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Removal of the Assumption of Cellular Targets in Computing Damage Aggregation to an Area Target from a Salvo of N Weapons

by

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In the computation of damage aggregation to an area target from a salvo of N weapons, the assumption of a cellular target (i.e., a target composed entirely of cells) is often made. This assumption together with the implicit associated assumption of weapon strikes only at the center of those cells constitutes a significant simplification to the real-world model with potentially significant errors in the computation.

It is the purpose of this paper to eliminate the assumption of cellular targets.

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Removal of the Assumption of Cellular Targets in Computing Damage Aggregation to an Area Target from a Salvo of n Weapons

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ABSTRACT

In the computation of damage aggregation to an area target from a salvo of N weapons, the assumption of a cellular target (i.e., a target composed entirely of cells) is often made. This assumption together with the implicit associated assumption of weapon strikes only at the center of those cells constitutes a significant simplification to the real-world model with potentially significant errors in the computation.

It is the purpose of this paper to eliminate the assumption of cellular targets.

Background

The analysis of the damage aggregation to an area target resulting from a salvo of n weapons fired at it, is a difficult topic. Under the assumption that each target point is equally likely to be hit and using a simple "cookie cutter" area of destruction (kill area) from a single hit, then this difficulty results in large part from the great geometric variety of the total damaged area that results from k ($k \leq n$) hits. The assumption of a cellular target, i.e. a target composed entirely of cells with predetermined locations greatly simplifies the problem. If it is further assumed that a hit **anywhere in a particular cell destroys that cell and no other cells** even if that hit occurs near the boundary of adjacent cells, then determining the geometry of the damaged area becomes more tractable.

For example, consider the target of figure 1 consisting of twelve cells. Then a hit, indicated by "X" near the boundary of the shaded cell would in the basic model of [1] and [2] damage the shaded cell and no other cell. Effectively this assumes that hits only occur at the centers of the predetermined cells.

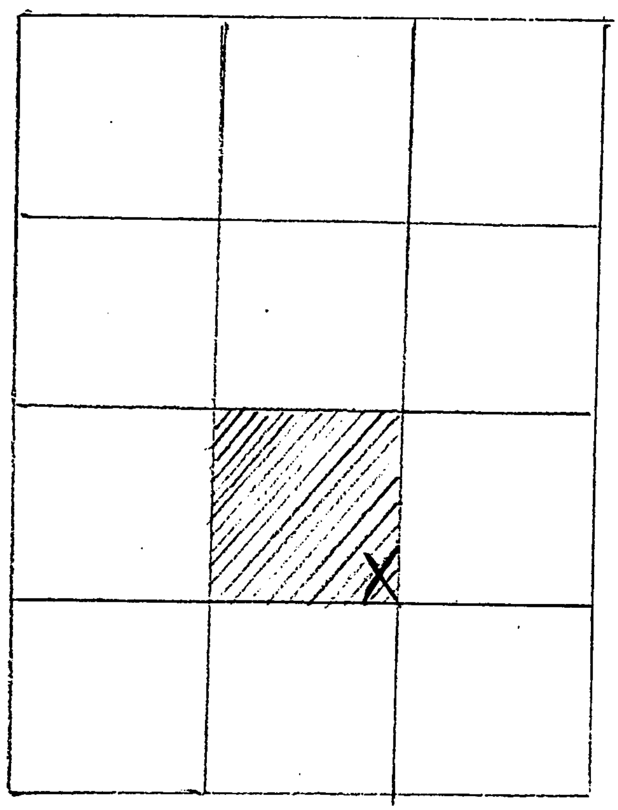


Figure 1

However since hits do not in reality always occur in this manner, then such a model does little to represent the real world situation.

It is the purpose of this work to remove the assumption of predetermined cell location and to thus allow hits to occur anywhere in the target. In order to do this we will first use predetermined location cells, forming in essence a partition of the target. We will then form a sequence of such partitions of smaller and smaller cells where in the limit, these cells reduce to points.

For example, using a target of size 100 square feet with square cells each having a cell size of $(2/3)^2$ square feet, then the partition will be as shown in Figure 2.

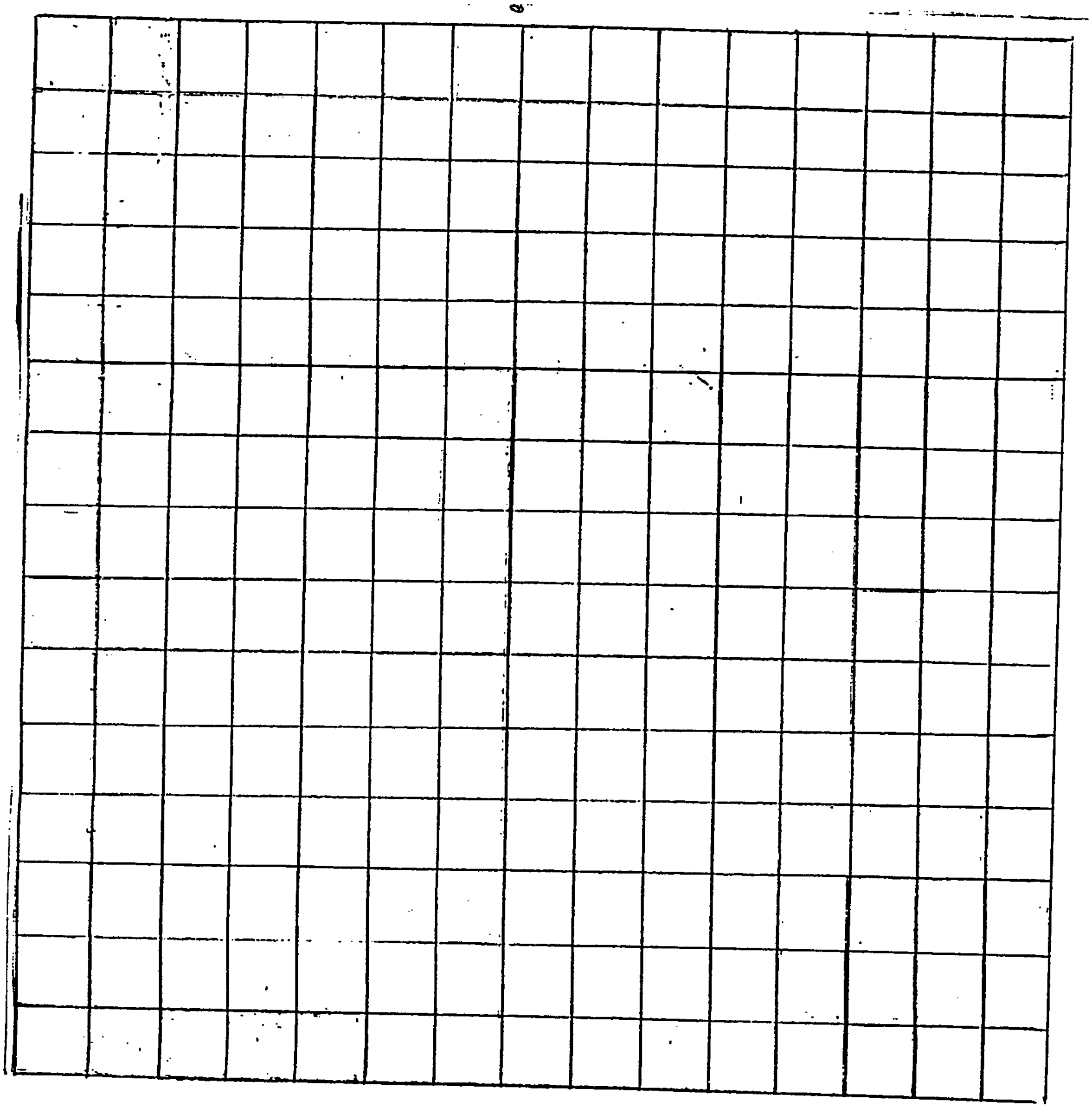
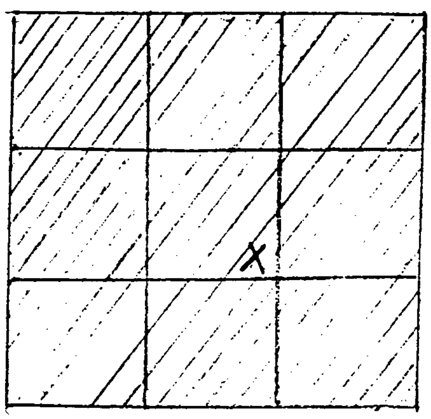


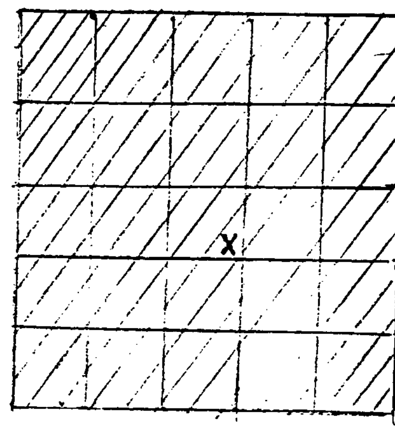
Figure 2

In the proposed limiting process, as the cells reduce in size, the above model error (in essentially assuming that a hit anywhere in a cell is a hit in the center of that cell) becomes less and less significant. Of course as the cell size reduction occurs, the kill area (which is a constant) will be composed of more than one cell and in fact an increasing number of cells. Thus for any fixed partition of the target, a hit to a cell will in general damage that cell in addition to a number of surrounding cells whose total area equals the kill area[1]. The actual cell that was hit will then be at the center of this kill area. For example, continuing with square cells and assuming a square kill area of 4 square feet and using two different partitions with cells of size $(2/3)^2$, $(2/5)^2$ square feet respectively, then the pictures are (not to scale)



kill area=4 square feet
cell size=$(2/3)^2$ square feet

Figure 3



kill area=4 square feet
cell size=$(2/5)^2$ square feet

Figure 4

Clearly, the error between the area actually damaged (which would be the kill area centered at the "X" ... this is not displayed) and the area damaged according to the model (which is the kill area centered at the center of the hit cell...this is displayed) is smaller in Figure 4 (with smaller cells) than in Figure 3[2]. In the limit, as the cell size becomes smaller and smaller (with limit being a point) then the above noted error should reduce to a limit of zero,

[1] The partitions are chosen so that an integral number of cells make up the kill area.
[2] The above assumes that the hit cell is sufficiently interior (i.e. within the target boundary) to the target so as to have a kill area (of which it is the center) entirely contained in the target. If this is not the case for a hit cell then appropriate modifications (to be explained later) are made.

Explanation of Solution Technique

We continue with the explanation of our partitions. For simplicity of presentation we assume a square target of side L and a weapon with a square kill area (cookie cutter area of destruction) of side 2R. We use n=1,2,3… as the index of our partitions. That is each value of n will represent a partition of the target. For any n our cell size will be $(2R/(2n+1))^2$ square feet.

The number of *interior* target cells along a side of the target is then L/(2R/(2n+1))[3] and then the total number of interior target cells is[4] $m_n=(L/(2R/(2n+1)))^2$. In our model we shall consider a hit to the target as a hit to an interior cell or to an "exterior" cell for we shall allow n rows and n columns of exterior cells (which surround the target) in the partition represented by n (see Figures 5 and 6 for the cases n=1 and n=2). Counting the interior and exterior cells, then there is a total of L/(2R/(2n+1)) + 2n cells along a side of the target. This results in a grand total of $T_n=(L/(2R/(2n+1)) + 2n)^2$ interior and exterior cells.

Now for a fixed value of n, we let

(1) $$C_i(k)=\begin{cases}1 \text{ if the } i^{th} \text{ cell is damaged after k hits}\\ 0 \text{ if the } i^{th} \text{ cell is not damaged after k hits}\end{cases}$$

Next, we have the following: according to our defined kill area and cell size, then, for an undamaged target, a hit to a cell will damage a square of $\left(2R/(2R/(2n+1))\right)^2=(2n+1)^2$ cells around (and including) the cell that was hit. We call this the "kill area around the cell." Also, if one examines Figures 5 and 6, it will be clear that any interior cell $\overline{C}$ may be damaged either by a direct hit or by a hit to any other cell in the kill area[5] around $\overline{C}$. Thus any interior cell can be damaged in $(2n+1)^2$ ways (all equally likely). Since there are a total of T_n (equally likely) cells to hit, then for the n^{th} cell partition

(2) $$P_n[C_i(1)=1]=(2n+1)^2/T_n=\left((2n+1)/\left(L/(2R/(2n+1))+2n\right)\right)^2$$
$$=\left(((2n+1)2R)/\left(L(2n+1)+4nR\right)\right)^2$$

where $P_n[C_i(1)=1]$ refers to the probability, in the n^{th} partition of the target, of the i^{th} cell being damaged after the first hit.

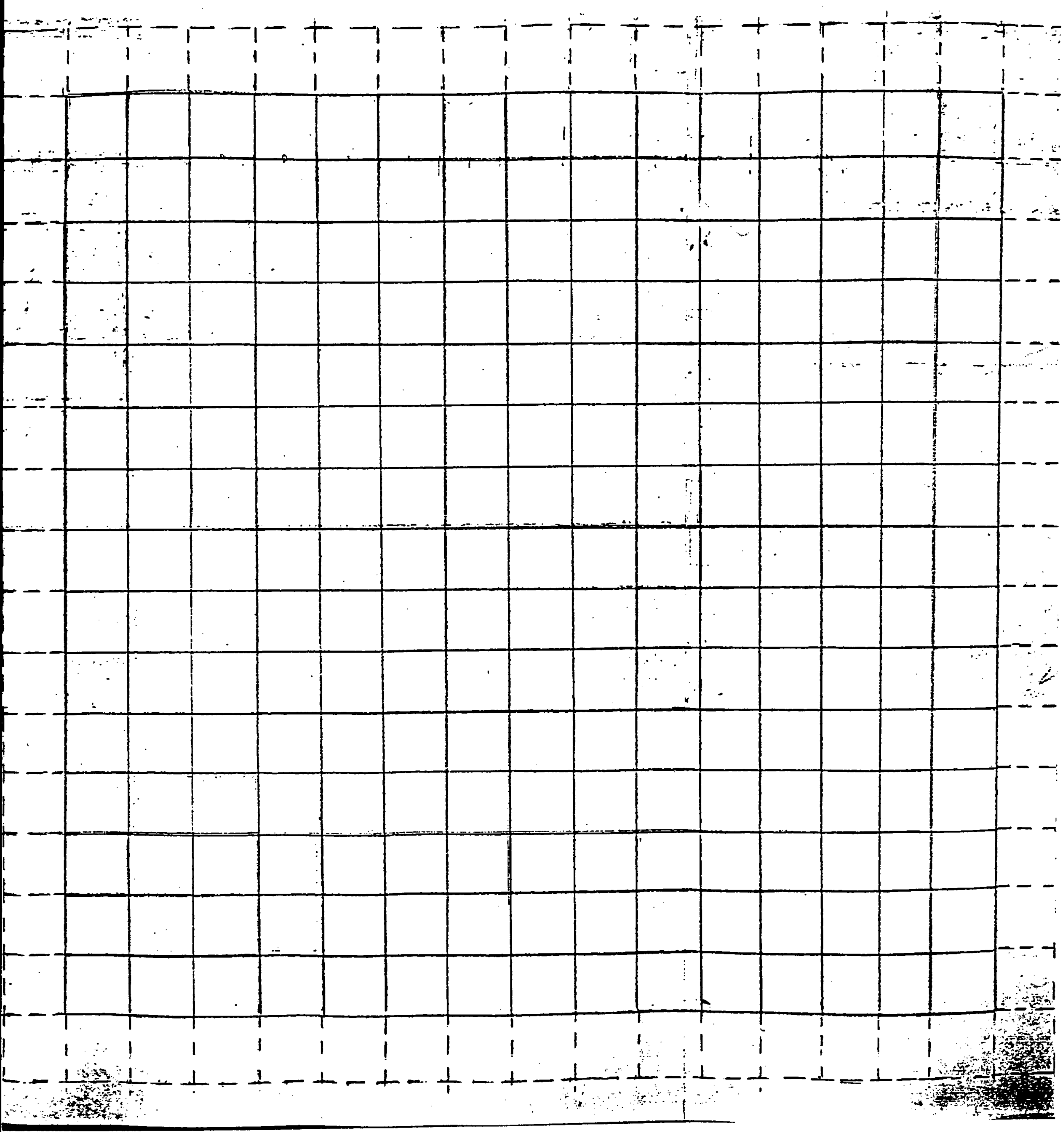
We now consider points (as well as interior or exterior cells) in the target. Then for the point with coordinates (x,y), let

---

[3] We shall assume that L/2R is an integer.

[4] We number our cells so that the first m_n cells are the interior cells.

[5] Where such kill area may include exterior cells (in the indicated n rows and n columns outside the target) as shown in Figures 5 and 6.



Exterior cell

Interior cell

Figure 5
Target Partitioning for n=1

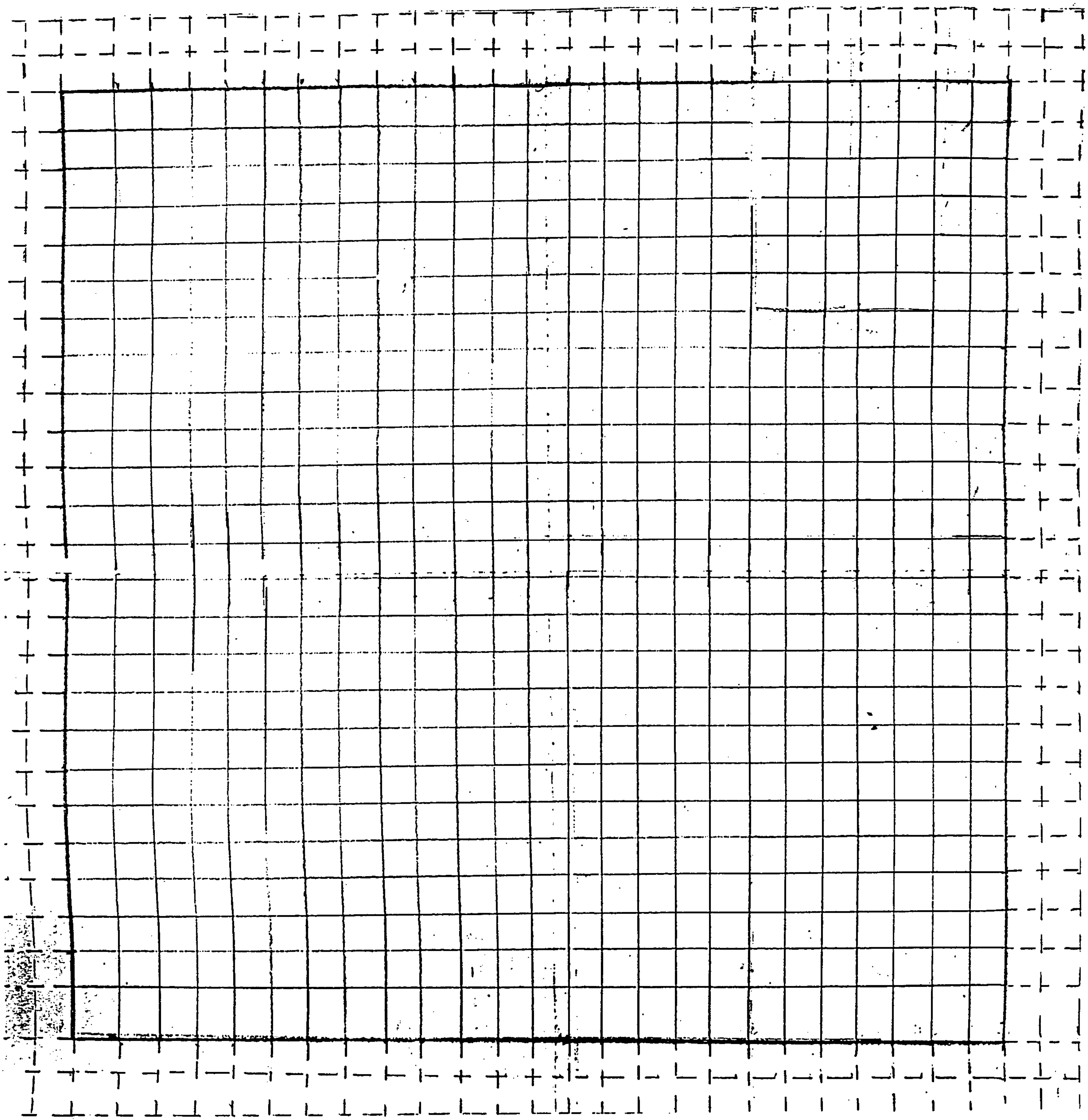


Figure 6
Target Partitioning for n=2

(3) $$(x,y)(k)=\begin{cases}1 \text{ if the point } (x,y) \text{ is damaged after k hits}\\ 0 \text{ if the point } (x,y) \text{ is not damaged after k hits}\end{cases}$$

Now for the n^{th} partition, the arbitrary point (x,y) is in some cell (perhaps more than one cell if it is on the boundary of a cell, in which case use any one of those cells) C_{i_n}. Thus

(4) $$P_n[(x,y)(1)=1]=P_n[C_{i_n}(1)=1]=\left((2n+1)^2\right)/T_n$$

where $P_n[(x,y)(1)=1]$ is the probability (in the n^{th} partition) of (x,y) being damaged after the first hit to the target. Taking the limit as $n\to\infty$ to get the true value of the probability of (x,y) being damaged after the first hit, i.e. P[(x,y)(1)=1], we have

(5) $$\begin{aligned}P[(x,y)(1)=1]&=\lim_{n\to\infty}P_n[C_{i_n}(1)=1]=\lim_{n\to\infty}\left((2n+1)2R/(L(2n+1)+4nR)\right)^2\\&=\left(\lim_{n\to\infty}(2n+1)2R/(L(2n+1)+4nR)\right)^2\\&=\left(\lim_{n\to\infty}(2+1/n)2R/(L(2+1/n)+4nR)\right)^2=\left(4R/(2L+4R)\right)^2\end{aligned}$$

Now let $\Delta(k)$ be the portion of the target damaged after k hits. Also, let

$$w(x,y)=\begin{cases}1/L^2 & \text{if } (x,y) \text{ is interior point (uniform weighting)}\\ 0 & \text{if } (x,y) \text{ is exterior point}\end{cases}$$

Then for k=1

(6) $$\Delta(1)=\iint_{Target} w(x,y)\,(x,y)(1)\,dydx$$

and with D(k) as the expected value of $\Delta(k)$, then again for k=1

(7) $$D(1)=E[\Delta(1)]=E\left[\iint_{Target} w(x,y)(x,y)(1)dydx\right]=\iint_{Target} w(x,y)E[(x,y)(1)]dydx$$

with

(8) $$E[(x,y)(1)]=1P[(x,y)(1)=1]+0P[(x,y)(1)=0]=P[(x,y)(1)=1]=\delta_{x,y}(1)$$

where $\delta_{x,y}(1)$ *is shorthand for* P[(x,y)(1)=1]. Since we assume that all points in our target are equally likely to be hit, then we drop the dependence upon (x,y) and write $\delta(1)$. Also by the above listed values for w(x,y) we have

(9) $$D(1) = \iint_{TargetInterior} 1/L^2\,\delta(1)dydx = \delta(1)/(L^2)L^2 = \delta(1) = (4R/(2L+4R))^2.$$

This is the expected value of the portion of the target damaged after the first hit (k=1). Continuing on for general k:

(10) $$P[(x,y)(1)=0] = 1 - P[(x,y)(1)=1]$$

and assuming independent hits

(11) $$P[(x,y)(k)=0] = (P[(x,y)(1)=0])^k = (1-P[(x,y)(1)=1])^k$$

so that by (5)

(12) $$P[(x,y)(k)=1] = 1-P[(x,y)(k)=0] = 1-(1-P[(x,y)(1)=1])^k = 1-(1-(4R/(2L+4R))^2)^k.$$

Then analogous to (6):

(13) $$\Delta(k) = \iint_{Target} w(x,y)(x,y)(k)dydx$$

and analogous to (8) and using (12):

(14) $$\begin{aligned} E[(x,y)(k)] &= 1P[(x,y)(k)=1] + 0P[(x,y)(k)=0] \\ &= P[(x,y)(k)=1] = 1-(1-(4R/(2L+4R))^2)^k \end{aligned}$$

so finally

(15) $$\begin{aligned} D(k) = E[\Delta(k)] &= E\left[\iint_{Target} w(x,y)(x,y)(k)dydx\right] = \iint_{Target} w(x,y)E[(x,y)(k)]dydx \\ &= \iint_{TargetInterior} \frac{1}{L^2}\left[1-(1-(4R/(2L+4R))^2)\right]^k dydx = 1-\left[1-(4R/(2L+4R))^2\right]^k \end{aligned}$$

This represents the expected portion of the target damaged after k hits. As a comparison exercise, we select L=10 and R=1 as representative numbers for our parameters and compute D(1):

(16) $$D(1) = 1-\left[1-(4/(20+4))^2\right] = .0277778.$$

In order to compute the corresponding values of D for some partitions, we define $\Delta_n(1)$, $D_n(1)$ respectively, as the portion of the target damaged after the first hit and its expected value both computed in the n^{th} partition of the target. We also define ${}_n w_i$ as the weighting given to the i^{th} cell in the n^{th} partition:

$$ {}_n w_i(C_i) = \begin{cases} \dfrac{1}{m_n} & \text{if } C_i \text{ is interior cell} \\ 0 & \text{if } C_i \text{ is exterior cell} \end{cases} $$

Then working with cells (rather than points) in the n^{th} partition and following a parallel development to 6)→9) gives by (4) and (2)

(17) $$\Delta_n(1) = \sum_{i=1}^{T_n} {}_n w_i(C_i) C_i(1) = \sum_{i=1}^{m_n} \frac{1}{m_n} C_i(1)$$

(18) $$D_n(1) = E[\Delta_n(1)] = E\left[\sum_{i=1}^{m_n} \frac{1}{m_n} C_i(1)\right] = \sum_{i=1}^{m_n} \frac{1}{m_n} E[C_i(1)]$$

where

(19) $$E[C_i(1)] = 1P_n[C_i(1)=1] + 0P_n[C_i(1)=0] = P_n[C_i(1)=1] = P_n[C(1)=1] = \frac{(2n+1)^2}{T_n}$$

where the next to last equality results from the equal likeliness of all cells (in the n^{th} partition) to be damaged after the first hit (so the dependency on i is removed). So then by (2):

(20) $$D_n(1) = \sum_{i=1}^{m_n} \frac{1}{m_n} \frac{(2n+1)^2}{T_n} = \frac{(2n+1)^2}{T_n} = \left(\frac{(2n+1)2R}{L(2n+1)+4nR}\right)^2.$$

Computing $D_n(1)$ for some values of n:

For n=1, $D_1(1) = (6/34)^2 = .031142$

For n=2, $D_2(1) = \left(\frac{5(2)}{10(5)+8}\right)^2 = (10/58)^2 = .029727$

For n=3, $D_3(1) = \left(\frac{7(2)}{10(7)+12}\right)^2 = (14/82)^2 = .029149.$

This is a decreasing sequence with limit .0277778 which represents the expected damage after one hit without the assumption of predetermined cell locations.

As noted above, the removal of predetermined cell location is a significant step towards making a more real-world model for damage aggregation.

Also, this is a first step in using the concept of a limiting process to delete the predetermined cellular construction of the target. Certain construction techniques used herein such as square targets, seem to be removable in generalizing this approach. For example, it seems that any target which is decomposable into a finite number of square sub-targets might be solvable by this approach. This is certainly an area for future work.

Bibliography

[1] Esary, J.D., Studies on Damage Aggregation for Weapons Salvos. NPS Technical Report NPS-55-90-16

[1] Esary, J.D., Studies on Damage Aggregation for Weapons Salvos (2). NPS Technical Report NPSOR 92-007

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