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THE DIOCOTRON INSTABILITY IN ANNULAR RELATIVISTIC ELECTRON BEAMS

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ABSTRACT

The instability is shown to have a finite k bandwidth with the fastest growing mode in the lab frame shifted to nonzero k. The thin beam dispersion relation, the appearance of several new parameters, and the instability mechanism are discussed.

The diocotron instability in nonrelativistic annular electron beams has long been utilized for the generation of microwaves in the crossed field magnetron.[1,2] Allowed to evolve to a nonlinear state, the instability can disrupt the beam by causing filamentation.[3,4] The recent interest in areas employing intense annular relativistic electron beams, such as free electron lasers,[5] multi-gapped accelerators,[6] and the Anomalous Intense Driver fusion concept,[7] makes it particularly important to understand the diocotron instability in the relativistic regime. The present analysis of the instability provides a means of determining constraints necessary for propagation of these beams down long conducting drift tubes and for evaluating the feasibility of using intense relativistic beams for high-power microwave generation.

Previous relativistic analyses of the diocotron instability have been limited to the special cases where either the wave frequency ω or the axial wavenumber k have been set equal to zero.[3,8] A complete relativistic theory must of necessity include ω and k both nonzero. In the analysis presented here the dispersion relation is calculated in the rest frame of the beam. The dispersion relation in the lab frame is then obtained by applying a Lorentz transformation:

$$\omega_\ell = \gamma(\omega_b + k_b v) \quad , \quad k_\ell = \gamma(k_b + \omega_b v/c^2) \tag{1}$$

where the ℓ and b subscripts denote quantities evaluated in the lab and beam reference frames respectively. The velocity of the beam is v and $\gamma \equiv (1-v^2/c^2)^{-\frac{1}{2}}$ where c is the speed of light. From Eqs. (1) it is clear that in order to obtain the lab frame quantities it is necessary to know ω as a function of k in the beam frame. Therefore, much of the present analysis concerns the extension of the beam frame problem to include finite k.

The electron beam is assumed to be initially in an azimuthally symmetric equilibrium inside a conducting drift tube of radius d as illustrated in Fig. 1(a). A large externally applied axial magnetic field is required for equilibrium. In the beam frame the only equilibrium motion of the electrons is an $\underset{\sim}{E} \times \underset{\sim}{B}$ rotational drift induced by the applied magnetic field B_z and the radial electric self-field E_r of the unneutralized beam. If the azimuthal motion is nonrelativistic, then the beam frame problem reduces to the case considered by Levy.[1]

The electrostatic approximation is used in the beam frame. This does not mean that the perturbed magnetic field is zero in the lab frame, however, since the Lorentz transformation of the electric field of the beam frame will give rise to a magnetic component in the lab frame. Using Maxwell's equations and the linearized cold fluid equations for the electrons, solutions of the form $\underset{\sim}{E}(r)\exp\{i(kz+\ell\theta-\omega t)\}$ yield the following equation

$$\underset{\sim}{\nabla} \cdot (\underset{\approx}{\varepsilon} \cdot \underset{\sim}{E}) = 0 \tag{2}$$

where $\underset{\sim}{\nabla}$ denotes the operator $\hat{e}_r r^{-1}(d/dr)r + \hat{e}_\theta i\ell/r + \hat{e}_z ik$, and $\underset{\approx}{\varepsilon}$ is the dielectric tensor whose only nonzero components for $\omega \ll \omega_c$ and $\omega_0 \ll \omega_c$ are $\varepsilon_{rr} = 1$, $\varepsilon_{r\theta} = -\varepsilon_{\theta r} = -i\omega_p^2/\omega\omega_c$, $\varepsilon_{\theta\theta} = 1+\omega_p^2\omega_0/\omega(\ell\omega_0-\omega) \times [(r/\omega_c)d(\ln \omega_p^2)/dr-2k^2r^2/\ell(\ell\omega_0-\omega)]$, and $\varepsilon_{zz} = 1+\omega_p^2/\omega(\ell\omega_0-\omega)$. The plasma frequency of the beam is denoted by ω_p, and ω_c is the cyclotron frequency associated with the applied magnetic field. The equilibrium rotational frequency ω_0 of the electrons is a function of the radial position r. Introducing the electrostatic potential Φ such that $\underset{\sim}{E} = -\underset{\sim}{\nabla}\Phi$, Eq. (2) becomes

$$r^{-1}d(rd\phi/dr)/dr - (\ell^2/r^2 + k^2)\phi =$$

$$\frac{\phi}{(\ell\omega_0 - \omega)} \left\{ \frac{\ell}{r} \frac{\omega_p^2}{\omega_c} \frac{d}{dr}(\ell n \ \omega_p^2) - \frac{k^2 \omega_p^2}{(\ell\omega_0 - \omega)} \right\} \tag{3}$$

which reduces to Levy's Eq. (10) for $k=0$.[1] A similar equation has been derived in a study of an instability driven by shear in the axial velocity.[9]

The diocotron dispersion relation is obtained by solving Eq. (3) in the three regions shown in Fig. 1(a). The beam density is nonzero only in region II ($a \leq r \leq b$). The potential ϕ is required to be continuous and to vanish at the outer conductor and at the origin. Jump conditions on $d\phi/dr$ are obtained by integrating the differential equation across the vacuum-beam interfaces.

Before solving for the complete dispersion relation, it is instructive to consider the physical mechanism of the instability. For the special case of a step-function radial density profile and $k=0$, the dispersion relation reduces to a quadratic equation in ω corresponding to two modes.[1] These waves cause density perturbations only at the edges of the beam and hence are surface modes. By considering the even more special case in which the outer surface of the beam is in contact with the conductor ($d=b$), one finds that the dispersion relation yields one mode which corresponds to a wave with frequency $\Omega=1-(a/b)^{2\ell}$ propagating on the inside surface of the beam. Here Ω is the frequency ω normalized to the diocotron frequency, $\omega_d \equiv \omega_p^2/2\omega_c$. Similarly, placing a conductor in contact with the inner surface of the beam shorts out perturbations on the inner surface and the beam again has only one

normal mode corresponding to a wave with frequency $\Omega = \ell(1-a^2/b^2) - [1-(a/b)^{2\ell}][1-(b/d)^{2\ell}]/[1-(a/d)^{2\ell}]$ which propagates on the outside surface of the beam. The wave energy E of these modes is

$$E = \frac{1}{8} \int_0^d r\, dr\, \{\underset{\sim}{E}^* \cdot \partial(\omega \underset{\approx}{\varepsilon}_r)/\partial\omega \cdot \underset{\sim}{E}\} \tag{4}$$

where $\underset{\approx}{\varepsilon}_r$ is the real part of the dielectric tensor.[10] From this expression, it is easy to show that the inside surface wave has positive energy and the outside surface wave has negative energy. The dispersion diagram for these modes along with the k=0 diocotron dispersion diagram is shown in Fig. 1(b). Thus, it becomes clear that the diocotron instability results from the coupling of a positive energy wave with a negative energy wave in a manner analogous to the mechanism of the two-stream instability.[11]

It has been shown that if a sufficiently large amount of charge[1] or current[12] (which has an equivalent effect as charge in the beam frame) is placed inside the annular beam the diocotron modes can be stabilized. The addition of a charge Q increases the difference between the frequencies of the outside and inside surface modes by an amount $\Delta\Omega = (4\ell Qe/m\omega_p^2)(a^{-2}-b^{-2})$. Thus, if these modes are near resonance for Q=0, then the addition of a large amount of either positive or negative charge results in a frequency shift which precludes the possibility of the coupling of the positive energy wave with the negative energy wave.

The right-hand side of Eq. (3) contains resonance terms which make the differential equation singular when $\ell\omega_0(r)=\omega$. By considering only

the special case of a step function density profile and k=0, one avoids

the question of these resonances since the coefficients of these terms

vanish. However, the solution for this case reveals that the diocotron

mode frequencies are precisely in the right range of values to be reso-

nant. It has been shown that for a smoothly varying density profile

the first term of the right-hand side of Eq. (3) can lead to growth or

damping depending on the sign of the term.[10] The physical mechanism

for this process is a wave-particle interaction analogous to Landau

damping. The second term on the right-hand side cannot be avoided by

choosing a specific density profile. Because this term is always nega-

tive, one would expect from the analysis of the contribution of the

first term that the second term always provides a damping effect for

finite k. For large k this wave-particle interaction should dominate

the wave-wave interaction and the instability will be stabilized. Thus,

the diocotron instability has a finite bandwidth in k.

An upper bound on the growth rate and bandwidth may be obtained by

the means of quadratic forms.[9] Defining ψ by $\Phi = (\ell w_0 - w)^{\frac{1}{2}} \psi$, multiply-

ing Eq. (3) by ψ^* and integrating from r = 0 to r = d one obtains

$$w_i^2 < \{(\ell w_0'/2)^2 - k^2 w_p^2\}/\{k^2 + (j_\ell/d)^2\} \tag{5}$$

where w_i is the imaginary part of w, j_ℓ is the first zero of the

ordinary Bessel function of the first kind and order ℓ, and w_0' is the

derivative of w_0 evaluated at its largest value. This expression

demonstrates that the diocotron modes become stabilized if the <u>shear</u> in

the equilibrium rotation frequency, which provides the free energy for

the instability, vanishes. Equation (5) also shows that the instability must stabilize for large k (the expression is valid only for $\omega_i^2 \geq 0$). For the step-function beam density profile, Eq. (5) restricts the range of k over which instability can occur to values such that $\Delta\kappa < 2\ell/\eta$ where $\kappa \equiv ka$ and $\eta \equiv 2\omega_c/\omega_p$.

In what follows, Eq. (3) is analyzed for the step-function beam density profile. The generalization to other density profiles is straightforward. It should be pointed out, however, that other profiles may be more or less unstable owing to the Landau damping (or growth) contribution of the first term on the right-hand side. In regions I and III the right-hand side of Eq. (3) vanishes and the eigenfunctions in these regions are expressed in terms of the modified Bessel functions $I_\ell(x)$ and $K_\ell(x)$ where $x \equiv kr$. For the step-function density profile, $\omega_0 = \omega_d(1-a^2/r^2)$. Thus, in region II, Eq. (3) has a second-order pole at $r/a = \rho_0 \equiv \ell/(\ell-\Omega)$. Using a change of variables, $u \equiv \ell n(r/a)$, Eq. (3) in region II can be cast in the form of a Schrödinger equation: $d^2\Phi/du^2 - V(u)\Phi = 0$. Because most of the new physics for finite k results from the resonance term or pole, V(u) can be approximated by expanding about this pole. This expansion should be particularly good for a thin beam, $(b-a)/a \ll 1$, since all points in region II are then close to the pole. Defining a new variable $z = \delta \ell n(r/a\rho_0)$, then this equation, neglecting terms of $O(z)$, becomes

$$d^2\Phi/dz^2 + \{-\tfrac{1}{4}+\lambda/z+(\tfrac{1}{4}-\mu^2)/z^2\}\Phi = 0 \tag{6}$$

where $\delta \equiv 2\{\ell^2+\kappa^2\rho_0^2(1-23\alpha^2/12)\}^{\frac{1}{2}}$, $\lambda \equiv \kappa^2\rho_0^2\alpha^2/\delta$, $\mu^2 \equiv (1-\kappa^2\rho_0^2\alpha^2)/4$, and $\alpha^2 \equiv \eta^2/(\ell-\Omega)^2$.

Equation (6) is Whittaker's equation whose general solution can be expressed in terms of the functions $W_{\lambda,\mu}(z)$ and $W_{-\lambda,\mu}(-z)$. Application of the boundary and jump conditions yields the following dispersion relation:

$$W[\delta W_{\lambda,\mu}'(z_a)-hW_{\lambda,\mu}(z_a)]+g[hU+\delta V] = 0 \tag{7}$$

where $W \equiv W_{\lambda,\mu}'(z_b)W_{-\lambda,\mu}(-z_b)-W_{\lambda,\mu}(z_b)W_{-\lambda,\mu}'(-z_b)$, $U \equiv W_{\lambda,\mu}(z_a)W_{-\lambda,\mu}(-z_b)$
$-W_{\lambda,\mu}(z_b)W_{-\lambda,\mu}(-z_a)$, $V \equiv W_{\lambda,\mu}(z_b)W_{-\lambda,\mu}'(-z_a)-W_{\lambda,\mu}'(z_a)W_{-\lambda,\mu}(-z_b)$,
$h \equiv x_a I_\ell'(x_a)/I_\ell(x_a)-2\ell/\Omega$ and

$$g \equiv W_{\lambda,\mu}'(z_b)-2\ell \ W_{\lambda,\mu}(z_b)/\delta[\ell(1-a^2/b^2)-\Omega]$$

$$-W_{\lambda,\mu}(z_b)\frac{x_b}{\delta} \ \frac{I_\ell'(x_b)K_\ell(x_d)-I_\ell(x_d)K_\ell'(x_b)}{I_\ell(x_b)K_\ell(x_d)-I_\ell(x_d)K_\ell(x_b)} \ .$$

The prime denotes differentiation with respect to the argument. The subscripts a, b, and d denote evaluation of x and z at r = a, b, and d, respectively. A numerical evaluation of this dispersion relation exhibits the finite bandwidth in k discussed earlier, and it is found that $\Delta\kappa$ scales as $1/\eta$ and is bounded by $\Delta\kappa<2\ell/\eta$.

The dispersion relation given in Eq. (7) is for the beam frame. To obtain the dispersion relation in the lab frame, the converse relation of Eqs. (1) can be used to express Eq. (7) in terms of the lab frame quantities. In dimensionless variables this Lorentz transformation is $\Omega=\gamma^2(\Omega_\ell-\kappa_\ell \ \beta\gamma^{-1}c/\omega_d a)$ and $\kappa=\gamma(\kappa_\ell-\Omega_\ell \ \beta\gamma \ \omega_d a/c)$, where $\beta\equiv v/c$. The effect of this transformation is to shift the band of unstable

modes to larger κ values and to cause an asymmetry of the spectrum about the most unstable mode.

The most unstable mode in the beam frame is the k=0 mode. This is true because the damping term in Eq. (3) vanishes. Also, it can be argued that the frequency of the k=0 mode is either a maximum or minimum because in the beam frame there is no preferred z-direction and the dispersion relation must, therefore, be an even function of k. Equation (1) shows that if the most unstable mode occurs at $k_b=0$, it must occur at $k_\ell \neq 0$. An approximate expression for ω and k of the most unstable mode in the lab frame can be found by expanding Eqs. (1) about $k_b=0$ and requiring k_ℓ to be real. Neglecting terms of order $[Im(k_b)]^2$ one finds $k_\ell = \gamma Re(\omega_b)\beta/c$ and $\omega_\ell = \gamma Re(\omega_b) + iIm(\omega_b)/\gamma$. Because the diocotron mode frequencies are proportional to the beam density, another factor of $1/\gamma$ enters; i.e., the most unstable mode in the lab frame occurs for $k_\ell \cong Re(\omega_L)\beta/c$ and $\omega_\ell \cong Re(\omega_L) + iIm(\omega_L)/\gamma^2$ where ω_L denotes the diocotron mode frequency of Levy's quadratic dispersion relation evaluated using the lab frame density. These equations suggest that as far as propagation is concerned the instability rapidly ($\propto 1/\gamma^2$) becomes less of a problem as the beam energy is increased. Furthermore, because the scaling of the real part of the frequency is not degraded by increasing γ, high frequency microwave generation using intense high energy beams appears to be possible.

It should be noted that the relativistic, finite k analysis presented here expands the parameter space of the diocotron instability problem. In addition to the mode number ℓ and the geometrical factors b/a and d/a, new parameters now include κ, γ, η and $\omega_d a/c$. The range

of κ for unstable waves, as mentioned earlier, is determined by η. The parameter $\omega_d a/c$, which was introduced by requiring $(\underset{\sim}{k}, i\omega)$ to be a four-vector, is a measure of how strongly the relativistic effects modify the characteristics of the instability. A complete parameter study is too involved to be presented in a letter and will be presented elsewhere along with the results of particle-in-cell simulations of the diocotron instability.

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Fig. 1.   (a) Beam configuration; (b) Real part of frequency vs.

azimuthal mode number showing surface wave coupling for

b/a = 1.1, d/a = 1.5, k = 0.

(a) CONDUCTOR, III, E-BEAM, II, I, a, b, d

(b) NEGATIVE ENERGY WAVE (OUTSIDE), DIOCOTRON MODES, POSITIVE ENERGY WAVE (INSIDE)