OPTICAL SCIENCE CONSULTANTS
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ABSTRACT

The ability of an adaptive optics system to compensate for turbulence-induced random wavefront distortion is studied in terms of the ensemble-average, aperture-average squared residual wavefront error. It is shown that if the adaptive optics are controlled in terms of the appropriate set of Karhunen-Loève functions, the minimum possible number of degrees-of-freedom will be required for a given performance. Results are presented for the required number of degrees-of-freedom for any level of performance if the adaptive optics control utilizes the Karhunen-Loève functions, the Zernike-function, a set of local piston/tilt-only functions, or a set of local piston-only functions. With the exception of the last of these, which requires almost seven-times the optimum number of degrees-of-freedom, there is no more than a factor of two spread in required number of degrees-of-freedom amongst the other possibilities.

Barber, barber, shave a pig
How many hairs to make a wig?

Introduction

In the design of an adaptive optics system to compensate for wavefront distortion produced by atmospheric turbulence, the primary concern must, of course, be the achievement of an adequate degree of correction. If we can incorporate a sufficient number of degrees-of-freedom into the adaption mechanism, we can be assured of achieving any desired level of correction. However, in most practical cases, it is desirable to minimize the number of degrees-of-freedom utilized.* This might be, for example, to minimize noise effects† or more significantly, because we can not afford/support the requirements for data collection and processing to control a large number of degrees-of-freedom. For a multi-dither COAT adaptive optics system we might, for example, have difficulty obtaining a wide enough range of dither frequencies to control more than a rather limited number of degrees-of-freedom.

Given this limitation in the number of degrees-of-freedom which can be utilized, it is obviously advantageous to design the adaptive optics system so that these degrees-of-freedom give, in a statistical sense, the best possible fit to the turbulent wavefront distortion. This paper will be concerned with an investigation of this subject. We shall first show that the best possible fit is obtained for a finite number of degrees-of-freedom if the correction mode corresponding to each degree-of-freedom is chosen from the set of Karhunen-Loève orthonormal functions "matching" the wavefront distortion statistics. We shall then consider in quantitative terms the quality of the fit that can be obtained using the Karhunen-Loève-functions,

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* We should be careful to distinguish between the number of actuators on an adaptive optics device and the number of degrees-of-freedom. The former can be considerably greater than the latter. If the same set of signals combined in (potentially distinct but) only deterministically different ways are used to control a large number of actuators, the number of degrees-of-freedom corresponds to the number of signals and not to the number of actuators.

† An unlimitedly large number of degrees-of-freedom can be utilized without deleterious noise effects if we utilize a priori information on the turbulence statistics to influence the calculated control signals for each degree-of-freedom.

the Zernike-functions, and piston and piston/tilt-functions. We shall carry out this evaluation for a circular aperture of diameter D . Results will be presented in such a form that we can easily compare the number of degrees-of-freedom required to achieve a desired level of fit, using each of these mode types.

Orthonormal Mode Decomposition

Each degree-of-freedom will control a mode of deformation of the adaptive optics system. Any finite set of these modes can be represented in terms of an equal or lesser number of suitably chosen orthonormal modes. It is therefore appropriate to couch our considerations in terms of various orthonormal modes.

If we let $W(\vec{r})$ be a function defining the adaptive optics aperture, i.e.,

$$W(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r} \text{ is inside the aperture} \\ 0 & \text{if } \vec{r} \text{ is outside the aperture,} \end{cases} \tag{1}$$

then a set of functions $\{G_i(\vec{r})\}$ are orthonormal over the aperture if

$$\int d\vec{r}\, W(\vec{r})\, G_i(\vec{r})\, G_{i'}(\vec{r}) = \delta_{i,i'} . \tag{2}$$

The well-known Zernike functions are orthonormal over an unobstructed circular aperture. If the aperture is divided up into a set of non-overlapping subapertures (of any shape) and a set of orthonormal functions are defined in each subaperture, then the sum of all of these functions forms an orthonormal set covering the full aperture. Since the so-called piston and tilt modes are samples from a set of orthonormal functions on the subaperture,*

* If the subaperture is circular, the piston and tilt modes are just the first three of the Zernike-functions for that circular region.

the combination of all the piston functions or all the piston and tilt functions for each subaperture of a full aperture are part of a more extensive set of orthonormal functions covering the full aperture.

To see how well a finite set of modes from an orthonormal set of functions can compensate for wavefront distortion, we consider the random function $f(\vec{r})$ corresponding to the turbulence distorted wavefront at the aperture plane. The adaptive optics would seek to match this random function in a way that minimized the discrepancy-squared integrated over the aperture. With an unlimited set of orthonormal modes available, the fitting function would be

$$\hat{f}_G(\vec{r}) = \sum_{i=1}^{\infty} g_i \, G_i(\vec{r}) \quad , \tag{3}$$

where

$$g_i = \int d\vec{r} \, W(\vec{r}) \, f(\vec{r}) \, G_i(\vec{r}) \quad . \tag{4}$$

It can be shown that this minimizes the quantity

$$\Delta_G = \int d\vec{r} \, W(\vec{r}) \, [f(\vec{r}) - \hat{f}_G(\vec{r})]^2 \quad . \tag{5}$$

With only control of the first-N orthonormal modes available as degrees-of-freedom, the adaptive optics would fit the function

$$\hat{f}_G(N;\vec{r}) = \sum_{i=1}^{N} g_i \, G_i(\vec{r}) \quad . \tag{6}$$

It can be shown that this minimizes the quantity

$$\Delta_G(N) = \int d\vec{r} \, W(\vec{r}) \, [f(\vec{r}) - \hat{f}_G(N;\vec{r})]^2 \quad . \tag{7}$$

In a statistical sense, what we are interested in is the ensemble average value of $\Delta_G(N)$. The smaller this is for a given value of N (or the smaller we can make N for a desired value of the ensemble average), the better we would judge the orthonormal series to be for matching the wavefront distortion. If we take the ensemble average value of Eq. (7), and substitute Eq. (6), we get

$$\begin{aligned}\langle \Delta_G(N) \rangle &= \langle \int d\vec{r}\ W(\vec{r})\ [f(\vec{r}) - \sum_{i=1}^{N} g_i\ G_i(\vec{r})]^2 \rangle \\ &= \langle \int d\vec{r}\ W(\vec{r}) \Big\{ [f(\vec{r})]^2 - 2\ f(\vec{r}) \sum_{i=1}^{N} g_i\ G_i(\vec{r}) \\ &\quad + \Big[\sum_{i=1}^{N} g_i\ G_i(\vec{r})\Big]\Big[\sum_{i'=1}^{N} g_{i'}\ G_{i'}(\vec{r})\Big]\Big\} \rangle \quad . \end{aligned} \tag{8}$$

Making the product of sums into a double sum, interchanging the order of integration and summation, and then recognizing that only $f(\vec{r})$ and g_i (and $g_{i'}$) are random quantities, and accordingly manipulating the ensemble average brackets, we get from Eq. (8)

$$\begin{aligned}\langle \Delta_G(N) \rangle &= \langle \int d\vec{r}\ W(\vec{r})\ [f(\vec{r})]^2 \rangle \\ &\quad - 2 \sum_{i=1}^{N} \langle g_i \Big[\int d\vec{r}\ W(\vec{r})\ f(\vec{r})\ G_i(\vec{r})\Big] \rangle \\ &\quad + \sum_{i,i'=1}^{N} \langle g_i\, g_{i'} \rangle \Big[\int d\vec{r}\ W(\vec{r})\ G_i(\vec{r})\ G_{i'}(\vec{r})\Big] \quad . \end{aligned} \tag{9}$$

If we substitute Eq.'s (2) and (4) into Eq. (9), we obtain

$$\begin{aligned}\langle \Delta_G(N) \rangle &= \mathcal{F} - 2 \sum_{i=1}^{N} \langle g_i [g_i] \rangle + \sum_{i,i'=1}^{N} \langle g_i\, g_{i'} \rangle\, \delta_{i,i'} \\ &= \mathcal{F} - \mathcal{G}(N) \quad , \end{aligned} \tag{10}$$

where

$$\mathcal{F} = \langle \int d\vec{r}\ W(\vec{r})\ [f(\vec{r})]^2 \rangle \quad , \tag{11}$$

and

$$\mathcal{G}(N) = \sum_{i=1}^{N} \langle (g_i)^2 \rangle \quad . \tag{12}$$

We identify $\mathcal{F}$ with the "magnitude" of the wavefront distortion to be corrected by the adaptive optics system, and $\mathcal{G}(N)$ with the "magnitude" of the useful effort of the system to achieve the correction. It is obvious from Eq. (7) that $\Delta_G(N)$ and thus $\langle \Delta_G(N) \rangle$ are positive quantities. In view of this, it follows from Eq. (10) that $\mathcal{G}(N)$ will never be greater than $\mathcal{F}$. Obviously, then, that choice of the orthonormal set of functions $\{G_i(\vec{r})\}$ which maximizes $\mathcal{G}(N)$ represents the best set to use in designing the adaptive optics system if only N degrees-of-freedom can be utilized (and our objective is to achieve the best possible wavefront distortion.) We shall now show that in this sense the best possible set of orthonormal functions is the set of so-called Karhunen-Loève functions. With this set of orthonormal functions, fewer degrees-of-freedom are required (i.e., a smaller value of N is needed) to make $\mathcal{G}(N)$ equal or exceed any particular value than with any other possible set of orthonormal functions.

Karhunen-Loève Functions

The so-called Karhunen-Loève functions are specific to a region and to the statistics (in particular the second moment) of a random function. In all of our discussions, we shall understand the region to be that defined by the aperture function, $W(\vec{r})$, and the random function to be the random wavefront distortion function, $f(\vec{r})$. We shall denote the set of Karhunen-Loève functions by the notation $\{K_i(\vec{r})\}$. These functions are orthonormal over the aperture, i.e.,

$$\int d\vec{r}\ W(\vec{r})\ K_i(\vec{r})\ K_{i'}(\vec{r}) = \delta_{i,i'} \quad . \tag{13}$$

Decomposing the random function $f(\vec{r})$ in terms of the set of Karhunen-Loève functions, we would write

$$\hat{f}_K(\vec{r}) = \sum_{i=1}^{\infty} k_i\ K_i(\vec{r}) \quad , \tag{14}$$

where

$$k_i = \int d\vec{r}\ W(\vec{r})\ f(\vec{r})\ K_i(\vec{r}) \quad . \tag{15}$$

The function $\hat{f}_K(\vec{r})$ is formed in such a way as to minimize the integral over the aperture of the square of the deviation between $f(\vec{r})$ and $\hat{f}_K(\vec{r})$, i.e., it minimizes

$$\Delta_K = \int d\vec{r}\ W(\vec{r})\ [f(\vec{r}) - \hat{f}_K(\vec{r})]^2 \quad . \tag{16}$$

Utilizing only the first-N function from the set, we would approximate $f(\vec{r})$ by

$$\hat{f}_K(N;\vec{r}) = \sum_{i=1}^{N} k_i\ K_i(\vec{r}) \quad , \tag{17}$$

which is such that it minimizes the quantity

$$\Delta_K(N) = \int d\vec{r}\ W(\vec{r})\ [f(\vec{r}) - \hat{f}_K(N;\vec{r})]^2 \quad . \tag{18}$$

The special feature of the set of Karhunen-Loève functions which distinguishes it from all other sets of orthonormal functions is the fact that the random variable, k_i (corresponding to g_i for the other sets of orthonormal functions) has the statistical property that

$$\langle k_i k_{i'} \rangle = \delta_{i,i'}\ \varkappa_i \quad . \tag{19}$$

In fact, a homogeneous (eigenvalue/eigenfunction) integral equation which provides the basis for developing the Karhunen-Loève functions can be developed from Eq. (19) in conjunction with Eq.'s (13) and (15). [The second moment of $f(\vec{r})$ appears as the kernel in the integral equation — the bounds of the integral corresponding to the aperture function, $W(\vec{r})$.] We shall assume that the set of Karhunen-Loève functions are so ordered that if $i < i'$, then $\kappa_i \geq \kappa_{i'}$.

Following exactly the same procedure as was utilized in the previous section, we can show that corresponding to Eq.'s (10) and (12), in this case we get

$$\langle \Delta_K(N) \rangle = \mathcal{F} - \mathcal{K}(N) \quad , \tag{20}$$

where

$$\mathcal{K}(N) = \sum_{i=1}^{N} \langle (k_i)^2 \rangle$$

$$= \sum_{i=1}^{N} \kappa_i \quad . \tag{21}$$

Now, to prove that the Karhunen-Loève functions are the optimum functions for our purpose, we simply have to prove that for all N

$$\mathcal{K}(N) \geq \mathcal{S}(N) \quad , \qquad \text{(to be proved)} \quad . \tag{22}$$

We start by defining the quantity

$$\alpha_{i,j} = \int d\vec{r}\; W(\vec{r})\; G_i(\vec{r})\; K_j(\vec{r}) \quad , \tag{23}$$

and noting that since $\{K_j(\vec{r})\}$ is a complete set of orthonormal functions, we can write

$$G_i(\vec{r}) = \sum_{j=1}^{\infty} \alpha_{i,j} \; K_j(\vec{r}) \qquad , \tag{24}$$

while as a consequence of the fact that $\{G_i(\vec{r})\}$ is also a complete set of orthonormal functions, we can equally well write

$$K_i(\vec{r}) = \sum_{j=1}^{\infty} \alpha_{j,i} \; G_j(\vec{r}) \qquad . \tag{25}$$

By virtue of the fact that $G_i(\vec{r})$ is a normalized function, we can write

$$1 = \int d\vec{r} \; W(\vec{r}) \; G_i(\vec{r}) \; G_i(\vec{r}) \qquad . \tag{26}$$

If we substitute Eq. (24) into Eq. (26) twice (once for each G_i), and then make the product of sums into a double sum and then interchange the order of integration and summation, we get

$$1 = \sum_{j,j'=1}^{\infty} \alpha_{i,j} \; \alpha_{i,j'} \int d\vec{r} \; W(\vec{r}) \; K_j(\vec{r}) \; K_{j'}(\vec{r}) \qquad . \tag{27}$$

Now making use of Eq. (13), we can perform the integration and then the j'-summation, yielding the result that

$$\sum_{j=1}^{\infty} (\alpha_{i,j})^2 = 1 \qquad . \tag{28}$$

It follows in a trivial way from Eq. (28) that

$$\sum_{i=1}^{N} \sum_{j=1}^{\infty} (\alpha_{i,j})^2 = N \qquad . \tag{29}$$

Working in the same way, but with the roles of K_i and G_i interchanged, and using Eq. (25) in place of Eq. (24), we can also develop the results that

$$\sum_{i=1}^{\infty} (\alpha_{i,j})^2 = 1 \quad , \tag{30}$$

and

$$\sum_{j=1}^{N} \sum_{i=1}^{\infty} (\alpha_{i,j})^2 = N \quad . \tag{31}$$

If we define A as

$$A = \sum_{i=1}^{N} \sum_{j=1}^{\infty} (\alpha_{i,j})^2 \quad , \tag{32}$$

then it follows from Eq. (29) that

$$\sum_{i=1}^{N} \sum_{j=N+1}^{\infty} (\alpha_{i,j})^2 = N - A \quad . \tag{33}$$

and from Eq. (31), that

$$\sum_{j=1}^{N} \sum_{i=N+1}^{\infty} (\alpha_{i,j})^2 = N - A \quad . \tag{34}$$

If we start with Eq. (12) and substitute Eq. (4) into that and then substitute Eq. (24) into that result, we get

$$\mathscr{S}(N) = \sum_{i=1}^{N} \langle \int d\vec{r}\ W(\vec{r})\, f(\vec{r})\, G_i(\vec{r}) \int d\vec{r}'\ W(\vec{r}')\, f(\vec{r}')\, G_i(\vec{r}') \rangle$$

$$= \sum_{i=1}^{N} \langle \int d\vec{r}\ W(\vec{r})\, f(\vec{r}) \sum_{j=1}^{\infty} \alpha_{i,j}\ K_j(\vec{r})$$

$$\times \int d\vec{r}\,' \, W(\vec{r}\,') \, f(\vec{r}\,') \sum_{j'=1}^{\infty} \alpha_{i,j'} \, K_{j'}(\vec{r}\,') \rangle \quad . \tag{35}$$

At this point, if we interchange the order of integrations, summations, and ensemble averaging, and make use of Eq. (15), we get

$$\mathscr{G}(N) = \sum_{i=1}^{N} \sum_{j,j'=1}^{\infty} \alpha_{i,j} \, \alpha_{i,j'} \langle \int d\vec{r} \, W(\vec{r}) \, f(\vec{r}) \, K_j(\vec{r})$$

$$\times \int d\vec{r}\,' \, W(\vec{r}\,') \, f(\vec{r}\,') \, K_{j'}(\vec{r}\,') \rangle$$

$$= \sum_{i=1}^{N} \sum_{j,j'=1}^{\infty} \alpha_{i,j} \, \alpha_{i,j'} \langle k_j \, k_{j'} \rangle \quad . \tag{36}$$

Making use of the basic property of the Karhunen-Loève functions as defined by Eq. (19), we can rewrite this result as

$$\mathscr{G}(N) = \sum_{i=1}^{N} \sum_{j,j'=1}^{\infty} \alpha_{i,j} \, \alpha_{i,j'} \, \delta_{j,j'} \, \varkappa_j$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{\infty} (\alpha_{i,j})^2 \, \varkappa_j$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} (\alpha_{i,j})^2 \, \varkappa_j + \sum_{i=1}^{N} \sum_{j=N+1}^{\infty} (\alpha_{i,j}) \, \varkappa_j \quad . \tag{37}$$

If we start with Eq. (21) and make use of Eq. (30), we can write

$$\mathscr{K}(N) = \sum_{i=1}^{N} \varkappa_i$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{\infty} (\alpha_{j,i})^2 \, \varkappa_i$$

$$= \sum_{j=1}^{N} \sum_{i=1}^{\infty} (\alpha_{i,j})^2 \varkappa_j$$

$$= \sum_{j=1}^{N} \sum_{i=1}^{N} (\alpha_{i,j})^2 \varkappa_j + \sum_{j=1}^{N} \sum_{i=N+1}^{\infty} (\alpha_{i,j})^2 \varkappa_j \quad . \tag{38}$$

Comparing Eq.'s (37) and (38), we see that the problem of proving Eq. (22) reduces to that of proving that

$$\sum_{j=1}^{N} \sum_{i=N+1}^{\infty} (\alpha_{i,j})^2 \varkappa_j \geq \sum_{i=1}^{N} \sum_{j=N+1}^{\infty} (\alpha_{i,j})^2 \varkappa_j \, , \quad \text{(to be proved)} . \tag{39}$$

By virtue of the ordering of the Karhunen-Loève functions in terms of the magnitudes of the $\varkappa_i$'s, we see that in the sum on the left-hand-side of Eq. (39) all of the $\varkappa_j$'s are greater than or equal to $\varkappa_N$. Taking advantage of Eq. (34), we can thus write

$$\sum_{j=1}^{N} \sum_{i=N+1}^{\infty} (\alpha_{i,j})^2 \varkappa_j \geq \left(\sum_{j=1}^{N} \sum_{i=N+1}^{\infty} (\alpha_{i,j})^2 \right) \varkappa_N$$

$$\geq (N-A) \varkappa_N \quad . \tag{40}$$

In the sum on the right-hand-side of Eq. (39), it follows from the ordering of the $\varkappa_j$'s that all of the $\varkappa_j$'s are less than or equal to $\varkappa_{N+1}$. Thus, making use of Eq. (33), we get

$$\sum_{i=1}^{N} \sum_{j=N+1}^{\infty} (\alpha_{i,j})^2 \varkappa_j \leq \left(\sum_{i=1}^{N} \sum_{j=N+1}^{\infty} (\alpha_{i,j})^2 \right) \varkappa_{N+1}$$

$$\leq (N-A) \varkappa_{N+1} \quad . \tag{41}$$

Since $\kappa_{N+1} \le \kappa_N$, Eq. (39) follows, and from that Eq. (22) follows. We can write

$$\mathcal{K}(N) \ge \mathcal{J}(N) \quad . \tag{42}$$

As we noted before, this relationship means that the Karhunen-Loève functions are the most efficient functions for representation of random wavefront distortion. This is in the sense that for any given number of terms (i.e., number of degrees-of-freedom), a Karhunen-Loève approximation to the random wavefront distortion will have the least mean* square error of any orthonormal series (or any other kind of series) approximation to the random wavefront using that number of degrees-of-freedom.

Having established that the Karhunen-Loève functions are the most efficient at approximating random wavefront distortion, we now turn our attention to the quantitative aspects of the problem. How efficient is the Karhunen-Loève series approximation, and how efficient are approximations based on other series? We take this up in the next section.

Quantitative Results for Various Series Approximations

As examples of various types of series representations of random wavefront distortion, we shall consider, in addition to our optimum case of the Karhunen-Loève functions, the following four other cases. These four cases represent the use of 1) local piston-only modes, 2) local piston/tilt-only modes, 3) Zernike polynomials, and 4) a variation of the Karhunen-Loève functions in which tilt is treated separately. It is convenient to start with the local piston-only modes.

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* The mean is taken in the sense of <u>both</u> an aperture average and an ensemble average.

We consider a circular aperture of diameter D , and consider it subdivided in a covering set of non-overlapping circular regions of diameter d . (Obviously, it is impossible to achieve such a decomposition, and in practice one may consider each of these regions to be a regular hexagon of area $\frac{1}{4}\pi d^2$, arranged in a near circular tile pattern of area $\frac{1}{4}\pi D^2$.) If each of these subregions is allowed to move up and down, i.e., piston motion only, to provide phase shift to match (compensate) the random wavefront distortion, then we are considering a representation in terms of

$$N = (D/d)^2 - 1 \qquad \text{(piston-only)} \quad , \tag{43}$$

degrees-of-freedom. The subtraction of the factor of one takes account of the fact that one of the pistons, or the average of several or all of the pistons must be taken as a reference for the amount of vertical displacement of the pistons. This reference is not to be considered as a degree-of-freedom.

It has been shown[1] that for a random wavefront distortion with a phase structure function that can be written as

$$\mathcal{D}(r) = 6.88\ (r/r_0)^{5/3} \qquad (\text{rad}^2) \quad , \tag{44}$$

the mean square wavefront distortion relative to the average phase, over a circular aperture of diameter D is

$$\sigma_D^2 = 1.0299\ (D/r_0)^{5/3} \qquad (\text{rad}^2) \quad . \tag{45}$$

Over each of the sub-elements, the piston motion can follow the average phase, but there will remain higher order wavefront distortion which the piston motion is unable to accommodate. This higher order aberration, by virtue of the same considerations as led to Eq. (45), will have a mean square value

$$\sigma_d^2 = 1.0299\ (d/r_0)^{5/3} \qquad (\text{rad}^2) \quad . \tag{46}$$

This expression gives not only the remaining aberration over the subelement after piston motion has done as well as it can to match the local wavefront distortion, but also, since it is the same on each of the set of subapertures covering the full aperture, represents average residual wavefront distortion over the full aperture with local piston-only wavefront correction with N degrees-of-freedom.

It is convenient to consider mean square wavefront distortion in terms of $\sigma^2/(D/r_0)^{5/3}$. The limit of $\sigma^2/(D/r_0)^{5/3}$ for no correction, i.e., for $N = 0$ degrees-of-freedom, we see from Eq. (45) is

$$\sigma^2/(D/r_0)^{5/3} = 1.0299 \quad , \quad (N = 0) \tag{47}$$

In Fig. 1, we plot the residual error for $N \neq 0$, which we see from combining Eq.'s (43) and (46) has the form

$$\sigma^2/(D/r_0)^{5/3} = 1.0299\,(N = 1)^{-5/6} \quad , \quad \text{(piston-only)} \quad . \tag{48}$$

If we consider a set of degrees-of-freedom corresponding to modes localized to a set of circular subregions of diameter d , covering the full aperture, but allow the degrees-of-freedom to represent not only piston motion but also two components of tilt, then the number of degrees-of-freedom is

$$N = 3\,(D/d)^2 - 1 \quad , \quad \text{(piston/tilt-only)} \quad , \tag{49}$$

where in this case there are three degrees-of-freedom for each subregion, except that one of the piston motions must be taken as a reference for all the rest, and so should not be considered to be a degree-of-freedom. It has been shown[1] that for a subaperture of diameter d over which there is both piston and two axes of tilt adjustable to match the wavefront distortion, the higher order aberrations in the wavefront which can not be accommodated by the piston/tilt adjustments will have a mean square value of

$$\sigma_d^2 = 0.13433\ (d/r_0)^{5/3} \qquad (\mathrm{rad}^2) \quad . \tag{50}$$

Since this mean square residual wavefront error applies equally well to all of the subapertures that cover the full aperture, Eq. (50) also defines the mean square residual error over the full aperture for piston/tilt-only corrections. Combining Eq.'s (49) and (50), we see that for N degrees-of-freedom with piston/tilt-only wavefront correction, the residual wavefront error on a circular aperture of diameter D will be such that

$$\sigma^2/(D/r_0)^{5/3} = 0.13433 \times 3^{5/6}\ (N+1)^{-5/6}$$

$$= 0.33556\ (N+1)^{-5/6} \qquad (\text{piston/tilt-only}) \ . \tag{51}$$

In Fig. 1 we also show this dependence.

The use of the Zernike polynomials to define a set of functions over a circular region of diameter D , as a basis for representing random wavefront distortion, has been studied by Noll.[2] He has been able to show that for a fit to the random wavefront distortion over the full aperture with N degrees-of-freedom in the Zernike function set, the mean square residual wavefront error can be written as

$$\sigma^2 = Z(N)\ (D/r_0)^{5/3} \quad , \qquad (\text{Zernike-functions}) \ , \tag{52}$$

where the function Z(N) has the values listed in Table 1. Here, although a constant value is generally taken as the first of the Zernike-functions, following the same argument as before, we do not consider this to be a degree-of-freedom. In Fig. 1, we also show the quantity

$$\sigma^2/(D/r_0)^{5/3} = Z(N) \quad , \qquad (\text{Zernike-functions}) \ . \tag{53}$$

Before considering the pure Karhunen-Loève function's ability to match the random wavefront distortion over the circular aperture of diameter D , we consider a minor variant case in which tilt is treated separately. This allows us to consider a system in which the adaptive optics are mounted in gimbals to track a target. In this case, aperture averaged angle-of-arrival fluctuations will be tracked by the two degrees-of-freedom inherent in the two gimbal axes and only the higher order aberrations will have to be accommodated by the adaptive optics.* The performance of the "tilt-free" Karhunen-Loève functions in this role has been modeled by us,[3] and the results quoted there for

$$\mathcal{K}_T(n) = (D/r_0)^{5/3} \sum_{i=1}^{n} \kappa_{T,i} \tag{54}$$

for the sum of the variances (eigenvalues) associated with the first n tilt-free Karhunen-Loève functions (eigenfunctions) can be used to calculate the residual wavefront error variance. Considering that there are two more degrees-of-freedom than n , i.e., if we use an n-term tilt-free Karhunen-Loève series, we are dealing with

$$N = n + 2 \tag{55}$$

degrees-of-freedom, and that the tilt-free mean square wavefront distortion over an aperture of diameter D is

$$\sigma_T^2 = 0.13433\,(D/r_0)^{5/3} \quad , \tag{56}$$

we see that the mean-square residual wavefront distortion over the aperture will be such that

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* In treating the Zernike-functions, it was not necessary to separately consider the tilt degrees-of-freedom provided by the two gimbal axes as these are identical to the degrees-of-freedom provided by the first two Zernike functions. It therefore made no difference in our results whether we considered the adaptive optics per se, or the gimbals to supply these two degrees-of-freedom.

$$\sigma^2/(D/r_0)^{5/3} = K_T(N)$$

$$= 0.13433 - \sum_{i=1}^{N-2} \varkappa_{T,i} \qquad \left(\begin{array}{c}\text{tilt-free} \\ \text{Karhunen-Loève} \\ \text{functions}\end{array}\right) . \quad (57)$$

The function $K_T(N)$ has the values listed in Table 1. The corresponding residual wavefront aberration values are plotted in Fig. 1.

The optimum performance in adaptive optics for a limited number of degrees-of-freedom is achieved if there is no separate tilt removal (using up the first two degrees-of-freedom in a not quite optimal, though generally quite practical manner), and the Karhunen-Loève functions are used to model all aspects of the wavefront distortion. In this case, we have shown that[3]

$$\mathcal{K}(N) = (D/r_0)^{5/3} \sum_{i=1}^{N} \varkappa_i \quad , \qquad (58)$$

i.e., the sum of the first N variances is such that the residual wavefront aberration with N degrees-of-freedom has the form

$$\sigma^2/(D/r_0)^{5/3} = K(N)$$

$$= 1.0299 - \sum_{i=1}^{N} \varkappa_i \quad , \quad \left(\begin{array}{c}\text{Karhunen-Loève} \\ \text{functions}\end{array}\right) , \qquad (59)$$

with K(N) taking the values listed in Table 1. In Fig. 1, we plot these values.

As can be seen, when we consider large numbers of degrees-of-freedom, the tilt-free Karhunen-Loève functions (which take advantage of the two degrees-of-freedom that can be provided by the gimbal axes) allow adaptive optics performance that only requires about 4% more degrees-of-freedom than the optimum series to achieve a given residual error level.

The Zernike-functions require about 29% more degrees-of-freedom to deliver the same residual wavefront aberration performance. The local piston/tilt-only functions require about an additional > 75% more degrees-of-freedom than the optimum to allow this same level of performance. The local piston-only function requires an additional 583% more degrees-of-freedom than the optimum to allow this same level of performance. Overall, we see that the required number of degrees-of-freedom for a given residual error, compared to the optimum (Karhunen-Loève) case is as indicated in Table 2. Clearly, we generally would wish to avoid use of the local piston-only functions for adaptive optics, but once we advance to the level of sophistication involved in local piston/tilt degrees-of-freedom in the design of adaptive optics, consideration of more sophisticated representations of the wavefront distortion will never yield more than a modest further reduction in the number of degrees-of-freedom required to achieve a desired residual aberration performance.

References

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3. D. L. Fried, "Theoretical Study of Non-Standard Imaging Concepts," Technical Report by Optical Science Consultants under Air Force Contract F30602-74-C-0115, published by Rome Air Development Center as RADC-TR-75-182, July 1975.

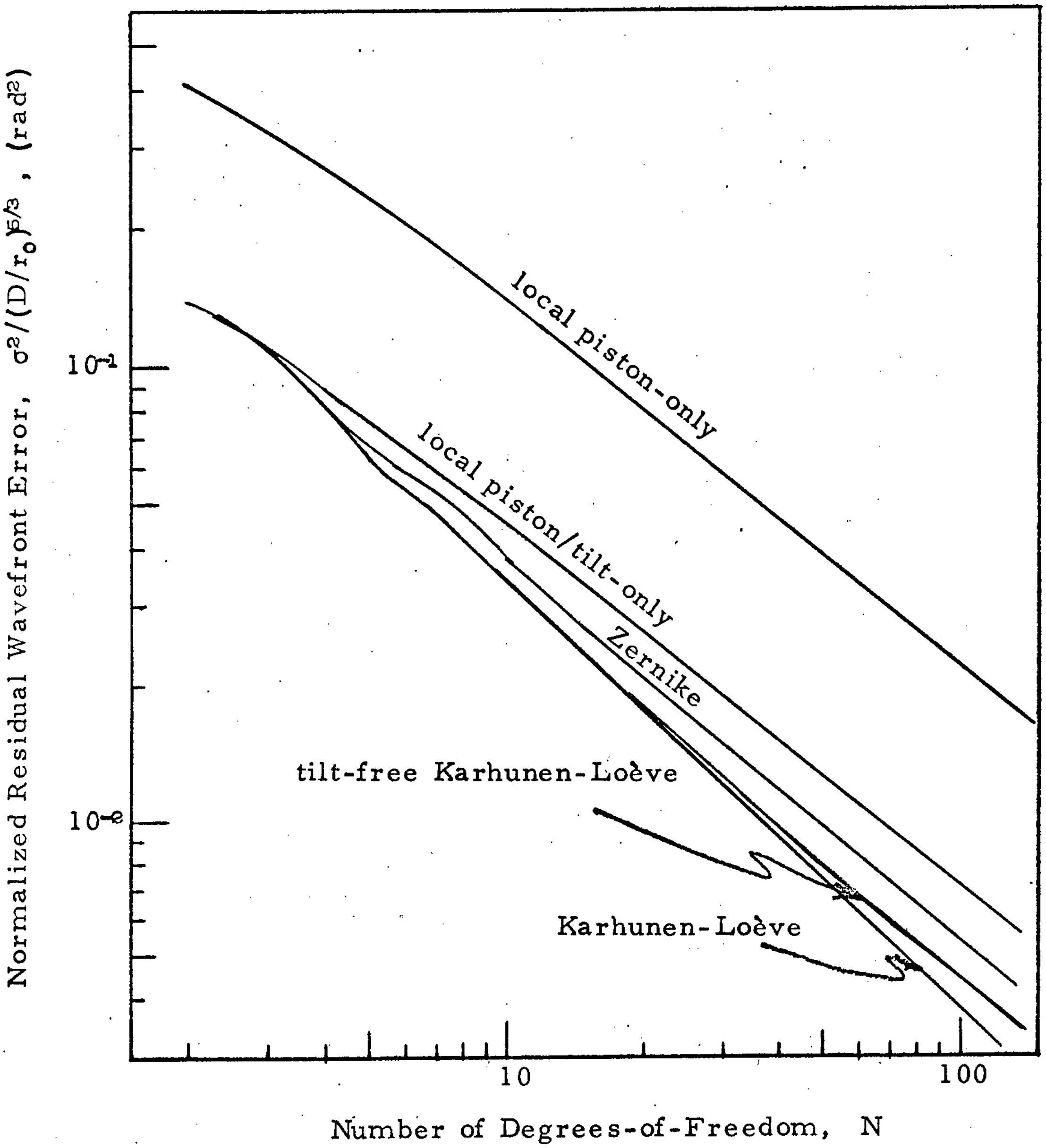

Figure 1. Relationship Between Degrees-of-Freedom and the Residual Wavefront Error. Results are shown for the five cases, in descending order on the graph, for 1) local piston-only modes, 2) for local piston/tilt-only modes, 3) for Zernike-functions, 4) for tilt-free Karhunen-Loève functions, and 5) for Karhunen-Loève functions.

Table 1

Residual Mean-Square Discrepancy for an N-Term Series

(For N greater than about 50, the K(N) results may suffer from significant loss of accuracy due to round-off errors.)

N	Z(N)	$K_T(N)$	K(N)
2	1.3433×10^{-1}	1.3433×10^{-1}	1.3290×10^{-1}
3	1.1117	1.1044	1.0901
4	8.8009×10^{-2}	8.6545×10^{-2}	8.5118×10^{-2}
5	6.4849	6.2694	6.1267
7	5.2497	4.9413	4.7985
10	3.7697	3.3480	3.2925
15	2.6718	2.3088	2.2533
20	2.0781	1.7809	1.7296
25	1.7513	1.4659	1.4145
30	1.5029	1.2389	1.1876
40	1.1811	9.6327×10^{-3}	9.1229×10^{-3}
50	9.7908×10^{-3}	7.9214	7.4114
70	7.3845	5.9526	5.4426
100	5.4770	4.4086	3.8988
150	3.8998	3.1476	2.6375
200		2.4822	1.9722

Table 2

Required Number of Degrees-of-Freedom for a Given Mean-Square Error

(Results here are based on matching the performance of the optimum series with 25 degrees-of-freedom.)

Type Series	Relative Number of Degrees-of-Freedom
Karhunen-Loève	100:100
Tilt-Free Karhunen-Loève	104:100
Zernike	129:100
Local Piston/Tilt-Only	175:100
Local Piston-Only	683:100