REPORT DOCUMENTATION PAGE

Form Approved
OMB NO. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*	2. REPORT DATE July/September 1997	3. REPORT TYPE AND DATES COVERED Technical Report

4. TITLE AND SUBTITLE Conservative Delta Hedging	5. FUNDING NUMBERS DAAH04-95-1-0105
6. AUTHOR(S) Per Aslak Mykland	

7. PERFORMING ORGANIZATION NAMES(S) AND ADDRESS(ES) Univeristy of Chicago Chicago, IL 60637	8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES) U.S. Army Research Office P.O. Box 12211 Research Triangle Park, NC 27709-2211	10. SPONSORING / MONITORING AGENCY REPORT NUMBER ARO 31383.8-MA

11. SUPPLEMENTARY NOTES

The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.

12a. DISTRIBUTION / AVAILABILITY STATEMENT Approved for public release; distribution unlimited.	12 b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

ABSTRACT IN INTRODUCTION

DTIC QUALITY INSPECTED 8

14. SUBJECT TERMS	15. NUMBER IF PAGES
	16. PRICE CODE

17. SECURITY CLASSIFICATION OR REPORT UNCLASSIFIED	18. SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED	19. SECURITY CLASSIFICATION OF ABSTRACT UNCLASSIFIED	20. LIMITATION OF ABSTRACT UL

Conservative Delta Hedging [1]

by

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TECHNICAL REPORT NO. 456

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July/September 1997

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ABSTRACT

It is common to have interval predictions for volatilities and other quantities governing securities prices. The purpose of this paper is to provide an exact method for converting such intervals into arbitrage based prices of financial derivatives or industrial or contractual options. We call this procedure *conservative delta hedgning*. As existing procedures are of an ad hoc nature, the proposed approach will permit an institution's management a greater oversight of its exposure to risk.

1. Introduction. How should one hedge an option when there is time variability about the parameters governing the process? Apart from the possibility of substantial jumps in prices, this is perhaps the most major conceptual issue facing institutions that sell and hedge options.

To focus the mind, one can consider an instrument governed by

$$dS_t = rS_t dt + \sigma S_t dW_t \qquad (1.1)$$

under the risk neutral distribution (Harrison and Kreps (1979), Harrison and Pliskà (1981)). Typically, the magnitude of the risk free interest rate r and the volatility σ vary with time, and successfully delta hedging an option (for a basic description of delta hedging, see, for example, Hull (1996), Chapter 14.5, p. 312-321) involves the distribution of future values of these two quantities. What values or distributions should one use?

In many circumstances, the answer is that the uncertainty in future values of r and σ can also be hedged away. If there are already market traded options on S_t, one can deal with the volatility this way. For example, in a stochastic volatility model (see, *e.g.* Ball and Roma (1994), Hoffman, Platen and Schweizer (1992), Hull and White (1987), Pham and Touzi (1996), and Renault and Touzi (1996); see also Duffie (1996). Chapter 8.H),

$$d\sigma_t = \nu(\sigma_t)dt + \gamma(\sigma_t)dB_t, \qquad (1.2)$$

one needs only one derivative instrument to hedge future variability in σ. Similarly, r can be hedged in bonds, Eurodollar futures, or similar instruments.

This approach, however, is not always available. Market traded derivatives may not exist. A common instance of this, is intermarket correlations: the options may exist in each market, so you can take care of the volatility, but the dependence may not be hedgeable. Even where market traded derivatives do exist. supply and demand typically do not balance, so somebody has an interest in hedging options without in turn hedging away risk due to time variability. Finally, even when such hedging is available, there can be substantial uncertainty about what model to use, and a number of financial institutions are putting a substantial amount of research effort into this matter (particularly on the fixed income side).

This is where it becomes natural to specify prediction regions for the parameters. This can be done with statistical methods, by using implied quantities, or by intelligent guesswork. As far as statistics is concerned, there is a substantial literature on estimating parameters in models for time varying interest rates and volatilities. Important references include Aït-Sahalia (1996), Bibby and Sørensen (1994, 1995), Bollerslev, Chou and Kroner (1992), Dachuna-Castelle and Florens-Zmirou (1986), Danielsson (1994), Duan (1995), Gallant (1996), Hansen and Scheinkman (1995), Jacquier, Polson and Rossi (1994), Kessler and Sørensen (1995), Liptser and Shiryayev (1978), and Renault (1995).

There is a missing link, however, between prediction regions and derivatives pricing. The former give rise to a family of probability distributions. Derivatives pricing relies, however, mainly on the construction of replicating strategies in the presence of a specific probability. It is not, except in special cases (Avellaneda, Levy and Paras (1995), cf. Example 3 in Section 3 of this paper), known how to convert families of probabilities into replicating strategies. And so, prediction regions can only be used in an ad hoc fashion. The value-at-risk literature is an example of this (see Duffie and Pan (1997) for a review).

There are two important consequences of this:

(1) There is uncertainty in the hedging operation itself – what "deltas" should one use; how should one deal with the incompleteness.

(2) There is difficulty in putting a value on an institution's portfolio. In some ways, this is a more serious problem. It prevents management oversight, and is undoubtedly a factor behind recent "scandals" involving derivative securities. Headline-grabbing cases include those of Orange County, the Bank of Tokyo-Mitsubishi (see McGee (1997), for example), and the National Westminster Bank (Finch (1997)). The financial operations of the Government of Belgium constitutes a less high profile, but equally (if not more) astonishing case (see King (1997)). On top of that, there is the room which the lack of oversight gives to straightforward fraud, as in the Barings case, or that of the three billion dollar copper scheme.

An aspect of this problem is that accounting rules tend to try to mark the portfolio to market as much as possible. For illiquid instruments, this sometimes involves using prices arising from

reported transactions where the price is quite wrong. Somebody had a bad day. Nonetheless, such a price is forced on others for accounting purposes.

The question of book value also comes up in settings other than derivative financial securities. As demonstrated by recent work reported, *e.g.*, in Dixit and Pindyck (1993), option theory can be helpful in valuing anything from a firm's expansion possibilities to a government's contracts and obligations. In this only partly financial setting, the current approach should be particularly valuable.

The intent of this article is to provide an approach to this problem. We shall supply a device for valuing and hedging which permits an exact quantification of the impact on prices of uncertainty about the future when this uncertainty is expressed via intervals or other regions for parameters. The main qualification is that we remain in the standard setup where securities prices are continuous.

The main antecedent in the literature for the current article is the work of the 'Columbia group' (in particular, Cvitanić and Karatzas (1992, 1993), Karatzas (1996), and Karatzas and Kou (1996)) on constrained delta hedging, which has been an inspiration to the author. Unlike the setting of the Columbia group, we consider incompleteness through uncertainty about probabilities, and not only through non-hedgeability in certain securities. Nontheless, there are substantial similarities in approach and techniques.

The basic idea in the paper is spelled out in section 2, a general framework is developed in section 3, and in section 4 we discuss European options. Section 5 defines self-financing strategies in this context. Section 6 and the Appendix contain relevant proofs.

2. Conservative delta hedging. The problem of accurately determining process parameters such as r and σ is really twofold: (1) prediction (what is going to be the value of these parameters in the future?) and (2) interpolation (what is the current value of these parameters?). The difference between the two is that the latter, though intricate, is quite well posed. The former is not.

For interpolation there is a substantial number of sources of information – implied (Beckers (1981), Bick and Reisman (1993)) and observed volatility; different strike prices for the former, different schemes for the latter (see, for example, Chesney, Elliot, Madan and Yang (1993), Florens-Zmirou (1993), Foster and Nelson (1996) and Pastorello (1996)), and a slew of possible interest rates and measures of yield. The problem of analyzing such data is fairly robust to model assumptions; in particular, observed volatility is mostly a nonparametric problem (just use some smoothed version of the quadratic variation). There are a number of conceptual problems that have an impact on the analysis (see, *e.g.*, Mykland (1997) for an attempt to think about these problems), but the difficulties are nonetheless mainly ones of fine tuning.

Prediction, on the other hand, is a nightmare. It depends critically on what model is used for the evolution of process parameters. One is caught is a trade-off where models tend to either be too simple to be adequate descriptions of reality, or too complex for the amount of data available. This is most clearly visible in the case of interest rates. The typical parametric one factor models (see Duffie (1996) and Hull (1996) for an overview) belong to the first category; nonparametric models (see Aït-Sahalia (1996) and Heath, Jarrow and Morton (1992) for very different instances of what that might mean) tend to be in the second one. Striking a balance (multi-factor parametric models?) is a delicate issue. Also, obviously, the past may not be a good predictor of the future.

The approach we argue in this paper is the following: take a conservative stab at predicting the cumulative volatility. and then hedge according to the (approximate) actual volatility. This uses sharply the well posed component of the model, but guards against excessive reliance on the ill posed ones.

The basic idea – and the fact that it works – can be illustrated in the context of European call options.

EXAMPLE 1. Consider a European call with strike price K that pays off at time T. Let

$$C(S, R, \Xi) = S\Phi(d_1) - X\exp(-R)\Phi(d_2), \tag{2.1}$$

where

$$d_1 = (\log(S/K) + R + \Xi/2)/\sqrt{\Xi} \tag{2.2}$$

and $d_2 = d_1 - \sqrt{\Xi}$. In other words, this is the Black-Scholes-Merton (Black and Scholes (1973), Merton (1973)) price of the option at time t and stock price S if the r_u and σ_u processes are nonrandom, and if if the cumulative interest rate from t to T is R, and the cumulative volatility in the same period is Ξ.

Consider the instrument whose value at time t is

$$V_t = C(S_t, R_t, \Xi_t), \tag{2.3}$$

where

$$R_t = R - \int_0^t r_u du \text{ and } \Xi_t = \Xi - \int_0^t \sigma_u^2 du. \tag{2.4}$$

In equation (2.4), r_t and σ_t are the actual observed quantities. Now suppose that

$$R_0 \geq \int_0^T r_u du \text{ and } \Xi_0 \geq \int_0^T \sigma_u^2 du. \tag{2.5}$$

Then an easy calculation will show that no matter how r_t and σ_t actually otherwise behave (and whether they are random or nonrandom), there is a self financing strategy for V_t, hedging in the security S_t and money market bond

$$\beta_t = \exp\left(\int_0^t r_u du\right). \tag{2.6}$$

Furthermore,

$$V_T \geq (S_T - K)^+ \tag{2.7}$$

almost surely. In other words, one can both synthetically create the security V_t, and one can use this security to cover one's obligations. ∎

The above is sufficiently innocent looking that is should be emphasized that it is not trivial. Keep in mind that it is true for all probability distributions under which (2.5) is satisfied with probability 1.

Unless the cumulative interest is actually known beforehand, however, the scheme given above is not quite right. This is in the sense that a lower price can usually be found as the starting point for a self financing strategy satisfying (2.7). As follows.

EXAMPLE 2. What the procedure in Example 1 overlooks, is that the price Λ_0 of the zero coupon bond maturing at T is actually known (at least in most markets). This bond satisfies

$$\Lambda_t = E^* \left[\exp\left(- \int_t^T r_u du \right) \mid \mathcal{F}_t \right], \tag{2.8}$$

where P^* is the risk neutral distribution, and the procedure we have given ignores that. A consequence, for example, is that if one prices a forward contract the same way, one gets the price wrong.

It will sometimes be the case that the existence of security Λ_t will also make lower bounds for r and σ active, and one can consider replacing (2.5) by

$$R^+ \geq \int_0^T r_u du \geq R^- \text{ and } \Xi^+ \geq \int_0^T \sigma_u^2 du \geq \Xi^-. \tag{2.9}$$

We shall see in Section 4 how to compute the value of the call option in this case. We show that if r_0 solves

$$\Phi(d_2(S_0, r_0, \Xi^+)) = \frac{\exp(-R^-) - \Lambda_0}{\exp(-R^-) - \exp(-R^+)}, \tag{2.10}$$

where Φ is the cumulative normal distribution and in the same notation as in (2.1)–(2.2), then one can start a super-replicating strategy with the price at time zero given in the following:

$$r_0 \geq R^+ : C(S_0, R^+, \Xi^+)$$
$$R^+ > r_0 > R^- : C(S_0, r_0, \Xi^+) + K \left(\exp(-r_0) - \exp(-R^+) \right) \Phi \left(d_2(S_0, r_0, \Xi^+) \right) \tag{2.11}$$
$$r_0 \leq R^- : C(S_0, R^-, \Xi^+) + K \left(\exp(-R^-) - \Lambda_0 \right)$$

An illustration of the improvement over Example 1 is given in figure 1.

[figure 1 here]

∎

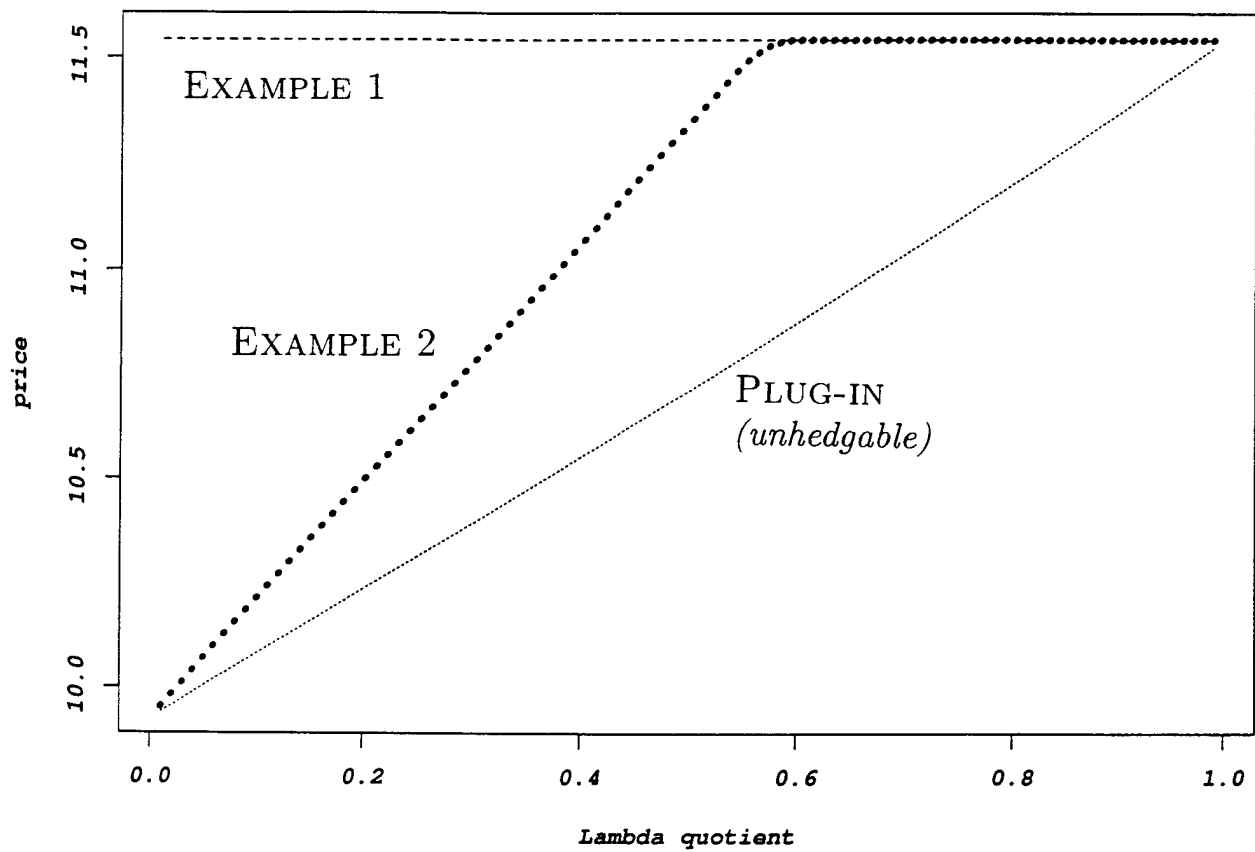FIG. 1. *Option price as* Λ_0 *varies. Three ways of pricing European call options in the presence of restrictions (2.9). The Λ-quotient is the number in (2.10). The plug-in price is $C(S_0, -\log \Lambda_0, \Xi^+)$. Parameter values are $\Xi^+ = 0.04$, $R^- = 0.04$, and $R^+ = 0.07$. The option is at the money ($K = S_0$).*

In fact, in the presence of market traded derivatives, there are additional restrictions that one can consider; we comment on such possibilities in Section 7.

3. General Theory.

DEFINITION. $\mathcal{N}$ is a σ-ideal in a filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t)$ provided $\mathcal{N}$ is a subset of $\mathcal{F}$ that is closed under countable union, and that satisfies that any $\mathcal{F}$ measurable subset of a set in $\mathcal{N}$ is also in $\mathcal{N}$.

A property holds $\mathcal{N} - a.s.$ if it fails only for a set A in $\mathcal{N}$. If $S_t^{(1)}, ..., S_t^{(p)}$ are adapted continuous processes (representing traded securities paying no dividends), and if η is a payoff to be made at a (non random or stopping) time τ, $V_u, t \leq u \leq \tau$ is said to be a super-replication of this payoff (from t to τ) if

(i) one can cover one's obligations:

$$V_\tau \geq \eta \tag{3.1}$$

$\mathcal{N} - a.s.$; and

(ii) there are processes H and D, defined $\mathcal{N} - a.s.$, so that

$$V_u = H_u + D_u, \quad t \leq u \leq \tau, \tag{3.2}$$

$\mathcal{N} - a.s.$, where D_t is a nonincreasing adapted *càdlàg* process (except for a set in $\mathcal{N}$), and where H_t is self financing $\mathcal{N} - a.s.$ (in a sense to be defined below) in the traded securities $S_t^{(1)}, ..., S_t^{(p)}$.

There is some variation in how to technically define "self financing" (see, e.g., Duffie (1996), Chapter 6.C (p. 103-105)), and the issues get somewhat compounded in the case of σ-ideals representing several possible probabilities. For this reason, the definition is deferred until after the main theorem and some examples (see Section 5). ∎

A common way of defining a σ-ideal would be to start with a set $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F}, \mathcal{F}_t)$, and then set

$$\mathcal{N} = \{A \epsilon \mathcal{F} : P(A) = 0 \text{ for all } P \epsilon \mathcal{P}\}. \tag{3.3}$$

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This is often conceptually easier than to state the σ-ideal directly.

Consider some instances of this:

EXAMPLE 1. (European call; hedging in the stock only). In this case, $p = 1$, and $\mathcal{P}$ is the set of all probability distributions so that $S_0 = s_0$ (the actual value),

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \tag{3.4}$$

and so that (2.5) holds. We also assume that

$$\exp\left\{\int_0^t \lambda_u dW_u - \frac{1}{2}\int_0^t \lambda_u^2 du\right\} \qquad \text{is a } P\text{-martingale}, \tag{3.5}$$

where $\lambda_u = (\mu_u - r_u)/\sigma_u$. The strategy given is clearly a super-replication of a European call. The condition (3.5) is what one needs for Girsanov's Theorem (see, for example, Karatzas and Shreve (1991), Theorem 3.5.1) to hold. Without this theorem, there would be no equivalent martingale measure, and we would get in trouble further down the line. See Remark 3.1. ∎

EXAMPLE 2. (European call; hedging in the stock and a zero coupon bond). Here, $p = 2$, $S_t^{(1)} = S_t$, $S_t^{(2)} = \Lambda_t$. Further details are explored in Section 4. ∎

EXAMPLE 3. (Avellaneda, Levy, and Paras (1995)). Here $p = 1$ and the interest rate is fixed. $\mathcal{P}$ is the set of probabilities for which $S_0 = s_0$, and for which (3.4)-(3.5) and

$$\sigma_t \epsilon [\sigma^-, \sigma^+] \text{ for all } t\epsilon[0, T] \tag{3.6}$$

hold. With this assumption, a super-replicating strategy is constructed for European options based on the "Black-Scholes-Barenblatt" equation (cf. Barenblatt (1978)). ∎

What should the price be in general? The natural hedging based approach is as follows, cf. Cvitanić and Karatzas (1992):

DEFINITION. The *conservative ask price* (or offer price) at (a nonrandom or stopping) time t for a payoff η to be made at a time $\tau \geq t$ (relative to securities $S_t^{(1)}, ..., S_t^{(p)}$ and the σ-ideal $\mathcal{N}$)

is

$$A_t = \underset{\mathcal{N}}{\text{ess inf}}\{V_t : (V_u)_{t \le u \le \tau} \text{ is a super-replication of the payoff }\}, \tag{3.7}$$

which we shall take to mean that (i) $A_t \le V_t, \mathcal{N} - a.s.$ for any V_t in the set on the right hand side of (3.7), and (ii) that any other random variable $\widetilde{A}_t$ satisfying the same criterion must be equal to $A_t, \mathcal{N} - a.s.$ This is in analogy to the definition in Proposition VI-1-1 (p. 121) of Neveu (1975), but because of the apparent absence of a dominating measure, the properties stated in that result may not apply. A_t need not exist, but is clearly unique ($\mathcal{N} - a.s.$) if it does.

Similarly, the *conservative bid price* can be defined as the supremum over all sub-replications of the payoff, in the obvious sense. ∎

The conservative bid price B_t equals

$$B_t(\eta) = -A_t(-\eta), \tag{3.8}$$

in obvious notation, and subject to mild regularity conditions. For this reason, it is enough to study ask prices. Note that put-call parity (see Hull (1996), Chapter 7.6, p. 167-170) holds for European options,

$$A_t((K - S_T)^+) + S_t = A_t((S_T - K)^+) + \Lambda_t K, \tag{3.9}$$

and similarly for bid prices. This, obviously, is provided one assumes the existence of a risk free zero coupon bond Λ_t maturing at T.

We shall see that in the case under study, A_t is, in fact, a super-replication of the payoff.

EXAMPLE 1. The solution to this problem represents the price under one of the probability distributions $P \epsilon \mathcal{P}$. Hence, if one goes below that price, there would be no way of covering the final payoff if this distribution occurred. In other words, the quoted price is the conservative ask price.

∎

EXAMPLE 3. For the same reasons as in Example 1, the price given by Avellaneda, Levy and Paras (1995) is the conservative ask price. Note, however, that the hedging strategy presented by that paper is not the one used here. Instead of basing themselves on the actually accrued volatility

(as in (2.4)), they hedge based on a worst case non-observed volatility. The reason why they can get away with this is that the method of specifying conservativeness is particularly conservative; if one supposes a volatility model like (1.2), for example, a 95 % (say) bound of the type (3.6) yields a much higher price than one of the type (2.5), since the latter involves the average of σ_t^2 while the former depends on the maximum and the minimum of this process. The high prices that are inherent in the Avellaneda, Levy and Paras (1995) approach has been commented on by Ahn, Muni and Swindle (1996a, b), who suggest ways of alleviating the effect. ∎

EXAMPLE 4. (American and Bermudan options, r and σ fixed). The price of a payoff $f(S_\tau, \tau)$ at a time τ to be determined by the owner of the option (between 0 and T) is given by (see, *e.g.*, Myneni (1992), or Duffie (1996), Chapter 8.E)

$$\sup_\tau E^* \exp\{-(\tau - t)r\} f(S_\tau, \tau), \tag{3.10}$$

where the supremum is over all stopping times τ between t and T. P is fixed. The justification is exactly that a super replication can be constructed with starting value (3.10), and also that the supremum represents the price under one of the possible choices for τ. In our terminology, set $P_{\text{ext}} = $ (law of τ given the S_t-process) $\otimes$ P. $\mathcal{P}$ is the collection of such P_{ext}s for which τ is a stopping time between 0 and T. ∎

To give the general form of A_t, we consider an appropriate set $\mathcal{P}^*$ of "risk neutral" probability distributions P^*.

DEFINITION. If S_t is a process, $S_t^* = \beta_t^{-1} S_t$, and vice versa. Recall that β_t is the money market account process from (2.6). $\mathcal{P}^*$ is now defined as the set of probability measures P^* on $\mathcal{F}$ whose null sets include those in $\mathcal{N}$, and for which $S_t^{(1)*}, ..., S_t^{(p)*}$ are martingales. In analogy with (3.3), we also set

$$\mathcal{N}^* = \{A \epsilon \mathcal{F} : P^*(A) = 0 \text{ for all } P^* \epsilon \mathcal{P}^*\} \tag{3.11}$$

and

$$\mathcal{N}^e = \{A \epsilon \mathcal{F} : P^e(A) = 0 \text{ for all } P^e \text{ extremal in } \mathcal{P}^*\} \tag{3.12}$$

Extremal here refers to the usual meaning of the word: P^e is extremal in $\mathcal{P}^*$ if $P^e \in \mathcal{P}^*$ and if, whenever $P^e = a_1 P_1^e + a_2 P_2^e$ for a_1, $a_2 > 0$ and P_1^e, $P_2^e \in \mathcal{P}^*$, it must be the case that $P^e = P_1^e = P_2^e$. ∎

Clearly, $\mathcal{N} \subseteq \mathcal{N}^* \subseteq \mathcal{N}^e$. We shall require that $\mathcal{N} = \mathcal{N}^e$. The reason for $\mathcal{N} = \mathcal{N}^*$ is given by Remark 3.1. Conditions for $\mathcal{N}^* = \mathcal{N}^e$ are discussed after Theorem 3.2. As we shall see, the condition $\mathcal{N} = \mathcal{N}^e$ is satisfied in Examples 1, 3 and 4. In Section 4, we also impose conditions to make this be the case in Example 2.

REMARK 3.1. *Suppose $\mathcal{N}$ is formed as in (3.3), and that every $P \epsilon \mathcal{P}$ has an equivalent martingale measure, that is to say, a measure $P^* \epsilon \mathcal{P}^*$ that is mutually absolutely continuous with P. Then $\mathcal{N} = \mathcal{N}^*$.*

The above is trivial, of course, but we state it formally to emphasize the implications of requiring equivalent martingale measures.

Our basic result is that, subject to regularity conditions, at nonrandom times t,

$$A_t^* = Z_{t+}^* = \lim_{r \downarrow t, \, r \in \mathbb{Q}} Z_r^*, \qquad (3.13)$$

where

$$Z_t^* = \text{ess sup}\{E^*(\eta^* \mid \mathcal{F}_t) : P^* \epsilon \mathcal{P}^*\}, \qquad (3.14)$$

and where

$$\eta^* = \exp\{-\int_0^T r_u du\}\eta. \qquad (3.15)$$

In other words, this is s minimax type solution. The results obtained by Cvitanić and Karatzas (1992, 1993) for their problem (constrained hedging) are also of this form.

Formula (3.13) may at first glance appear somewhat disturbing, as it involves the supremum over uncountably many random variables. One runs into the same problem as that mentioned in connection with the definition (3.7). We show in the Appendix, however, that (3.13) can be given a definition with reasonable properties. This can be done for a broad class of models, namely filtered measurable spaces where all the $\mathcal{F}_t$s are countably generated.

This is a weak requirement, since spaces of finite or countable dimensional *càdlàg* processes on $[0, T]$ satisfy this condition (see, *e.g.*, Jacod and Shiryaev (1987), Theorem VI.1.14). In other words, the present theory applies so long as the derivative security depends only on finitely many *càdlàg* underlying securities. The traded ones also need to be continuous; the non traded ones need not be.

For the formal result, we need the following norm:

$$||X|| = \sup\{E^*|X| : P^* \epsilon \mathcal{P}^*\}. \tag{3.16}$$

THEOREM 3.2. *Assume that $(\mathcal{F}_t)$ is right continuous, and that each $\mathcal{F}_t$ is countably generated. Suppose that $\mathcal{N} = \mathcal{N}^e$. Also suppose that the $S_t^{(i)}$ are $\mathcal{N} - a.s.$ continuous and adapted, and that the short rate process r_t is nonnegative and adapted. Also assume that $0 \leq \tau \leq T$, where T is nonrandom and finite, and let η be $\mathcal{F}_\tau$-measurable. Finally suppose that $||\eta^*||$ is finite. Then the process A_t^* is defined by (3.13) up to indistinguishability $\mathcal{N}$-a.s. Assume that this process is locally bounded. Then A_t is a super-replication of η, and it is also the conservative ask price (3.7) for all (non random or stopping) time t, $0 \leq t \leq \tau$.*

The assumption $\mathcal{N}^* = \mathcal{N}^e$ is satisfied if $\mathcal{P}^*$ is the convex hull of its extreme points. Sufficient conditions for a requirement of this type are given in Theorems 15.2, 15.3 and 15.12 (p. 496-498) in Jacod (1979). For example, the first of these results gives the following as a special case (see Section 6). This covers our examples.

PROPOSITION 3.3. *Assume the conditions of Theorem 3.2, except replace $\mathcal{N} = \mathcal{N}^e$ by $\mathcal{N} = \mathcal{N}^*$. Suppose that $(\mathcal{F}_t)$ is the filtration generated by $(S_t^{(1)}, ..., S_t^{(p)}, r_t)$. Also suppose that $\mathcal{N}^* = \widetilde{\mathcal{N}}^*$, where $\widetilde{\mathcal{N}}$ is countably generated. Then $\mathcal{N} = \mathcal{N}^e$.*

EXAMPLES 1, 3 AND 4. The restrictions imposed can clearly be expressed in the form of a countably generated σ-ideal. Hence, the above solutions coincide with those given by Theorem 3.2. ∎

Finally note that what price one should use to value a security will depend on the purpose of the valuation. The basic operating principles are as follows:

(i) Use statistical or other techniques to find a prediction region for the relevant quantities.

(ii) If you are selling a security, use the conservative ask price (plus a fee, probably). If you already have a portfolio, you may wish to charge

$$A(\text{portfolio} + \eta) - A(\text{portfolio}) \tag{3.17}$$

for the security with payoff η. In other words, if η counterbalances some unhedgeable risk in your portfolio, you may wish to sell it fairly cheaply. Incidentally, the theory developed in here is for payoffs at one single time τ, but this can obviously be expanded to payoffs at several different times.

(iii) If you are buying a security η, this is just like selling $-\eta$.

(iv) For book value purposes, one may wish to state both the ask and bid prices of the whole portfolio, but with particular emphasis on the bid value.

4. A case study: Convex European options hedged in a stock and a zero coupon bond. As in Example 2, $p = 2$, $S_t^{(1)} = S_t$, $S_t^{(2)} = \Lambda_t$. $\mathcal{P}$ is assumed to be the set of probability measures where

- S_t and Λ_t are continuous and adapted semi-martingales, with $S_0 = s_0$ and $\Lambda_0 = \lambda_0$ (the actual values);

- $[\log S, \log S]_t$ is absolutely continuous (w.r.t. Lebesgue measure), with derivative σ_t^2;

- (2.9) holds; and

- every $P \in \mathcal{P}$ has an equivalent martingale measure P^*, i.e., $P^* \sim P$, so that S_t^* and Λ_t^* are P^*-martingales.

In view of Girsanov's Theorem, the last condition is tantamount to imposing conditions along the lines of Example 1. By Remark 3.1, $\mathcal{N} = \mathcal{N}^*$ is satisfied, and hence so is $\mathcal{N} = \mathcal{N}^e$.

We consider ask prices of options with payoff $\eta = f(S_T)$ at (nonrandom) time T, where f is convex. This includes calls and puts.

First of all, as far as (2.9) is concerned, Ξ^+ is attained. This is because, if not, one can rescale time $\tilde{r}_t = r_{t(1-\epsilon)}$, $\tilde{S}_t = S_{t(1-\epsilon)}$ for $0 \leq t \leq T$, with $\tilde{r}_t = 0$ for $(1-\epsilon)T \leq t \leq T$ and let $\tilde{S}_t$ run up the full unused volatility on the same interval. Since $\tilde{S}_t^*$ is a martingale, it follows that $\exp(-\int_0^t \tilde{r}_u du)f(\tilde{S}_t)$ is a submartingale from $(1-\epsilon)T$ to T.

It follows that the conservative ask price at time 0 is in this case

$$A_0 = \sup_X E^* X f(S_T^*/X), \qquad (4.1)$$

where $\log S_T^* - \log S_0^*$ is normal $N(-\frac{1}{2}\Xi^+, \Xi^+)$, and the supremum is over all random variables X for which

$$\exp(-R^+) \leq X \leq \exp(-R^-) \qquad (4.2)$$

and satisfying

$$E^* X = \Lambda_0. \qquad (4.3)$$

This is provided we can show that $\beta_T = X^{-1}$ gives a β_t and a Λ_t with the desired properties.

Since $x \to xf(s/x)$ is convex (for $x, s > 0$), the functional $X \to E^* X f(S_T^*/X)$ is also convex, and so the supremum in (4.1) is attained for an X which only takes the values $\exp(-R^-)$ and $\exp(-R^+)$.

Let $B^\pm = \{\omega : X(\omega) = \exp(-R^\pm)\}$ for $\pm = +$ or $-$. It follows from the above that we wish to maximize

$$E^* \exp(-R^+)f(\exp(R^+)S_T^*)I_{B+} + E^* \exp(-R^-)f(\exp(R^-)S_T^*)I_{B-} \qquad (4.4)$$

subject to

$$\exp(-R^+)P(B^+) + \exp(-R^-)P(B^-) = \Lambda_0. \qquad (4.5)$$

Now let

$$B^+ = \{\omega : S_T^* \geq \tilde{K}\}, \qquad (4.6)$$

where $\widetilde{K}$ is determined by (4.5), and let $\widetilde{B}^{\pm}$ be any other pair of sets satisfying (4.5). Since f is convex, $s \to \exp(-R^-)f(\exp(R^-)s) - \exp(-R^+)f(\exp(R^+)s)$ is a nondecreasing function. Hence,

$$\exp(-R^+)f(\exp(R^+)S_T^*)(I_{B^+} - I_{\widetilde{B}^+}) + \exp(-R^-)f(\exp(R^-)S_T^*)(I_{B^-} - I_{\widetilde{B}^-})$$

$$\geq (\exp(-R^-)f(\exp(R^-)\widetilde{K}) - \exp(-R^+)f(\exp(R^+)\widetilde{K}))(I_{B^+} - I_{\widetilde{B}^+}),$$

and so the choice (4.6) maximizes (4.4) subject to (4.5). In the notation (2.1)-(2.2), $\widetilde{K}$ is found from (4.5) by

$$\Phi(d_2(S_0, \widetilde{K}, 0, \Xi^+)) = \frac{\exp(-R^-) - \Lambda_0}{\exp(-R^-) - \exp(-R^+)}, \tag{4.7}$$

Note that this equation always has a (unique) solution because $\exp(-R^-) \geq \Lambda_0 \geq \exp(-R^+)$.

To see that the maximum is indeed the ask price, we now construct a $P^* \in \mathcal{P}^*$ as follows. Fix t_0 so that $0 < t_0 < T$. On $[0, t_0)$, we let $dS_t^* = \sigma S_t^* dW_t$, with $\sigma^2 = \Xi^+/t_0$, and we set $r_t = R^-/t_0$. From t_0 to T, S_t^* is constant $(= S_{t_0-}^*)$. If $S_{t_0}^* \in B^-$, then $r_t = 0$, otherwise $r_t = (R^+ - R^-)/(T - t_0)$. This gives us an allowable β_t, and Λ_t^* is obviously

$$\Lambda_t^* = \exp(-R^+)P(B^+ \mid S_t) + \exp(-R^-)P(B^- \mid S_t). \tag{4.8}$$

We have thus proved the following

PROPOSITION 4.1. *Subject to the assumptions at the beginning of Section 4, the conservative ask price at time zero for payoff $f(S_T)$ at time T is*

$$A_0 = E^* \exp(-R^+)f(\exp(R^+)S)I_{\{S \geq \widetilde{K}\}} + E^* \exp(-R^-)f(\exp(R^-)S)I_{\{S < \widetilde{K}\}}, \tag{4.9}$$

where $\widetilde{K}$ is given by (4.7), and where $\log S$ is normal $N(\log S_0 - \frac{1}{2}\Xi^+, \Xi^+)$.

Alternatively, one can obviously write

$$A_0 = C(S_0, R^+, \Xi^+, f(s)I(s \geq \widetilde{K}\exp(R^+)) + C(S_0, R^-, \Xi^+, f(s)I(s < \widetilde{K}\exp(R^+)), \tag{4.10}$$

where $C(S_0, R, \Xi, h(s))$ is the Black-Scholes-Merton price for payoff $h(S_T)$ at time T.

Note that if one wishes to disregard the bond Λ_t (as in Example 1), one just maximizes (4.4) without (4.5).

And now for the European call.

EXAMPLE 2 (continued).   If K is the strike price, $f(s) = (s - K)^+$. The solution given in (2.10)–(2.11) follows directly by setting $r_0 = \log(K/\widetilde{K})$. ∎

5. Defining self-financing strategies.

In essence, H_t being self-financing means that we can represent H_t^* by

$$H_t^* = H_0^* + \sum_{i=1}^{P} \int_0^t \theta_s^{(i)} dS_s^{(i)*}. \qquad (5.1)$$

This is in view of numeraire invariance (see, e.g., Section 6.B of Duffie (1996)).

There are two problems associated with making this definition operational.

The first issue is what exactly we mean by the stochastic integral. This object is usually only defined with reference to a probability distribution, and here we have several, none of which dominates (in the sense of absolute continuity) all the other ones (except in special cases). For this reason, we shall briefly define the integral in our situation.

DEFINITION. $L_{\text{loc}}^2(\mathcal{P}^*)$ is the set of p-dimensional predictable processes $\theta_t^{(1)}, \ldots, \theta_t^{(p)}$, so that $\int_0^t \theta_u^{(i)2} d\langle S^{(i)*}, S^{(i)*} \rangle_u$ is locally integrable $\mathcal{N}^*$-a.s. The stochastic integral (5.1) is then defined by the process in Theorems I.4.31 and I.4.40 (p. 46–48) in Jacod and Shiryaev (1987). The integral can be taken to be independent of $P^* \in \mathcal{P}^*$ because, in Jacod and Shiryaev's notation, if one lets B be the set of ω for which every sequence has a subsequence for which $H^n \cdot X_t(\omega)$ has a limit (in the supremum norm), then the conclusions in I.4.31 (iii) and I.4.40 (iii') are that $P^*(B^C) = 0$ for all $P^* \in \mathcal{P}^*$, and so $B^C \in \mathcal{N}^*$. ∎

In other words, we replace convergence in probability with a convergence one could denote by $\overset{\mathcal{N}^*}{\to}$ to distinguish it from convergence $\mathcal{N}^*$-a.s.

The other issue concerning (5.1) is that one must be able to rule out doubling strategies. The two most popular ways of doing that are to insist either that H_t^* be in an L^2-space, or that it is bounded below (Duffie (1996), Section 6.C; see also Harrison and Kreps (1979), Dybvig and Huang (1988), and Karatzas (1996)). We shall here go with a criterion that encompasses both.

DEFINITION. Suppose $\mathcal{N}^* = \mathcal{N}$. A process H_t, $0 \leq t \leq T$, is *self-financing* with respect to $S_t^{(1)}, \ldots, S_t^{(p)}$ if the process is uniquely defined up to a set in $\mathcal{N}$, if H_t^* satisfies (5.1) $\mathcal{N}$-a.s., and if $\{H_\lambda^{*-}, 0 \leq \lambda \leq T, \lambda$ stopping time$\}$ is uniformly integrable under all $P^* \in \mathcal{P}^*$. $\mathcal{H}$ is the set of such (H_t)s; $\mathcal{H}^*$ is the set of corresponding (H_t^*)s. ∎

We need $\mathcal{N}^* = \mathcal{N}$ since we have involved $\mathcal{P}^*$ and since we want the hedge to work $\mathcal{N}$-a.s. There is no particular loss in this, however, since Theorem 3.2 depends on $\mathcal{N}^* = \mathcal{N}$ anyway.

The reason for seeking to avoid the requirement that H_t^* be bounded below is that, to the extent possible, the same theory should apply equally to bid and ask prices. Since the bid price is normally given by (3.8), securities that are unbounded below will be a common phenomenon. For example, $B((S - K)^+) = -A(-(S - K)^+)$, and $-(S - K)^+$ is unbounded below.

It should be emphasized that our definition does, indeed, preclude doubling type strategies:

PROPOSITION 5.1. *Let $\mathcal{N} = \mathcal{N}^*$. Suppose that H_t^* is self financing in the sense given above. Then, if there are stopping times λ and μ, $0 \leq \lambda \leq \mu \leq T$, so that $H_\mu^* \geq H_\lambda^*$, $\mathcal{N}^*$-a.s., then $H_\mu^* = H_\lambda^*$, $\mathcal{N}^*$-a.s.*

Proof of Proposition 5.1. H_t^* is continuous by (5.1), and is therefore a locally bounded martingale. Let $\lambda_n \uparrow \lambda$ in such a way that each $H_{t \wedge \lambda_n}^*$ is a martingale. Now fix $P^* \in \mathcal{P}^*$. By optional stopping, $E^*(H_{\lambda_n}^* \mid \mathcal{F}_\mu) = H_{\mu \wedge \lambda_n}^*$. By continuity, the r.h.s. goes to H_μ^*. By Fatou's Lemma and the uniform integrability condition, H_t^* is a supermartingale for all $P^* \in \mathcal{P}^*$. This takes care of the left hand side. The result follows. ∎

Suppose that V_t^* is a *càdlàg* process which is a semimartingale under all $P^* \epsilon \mathcal{P}^*$. We now show how to determine hedge ratios for V_t^*.

Set

$$C_t = \sum_{i=1}^{p} \left[S^{(i)^*}, S^{(i)^*} \right]_t.$$

(5.2)

We use the "naive" definition of quadratic variation and covariation, limits of sums of squared differences, because this does not refer to probabilities. Theorem I.4.47 (p. 52) in Jacod and Shiryaev (1987) assures that this limit exists $\mathcal{N}^*$-a.s.

We similarly define $\underline{\underline{G}}_t$ to be the matrix with elements $d[S^{(i)^*}, S^{(j)^*}]_t/dC_t$, and $\underline{f}_t$ the vector with elements $d[V^*, S^{(i)^*}]_t/dC_t$. Clearly, $\underline{f}_t = \underline{\underline{G}}_t \underline{\theta}_t$ has at least one predictable solution. Note that this is done for every ω except on a set $N \in \mathcal{N}^*$: the set where the above quantities are not defined have P^*-probability 0 for all $P^* \in \mathcal{P}^*$, hence $N \in \mathcal{N}^*$. Here is the crux of the argument in the paper:

PROPOSITION 5.2. *Let $(\Omega, \mathcal{F}, \mathcal{F}_t)$ be a filtered measurable space with $(\mathcal{F}_t)$ right continuous. Let $\mathcal{N}$ be a σ-ideal in $\mathcal{F}$, let $\mathcal{P}^*$ $\mathcal{N}^*$, and $\mathcal{N}^e$ be defined as in Section 3. Suppose that $\mathcal{N}^* = \mathcal{N}^e$. Let $S_t^{(i)^*}$, $i = 1, \ldots, p$ be continuous martingales for all $P^* \in \mathcal{P}^*$. Suppose V_t^* is a locally bounded càdlàg process ($\mathcal{N}^*$-a.s.) which is a super-martingale for all $P^* \in \mathcal{P}^*$. Let $\theta_t^{(i)}$ be constructed as above. Then $(\theta_t^{(i)}, \ldots, \theta_t^{(p)})$ is in $L^2_{\text{loc}}(\mathcal{P}^*)$. Furthermore, let (H_t^*) be given by (5.1), and set $D_t^* = V_t^* - H_t^* - V_0^*$. Then, (D_t^*) is a nonincreasing process, $\mathcal{N}^*$-a.s.*

The proof is given in the next section.

6. Proofs for Sections 3 and 5.

Since Proposition 5.2 logically preceeds Theorem 3.2, we take its proof before that of the theorem

Proof of Proposition 5.2. Define $\theta_t^{(i)}$ as indicated. To see that $\underline{\theta} \epsilon L^2_{\text{loc}}(\mathcal{P}^*)$, fix $P^* \epsilon \mathcal{P}^*$, and do a Doob-Meyer decomposition of V_t^* under P^* (valid by Theorem 8.22 (p. 83) in Elliot (1982)). Call the resulting local martingale part M_t^*, and note that $\underline{\theta}$ would be the same if defined with M_t^* rather than V_t^*. Since the $S_t^{(i)}$ are continuous, it follows from Theorem III.4.11 (p. 169) in Jacod and Shiryaev that $\underline{\theta} = (\theta^{(1)}, \ldots, \theta^{(p)}) \in L^2_{\text{loc}}(P^*)$. Since P^* was arbitrary in $\mathcal{P}^*$, $\underline{\theta} \in L^2_{\text{loc}}(\mathcal{P}^*) = \bigcap_{P^* \epsilon \mathcal{P}^*} L^2_{\text{loc}}(P^*)$.

By the same result, for each i,

$$[D^*, S^{(i)*}]_t = 0 \quad \text{for all t}, \; \mathcal{N}^* - a.s. \tag{6.1}$$

Suppose that P^e is an extremal element of $\mathcal{P}^*$. By Proposition 11.14 (p. 345) in Jacod (1979), P^e is also extremal in the set $M(\{S^{(1)*}, ..., S^{(p)*}\})$ (in Jacod's notation). Since H_t^* is a locally bounded martingale under P^e, one can let $D_t^* = M_t^* + R_t^*$ be a Doob-Meyer decomposition of D_t^* under P^e (valid for the same reason as that of V_t^* under P^*), where R_t^* is a predictable nonincreasing process, and M_t^* is locally a bounded martingale. (6.1) now implies that $[M^*, S^{(i)*}]_t$ is zero for all t and i, $\mathcal{N}^*$-a.s. This, along with the extremality of P^e in $M(\{S^{(1)*}, ..., S^{(p)*}\})$, however, implies that M_t^* must be constant under P^e. (Refer to Jacod (1979), Theorem 11.2 (p. 338) and the remarks following Definition 4.1 (p. 113); alternatively, see Chapter V.4 of Revuz and Yor (1994)).

In other words, D_t^* must be a nonincreasing process under P^e. Since this is true, however, for all extremal elements P^e of $\mathcal{P}^*$, the assumption $\mathcal{N}^* = \mathcal{N}^e$ implies that this property must be true $\mathcal{N}^*$-a.s. ∎

Proof of Theorem 3.2. Define Z_t^* and A_t^* for nonrandom t, $0 \le t \le T$, by (3.13)-(3.14). Z_t^* exists by Proposition A.1, and is a supermartingale (for all $P^* \in \mathcal{P}^*$) by Corollary A.2, both in the Appendix. The proof of Proposition 1.3.14 (p. 16) of Karatzas and Shreve (1991) shows that A_t^* is well defined and *càdlàg* ($\mathcal{N}$-a.s.), and that this process is also a supermartingale for all $P^* \in \mathcal{P}^*$.

By the assumptions of Theorem 3.2, A_t^* satisfies the assumptions of Proposition 5.2. A_t^{*-} is also uniformly integrable for all $P^* \in \mathcal{P}^*$, and hence so is H_t^{*-} (since (D_t) is nonincreasing). It follows that (A_t) is a super-replication of η.

On the other hand, if (V_t) is a super-replication of η, then, for a given stopping time t, $0 \le t \le \tau$,

$$V_t^* \ge E^*(\eta^* \mid \mathcal{F}_t)$$

for all $P^* \epsilon \mathcal{P}^*$. Hence, $V_t^* \ge Z_t^*$ $\mathcal{N}$-a.s. Since (V_u^*) is *càdlàg*, $V_t^* \ge A_t^*$, $\mathcal{N}$-a.s. ∎

Proof of Proposition 3.3. Redefine $\mathcal{F}_t$ as the smallest σ-field containing $\widetilde{\mathcal{N}}$ and the original $\mathcal{F}_t$. This does not change the problem. Set $\mathcal{G} = \sigma(\widetilde{\mathcal{N}})$.

We use Theorem 15.2c (p. 496) in Jacod (1979), with (in Jacod's notation) $S = \bigcap_{X \in \mathcal{X}} Ss^1(X)$, where $\mathcal{X}$ consists of our processes $S_t^{(1)}, ..., S_t^{(p)}, -S_t^{(1)}, ..., -S_t^{(p)}, \beta_t^{-1}$. It then follows that, if $\mathcal{N}_S^*$ and $\mathcal{N}_S^\varepsilon$ are found by (3.11)-(3.12) for (Jacod's) S, then these two σ-ideals are equal. However, $S = \widetilde{M}(\{S^{(1)}, ..., S^{(p)}\})$ (again in Jacod's notation, see p. 345 of that work). This is because r cannot be negative. The result follows. ∎

7. Further questions.

The above has provided a general methodology for converting prediction regions into prices and hedging strategies. Many questions of implementation remain, however, in the realm of future research.

An important issue is what form one should choose for the prediction region, and what instruments to hedge in. Some possibilities have been provided by the examples, but the imagination is really the limit here. One can include market traded options and other derivatives among the S_t considered. One can bound (traded or untraded) securities themselves, or one can consider quadratic covariations between the various instruments (including the short rate). Example 2 is an instance of mixing those two approaches.

Computation of the ask and bid prices and of the hedge ratios are also going to be a major issue. The more exotic the derivative, and the more varied the traded securities one wishes to hedge in, the more likely it is that numerical techniques will be required. Examples 1 and 2 show that there are cases where analytic solutions exist, but this will probably be a minority of cases.

On the more statistical front, it is also worth pursuing a stronger result than our Theorem 3.2, as follows. If one has a prediction interval of the form (2.5), and one uses conservative delta hedging as described, the probability of failure is the same as the probability of the prediction interval not covering the realized values. Is remains unsolved whether this is the case in the genral setting of Theorem 3.2.

Dynamically adjusted contingency reserves is worth further investigation. If one has a bound of type (2.5) or (2.9), but then it turns out half way that only a fraction of the provided for

volatility has been used, one may wish to harvest some profit right away. Similarly, if it looks like one is going to exceed the budgeted volatility, one may wish to take a loss immediately so as to increase the bound. It is pretty straightforward to extend the theory in this paper to let $\mathcal{P}$ be time varying; the main question is how to formulate good strategies for the harvesting.

Finally, there are all the issues arising from non-statistical incompleteness. If the prices of underlying securities can jump, is there still some way to use this procedure without getting impossibly big bid-ask spreads? Also. if one lives with the fiction of continuous prices, there is still the question of interpolating to assess the accrued volatility (or other quadratic variations). Some references are given at the beginning of Section 2, but there would seem to be a need for further exploration.

A more subtle form of incompleteness arises if one lets the underlying securities prices be continuous, but one has a non-continuous model for the volatility (an α-stable process, say) or some other untraded security. One can use the results in this paper if one finds a prediction interval on the basis of the model. A tantalizing question, however, is whether one can hedge (in market traded derivatives) the continuous part and use intervals for for the discontinuous part. One can envisage a number of scenarios where some sort of partial hedging and partial interval use would lower bid-ask spreads without reducing the safety margin.

There is a lot of unanswered questions.

APPENDIX

We here deal with the measure theoretic details having to do with the essential supremum of conditional expectations, in the (apparent) absence of a dominating probability distribution. If such a distribution were to exist, one could, for example, use the result in Proposition VI-1-1 (p. 121) of Neveu (1975). In our case, we shall define the relevant objects in analogy to Neveu, but we need new proofs motivating the definitions.

DEFINITION. Suppose that $\mathcal{Q}$ is a collection of probability distributions on $\mathcal{F}$, and let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Let X be a random variable satisfying

$$\sup_{P \epsilon \mathcal{Q}} E|X| < \infty. \tag{A.1}$$

Then ess $\sup\limits_{P \epsilon \mathcal{Q}} E(X \mid \mathcal{G})$ is the random variable Z for which

(i) Z is $\mathcal{G}$-measurable;

(ii) $Z \geq E(X \mid \mathcal{G})$, P-a.s., for all $P \epsilon \mathcal{Q}$; and

(iii) if $\widetilde{Z}$ also satisfies (i) and (ii), then $\widetilde{Z} \geq Z$. P-a.s., for all $P \epsilon \mathcal{Q}$. ∎

Provided this quantity exists, it is, obviously, unique up to joint null sets of $\mathcal{Q}$. For existence and meaningfulness, we assume in the following that $\mathcal{G}$ is countably generated. There is, therefore, a nested sequence of finite σ-fields $\mathcal{G}_n$ so that

$$\mathcal{G} = \bigvee_{n=1}^{\infty} \mathcal{G}_n. \tag{A.2}$$

PROPOSITION A.1. Let X satisfy (A.1) and suppose that $\mathcal{G}$ is a countably generated σ-field. Let $\mathcal{G}_n$ be finite nested σ-fields satisfying (A.2). Suppose that $\mathcal{Q}$ is closed in the total variation distance. Set

$$Z_n(\omega) = \sup_{P \epsilon \mathcal{Q}} E(X \mid \mathcal{G}_n)(\omega). \tag{A.3}$$

In other words, because of the closedness of $\mathcal{Q}$, if $C_{n,k}$ are the atoms of $\mathcal{G}_n$, then

$$Z_n(\omega) = E_{n,k}(X \mid C_{n,k}) \text{ for } \omega \epsilon C_{n,k} \tag{A.4}$$

where $P_{n,k}$ is an element of $\mathcal{Q}$ which maximizes the right hand side of (A.4). Then,

$$\limsup_{n \to \infty} Z_n = \operatorname*{ess\,sup}_{P \in \mathcal{Q}} E(X \mid \mathcal{G}) \quad P - a.s., \text{ for all } P \epsilon \mathcal{Q}. \tag{A.5}$$

Furthermore, there is at least one increasing sequence $\mathcal{G}_n$ so that, as $n \to \infty$,

$$Z_n \to \operatorname*{ess\,sup}_{P \in \mathcal{Q}} E(X \mid \mathcal{G}) \quad P - a.s., \text{ for all } P \epsilon \mathcal{Q}. \tag{A.6}$$

Proof of Proposition A.1. Let Z^* and Z_* be, respectively, the limits superior and inferior of the sequence Z_n. Also, let Z be the essential supremum itself.

First of all, by the Martingale Convergence Theorem (see, *e.g.*, Billingsley (1995)), Z_* satisfies criteria (i) and (ii) in the definition above, whence $Z_* \geq Z$, P-a.s., for all $P \epsilon \mathcal{Q}$.

To get the opposite inequality, let $P \epsilon \mathcal{Q}$. Let $\mathcal{G}_n$ be generated by the $\mathcal{G}_n$ in (A.2) along with the sets $\{a_i^{(n)} \leq Z < a_{i+1}^{(n)}\}$, where the set $\alpha^{(n)} = \{a_0^{(n)}, a_1^{(n)}, \dots\}$ are finite and nested, starting with $\{-\infty, \infty\}$, and becoming dense uniformly on any compact in $\mathbb{R}$ as $n \to \infty$. Let $\omega \epsilon Z^{-1}(K)$, where K is compact, and let E_n be the set $C_{n,k}$ where ω is a member.

By definition of Z. $Z \geq E_{n,k}(X \mid \mathcal{G})$ on $C_{n,k}$, $P_{n,k}$-a.s. Hence, for some sequence i_n

$$Z_n(\omega) \leq \sup_{\omega' \epsilon E_n} Z(\omega')$$
$$\leq Z(\omega) + a_{i_n+1}^{(n)} - a_{i_n}^{(n)}$$
$$\to Z(\omega) \quad \text{as } n \to \infty. \tag{A.7}$$

It follows that $Z^* \leq Z$ on $Z^{-1}(K)$, and hence P-a.s. The result follows. ∎

It is easy to see that (A.6) is not satisfied with an arbitrary sequence $\mathcal{G}_n$.

To get at the supermartingale property needed in the proof of Theorem 3.2, we can further characterize $\operatorname{ess\,sup}_{P \epsilon \mathcal{Q}} E(X \mid \mathcal{G})$ as follows. Keep in mind that if $Q_1, Q_2, \dots$ is a countable set of probabilities, there is a probability Q dominating (in the sense of absolute continuity) all the Q_is.

For example, take

$$Q = \sum_{m=1}^{\infty} \frac{1}{2^m} Q_m, \tag{A.8}$$

COROLLARY A.2. *Assume the conditions of Proposition A.1. For all $P \epsilon \mathcal{Q}$, there is a countable collection $\mathcal{D}$ of elements in $\mathcal{Q}$, with $P \epsilon \mathcal{D}$, so that the following is satisfied. Let Q dominate $\mathcal{D}$. For all $R \epsilon \mathcal{D}$, define $\widetilde{E}_R(X \mid \mathcal{G})$ Q-a.s. by setting it to $-\infty$ on the set $\Omega - C_R$ and to $E_R(X \mid \mathcal{G})$ Q-a.s. on C_R, where $Q << R$ on C_R and $R(\Omega - C_R) = 0$. Then*

$$\operatorname*{ess\,sup}_{R \epsilon \mathcal{Q}} E_R(X \mid \mathcal{G}) = \sup_{R \epsilon \mathcal{D}} \widetilde{E}_R(X \mid \mathcal{G}) \quad Q - a.s. \tag{A.9}$$

Note that the right hand side of (A.9) is measurable since $\mathcal{D}$ is countable.

Proof of Corollary A.2. Set $\mathcal{D} = \{P\} \bigcup \{P_{n,k}, \text{ all } n, k\}$, where the $P_{n,k}$ are the ones occurring in the second part of the proof of Proposition A.1. Define Y to be the right hand side of (A.9). The result then follows by the same argument as in the proof of Proposition A.1. ∎

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