation (e.g., __ = 4 x 7 at practice, 28 = _ x 4 at test).

How will practice on one version of a problem, such as "__ = 4 x 7", impact performance on the various altered versions presented at test, such as those in the example above? The Fendrich et al. (in press) study suggests significant positive transfer across operand order. It is less clear, however, whether there will be positive transfer across a change in operation, or across a change in both operation and operand order. Although Campbell (1987a) showed no positive transfer (indeed, negative transfer) across problems within multiplication that have different operands, it is not at all clear whether this result can be used to predict transfer across operation, because there may be dependencies in memory for corresponding problems in two operations (e.g., __ = 4 x 7 and 28 = _ x 7) that would not be present when considering two problems within the same operation (e.g., 3 x 7 and 6 x 8).

To generate hypotheses about the relative amount of transfer that will be obtained across the various test conditions, one must speculate on the detailed characteristics of the representation(s) that are formed/strengthened with practice. One straightforward possibility is that arithmetic problems are represented in memory in an essentially abstract form, and that practice causes a strengthening of an association between an abstract representation for the problem, and an abstract representation for the answer. According to this model, which will be refered to straight-forwardly the abstract representations model, only the mathematically essential characteristics of arithmetic problems are represented in memory, i.e., the specific sensory details of the contexts in which the problems are solved are assumed not to be a part of the representation. It follows from this model that there should be a unique representation in memory for each of the three mathematically unique problems that make up a number relation in arithmetic, where number relation refers to a triplet of numbers that are tied together by arithmetic operations. For example, (4,7,28) constitutes a number relation in multiplication/division. There are three mathematically distinct ways of presenting any number relation as an arithmetic problem. For the example above, these are, (4,7,x) = 28, (4,28,÷) = 7, and (7,28,÷) = 4, where the order and perceptual characteristics of the symbols in parentheses is unimportant. According to the abstract representations model, there should be unique representations in memory for each of these mathematically unique problems.

The abstract representation model makes clear predictions with respect to the current experiments. First, there should be excellent performance on any problems at test that are mathematically equivalent to the corresponding problems presented at practice. Thus, for example, the model predicts complete transfer of learning across a simple change in operand order. The previous findings of positive transfer across operand order (Fendrich et. al, in press) is thus basically consistent with this simple model, although, strictly speaking, the model would predict equivalent performance on the practiced and unpracticed operand orders, whereas Fendrich et. al (in press) found significantly longer RTs for the unpracticed operand order. The model could be modified, however, to account for this finding by assuming that access to the abstract underlying representation becomes faster with practice on a specific format (e.g., the practiced operand order), and thus, at test, there is an advantage for the practiced format. An elaboration of the model along these lines is discussed in the following paragraph. In contrast to the prediction of good transfer across operand order, the abstract representations model would predict no transfer to problems at test that are mathematically unique from the corresponding problems at practice. Thus, for example, practice on __ = 4 x 7 would not be expected to transfer to 28 = _ x 7, or to 28 = _ x 4.

The abstract representations model fits naturally within a framework for number processing and arithmetic proposed by McCloskey, Caramazza, & Basili (1985; see also McCloskey, in press; Sokol, Goodman-Schulman, & McCloskey, 1989). Briefly, this model holds that skilled arithmetic performance involves the operation of three distinct and sequentially processing systems: (1) an encoding system, which translates the input from format specific representations to an abstract numerical representation, (2) a calculational system that operates on the abstract representations of the problem and generates an answer, and (3) a response system that converts the abstract representation of the answer into an appropriate code for the required mode of output (e.g., a motor program for entering the answer on a key pad). The calculational system is in turn composed of two sub-systems, a fact retrieval sub-system, which contains arithmetic facts, and a procedural sub-system, which stores procedures and algorithms such as the repeated adding algorithm that children often use to solve multiplication problems. According to the abstract representations model, the sub-system from which arithmetic facts are retrieved should contain three seperate facts for each number relation. Again, for the number relation (4,7,28), these facts would be (4,7,x) = 28, (4,28,÷) =7, and (7,28,÷)=4. The encoding system in the McCloskey et al. (1985) model provides a potential locus for the increase in RT across a change in operand order in multiplication found in the Fendrich et al (in press) study; with practice, the encoding system might simply become more efficient at encoding the practiced operand order.

The abstract representations model ignores the possibility that representations that form with practice will incorporate the perceptual characteristics of the specific formats in which the problems are presented. In the current experiments, subjects first saw the problems on the computer screen without the answer (e.g., " __ = 4 x 7"), and then, after they responded, the answer (which was the correst on the vast majority of trials), appeared on the screen with the problem (e.g., "28 = 4 x 7").

Repeated experience with these problems might result in a very specific sensory-based representation, "28 = 4 x 7", forming in memory. This specific representations model predicts that relative performance in the various conditions at test will be determined by the extent to which the test problem reinstates exactly the specific representation formed with practice. Thus, for example, practice on "__ = 4 x 7" would be predicted to transfer better to "28 = _ x 7" than to either "__ = 7 x 4", or "28 = _ x 4", because "28 = _ x 7" would be a closer match to the specific representation, "28 = 4 x 7", that formed with practice.

The specific representation model maps closely onto the recent network-interference model of arithmetic fact representation set forth by Campbell and Oliphant (in press; see also Campbell & Graham, 1985). A central assumption of the network-interference model that is most relevant to the present discussion is that representations of arithmetic facts reflect the perceptual characteristics of the format(s) in which the problem has been encountered (see also Campbell & Clark, 1988). Thus, if a problem is presented visually in the form "__ = 4 x 7", a physical code for that specific visual format forms in memory (e.g., "28 = 4 x 7"). Practice on any problem that corresponds to this visual format (i.e., __ = 4 x 7, 28 = _ x 7, 28 = 4 x _) strengthens the overall "unitized" representation, "28 = 4 x 7". Thus, practice on any problem corresponding to a given unitized representation will transfer positively to other problems corresponding to the same unitized representation, and poorly to any other problem, as is demonstrated for the example problem "__ = 4 x 7" in the previous paragraph.

It is important to note that both the abstract representations model and the specific representations model are in principle with the interative-activation and competition retrieval processes which are assumed to underlie the interference effects that have been the focus of attention in the mental arithmetic literature to date. Under the abstract representations model, for example, there would have to be a mapping of the perceived stimulus item (e.g., "__ = 4 x 7") onto the the appropriate abstract representation, e.g., (4,7,x). If this mapping is sensitive to the combinatorial nature of arithmetic, then additional representations that share an operand with the correct representation would also receive some activation. Each of the activated representations would then compete in an interactive-activation retrieval process, resulting in the table-related error and priming effects as outlined earlier. A very similar scenario would account for the same interference effects under the specific representations model. For the problem, "__ = 4 x 7", the specific representation, "28 = 4 x 7", would become active, as well as other representations that have a common operand with "__ = 4 x 7". Each of these active representations would then compete for activation. Other types of interference effects, such as the effect of magnitude similarity, could similarly be incorporated into either system of representation. In sum, the issues of representation being explored in these studies can be considered to be essentially orthogonal to the issue of how to account for the vast majority of the priming and related interference effects that have been explored in the literature to date (but see Campbell, 1990, for a priming effect which may be more consistent with the specific representations model).

In both Experiments 1 and 2, subjects were tested both immediately after practice, and after a one month delay. In the study by Fendrich, et. al (in press), there was good retention over a 1 month interval of improvements in multiplication RT gained through practice. One additional goal of the present experiments was to replicate this finding, and to generalize it to division problems and to problems with the symbol "÷".

CHAPTER 2

EXPERIMENT 1

Subjects received practice for four sessions on a set of simple multiplication and division problems, and were then tested on each of the practice problems, and also on problems representing a change in operand order, a change in operation, and a change in both operand order and operation. Table 1 shows an example of a multiplication and division problem at practice, and each of the corresponding conditions at test.

According the the abstract representations hypothesis, performance should be good whenever the test problem is mathematically equivalent to the corresponding practice problem. This includes problems in the no-change conditions for both multiplication and division, and problems in the operand order condition for multiplication. Performance should be relatively poor when the test problem is mathematically unique from the corresponding practice problem. This includes problems in each of the remaining conditions; the operation and operation plus operand order conditions for multiplication, and the operand order, operation, and operation plus operand order conditions for division.

In contrast, the specific representations model predicts that performance will be best when the sensory-specific representation of the number relation as a whole that was acquired during practice is reinstated at test. Thus, performance in the no-change and operation change conditions for both multiplication and division should be relatively good, and performance in the operand order and operand order plus operation change conditions for both multiplication and division should be relatively poor .

Method

Subjects

Twelve subjects from an introductory psychology course received credit for participating in the experiment.

Apparatus and Materials

Subjects were tested on Zenith Data Systems personal computers, programmed with the Micro Experimental Language (MEL) software (Schneider, 1988).

Four problem sets were constructed to allow counterbalancing across operand order and operation, and to control for possible effects of ascending-descending operand order, and problem difficulty. An example practice set (practice set 1) is shown in Table 2. The practice sets were constructed in the following manner. Excluding squares problems (e.g., 4 x 4), there are 72 problems between 1 = 1 x 1, and 81 = 9 x 9. These problems were divided into two subsets of 36 problems (practice sets 1 and 2), such that problems differing only in operand order were in different practice sets (i.e., 3 x 2 and 2 x 3 were in different practice sets). Within each of these two practice sets, 18 of the problems were multiplication problems, and 18 of the problems were division problems. Half of the problems of both operations (multiplication and division) had ascending operand order (e.g., __ = 3 x 6), and half had descending operand order (e.g., __ = 7 x 4). The multiplication and division problems in each practice set were also roughly equated on problem difficulty. Practice sets 3 and 4 were then constructed by simply reversing the operation of each of the problems in practice sets 1 and 2 (i.e., if __ = 4 x 5 was a problem in practice set 1, that problem became 20 = __ x 5 in problem set 3). In sum, four practice sets were created such that there was exactly one problem from each number relation in each set (i.e., sets 1, 2, 3 and 4 contained __ = 2 x 3, __ = 3 x 2, 6 = __ x 3, and 6 = __ x 2, respectively). Each one of these four sets was then used equally often across the twelve subjects during training. The immediate and delayed tests consisted of all 144 problems that made up the four practice sets.

Procedure

Each subject was tested for four sessions. Each session lasted about 40 minutes. The first three sessions were held on Monday, Wednesday, and Friday of one week, and the fourth session was held on Friday approximately one month later. During practice, each subject was exposed to 40 blocks of problems; 15 in both the first and second sessions, and 10 in the third session. Each block contained one instance of each of the 36 problems in a given subject's practice set. The order in which problems were presented was randomly determined for each block of practice and for each subject. Problems were presented one at a time, centered on the computer screen. As each problem appeared on the screen, the subject typed the answer using the numeric keypad and then pressed the "enter" key. As the subject typed the answer, it appeared on the computer screen, replacing the underlined spaces. Subjects were told to answer each problem as quickly and accurately as possible. If the subject entered the correct answer, a "correct answer" notice was displayed below the problem for 1 second. If the subject entered an incorrect answer, an "incorrect answer" notice, and the correct answer, was displayed for 1.5 seconds. The screen was then blank for 1 second, and then the next problem was displayed. After each block of 36 trials, a message was displayed on the screen requesting subjects to press the "enter" key to begin the next block. There was no limit on the time subjects were allowed before proceeding to the next block.

The immediate test was given at the end the third session, following the last practice block. Eight blocks of 36 problems were presented in exactly the same fashion as the practice problems. Across the first four blocks, each of the 144 problems which made up the four practice sets was presented once. Each of these problems was presented a second time across the final four blocks. Each block of 36 problems contained nine problems from each of the four practice sets, with each block having an equal number of multiplication and division problems, and an equal number of problems with ascending and descending operand orders. Subject to the constraints above, the order of presentation of blocks and the order of problem presentation within each block was determined randomly. There was no direct indication to the subjects that the eight immediate test blocks were any different from the practice blocks. The delayed test was structured in exactly the same way as the immediate test, except that each of the 144 problems was presented four times across 16 blocks of 36 problems.

Results

Problems with single-digit products (e.g., __= 4 x 2; 9 = _ x 9) were analyzed separately from problems with double-digit products (e.g., __ = 4 x 7; 56 = _ x 8). This seperation was motivated by previous findings suggesting that most multiplication problems with single-digit products are solved by rules (e.g., if a multiplication problem has 1 as one of the operands, the answer is the other operand) rather than by retrieval of facts from memory (e.g., McCloskey, Aliminosa, & Sokol, in press; Sokol, et al., 1991). The primary RT analyses were performed using the initiate RT (interval between the onset of the problem on the computer screen, and the pressing of the first digit of the answer). Previous research shows initiate RT to be highly correlated with both total RT (interval between the onset of the problem, and the pressing of the 'Enter' key), and with RT patterns produced through use of a voice key (A. F. Healy, personal communication, Feburary 4, 1992). Trials on which errors were made were excluded from all RT analyses.

Practice

The error rate for multiplication problems with single-digit products was .9%, 1.5%, and .8% in Sessions 1, 2, and 3, respectively. The same values for division problems with single-digit products were 2.1%, 1.1%, and 2.2%. For multiplication problems with double-digit products, the error rates were 4.0%, 3.7%, and 2.9%, in sessions 1, 2, and 3, respectively. For division problems, these values were 3.5%, 2.8%, and 2.8%.

The power law of practice predicts an essentially linear decrease in log RT as a function of log block (see Newell & Rosenbloom, 1981). That the practice data strongly conform to this prediction can be seen in Figure 1, which shows log RT for correct problems plotted by log block and operation (multiplication or division). Figure 1 a shows results for problems with single-digit products, and Figure 1 b shows the results for problems with double-digit products. In these figures, data are collapsed across subjects and problems. Thus, for problems with single-digit products, each data point represents up to 60 observation, and for problems with double-digit products, each data point represents up to 156 observations. An analysis of covariance (ANCOVA) was performed to confirm the effects of practice (the improvement in log RT as a function of log block), and operation (the advantage of multiplication over division) suggested in Figure 1. Preliminary analyses included the counterbalanced practice sets as a between subjects variable, log block as a continuous within subjects variable, and operation (multiplication or division) as a categorical within subjects variable. The results of these analyses showed no significant main effects or interactions involving the counterbalanced practice sets. Thus, the results reported below are from analyses collapsing across practice sets.

In the analysis for problems with single-digit products, the overall r^2 was .83. The effect of log block was strongly significant, $F(1, 11) = 212.59$, $p < .01$, MSe=.0493, confirming that practice was effective in improving log RT. The effect of operation was also significant both at the beginning of practice, $F(1, 11) = 50.4$, $p < .001$,MSe=.0613, and at the end of practice, $F(1, 11) = 7.8$, $p < .02$, MSe=.1247, showing that, throughout practice, multiplication problems were easier than division problems. The advantage for multiplication over division (in anti-log RT) was 245 ms at the beginning of practice, and 37 milliseconds at the end of practice. The interaction between log block and operation was also significant, $F(1,11) = 19.5$, $p < .01$, MSe=.0551, indicating that the proportional difference in RT (or, equivalently, the interval difference in log RT) between division and multiplication decreased with practice.

In the analysis of problems with double-digit products, the overall r^2 was .90. The improvement in log RT with practice was again strongly significant, $F(1,11) = 242.6$, $p<.01$, MSe = .0828, as was the advantage for multiplication over division, both at the beginning of practice, $F(1,11) = 56.6$, $p < .01$, MSe = .0494, and at the end of practice, $F(1,11) = 39.1$, $p < .01$, MSe = .1711. The anti-log RT advantage for multiplication was 341 ms at the beginning of practice, and 109 ms at the end of practice. The interaction between operation and log block was again significant, $F(1,11) = 5.75$, $p < .05$, MSe = .0308.

In both of the above analyses, improvement in RT with practice reflects not only improvement in time to encode and retrieve the answer, but also improvement in time to execute the motor program to enter the answer on the key pad. Thus, strictly speaking, it is not possible to infer from the above analyses that practice resulted in a speed-up in the fact retrieval component of the task. Rather, the results could reflect speed-up in motor response. Secondary analyses on the second digit entry RT (the latency between pressing the first digit of the answer and pressing the second digit for multiplication problems), however, argue strongly against this possibility. There was only a 45 anti-log ms improvement (from 188 ms to 143 ms) in second digit entry RT from the beginning to the end of practice, and the course of speed-up followed a power law. This speed-up is minor compared to the 580 ms overall improvement in the initiate RT for problems with double-digit answers. Assuming that the motor response component of the initiate RT is similar to the second digit entry RT in terms of both overall magnitude, and the course of speed-up with practice, the speed-up in initiate RT would appear to reflect mostly improvement in time to encode and retrieve the answer from memory. It should be noted that second digit entry RT may not in fact be a reasonable approximation of the motor component of the intitiate RT. It is, however, the best estimate availabe in the present experiments. Further research would be necessary to provide more precise estimates of the speed-up in fact retrieval with practice.

Immediate and Delayed Tests

Data from both the immediate and delayed tests were analyzed separately for problems single-digit products and problems with double-digit products. Also, because patterns of performance across test conditions were different for multiplication and division problems, analyses were performed separately for these two classes of problems. Preliminary analyses showed no effects of the counterbalanced practice sets. Thus, all analyses reported below are collapsed across this variable.

Multiplication. The error percentages on problems with single-digit products was uniformly low, averaging 3.2%. The error percentages on problems with double-digit products at the immediate test were 1.3%, 3.2%, 11.2%, and 11.5% for the no-change, operand order, operation, and operand order plus operation test conditions, respectively. Comparable values at the delayed test were 4.3%, 3.5%, 8.8%, and 8.0%.

The anti-log of the mean log initiate RT (averaged across subjects and correctly solved problems) is plotted in Figure 2 by test (immediate and delayed), block within test, and test condition (no-change, operand order, operation, and operation plus operand order). Figure 2 a shows the results for problems with single-digit products, and Figure 2 b shows the results for problems with double-digit products. Table 3 shows a specific example of each practice condition for the sample double-digit test problem __ = 4 x 7. Also shown in Figure 2 are the multiplication RTs extrapolated from the power law equations from practice, included to allow comparision of these RTs with the actual performance on no-change problems at the immediate test.

Inspection of Figure 2 (a and b) shows that performance was poorer in the No-change conditions than in the Extrapolated conditions. A 2 (condition; extrapolated vs. no-change) x 2 (block; 1 or 2) within subjects alalysis of variance (Anova) was performed seperately for problems with single-digit and double-digit products in an effort to investigate the reliability of this effect. For problems with single-digit products, the effect of Condition was reliable, $F(1,11) = 18.4$, $Mse = .00204$, $p = .0013$. There was no reliable effect of Block, $F(1,11) = .21$, $Mse = .000949$, $p = .6578$, nor was there an interaction between condition and block, $F(1,11) = .07$, $Mse = .000988$, n.s. For problems with double-digit products, Condition was again relaible, $F(1,11) = 20.3$, $Mse = .00114$, $p = .0009$, but there was no effect of Block, $F(1,11) = .3$, $Mse = .00073$ n.s., and no interaction, $F(1,11) = .58$, $Mse = .000726$, n.s. These results confirm similar increases in RT from practice to test reported by Campbell (1987). One account for this effect is that exposure to problems in the various transfer conditions causes activation of problem representations that were not activated during practice. These newly active representations may then compete in the retrieval process, slowing retrieval times for practiced problems (see Campbell, 1987).

Additional analyses were performed comparing performance among the various test conditions (no-change, operand-order, operation, and operand order plus operation). Separately for both the immediate and the delayed test data, and for problems with single-digit and double-digit products, a 2 (operation, same or different) x 2 (operand order, same or different) within subjects ANOVA was performed on the log initiate RT. The unit of analysis was the log initiate RT averaged across blocks and correctly solved problems. For problems with single-digit products on the immediate test, the main effect of operation was reliable, $F(1,11) = 7.89$, $p < .02$, MSe = .01448, indicating much poorer performance in the operation and operation plus operand order conditions. Neither the main effect of operand order, $F(1,11) = .19$, MSe = .00796, nor the interaction between operation and operand order, $F(1,11) = 1.27$, MSe = .00582, were reliable. On the delayed test, there was again an effect of operation $F(1,11) = 5.78$, $p < .05$, MSe = .00336, but there were no reliable effects due to either operand-order, $F(1,11) = 1.04$, MSe = .00327, or the interaction between operaion and operand-order, $F(1,11) = .001$, MSe = .0022.

In the analysis of problems with double-digit products on the immediate test, there was a main effect of operation, $F(1,11) = 35.1$, $p < .001$, MSe = .0882, as well as operand-order, $F(1, 11) = 5.34$, $p < .05$, MSe = .01736. The interaction was not significant, $F(1,11) = .96$, MSe = .01163. A more focused analysis showed no effect of operand order given a change in operation, $F(1,11) = 1.32$, MSe = .0149, indicating that the main effect of operand order can be attributed to the difference between the no-change and operand order change conditions. Analysis of the delayed test data again revealed an overall increase in log RT with a change in operation, $F(1,11) = 23.16$. $p < .01$, MSe = .03095, and also with a change in operand order, $F(1,11) = 4.94$, $p < .05$, Mse = .00762. There was no interaction, $F(1,11) = .1$, MSe = .00778. As with the immediate test, there was no significant effect of operand order given a change in operation, $F(1,11) = 1.36$, Mse = .01009.

It is important to consider the effect that improvement in motor response time may have had on relative levels of performance in the various test conditions. Because the sequence of digits corresponding to the answers to problems in the no-change and operand order change conditions were entered on the keypad during practice, and in most cases the sequence corresponding to the answers to operation and operation plus operand order change problems were not, some of the RT advantage for no-change and operand order change problems at test might be attributable to faster execution of the the motor response for these problems. To investigate this possibility, supplementary analyses were performed on a subset of double-digit problems that allow control for speed-up in motor RT with practice. Specifically, three pairs of problems used across the practice sets share the same product (2 x 6 and 3 x 4; 2 x 9 and 3 x 6; 3 x 8 and 4 x 6). The practice sets were constructed such that one member from each pair was presented as a multiplication problem in each set uring practice, and the other member was presented as a division problem. Thus, when the member of each pair that was presented as a division problem at practice was presented as a multiplication problem (in either operand order) at test, the subject was already experienced at executing the motor response for the answer. Analyses limited to these problem pairs again showed a strong main effect of operation, $F(1,11) = 12.5$, $p < .01$, MSe = .2898, and $F(1,11) = 8.17$ ($p < .05$, MSe = .0437), on the immediate and delayed tests, respectively.

To investigate retention of skill across the various test conditions, pairwize comparisons of the last block of the immediate test and the first block of the delayed test were performed for each test condition (no-change, operand order, operation, and operation plus operand order). There were no relaible differences obtained from these comparisions for problems with single-digit products. For problems with double-digit products, performance in the no-change condition was reliably better on the last block of posttest than on the first block of retention, $F(1,11) = 5.94$, Mse = .001398, $p = .033$. There were no reliable differences for the remaining conditions; operand order, $F(1,11) = 2.73$, Mse = .00362, n.s., operation, $F(1,11) = 5.2$, Mse = .001569, n.s., operation plus operand order, $F(1,11) = .001$, Mse = .001343, n.s.

<u>Division</u>. The error percentages for problems with single-digit products on both the immediate and delayed tests was uniformly low, averaging 2.1%. The error percentages for problems with double-digit products on the immediate test were

2.2%, 10.5%, 10.8%, and 9.2%, in the no-change, operand order, operation, and operand order plus operation change conditions, respectively. The same values at the delayed test were 3.4%, 7.6%, 9.3%, and 8.0%.

The anti-log of the mean log initiate RT for correctly solved division problems is shown in Figure 3. Each of the four test conditions is shown, as well as the predicted RT for problems in the no-change condition extrapolating from the power law fits for the practice data. Figure 3 a shows results for problems with single-digit products, and Figure 3 b shows results for problems with double-digit products. Table 4 shows a specific example of each test condition for the sample test problem 28 = _ x 7.

Within subjects ANOVAs (performed seperately for problems with single-digit and double-digit products) showed a reliably faster RT for the extrapolated condition over the no-change copndition both for problems with single-digit products, $F(1,11) = 15.7$, Mse = .0020466, $p = .0022$, and for problems with double-digit products, $F(1,11) = .0036$, Mse = .00194, $p = .0036$. There were no significant effects of block, nor were there relaible interactions between condition and block.

Addional analyses were performed to evaluate performance in the various test conditions. For problems with single-digit products on the immediate test, there was an effect of both operation, $F(1,11) = 4.54$, $P = .057$, MSe = .0177, and operand order, $F(1,11) = 32.87$, $p > .01$, MSe = .0143. The interaction was not reliable, $F(1,11) = 2.85$, MSe = .0049. On the delayed test, the was no effect of operation $F(1,11) = 1.5$, MSe = .0116, but there was a significant effect of operand order $F(1,11) = 32.6$, $p < .01$, MSe = .00417, and a reliable interaction, $F(1,11) = 22.1$, $p < .01$, MSe = .00609.

For problems with double-digit products on the immediate test, there was a main effect of both operation, $F(1,11) = 45.8$, $p < .01$, MSe = .0254, and operand-order, $F(1,11) = 24.9$, $p < .01$, MSe = .02879, and there was a reliable interaction, $F(1,11) = 19.2$, $p < .01$, MSe = .0243. Post hoc analyses showed no significant differences among the operand order, operation, and operand order plus operation conditions; operation versus operation plus operand order, F(1,11) = .37, MSe = .0361, operand order versus operation plus operand order, F(1,11) = 2.95, MSe = .0264, or operand order versus operation, F(1,11) = 1.09, MSe = .0247. In sum, these results reflect the pattern in Figure 3 b which shows performance in the no-change condition to be substantially better than performance in each of the other conditions. On the delayed test, both of the main effects and the interaction were again reliable; operation, F(1,11) = 17.4, p < .01, MSe = .0276, operand order, F(1,11) = 56.2, p < .01, MSe = .003568, and the interaction , F(1,11) = 18.7, p < .01, MSe = .00714. There were again no significant differences among the operand order, operation, and operand order plus operation conditions; operation versus operation plus operand order, F(,11) = .73, MSe = .00461, operand order versus operation plus operand order, F(1,11) = 2.48, MSe = .0216, or operand order versus operation, F(1,11) = 2.0, MSe = .0149.

As with multiplication problems, pairwize comparisons of the last block of the immediate test and the first block of the delayed test were performed on problems in each test condition (no-change, operand order, operation, and operation plus operand order) in order to investigate the relative retention of skill in the various test conditions. There were no reliable differences for problems with single-digit products. For problems with double-digit products, there was better performance on the last block of immediate test than on the first block of delayed test in the no-change condition, F(1,11) = 3.8, Mse = .00297, p = .077. There were no other reliable differences; operand order, F(1,11) = 2.46, Mse = .01136 n.s., operation, F(1,11) = .35, Mse = .01156, n.s., and operation plus operand order, F(1,11) = 1.39, Mse = .00467, n.s.

Discussion

Practice

For all problems, there was substantial improvement in RT with practice, and the course of speed-up was well described by a power law. The good power law fit does not, however, rule out other possible learning laws (e.g., the exponential function). In Appendix A, a more general form of the power law (including free parameters allowing for non-zero asymptotic RT, and previous learning experience) will be compared directly with an exponential function in an attempt to establish the validity of the power law in the domain of arithmetic in a more rigorous fashion.

There was an RT advantage for multiplication over division which was significant both at the beginning and at the end of practice. This advantage for multiplication could be related to one or more of the following factors: (a) in formal education and/or in everyday life, multiplication may be performed more frequently than division, and thus the answers may be more quickly accessed, (b) division may be mediated by knowledge of the corresponding multiplication problems, and (c) multiplication facts may be more efficiently encoded and/or retrieved from a generic underlying memory structure in which both multiplication and division facts are stored. The data do not allow descrimination among these possibilities.

One further alternative is examined in Experiment 2. The advantage for multiplication may simply reflect the fact that the symbol "x" was used for both multiplication and division problems in Experiment 1; because the "x" symbol directly implies multiplication, its presence may have facilitated performance on multiplication problems, and/or interfered with performance on division problems. Clearly, this factor will need to be investigated before any other hypotheses (such as those outlined above) can be seriously considered. Such an investigation is one purpose of Experiment 2, where symbol ("x" and "÷") will be manipulated orthogonally to operation (multiplication and division).

Immediate and Delayed Tests

Consistent with the abstract representations hypothesis, performance on both the immediate test and the delayed test was relatively good in the no-change conditions for both multiplication and division, and also in the operand order condition for multiplication. There was relatively poor performance in each of the remaining conditions. This pattern was especially for problems with double-digit products. The predictions of the specific representations model, in contrast, are inconsistent with the obtained results. This model predicts that, for both multiplication and division, performance should be better in the no-change and operation change conditions, and relatively poor in the operand order and operation plus operand order condition. The results are in clear contradiction to this strong prediction of the specific represnetations model.

There was, however, was some evidence suggesting second order effects which are consistent with the specific representations model. For both multiplication and division problems, there was a trend (although not statistically significant) toward better performance in the operation change condition than in the operation plus operand order change condition. This is not predicted by the abstract representations model, but it is consistent with the possibility that specific representations such as those proposed by Campbell and Oliphant (in press) are exerting a small second order effect.

A second order influence of sensory-specific representations also provides one possible account for the reliable RT advantage for the no-change condition over the operand order condition for multiplication. There are also, however, other possible accounts for the RT differences between these conditions. First, as discussed earlier, the abstract representations model, considered within the framework of the McCloskey, et al. (1985) framework for number processing, can account for this effect under the assumption that, while a single abstract represetation mediates fact retrieval for both operand orders, the practiced operand order can be encoded more quickly.

Two additional possibilities are not directly consistent with either the abstract represetations or the specific represntations model. First, the practiced operand order may serve as a canonical case at test, such that subjects mentally transpose the problem into the practiced operand order (through some unspecified process) before retrieving the answer. The time required to execute this transposing process could then account for the increased RT in the operand order change test conditions. Second, different operand orders may in fact access partially overlapping (i.e., distributed) representations in memory. Clearly, differentiating among these possible accounts is a complex issue and will require additional research.

The strong similarity of the results on the immediate and delayed tests indicates that the improvement of skill with practice was stable across the long term (1 month). There was only a slight increase in RT for no-change problems (for both multiplication and division) across the retention interval, indicating good retention of the skill acquired during practice. For problems in the other conditions conditions, there was no reliable increase in RT over the interval.

The finding of good retention replicates results of Fendrich et al. (in press) for multiplication, and also extends their general finding to division. These results are generally consistent with the procedural reinstatement account of long term retention proposed by Healy, Fendrich, Crutcher, Wittman, Gesi, Ericsson, and Bourne (in press). According to this account, performance will be best after a retention interval when procedures developed during learning are reinstated at test. Procedures developed during practice can be viewed as being most completely reinstated for the no-change problems in this experiment, and indeed performance on the delayed test is best for these problems.

CHAPTER 3

EXPERIMENT 2

One limitation of Experiment 1 is that all problems included the symbol "x", regardless of whether the actual mathematical operation required was multiplication or division. Thus, there was no way to assess the influence that symbols for different operations might have on the retrieval process. In Experiment 2, the effect of symbol was explored by testing subjects on problems requiring the operation of multiplication, but having the symbol for either multiplication or division (e.g. __ = 4 x 7 and __ ÷ 4 = 7), and on problems requiring the operation of division, but again having the symbol for either multiplication or division (e.g., 21 = _ x 4, 21 ÷ _ = 4). Performance was then evaluated on the practice problems themselves, and across a change in symbol, a change in operation, and a change in both symbol and operation (see Table 5).

As previously discussed, one interpretation of the operation effect at practice in Experiment 1 is that multiplication is fundamentally easier than division. If this interpretation is correct, then multiplication should be easier than division throughout practice in Experiment 2, regardless of the symbol employed (x or ÷). Alternatively, performance may depend intimately on the consistency between the mathematical operation required, and the symbol employed. According to this consistency hypothesis, performance on consistent problems (e.g., __ = 4 x 7 and 40 ÷ _ = 8) should be better than performance on inconsistent problems (e.g., __ ÷ 7 = 4; 40 = _ x 8). The orthognonal manipulation of symbol and operation in this experiment allows for an investigation of this possibility.

The test conditions in this experiment also provide an opportunity to replicate and extend the findings from the immediate and delayed tests in Experiment 1. Based on the abstract representations model, which was supported by the Experiment 1 results, there should be relatively good performance in the no-change and symbol change conditions for both multiplication and division, because the problems in these conditions are mathematically equivalent to the corresponding problems seen at practice. There should be relatively poor performance in the operation and operation plus symbol conditions for both multiplication and division, because these problems are mathematically unique from the problems seen at practice.

The specific representations model, on the other hand, predicts good performance in the no-change and operation change conditions for both multiplication and division problems, and relatively poor performance in the symbol change and operation plus symbol change conditions for both multiplication and division.

Method

Subjects

Twelve subjects from an introductory psychology course received credit for participating in the experiment.

Apparatus Materials and Procedure

Subjects were tested on Zenith Data Systems personal computers, programmed in the MEL (Schneider, 1988).

The materials and procedure for Experiment 2 was the same as that for Experiment 1, with the following exceptions. First, on half of the multiplication and division problems, the symbol was "x", and on the other half the symbol was "÷". Thus, an example practice set in this experiment can be derived from table 2 by swithing the symbol to "÷ "for half of the multiplication and half of the division problems. This manipulation yielded four problem types at practice; multiplication problems with symbol "x", multiplication problems with symbol "÷", division problems with symbol "x", and division problems with symbol "÷". On the immediate and delayed tests, each problem was presented again exactly as it was at practice (the no-change condition), with a change in symbol, with a change in operation, and with a change in both operation and symbol (see Table 5 for an example of each of the four problem types at practice, and an example of each of the corresponding conditions at test).

Results

Practice

To simplify and facilitate discussion, problems with single-digit and double-digit products were collapsed together for the analyses of practice data. Preliminary analyses showed overall faster RTs and lower error rates for problems with single-digit products. Beyond this, there were no relevant differences between the two classes of problems.

As with Experiment 1, the overall error percentages dropped slightly from Session 1 to Session 3 (from 5.0% for Session 1 to 2.1% for Session 3). Averaged across sessions, the error percentages for the four problem types were: multiplication problems with symbol "x", 3.1%, multiplication problems with symbol "÷", 3.9%, division problems with symbol "x", 2.4%, and division problems with symbol "÷", 4.7%.

In Figure 4, the log RT (averaged across subjects and correctly solved problems) is plotted as a function of log block and problem type, with least squares regression curves fit to each problem type. Unlike Experiment 1 practice analyses, the counterbalanced practice sets were involved in several significant two and three-way interactions. Thus, the analyses reported below are from an ANCOVA with practice set as a between subjects factor, and log block, operation, and symbol as within subjects factors. In ANCOVA models, effects involving continuous variables can be summarized with a single significance test. The effects involving only categorical variables, however, must be evaluated at specified levels of the continuous variable. Accordingly, the results of the ANCOVA will be divided into three parts for discussion: (a) effects involving the continuous variable (log block), (b) effects involving variables other than log block at the beginning (first block) of practice, and (c) the effects involving variables other than log block at the end (last block) of practice.

There was a strong main effect of log block, $F(1,11) = 133.6$, $p < .01$, MSe = .1796, reflecting an overall improvement in log RT with practice. There was also an interaction of log block and symbol, $F(1,11) = 43.3$, $p < .01$, MSe = .0666. Both of the above effects, however, were qualified by a three way interaction among log block, symbol, and operation, $F(1,,1) = 21.8$, $p < .01$, MSe = .0477. The three-way interaction is clearly shown in Figure 4. Three of the four problem types exhibit essentially the same slope, while the slope of the fourth problem type (multiplication with symbol "÷") was much steeper. This steeper learning curve for multiplication problems with symbol "÷" (e.g., __ ÷ 6 = 7) likely reflects the relative unfamiliarity of this format for presenting arithmetic problems. It may have taken subjects several blocks of practice to realize that these problems are mathematically equivalent to more tradition formats for multiplication problems, such as the multiplication with symbol "x" (e.g., __ = 4 x 7) problems in this experiment.

Several effects not involving log block were significant at the beginning of practice. First, the main effects of both operation $F(1,11) = 9.33$, $p < .02$, MSe = .0795, and symbol, $F(1,11) = 72.8$, $p < .01$, .0999 were significant. These main effects were qualified, however, by an interaction between operation and symbol, $F(1,11) = 14.9$, $p < .01$, MSe = .1545. This interaction again reflects the exceptionally poor performance at the beginning of practice on multiplication problems with symbol "÷". Because of the poor performance in this condition, there was no clear overall advantage for either multiplication or division problems at the beginning of practice. For <u>both</u> multiplication and division problems, however, problems with the symbol "x" were responded to more rapidly than problems with the symbol "÷", as reflected in the strongly significant main effect of symbol.

There were three interactions involving the counterbalanced practice sets at the beginning of practice; practice set x operation, $F(1,11) = 5.8$, $P < .05$, $MSe = .0795$, practice set x symbol, $F(1,11) = 7.6$, $p < .01$, $MSe = .0999$, and practice set x operation x symbol, $F(1,11) = 14.9$, $p < .01$, $MSe = .1544$. These interactions reflect varying relative difficulty of the four problem types across the practice sets. Recall that practice sets in the present experiment were derived from the practice sets in Experiment 1 by simply switching the symbol from "x" to "÷" on half of the multiplication and half of the division problems. The interactions involving practice set likely reflect failure to equate the four problem types with respect to problem difficulty when deriving the new problem sets. The nonsignificant main effect of practice set, $F(1,11) = 2.9$, $MSe = .398$, is consistent with this account, because switching the symbol on half the problems within each practice set would not be expected to affect the overall relative difficulty of the four practice sets.

At the end of practice,The main effects of operation, $F(1,11) = 7.7$, $p < .05$, $MSe = .2598$, and symbol, $F(1,11) = 7.1$, $P < .05$, $MSe = .2034$, persisted at the end of practice. The interaction between operation and symbol, which was significant at the beginning of practice, was no longer significant at the end of practice, $F(1,11) = 1.1$, $MSe = .0477$. Because there was no interaction, the main effect of operation shows that multiplication problems were solved more quickly than division problems at the end of practice, regardless of symbol. As with the beginning of practice, performance was better on problems with the symbol "x" than on problems with symbol '÷', regardless of operation. Each of the interactions involving practice set that were significant at the beginning of practice were again significant (or nearly so) at the end of practice; practice set x operation, $F(1,11) = 3.3$, $p = .077$, $MSe = .2598$, practice set x symbol, $F(1,11) = 4.6$, $P < .05$, MSe = 1.49, and practice set x operation x symbol, $F(1,11) = 4.71$, $P < .05$, MSe = .2182.

Immediate and Delayed Tests

The error rate for probvlems with single-digit products was uniformly low, averaging 2.1%. For problems with doulbe-digit products, the overall error percentages on the immediate test for same operation conditions (the no-change and symbol conditions) and different operation conditions (the operation and operation plus symbol conditions) were 5.9% and 13.5%, respectively. These same values on the delayed test were 6.3% and 9.2%.

Preliminary analyses showed no notable differences in test performance across the four problems types, and thus the data were collapsed across problems type in all analyses reported below. Preliminary analyses did show, however, substantial differences in the RT results for problems with double-digit and single-digit products, and thus problems falling into these two classes were analyzed separately. The anti-log of the mean log initiate RT (averaged across problems and subjects) is plotted in Figure 5 by test (immediate or delayed), and test condition (no-change, symbol change, operation change, or symbol plus operation change). Figure 5 a shows results for problems with single-digit products, and Figure 5 b shows results for problems with double-digit products. Also shown in Figure 5 are the expected RTs in the no-change conditions on the immediate test extrapolating from the power law equations which were fit to the practice data.

ANOVAs comparing the Extrapolated and No-change conditions (see the Results section in Experiment 1 for details) revealed reliable RT advantage for the extrapolated condition both for problems with single-digit products, $F(1,11) = 8.9$, Mse = .02348, $p = .012$, and for problems with dublee-digit products, $F(1,11) = 8.15$, Mse = .00736, $p = .016$. There were no reliable effects involving Block, or the interaction between Condition and Block.

Separately for the immediate and the delayed test, an ANOVA with one between subjects factor (practice set) and two within subjects factors (operation, same or different, and symbol, same or different) was performed on the log initiate RT. This preliminary ANOVA showed no main effect or interactions involving the counter-balanced practice sets. The data were therefore collapsed across practice set and analyzed on the two within subjects variables only. For problems with single-digit products on the immediate test, the main effect of operation was significant, $F(1,11) = 6.3$, $P < .05$, $MSe = .0188$, although there was no effect of symbol, $F(1,11) = 1.76$, $MSe = .0324$. This effect was qualified by an interaction between operation and symbol, $F(11,1) = 5.99$, $p < .05$, $MSe = .0348$. On the delayed test there were no significant differences between the conditions.

For problems with double-digit products on the immediate test, there was an effect of both operation, $F(11,1) = 127.2$, $p < .01$, $MSe = .0249$, and symbol, $F(11,1) = 18.8$, $p < .01$, $MSe = .00671$. There was no reliable interaction $F(1,11) = 3.27$, $MSe = .008$. On the delayed test, there was again an effect of operation, $F(1,11) = 59.7$, p , .01, $MSe = .0139$, there was a trend toward a significant effect of symbol, $F(1,11) = 2.65$, $p = .13$, $MSe = .00982$, and there was no interaction, $F(1,11) = 1.03$, $MSe = .0036$.

In Experiment 1, there was a trend toward an RT advantage for the operation condition over the operation plus operand order condition, suggesting that specific, sensory based representations may be exerting a second order influence on the retrieval processes. The analogous comparison in this experiment is between the operation condition and the operation plus symbol condition for problems with double-digit products. Collapsing across the immediate and delayed tests, the RT advantage for operation condition over the operation plus symbol condition (43 ms) was marginally significant, $F(1,11) = 4.09$, $MSe = .01059$, $p = .0$

Pairwize comparisons between the last block of the immediate test and the first block of the delayed test for each test condition showed no reliable differences for problems with single-digit products. For problems with double-digit produts, performance was reliably better in the no-change condition on the last block o the immediate test than on the first block of the delayed test, $F(1,11) = 20.0$, Mse = .00166, $p = .0009$. There were no other reliable diferences; symbol, $F(1,11) = 2.07$, Mse = .00234 n.s., operation, $F(1,11) = .50$, Mse = .00147 n.s., operation plus symbol, $F(1,11) = 1.32$, Mse = .00131, n.s.

Discussion

Practice

As with Experiment 1, the course of improvement in RT with practice was shown to follow closely a simple power law. The practice results also replicate the operation effect (the RT advantage for multiplication over division) found in Experiment 1, and further demonstrate that this effect in Experiment 1 was not an artifact of the use of the "x" symbol for both multiplication and division problems; by the end of practice, multiplication problems were being solved more quickly than division problems regardless whether the symbol employed was "x" or "÷". This finding was true despite the fact the performance on multiplication problems with symbol "÷" was the worst of all problems types at the beginning of practice. The fact that performance in this condition was better than performance in the two division condition conditions by the end of practice attests to the robustness of the operation effect.

The consistency hypothesis (the hypotheses that performance will be best on problems in which the symbol is consistent with the operation) was, in contrast, clearly refuted by these results. To the contrary, there was an unexpected symbol effect, such that, by the end of practice, problems with the symbol "x" were uniformly easier than problems with the symbol "÷", regardless of whether the actual operation is multiplication or division.

The various mechanisms which might underlie both the operation and symbol effects will be discussed in detail in the General Discussion.

Immediate and Delayed Tests

The results for problems with double-digit products are most relevant to the theoretical issues concerning representation of arithemtic facts, and are most consistent with the results from Experiment 1. Therefore, the immediately following discussion will be limited to these problems, with discussion of results for problems with single-digit products delayed to a later point.

As expected, for both multiplication and division problems there was relatively good performance in no-change and symbol test conditions, and relatively poor performance in operation, and operation plus symbol test conditions. Both of these results converge on the results of Experiment 1, and provide further support for the specific representations model. However, as with Experiment 1, there was also some evidence of a second-order effect which is consistent with the specific representations model; there was a strong trend toward better performance in the operation test condition than in the operation plus symbol test condition. The analogous trend in Experiment 1 for both multiplication and division problems was toward better performance in the operation condition than in the operation plus operand order condition. It may be difficult to account for these recurring trends without assuming that sensory-specific representations of the sort specified by Campbell and Oliphant (in press) are exerting a second-order influence of the retrieval process. Additional research will be needed, however, to confirm these effects.

The test results for problems with single-digit products were very different from those for problems with double-digit products. For these problems, the largest gap in RT was between the no-change condition and all other conditions. This contrasts sharply with the multiplication problems with single-digit products in

Experiment 1, where the largest gap in performance was between the same operation conditions and the different operation conditions. It is unclear why a change in symbol (Experiment 2) and a change in operand order (Experiment 1) for multiplication should have essentially the same effect on performance for problems with double-digit products, but a very different effect on performance for problems with single-digit products. In order to make sense of this finding, it is likely that more detailed analyses of subsets of problems with single-digit products would have to be undertaken. Three distinct subsets of these problems in these experiments are: (1) N x 1 = N, (2) 1 x N = N, and (3) 2 x 3 = 6, 3 x 2 = 6, 2 x 4 = 8, and 4 x 2 = 8. The patterns of performance at test might be very different for these three subsets of problems. Limitations in statistical power, however, prevent meaningful analyses of these subsets to be undertaken with the current data.

The difficulty in interpreting results for problems with single-digit products should not detract from the theoretical inferences that can be made about representation based on the results for problems with double-digit products. Recall that N x 1 and 1 x N problems, which constitute the majority of problems with single-digit products in this experiment, are believed to be solved using rules rather than retrieval of answers from memory. Both the abstract and specific representations models are models of representation of arithmetic facts in memory, and thus may not be applicable to these problems.

CHAPTER 4

GENERAL DISCUSSION

The results of these experiments provide new data on several aspects of skilled mental arithmetic performance. First, the course of improvement in RT with practice was shown to follow closely a power law for all tested problems types. A more detailed investigation of the capacity of the power law, relative to the exponential family of functions, to fit the practice data is described in Appendix A. Second, the relative difficulty of multiplication and division problems presented in a variety of formats was documented. One robust finding across both experiments was an operation effect, such that multiplication problems were solved more quickly than division problems. There are at least three possible explanations for the operation effect. First, division performance may be mediated by multiplication knowledge. That is, if subjects come into the study with well developed multiplication skill, but relatively poor division skill, they may attempt to solve the division problems by matching each division problem with the corresponding multiplication problem of the same number relation. For example, subjects might solve 42 = _ x 7 by accessing pre-existing knowledge of 6 x 7 = __. This process might involve random (or perhaps some form of systematic) accessing of multiplication problems for which 7 is one of the operands, until a match is found with the correct product, 42. This mediation of division responses by way of multiplication might simply become more efficient or automatized with practice. Since solving division by way of multiplication would always take more time than simply solving a multiplication problem, we would expect to find poorer performance on division than multiplication regardless of the amount of practice. This hypothesis may indeed approximate a strategy used for division at the beginning of practice, however, given the strong evidence in the arithmetic literature that skilled multiplication and addition involves direct retrieval from memory, it would be unparsimonious to assume that the highly analogous domain of skilled division represents an exception (additional evidence that post-practice division performance dominantly reflects fact retrieval is discussed in Appendix B, where error patterns on the immediate and delayed test are explored in detail).

A second possible explanation for the operation effect draws on the same frequency mechanism often proposed as an explanation for the problem-size effect (e.g., Campbell & Graham, 1985). According to this account, multiplication performance is better at the beginning of practice simply because subjects have been exposed to multiplication more frequently. Because multiplication and division problems were presented with equal frequently during practice, this advantage would be expected to decrease with practice, but might not completely disappear.

Still a third possibility is that division facts are more difficult to store and/or retrieve from memory. This could be the case, for example, because all division answers are answers to more than one problem (e.g., 24 = _3_ x 8, 12 = _3_ x 4, 27 = _3_ x 9), whereas most multiplication answers are answers to only one problem. From a memory network perspective (see Anderson, 1990), the multiple problems associated with each division answer may result in a sort of reverse spreading activation effect, such that, when the correct answer receives activation, this activation in turn partially "drains" back to the other associated problem representations, thus increasing the time necessary for the correct answer to reach some activation threshold. This account, however, ignores the fact that while there would be more reverse spreading activation for division problems, there would be more forward spreading activation for multiplication problems. This would be the case because each of the operands of a multiplication problems is associated with many answers. In contrast, for division, the presented operand is also associated with many answers, but the product is typically only associated with one answer. Thus, in order for the reverse spreading activation to yield longer RTs for division, it would have to have a strong enough influence to override the greater amount of forward spreading activation for multiplication. Clearly this, and the other proposed accounts of the operation effect, are largely speculative. More research is needed to address this issue.

A symbol effect (the findings that problems with symbol "x" were solved faster than problems with symbol "÷", regardless of the arithmetic operation being performed) was also found in Experiment 2. This effect may be explainable through a frequency account assuming that the "x" symbol is used more frequency than the "÷" symbol, as typically seems to be the case in format educational contexts. The "÷" symbol is typically used during initial learning of division in order to differentiate division from the other operations. When students begin learning algebra, however, this symbol becomes essentially redundant with the "x" symbol, and thus is no longer needed. Further, where an explicit division symbol is used in high school and college mathematics courses, it is often the "/" symbol rather than the "÷" symbol. Thus, for the typical college student, the symbol "x" may actually be at least as strongly associated with division as is the symbol "÷".

A frequency account for the symbol effect is essentially the same frequency hypothesis as proposed by previous researchers to explain the problem-size effect, and also proposed in this paper as one possible account for the operation effect. Unlike these other effects, it is unclear what mechanisms other than frequency might plausibly underlie the symbol effect. The frequency hypothesis, then, appears to have the unique capacity to account for the problem-size effect, the operation effect, and the symbol effect, and should perhaps for this reason be considered the most promising account for each of these phenomena.

The test results for problems with double-digit products across both experiments generally support the abstract representations model. Indeed, this model, placed in the context of the more general model of number processing developed by McCloskey et al. (1985), can account for every statistically significant finding in the test data for problems with double-digit products. More specifically, the abstract representations model predicts that, whenever the problem presented at test is mathematically equivalent to the corresponding problem presented at practice, there will be significant positive transafer of learning. When the version of the problem at test, however, is mathematically unique from the corresponding problems at practice, there will be no transfer. This simple model accounts for the majority of the findings in both experiments. The only additional reliable finding was that of a slight increase in RT across a change in operand order (Experiment 1; see also Fendrich et al., in press) and across a change in symbol (Experiment 2). Within the context of the McCloskey et al. (1985) model of number processing, the increased RT in these conditions can be accounted for by longer encoding time for the novel operand order (see the Discussion for Experiment 1).

Future Directions

Despite the evidence supporting the abstract representations model, there are reasons to suspect that this simple model may not be entirely adequate. First, although the slightly poorer performance in the operand order condition for multiplication (Experiment 1), and the symbol condition (Experiment 2) relative to the no-change conditions can be accounted for as an encoding effect, there is no evidence to support directly this account over other potential accounts such as a transformation into the practiced operand order at test, or the existence of more specific, partially overlapping, or distributed, representations in memory. Both of these alternative accounts represent potential challenges to the the abstract representations model, and one aim of future work in this area should be to differentiate among these accounts. One prospective experiment in this direction would involve training subjects extensively on problems presented in one format (e.g., "six times nine equals ?"), and then testing them on both operand orders of the same problems presented in a different format (e.g., "6 x 9 = ?", and "9 x 6 = ?"). This manipulation should equate performance on the two operand orders at test with respect to encoding, as conceptualized under the framework of the McCloskey et al. (1985) model. Any differences in RT at test would thus be attributable to processing in the retrieval network itself, or, perhaps, to utilization of a canonical form of representation that develops with practice.

There was also some evidence from both experiments that the type of representation assumed in the specific representations model does develop with practice, and does exert some secondary influences on the retrieval process. This was suggested by the trend toward better performance across a change in operation than across either a change in both operation and operand order (Experiment 1), or across a change in both operation and symbol (Experiment 2). Other than sensory specific representations of the type proposed by Campbell & Oliphant (in press), it is difficult to conceive of what factors might underlie these differences. None of these effects, however, reached the .05 significance level, and further studies with more statistical power will have to be conducted to confirm these patterns.

An additional issue which has not been addressed is whether there was any positive transfer to the test conditions in which the problems presented at test were mathematically unique from the corresponding problems presented at practice (i.e., the operation change conditions for all experiments, and also the operand order change condition for division in Experiment 1). While it is clear that performance in these conditions was much worse than performance in the mathematically equivalent test conditions, it is less clear whether performance in these conditions was at all facilitated (or perhaps worsened) by practice. As a very rough index of this, RT for multiplication problems in the worst multiplication condition (the operation plus operand order condition) on the first block of the immediate test in Experiment 1 was roughly equivalent to RT for multiplication on the firsat block of practice. For division in Experiment 1, RT in the worst test condition (again, the operation plus operand order condition) on the first block of the immediate test was about 150 ms slower than RT on the first block of practice. This suggests no transfer, or at best minimal transfer of learning to these test conditions, a finding generally consistent with the abstract representations model. To address this issue more rigorously, however, an additional control is needed in which some problems are not seen during practice in any form. These problems could then be presented as both multiplication and division problems at test, and serve as a baseline along which to evaluate performance in each of the test conditions.

Conclusion

Some of the most exciting recent work in mental arithmetic has been the development of several computational models (e.g., Anderson, et al., in press; Campbell & Oliphant, in press; McCloskey & Lindemann, in press) intended to account for the well established performance phenomena, such as the finding that most errors are table-related and close in magnitude to the correct answer. One consequence of efforts to develop such models is that detailed assumptions of representation, which in earlier verbal descriptions of models (e.g., Campbell & Graham, 1985) were vague or entirely ignored, must be made explicit. These explicit assumptions about representation promise to make these and future models of arithmetic performance more testable than earlier models, and investigations of practice and transfer should prove valuable in this respect. As a case in point, the current studies appear to refute the basic assumptions about representation in one of the new computational models, the Campbell & Oliphant (in press), model. Instead, the results can be accounted for, to a first-order approximation, by the abstract representations model proposed in the introduction of this paper. The follow-up research outlined above, and other investigations into practice and transfer, should be instrumental in further elaborating, qualifying, or perhaps ultimately rejecting this model.

APPENDIX A

SUPPLEMENTARY ANALYSIS:

WHICH LAW OF LEARNING IS MOST POWERFUL?

In preceding analyses, it was demonstrated that a simple power law (that is, a power law assuming no previous learning and a value of zero for asymptotic RT, see Newell & Rosenbloom, 1981) provides a good fit to the practice data for all problem types in both Experiment 1 and Experiment 2. The good fits to the simple power law, however, do not show that this family of functions is better than other families that could be applied to the data. Specifically, the exponential family could conceivably fit the data as well as, or better than, the power law. To test this, a three parameter exponential function having the form, RT = A+B*EXP(-D*block), and a four parameter power law function having the form, RT = A+B*(block + C)**(-D), were fit to the practice data for problems with double-digit products in Experiment 1 (separately for multiplication and division) using the Gauss-Newton nonlinear regression technique. In both of the equations above, the parameter A represents the asymptotic RT (the RT that would be predicted with infinite practice), B represents the total amount of improvement in RT that would be predicted with infinite practice, and C describes the rate of learning. The parameter D in the power law equation allows previous learning experience to be estimated (see Newell and Rosenbloom, 1981, for a more detailed discussion of exponential and power law functions). The best fitting parameter estimates from the regression analyses employing these models, and the corresponding r^2 values, are shown in Table 6 for both multiplication and division problems. Two results from Table 6 indicate that the power law provides a better fit to the data than does the exponential law. First, the r^2 is larger (although not overwhelmingly so) for the power law for both multiplication and division problems. This larger r^2 value must be weighted against the fact that the power law function has one more free parameter than does the exponential function. This additional parameter, however, allowed the power law function to account for substantially more variance than could be accounted for by the exponential functions; the associated F values were; $F(1,36) = 30.0$, for multiplication, and $F(1,36) = 34.6$, for division. A second factor supporting the power law is that, for both multiplication and division, several of the RT values actually observed were smaller than the asymptotes predicted by the exponential equation. For multiplication, 8 of the 9 observed mean RTs in the last 9 blocks of practice were smaller than the asymptote of 655 ms predicted by the best fitting exponential function. For division, 8 observed RTs (including 7 of the last 9) were smaller than the predicted asymptote of 764 ms. In contrast, the asymptotes predicted by the power law function were well below the minimum RT value in the data (by 300 to 400 ms). The clearly invalid asymptotes predicted by the exponential function speak against that function as a candidate law of learning in skilled arithmetic.

Some additional evidence in favor of the power law is provided by visual inspection of the plots of the optimal fits for the exponential and power law functions. This approach is complicated somewhat by the fact that there is no transformation space in which the three parameter exponential function or the four parameter power law function plot linearly. It is desirable to have linear plots because systematic deviations of the data from the predicted function are easier to spot when the predicted function is linear. Fortunately, a two parameter version of the exponential function of the form RT = B*EXP(-D*block), when restated in logarithmic form, log(RT) = log(B) - D*(block), does plot linearly. Also, a two parameter version of the power law function of the form RT = B*block**(-D), when restated in logarithmic form, log(RT) = log(B) - D*log(block), plots linearly. The exponential function, then, can be made to plot linearly when A, the parameter for the asymptote, is zero. Similarly, the power law function can be made to plot linearly when both the parameter for A, and the parameter for previous learning, C, are zero. Thus, by simply subtracting the best fitting value for A from the RT data, adding the value for D (in the case of the power law fit) to the independent variable (block), and making the appropriate log transformations, the data can be plotted in a form such that the best fitting exponential and power law functions correspond to simple linear regression curves.

Figure 6 shows the practice data, with the asymptotic RT predicted by the exponential function subtracted, plotted in exponential space. Figure 6 a shows multiplication problems, and Figure 6 b shows division problems. The best fitting two parameter exponential functions are also shown. For both multiplication and division, the observed values are first slightly higher than the values predictions by the best fitting exponential function, then slightly lower, and then perhaps slightly higher again. This is exactly the pattern of deviations that Newell and Rosenbloom (1981) showed to be characteristic of data which follow a power law, when plotted in exponential space. Compare these results with the data (with the appropriate values for the asymptotes subtracted) plotted in the best fitting power law space, as shown in Figure 7 a (multiplication) and b (division). Here, there is clearly less systematic deviation from the predicted linear function.

One complication which was encountered in plotting the best fitting exponential function in linear form should be noted. Because the asymptotes predicted by the exponential function for both multiplication and division problems were actually larger than some of the observed data points, subtracting the value for the asymptotes from the data resulted in negative RT values in some cases. Negative RT values are undefined in the log transformation spaces that were used in the plots. Thus, to allow the data to be plotted in a log scale, the actual value for the parameter A that was subtracted from the data was 100 ms smaller than the smallest observed RT for multiplication and division, respectively. A consequence of this is that the spaces in which the multiplication and division data are plotted in Figure 6 are not actually the best fitting exponential spaces. They can be considered approximations to the most reasonable space, however, for at least three reasons. First, as discussed, the best fitting exponential space predicts an asymptotic RT which is larger than some of the observed RTs, which must clearly be incorrect. Second, by subtracting from the data the value of the smallest RT minus 100 ms, an implicit assumption is being made that, with infinite practice, RT will improve by no more than 100 ms over the best observed performance over 40 blocks of practice. If anything, this is likely to be an underestimate of the asymptotic RT. Finally, additional plots of the data for which the value subtracted as the asymptote was the value of the smallest observation minus 10 ms also showed essentially the same pattern of deviation as shown in Figure 6 a and b.

Discussion

While not overwhelming, the evidence above clearly points toward the generalized power law over the exponential function as a law of speed-up with practice in skilled mental arithmetic. This finding replicates many findings supporting the power law for tasks other than arithmetic (see Newell & Rosenbloom, 1981). A unique aspect of these analyses is that they involve data from what can be thought of as a retraining task. Presumably, many years preceding these experiments, subjects learned multiplication and division skill to a relatively high level of proficiency. The current findings thus extend the applicability of the power law into contexts where additional practice is given on a previously well learned skill. It is worth noting, however, that the estimated value for the parameter C (the amount of previous learning) was zero for both multiplication and division (the value of C was constrained in the model fitting process so that it could not be less than zero). This result is odd considering that these simple arithmetic problems were likely solved by subjects several hundred (perhaps several thousand) times preceding the experiment. There are, however, at least three factors that might account for this anomaly. First, the improvement in initiate RT (the RT value used in these analyses) partly reflects improvement in motor response RT. It is reasonable to assume that the amount of previous learning for this component of the task is minimal, perhaps zero. As was previously shown, however, only a modest proportion of the overall speed-up in initiate RT can be attributed to motor response speed-up (see Chapter 2). Thus, there would appear to be other factors underlying the prediction of zero previous learning. Another possibility is that a significant amount of improvement in RT reflects improvement that is specific to idiosyncratic aspects of the task environment (e.g., getting familiar with the computer display, overcoming nervousness, etc.). Again, the amount of previous learning for these factors could plausibly be zero. Finally, if a significant amount of the learning which occurred during practice was actually relearning of a partially forgotten skill, and if relearning is faster than initial learning, then the rate of speed-up in the early portion of the learning curve would be artificially high, and this would attenuate the estimate for C, the amount of previous learning. Further empirical work would be needed to differentiate among the above possibilities.

The above results also have some direct relevance for modeling efforts in skilled arithmetic. Most of the models developed to date (e.g., Campbell & Oliphant, in press; McCloskey et al., in press) focus on performance, and do not directly address issues of practice. It is pervasively true in these models, however, that practice is assumed to strengthen connections among relevant representations. Ultimately, these and any future models will need to incorporate detailed learning algorithms which explicitly define the rate at which connections are strengthened with practice, and the impact that this stregthening of connections has on RT. The demonstrations of power law speed-up with practice in these experiments provide a useful empirical constraint for efforts to develop such models.

APPENDIX B

SUPPLEMENTARY ANALYSIS:

CONFUSION ERRORS AT TEST IN EXPERIMENT 1

Campbell (1987a) pretested college subjects on single-digit multiplication problems (2x2 to 9x9), gave the subjects extended practice on a subset of these problems, and then tested them again on all problems. Among the findings were that RT to unpracticed problems at posttest was slower (the error rate higher) than for these same problems at pretest. Also, errors made on unpracticed problems at test were predominantly answers to problems seen at practice. This result can be taken as support for the claim that skilled multiplication reflects retrieval of facts from a memory network in which multiple problem representations become active on each retrieval attempt. When a problem receives extended practice, the associations among the representations corresponding to that problem become stronger. Because of this fact, the practiced problems create even more interference than usual (i.e., than would have been the case without practice) when unpracticed problems are encountered at post test. This increased interference results in slower RTs and increased errors on unpracticed problems at post test relative to pretest. Further, the errors to unpracticed problems at post test would be expected to be predominantly answers to problems seen at practice. In this chapter, similar error analyses on the Experiment 1 immediate and delayed test data are reported in an effort to: (a) replicate the basic effect (which will be termed the practice error effect) for multiplication found in the Campbell (1987a) study, (b) evaluate whether an analogous effect can be found for skilled division, and (c) establish whether these effects are stable across significant delays between practice and test.

In the Campbell (1987a) study, the practice error effect was computed by comparing the percentage of errors on unpracticed problems at post test that were answers to practiced problems, and comparing this with the percentage of errors that would be expected by chance to be answers to practice problems. Campbell (1987a) found that 65% of errors were answers to practiced problems, whereas only 28% of errors were answers to other unpracticed problems. Because of the approach Campbell used to calculate the practice error effect, the percentage above reflects a combined effect across table-related, table-unrelated, and miscellaneous errors (errors that are not valid multiplication answers, e.g., 17). An even larger effect might be obtained if analyses were limited to table-related errors, because representations for all table-related problems are believed compete most strongly with one another (see Campbell and Graham, 1985), and table-related errors to unpracticed problems may thus be more likely than either table-unrelated or miscellaneous errors to reflect enhanced interference from the practice (i.e., answers to table-related practice problems may be most likely to intrude as errors at test). This is the approach taken in the present analyses. As will become clear later, an additional advantage of this approach is that it allows an effect analogous to the multiplication practice error effect to be computed for division errors.

Multiplication Errors

Error analyses on both the immediate and delayed test results were restricted to the operation change and the operation plus operand order change conditions for two reasons. First, there were relatively few errors in the no-change and operand order change conditions, making meaningful analysis of these conditions difficult. Second, operation and operation plus operand order conditions provide the most promising data for eliciting the practice error effect, because these problems were not seen as multiplication problems during practice. Thus the correct answers to these problems were not strengthened with practice, making these problems relatively more susceptible to interference from problems that were seen at practice. In other words, problems in these conditions are most analogous to the unpracticed problems in the Campbell (1987a) study.

The majority of errors (69%) were table-related, replicating findings from several earlier studies (e.g., Campbell & Graham, 1985). Of primary interest was whether table-related errors are more likely to be answers to problems in a given subject's practice set than would be expected by chance. The chance rate was determined for each error by summing the number of table-related problems that were in the subject's practice set and dividing this value by the total number of table-related problems across all practice sets (recall that all problems from all practice sets were seen by each subject at test). Operand order was ignored in these calculations. The results, aggregated across the operation and the operation plus operand order test conditions, are shown on the left side of Figure 8. On the immediate test, 91% (40 of 44) of the table-related errors were answers to problems in the subject's practice sets, this compared with the 54% that would be expected by chance. In order to analyze the significance of this effect, it was necessary to adjust for non-independence caused by occurrence of the same error for a given subject in both blocks of the immediate test. This was done by simply removing the redundant errors that occurred in block two (there were 5 of these) from the analysis. Even in this conservative test, the effect was highly significant, chi^2(1, N = 39) = 20.1, $p < .001$. Analysis of errors at retention yielded analogous results. Of the 74 table-related errors across the four blocks at retention, 58 (78.4%) of them were answers to problems in the subject's practice sets, compared to 55% as would be expected by chance. With the 12 redundant errors removed, this effect was also significant, chi^2(1, N=39) = 9.34, $p < .01$.

Division Errors

Errors in all test conditions except the no-change condition were analyzed for division. As with multiplication, there were few errors in the no-change condition, making meaningful analyses of this condition difficult. In contrast to multiplication, however, there was a large number of errors in the operand order change condition, and this condition was thus included in the analyses. Problems were divided into two groups for analysis. The first group included problems for which the practice error effect could be mediated by way of either the product or the presented operand.[1] This includes all problems in the operand order test condition. For example, if 21 = _ x 3 was seen seen during practice, the operand order test condition for this problem would be 21 = _ x 7. According to some arithemtic models (e.g., Campbell & Graham, 1985), an association should have been strengthened during practice linking the 21 and the 7 (the answer to 21 = _ x 3), and this might result in 7 being a frequent mistake on 21 = _ x 7 test. Thus, errors to problems in this test condition may be product mediated. Also, the 7 may have formed associations with answers to other division problems involving 7 that were seen at practice (e.g., 63 = _ x 7), and these answers (e.g., 9) might also be frequent errors to 21 = _ x 7. Thus, in this group, there may be both product mediated and operand mediated interference from practice problems.

The second grouping for this analysis consists of all remaining errors to problems in the operation and operation plus operand order test conditions. For this group, interference from practice problems could only be operand-mediated, since the product of problems in these test conditions was not seen in the context of any problem at test.

This separation of division errors into two groups was motivated by the possibility that interference from practice could occur by way of both the product and the presented operand for group 1 problems, but could occur only by way of the presented operand for group 2 problems. This could result in differences in the magnitude of the practice error effect for these two groups of problems.

There were 51 and 85 errors that fell in the first group on the immediate and delayed test, respectively. On the immediate test, 34, or 67%, of these errors were answers to problems in the subjects' practice sets. Nineteen of these errors were traceable to the subject's practice sets by way of the product (i.e., 19 of the errors were answers to the reversed operand order problem seen during practice), an 15 of the errors were traceable by way of the presented operand (i.e., 15 of the errors were answers to problems seen during practice that had the presented operand). This value of 67% compares to the 28% that would be expected by chance for this group (see Figure 8). After eliminating the 6 redundant errors, 28 of the 44 remaining errors were answers to problems in the subjects' practice sets. This was significantly greater that chance expectation, chi^2(1, N=44) = 27.7, p < .01. On the delayed test, 58 of 85 errors (68%) were answers to problems in the subjects' practice sets. Thirty-one of the these errors were traceable by way of the product, and 27 of them were traceable by way of the presented operand. After adjustment for redundancy (11 instances), this proved significantly different from chance (chance rate on the delayed test was also .28), chi^2 (1, N = 74) = 46.5, p < .001.

For the second group of error instances (see Figure 8), only the presented operand could be responsible for a practice error effect. On the immediate test, 9 of 46 (20%) of these errors were answers to problems in the subjects' practice sets having the presented operand. The chance rate for these problems was .23. A chi square analysis showed no significant difference chi^2 (1, N=46) = .306, n.s. On the delayed test, there were 70 errors, 11 of them answers to problems in the subjects' practice sets having the presented operand. The chance rate for these problems was .20. Again, there was no significant difference, Chi^2 (1, N=70) = .736, n.s.

Discussion

The practice error effect previously found for multiplication (Campbell, 1987a) was replicated in these analyses. Indeed, by limiting analyses to errors that were table-related, the effect was even more pronounced that the found by Campbell (1987a). Results for Group 1 division problems (where errors could be either product or operand mediatied) also showed a strong practice error effect on both the immediate and delayed tests. There was no evidence of a practice error effect, however, for Group 2 division problems (where the practice error effect could be only operand mediated). It would not have been surprising to find a smaller effect for Group 2 problems than for Group 1 problems, sbecause only the presented operand could contribute to the effect for Group 2 problems. However, the finding of absolute no practice error effect for Group 2 problems is somewhat odd considering therelatively strong effect found for Group 1 problems. A possible explanation for this finding is that there is a greater role played by the product (relative to the presented operand) in division fact retrieval. This greater influence of the product could be related to the fact that a given product was seen in at most one division problem during practice. In contrast, each operand was seen in many multiplication and divisions problems during practice. Because of this difference, the product was a deterministic, rather than a probabilistic, predictor of the answer. Thus, at test, the product may have had strong enough associations to answers with which it was paired at practice for these answers to intrude as errors. In contrast, associations formed during practice between the presented operand and answers may have been too weak to elicit a detectable practice error effect. This account, however, is inconsistent with the fact that a large portion of the practice error effect for Group 1 problems involves operand mediated errors. This account is also inconsistent with the findings of a strong practice error effect for multiplication problems. An alternative and perhaps more likely account for the null practice error effect for Group 2 problems is that for these problems, the product was never seen as an input element during practice. Thus, at test, the novel product could have signaled to the subject a novel problem, allowing the subject to suppress any answers that were associated with the presented operand during practice, and producing the null practice error effect for Group 2 problems. Further investigation of the practice error effect for division would be necessary to differentiate between the explanations proposed above.

The finding of a practice error effect for Group 1 division problems provides one of the first direct sources of evidence for interference effects in skilled division analogous to the well established interference effects for skilled multiplication (see also Campbell, 1985). A second new finding in these analyses is that the practice error effect is stable across an interval of one month, indicating that the effect cannot be accounted for by some form of short-term priming, but rather reflects stable long-term changes to the memory structure resulting from practice.

[1]Some problems in the operation transfer conditions were also potentially subject to interference by way of the product and the presented operand, and thus fall under the first group: There are three pairs of number relations used in the study for which the product is the same for both pairs: (2,9,18) and (3,6,18); (3,4,12) and (2,6,12); and (3,8,24) and (4,6,24). One number relation from each pair was presented as a multiplication problem at practice, and the other as a division problem. For example, in one practice set, 18 = _ x 3 and __ = 9 x 2 were presented. Thus, in the operation transfer condition for division for subjects having that practice set, 18 = _ x 2 would be presented. The 6 is a potential product mediated answer here due to the associations that may have formed between 18 and 6 during practice. Thus, errors to these problems were included in the group 1, rather than the group 2, analysis.

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Table 1

Examples of a multiplication and a division problem at practice and the corresponding test conditions in Experiment 1

	Test Condition			
Practice	No change	Operand-order	Operation	Operation & Op-order
__ = 4 x 7	__ = 4 x 7	__ = 7 x 4	28 = _ x 7	28 = _ x 4
48 = _ x 6	48 = _ x 6	48 = _ x 8	__ = 8 x 6	__ = 6 x 8

Table 2

A sample practice set (practice set 1) used in Experiment 1

Multiplication	Division
__ = 6 x 9	36 = _ x 4
__ = 7 x 9	27 = _ x 9
__ = 4 x 8	48 = _ x 8
__ = 8 x 7	42 = _ x 7
__ = 8 x 9	28 = _ x 4
__ = 7 x 5	40 = _ x 8
__ = 6 x 5	45 = _ x 9
__ = 2 x 8	21 = _ x 7
__ = 5 x 4	14 = _ x 7
__ = 3 x 5	10 = _ x 5
__ = 3 x 4	12 = _ x 2
__ = 6 x 3	18 = _ x 9
__ = 8 x 3	24 = _ x 6
__ = 4 x 2	6 = _ x 2
__ = 1 x 6	8 = _ x 8
__ = 9 x 1	7 = _ x 1
__ = 1 x 4	3 = _ x 1
__ = 5 x 1	2 = _ x 1

Table 3

Test conditons for an example multiplication problems on the immediate and delayed tests in Experiment 1

Practice	Imm./delayed tests	Test condition
__ = 4 x 7	__ = 4 x 7	no-change
__ = 7 x 4	__ = 4 x 7	operand-order
28 = _ x 7	__ = 4 x 7	operation
28 = _ x 4	__ = 4 x 7	operand-order & operation

Table 4

Test conditions for an example division problem on the immediate and delayed tests in Experiment 1

Practice	Imm./delayed tests	Test condition
28 = _ x 7	28 = _ x 7	no-change
28 = _ x 4	28 = _ x 7	operand-order
__ = 4 x 7	28 = _ x 7	operation
__ = 7 x 4	28 = _ x 7	operand-order & operation

Table 5

<u>Examples of each of the four problem types seen at practice and the corresponding test conditions in Experiment 2.</u>

	Test Condition			
Practice	No change	Symbol	Operation	Operation & Symbol
__ = 4 x 7	__ = 4 x 7	__ ÷ 4 = 7	28 = _ x 7	28 ÷ _ = 7
__ ÷ 9 = 5	__ ÷ 9 = 5	__ = 9 x 5	45 ÷ _ = 5	45 = _ x 6
48 = _ x 6	48 = _ x 6	48 ÷ _ = 6	__ = 8 x 6	__ ÷ 8 = 6
18 ÷ _ = 3	18 ÷ _ = 3	18 = _ x 3	__ ÷ 6 = 3	__ x 6 = 3

Note: From top to bottom, the four types of problems at practice~~d~~ represented above are multiplication with symbol "x", multiplication with symbol "+", division with symbol "X", and division with symbol "+".

Table 6

Parameters for optimal exponential and power law fits to the practice data for multiplication and division problems with double-digit products in Experiment 1

	Exponential $RT=A+B*e^{(D*block)}$				Power law $RT=A+B*(block+C)^{D}$				
Problem type	A	B	D	r^2	A	B	C	D	r^2
Multiplication	655	567	-.14	.89	296	951	0	-.28	.94
Division	764	770	-.13	.91	123	1453	0	-.23	.96

Note: RT in milliseconds.

Figure Captions

Figure 1. Log RT plotted as a function of log block and operation (multiplication or division). Figure 1 a shows problems with single-digit products, and Figure 1 b shows problems with double-digit products.

Figure 2. Anti log RT (averaged across subjects and correctly solved problems) for multiplication problems on the immediate and delayed tests plotted as a function of test condition. (no change = no change from practice to test; operand order = operand order change from practice to test; operation = operation change from practice to test; operation & op ord = operation plus operand order change from practice to test.) Figure 2 a shows problems with single-digit products, and Figure 2 b shows problems with double-digit products.

Figure 3. Anti log RT (averaged across subjects and correctly solved problems) for division problems on the immediate and delayed tests plotted as a function of test condition. (no-change = no-change from practice to test; operand order = operand order change from practice to test; operation = operation change from practice to test; operation plus op ord = operation plus operand order change from practice to test.) Figure 3 a shows problems with single-digit products, and Figure 3 b shows problems with double-digit products.

Figure 4. Log RT for all correctly solved problems plotted as a function of log block and problem type. (multiplication (x) = multiplication problems with symbol "x"; multiplication (÷) = multiplication problems with symbol "÷"; division (x) = division problems with symbol "x"; division (÷) = division problems with symbol "÷".)

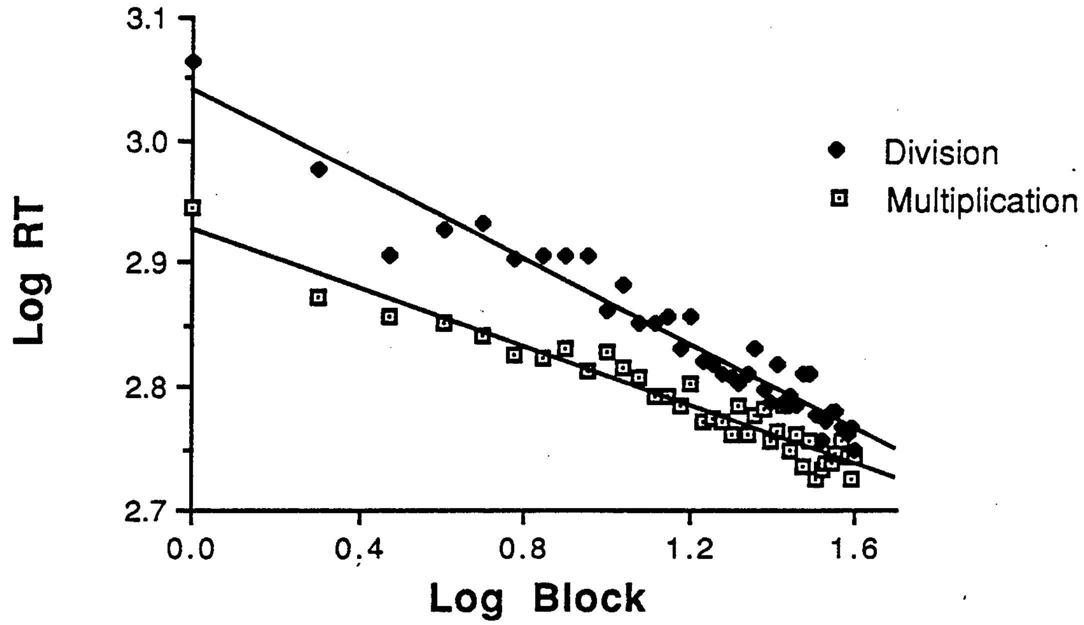
Figure 5. Anti log RT (averaged across subjects, problem type, and correctly solved problems) on the immediate and delayed tests plotted as a function of test condition. (no-change = no-change from practice to test; symbol = symbol change from practice to test; operation = operation change from practice to test; and operation & symbol = operation plus symbol change from practice to test. Figure 5 a shows problems with single-digit products, and Figure 5 b shows problems with double-digit products.

Figure 6. Best fitting exponential functions to the practice data for problems with double-digit products in Experiment 1. Figure 6 a shows the results for multiplication problems, and Figure 6 b shows the results for division problems.

Figure 7. Best fitting generalized power law functions to the practice data for problems with double-digit products in Experiment 1. Figure 7 a shows the results for multiplication problems, and Figure 7 b shows the results for division problems.

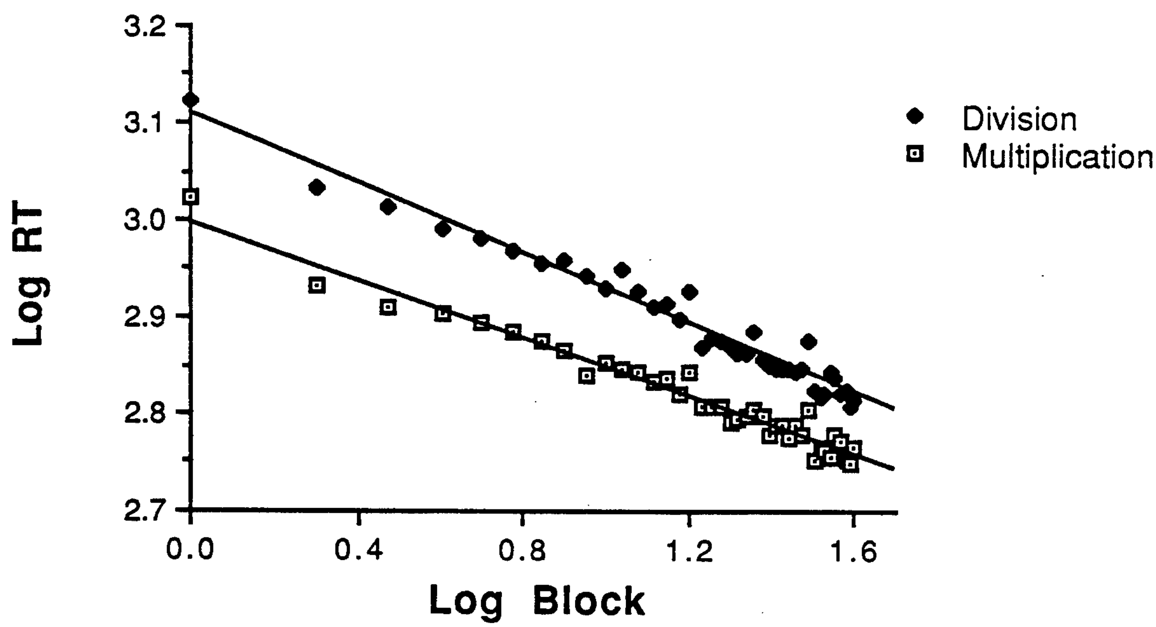
Figure 8. Proportion of table-related errors on the immediate and delayed test that were the answers to problems in the subject's practice set compared to the proportion expected by chance. (Mult imm = multiplication problems on the immediate test; Mult delay = multiplication problems on the delayed test; Div imm Gp 1 = group 1 division problems on the immediate test; Div delay Gp 1 = group 1 division problems on the delayed test; Div Gp 2 = group 2 division problems on the immediate test; Div delay Gp 2 = group 2 division problems on the delayed test.)

a.



3.1
3.0
2.9
2.8
2.7
Log RT
0.0
0.4
0.8
1.2
1.6
Log Block
Division
Multiplication

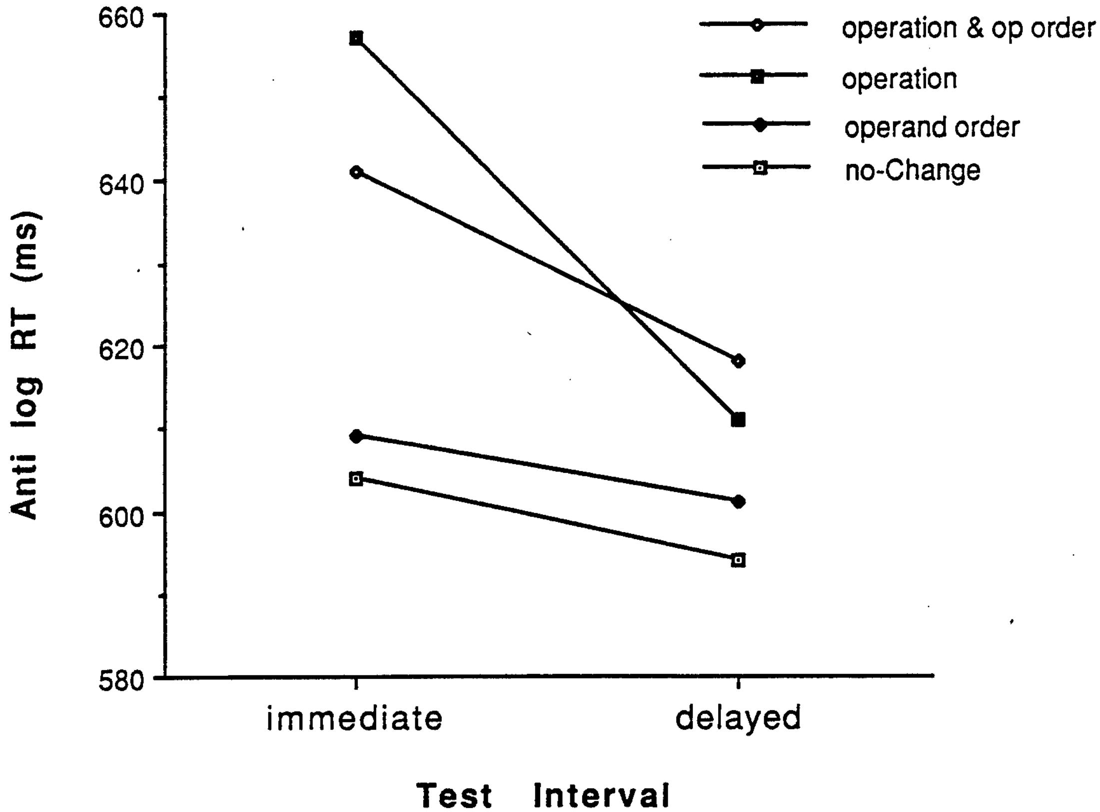
b.



3.2
3.1
3.0
2.9
2.8
2.7
Log RT
0.0
0.4
0.8
1.2
1.6
Log Block
Division
Multiplication

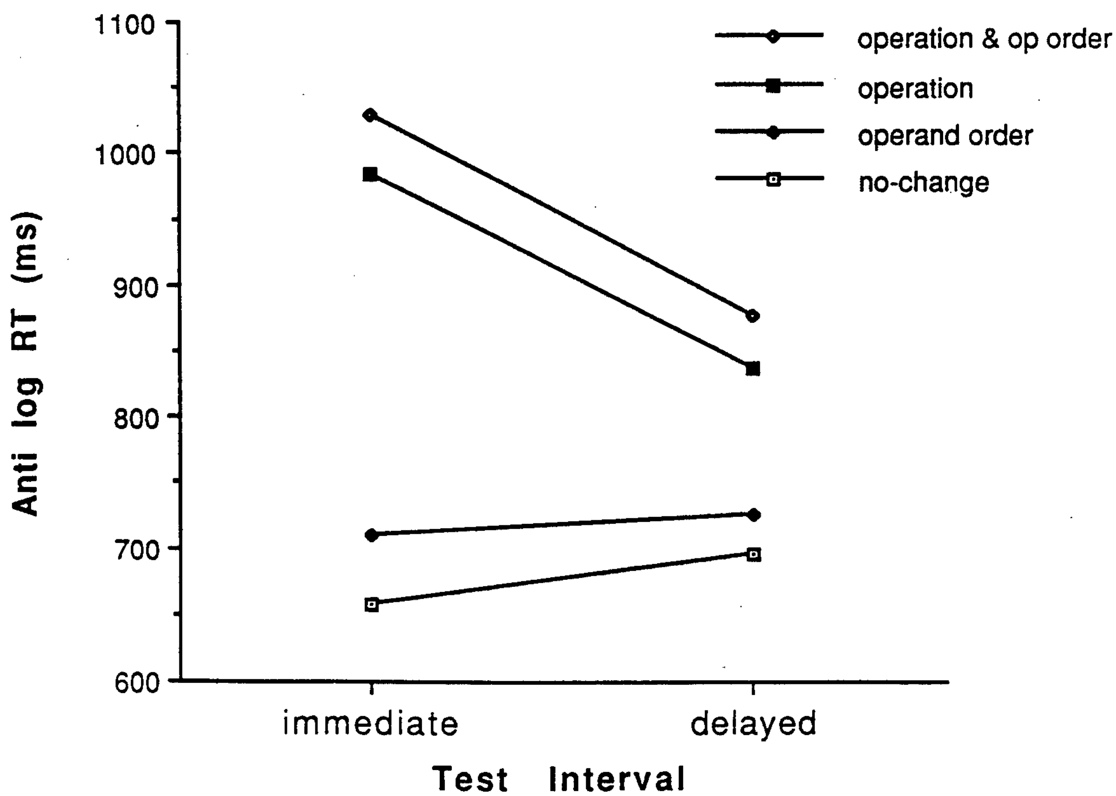
Fig 1

a.



660
640
620
600
580
Anti log RT (ms)
operation & op order
operation
operand order
no-Change
immediate
delayed
Test Interval

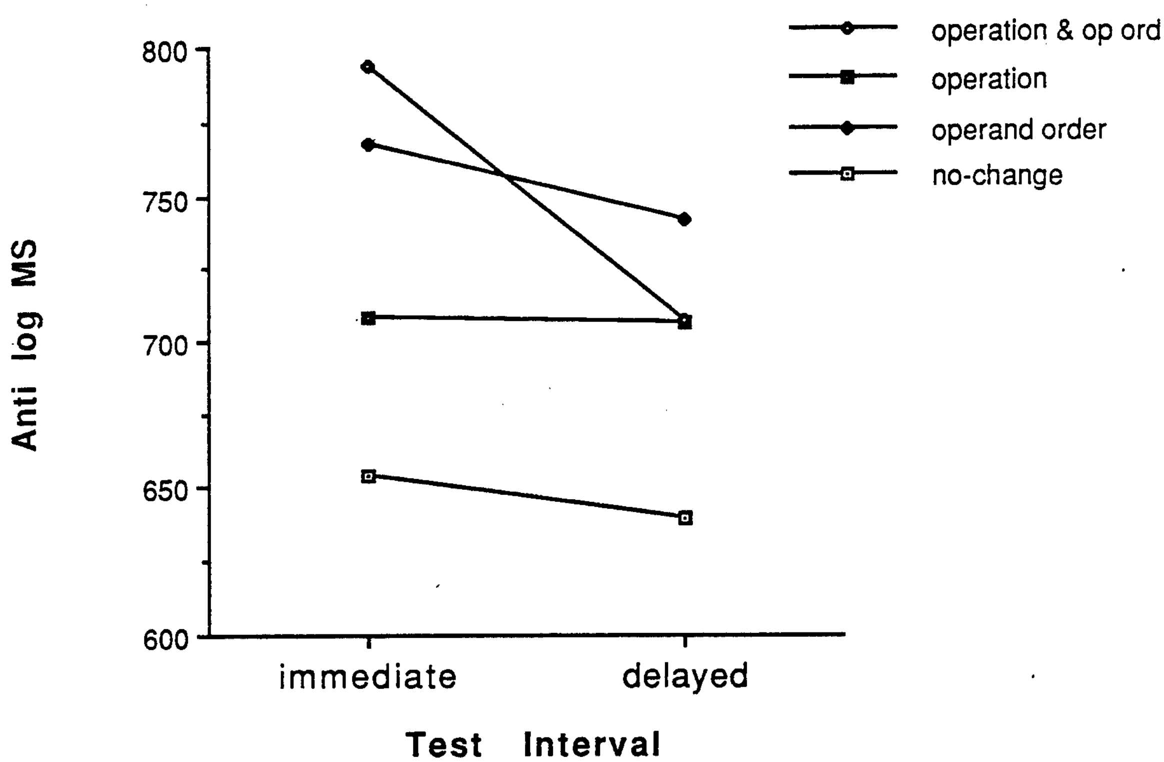
b.



1100
1000
900
800
700
600
Anti log RT (ms)
operation & op order
operation
operand order
no-change
immediate
delayed
Test Interval

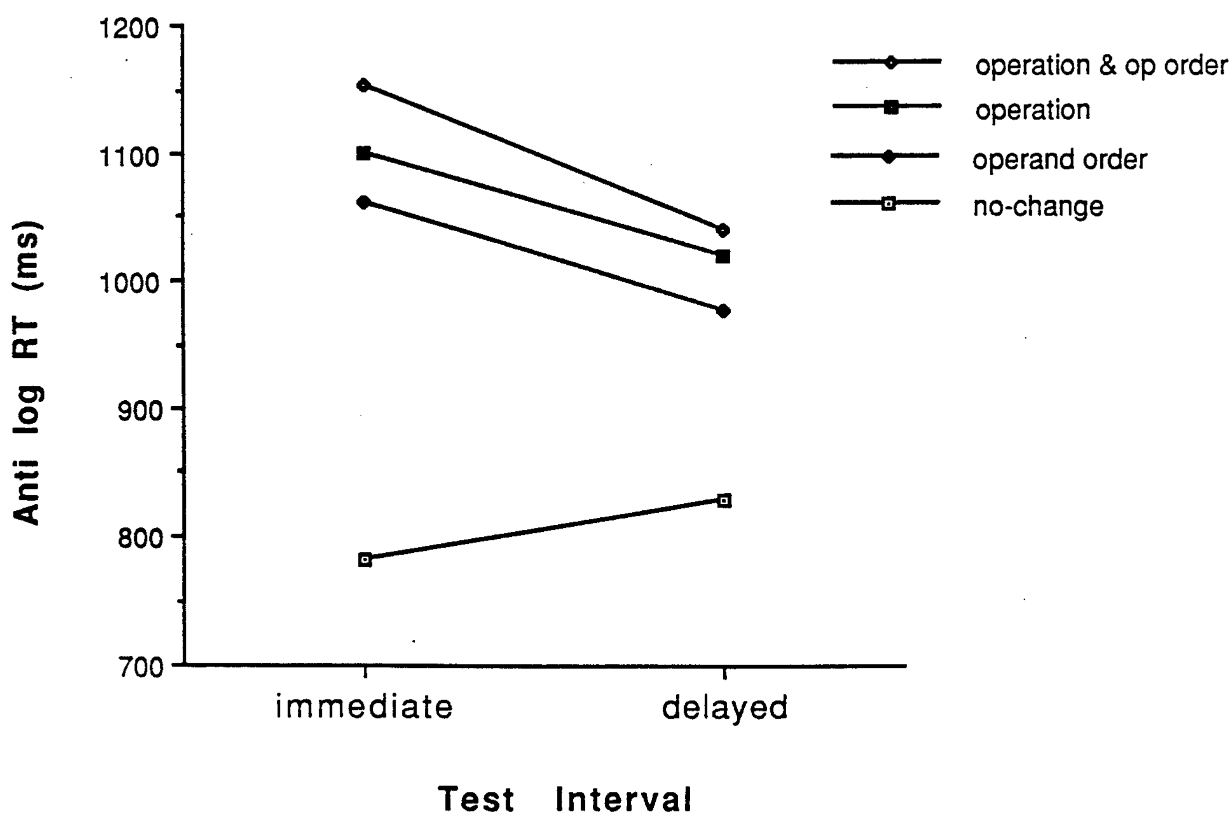
Fig 2

a.



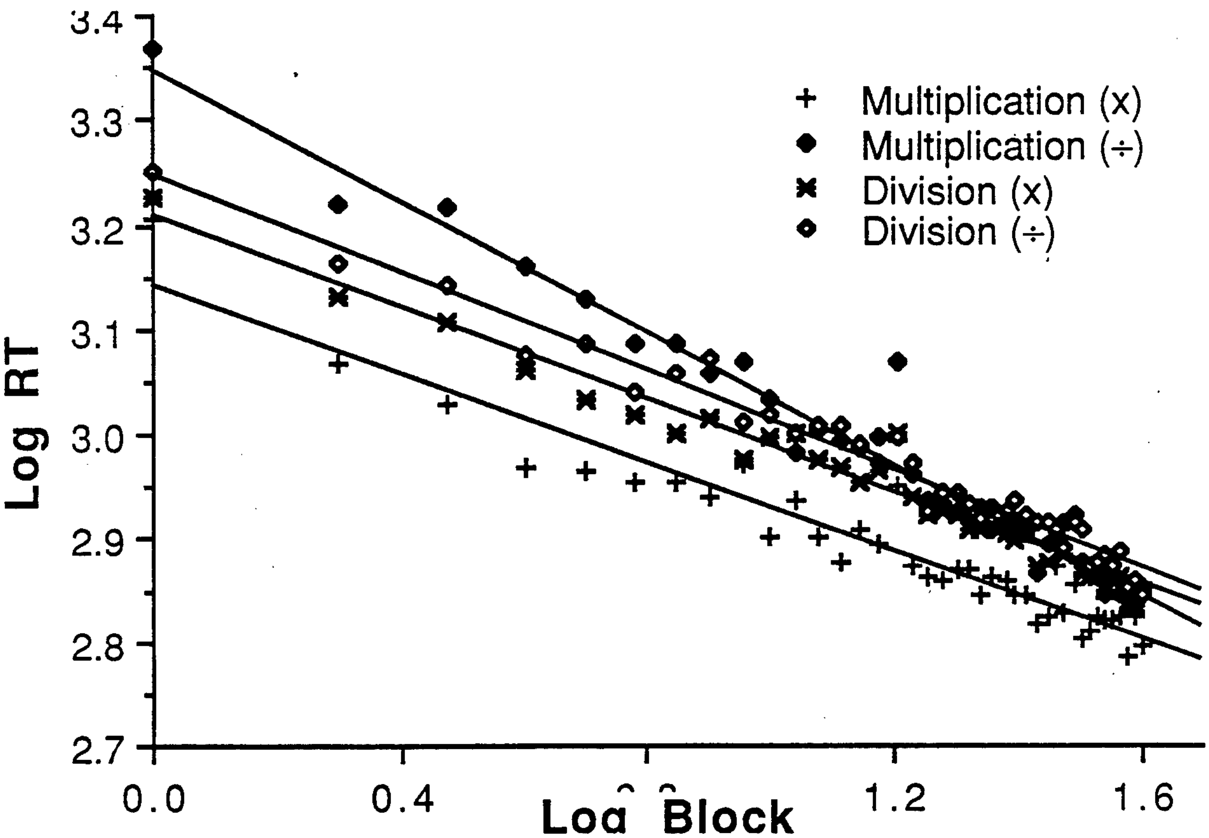
operation & op ord
operation
operand order
no-change
800
750
700
650
600
Anti log MS
immediate
delayed
Test Interval

b.



operation & op order
operation
operand order
no-change
1200
1100
1000
900
800
700
Anti log RT (ms)
immediate
delayed
Test Interval

Fig 3



3.4
3.3
3.2
3.1
3.0
2.9
2.8
2.7
Log RT
0.0
0.4
1.2
1.6
Log Block
Multiplication (x)
Multiplication (÷)
Division (x)
Division (÷)

Fig. 4

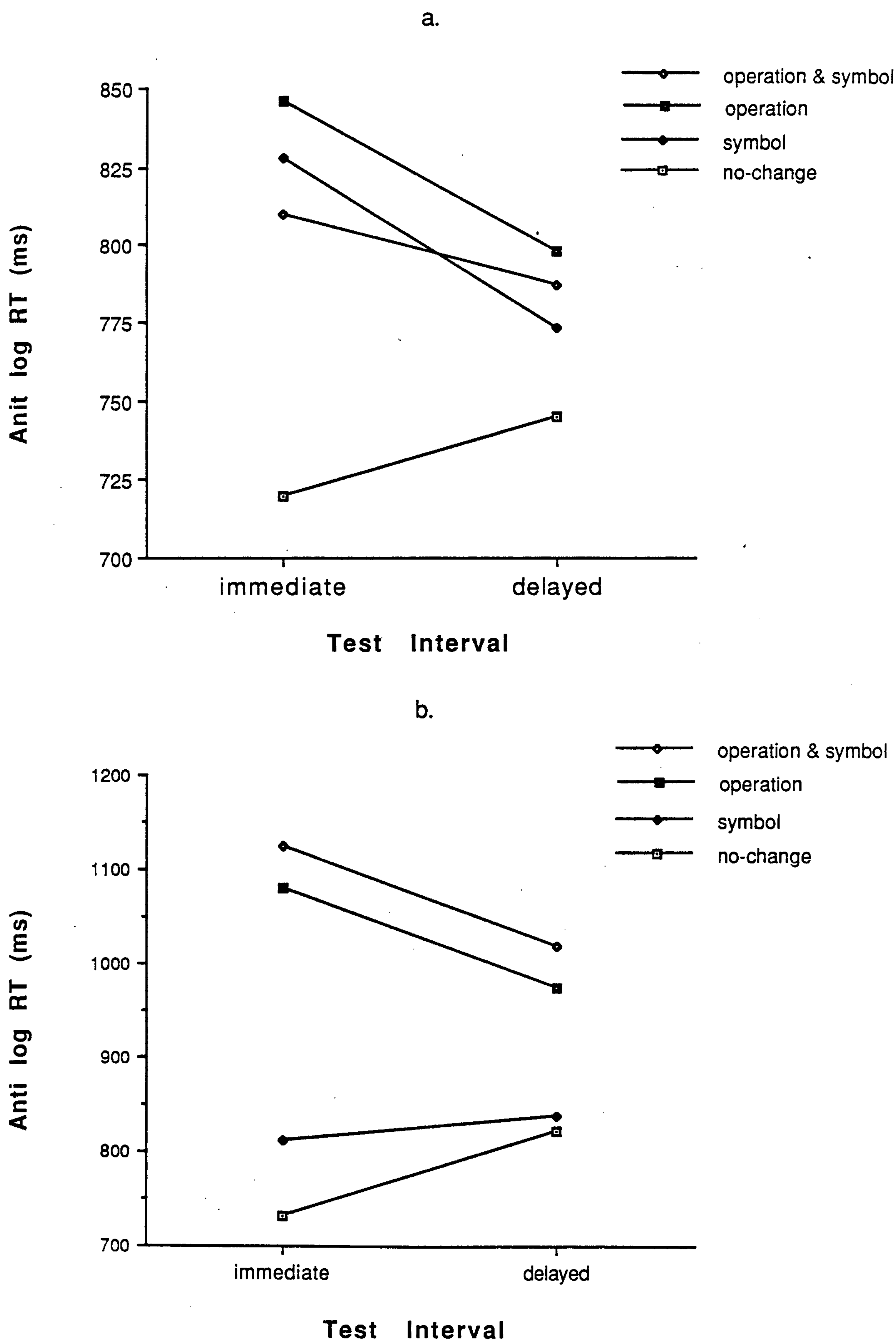
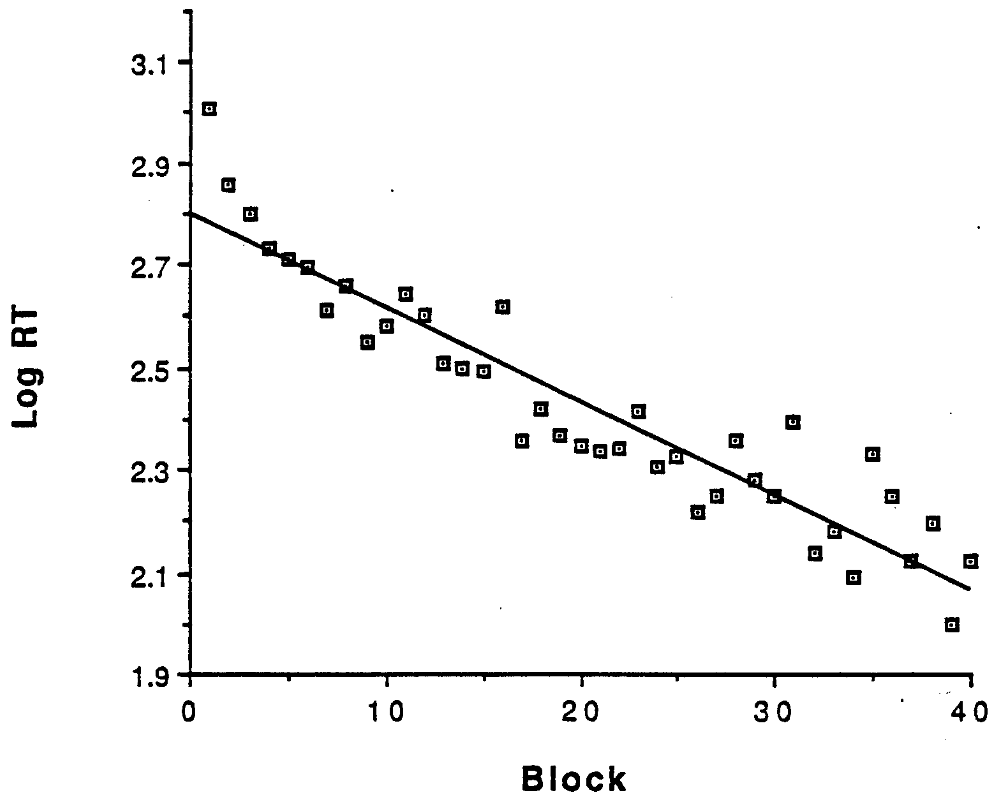

a.
operation & symbol
operation
symbol
no-change
850
825
800
775
750
725
700
Anit log RT (ms)
immediate
delayed
Test Interval
b.
operation & symbol
operation
symbol
no-change
1200
1100
1000
900
800
700
Anti log RT (ms)
immediate
delayed
Test Interval

Fig. 5


3.1
2.9
2.7
2.5
2.3
2.1
1.9
Log RT
0
10
20
30
40
Block

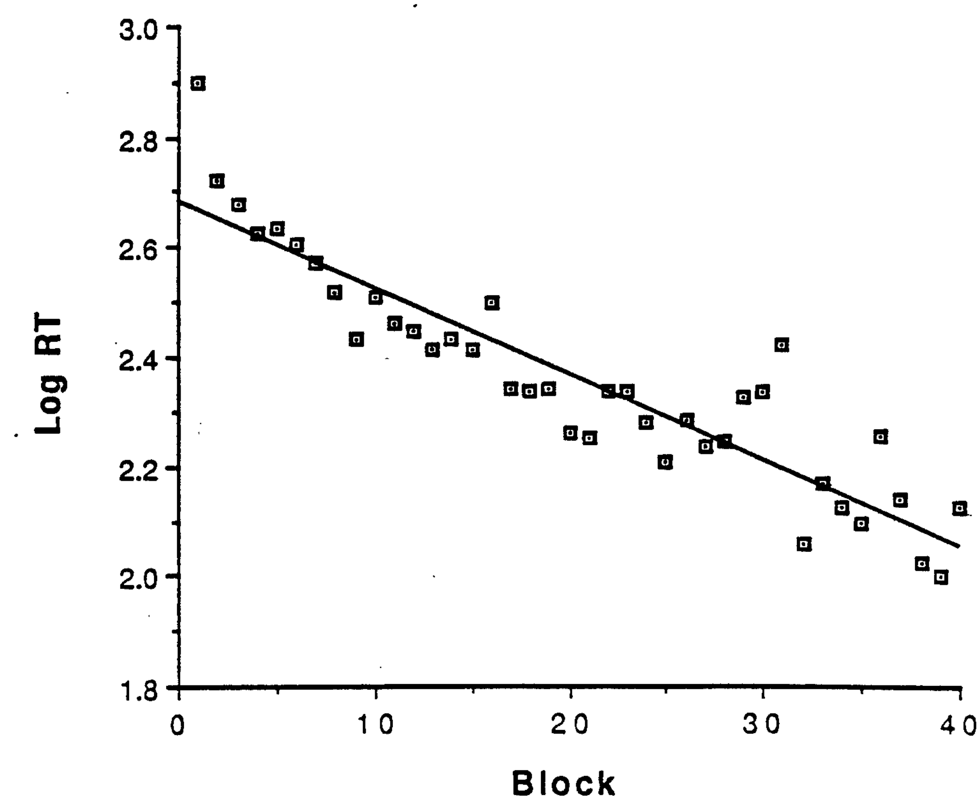
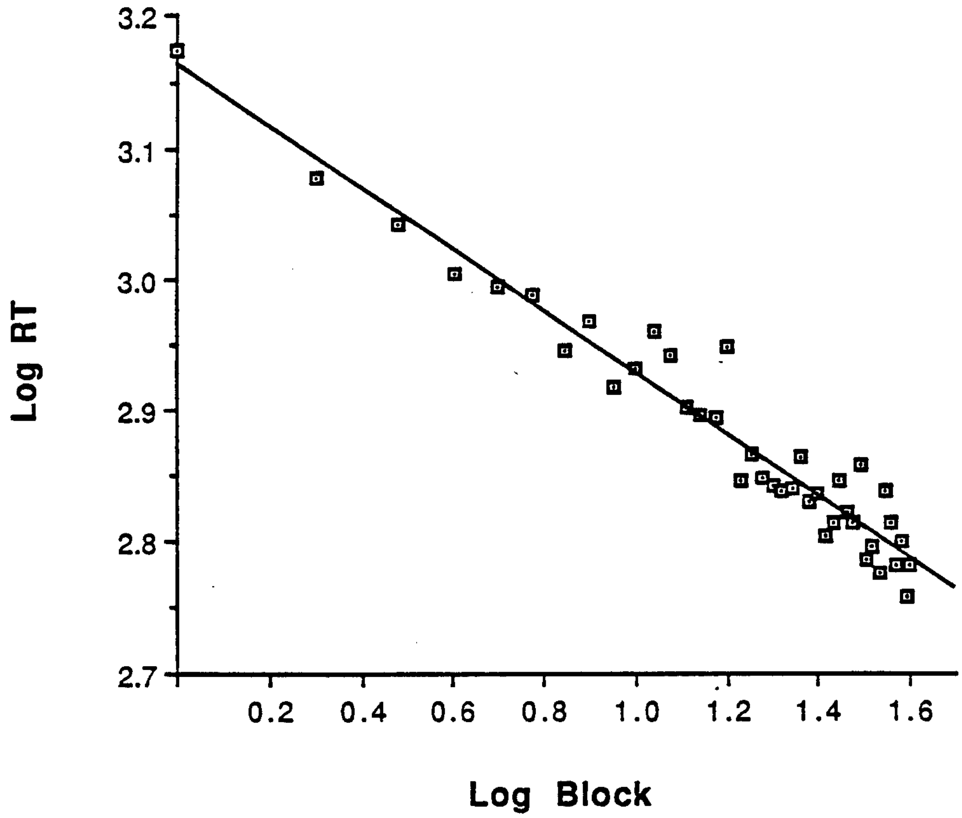

3.0
2.8
2.6
2.4
2.2
2.0
1.8
Log RT
0
10
20
30
40
Block

Figure 6


3.2
3.1
3.0
2.9
2.8
2.7
Log RT
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
Log Block

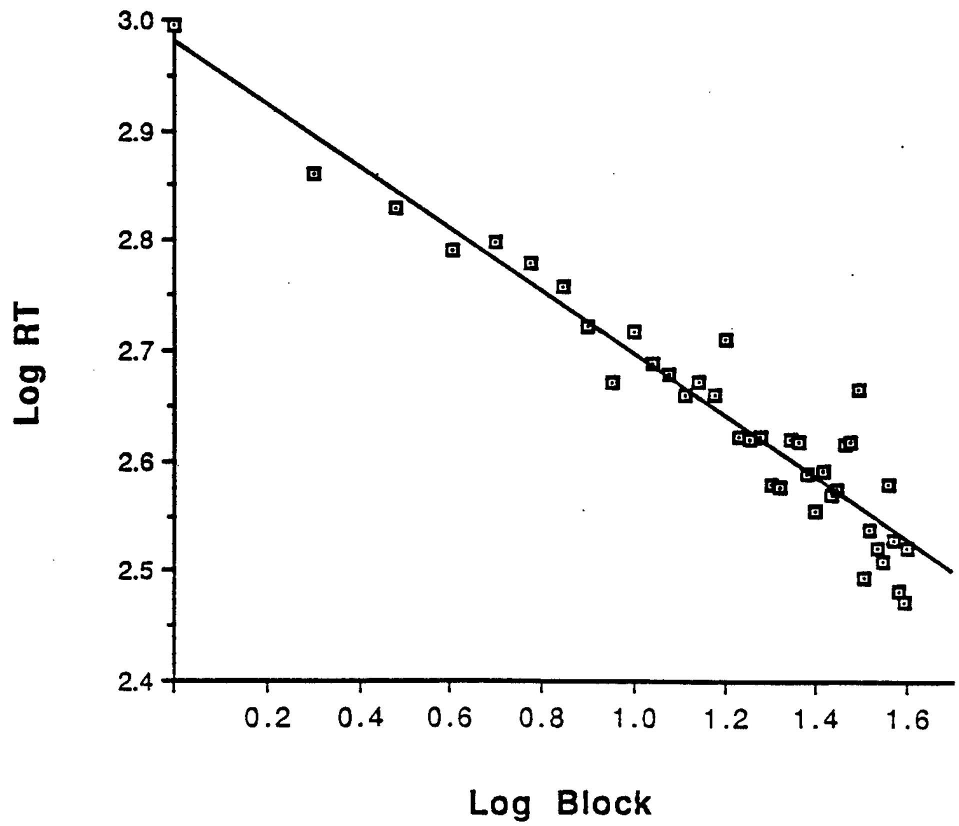

3.0
2.9
2.8
2.7
2.6
2.5
2.4
Log RT
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
Log Block

Figure 7

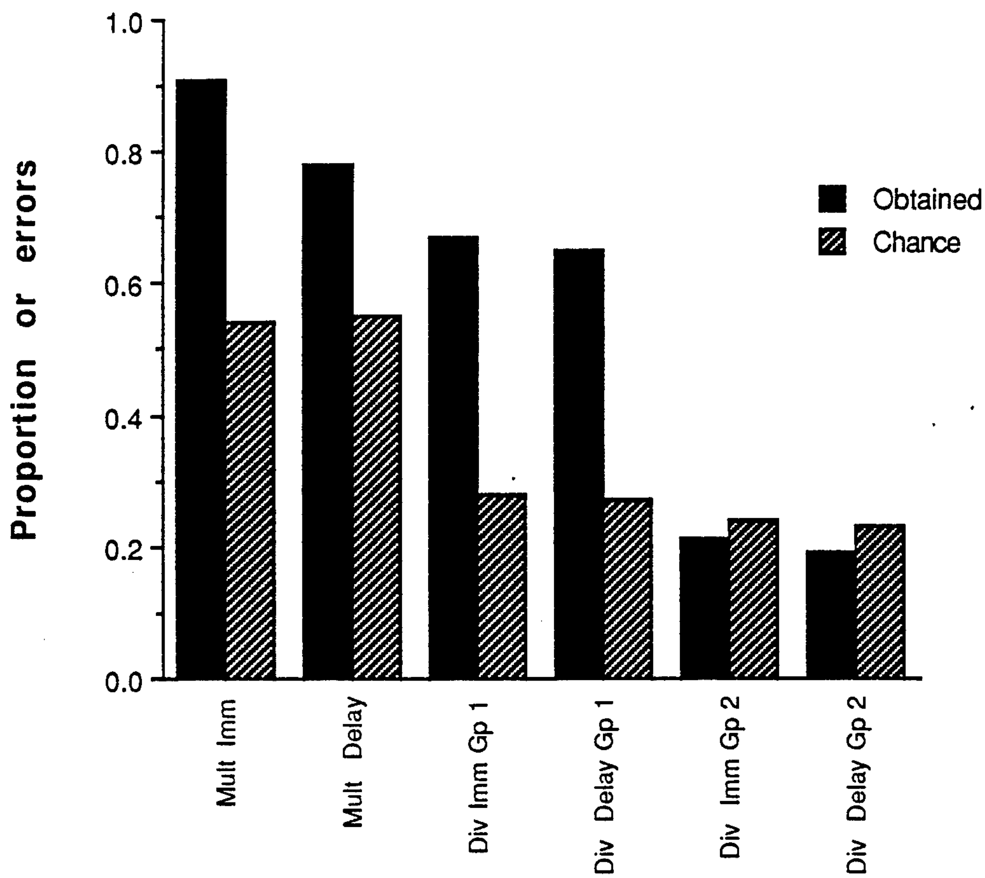

1.0
0.8
0.6
0.4
0.2
0.0
Proportion or errors
Obtained
Chance
Mult Imm
Mult Delay
Div Imm Gp 1
Div Delay Gp 1
Div Imm Gp 2
Div Delay Gp 2

Fig. 8