TWI-Soliton Decomposition of the Pump Wave in the Theory of Optical Parametric Amplification

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Abstract

Using the inverse scattering transform solution for parametric amplification, we show that the result of the interaction of a small signal pulse with an intense pump wave is essentially totally independent of the shape of the pump. Instead, the generated output is only determined by the amplitudes of the TWI-solitons contained in the pump.

1 Introduction

A new type of soliton generated by a quadratic nonlinearity has recently been observed in an experiment with an synchronously pumped optical parametric oscillator [1]. Applications of these solitons for second harmonic generation, sum frequency generation and for all-optical switching have been considered in references [2] - [4]. In Ref. [3], these solitons were termed TWI-solitons. An analytical theory of TWI-solitons, based on the inverse scattering transform (IST), was developed more then 20 years ago [5], [6], but because of the mathematical complexity of the required IST, these solitons are not familiar to the optical community. In this letter, we use the TWI-soliton theory to analyze the process of parametric amplification of ultrashort pulses, and to give a simple interpretation of the process.

The equations describing a quadratic interaction of three waves in a nonlinear medium can be written as:

$$\begin{aligned} \frac{\partial A_1}{\partial z}+\frac{1}{v_1}\frac{\partial A_1}{\partial t} &= iA_3A_2^*e^{i\Delta kz} \\ \frac{\partial A_2}{\partial z}+\frac{1}{v_2}\frac{\partial A_2}{\partial t} &= iA_3A_1^*e^{i\Delta kz} \\ \frac{\partial A_3}{\partial z}+\frac{1}{v_3}\frac{\partial A_3}{\partial t} &= iA_1A_2e^{-i\Delta kz} \end{aligned} \tag{1}$$

where A_j are the normalized amplitudes: $A_j=(E_j/E_0)\sqrt{n_j\omega_3/n_3\omega_j}$, ω_j are the frequencies, v_j are the group velocities, $\Delta k=k_3-k_1-k_2$ is the phase velocity mismatch, and n_j are the refractive indexes. E_0 is a normalization amplitude, defined by $E_0=\sqrt{n_1n_2\lambda_1\lambda_2}\chi_{nl}/(2\pi)^2$, where χ_{nl} is the coefficient of the nonlinear dielectric susceptibility. We also consider the pulses to be sufficiently long that all second–order dispersion effects are negligible and we also take the amplitudes, A_j, to be real. In this paper we treat the case where the group velocity of the wave with the highest frequency, v_3, lies *between* the group velocities of the daughter waves, v_1 and v_2: $v_1>v_3>v_2$. This case, we call the FSF interaction (Fundamental - Sum frequency - Fundamental) interaction. (In Ref. [7], this interaction was termed the "soliton-decay case").

A consideration of the Eqs. (1) for the FSF interaction, based on IST [5]-[8], shows that if all three pulses are initially separated from each other, then under rather general conditions (such as the amplitude never crossing zero, etc. [7]), whenever the area under the amplitude curve of any of these waves is greater than $\pi/2$, then that wave will contain solitons. In fact, the number, n, of solitons contained in that envelope follows directly from the area of the envelope:

$$\Im_i=\int_{-\infty}^{\infty}A_{i,0}(t)dt=\pi n+\epsilon_i \tag{2}$$

where $|\epsilon_i|<\pi/2$, $A_{i,0}$ is the amplitude of the i-th wave and n is the number of TWI-solitons contained in the wave. The quantity ϵ_i represents the non-soliton (radiation) part of the area.

We will consider only the case of parametric amplification, where in the beginning of the interaction, a small trigger pulse (a signal wave) at the frequency ω_1 is placed behind some intense pump pulse, which is at the frequency $\omega_3 = \omega_1 + \omega_2$. After initialization, the signal wave will propagate toward the pump, will overtake it and then interact with it. Any time the high-frequency pump contains a soliton, it is unstable. When it decays, it emits exactly one soliton into each daughter wave, for each soliton in the pump. Thus each soliton in the pump can be thought of as a bound state of zero binding energy, and consisting of a signal soliton and an idler soliton, wherein the energies of each are perfectly matched. This matching is exactly the relation that is prescribed by the Manley-Rowe relations. When this exact balance is disturbed, such as a small trigger pulse overtaking the pump and interacting with it, a decoupling of the signal and idler solitons occurs. As a result, the pump decays by emitting all of its solitons into the daughter waves (idler and signal) [5], [6].

As it follows from (2) the area of an intense pump wave ($\Im_3 >> \pi/2$) will consist almost exclusively of the TWI-solitons. Thus we shall neglect the radiation (non-soliton) part of the pump (since $\epsilon_3 << \Im_3$), and from now on, shall assume that the pump wave consists only of TWI-solitons.

In many ways, a solution of nonlinear equations using the IST is similar to a solution of the wave equation using the Fourier Transform [7]-[10]. In order to solve a linear differential equation using the Fourier transform, one has to decompose an arbitrary shaped wave into a superposition of simple plane waves. The amplitudes of such plane waves do not change during the propagation. Therefore the spectrum of the wave at the end of propagation is exactly the same as it was in the beginning. All that has changed is the phases between the spectral components. To solve a propagation problem, it is only necessary to take a superposition of the original plane waves, but with new phases as determined by the linear propagation.

Analogous to the linear Fourier transform, in the nonlinear IST, the wave with an arbitrary shape can be represented as a *nonlinear* superposition of solitons with different amplitudes and phases. Similar to the linear case, the number of solitons does not change during the interaction, but their phases do. Therefore to find the shapes of the waves after an nonlinear interaction, one needs to construct the final envelopes, using the same solitons, but with the new phases which they have accrued during the nonlinear interaction.

Let $A_3^{(i)}$ be a time profile of the pump before the interaction (index 'i' here stands for 'initial'). Let us also suppose that the pump initially consist of n TWI-solitons with the amplitudes $2\eta_i$ and the phases D_i. Because of the nonlinear nature of the three-wave interaction, a nonlinear superposition can not be obtained merely by adding together all solitons contained in the pump. For the TWI interaction, the law of superposition of n TWI-solitons is given by what is called the n-soliton formula. For TWI-solitons, this is [7]:

$$A_3^{(i)}(t) = 2 \sum_{j,k=1}^{n} D_j \exp[-(\eta_j + \eta_k)t](I + N)^{-1}_{j,k} \tag{3}$$

where I is the identity matrix and the matrix N is given by

$$N_{k,j} = \frac{D_j \exp[-(\eta_k + \eta_j)t]}{\eta_k + \eta_j} \tag{4}$$

In (3) the negative power denotes the inverse matrix, and $2\eta_i$ are the amplitudes of the solitons contained in the pump. As can be seen from (3) and (4), the result of a nonlinear superposition of n TWI-solitons depends not only on the amplitudes of solitons $2\eta_i$, but also on their phases, D_i. (Strictly speaking, the coefficients D_j are not the phases, but they are functions of the phases. Still, we shall refer to them loosely as the "phases".)

When the TWI-solitons are far apart from each other, the coefficients D_i have a simple physical meaning: they determine the positions and phases of the TWI-solitons. To understand this, let us consider a simple case when the pump contains only one soliton. For n=1, (3) and (4) give:

$$A_3^{(i)}(t) = 2\eta_1 sgn(D_1) sech[2\eta_1(t - t_0)] \tag{5}$$

where $t_0 = \ln(|D_1|/2\eta_1)/2\eta_1$. As can be seen from (5), t_0 gives the position of the center of the soliton with respect to the origin. However this simple physical interpretation of the phases D_i is legitimate only when the solitons are well-separated from each other. When the solitons are too close to each other and overlap, then it does not make sense to assign a position to each individual soliton within the group, since the individual solitons' shapes merge and "mix".

Let us suppose that initially there were only the pump and a small signal wave, and that the idler wave is absent before the interaction. During the interaction, all solitons in the pump, will transform into idler and signal wave solitons [5], [6]. The number of solitons in each of these waves will therefore be equal to the number of solitons initially contained in the pump. However the phases of each soliton will be changed by the interaction. (If the phases did not change, the shapes of the signal and idler waves on the output would exactly repeat the shape of the pump before the interaction). The final phases in these envelopes can be easily obtained using the analytical theory in Ref. [7]. For the initial conditions considered above, after some algebra, one obtains the following expression for the phases $D_{1,k}^{(f)}$ (the index 'f' stands here for 'final') of the $k-th$ soliton in the signal wave after the interaction:

$$D_{1,k}^{(f)} = \frac{2\rho_1(\eta_k)\alpha\eta_k}{\prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{\eta_k - \alpha\eta_j}{\eta_k + \alpha\eta_j}} \exp\left[2\frac{\eta_k}{v_1}t\right] \tag{6}$$

The index '1' here corresponds to the signal wave, $\alpha = (v_1 - v_3)/(v_1 - v_2)$, $2\eta_k$ is the amplitude of the $k-th$ soliton in the signal pulse (which is α times the corresponding amplitude in the pump), $\rho(\eta_k)$ is the reflection coefficient of the IST theory for the initial signal wave, and η_j are the η's that were initially in the pump pulse. When the intensity of the signal pulse is small initially, then we have $\rho(\eta_k) \simeq \int_{-\infty}^{\infty} A_{1,0}(t)\exp(2\eta_k t)dt$, where $A_{1,0}$ is the initial amplitude of the small signal wave, and provided the integral makes sense.

Formula (6) shows a remarkable quality of the soliton interaction: the final phases, $D_{1,k}^{(f)}$, depends *only* on the initial soliton content of the pump and *do not* depend on the initial phases between the solitons in the pump $D_{3,k}^{(i)}$. This means that the result of the nonlinear interaction does not depend on the exact pump shape, but depends only on the soliton content of the pump. An illustration of this fact, obtained by a numerical calculation of the system (1), is given in Fig 1. The first three pictures in this figure show the shape of the large pump and the small signal pulses before the interaction. Although the shapes of the pump pulses in Fig 1 (a,b, and c) are very different, they all contain exactly two TWI-solitons, with η's, $\eta_1 = 1$ and $\eta_2 = 2.7$. Because of the identical soliton content, all three waves in Fig 1 (a,b,c) have the same energy and the same area under the amplitude curve. The expression (6) predicts that the output for the signal wave will be exactly the same for all three cases. The numerical computations are in complete agreement with this conclusion: the output for all three processes is shown in the Fig 1 (d), which consists of only one of the results, with the other two being so close to this one, that no differences in the graphs could be seen. As can be seen from this picture, the two solitons which initially belonged to the pump finally moved to the signal and idler envelopes.

Let us consider the following physical consequence of (6). The sign of this expression is determined by the product in the denominator. Without loss of generality, we can take $\eta_j \leq \eta_{j+1}$. For the largest soliton, this product is positive. After that, $D_{1,k}^{(f)}$ will change sign every time j is increased by one. Now, according to (5), this means that the largest TWI-soliton in the signal wave will have a positive phase, the next TWI-soliton will have a negative phase, and the sign will change from one soliton to another, as one can clearly see in the example of the numerical calculations for two TWI-solitons in Fig 1 (d).

There are further physical consequences of (6). Consider the position of each TWI-soliton in the output pulse. We can estimate those by taking the logarithm of $D_{1,k}^{(f)}$ and dividing the result by $2\eta_k$:

$$t_0 = \frac{z}{v_1} + \frac{\ln \rho(\eta_k)}{2\eta_k} + Q(\eta_k) \qquad (7)$$

where in $Q(\eta_k)$, we have collected all the terms which depend only on η_k in a nonsingular fashion. The first term in this expression shows that all TWI-solitons in the final signal envelope move with the same speed v_1. The second term in (7) shows how much this TWI-soliton will be delayed by the nonlinear interaction, this term will be negative for small $A_{1,0}$. The smaller the trigger pulse, the longer the delay. The small solitons corresponding to the smaller η_k experience the larger delays. Therefore, the smaller the amplitude of the signal wave, the farther apart from each other will be TWI-solitons in the output signal wave.

In conclusion, using an analogy between the nonlinear IST and linear Fourier transform, we represent an intense pump pulse as a nonlinear superposition of the TWI-solitons. We have shown that the result of the process of parametric amplification of the small signal wave by the intense pump does not depend on the shape of the pump. The resulting train of TWI-solitons in the output signal wave is determined only by the number and the intensity of the TWI-solitons initially contained in the pump. Simple calculations show that the separation between the emerging solitons become larger when the intensity of the trigger pulse is small. The numerical calculations are in a remarkably excellent agreement with the analytical theory.

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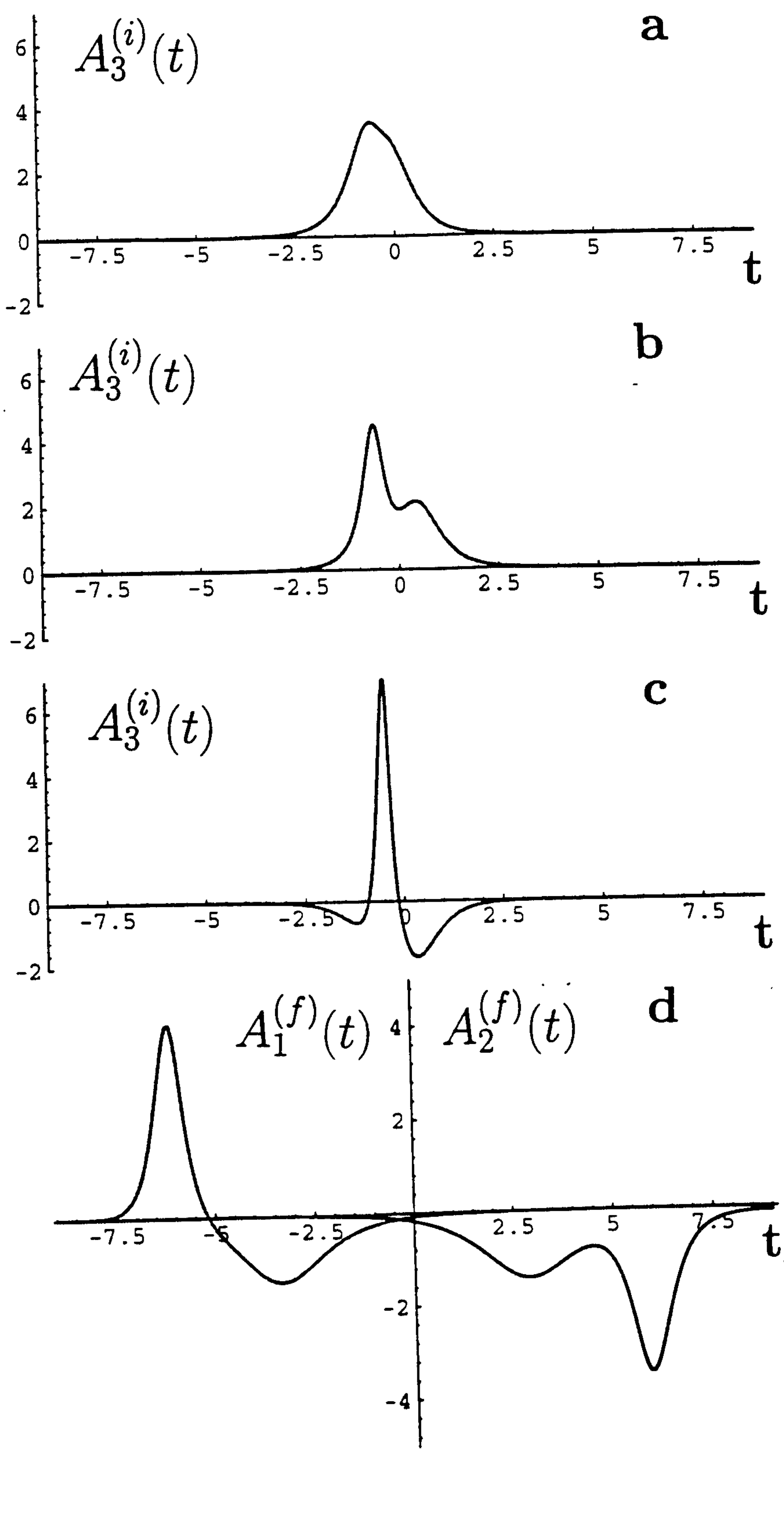
3 Figure captions

1. Figure 1. Numerical verification of analytical calculations. Pictures (a,b,c) represent a temporal distribution of the pump wave calculated using (5). Each pulse contains exactly two TWI-solitons with the amplitudes $\eta_1 = 2.7$ and $\eta_2 = 1$, but taken with different phases: $D_1 = 2$, $D_2 = 1$ (a), $D_1 = 5$, $D_2 = 1$ (b), $D_1 = -3$, $D_2 = 1$ (c). The output of the nonlinear interaction for all three cases is given in the Fig1(d): the signal wave is on the left and the idler wave is on the right. Both TWI-solitons, initially contained in the pump, now belong to the signal and idler waves. Since the pump did not contain any radiation, it has completely disappeared by the end of the interaction.


a
$A_3^{(i)}(t)$
b
$A_3^{(i)}(t)$
c
$A_3^{(i)}(t)$
d
$A_1^{(f)}(t)$
$A_2^{(f)}(t)$
t
-7.5
-5
-2.5
0
2.5
5
7.5
6
4
2
-2
-4