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Continuous Lattices and Mathematical Morphology

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Abstract

I generalize the topological structure of the concrete forms of mathematical morphology to the lattice-algebraical framework using the theory of *continuous lattices*. I show that when a complete lattice, $\mathcal{L}$, exhibits the dual of the property that defines a continuous lattice, then $\mathcal{L}$ together with a certain intrinsic lattice topology, $\mathrm{m}(\mathcal{L})$, which is related by duality to the *Lawson topology*, has almost all the familiar properties, suitably generalized, of the topologized lattices that constitute the basic mathematical structure of the concrete forms of mathematical morphology; for instance, the complete lattice of closed subsets of the Euclidean plane topologized with Matheron's *hit-miss topology*.

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1. Introduction

This report details the development of mathematical tools intended for use in designing improved or more nearly optimal algorithms for automatic/aided target recognition (ATR) in systems employing such target sensors as synthetic aperture and laser radars (SARS and LADARS), and forward-looking infrared sensors (FLIRS). It specifically concerns the development of such tools within the field of digital image processing and analysis known as *mathematical morphology*, a field that emerged in the early sixties in the *Fontainebleau School* of Serra and Matheron as a bottom-up, hierarchical approach to image analysis. Mathematical morphology has since found numerous practitioners throughout Europe, the United States, and South America, has been successfully applied in such diverse fields as materials science, microscopic imaging, pattern recognition, medical imaging, and computer vision, and is today one of the principal systematic methodologies employed in image-recognition research and practice, including ATR. The work reported here generalizes the topological-algebraical structure characteristic of the concrete forms of mathematical morphology. This, in turn, broadens the theoretical base of mathematical morphology and enhances its power for systematic applications.

In previous reports [1,2], I have reviewed and added to the theory that supports and provides the applications of *Euclidean* mathematical morphology to the processing of both binary and greyscale imagery. The objects of Euclidean morphology are the closed subsets of two- or three-dimensional Euclidean space, where these objects are regarded as having the lattice-algebraical structure furnished by the set operations of union and intersection, and a certain morphologically relevant topological structure called the *hit-miss topology*. Much of the recent work in mathematical morphology has been in generalizing its lattice-algebraical aspect. Prominent examples of this work can be found in Serra [3], Heijmans and Ronse [4], and Banon and Barrera [5]. Following a suggestion of Heijmans and Serra [6], this report seeks in the same spirit to generalize the topological aspect of mathematical morphology. To be more specific, I must first describe the object of generalization in some technical detail.

1.1 Technical Background

The more concrete theories included under the name mathematical morphology [7] feature a topology known as Matheron's [8] hit-miss topology.

This topology is defined on the complete lattice $\mathbf{F}(S)$ of closed subsets of a locally compact, second countable Hausdorff (LCS) space, S (e.g., a Euclidean plane), where the join and meet operations of the lattice are set union and intersection, respectively. Letting $\mathbf{G}(S)$ and $\mathbf{K}(S)$ denote the classes of open and compact subsets of S, the hit-miss topology of $\mathbf{F}(S)$ is defined as follows.

DEFINITION 1 *Let $\mathbf{F}^A = \{F \in \mathbf{F}(S) : F \cap A = \emptyset\}$ and let $\mathbf{F}_A = \{F \in \mathbf{F}(S) : F \cap A \neq \emptyset\}$, where $A \subset S$ is arbitrary. Then the hit-miss topology $\mathbf{m}$ of $\mathbf{F}(S)$ is the topology generated by the collection $\{\mathbf{F}^K : K \in \mathbf{K}(S)\} \cup \{\mathbf{F}_G : G \in \mathbf{G}(S)\}$.*

Noting the identity $\cap \mathbf{F}^{A_\alpha} = \mathbf{F}^{\cup A_\alpha}$, and defining the notation $\mathbf{F}_{\{A_\alpha\}} \equiv \cap \mathbf{F}_{A_\alpha}$, we see that the typical finite intersection of sets from the above generating class has the form

$$\mathbf{F}^{K_1} \cap ... \cap \mathbf{F}^{K_m} \cap \mathbf{F}_{G_1} \cap ... \cap \mathbf{F}_{G_k} = \mathbf{F}^{K_1 \cup ... \cup K_m} \cap \mathbf{F}_{G_1,...,G_k}.$$

Letting $K_1 \cup ... \cup K_m = K$, Matheron uses the notation $\mathbf{F}^K_{G_1,...,G_k} \equiv \mathbf{F}^K \cap \mathbf{F}_{G_1,...,G_k}$.

REMARK 1 *A base for Matheron's hit-miss topology is given by the collection of sets of the form $\mathbf{F}^K_{G_1,...,G_k}$, where K is an arbitrary compact subset of S, and $\{G_1, ..., G_k\}$ is an arbitrary* (possibly empty) *finite set of open subsets of S. That is, the elements of the set $\mathbf{F}^K_{G_1,...,G_k}$ are the closed subsets of S that "hit" all the G_i and "miss" K.*

DEFINITION 2 *If $\mathcal{B} \subset \mathbf{F}(S)$, then $\mathcal{B}$ is called a "lower set" if $F \in \mathcal{B}$ and $E \subset F \Longrightarrow E \in \mathcal{B}$; if $F \in \mathcal{B}$ and $F \subset E \Longrightarrow E \in \mathcal{B}$, then $\mathcal{B}$ is called an "upper set."*

REMARK 2 *The hit-miss topology can be resolved into "upper" and "lower" topologies as follows. Let $\mu(\mathbf{F}(S))$ and $\lambda(\mathbf{F}(S))$ denote the topologies respectively generated on $\mathbf{F}(S)$ by $\{\mathbf{F}^K : K \in \mathbf{K}(S)\} \cup \emptyset$ and $\{\mathbf{F}_G : G \in \mathbf{G}(S)\} \cup \mathbf{F}(S)$; we call $\mu(\mathbf{F}(S))$ and $\lambda(\mathbf{F}(S))$ the upper and lower topologies of $\mathbf{F}(S)$. Then we have the following:*

1. *$\mu(\mathbf{F}(S))$ and $\lambda(\mathbf{F}(S))$ are each contained in* $\mathbf{m}$.
2. *$\mu(\mathbf{F}(S)) \cup \lambda(\mathbf{F}(S))$ generates* $\mathbf{m}$.
3. *$\mu(\mathbf{F}(S))$ consists precisely of the* $\mathbf{m}$*-open lower sets.*
4. *$\lambda(\mathbf{F}(S))$ consists precisely of the* $\mathbf{m}$*-open upper sets.*

Some properties of $\mathbf{m}$ are summarized in the following composite theorem.

THEOREM 1 (Matheron) *The hit-miss topology $\mathbf{m}$ is compact, second countable, and Hausdorff. Convergence criteria relative to $\mathbf{m}$ are as follows:*

(A): *A sequence* $\{F_i\}$ *in* $\mathbf{F}(S)$ *converges to* $F \in \mathbf{F}(S)$ *if and only if* (1) $G \subset S$ *is open and* $G \cap F \neq \emptyset \Longrightarrow G \cap F_i \neq \emptyset$ *for all but at most finitely many* F_i, *and* (2) $K \subset S$ *is compact and* $K \cap F = \emptyset \Longrightarrow K \cap F_i = \emptyset$ *for all but at most finitely many* F_i.

(B): *A sequence* $\{F_i\}$ *in* $\mathbf{F}(S)$ *converges to* $F \in \mathbf{F}(S)$ *if and only if* (a) *for each* $x \in F$ *there exist* $x_i \in F_i$ *for all but at most finitely many* i *such that* $x_i \to x$, *and* (b) *if* $\{F_{i_k}\}$ *is a subsequence of* $\{F_i\}$, *then every convergent sequence* $x_{i_k} \in F_{i_k}$ *has its limit in* F. *In addition, conditions* (a) *and* (b) *are respectively equivalent to* (1) *and* (2) *of* (A).

Furthermore, if $\{F_i\}$ *is a monotone sequence in* $\mathbf{F}(S)$, *then we have that*

(C): $\{F_i\}$ *is decreasing* $\Longrightarrow \{F_i\}$ m*-converges to* $\bigcap_{k=1}^{\infty} F_k$.

(D): $\{F_i\}$ *is increasing* $\Longrightarrow \{F_i\}$ m*-converges to* $\overline{\bigcup_{k=1}^{\infty} F_k}$.

The local and global *upper* and *lower semicontinuity* of mappings from a general topological space to $\mathbf{F}(S)$ are defined in terms of the upper and lower topologies of $\mathbf{F}(S)$ as follows.

DEFINITION 3 *If* X *is a topological space,* $x \in X$, *and* Ψ *maps* X *to* $\mathbf{F}(S)$, *then we say*

1. Ψ *is upper semicontinuous* (USC) *if* Ψ *is continuous relative to the upper topology.*
2. Ψ *is USC at* x *if* Ψ *is* μ*-continuous at* x.
3. Ψ *is lower semicontinuous* (LSC) *if* Ψ *is continuous relative to the lower topology.*
4. Ψ *is LSC at* x *if* Ψ *is* λ*-continuous at* x.

Clearly, Ψ is m-continuous (at x) if and only if Ψ is both USC and LSC (at x), Ψ is USC $\Longleftrightarrow$ Ψ is USC at every $x \in X$, and Ψ is LSC $\Longleftrightarrow$ Ψ is LSC at every $x \in X$.

THEOREM 2 (Matheron) *The mapping* $(E, F) \longmapsto E \cup F$ *of* $\mathbf{F}(S) \times \mathbf{F}(S)$ *onto* $\mathbf{F}(S)$ *is* m*-continuous. On the other hand, the mapping* $(E, F) \longmapsto E \cap F$ *is USC but not LSC.*

Another important feature of the hit-miss topology relates to the concepts of *upper* and *lower limits*. Matheron defines the upper and lower limits of sequences in $\mathbf{F}(S)$ as follows.

DEFINITION 4 *Let* $\{F_i\}$ *be a sequence in* $\mathbf{F}(S)$ *and let* $L(\{F_i\})$ *denote its set of limit points. Then the lower and upper limits of* $\{F_i\}$ *are defined by*

$$\underline{Lim}\, F_i = \bigcap\{F : F \in L(\{F_i\})\} \text{ and } \overline{Lim}\, F_i = \bigcup\{F : F \in L(\{F_i\})\}.$$

THEOREM 3 (Matheron) *If $\{F_i\}$ is a sequence in $\mathbf{F}(S)$, then (a) $\underline{Lim}\ F_i$ is the largest $F \in \mathbf{F}(S)$ that satisfies (a) of THEOREM 1–B, (b) $\overline{Lim}\ F_i$ is the smallest $F \in \mathbf{F}(S)$ that satisfies (b) of THEOREM 1–B, and (c) $F_i \to F$ relative to $\mathbf{m}$ if and only if*

$$\underline{Lim}\ F_i = \overline{Lim}\ F_i = F.$$

THEOREM 4 (Matheron) *Let $\Psi : X \longrightarrow \mathbf{F}(S)$, where X is a first countable Hausdorff space. Then we have the following characterizations.*

(1): *Ψ is USC at $x \iff \Psi(x) \supset \overline{Lim}\ \Psi(x_i)$ for all sequences $\{x_i\}$ in X that converge to x.*

(2): *Ψ is LSC at $x \iff \Psi(x) \subset \underline{Lim}\ \Psi(x_i)$ for all sequences $\{x_i\}$ in X that converge to x.*

To conclude this description of Matheron's morphological topology, I introduce a two-part theorem whose first part was proved by Matheron. In the proof of the second part, I use the term *subsequence* in the unconventional manner of Kelley [9]. Generally, a subsequence $\{i_k\}$ is a strictly increasing function $k \longmapsto i_k$ whose domain and range are the positive integers (or natural numbers). For Kelley, a *subsequence* $\{i_k\}$ is a nondecreasing function $k \longmapsto i_k$ that eventually goes to infinity; i.e., given a positive integer N, there is a k such that $i_k \geq N$. Another term for such a *subsequence* is *cofinal subset of* $\{1, 2, 3, ...\}$. I indicate the term *subsequence* in Kelley's sense by italics.

THEOREM 5 *If $\{F_i\}$ is a sequence in $\mathbf{F}(S)$, then*

$$\overline{Lim}\ F_i = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i \geq n} F_i}$$

where the last overbar denotes topological closure, and

$$\underline{Lim}\ F_i = \bigcap_{\{i_k\}} \overline{\bigcup_{k \geq 1} F_{i_k}}$$

where $\{i_k\}$ ranges over the cofinal subsets of $\{1, 2, 3, ...\}$.

Proof: For the first part, see Matheron [8], Proposition 1–2–3. For the second part, consider the following. If $\{F_{i_k}\}$ has a *subsequence* that converges to F, then $F \subset \overline{\bigcup_{k=1}^{\infty} F_{i_k}}$ by Matheron's convergence criteria (THM. 1–B). Since every $\{F_{i_k}\}$ has a convergent subsequence, $\underline{Lim}$ $F_i \subset \bigcap \overline{\bigcup_{k=1}^{\infty} F_{i_k}}$ where the intersection extends over all *subsequences* $\{i_k\}$. It remains to prove the reverse inclusion. Let $x \in \overline{\bigcup_{k=1}^{\infty} F_{i_k}}$ for every *subsequence* $\{i_k\}$. We show that this implies that x lies in every *subsequential* limit of $\{F_i\}$. Let $F_{i_k} \to F$. It is sufficient to prove that $x \in F$. In particular, we know that $x \in \overline{\bigcup_{k=1}^{\infty} F_{i_k}}$. Let $\{x_{1,j}\}$ in $\overline{\bigcup_{k=1}^{\infty} F_{i_k}}$ have limit x. If $\{x_{1,j}\}$ has a *subsequence* $x_{1,j_\ell} \in F_{i_{k_\ell}}$, we are done. If not, then $\{x_{1,j}\}$ along with its limit x lies entirely in $\overline{\bigcup_{k=1}^{m_1} F_{i_k}}$ for some positive integer m_1. Since x must lie as well in $\overline{\bigcup_{k=m_1+1}^{\infty} F_{i_k}}$, there is a sequence $\{x_{2,j}\}$ with limit x that lies in $\overline{\bigcup_{k=m_1+1}^{\infty} F_{i_k}}$. If $\{x_{2,j}\}$ has a *subsequence* $x_{2,j_\ell} \in F_{i_{k_\ell}}$, we are again done. If not, then $\{x_{2,j}\}$ along with its limit x lies entirely in $\bigcup_{k=m_1+1}^{m_2} F_{i_k}$ for some positive integer $m_2 > m_1$. Since $x \in \overline{\bigcup_{k=m_2+1}^{\infty} F_{i_k}}$ $\exists$ $\{x_{3,j}\}$ with limit x that lies in $\overline{\bigcup_{k=m_2+1}^{\infty} F_{i_k}}$. The theorem thus follows by induction.

The following definitions are introduced here in order to make a point about this theorem.

Definition 1 *A nonempty set* $\mathbf{D}$ *with a transitive and reflexive binary relation* $\trianglerighteq$ *is called a directed set if it has the Moore-Smith property, namely, for all* $\alpha, \beta \in \mathbf{D}$, *there exists a* $\gamma \in \mathbf{D}$ *such that* $\gamma \trianglerighteq \alpha$ *and* $\gamma \trianglerighteq \beta$.

Definition 2 *If* X *is a set and* $(\mathbf{D}, \trianglerighteq)$ *is a directed set, then a function* $\mathcal{N} : \mathbf{D} \longrightarrow X$ *is called a net in* X. *The values* $\{\mathcal{N}(\alpha) : \alpha \in \mathbf{D}\}$ *of a net* $\mathcal{N}$ *are frequently denoted by* $\{x_\alpha : \alpha \in \mathbf{D}\}$ *or more simply by* $\{x_\alpha\}$; *indeed, one often speaks of the net* $\{x_\alpha\}$.

Definition 3 *Let* $\mathcal{L}$ *be a complete lattice, let* $(\mathbf{D}, \trianglerighteq)$ *be a directed set, and let* $\{x_\alpha : \alpha \in \mathbf{D}\}$ *be a net in* $\mathcal{L}$. *Then the inferior and superior limits of* $\{x_\alpha : \alpha \in \mathbf{D}\}$, *denoted by* $\liminf x_\alpha$ *and* $\limsup x_\alpha$, *respectively, are defined by* [10]

$$\liminf x_\alpha = \sup_\alpha \inf\{x_\beta : \beta \trianglerighteq \alpha\} \quad \text{and} \quad \limsup x_\alpha = \inf_\alpha \sup\{x_\beta : \beta \trianglerighteq \alpha\}.$$

THEOREM 5 shows that the definition of the lower and upper limits of a sequence in $\mathbf{F}(S)$ does not coincide with the sequence version of the definition of the inferior and superior limits of a net in a complete lattice. The theorem does show that $\overline{Lim}\, F_i = \limsup F_i$, but it also shows that in general $\underline{Lim}\, F_i \neq \liminf F_i$. Frink [11] has given the following net definition of upper and lower limits:

$$\overline{Lim}\, x_\alpha \equiv \limsup x_\alpha \quad \text{and} \quad \underline{Lim}\, x_\alpha \equiv \inf_{\mathbf{C}} \sup\{x_\beta : \beta \in \mathbf{C}\}$$

where $\mathbf{C}$ ranges over the cofinal subsets of $\mathbf{D}$. Since the directed set associated with a sequence is $(\{1, 2, 3, \ldots\}, \geq)$, THEOREM 5 shows that Frink's definition reduces in the case of sequences to that of DEFINITION 4. For an arbitrary complete lattice $\mathcal{L}$, the class $\mathcal{S}$ of pairs $(\{x_\alpha\}, x)$, where $\{x_\alpha\}$ is a net in $\mathcal{L}$ and $x \in \mathcal{L}$, such that $\overline{Lim}\, x_\alpha = \underline{Lim}\, x_\alpha = x$, defines a topology $\mathcal{O}(\mathcal{S})$ on $\mathcal{L}$ given by

$$\mathcal{O}(\mathcal{S}) = \{U \subset \mathcal{L} : (\{x_\alpha\}, x) \in \mathcal{S} \text{ and } x \in \mathcal{U} \Longrightarrow x_\alpha \text{ is eventually in } \mathcal{U}\}.$$

Frink [11] named this topology the *convergence topology* of $\mathcal{L}$. We will later see that the convergence topology of $\mathbf{F}(S)$ and Matheron's hit-miss topology are in fact the same.

1.2 Outline of Report

With the foregoing, the substance of this report can be stated with clarity. What I propose to do here is display a generalization of Matheron's system—as above outlined by the fully capitalized remarks, theorems, etc—in the form of a certain type of abstract complete lattice, what I call an *upper-continuous* (UC) *lattice*. This generalization was, as mentioned, suggested by [6] and is based on the dual aspect of *continuous lattice theory* [12]. This theory is used to expose a type of intrinsic complete-lattice topology—which I call an M-topology—that has most of the outlined properties of Matheron's hit-miss topology when the lattice is UC. This M-topology is defined on any complete lattice in terms of "upper" and "lower" topologies as follows.

In a complete lattice, $\mathcal{L}$, the set of pairs $\mathcal{C}_\mu = (\{x_\alpha\}, x)$, where $\{x_\alpha\}$ is a net in $\mathcal{L}$ and x is an element of $\mathcal{L}$ that satisfies $\limsup x_\alpha \preceq x$, defines a topology $\mu(\mathcal{L})$ on $\mathcal{L}$ given by

$$\mu(\mathcal{L}) = \{U \subset \mathcal{L} : (\{x_\alpha\}, x) \in \mathcal{C}_\mu \text{ and } x \in \mathcal{U} \Longrightarrow x_\alpha \text{ is eventually in } \mathcal{U}\}.$$

I call $\mu(\mathcal{L})$ the upper topology of $\mathcal{L}$. Adopting the notation $\downarrow x \equiv \{x' \in \mathcal{L} : x' \preceq x\}$, we may then define the lower topology and M-topology as follows.

Definition 4 *If $\mathcal{L}$ is a lattice, then the topology generated on $\mathcal{L}$ by $\{\mathcal{L} \setminus \downarrow x : x \in \mathcal{L}\}$ will be denoted $\lambda = \lambda(\mathcal{L})$ and called the lower topology of $\mathcal{L}$. The topology generated on a complete lattice $\mathcal{L}$ by $\mu(\mathcal{L}) \cup \lambda(\mathcal{L})$ will be called the* M*-topology of $\mathcal{L}$ and denoted* $\mathrm{m} = \mathrm{m}(\mathcal{L})$.

In any complete lattice, there is an intrinsically definable binary relation, namely, the dual of the "way below" relation discussed at length in Gierz [12] (henceforth denoted *GZ*), that I denote $\gg$ and call the "way above" relation. In terms of this "way above" relation, a complete lattice $\mathcal{L}$ will be called UC when for each $x \in \mathcal{L}$ we have $x = \inf\{x' \in \mathcal{L} : x' \gg x\}$. To repeat then, this report will show (1) that the foregoing system of Matheron is a UC lattice whose M-topology is the hit-miss topology and (2) that almost all the outlined results for Matheron's system have generalized versions in an M-topologized UC lattice. As part of the latter demonstration, it will be shown that the M-topology of a UC lattice is precisely Frink's convergence topology.

An outline of the rest of the report follows. Section 2 gives and to some extent repeats the basic definitions and some of the properties of (1) UC lattices and (2) the M-topologies of complete and UC lattices. Very little in the way of proofs are offered here because most of the results presented follow by duality from *GZ*. After a brief diversion to consider some general topological issues relative to posets and lattices (for perspective), I proceed to give a detailed proof of the proposition that $\mathbf{F}(S)$ is a UC lattice whose M-topology coincides with Matheron's hit-miss topology. By the end of section 2, it is established that an M-topologized UC lattice holds generalizations of REMARK 1, REMARK 2, and parts of THEOREM 1 and THEOREM 2. Section 3 develops the convergence theory of M-topologized UC lattices. Here I offer complete and detailed proofs because, to my knowledge, a number of the results obtained are new. I obtain generalizations of THEOREMS 2, 3, 4, and 5, prove that the M-topology of a UC lattice is precisely Frink's convergence topology, and develop results that I claim generalize most of THEOREM 1. (The full generalization of THEOREM 2 holds only in an M-topologized UC lattice that is not a *topological lattice*, what I call a *Matheron space*.) Section 4 finally establishes that the claimed generalization of (most of) THEOREM 1 is in fact such a generalization; it does this by means of interpretation in the context of Matheron's system.

Some comment here on background theory and terminology is appropriate.

I use the terms *lattice* and *poset* (partially ordered set) with their usual meanings. For lattice and poset theory in general, I follow Birkhoff [10]. The definitions below are for reference.

Definition 5 *If B is a subset of a poset $(X, \preceq)$, then we say*

(1): *B is a "lower set" if $x \in B$ and $y \preceq x \Longrightarrow y \in B$.*

(2): *B is an "upper set" if $x \in B$ and $x \preceq y \Longrightarrow y \in B$.*

(3): *B is "convex" if $x, z \in B$ and $x \preceq y \preceq z \Longrightarrow y \in B$.*

A nonempty subset $\mathcal{D}$ of a lattice $\mathcal{L}$ is called a *directed subset of $\mathcal{L}$* if $(\mathcal{D}, \succeq)$ is a directed set. Here $\succeq$ is the reverse of $\preceq$. Thus, a directed subset of a lattice is a nonempty subset that contains an upper bound of each of its finite subsets. Dually, a *filtered subset* of a lattice is a nonempty subset $\mathcal{S}$ that contains a lower bound of each of its finite subsets.

Definition 6 *A complete lattice $\mathcal{L}$ is called join-continuous if for all $x \in \mathcal{L}$ and all filtered subsets $\mathcal{S}$ of $\mathcal{L}$ we have $x \vee \inf \mathcal{S} = \inf(x \vee \mathcal{S}) \equiv \inf\{x \vee \delta : \delta \in \mathcal{S}\}$. Dually, a complete latttice $\mathcal{L}$ is called meet-continuous if for all $x \in \mathcal{L}$ and all directed subsets $\mathcal{D}$ of $\mathcal{L}$ we have $x \wedge \sup \mathcal{D} = \sup(x \wedge \mathcal{D}) \equiv \sup\{x \wedge \delta : \delta \in \mathcal{D}\}$.*

For the terminology and theory of general topology I follow Kelley [9]. The following remark, which succinctly states the relation between convergence and topology, is for reference.

Remark 1 *Given a set X and a class $\mathcal{S}$ consisting of pairs $(\{x_\alpha\}, x)$ where $\{x_\alpha\}$ is a net in X and $x \in X$, it follows that the family of sets*

$$\mathcal{O}(\mathcal{S}) = \{U \subset X : (\{x_\alpha\}, x) \in \mathcal{S} \text{ and } x \in \mathcal{U} \Longrightarrow x_\alpha \text{ is eventually in } \mathcal{U}\}$$

is a topology on X such that $(\{x_\alpha\}, x) \in \mathcal{S}$ implies that $\{x_\alpha\}$ $\mathcal{O}(\mathcal{S})$-converges to x. This topology is equivalently defined by letting the closed sets be those subsets F of X such that $x \in F$ whenever $(\{x_\alpha\}, x) \in \mathcal{S}$ and $\{x_\alpha\} \subset F$. $\mathcal{S}$ is called a convergence class for $\mathcal{O}(\mathcal{S})$ if $(\{x_\alpha\}, x) \in \mathcal{S} \iff \{x_\alpha\}$ $\mathcal{O}(\mathcal{S})$-converges to x.

2. Upper-Continuous Lattices

In this section I will have frequent need to refer to results in *GZ*. Such references will be of the form *GZ* IV–3.1, where the roman numeral indicates the chapter and the decimal number indicates the chapter section and article.

Definition 7 *If $\mathcal{L}$ is a lattice and $x, y \in \mathcal{L}$, then we say that "y is way above x" if for each filtered subset $\mathcal{S}$ of $\mathcal{L}$ such that $\inf \mathcal{S} \preceq x$, there is a $\delta \in \mathcal{S}$ such that $\delta \preceq y$. To denote that y is way above x, we write $y \gg x$.*

The dual definition of the "way below" relation of *GZ* is given below for comparison.

Definition 8 *If $\mathcal{L}$ is a lattice and $x, y \in \mathcal{L}$, then we say that "x is way below y" if for each directed subset $\mathcal{D}$ of $\mathcal{L}$ such that $y \preceq \sup \mathcal{D}$, there is a $\delta \in \mathcal{D}$ for which $x \preceq \delta$. To denote that x is way below y we write $x \ll y$.* (Note that $\ll$ is not the reverse of $\gg$.)

We will make use of the notations $x \Downarrow = \{x' \in \mathcal{L} : x \gg x'\}$ and $x \Uparrow = \{x' \in \mathcal{L} : x' \gg x\}$.

Definition 9 *A complete lattice $\mathcal{L}$ is said to be upper-continuous* (UC) *if*

$$x = \inf\{x' \in \mathcal{L} : x' \gg x\} = \inf(x \Uparrow) \textit{ for all } x \in \mathcal{L}.$$

The dual definition of a "continuous lattice" in *GZ* is

Definition 10 *A complete lattice $\mathcal{L}$ is said to be continuous if*

$$x = \sup\{x' \in \mathcal{L} : x' \ll x\} \textit{ for all } x \in \mathcal{L}$$

Continuous lattices are meet-continuous, but meet-continuous lattices are not necessarily continuous (*GZ* I–1.14). Dually, UC lattices are join-continuous, but join-continuous lattices are not necessarily UC. The following proposition and its corollary are immediate.

Proposition 1 *If $\mathcal{L}$ is a complete lattice and $u, x, y, z \in \mathcal{L}$, then*

1. $y \gg x \Longrightarrow y \succeq x$,
2. $z \succeq y \gg x \succeq u \Longrightarrow z \gg u$.
3. $x \gg z$ *and* $y \gg z \Longrightarrow x \wedge y \gg z$.

4. $\sup \mathcal{L} \equiv e \gg x$.

Corollary 1 *In any complete lattice, $\gg$ is transitive ($x, y, z \in \mathcal{L}$, $y \gg x$,* and *$z \gg y \Longrightarrow z \gg x$) and antisymmetric ($x, y \in \mathcal{L}$, $y \gg x$,* and *$x \gg y \Longrightarrow x = y$).*

The next proposition follows by duality from GZ I–1.19.

Proposition 2 *In a UC lattice $\mathcal{L}$, $y \gg x$ is equivalent to the following: For each filtered subset $\mathcal{S}$ of $\mathcal{L}$ such that $x \succeq \inf \mathcal{S}$, there is a $\delta \in \mathcal{S}$ such that $y \gg \delta$.*

2.1 Upper and Lower Topologies

Definition 11 *If $\mathcal{L}$ is a complete lattice, then let $C_\mu = C_\mu(\mathcal{L})$ denote the set of pairs $(\{x_\alpha\}, x)$ where $\{x_\alpha\}$ is a net in $\mathcal{L}$ and x is an element of $\mathcal{L}$ that satisfies* $\limsup x_\alpha \preceq x$. *The topology $\mu(\mathcal{L}) \equiv \mathcal{O}(C_\mu)$ will be called the upper topology of $\mathcal{L}$.*

The next theorem follows by duality from GZ II–1.8.

Theorem 1 *C_μ is a convergence class for a complete lattice $\mathcal{L}$ if and only if $\mathcal{L}$ is UC.*

Definition 12 *The lower topology $\lambda(\mathcal{L})$ of a lattice $\mathcal{L}$ is generated by $\{\mathcal{L} \setminus \downarrow x : x \in \mathcal{L}\}$.*

Note that $\mathcal{L} \setminus \downarrow x = \{x' \in \mathcal{L} : x' \not\preceq x\}$. We will often denote this set by $\mathcal{F}_x$. The set of finite intersections of members of $\{\mathcal{F}_x : x \in \mathcal{L}\}$ is a base for a topology λ' on $\bigcup\{\mathcal{F}_x : x \in \mathcal{L}\}$. It is clear that $\lambda' \subset \lambda(\mathcal{L})$. Suppose that $\mathcal{L}$ has a universal lower bound o. Then $o \notin \mathcal{F}_x$ for all $x \in \mathcal{L}$ and $\mathcal{F}_o = \mathcal{L} \setminus \{o\}$. Thus it follows in this case that $\bigcup\{\mathcal{F}_x : x \in \mathcal{L}\} = \mathcal{L} \setminus \{o\} \equiv \mathcal{L}'$ and that λ' is the smallest topology on $\mathcal{L}'$ that contains $\{\mathcal{F}_x : x \in \mathcal{L}\}$. Since $\mathcal{L}'$ is λ'-open and $\lambda' \subset \lambda(\mathcal{L})$, it follows that $\mathcal{L}'$ is λ-open and $\{o\}$ is λ-closed.

Lemma 1 *If $\mathcal{L}$ is a lattice, then $\{\mathcal{F}_x : x \in \mathcal{L}\} \cup \mathcal{L}$ is a subbase for the lower topology $\lambda(\mathcal{L})$.*

Proof: The finite intersections of the members of $\{\mathcal{F}_x : x \in \mathcal{L}\} \cup \mathcal{L}$ is a base for the unique smallest topology on $\mathcal{L}$ such that $\mathcal{F}_x$ is open for all $x \in \mathcal{L}$ and $\mathcal{L}$ is open. Moreover, every topology on $\mathcal{L}$ that contains $\mathcal{F}_x$ for all $x \in \mathcal{L}$ also contains $\mathcal{L}$. Hence the finite intersections of the members of $\{\mathcal{F}_x : x \in \mathcal{L}\} \cup \mathcal{L}$ is a base for the unique smallest topology on $\mathcal{L}$ that contains $\mathcal{F}_x$ for all $x \in \mathcal{L}$, and this is precisely $\lambda(\mathcal{L})$.

2.2 M-Topology

Definition 13 *The topology generated on a complete lattice $\mathcal{L}$ by $\mu(\mathcal{L}) \cup \lambda(\mathcal{L})$ will be called the* M-topology *of $\mathcal{L}$ and denoted $\mathbf{m} = \mathbf{m}(\mathcal{L})$. The space $(\mathcal{L}, \mathbf{m}(\mathcal{L}))$ will be denoted $\mathbf{M}(\mathcal{L})$.*

Thus the finite intersections of sets of either of the two forms

$$\mathcal{F}_x \equiv \{x' \in \mathcal{L} : x' \not\preceq x\} \qquad \text{and} \qquad \mathcal{F}^y \equiv \{x' \in \mathcal{L} : y \gg x'\},$$

where x and y are any elements of $\mathcal{L}$, is an open-set base $\mathcal{B}_{\mathbf{m}}$ for $\mathbf{m}(\mathcal{L})$. The typical member of $\mathcal{B}_{\mathbf{m}}$ has the form $\mathcal{F}_{x_1} \cap ... \cap \mathcal{F}_{x_n} \cap \mathcal{F}^{y_1} \cap ... \cap \mathcal{F}^{y_m}$, where $\{x_1, ..., x_n\}$ and $\{y_1, ..., y_m\}$ are finite subsets of $\mathcal{L}$. We will more briefly denote $\mathcal{F}_{x_1} \cap ... \cap \mathcal{F}_{x_n}$ by $\mathcal{F}_{x_1,...,x_n}$.

Lemma 2 *If $\mathcal{L}$ is a complete lattice and $y_1, ..., y_m \in \mathcal{L}$, then $\mathcal{F}^{y_1} \cap ... \cap \mathcal{F}^{y_m} = \mathcal{F}^{\inf\{y_1,...,y_m\}}$.*

Proof: $\mathcal{F}^{y_1} \cap ... \cap \mathcal{F}^{y_m} = \{x' \in \mathcal{L} : y_1 \gg x', ..., y_m \gg x'\}$ and $\mathcal{F}^{\inf\{y_1,...,y_m\}} = \{x' \in \mathcal{L} : \inf\{y_1, ..., y_m\} \gg x'\}$. By Proposition 1 we have that $y_1 \gg x', ..., y_m \gg x' \Longrightarrow \inf\{y_1, ..., y_m\} \gg x'$. Hence $\mathcal{F}^{y_1} \cap ... \cap \mathcal{F}^{y_m} \subset \mathcal{F}^{\inf\{y_1,...,y_m\}}$. Suppose, on the other hand, that $\inf\{y_1, ..., y_m\} \gg x'$. Since $y_i \succeq \inf\{y_1, ..., y_m\} \gg x' \succeq x'$ for all $i = 1, ..., m$, it follows, again by Proposition 1, that $y_i \gg x'$ for all $i = 1, ..., m$. Hence $\mathcal{F}^{y_1} \cap ... \cap \mathcal{F}^{y_m} \supset \mathcal{F}^{\inf\{y_1,...,y_m\}}$. This completes the proof.

Corollary 2 *The typical member of $\mathcal{B}_{\mathbf{m}}$ has the form*

$$\mathcal{F}^y \cap \mathcal{F}_{x_1,...,x_n} \equiv \mathcal{F}^y_{x_1,...,x_n}$$

for some $y \in \mathcal{L}$ and finite subset $\{x_1, ..., x_n\}$ of $\mathcal{L}$. Conversely, if $y \in \mathcal{L}$ and $\{x_1, ..., x_n\}$ is a finite subset of $\mathcal{L}$, then $\mathcal{F}^y_{x_1,...,x_n}$ is a member of $\mathcal{B}_{\mathbf{m}}$.

The typographic similarity of this result and REMARK 1 should not go unnoticed. We will indeed later see, with the proof of Proposition 11, that the base of Corollary 2 and that of REMARK 1 coincide on the lattice $\mathbf{F}(S)$. In the next remark, (1) follows from Lemma 1 and (2) follows by duality from GZ II–1.14.

Remark 2 *Let $\mathcal{L}$ be a complete lattice.*

1. *An open-set base for the lower topology $\lambda(\mathcal{L})$ is given by the collection*

   $$\{\mathcal{F}_{x_1,...,x_n} : x_1, ..., x_n \in \mathcal{L}, n \text{ a nonnegative integer}\}.$$

   (Note that $\mathcal{F}_e = \emptyset$ and that $\mathcal{F}_\emptyset = \mathcal{L}$ by the usual convention.)

2. *If $\mathcal{L}$ is UC, then $\{\mathcal{F}^y : y \in \mathcal{L}\}$ is an open-set base for the upper topology $\mu(\mathcal{L})$.*

In the next proposition, (1) through (4) follow by duality from GZ III–1.6, III–3.16, III–3.20, and VI–2.3, respectively.

Proposition 3 *Let $\mathcal{L}$ be a UC lattice.*

1. *A lower set $\mathcal{G}$ of $\mathcal{L}$ is* $\mathbf{m}$*-open if and only if $\mathcal{G}$ is μ-open.*
2. *A lower set $\mathcal{E}$ of $\mathcal{L}$ is* $\mathbf{m}$*-closed if and only if $\mathcal{E}$ is λ-closed.*
3. *An upper set $\mathcal{G}$ of $\mathcal{L}$ is* $\mathbf{m}$*-open if and only if $\mathcal{G}$ is λ-open.*
4. *An upper set $\mathcal{E}$ of $\mathcal{L}$ is* $\mathbf{m}$*-closed if and only if $\mathcal{E}$ is μ-closed.*

That is, the lower (upper) sets in $\mathbf{m}(\mathcal{L})$ are precisely the sets of $\mu(\mathcal{L})$ ($\lambda(\mathcal{L})$), and the upper (lower) $\mathbf{m}$-closed sets are precisely the μ-closed (λ-closed) sets. The similarity of the sum of Remark 2 and Proposition 3 with REMARK 2 should be duly noted. It will indeed later be seen—again, with Proposition 11—that the upper topology, lower topology, and $\mathbf{M}$-topology of the complete lattice $\mathbf{F}(S)$, in fact and in turn, coincide with the upper, lower, and hit-miss topologies of Matheron.

The first part of the next theorem follows by duality from GZ III–1.10; the second part follows by duality from GZ II–1.14 and III–2.3.

Theorem 2 *Let $\mathcal{L}$ be a UC lattice. Then* (1) *the* $\mathbf{M}$*-topology of $\mathcal{L}$ is compact and Hausdorff, and* (2) *the mapping $(x, y) \longmapsto x \vee y : \mathbf{M}(\mathcal{L}) \times \mathbf{M}(\mathcal{L}) \longrightarrow \mathbf{M}(\mathcal{L})$ is continuous.*

Thus the $\mathbf{M}$-topology of a UC lattice, like the hit-miss topology of $\mathbf{F}(S)$, is compact and Hausdorff (cf. THEOREM 1), and the join operation of a UC lattice, like the union (join) operation of $\mathbf{F}(S)$, is continuous (cf. THEOREM 2). We now define the upper and lower semicontinuity of maps from a general topological space into a UC lattice by essentially repeating DEFINITION 3.

Definition 14 *Let X be a topological space, let $\mathcal{L}$ be a UC lattice, let $f : X \longrightarrow \mathcal{L}$, and let $x \in X$. Then f is said to be* USC (LSC) *[at x] if f is continuous [at x] with respect to μ (λ).*

The next theorem is immediate.

Theorem 3 *Let X be a topological space, let $\mathcal{L}$ be a UC lattice, let $f : X \longrightarrow \mathcal{L}$, and let $x \in X$. Then we have the following.*

(a): *f is* USC (LSC) $\Longleftrightarrow$ *f is* USC (LSC) *at every x.*

(b): *f is continuous [at x] $\Longleftrightarrow$ f is both* USC *and* LSC *[at x].*

Proposition 4 *If X is a topological space, $\mathcal{L}$ is a UC lattice, and $f : X \longrightarrow \mathcal{L}$, then f is USC if and only if $f(\lim x_\alpha) \succeq \limsup f(x_\alpha)$ for all convergent nets $\{x_\alpha\}$ in X.*

Proof: According to the last definition and theorem, f is USC if and only if f is μ-continuous at every $x \in X$. Since the latter means that $\{f(x_\alpha)\}$ is μ-convergent to $f(\lim x_\alpha)$ for every convergent net $\{x_\alpha\}$ in X, and since this is equivalent to $f(\lim x_\alpha) \succeq \limsup f(x_\alpha)$, the proposition follows.

Here we see something similar to part (1) of THEOREM 4.

2.3 Poset and Lattice Topologies in General

In order to put the foregoing material in perspective, I will now enter upon a brief diversion to consider some general topological issues relative to posets and lattices. In the main I follow the monograph of Nachbin [13]. First we consider the following definition.

Definition 15 *If $\mathcal{L}$ is a pospace, i.e., a poset with a topology, then we say that*

(a): *$\mathcal{L}$ is a topological sup-lattice if $\mathcal{L}$ is a sup-lattice and $\vee : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$ is continuous.*

(b): *$\mathcal{L}$ is a topological inf-lattice if $\mathcal{L}$ is a inf-lattice and $\wedge : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$ is continuous.*

(c): *$\mathcal{L}$ is a topological lattice if $\mathcal{L}$ is a lattice and both $\wedge$ and $\vee$ are continuous as above.*

As an immediate consequence of this definition and Theorem 2, we note the following.

Remark 3 *If $\mathcal{L}$ is UC, then $\mathrm{M}(\mathcal{L})$ is a topological sup-lattice.*

Let us now make a few observations that concern the notion of convexity in a poset and its possible compatibilities with a given poset topology. If $(X, \tau, \preceq)$ is a pospace, then there are several ways in which τ might be compatible with the convexity concept of $(X, \preceq)$.

Definition 16 *If $(X, \tau, \preceq)$ is a pospace, then τ is called*

1. *Locally convex if each $x \in X$ has a convex local base.*
2. *Weakly convex if the set of open convex subsets of X is an open-set base for τ.*

3. *Convex if the open upper and open lower sets of X form an open-set subbase for τ.*

Remark 4 *A pospace $(X, \tau, \preceq)$ is locally convex if and only if given nets $\{x_\beta\}$, $\{y_\beta\}$, and $\{z_\beta\}$ in X such that $x_\beta \to \xi$, $z_\beta \to \xi$, and $x_\beta \preceq y_\beta \preceq z_\beta$ for all β, it follows that $y_\beta \to \xi$.*

Remark 5 *In a pospace, convexity $\Longrightarrow$ weak convexity $\Longrightarrow$ local convexity.*

If $(X, \tau, \preceq)$ is a pospace, then $\preceq$ may have one or more of the following closure properties.

CP1: If $\xi \in X$, $\{x_\beta\}$ is a convergent net in X with limit x, and $\xi \preceq x_\beta \ \forall \ \beta$, then $\xi \preceq x$.

CP2: If $\xi \in X$, $\{x_\beta\}$ is a convergent net in X with limit x, and $x_\beta \preceq \xi \ \forall \ \beta$, then $x \preceq \xi$.

CP: If $\{x_\beta\}$ and $\{y_\beta\}$ are convergent nets in X with limits x and y, respectively, and if $x_\beta \preceq y_\beta$ for all β, then $x \preceq y$.

If **CP1** and **CP2** hold, then $\preceq$ is called a *semi-closed order in X* or a *semi-closed ordering of X*; if **CP** holds, then $\preceq$ is called a *closed order in X* or a *closed ordering of X*.

Remark 6 *For any pospace, $(X, \tau, \preceq)$, we have the following.*

1. *$\preceq$ is semi-closed if and only if $\{x \in X : \xi \preceq x\}$ and $\{x \in X : x \preceq \xi\}$ are closed subsets of X for all $\xi \in X$; hence, $\preceq$ is semi-closed if and only if every "closed interval" of X is a τ-closed subset of X.*

2. *$\preceq$ is closed if and only if the graph $\{(x, y) \in X \times X : x \preceq y\}$ of $\preceq$ is a closed subset of the product space $X \times X$.*

3. *If $\preceq$ is closed, then $\preceq$ is semi-closed.*

Definition 17 *A pospace $(X, \tau, \preceq)$ is called topological if $\preceq$ is a closed ordering of X. A topological pospace $(X, \tau, \preceq)$ is called a compact pospace if τ is compact.*

We note that the "closed intervals" of a poset form a closed-set subbase for what is usually called the *interval topology* of the poset.

Remark 7 *The "closed intervals" of a topological pospace $(X, \tau, \preceq)$ are τ-closed; hence τ contains the interval topology of $(X, \preceq)$. In fact, if (X, τ) is a topological space with a merely semi-closed order $\preceq$, then τ contains the interval topology of $(X, \preceq)$.*

Proposition 5 (Nachbin) *A pospace $(X, \tau, \preceq)$ is a topological pospace if and only if for every $x, y \in X$ such that $x \npreceq y$ there exist disjoint open neighborhoods U and V of x and y, respectively, such that U is an upper set and V is a lower set.*

Corollary 3 *Every topological pospace is Hausdorff.*

Definition 18 *A pospace $(X, \tau, \preceq)$ is called normal if for every two disjoint closed subsets F and F' of X, where F is a lower set and F' is an upper set, there exist disjoint open sets U and U', where U is a lower set and U' is an upper set, such that $F \subset U$ and $F' \subset U'$.*

Remark 8 *Let (X, τ) be a topological space and let $\preceq$ be the partial ordering of X defined by $x \preceq y \iff x = y$. Then the pospace $(X, \tau, \preceq)$ is normal as a pospace if and only if the topological space (X, τ) is normal as a topological space.*

With regard to normal topological spaces, let us recall the following results of Urysohn.

Theorem 4 (Urysohn) *If (X, τ) is a topological space, then the following are equivalent.*

1. *(X, τ) is a normal topological space.*
2. *For any two disjoint closed subsets F and F' of X, there exists a continuous real valued function f on X such that $f = 0$ on F, $f = 1$ on F', and $0 \leq f \leq 1$ on X.*
3. *For every closed subset F of X, and every continuous real valued function f on F, there exists a continuous real valued function $f^\star$ on X such that $f^\star = f$ on F.*

Characterization (2) has been generalized to normal pospaces as follows.

Theorem 5 (Nachbin) *A pospace $(X, \tau, \preceq)$ is normal if and only if for every two disjoint closed subsets F and F' of X, where F is a lower set and F' is an upper set, there exists a continuous increasing real valued function f on X such that*

$$f = 0 \text{ on } F,\ f = 1 \text{ on } F', \text{and } 0 \leq f \leq 1 \text{ on } X.$$

Theorem 6 (Nachbin) *Let $(X, \tau, \preceq)$ be a compact pospace and let F and F' be closed subsets of X such that $x \in F$ and $x' \in F' \Longrightarrow x \npreceq x'$. Then there exist disjoint open sets U and U', where U is an upper set and U' is a lower set, such that $F \subset U$ and $F' \subset U'$.*

Corollary 4 *Every compact pospace is normal.*

Theorem 7 (Nachbin) *Every compact pospace is convex.*

Characterization (3) of Theorem 4 has the following generalization to compact pospaces.

Theorem 8 (Nachbin) *If K is a compact subset of a compact pospace X, then every continuous increasing real valued function defined on K can be extended to the entirety of X as a continuous increasing real valued function.*

We now begin to remake contact with the main material of this report.

Remark 9 *If $(X, \tau, \preceq)$ is a compact pospace, then we have the following.*

1. *The class $\mu(X)$ of open lower sets of X and the class $\lambda(X)$ of open upper sets of X are topologies on X; $\mu(X)$ and $\lambda(X)$ are called the upper and lower topologies, respectively, of X relative to τ.*
2. *By the convexity of $(X, \tau, \preceq)$, $\mu(X) \cup \lambda(X)$ is an open-set subbase for τ.*

Proposition 6 $(\mathcal{L}, \tau, \vee, \preceq)$ *is a topological sup-lattice* $\Longrightarrow$ $(\mathcal{L}, \tau, \preceq)$ *is a topological pospace.*

Proof: Since $x_\alpha \vee y_\alpha = y_\alpha$ for all α, it follows that $x_\alpha \vee y_\alpha \to y$. Since $(x_\alpha, y_\alpha) \to (x, y)$ in $\mathcal{L} \times \mathcal{L}$ and $\vee$ is a continuous operation in $\mathcal{L}$, we also have that

$$x_\alpha \vee y_\alpha \to x \vee y.$$

Hence $x \vee y = y$, and it follows that $x \preceq y$.

Corollary 5 *Every compact topological sup-lattice is a compact pospace and is hence convex.*

In particular, if $\mathcal{L}$ is a UC lattice, then $\mathrm{M}(\mathcal{L})$ is compact and a topological sup-lattice, is therefore a compact pospace, and is consequently convex. This and Proposition 3 show that *the upper and lower topologies of a UC lattice $\mathcal{L}$ relative to the compact-pospace structure of $\mathrm{M}(\mathcal{L})$ respectively coincide with the upper and lower topologies defined in section 2.1.*

The perspective just outlined can be deepened still further with the aid of the following.

Definition 19 *A sup-lattice $(\mathcal{L}, \tau, \vee, \preceq)$ with a topology τ will be said to be "locally joined" if every point of $\mathcal{L}$ has a local base consisting entirely of open sub-sup-lattices of $\mathcal{L}$.*

With this we obtain the theorem below, which follows by duality from GZ VI–3.4.

Theorem 9 (Fundamental Compact Topological Sup-Lattice Theorem) *For every UC lattice $\mathcal{L}$, $\mathrm{M}(\mathcal{L})$ is a locally joined compact topological sup-lattice with a universal lower bound. Conversely, if $(\mathcal{L}, \tau, \vee, \preceq)$ is a locally joined compact topological sup-lattice with a universal lower bound, then $\mathcal{L}$ is a UC lattice and τ is the M-topology of this lattice.*

2.4 Hit-Miss and M-topology

I will now show, among other things, that the complete lattice $\mathbf{F}(S)$ is UC and that its M-topology coincides with Matheron's hit-miss topology.

Let X be a topological space and let $\mathbf{G}(X)$ be the set of open subsets of X. $\mathbf{G}(X)$ is a complete distributive lattice relative to $\wedge = \cap$, $\vee = \cup$, and $\preceq = \subset$; indeed, the infimum of an arbitrary collection of open subsets of X is the interior of their intersection. Proposition 8 below (which is proved in *GZ* I–1.4) characterizes the way below relation in $\mathbf{G}(X)$ when X is locally compact and either Hausdorff or regular. We note the following.

Definition 20 *A topological space X is called regular if for each $x \in X$ and each neighborhood N of x there is a closed neighborhood F of x such that $F \subset N$.*

Proposition 7 *If X is a locally compact topological space that is either Hausdorff or regular, then for each $x \in X$ and each neighborhood N of x, there is a closed compact neighborhood K of x such that $K \subset N$.*

Proposition 8 *If X is a topological space, if $U, V \in \mathbf{G}(X)$, and if there is a compact subset K of X such that $U \subset K \subset V$, then $U \ll V$. If X is a locally compact space that is either Hausdorff or regular, and if $U, V \in \mathbf{G}(X)$, then*

$$U \ll V \iff U \subset K \subset V \text{ for some compact } K \subset X.$$

Proof: For the first part, note that a directed open cover $\mathcal{D}$ of V is also a directed open cover of K. Since K is compact, finitely many of the covering sets in $\mathcal{D}$ cover K. There is, therefore, an element W' of $\mathcal{D}$ such that $U \subset K \subset W'$. Hence $U \ll V$ by definition.

For the second part, since we already have the implication

$$U \ll V \Longleftarrow U \subset K \subset V \text{ for some compact } K \subset X,$$

it is enough to prove the converse. Assume, then, that $U \ll V$ and note that every point $\xi \in V$ has a compact neighborhood K_ξ such that $K_\xi \subset V$. Let W_ξ denote the interior of K_ξ. Then $\xi \in W_\xi$ and

$$V = \bigcup \{W_\xi : \xi \in V\}.$$

Let $\mathcal{D}$ denote the set of finite unions of the W_ξ. Then $\mathcal{D}$ is a directed set that covers V. Since $U \ll V$, there are finitely many W_ξ, say $W_{\xi_1}, \ldots, W_{\xi_n}$, such that $U \subset W_{\xi_1} \cup \ldots \cup W_{\xi_n} \subset V$. Therefore

$$U \subset K_{\xi_1} \cup \ldots \cup K_{\xi_n} \subset V,$$

and $K = K_{\xi_1} \cup \ldots \cup K_{\xi_n}$ is the required compact set.

Let $\mathbf{F}(X)$ denote the distributive lattice of closed subsets of a topological space X; again, $\wedge = \cap$, $\vee = \cup$, and $\preceq = \subset$. This lattice is also complete; the supremum of a collection of closed subsets of X is the closure of their union. Moreover, $\mathbf{F}(X)$ is dual-lattice isomorphic to $\mathbf{G}(X)$ under the complementation mapping. More specifically, we have the following.

Remark 10 *The mapping $U \longmapsto X \setminus U \equiv U^c$ of $\mathbf{G}(X)$ is a bijection onto $\mathbf{F}(X)$; moreover, letting U and V be open subsets of X, this mapping satisfies the following.*

1. $V^c \subset U^c \iff U \subset V$.
2. $U \cup V \longmapsto U^c \cap V^c$.
3. $U \cap V \longmapsto U^c \cup V^c$.

Let $\gg$ denote the way above relation in $\mathbf{F}(X)$. Then by definition, if $E, F \in \mathbf{F}(X)$, then $F \gg E$ if and only if for each filtered subset $\{F_\alpha\}$ of $\mathbf{F}(X)$ such that $\bigcap_\alpha F_\alpha \subset E$, there is an α_o such that $F_{\alpha_o} \subset F$.

Proposition 9 *If $E, F \in \mathbf{F}(X)$, then $F \gg E$ in $\mathbf{F}(X) \iff F^c \ll E^c$ in $\mathbf{G}(X)$.*

Proof: Suppose that $F^c \ll E^c$ in $\mathbf{G}(X)$ and let $\{F_\alpha\}$ be a filtered subset of $\mathbf{F}(X)$ such that $\bigcap_\alpha F_\alpha \subset E$. Put $U_\alpha = F_\alpha^c$. Then $\{U_\alpha\}$ is a directed subset of $\mathbf{G}(X)$ such that $E^c \subset \bigcup_\alpha U_\alpha$. Since $F^c \ll E^c$ in $\mathbf{G}(X)$, there is an α_o such that $F^c \subset U_{\alpha_o}$; hence, $F_{\alpha_o} \subset F$. This proves $\Longleftarrow$.

Suppose, conversely, that $F \gg E$ in $\mathbf{F}(X)$ and let $\{U_\alpha\}$ be a directed subset of $\mathbf{G}(X)$ such that $E^c \subset \bigcup_\alpha U_\alpha$. Put $F_\alpha = U_\alpha^c$. Then $\{F_\alpha\}$ is a filtered subset of $\mathbf{F}(X)$ such that $\bigcap_\alpha F_\alpha \subset E$. Since $F \gg E$ in $\mathbf{F}(X)$, there is an α_o such that $F_{\alpha_o} \subset F$; hence, $F^c \subset U_{\alpha_o}$. This proves $\Longrightarrow$ and completes the proof.

We thus obtain the following corollary of Proposition 8.

Corollary 6 *If X is a topological space, if $E, F \in \mathbf{F}(X)$, and if there is a compact subset K of X such that $E \subset K^c \subset F$, then $F \gg E$. If X is a locally compact topological space that is either Hausdorff or regular, and if $E, F \in \mathbf{F}(X)$, then*

$$F \gg E \iff E \subset K^c \subset F \text{ for some compact } K \subset X.$$

We also obtain the following corollary of this corollary.

Corollary 7 *If X is a locally compact Hausdorff space, and if $E, F \in \mathbf{F}(X)$, then*

$$F \gg E \iff E \text{ is contained in the interior of } F \text{ and } F^c \text{ is relatively compact.}$$

Having identified the way above relation in $\mathbf{F}(X)$ and the way below relation in $\mathbf{G}(X)$, when X is locally compact and either Hausdorff or regular, it will now be shown that these lattices are, respectively, UC and continuous.

Proposition 10 *Let X be a locally compact topological space that is either Hausdorff or regular. Then $\mathbf{G}(X)$ is continuous and $\mathbf{F}(X)$ is UC.*

Proof: Let V be an arbitrary open subset of X. It is clear that $V \supset \bigcup\{U \in \mathbf{G}(X) : U \ll V\}$, because $U \ll V \Longrightarrow U \subset V$. Therefore let x be an arbitrary point in V. Because X is locally compact and either Hausdorff or regular, and V is open, there is a closed compact neighborhood K_x of x such that $K_x \subset V$. Since x lies in the interior W_x of K_x, it follows that $W_x \subset K_x \subset V$, and therefore that $W_x \ll V$. Since this implies that $x \in \bigcup\{U \in \mathbf{G}(X) : U \ll V\}$, it follows that

$$V \subset \bigcup\{U \in \mathbf{G}(X) : U \ll V\},$$

and hence that $V = \bigcup\{U \in \mathbf{G}(X) : U \ll V\}$. Thus $\mathbf{G}(X)$ is continuous.

To see that $\mathbf{F}(X)$ is UC, let F be an arbitrary closed subset of X. Because $\mathbf{G}(X)$ is continuous we have $F^c = \bigcup\{U \in \mathbf{G}(X) : U \ll F^c\}$. By complementation and Proposition 9 we therefore obtain $F = \bigcap\{E \in \mathbf{F}(X) : E \gg F\}$.

Definition 21 *Let X be a general topological space and let $A, A_\alpha \subset X$. As before we define the notations: $\mathbf{F}^A \equiv \{F \in \mathbf{F}(X) : F \cap A = \emptyset\}$, $\mathbf{F}_A \equiv \{F \in \mathbf{F}(X) : F \cap A \neq \emptyset\}$, $\mathbf{F}_{\{A_\alpha\}} \equiv \cap \mathbf{F}_{A_\alpha}$, and $\mathbf{F}^A_{\{A_\alpha\}} \equiv \mathbf{F}^A \cap \mathbf{F}_{\{A_\alpha\}}$.*

Proposition 11 *If X is a locally compact Hausdorff space, then a base for the M-topology of $\mathbf{F}(X)$ is given by the collection*

$$\{\mathbf{F}^K_{G_1,\ldots,G_n} : K \textit{ compact}, G_1, \ldots, G_n \textit{ open, and } n \textit{ a nonnegative integer}\}.$$

Hence if X is second countable, then $\mathrm{m}(\mathbf{F}(X))$ coincides with Matheron's hit-miss topology.

Proof: By Proposition 10, $\mathbf{F}(X)$ is a UC lattice. Hence it follows by Corollary 2 that a base for the M-topology of $\mathbf{F}(X)$ is given by

$$\{\mathcal{F}^F_{F_1,\ldots,F_n} : F, F_1, \ldots, F_n \in \mathbf{F}(X), n \text{ a nonnegative integer}\}$$

Denoting $F_1^c = G_1, \ldots, F_n^c = G_n$, and $\overline{F^c} = K$, we have by Corollary 7 that $E \in \mathcal{F}^F_{F_1,\ldots,F_n} \iff E \cap G_i \neq \emptyset$ for all $i = 1, \ldots, n$, $E \cap K = \emptyset$, and K is compact.

If K is not compact, then $\mathcal{F}^{F}_{F_1,\ldots,F_n} = \emptyset$, and since $\mathbf{F}^{K}_{\emptyset} = \emptyset$ for all compact K, it follows that $E \in \mathcal{F}^{F}_{F_1,\ldots,F_n} \iff E \in \mathbf{F}^{K}_{G_1,\ldots,G_n}$. Hence $\{\mathbf{F}^{K}_{G_1,\ldots,G_n} : K$ compact, $G_1, \ldots, G_n$ open, and n a nonnegative integer$\}$ is a base for the M-topology of $\mathbf{F}(X)$. This completes the proof.

3. Convergence Theory for $\mathrm{M}(\mathcal{L})$

I begin by giving an equivalent characterization of the M-topology of a UC lattice (which follows by duality from *GZ* III–3) that will prove useful for the developments of this section.

Proposition 12 *If $\mathcal{L}$ is a UC lattice, $x \in \mathcal{L}$, and $\{x_\alpha\}$ is a net in $\mathcal{L}$, then $x_\alpha \longrightarrow x$ relative to $\mathrm{m}(\mathcal{L})$ if and only if $x = \limsup y_\beta$ for all subnets $\{y_\beta\}$ of $\{x_\alpha\}$.*

The technical definition of a subnet, which will come up in some proofs, is as follows.

Definition 22 *Let $(\mathbf{D}, \trianglerighteq)$ and $(\mathbf{D}', \trianglerighteq')$ be directed sets and let $\mathcal{N}$ and $\mathcal{N}'$ be nets in X relative to $\mathbf{D}$ and $\mathbf{D}'$, respectively. We say that $\mathcal{N}'$ is a subnet of $\mathcal{N}$ if there is a function $\Sigma : \mathbf{D}' \longrightarrow \mathbf{D}$ such that $\mathcal{N}' = \mathcal{N} \circ \Sigma$ and for which the following "subnet condition" holds.*

$$\forall\, \alpha \in \mathbf{D}\; \exists\, \alpha' \in \mathbf{D}' \text{ such that } \beta' \trianglerighteq' \alpha' \Longrightarrow \Sigma(\beta') \trianglerighteq \alpha.$$

I also state the following summary remark whose items will be used in the proofs that follow without particular notice.

Remark 11 *Let $\mathcal{L}$ be a lattice.*

1. *If $x \in \mathcal{L}$ and $\mathcal{A}_x$ is the subset of $\mathcal{L}$ whose elements are not successors of x, then a local base at x for $\lambda(\mathcal{L})$ is given by $\{\mathcal{F}_{x_1,\ldots,x_n} : x_1, \ldots, x_n \in \mathcal{A}_x$, n a nonnegative integer$\}$.*
2. *If $\{x_\alpha\}$ is a net in $\mathcal{L}$, and if $x \in \mathcal{L}$, then $x_\alpha \longrightarrow x$ relative to $\lambda(\mathcal{L})$ if and only if for each finite subset $\{x_1, \ldots, x_n\}$ of $\mathcal{L}$ such that $x \not\trianglelefteq x_i$ for all $i = 1, \ldots, n$, there exists a $\beta = \beta(x_1, \ldots, x_n)$ such that $x_\alpha \not\trianglelefteq x_i$ for all $i = 1, \ldots, n$ for all $\alpha \trianglerighteq \beta$.*
3. *If $\mathcal{L}$ is UC, $x \in \mathcal{L}$, and $\mathcal{A}^x$ is the subset of $\mathcal{L}$ whose elements are way above x, then a local base at x for $\mu(\mathcal{L})$ is given by $\{\mathcal{F}^y : y \in \mathcal{A}^x\}$.*
4. *If $\mathcal{L}$ is UC, $\{x_\alpha\}$ is a net in $\mathcal{L}$, and $x \in \mathcal{L}$, then $x_\alpha \longrightarrow x$ relative to $\mu(\mathcal{L})$ if and only if $\forall\, y \in \mathcal{L}$ such that $y \gg x$ there exists a $\beta = \beta(y)$ such that $y \gg x_\alpha\ \forall\, \alpha \trianglerighteq \beta$.*
5. *If $\mathcal{L}$ is UC, $\{x_\alpha\}$ is a net in $\mathcal{L}$, and $x \in \mathcal{L}$, then $x_\alpha \longrightarrow x$ relative to $\mathrm{m}(\mathcal{L})$ if and only if $x_\alpha \longrightarrow x$ relative to both $\mu(\mathcal{L})$ and $\lambda(\mathcal{L})$.*

It will become clear in section 4 that the next result generalizes THEOREM 1-A.

Proposition 13 *If $\mathcal{L}$ is a UC lattice, if $\{x_\alpha\}$ is a net in $\mathcal{L}$, and if $x \in \mathcal{L}$, then $x_\alpha \to x$ relative to the* M*-topology of $\mathcal{L}$ if and only if the following hold.*

1. *$\forall\, z \in \mathcal{L}$ such that $x \not\preceq z$, there exists a $\beta = \beta(z)$ such that $x_\alpha \not\preceq z \;\forall\, \alpha \trianglerighteq \beta$.*
2. *$\forall\, y \in \mathcal{L}$ such that $y \gg x$, there exists a $\beta = \beta(y)$ such that $y \gg x_\alpha \;\forall\, \alpha \trianglerighteq \beta$.*

Proof: If $x_\alpha \to x$ relative to the M-topology of $\mathcal{L}$, then (1) and (2) are clearly valid. For the converse, assume that (1) and (2) are valid and let $\mathcal{U}$ be an m-open set containing x. By Corollary 2, there are $y, x_1, ..., x_n \in \mathcal{L}$ such that

$$x \in \mathcal{F}^y_{x_1,...,x_n} \subset \mathcal{U}.$$

Hence there are $\beta(y), \beta(x_1), ..., \beta(x_n)$ such that $y \gg x_\alpha$ for all $\alpha \trianglerighteq \beta(y)$ and $x_\alpha \not\preceq x_i$ for all $i = 1, ..., n$ and all $\alpha \trianglerighteq \beta(x_i)$. By the Moore-Smith property of directed sets, there is a $\beta = \beta(y, x_1, ..., x_n)$ such that $\beta \trianglerighteq \beta(y)$ and $\beta \trianglerighteq \beta(x_i)$ for all $i = 1, ..., n$. Hence $x_\alpha \in \mathcal{U}$ for all $\alpha \trianglerighteq \beta$. This completes the proof.

Corollary 8 *Let $\mathcal{L}$ be a UC lattice, let $\{x_\alpha\}$ be a net in $\mathcal{L}$, and let $x \in \mathcal{L}$.*

1. *$x_\alpha \to x$ relative to $\lambda(\mathcal{L})$ is equivalent to* (1) *above.*
2. *$x_\alpha \to x$ relative to $\mu(\mathcal{L})$ is equivalent to* (2) *above.*

The following conjecture would be a generalization of THEOREM 1-B.

Conjecture 1 *Let $\mathcal{L}$ be a UC lattice, let $\{x_\alpha\}$ be a net in $\mathcal{L}$, and let $x \in \mathcal{L}$. Then $\{x_\alpha\}$* m*-converges to x if and only if the following hold.*

(A): *For each $\xi \preceq x$ there exists $\beta = \beta(\xi)$ and there exist $\xi_\alpha \preceq x_\alpha$ for all $\alpha \trianglerighteq \beta$ such that $\{\xi_\alpha\}$* m*-converges to ξ.*

(B): *If $\{y_\beta\}$ is a subnet of $\{x_\alpha\}$, $\xi_\beta \preceq y_\beta \;\forall\, \beta$, and $\{\xi_\beta\}$* m*-converges to $\xi \in \mathcal{L}$, then $\xi \preceq x$.*

Furthermore, (A) *and* (B) *are respectively equivalent to* (1) *and* (2) *of Proposition 13.*

The next proposition is as far as I could get in attempting to prove this conjecture.

Proposition 14 *Let $\mathcal{L}$ be a UC lattice, let $\{x_\alpha\}$ be a net in $\mathcal{L}$, and let $x \in \mathcal{L}$.*

(a): $\{x_\alpha\}$ *λ-converges to x whenever the following holds: For each $\xi \preceq x$ there exists $\beta = \beta(\xi)$ and there exist $\xi_\alpha \preceq x_\alpha$ for all $\alpha \trianglerighteq \beta$ such that $\{\xi_\alpha\}$* m*-converges to ξ. In other words, if* (A) *of the conjecture holds, then $\{x_\alpha\}$ λ-converges to x.*

(b): *If $\{x_\alpha\}$ μ-converges to x, $\{y_\beta\}$ is a subnet of $\{x_\alpha\}$, $\xi_\beta \preceq y_\beta$ for all β, and $\{\xi_\beta\}$* m*-converges to $\xi \in \mathcal{L}$, then $\xi \preceq x$. In other words, if $\{x_\alpha\}$ μ-converges to x, then* (B) *of the conjecture holds.*

Proof: (a) If $x = o \equiv \inf \mathcal{L}$, then there is no $z \in \mathcal{L}$ such that $x \not\preceq z$ and (1) of Proposition 13 is trivially satisfied; hence $\{x_\alpha\}$ λ-converges to o by Corollary 8. Suppose, then, that $x \neq o$, let z be such that $x \not\preceq z$, and let $\xi = x$. Then there is a β' and there are $\xi_\alpha \preceq x_\alpha$ for all $\alpha \trianglerighteq \beta'$ such that $\{\xi_\alpha\}$ m-converges to x. Since $\mathcal{F}_z$ is an m-open neighborhood of $\xi = x$, there is a $\beta = \beta(z)$ such that $\xi_\alpha \in \mathcal{F}_z$ for all $\alpha \trianglerighteq \beta$. Also, since $\xi_\alpha \preceq x_\alpha$ for all $\alpha \trianglerighteq \beta'$, it follows that $x_\alpha \in \mathcal{F}_z$ (equivalently $x_\alpha \not\preceq z$) for all $\alpha \trianglerighteq \beta$. This shows that $x_\alpha \to x$ relative to $\lambda(\mathcal{L})$.

(b) Before proceeding, first note that an equivalent statement of (b) is as follows: *If $\{x_\alpha\}$ μ-converges to x and $\xi \not\preceq x$, then for every subnet $\{z_\gamma\}$ of $\{x_\alpha\}$ there is no net $\{\xi_\gamma\}$ with $\xi_\gamma \preceq z_\gamma$ for all γ such that $\{\xi_\gamma\}$* m*-converges to ξ*. This is the form of (b) that we will prove. Assume, then, that $\{x_\alpha\}$ μ-converges to x and without loss of generality that $x \neq e$. (If $x = e$, then (b) is trivially satisfied.) Let $\xi \in \mathcal{L}$ be such that $\xi \not\preceq x$. There is a $y \in \mathcal{L}$ such that $y \gg x$ and $\xi \preceq y$ (for instance, e). Hence there is a $\beta(y)$ such that $y \gg x_\alpha$ for all $\alpha \trianglerighteq \beta(y)$. Let $\{z_\gamma\}$ be a subnet of $\{x_\alpha\}$ and let $\{\xi_\gamma\}$ be any net that satisfies $\xi_\gamma \preceq z_\gamma$ for all γ. Then $y \gg \xi_\gamma$ eventually and $\{\xi_\gamma\}$ cannot, therefore, m-converge to ξ. This completes the proof.

The next proposition generalizes THEOREM 1-C and 1-D.

Proposition 15 *Let $\mathcal{L}$ be a UC lattice and let $\{x_\alpha\}$ be a monotone net in $\mathcal{L}$.*

1. *If $\{x_\alpha\}$ is decreasing, then $\{x_\alpha\}$* m*-converges to* $\inf\{x_\alpha\}$.
2. *If $\{x_\alpha\}$ is increasing, then $\{x_\alpha\}$* m*-converges to* $\sup\{x_\alpha\}$.

Proof: (1) Suppose that $\inf\{x_\alpha\} \not\preceq z$. Since $\inf\{x_\alpha\} \preceq x_\alpha$ for all α, it follows for all α that $x_\alpha \not\preceq z$. Hence $\{x_\alpha\}$ λ-converges to $\inf\{x_\alpha\}$. On the other hand, suppose that $y \gg \inf\{x_\alpha\}$. It follows that $\{x_\alpha\}$ is a filtered subset of $\mathcal{L}$, because $\{x_\alpha\}$ is nonempty and a monotone decreasing net. Since $y \gg \inf\{x_\alpha\}$, it follows by Proposition 2 that there is a β such that $y \gg x_\beta$; hence, $y \gg x_\alpha$ for all $\alpha \trianglerighteq \beta$ and it follows that $\{x_\alpha\}$ μ-converges to $\inf\{x_\alpha\}$. This completes the proof of (1).

(2) Suppose that $\sup\{x_\alpha\} \not\preceq z$. There is then a β such that $x_\beta \not\preceq z$; indeed, $x_\alpha \preceq z$ for all α implies that $\sup\{x_\alpha\} \preceq z$. Because $\{x_\alpha\}$ is monotone increasing, then, it follows that $x_\alpha \not\preceq z$ for all $\alpha \trianglerighteq \beta$. Hence $\{x_\alpha\}$ λ-converges to $\sup\{x_\alpha\}$. On the other hand, if $y \gg \sup\{x_\alpha\}$, then it is clear that $y \gg x_\alpha$ for all α. Hence $\{x_\alpha\}$ μ-converges to $\sup\{x_\alpha\}$. Thus (2) is proved and the theorem follows.

Corollary 9 *Let $\mathcal{L}$ be a UC lattice and let $\{x_\alpha\}$ be a net in $\mathcal{L}$.*

1. $\{u_\alpha\} \equiv \{\sup\{x_\beta : \beta \trianglerighteq \alpha\}\}$ m*-converges to* $\limsup x_\alpha$.
2. $\{v_\alpha\} \equiv \{\inf\{x_\beta : \beta \trianglerighteq \alpha\}\}$ m*-converges to* $\liminf x_\alpha$.

Proof: Since $\alpha' \trianglerighteq \alpha \Longrightarrow u_{\alpha'} \preceq u_\alpha$ and $v_\alpha \preceq v_{\alpha'}$, we see that $\{u_\alpha\}$ and $\{v_\alpha\}$ are, respectively, monotone decreasing and monotone increasing. Hence (1) and (2) follow.

3.1 Upper and Lower Limits

Definition 23 *If $\mathcal{L}$ is a UC lattice and $\{x_\alpha\}$ is a net in $\mathcal{L}$, then we define the upper and lower limits $\overline{Lim}\ x_\alpha$ and $\underline{Lim}\ x_\alpha$, respectively, to be the supremum and infimum of the set of $x \in \mathcal{L}$ such that $x = \lim y_\beta$* (relative to the M-topology) *for some subnet $\{y_\beta\}$ of $\{x_\alpha\}$.*

Lemma 3 *If $\mathcal{L}$ is a UC lattice and $\{x_\alpha\}$ is a net in $\mathcal{L}$, then*

$$\liminf x_\alpha \preceq \underline{Lim}\ x_\alpha \preceq \overline{Lim}\ x_\alpha \preceq \limsup x_\alpha.$$

Proof: Let $\{x_{\Sigma(\gamma)}\}$ be an m-convergent subnet of $\{x_\alpha\}$ and, as above, put

$$u_\alpha = \sup\{x_\beta : \beta \trianglerighteq \alpha\} \quad \text{and} \quad v_\alpha = \inf\{x_\beta : \beta \trianglerighteq \alpha\}.$$

Then $v_\alpha \preceq x_\alpha \preceq u_\alpha$ and $v_{\Sigma(\gamma)} \preceq x_{\Sigma(\gamma)} \preceq u_{\Sigma(\gamma)}$ for all α and γ. Thus we have that

$$\lim v_{\Sigma(\gamma)} \preceq \lim x_{\Sigma(\gamma)} \preceq \lim u_{\Sigma(\gamma)}$$

where lim denotes the m-limit. Hence we obtain $\liminf x_\alpha \preceq \lim x_{\Sigma(\gamma)} \preceq \limsup x_\alpha$. Since $\underline{Lim}\ x_\alpha$ and $\overline{Lim}\ x_\alpha$ are, respectively, the infimum and supremum of the set of all such limits as $\lim x_{\Sigma(\gamma)}$, we see that the lemma follows.

Remark 12 *If $\mathcal{L}$ is a UC lattice and $\{x_\alpha\}$ is a net in $\mathcal{L}$, then the set of μ-limits of $\{x_\alpha\}$ has a least element and that least element is $\limsup x_\alpha$, i.e., $\limsup x_\alpha \equiv x$ is the smallest $x \in \mathcal{L}$ that satisfies* (2) *of Proposition 13.*

Definition 24 *If $\mathcal{L}$ is a UC lattice and $\{x_\alpha\}$ is a net in $\mathcal{L}$, then let $\mathcal{E} = \mathcal{E}(\{x_\alpha\})$ denote the set of $x \in \mathcal{L}$ such that x is the* m*-limit of a subnet of $\{x_\alpha\}$.*

Thus $\mathcal{E}(\{x_\alpha\}) \subset [\inf \mathcal{E}(\{x_\alpha\}), \sup \mathcal{E}(\{x_\alpha\})] = [\underline{Lim}\ x_\alpha, \overline{Lim}\ x_\alpha] \subset [\liminf x_\alpha, \limsup x_\alpha]$. The next proposition shows that the upper limit and superior limit are the same in a UC lattice, and will be seen in section 4 to generalize the first part of THEOREM 5.

Proposition 16 *If $\mathcal{L}$ is a UC lattice and $\{x_\alpha\}$ is a net in $\mathcal{L}$, then $\overline{Lim}\ x_\alpha = \limsup x_\alpha$.*

Proof: It is sufficient to show that $\{x_\alpha\}$ μ-converges to $\sup \mathcal{E}$. It is readily seen that each m-convergent subnet of $\{x_\alpha\}$ μ-converges to $\sup \mathcal{E}$: note that if $\{y_\beta\}$ is such a subnet, then $\{y_\beta\}$ μ-converges to precisely all the points of the closed interval $[\limsup y_\beta, e]$; then note that $\limsup y_\beta \preceq \sup \mathcal{E}$ for all the $\{y_\beta\}$ being considered. We will complete the proof by showing that every subnet of $\{x_\alpha\}$ μ-converges to $\sup \mathcal{E}$.

Suppose on the contrary, then, that $\{y_\beta : \beta \in \Delta\}$ is a subnet of $\{x_\alpha\}$ that does not μ-converge to $\sup \mathcal{E}$. Then $\exists\, y \in \mathcal{L}$ such that $y \gg \sup \mathcal{E}$ and there exists a function

$$\Sigma : \Delta \longrightarrow \Delta : \beta' \longmapsto \Sigma(\beta')$$

such that $\Sigma(\beta') \trianglerighteq \beta'$ and $y \not\gg y_{\Sigma(\beta')}$ for all $\beta' \in \Delta$. That Σ satisfies

$$\forall\, \beta' \in \Delta\ \exists\, \gamma \in \Delta \text{ such that } \beta \trianglerighteq \gamma \Longrightarrow \Sigma(\beta) \trianglerighteq \beta'$$

can be seen by putting $\gamma = \beta'$, for then it reads

$$\forall\, \beta' \in \Delta,\ \beta \trianglerighteq \beta' \Longrightarrow \Sigma(\beta) \trianglerighteq \beta'$$

and this is true because $\Sigma(\beta) \trianglerighteq \beta \trianglerighteq \beta'$. Therefore, $\{y_{\Sigma(\beta')} : \beta' \in \Delta\}$ is a subnet of $\{y_\beta : \beta \in \Delta\}$ and therefore a subnet of $\{x_\alpha\}$. Now note that $\{y_{\Sigma(\beta')} : \beta' \in \Delta\}$ has an m-convergent subnet $\mathcal{N}$, because $\mathbf{M}(\mathcal{L})$ is a compact space; according to what we have already shown, then, this subnet must μ-converge to $\sup \mathcal{E}$. But according to the foregoing, $y \gg \sup \mathcal{E}$ and $y \not\gg y_{\Sigma(\beta')}$ for all $\beta' \in \Delta$; hence, y is not way above any of the terms of $\mathcal{N}$. This contradiction completes the proof.

Corollary 10 *If $\mathcal{L}$ is a UC lattice and $\{x_\alpha\}$ is a net in $\mathcal{L}$, then $\overline{Lim}\ x_\alpha$ is the smallest $x \in \mathcal{L}$ that satisfies* (2) *of Proposition 13.*

This corollary and the next theorem together will be seen to generalize most of THEOREM 3.

Theorem 10 *If $\mathcal{L}$ is a UC lattice and $\{x_\alpha\}$ is a net in $\mathcal{L}$, then $\{x_\alpha\}$* m*-converges to $x \in \mathcal{L}$ if and only if $\underline{Lim}\ x_\alpha = \overline{Lim}\ x_\alpha = x$.*

Proof: For UC lattices, I will henceforth indicate that $\{x_\alpha\}$ m-converges to x by writing either $\lim x_\alpha = x$ or $x_\alpha \to x$. Now, it is clear that $\lim x_\alpha = x \Longrightarrow \overline{Lim}\ x_\alpha = x$ because $x = \limsup x_\alpha$ and we always have $\overline{Lim}\ x_\alpha = \limsup x_\alpha$. That $\lim x_\alpha = x$ also implies $\underline{Lim}\ x_\alpha = x$ follows from the evident fact that $\mathcal{E}(\{x_\alpha\})$ is the single element x when $\lim x_\alpha = x$. More briefly, $x_\alpha \to x \Longrightarrow \mathcal{E}(\{x_\alpha\}) = \{x\}$; hence

$$\underline{Lim}\ x_\alpha = \inf\{x\} = x = \sup\{x\} = \overline{Lim}\ x_\alpha.$$

On the other hand, if $\underline{Lim}\ x_\alpha = \overline{Lim}\ x_\alpha = x$, then $\mathcal{E}(\{x_\alpha\})$ is the single element x: (1) by compactness, $\{x_\alpha\}$ always has an m-convergent subnet, so that $\mathcal{E}(\{x_\alpha\})$ is not empty, and (2) every element $y \in \mathcal{E}(\{x_\alpha\})$ must satisfy $x \preceq y \preceq x$, which implies that $y = x$. Thus in this case we have that every m-convergent subnet converges to $x = \limsup x_\alpha$ and have finally to show that all subnets of $\{x_\alpha\}$ m-converge to x.

Let $\{y_\beta\}$ be an arbitrary subnet of $\{x_\alpha\}$. If $\{y_\beta\}$ does not converge to x, then $\{y_\beta\}$ must have a subnet, no subnet of which m-converges to x. By compactness, however, every subnet of $\{y_\beta\}$ has an m-convergent subnet, and by what we have already shown, this subnet must m-converge to x. This contradiction completes the proof.

The *Moore-Smith order topology* of a complete lattice $\mathcal{L}$ is defined via the concept of *order convergence*. If $(\mathrm{D}, \trianglerighteq)$ is a directed set and $\{x_\alpha : \alpha \in \mathrm{D}\}$ is a net in $\mathcal{L}$, then $\{x_\alpha\}$ is said to *order converge* to $x \in \mathcal{L}$ if $\liminf x_\alpha = \limsup x_\alpha = x$. The collection $\mathcal{S}_o$ of pairs $(\{x_\alpha\}, x)$, where $\{x_\alpha\}$ is a net in $\mathcal{L}$ that order converges to x, defines the topology $\mathcal{O}(\mathcal{S}_o)$ on $\mathcal{L}$ called the (Moore-Smith) order topology. Note that $\mathcal{S}_o$ is not necessarily a convergence class for $\mathcal{O}(\mathcal{S}_o)$. We now have the following corollary of the last theorem.

Corollary 11 *The order-convergent nets of a UC lattice* m*-converge to their order limits.*

This corollary does not imply that the M-topology of a UC lattice is weaker than the order topology; this is a fact, however. We may therefore incidentally note the following.

Remark 13 *The* M*-topology of a UC lattice is at once stronger than the interval topology and weaker than the order topology.*

For the next main result (Prop. 17) the following lemmas will be useful.

Lemma 4 *Let $\mathcal{L}$ be a UC lattice, let $\{x_\alpha\}$ be a net in $\mathcal{L}$, and let $\Lambda(\{x_\alpha\})$ denote the set of λ-limits of $\{x_\alpha\}$. Then we have the following.*

1. *If $x \in \mathcal{L}$ is a λ-limit of $\{x_\alpha\}$, then ξ is a λ-limit of $\{x_\alpha\}$ for all $\xi \preceq x$.*

2. $\sup \Lambda(\{x_\alpha\}) \in \Lambda(\{x_\alpha\})$.

3. $\Lambda(\{x_\alpha\}) = [o, \sup \Lambda(\{x_\alpha\})]$.

Proof: (1) For all $z \in \mathcal{L}$ such that $x \not\preceq z$, there exists a $\beta(z)$ such that $\alpha \trianglerighteq \beta(z) \Longrightarrow x_\alpha \not\preceq z$. Let $\zeta \in \mathcal{L}$ be such that $\xi \not\preceq \zeta$. Then, since $x \not\preceq \zeta$, it follows that there is a $\beta(\zeta)$ such that $\alpha \trianglerighteq \beta(\zeta) \Longrightarrow x_\alpha \not\preceq \zeta$. This proves (1).

(2) For brevity put $\sup \Lambda(\{x_\alpha\}) = \ell$ and suppose that $z \in \mathcal{L}$ satisfies $\ell \not\preceq z$. Since

$$y \preceq z \; \forall\, y \in \Lambda(\{x_\alpha\}) \Longrightarrow \ell \preceq z,$$

let $\eta \in \Lambda(\{x_\alpha\})$ be such that $\eta \not\preceq z$. Then there is a $\beta(\eta)$ such that $\alpha \trianglerighteq \beta(\eta) \Longrightarrow x_\alpha \not\preceq \eta$. Hence ℓ is a λ-limit of $\{x_\alpha\}$. This proves (2), and (3) is now obvious.

Lemma 5 *If $\mathcal{L}$ is a UC lattice and $\{x_\alpha\}$ is an* m*-convergent net in $\mathcal{L}$, then*

$$\underline{Lim}\, x_\alpha = \sup \Lambda(\{x_\alpha\}).$$

Proof: Since $\Lambda(\{x_\alpha\}) = [o, \sup \Lambda(\{x_\alpha\})]$ and $M(\{x_\alpha\}) = [\limsup x_\alpha, e]$ is the set of μ-limits of $\{x_\alpha\}$, it follows that $\Lambda(\{x_\alpha\}) \cap M(\{x_\alpha\})$ is exclusively either the single element $\limsup x_\alpha$ or $\emptyset$, depending on whether $\{x_\alpha\}$ m-converges or not. Since $x_\alpha \to \limsup x_\alpha$ if and only if $[o, \sup \Lambda(\{x_\alpha\})] \cap [\limsup x_\alpha, e] = \limsup x_\alpha$, and since the latter is true if and only if $\sup \Lambda(\{x_\alpha\}) = \limsup x_\alpha$, we see that $\sup \Lambda(\{x_\alpha\}) = \underline{Lim}\, x_\alpha$.

Lemma 6 *Let $\mathcal{L}$ be a UC lattice and let $\{x_\alpha\}$ be a net in $\mathcal{L}$.*

1. $\mathcal{E}(\{x_\alpha\}) \subset [\sup \Lambda(\{x_\alpha\}), \limsup x_\alpha]$.

2. *Equivalently,* $\sup \Lambda(\{x_\alpha\}) \preceq \underline{Lim}\, x_\alpha$.

Proof: Note that a subnet $\{y_\beta\}$ of $\{x_\alpha\}$ such that $y_\beta \to x \in \mathcal{E}(\{x_\alpha\})$ must both λ- and μ-converge to x. Since $\Lambda(\{y_\beta\}) \supset \Lambda(\{x_\alpha\})$ and $M(\{y_\beta\}) \supset M(\{x_\alpha\})$, it follows that

$$\sup \Lambda(\{x_\alpha\}) \preceq x \preceq \limsup x_\alpha.$$

Hence $\sup \Lambda(\{x_\alpha\}) \preceq \underline{Lim}\, x_\alpha$ and the proof is complete.

Lemma 7 *Let $\mathcal{L}$ be a UC lattice, let $\{x_\alpha\}$ be a net in $\mathcal{L}$, and let $\mathcal{T}$ denote the set of* m*-convergent subnets of $\{x_\alpha\}$.*

1. $\sup \Lambda(\{y_\beta\}) = \limsup y_\beta = \lim y_\beta$ *for all* $\{y_\beta\} \in \mathcal{T}$.

2. $\underline{Lim}\, x_\alpha = \inf\{\sup \Lambda(\{y_\beta\}) : \{y_\beta\} \in \mathcal{T}\}$.

3. $\sup \Lambda(\{x_\alpha\}) \preceq \underline{Lim}\, x_\alpha \preceq \sup \Lambda(\{y_\beta\})$ *for all* $\{y_\beta\} \in \mathcal{T}$.
4. $\bigcap\{\Lambda(\{y_\beta\}) : \{y_\beta\} \in \mathcal{T}\} = \Lambda(\{x_\alpha\})$.

Proof: (1) is clear. (2) By definition, $\underline{Lim}\, x_\alpha = \inf\{\lim y_\beta : \{y_\beta\} \in \mathcal{T}\}$. But for all $\{y_\beta\} \in \mathcal{T}$ we have $\sup \Lambda(\{y_\beta\}) = \lim y_\beta$. Hence

$$\underline{Lim}\, x_\alpha = \inf\{\sup \Lambda(\{y_\beta\}) : \{y_\beta\} \in \mathcal{T}\}.$$

(3) now follows from (1) and (2). For (4) we note that $\bigcap\{\Lambda(\{y_\beta\}) : \{y_\beta\} \in \mathcal{T}\}$ is an m-closed lower superset of the closed lower set $\Lambda(\{x_\alpha\})$. Suppose that there is an element $\xi \in \Lambda(\{y_\beta\})$ for all $\{y_\beta\} \in \mathcal{T}$, but that $\xi \notin \Lambda(\{x_\alpha\})$. Then there is a z_0 such that $\xi \not\preceq z_0$ for which to every α there corresponds a $\gamma(\alpha) \trianglerighteq \alpha$ such that $x_{\gamma(\alpha)} \preceq z_0$. Thus $\{x_{\gamma(\alpha)}\}$ is a subnet of $\{x_\alpha\}$ all of whose terms preceed or equal z_0. Since by compactness this subnet has an m-convergent subnet, we see that a contradiction has arisen. Thus $\bigcap\{\Lambda(\{y_\beta\}) : \{y_\beta\} \in \mathcal{T}\} = \Lambda(\{x_\alpha\})$.

Proposition 17 *If $\mathcal{L}$ is a UC lattice and $\{x_\alpha\}$ is a net in $\mathcal{L}$, then*

$$\underline{Lim}\, x_\alpha = \sup \Lambda(\{x_\alpha\}).$$

Proof: Again let $\mathcal{T}$ denote the set of m-convergent subnets of $\{x_\alpha\}$. By Lemma 7

$$\underline{Lim}\, x_\alpha = \inf\{\sup \Lambda(\{y_\beta\}) : \{y_\beta\} \in \mathcal{T}\}.$$

The term on the right of the above equation is the infimum of the upper endpoints of the closed intervals $[o, \sup \Lambda(\{y_\beta\})] = \Lambda(\{y_\beta\})$ arising from the $\{y_\beta\} \in \mathcal{T}$. Said infimum therefore lies in all these closed intervals and hence in their intersection. By Lemma 7 we therefore have that

$$\begin{aligned} \underline{Lim} x_\alpha &= \inf\{\sup \Lambda(\{y_\beta\}) : \{y_\beta\} \in \mathcal{T}\} \\ &\preceq \sup \bigcap\{\Lambda(\{y_\beta\}) : \{y_\beta\} \in \mathcal{T}\} \\ &= \sup \Lambda(\{x_\alpha\}), \end{aligned}$$

i.e., $\underline{Lim}\, x_\alpha \preceq \sup \Lambda(\{x_\alpha\})$. By Lemma 6, then, this completes the proof.

Corollary 12 *If $\mathcal{L}$ is a UC lattice and $\{x_\alpha\}$ is a net in $\mathcal{L}$, then $\underline{Lim}\, x_\alpha$ is the largest $x \in \mathcal{L}$ that satisfies* (1) *of Proposition 13.*

Proof: By definition and Corollary 8, $\Lambda(\{x_\alpha\})$ is the set of $x \in \mathcal{L}$ that satisfies (1) of Proposition 13. By Proposition 17 and Lemma 4, $\underline{Lim}\, x_\alpha$ is the largest element of $\Lambda(\{x_\alpha\})$.

We summarize Corollary 10, Theorem 10, and Corollary 12 as follows.

Theorem 11 *If $\mathcal{L}$ is a UC lattice and $\{x_\alpha\}$ is a net in $\mathcal{L}$, then* (a) $\underline{Lim}\, x_\alpha$ *is the largest $x \in \mathcal{L}$ that satisfies* (1) *of Proposition 13,* (b) $\overline{Lim}\, x_\alpha$ *is the smallest $x \in \mathcal{L}$ that satisfies* (2) *of Proposition 13, and* (c) $\{x_\alpha\}$ m*-converges to $x \in \mathcal{L}$ if and only if $\underline{Lim}\, x_\alpha = \overline{Lim}\, x_\alpha = x$.*

This theorem will be seen with Theorem 13 to fully generalize THEOREM 3.

3.2 USC and LSC Mappings

For the next main result (Prop. 18), the following lemmas will be useful.

Lemma 8 *If Ω is a topological space, $\mathcal{L}$ is a UC lattice, and $f : \Omega \longrightarrow \mathcal{L}$, then f is LSC relative to $\mathrm{M}(\mathcal{L})$ if and only if for each $x \in \mathcal{L}$ and for every convergent net $\{\omega_\alpha\}$ in Ω such that $f(\omega_\alpha) \preceq x$ for all α, we have that $f(\lim \omega_\alpha) \preceq x$.*

Proof: We have that f is LSC if and only if

$$f^{-1}(\mathcal{F}_x) = \{\omega \in \Omega : f(\omega) \not\preceq x\} \text{ is open in } \Omega \text{ for all } x \in \mathcal{L}.$$

Thus f is LSC if and only if $\{\omega \in \Omega : f(\omega) \preceq x\}$ is closed in Ω for all $x \in \mathcal{L}$. Finally, $\{\omega \in \Omega : f(\omega) \preceq x\}$ is closed in Ω if and only if for every convergent net $\{\omega_\alpha\}$ in Ω such that $f(\omega_\alpha) \preceq x$ for all α, it follows that $f(\lim \omega_\alpha) \preceq x$.

Lemma 9 *If $\mathcal{L}$ is a complete lattice and $\{S_\alpha : \alpha \in \mathcal{A}\}$ is a family of subsets of $\mathcal{L}$, then*

$$\inf \bigcup_{\alpha \in \mathcal{A}} S_\alpha = \inf\{\inf S_\alpha : \alpha \in \mathcal{A}\}.$$

Proof: For all $\alpha \in \mathcal{A}$ it is clear that $\inf \cup_\alpha S_\alpha \preceq \inf S_\alpha$; hence,

$$\inf \bigcup_{\alpha \in \mathcal{A}} S_\alpha \preceq \inf\{\inf S_\alpha : \alpha \in \mathcal{A}\}.$$

On the other hand, if $x \in \cup_\alpha S_\alpha$, then $x \in S_{\alpha_0}$ for some $\alpha_0 \in \mathcal{A}$. Therefore $\inf S_{\alpha_0} \preceq x$ and we see that $\inf\{\inf S_\alpha : \alpha \in \mathcal{A}\} \preceq x$. This shows that

$$\inf\{\inf S_\alpha : \alpha \in \mathcal{A}\} \preceq \inf \bigcup_{\alpha \in \mathcal{A}} S_\alpha,$$

and this completes the proof.

Lemma 10 *If $\mathcal{L}$ is a UC lattice and $\{x_\alpha\}$ is a net in $\mathcal{L}$, then*

$$\underline{Lim}\, x_\alpha = \inf\{\sup\{y_\beta\} : \{y_\beta\} \in \mathcal{T}(\{x_\alpha\})\}$$

where $\mathcal{T}(\{x_\alpha\})$ denotes the set of **m***-convergent subnets of* $\{x_\alpha\}$.

Proof: Let $\{y_\beta : \beta \in \mathbf{D}\}$ be any **m**-convergent subnet of $\{x_\alpha\}$. Then $\{y_\beta : \beta \trianglerighteq \gamma\}$ is an **m**-convergent subnet of $\{x_\alpha\}$ for all fixed γ. Thus the set

$$S = \{\sup\{y_\beta : \beta \in \mathbf{D}\}, \{\sup\{y_\beta : \beta \trianglerighteq \gamma\} : \gamma \in \mathbf{D}\}\}$$

is a subset of $\{\sup\{y_\beta\} : \{y_\beta\} \in \mathcal{T}(\{x_\alpha\})\}$; in fact, the latter is the union of the subsets S, one for each $\{y_\beta\} \in \mathcal{T}(\{x_\alpha\})$. By the last lemma we therefore have

$$\inf\{\sup\{y_\beta\} : \{y_\beta\} \in \mathcal{T}(\{x_\alpha\})\} = \inf_{\{y_\beta\} \in \mathcal{T}(\{x_\alpha\})} \{\sup\{y_\beta : \beta \trianglerighteq \gamma\} : \gamma \in \mathbf{D}\}.$$

Since the infimum on the right of the above is precisely $\underline{Lim}\, x_\alpha$, the proof is complete.

Lemma 11 *If Ω is a topological space, $\mathcal{L}$ is a UC lattice, $f : \Omega \longrightarrow \mathcal{L}$, and f is LSC (relative to $\mathbf{M}(\mathcal{L})$), then $f(\lim \omega_\alpha) \preceq \underline{Lim}\, f(\omega_\alpha)$ for every convergent net $\{\omega_\alpha\}$ in Ω.*

Proof: For all **m**-convergent subnets $\{f(\omega_{\Sigma(\beta)})\}$ of $\{f(\omega_\alpha)\}$, we have for all β that

$$f(\omega_{\Sigma(\beta)}) \preceq \sup_{\beta'}\{f(\omega_{\Sigma(\beta')})\}.$$

Since f is LSC, we have for all $\{f(\omega_{\Sigma(\beta)})\}$ that $f(\lim \omega_\alpha) \preceq \sup_{\beta'}\{f(\omega_{\Sigma(\beta')})\}$. Hence

$$f(\lim \omega_\alpha) \preceq \inf \sup_{\beta'}\{f(\omega_{\Sigma(\beta')})\},$$

where the infimum on the right extends over all convergent subnets of $\{f(\omega_\alpha)\}$. By the last lemma, this infimum is precisely $\underline{Lim}\, f(\omega_\alpha)$. This completes the proof.

Proposition 18 *If Ω is a topological space, $\mathcal{L}$ is a UC lattice, and $f : \Omega \longrightarrow \mathcal{L}$, then f is LSC if and only if $f(\lim \omega_\alpha) \preceq \underline{Lim}\, f(\omega_\alpha)$ for every convergent net $\{\omega_\alpha\}$ in Ω.*

Proof: Assume first that $f(\lim \omega_\alpha) \preceq \underline{Lim}\, f(\omega_\alpha)$ for every convergent net $\{\omega_\alpha\}$ in Ω. Given $x \in \mathcal{L}$ we restrict our attention to those of the $\{\omega_\alpha\}$ such that $f(\omega_\alpha) \preceq x$ for all α. For these $\{\omega_\alpha\}$ it follows that

$\underline{Lim}\ f(\omega_\alpha) \preceq x$ because all m-convergent subnets of $\{f(\omega_\alpha)\}$ have m-limits $\xi \preceq x$; moreover, since $f(\lim \omega_\alpha) \preceq \underline{Lim}\ f(\omega_\alpha)$, it follows that $f(\lim \omega_\alpha) \preceq x$. Hence f is LSC by Lemma 8. Since the converse of what we have just shown is precisely the last lemma, the proof is complete.

We summarize Propositions 4, 16, and 18 to obtain a generalization of THEOREM 4.

Theorem 12 *Let Ω be a topological space, let $\mathcal{L}$ be a UC lattice, and let $f : \Omega \longrightarrow \mathcal{L}$.*

1. *f is USC if and only if $f(\lim \omega_\alpha) \succeq \overline{Lim}\ f(\omega_\alpha)$ for every convergent net $\{\omega_\alpha\}$ in Ω.*
2. *f is LSC if and only if $f(\lim \omega_\alpha) \preceq \underline{Lim}\ f(\omega_\alpha)$ for every convergent net $\{\omega_\alpha\}$ in Ω.*

3.3 Lower Limits and Frink's Convergence Topology

For the next theorem I first recall some facts about *cofinal* and *residual* subsets.

Definition 25 *Let $(\mathbf{D}, \trianglerighteq)$ be a directed set. Then a subset $\mathbf{C}$ of $\mathbf{D}$ is called cofinal if for every $\alpha \in \mathbf{D}$ there exists a $\gamma \in \mathbf{C}$ such that $\gamma \trianglerighteq \alpha$. A subset $\mathbf{R}$ of $\mathbf{D}$ is called residual if there is a $\beta \in \mathbf{D}$ such that $\mathbf{R} = \{\alpha \in \mathbf{D} : \alpha \trianglerighteq \beta\}$.*

Remark 14 *Let $(\mathbf{D}, \trianglerighteq)$ be a directed set. If $\mathbf{C}$ is a cofinal subset of $\mathbf{D}$, then $(\mathbf{C}, \trianglerighteq)$ is a directed set; moreover, if $\{x_\alpha : \alpha \in \mathbf{D}\}$ is a net, then $\{x_\gamma : \gamma \in \mathbf{C}\}$ is a subnet of $\{x_\alpha : \alpha \in \mathbf{D}\}$. Finally, if a subset of $\mathbf{D}$ is not cofinal, then its complement in $\mathbf{D}$ is.*

Lemma 12 *If $\mathbf{D}$ is a directed set, if $\mathbf{C} \subset \mathbf{D}$ is a cofinal subset of $\mathbf{D}$, and if $\mathbf{R} \subset \mathbf{C}$ is a residual subset of $\mathbf{C}$, then $\mathbf{R}$ is a cofinal subset of $\mathbf{D}$.*

Proof: Let $\alpha \in \mathbf{D}$ be given. Then there is a $\beta' \in \mathbf{C}$ such that $\beta' \trianglerighteq \alpha$. Let $\mathbf{R} = \{\gamma \in \mathbf{C} : \gamma \trianglerighteq \lambda\}$ and let $\beta \in \mathbf{D}$ satisfy $\beta \trianglerighteq \beta'$ and $\beta \trianglerighteq \lambda$. Then $\beta \in \mathbf{R}$ and $\beta \trianglerighteq \alpha$.

Lemma 13 *Let $\mathcal{L}$ be a UC lattice, let $\{x_\alpha\}$ be a net in $\mathcal{L}$, and let $\{y_\beta\}$ be a subnet of $\{x_\alpha\}$ that* m*-converges to ξ. Then $\xi \preceq \sup\{x_\alpha\}$.*

Proof: $\sup\{x_\alpha\} \succeq \sup\{x_\gamma : \gamma \trianglerighteq \alpha\} = u_\alpha$ for all α. Since $\{u_\alpha\}$ m-converges to $\limsup x_\alpha$, it follows that $\sup \mathcal{E}(\{x_\alpha\}) = \limsup x_\alpha \preceq \sup\{x_\alpha\}$. Since $\xi \in \mathcal{E}(\{x_\alpha\})$, we are done.

Lemma 14 *If $\mathcal{L}$ is a UC lattice, $\mathbf{D}$ is a directed set, $\{x_\alpha : \alpha \in \mathbf{D}\}$ is a net in $\mathcal{L}$, and $\mathcal{C}$ denotes the cofinal subsets of $\mathbf{D}$, then*

$$\inf_{\mathbf{C}\in\mathcal{C}} \sup\{x_\gamma : \gamma \in \mathbf{C}\} \preceq \underline{Lim}\, x_\alpha.$$

Proof: Since $\underline{Lim}\, x_\alpha = \inf\{\sup\{y_\beta\} : \{y_\beta\} \in \mathcal{T}(\{x_\alpha\})\}$, it is sufficient to show for each $\{y_\beta\} \in \mathcal{T}(\{x_\alpha\})$ that there exists a $\mathbf{C} \in \mathcal{C}$ such that $\sup\{y_\beta\} = \sup\{x_\gamma : \gamma \in \mathbf{C}\}$.

Accordingly, let $\{y_\beta : \beta \in \mathbf{D}'\} \in \mathcal{T}(\{x_\alpha\})$. Thus there is a $\Sigma : \mathbf{D}' \longrightarrow \mathbf{D}$ that satisfies the subnet condition, $y_\beta = x_{\Sigma(\beta)}$, and $\{y_\beta\}$ m-converges to, say, y. Let $\mathbf{C}'$ be a cofinal subset of $\mathbf{D}'$. Then $\{y_\beta : \beta \in \mathbf{C}'\}$ is a subnet of $\{x_\alpha : \alpha \in \mathbf{D}\}$. Moreover, $\mathbf{C} = \Sigma(\mathbf{C}')$ is a cofinal subset of $\mathbf{D}$. It follows that $\{x_\gamma : \gamma \in \mathbf{C}\}$ is a subnet of $\{x_\alpha : \alpha \in \mathbf{D}\}$ that has precisely the same set of terms as $\{y_\beta : \beta \in \mathbf{D}'\}$. Hence

$$\sup\{y_\beta : \beta \in \mathbf{D}'\} = \sup\{x_\gamma : \gamma \in \mathbf{C}\}.$$

Theorem 13 *If $\mathcal{L}$ is a UC lattice, $\mathbf{D}$ is a directed set, $\{x_\alpha : \alpha \in \mathbf{D}\}$ is a net in $\mathcal{L}$, and $\mathcal{C}$ denotes the cofinal subsets of $\mathbf{D}$, then*

1. $\overline{Lim}\, x_\alpha = \limsup x_\alpha$.
2. $\underline{Lim}\, x_\alpha = \inf_{\mathbf{C}\in\mathcal{C}} \sup\{x_\gamma : \gamma \in \mathbf{C}\}$.

Proof: (1) is Proposition 16.

For (2) we first show that $\inf \mathcal{E}(\{x_\alpha\}) \preceq \inf_{\mathbf{C}\in\mathcal{C}} \sup\{x_\gamma : \gamma \in \mathbf{C}\}$, i.e.,

$$x \preceq y \,\forall\, y \in \mathcal{E}(\{x_\alpha\}) \Longrightarrow x \preceq \sup\{x_\gamma : \gamma \in \mathbf{C}\} \,\forall\, \mathbf{C} \in \mathcal{C}.$$

Assume that $x \preceq y$ for all $y \in \mathcal{E}(\{x_\alpha\})$ and let $\mathbf{C}$ be a cofinal subset of $\mathbf{D}$. Then $\{x_\gamma : \gamma \in \mathbf{C}\}$ is a subnet of $\{x_\alpha : \alpha \in \mathbf{D}\}$, $\{x_\gamma : \gamma \in \mathbf{C}\}$ has an m-convergent subnet $\{z_\beta\}$, the m-limit ξ of $\{z_\beta\}$ lies in $\mathcal{E}(\{x_\alpha\})$, $x \preceq \xi$ by assumption, and by Lemma 13 $\xi \preceq \sup\{x_\gamma : \gamma \in \mathbf{C}\}$. Since $\mathbf{C}$ was an arbitrary cofinal subset of $\mathbf{D}$, it follows that $x \preceq \sup\{x_\gamma : \gamma \in \mathbf{C}\}$ for all $\mathbf{C} \in \mathcal{C}$. To complete the proof we must show that $\inf_{\mathbf{C}\in\mathcal{C}} \sup\{x_\gamma : \gamma \in \mathbf{C}\} \preceq \inf \mathcal{E}(\{x_\alpha\})$. But this is precisely Lemma 14.

This theorem generalizes THEOREM 5, and, together with Theorem 10, shows that the M-topology of a UC lattice is identical with Frink's [11] convergence topology.

We may finally note that Theorem 2 generalizes the beginning of THEOREM 1, apart from the second countability property, and a portion of THEOREM 2. The next proposition generalizes still another part of THEOREM 2.

Proposition 19 *If $\mathcal{L}$ is a UC lattice, then the mapping of $\mathcal{L} \times \mathcal{L}$ to $\mathcal{L}$ given by the meet operation (i.e., by $(x, y) \longmapsto x \wedge y$) is USC.*

Proof: Let $\{x_\alpha\}$ and $\{y_\alpha\}$ be m-convergent nets in $\mathcal{L}$ with m-limits x and y, respectively. It is sufficient to show that $\limsup(x_\alpha \wedge y_\alpha) \preceq x \wedge y$. Since for all α we have that $x_\alpha \wedge y_\alpha \preceq x_\alpha$ and $x_\alpha \wedge y_\alpha \preceq y_\alpha$, it follows that $\limsup(x_\alpha \wedge y_\alpha) \preceq \limsup x_\alpha = x$ and $\limsup(x_\alpha \wedge y_\alpha) \preceq \limsup y_\alpha = y$. Hence $\limsup(x_\alpha \wedge y_\alpha) \preceq x \wedge y$.

3.4 Matheron Spaces

We will finally distinguish a species of $\mathrm{M}(\mathcal{L})$—where $\mathcal{L}$ is UC—that I call a *Matheron space*. We begin with a corollary of Proposition 19.

Corollary 13 *If $\mathcal{L}$ is a UC lattice, then $\mathrm{M}(\mathcal{L})$ is a topological lattice if and only if the mapping of $\mathcal{L} \times \mathcal{L}$ to $\mathcal{L}$ given by the meet operation is LSC.*

If $\mathcal{L}$ is UC and $\mathrm{M}(\mathcal{L})$ is not a topological lattice, then I call $\mathrm{M}(\mathcal{L})$ a *Matheron space*. Thus a Matheron space is exactly the type of M-topologized UC lattice for which a full generalization of THEOREM 2 holds, i.e., for which the mapping $(x, y) \longmapsto x \wedge y$ is USC and not LSC.

Because *a distributive UC lattice, $\mathcal{L}$, is meet-continuous if and only if $\mathrm{M}(\mathcal{L})$ is a topological lattice* (this follows by duality from *GZ* VII–2.4), we obtain the following final result.

Proposition 20 *If $\mathrm{M}(\mathcal{L})$ is a distributive Matheron space, then $\mathcal{L}$ is not continuous.*

4. Open and Closed Subset Lattices

I now detail the manner in which the foregoing theory of $\mathbf{M}(\mathcal{L})$ generalizes the more concrete topological theory outlined in the Introduction, i.e., Matheron's hit-miss topology. It was indeed pointed out in section 3 that results there asserted to be generalizations of those in the Introduction would become clearly so only in view of material in this section.

Our first theorem characterizes the m-convergence of nets in $\mathbf{F}(X)$ when X is locally compact and either Hausdorff or regular. It is at once a particular case of Proposition 13, which fact is indeed its proof, and (Prop. 12) an obvious generalization of THEOREM 1-A.

Theorem 14 *Let X be a locally compact space that is either Hausdorff or regular. A net $\{F_\alpha\}$ in $\mathbf{F}(X)$ converges to $F \in \mathbf{F}(X)$ if and only if both of the following hold.*

1. *$G \subset X$ is open and $G \cap F \neq \emptyset \Longrightarrow G \cap F_\alpha \neq \emptyset$ for a residual set of α's.*
2. *$K \subset X$ is compact and $K \cap F = \emptyset \Longrightarrow K \cap F_\alpha = \emptyset$ for a residual set of α's.*

Proof: Suppose first that $\{F_\alpha\}$ m-converges to F. (1) $G \subset X$ is open and $G \cap F \neq \emptyset$ is equivalent to G^c is closed and $F \not\subset G^c$. Hence $F \in \mathcal{F}_{G^c}$ and it follows that $F_\alpha \in \mathcal{F}_{G^c}$ for for a residual set of α's. (2) $K \subset X$ is compact and $K \cap F = \emptyset$ is equivalent to $\overline{K^c} \gg F$. Hence $F \in \mathcal{F}^{\overline{K^c}}$ and it follows that $F_\alpha \in \mathcal{F}^{\overline{K^c}}$ for a residual set of α's. For the converse, simply note that the conditions (A) $G \subset X$ is open and $G \cap F \neq \emptyset \Longrightarrow G \cap F_\alpha \neq \emptyset$ for a residual set of α's, and (B) $K \subset X$ is compact and $K \cap F = \emptyset \Longrightarrow K \cap F_\alpha = \emptyset$ for a residual set of α's, are respectively equivalent to (A) $E \subset X$ is closed and $F \not\subset E \Longrightarrow F_\alpha \not\subset E$ for a residual set of α's, and (B) $E \subset X$ is closed and $E \gg F \Longrightarrow E \gg F_\alpha$ for a residual set of α's.

Theorem 15 *Let X be a locally compact space that is either Hausdorff or regular, let $\{F_\alpha\}$ be a net in $\mathbf{F}(X)$, let $F \in \mathbf{F}(X)$, and suppose that the following hold.*

(a): *For each $x \in F$ there exist $x_\alpha \in F_\alpha$ for a residual set of α's such that $x_\alpha \to x$ in X.*

(b): *If $\{F_{\Sigma(\beta)}\}$ is a subnet of $\{F_\alpha\}$, $x_{\Sigma(\beta)} \in F_{\Sigma(\beta)}\ \forall\, \beta$, and $x_{\Sigma(\beta)} \to x$ in X, then $x \in F$.*

Then $\{F_\alpha\}$ converges to F relative to $\mathbf{m}(\mathbf{F}(X))$. In fact, (a) implies (1) of the last theorem and (2) of that theorem is equivalent to (b).

Proof: It is enough to prove the last statement.

(a) $\Longrightarrow$ (1): If F is empty there is nothing to prove. If $F \neq \emptyset$, let $G \subset X$ be open with $G \cap F \neq \emptyset$ and let $x \in G \cap F$. By (a) there exist $x_\alpha \in F_\alpha$ for a residual set of α's such that $x_\alpha \to x$ in X. Since x is also in G we have that G is an open neighborhood of x; hence, $x_\alpha \in G$ for a residual set of α's and $G \cap F_\alpha \neq \emptyset$ for the same residual set.

(2) $\Longrightarrow$ (b): If $F = X$ there is nothing to prove. Assume, then, that $F \neq X$, let $x \notin F$, and let K be a closed compact neighborhood of x disjoint from F. By (2), there is a γ such that $K \cap F_\alpha = \emptyset$ for all $\alpha \trianglerighteq \gamma$. Now let $\Sigma : \Delta \longrightarrow \mathbf{D}$ satisfy the subnet condition, where Δ is a directed set and $\mathbf{D}$ is the directed set of the α's; thus $\{F_{\Sigma(\beta)} : \beta \in \Delta\}$ is a subnet of $\{F_\alpha : \alpha \in \mathbf{D}\}$. Note that the mentioned subnet condition is

$$\forall\, \alpha \in \mathbf{D}\ \exists\, \beta'(\alpha) \in \Delta \text{ such that } \beta \trianglerighteq_\Delta \beta'(\alpha) \Longrightarrow \Sigma(\beta) \trianglerighteq \alpha.$$

Let $x_{\Sigma(\beta)} \in F_{\Sigma(\beta)}$ for all $\beta \in \Delta$ and suppose that $\{x_{\Sigma(\beta)}\}$ converges in X to x. Let $\Delta' = \{\beta \in \Delta : \Sigma(\beta) \trianglerighteq \gamma\}$. Then $K \cap F_{\Sigma(\beta)} = \emptyset$ for all $\beta \in \Delta'$. It follows from the subnet condition that $\{\beta \in \Delta : \beta \trianglerighteq_\Delta \beta'(\gamma)\} \subset \Delta'$, and hence that $K \cap F_{\Sigma(\beta)} = \emptyset$ for all $\beta \trianglerighteq_\Delta \beta'(\gamma)$. Since $\{x_{\Sigma(\beta)} : \beta \in \Delta\}$ converges in X to x, it follows that the subnet $\{x_{\Sigma(\beta)} : \beta \trianglerighteq_\Delta \beta'(\gamma)\}$ converges in X to x. But none of the $x_{\Sigma(\beta)}$ in this subnet lie in K and K is a neighborhood of x. This contradiction proves that (2) $\Longrightarrow$ (b).

(b) $\Longrightarrow$ (2): We assume that (2) is false and show that then (b) is false. Let $K \subset X$ be a compact set with $K \cap F = \emptyset$ such that for each α there is a $\Sigma(\alpha) \trianglerighteq \alpha$ such that $K \cap F_{\Sigma(\alpha)} \neq \emptyset$. Any of the maps $\Sigma : \mathbf{D} \longrightarrow \mathbf{D}$ so defined satisfy the subnet condition, for $\beta \trianglerighteq \Sigma(\alpha) \Longrightarrow \Sigma(\beta) \trianglerighteq \alpha$. For each α let $x_\alpha \in K \cap F_{\Sigma(\alpha)}$. By compactness, $\{x_\alpha\}$ has a subnet that converges to an $x \in K$; hence $x \notin F$ and (b) is false.

Lemma 15 *If X is a topological space that satisfies the first axiom of countability, then each $x \in X$ has a countable local base $\{G_i : i = 1, 2, ...\}$ such that $G_1 \supset G_2 \supset ... \supset G_k \supset ....$*

Proof: Let $\{U_i\}$ be a countable local base at x for the topology of X, and let $G_i = \cap_{k=1}^{i} U_k$ for all i. Then $\{G_i : i = 1, 2, ...\}$ is countable set of open neighborhoods of x such that

$$G_1 \supset G_2 \supset ... \supset G_k \supset ....$$

Moreover, if V is an open neighborhood of x, then let n be a positive integer such that $U_n \subset V$. Since $G_n \subset U_n \subset V$, the lemma can be seen to follow.

Lemma 16 *Let X be a locally compact space that satisfies the first axiom of countability and is either Hausdorff or regular, let $\{F_i\}$ be a sequence in $\mathbf{F}(X)$, and let $F \in \mathbf{F}(X)$. Then, referring to the statements below, we have that* (I) *implies* (A).

(I): *$G \subset X$ is open and $G \cap F \neq \emptyset \Longrightarrow G \cap F_i \neq \emptyset$ for all but finitely many i.*

(A): *For each $x \in F$ there exist $x_i \in F_i$ for all but finitely many i such that $x_i \to x$ in X.*

Proof: If F is empty there is nothing to prove. If $F \neq \emptyset$, let $x \in F$, and let $\{G_i : i = 1, 2, ...\}$ be a countable local base at x such that $G_1 \supset G_2 \supset ... \supset G_k \supset ....$ Let $\mathcal{Z}_+$ denote the set of positive integers. Since $G_k \cap F \neq \emptyset$ for all $k = \mathcal{Z}_+$, it follows from (I) that for each $k \in \mathcal{Z}_+$ there exists a positive integer N_k such that

$$G_k \cap F_i \neq \emptyset \text{ for all } i \geq N_k.$$

For each $k \in \mathcal{Z}_+$ and each $i = N_k, N_k + 1, ..., N_{k+1} - 1$, choose $x_i \in G_k \cap F_i$. This defines a sequence $\{x_i : i \geq N_1\}$ such that $x_i \in F_i$ for all $i \geq N_1$. If V is an open neighborhood of x, then there is a positive integer $k(V)$ such that $G_{k(V)} \subset V$; moreover, $x_i \in G_{k(V)} \subset V$ for all $i \geq N_{k(V)}$; hence $x_i \to x$ in X. This completes the proof.

We therefore obtain the following obvious generalization of THEOREM 1-A and 1-B.

Theorem 16 *Let X be a locally compact space that satisfies the first axiom of countability and is either Hausdorff or regular, let $\{F_i\}$ be a sequence in $\mathbf{F}(X)$, and let $F \in \mathbf{F}(X)$. Then $F_i \to F$ relative to $\mathbf{m}(\mathbf{F}(X))$ if and only if the following hold.*

(a): *For each $x \in F$ there exist $x_i \in F_i$ for all but finitely many i such that $x_i \to x$ in X.*

(b): *If $\{F_{i_k}\}$ is a subsequence of $\{F_i\}$, $x_{i_k} \in F_{i_k}\ \forall\, k$, and $x_{i_k} \to x$ in X, then $x \in F$.*

Moreover, (a) and (b) are respectvely equivalent to (1) and (2) below.

(1): *$G \subset X$ is open and $G \cap F \neq \emptyset \Longrightarrow G \cap F_i \neq \emptyset$ for all but finitely many i.*

(2): *$K \subset X$ is compact and $K \cap F = \emptyset \Longrightarrow K \cap F_i = \emptyset$ for all but finitely many i.*

Proposition 21 *If X is a locally compact first countable Hausdorff space, then the subset of $\mathbf{F}(X)$ that consists of all the one-point subsets of X has a relative* **M**-*topology that coincides with the given topology of X when each $x \in X$ is identified with $\{x\} \in \mathbf{F}(X)$.*

Proof: Another way to state the conclusion is: If $\{x_i\}$ is a sequence in X and $x \in X$, then $x_i \to x$ in X if and only if $\{x_i\} \to \{x\}$ relative to $\mathbf{m}(\mathbf{F}(X))$. Suppose, then, that $\{x_i\}$ is a sequence in X, $x \in X$, and $x_i \to x$ in X. Then $\{x_i\} \to \{x\}$ relative to $\mathbf{m}(\mathbf{F}(X))$ by the last theorem. On the other hand, if $\{x_i\}$ is a sequence in X, $x \in X$, and $\{x_i\} \to \{x\}$ relative to $\mathbf{m}(\mathbf{F}(X))$, then every subsequence of $\{x_i\}$ X-converges to x by (b) of the last theorem, and it follows that $\{x_i\}$ X-converges to x.

Theorem 15 has the following immediate corollary.

Corollary 14 *Let X be a locally compact Hausdorff space, let $\{x_\alpha\}$ be a net in X, and let $x \in X$. Then $x_\alpha \to x$ in X implies that $\{x_\alpha\} \to \{x\}$ relative to $\mathbf{m}(\mathbf{F}(X))$.*

Remark 15 *Let X be a locally compact topological space, let $x \in X$, and let $\mathbf{U}_x$ be the set of relatively compact open neighborhoods of x. $\mathbf{U}_x$ is a local base at x for the topology of X. Let $\mathcal{I}$ be an indexing set in one-to-one correspondence with $\mathbf{U}_x$. Thus $\mathbf{U}_x = \{G_\kappa : \kappa \in \mathcal{I}\}$. Define the relation $\trianglerighteq_{\mathcal{I}}$ in $\mathcal{I}$ by $\kappa \trianglerighteq_{\mathcal{I}} \kappa' \iff G_\kappa \subset G_{\kappa'}$. Then $\trianglerighteq_{\mathcal{I}}$ is a partial ordering of $\mathcal{I}$ and $(\mathcal{I}, \trianglerighteq_{\mathcal{I}})$ is a distributive lattice. Furthermore, if $\kappa, \kappa' \in \mathcal{I}$, and if $\gamma \in \mathcal{I}$ is the index belonging to $G_\kappa \cap G_{\kappa'}$, then $\gamma \trianglerighteq_{\mathcal{I}} \kappa$ and $\gamma \trianglerighteq_{\mathcal{I}} \kappa'$; hence $(\mathcal{I}, \trianglerighteq_{\mathcal{I}})$ is also a directed set.*

Lemma 17 *Let X be a locally compact topological space, let $x \in X$, and let $\{G_\kappa : \kappa \in \mathcal{I}\}$ be as in the above remark. For each $\kappa \in \mathcal{I}$ let $x_\kappa \in G_\kappa$ and $y_\kappa \in \overline{G_\kappa}$. Then both $\{x_\kappa : \kappa \in \mathcal{I}\}$ and $\{y_\kappa : \kappa \in \mathcal{I}\}$ converge to x.*

Proof: Let U be an open neighborhood of x and let $\kappa_o \in \mathcal{I}$ be such that $G_{\kappa_o} \subset U$. Then for all $\kappa \trianglerighteq_{\mathcal{I}} \kappa_o$ we have $G_\kappa \subset G_{\kappa_o} \subset U$, and hence that $x_\kappa \in U$. By Proposition 7, there is a $\kappa_0 \in \mathcal{I}$ such that $\overline{G_{\kappa_0}} \subset U$. Thus for all $\kappa \trianglerighteq_{\mathcal{I}} \kappa_0$ we have $\overline{G_\kappa} \subset \overline{G_{\kappa_0}} \subset U$. Therefore, $x_\kappa \in U$ for all $\kappa \trianglerighteq_{\mathcal{I}} \kappa_0$. This completes the proof.

If X is a T_1-space, then $x \in X$ is called *isolated* if $\{x\}$ is open.

Proposition 22 *If X is a locally compact Hausdorff space, and if X has a point that is not isolated, then $\mathbf{M}(\mathbf{F}(X))$ is a distributive Matheron space.*

Proof: We show that the mapping $(E, F) \longmapsto E \cap F$ of $\mathbf{F}(X) \times \mathbf{F}(X)$ to $\mathbf{F}(X)$ is not LSC, i.e., that there is a convergent net $\{(E_\kappa, F_\kappa)\}$ in $\mathbf{F}(X) \times \mathbf{F}(X)$ for whose limit (E, F) we have $E \cap F \not\subset \underline{Lim}\, E_\kappa \cap F_\kappa$. I claim that there are nets $\{\xi_\kappa\}$ and $\{\eta_\kappa\}$ in X such that $\xi_\kappa \neq \eta_\kappa$ for all κ, $\xi_\kappa \to x$

in X, and $\eta_\kappa \to x$ in X. Given this claim, we have $\{(\{\xi_\kappa\}, \{\eta_\kappa\})\} \to (\{x\}, \{x\})$ in $\mathbf{F}(X) \times \mathbf{F}(X)$, by the last corollary, and

$$\{x\} \cap \{x\} = \{x\} \not\subset \underline{Lim}\, \{\xi_\kappa\} \cap \{\eta_\kappa\} = \emptyset.$$

To prove the claim, let x be a non-isolated point of X, let $\{G_\kappa : \kappa \in \mathcal{I}\}$ be as in the last remark, and put $\xi_\kappa = x$ for all $\kappa \in \mathcal{I}$. For each $\kappa \in \mathcal{I}$, let $\eta_\kappa \in G_\kappa$ be such that $\eta_\kappa \neq x$. Then $\eta_\kappa \to x$ in X, by the last lemma, and this implies, by the last corollary, that $\{\eta_\kappa\} \to \{x\}$ relative to $\mathbf{m}(\mathbf{F}(X))$. This completes the proof.

Corollary 15 *If X is a locally compact Hausdorff space, and if X has a point that is not isolated, then $\mathbf{F}(X)$ is not continuous.*

5. Conclusion

This report has shown that an M-topologized upper-continuous lattice abstracts numerous aspects of the more concrete hit-miss topology introduced by Matheron in developing the topological fundamentals of mathematical morphology. It would thus now appear to be possible to attempt to enrich the abstract lattice-algebraic approach to morphology of Heijmans and Ronse [4] by investigating the general ways that the structure of an M-topology can be combined with their basic structural framework, which is that of a complete lattice, $\mathcal{L}$, with a sup-generating subset, ℓ, on which an abelian group, G, acts effectively and ℓ-admissably (i.e., ℓ is G-invariant and G acts transitively on ℓ) as a group of automorphisms. Success in this would result in an abstract mathematical system that exhibits most of the algebraic and topological properties of the more concrete morphologies, and would therefore potentially be a much more general morphological theory with a considerably wider range of applications than the existing theories envision. This, inturn, would make new and more effective applications to ATR and computer vision problems possible.

There are, however, two features of Matheron's system that are not present in the abstraction developed here. One is the apparent lack of a full generalization of THEOREM 1. I believe that to obtain such a generalization it will be necessary to introduce certain assumptions about the atomicity properties of the basic lattice; this is something that needs to be further investigated. The second arises from the fact that no assumptions have been made about the possible first- or second-countability properties of an M-topology. Indeed, throughout I have used the general topological tool of net convergence and have not made the simplifying assumptions that would render mere sequence convergence adequate for the topological arguments. It may, however, be necessary to introduce such assumptions to attempt to abstract the probability theoretic aspects of Matheron's system. Matheron's concept of a *random closed set*, which would have to be generalized as a random variable in an M-topologized upper-continuous lattice, was not considered in the work presented here. In view of the importance of the random variable concept for the applications potency of mathematical morphology to such fields as ATR and computer vision, the development of an M-topological generalization of the random closed set concept would be a high-priority task for the line of research initiated here.

Finally, it should be mentioned that Banon and Barrera [5] have developed pure lattice algebra along lines that suggest that mathematical morphology can be profitably viewed as essentially the general theory of mappings between complete lattices, or at least that the latter is a useful abstract perspective in which to view the various concrete forms of mathematical morphology. The relation between this approach and that of Heijmans and Ronse needs to be clarified and the approaches somehow combined. This is accomplished in another of my reports, called *Lattice-Algebraic Morphology*, which will be issued soon.

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I generalize the topological structure of the concrete forms of mathematical morphology to the lattice-algebraical framework using the theory of *continuous lattices.* I show that when a complete lattice, $\mathcal{L}$, exhibits the dual of the property that defines a continuous lattice, then $\mathcal{L}$ together with a certain intrinsic lattice topology, $\mathbf{m}(\mathcal{L})$, which is related by duality to the *Lawson topology,* has almost all the familiar properties, suitably generalized, of the topologized lattices that constitute the basic mathematical structure of the concrete forms of mathematical morphology; for instance, the complete lattice of closed subsets of the Euclidean plane topologized with Matheron's *hit-miss topology.*

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