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Axisymmetric Compression of a Mohr-Coulomb Medium with Arbitrary Dilatancy, Including Free-Field Yielding

David M. Kendall
Gary L. Wintergerst
Danny N. Burgess
BDM International, Inc.
1501 BDM Way
McLean, VA 22102-3204

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A closed-form analytical solution is developed for hole closure and stresses and strains in a Mohr-Coulomb medium surrounding a circular hole and loaded axisymmetrically in plane strain. This solution extends multiple-plastic-zone solutions developed by others to allow arbitrary dilatancy. It also extends the solution to include cases where particular combinations of friction angle, elastic properties, and free-field pressure cause the free field to yield before significant deformation of the opening occurs. Step-by-step procedures for applying the solution and numerical examples are presented.

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INTRODUCTION

An analytical solution is presented for the stresses and strains around a circular hole in a Mohr-Coulomb material under conditions of plane strain when subjected to a hydrostatic free-field pressure. The solution is a direct extension of the solution of Florence and Schwer (Florence, 1978) and (Florence, undated) to include arbitrary dilatancy and the case where out-of-plane yielding occurs in the free field.

1.1 BACKGROUND.

The basic approach to this problem was defined by (Newmark, 1969) in response to the need for a method of designing underground structures to resist the effects of ground shock due to nuclear weapons. In the development of his solution, Newmark assumed an elastic-perfectly-plastic medium with plastic volume constancy in a plane strain configuration. Newmark's work was expanded by Hendron and Aiyer (Hendron, 1972), who provided the solution for a dilatant material using an associated flow rule. Both the Newmark and the Hendron and Aiyer solutions were based on the assumption that the out-of-plane stress was the intermediate principal stress. There are, however, certain combinations of applied load and material properties where the out-of-plane stress is not the intermediate principal stress.

Florence and Schwer (Florence, 1978) and (Florence, undated) generalized the Hendron and Aiyer solution by eliminating the requirement that the out-of-plane stress be the intermediate principal stress. Their solution is considerably more complex, since it requires one to deal with multiple yield conditions in a single material. In fact, the solution is divided into two cases depending on the material properties and loads.

Detournay and others (Detournay, 1983) presented a more general solution which allowed arbitrary dilatancy through the use of a nonassociative flow rule, but this solution maintained the requirement that the out-of-plane stress be the intermediate principal stress. A similar solution was developed independently by (Merkle, 1982), although we have not found it in published form.

Wintergerst and others (Wintergerst, 1991) generalized both cases of the Florence and Schwer solution to allow arbitrary dilatancy through the use of a nonassociative flow rule, thus providing a solution with both arbitrary dilatancy and no limitation on the intermediacy of the out-of-plane stress.

The analytical solutions developed by Florence and Schwer and by Wintergerst, et al for the plane strain, axisymmetric compression of a circular opening in a Mohr-Coulomb material make the assumption that out-of-plane yielding of the free-field does not occur. These solutions assume that an elastic zone always exists outside the plastic annulus and therefore do not apply to the free-field yielding case (i.e., an infinite outer radius of the plastic zone). However, as will be shown herein, certain combinations of friction angle, elastic properties, and free-field pressure can cause the free-field to yield before significant deformation of the opening has occurred, rendering these solutions incomplete. Yielding of the free-field does not necessarily translate into excessive closure of the tunnel.

1.2 SCOPE AND ORGANIZATION OF THE REPORT.

The earlier solutions, including the provision for arbitrary dilatancy, are extended in this report to include yielding of the free field.

Section 2 contains a complete problem description, a discussion of the solution approach, the definitions of the Florence and Schwer solution cases, and a derivation of the conditions under which the free field will yield. In developing the solution it is convenient to separate the problem into two general cases based on how the concentric plastic zone(s) develops around the opening. These two cases were identified by Florence and Schwer as Case I and Case II. The difference between these two cases is that the initial plastic zone that develops at the edge of the opening is governed by two different yield conditions, depending on the specific combinations of material properties and loading. In Case I, the out-of-plane stress within the initial plastic zone is the intermediate principal stress. In Case II, however, the out-of-plane stress is the maximum (least compressive) principal stress. Section 2 also contains a derivation of the material and load parameters which demarcate Cases I and II. Since additional concentric plastic zones may form outside of this initial zone, these two cases are each subdivided into two subcases.

For Case I, there are conditions that cause only one initial plastic zone to form and no free-field yielding is possible. These conditions constitute Case Ia solutions. Cases where additional plastic zones develop with continued loading will constitute Case Ib solutions. The free field may or may not yield depending on the material properties. The Case I solutions are described in Section 3.

For Case II the order of the formation of the plastic zones differentiates the two solution subcases. In Case IIa, the initial plastic zone is followed by the formation of a second plastic zone, which also begins at the edge of the hole. Under certain conditions further loading will cause the free field to yield. In Case IIb the free field yields after formation of the initial plastic zone. Further loading causes a third plastic zone to form. This third plastic zone has the same stress conditions as the second plastic zone in Case IIa, and it also begins at the edge of the hole. The Case II solutions are described in Section 4.

Tables summarizing the calculational procedures applicable to the various solution cases and radial zones are presented in Section 5. Numerical examples of the extended solutions to the tunnel problem are presented in Section 6. These examples include comparisons with the results of finite element analyses. Finally, references are listed in Section 7.

Frequently used symbols are defined in Table 1-1 which is adapted from (Wintergerst, 1991).

Table 1-1. Definitions of frequently used symbols.

a	- interior radius of the opening
E	- Young's modulus for medium
f	- yield surface definition
g	- plastic potential surface corresponding to yield surface f
G	- shear modulus for medium
M	- arbitrary dilatancy parameter
N	- friction parameter
p_a	- internal pressure acting on hole boundary
p_a^*	- negative of the radial stress at the elastic-plastic boundary
p_b	- far-field pressure (compressive stress) at large radius
$\overline{p}_b$	- far-field pressure at initial yield
$\tilde{p}_b$	- far-field pressure at which $R = \tilde{R}$
p_b'	- far-field pressure when the inner plastic zone begins to form under Case II yield conditions
$\hat{p}_b$	- far-field yield pressure
$\hat{p}_b^*$	- pressure value used to discriminate Cases IIa and IIb
p_b''	- far-field pressure at which the third plastic zone forms in Case IIb
r	- arbitrary radius to any point in the medium
R	- radius to the elastic-plastic boundary
$\overline{R}$	- maximum radius at which the radial and out-of-plane stresses are equal
$\tilde{R}$	- minimum radius at which the radial and out-of-plane stresses are equal
R'	- radius to the elastic-plastic boundary when $p_b = p_b'$
σ_u	- unconfined compressive strength of medium
σ_r	- stress in radial direction
σ_θ	- stress in circumferential direction
σ_z	- stress in out-of-plane direction
ε_r	- strain in radial direction
ε_θ	- strain in circumferential direction
ε_z	- strain in out-of-plane direction
Note:	Strains are further distinguished by the superscripts (e) to denote elastic component and (p) to denote plastic component, and overdots to denote derivatives with respect to time
ν	- Poisson's ratio for medium ($0 < \nu < 0.5$)
ϕ	- friction angle for medium
λ	- flow constant of proportionality
ω	- dilatancy angle

SECTION 2
PROBLEM DESCRIPTION AND APPROACH

The purpose of this section is to furnish a complete description of the problem and the approach taken to solve it. In addition, we present derivations which: (1) define the two solution cases developed by Florence and Schwer; and (2) define the conditions under which the free field will yield. These derivations are included in this section since they are a part of the free-field yielding solutions for both Case I and Case II presented in Sections 3 and 4, respectively. The problem geometry, loading, material law, and solution restrictions are described in Section 2.1. The overall solution approach is discussed in Section 2.2. The Florence and Schwer Case I and Case II conditions are developed in Section 2.3. Finally, the conditions for free-field yielding are derived in Section 2.4.

2.1 PROBLEM DESCRIPTION.

The problem geometry consists of a circular hole with radius a in an infinite medium under conditions of plane strain. The loading of the material consists of: (1) a compressive pressure, p_a, applied at the inside edge of the hole (r equals a); and (2) a compressive pressure, p_b, applied at r equal to infinity (defined as the free field). Tensile values of p_a and p_b are precluded. The loading is assumed to be monotonic, with p_a equal to p_b until p_a reaches its final value. Subsequently, p_a is held constant while p_b is increased to its final value. The internal pressure, p_a, can be considered to represent the resistance provided by a structural liner. The final value of p_a would then represent yielding of the liner. The loading is assumed to be applied sufficiently slowly that inertial effects can be ignored. Finally, it is assumed that during the initial loading phase with p_a equal to p_b, the material response is linear.

The material is assumed to be governed by a Mohr-Coulomb elastic-plastic constitutive law with a failure surface defined by

$$f = \sigma_3 - N\sigma_1 + \sigma_u \tag{2.1}$$

where σ_1 is the maximum (least compressive) principal stress and σ_3 is the minimum (most compressive) principal stress. Throughout this report we adopt the sign convention that stresses are positive in tension, while the applied pressures, p_a

and p_b, are positive in compression. The material parameter σ_u represents the unconfined compressive strength of the material. The material parameter N is related to the friction angle, ϕ, by

$$N = \frac{1+\sin(\phi)}{1-\sin(\phi)} \tag{2.2}$$

The previous solutions developed for this problem are not valid for N equal to one, that is, it has been assumed that the friction angle was greater than zero. We make the same assumption for the solution presented herein.

Plastic flow of the material is assumed to be governed by the nonassociative flow rule

$$\dot{\varepsilon}_{ij}^{(p)} = \lambda \frac{\partial g}{\partial \sigma_{ij}} \tag{2.3}$$

where $\dot{\varepsilon}_{ij}^{(p)}$ represents the components of the plastic strain rate tensor, λ, is the plastic multiplier (a scalar), g represents the plastic potential surface, and σ_{ij} represents the components of the stress tensor. The form of the plastic potential surface is taken to be

$$g = \sigma_3 - M\sigma_1 + \kappa \tag{2.4}$$

where the material parameter M is related to the dilatation angle ω by

$$M = \frac{1+\sin(\omega)}{1-\sin(\omega)} \tag{2.5}$$

and κ is an arbitrary constant. The constant κ is arbitrary since only the derivatives of g with respect to the stresses are used in the flow rule. Note that if ω is equal to ϕ, then M is equal to N and the flow rule becomes associative. Furthermore, the value of M is restricted by

$$1 \le M \le N \tag{2.6a}$$

or

$$0 \le \omega \le \phi \tag{2.6b}$$

A dilatancy angle of zero (M = 1) indicates a non-dilatant material.

It should be noted that for the geometry and loading of this problem, the principal stresses coincide with the normal stresses σ_r, σ_θ, and σ_z, where the subscripts r, θ, and z represent the radial, circumferential, and out-of-plane components, respectively. The shear stress components, $\sigma_{r\theta}$, $\sigma_{\theta z}$, and σ_{rz}, are all zero.

Elastic material response is assumed to be isotropic and therefore governed by two independent elastic constants. Throughout this report we use Poisson's ratio, ν, Young's modulus, E, and shear modulus G, depending on which is most convenient. The relationship between G, E, and ν is given by

$$G = \frac{E}{2(1+\nu)} \tag{2.7}$$

We further restrict the value of Poisson's ratio to be greater than zero and less than one-half.

2.2 APPROACH.

The general solution strategy used by Florence and Schwer for nonassociative flow (Florence, 1978) and subsequently adopted by (Wintergerst, 1991) for arbitrary dilatancy is as follows. The medium is broken up into regions (zones) which depend on the stress state (both principal stress ordering and magnitudes) and hence on the governing yield condition (if any). The number of regions, and the stress state in each region is dependent on both the loading and material properties. The stress states include both those governed by elastic material response and those governed by plastic material response.

Florence and Schwer used numerical analysis to guide their postulations about the existence and conditions of the various plastic zones. After deriving their analytical solutions, they performed finite element analyses and obtained excellent agreement between analytical and numerical results. We have adopted the same solution philosophy in extending the analytical solution to include yielding of the free field.

Within each region, equilibrium conditions, strain-displacement compatibility, and the constitutive law (including yield conditions and flow rules, where applicable) were used to derive an ordinary differential equation (in terms of a primary unknown) which governs the response in that region. Depending on the region, the primary solution variable was taken to be either the radial displacement or the radial stress. The choice was based on ease of formulation and solution. The other unknowns (i.e., the remaining stress and strain components, or the displacement) are written in terms of the primary unknown.

A general solution for the governing differential equation in each region is then determined. These solutions typically contain unknown constants, which are determined either by applying continuity conditions at the region boundaries or by using applied loading conditions at the edge of the hole or at r equal to infinity.

2.3 CASE I AND CASE II DEFINITIONS.

In this section, the Florence and Schwer derivations of the Case I and Case II conditions (Florence, 1978) and (Florence, undated) are described.

The derivation of the elastic solution for a hole in an infinite medium under conditions of plane strain is well documented, so only the results are presented here.

$$\sigma_r = -p_b + (p_b - p_a)\frac{a^2}{r^2} \tag{2.8}$$

$$\sigma_\theta = -p_b - (p_b - p_a)\frac{a^2}{r^2} \tag{2.9}$$

$$\sigma_z = -2\nu p_b \tag{2.10}$$

$$2G\varepsilon_r = -(1-2\nu)p_b + (p_b - p_a)\frac{a^2}{r^2} \tag{2.11}$$

$$2G\varepsilon_\theta = -(1-2\nu)p_b - (p_b - p_a)\frac{a^2}{r^2} \tag{2.12}$$

For p_a equal to p_b during the initial loading, Equations 2.8 and 2.9 reduce to $\sigma_r = \sigma_\theta = -p_b$. For values of Poisson's ratio satisfying the inequality $0 < \nu < 0.5$, the stress ordering is $\sigma_r = \sigma_\theta < \sigma_z$. With increased loading, yielding of the material will be governed by

$$f_1 = \sigma_\theta - N\sigma_z + \sigma_u = 0 \tag{2.13a}$$

$$f_2 = \sigma_r - N\sigma_z + \sigma_u = 0 \tag{2.13b}$$

Substitution of the stress components in terms of p_b into the yield condition of Equation 2.13a gives

$$p_a = p_b = \frac{\sigma_u}{1-2N\nu} \tag{2.14}$$

For compressive values of p_b (equal to p_a), the material will not yield if $2N\nu > 1$. For the condition $2N\nu < 1$, the material will yield throughout when Equation 2.14 is satisfied. As discussed in Section 2.1, we assume that the internal pressure, p_a, does not cause yielding of the material, and thus we restrict the internal pressure to

$$p_a < \frac{\sigma_u}{1-2N\nu} \text{ when } 2N\nu < 1 \tag{2.15}$$

So far, there is no restriction on p_a for $2N\nu > 1$, but one will be developed later.

As discussed in Section 2.1, once the internal pressure reaches its maximum value, it is held constant, while the free-field pressure is increased. Once the external pressure is increased above the internal pressure, we have (according to Equations 2.8 through 2.10) $\sigma_\theta < \sigma_r$ and $\sigma_\theta < \sigma_z$ throughout the medium. At the edge of the hole (r equals a), $\sigma_r(a) > \sigma_z(a)$ if $p_a < 2\nu p_b$ and $\sigma_r(a) < \sigma_z(a)$ if $p_a > 2\nu p_b$. An inspection of Equations 2.8 through 2.10 shows that the maximum stress difference occurs at the edge of the hole, and therefore initial yielding will occur there.

We will denote the external pressure at which yielding begins as $\overline{p}_b$. The stresses at the edge of the hole (r equals a) when yielding commences are evaluated using Equations 2.8 through 2.10.

$$\sigma_r(a) = -p_a \tag{2.16a}$$

$$\sigma_\theta(a) = -2\overline{p}_b + p_a \tag{2.16b}$$

$$\sigma_z(a) = -2\nu\overline{p}_b \tag{2.16c}$$

An inspection of Equations 2.16 indicates that if $p_a < 2\nu\overline{p}_b$, then the stress order is $\sigma_\theta < \sigma_z < \sigma_r$ at r equals a, and therefore the governing yield condition is

$$f = \sigma_\theta - N\sigma_r + \sigma_u = 0 \tag{2.17}$$

Substitution of the stresses from Equations 2.16 into the yield condition of Equation 2.17 leads to the initial yield pressure

$$\overline{p}_b = \frac{1}{2}\left[(N+1)p_a + \sigma_u\right] \tag{2.18}$$

Substitution of Equation 2.18 for $\overline{p}_b$ into the inequality $p_a < 2\nu\overline{p}_b$ gives

$$p_a < \frac{\nu\sigma_u}{1-(N+1)\nu} \text{ when } (N+1)\nu < 1 \tag{2.19}$$

The restriction $(N+1)\nu < 1$ is imposed since we are only considering compressive values of p_a. For $(N+1)\nu > 1$ there is no restriction on p_a. The condition $\sigma_\theta < \sigma_z < \sigma_r$ at initial yielding constitutes Case I.

Inspection of Equations 2.16 also indicates that if $2\nu\bar{p}_b < p_a < \bar{p}_b$, then the stress order at the edge of the hole is $\sigma_\theta < \sigma_r < \sigma_z$ and therefore the governing yield condition is given by

$$f = \sigma_\theta - N\sigma_z + \sigma_u = 0 \tag{2.20}$$

Substitution of Equations 2.16 into Equation 2.20 leads to the initial yield pressure

$$\bar{p}_b = \frac{p_a + \sigma_u}{2(1-N\nu)} \text{ when } N\nu < 1 \tag{2.21}$$

The restriction $N\nu < 1$ is imposed since we are only considering compressive values of $\bar{p}_b$. The condition $\sigma_\theta < \sigma_r < \sigma_z$ at initial yielding constitutes Case II.

For the condition $N\nu > 1$, yielding governed by Equation 2.20 will not occur for compressive values of p_b. Instead, as p_b is increased it will eventually reach a value where $p_a < 2\nu p_b$ so that the stress ordering is $\sigma_\theta < \sigma_z < \sigma_r$ and yielding is governed by Equation 2.17. Thus, for $N\nu > 1$, we have a Case I condition and therefore $\bar{p}_b$ is given by Equation 2.18. Also, for the condition $N\nu > 1$, it follows that $(N+1)\nu > 1$ so no restriction is placed on p_a (as was done in Equation 2.19).

For Case II with $N\nu < 1$, the yield pressure of Equation 2.21 is substituted into the inequality $2\nu\bar{p}_b < p_a < \bar{p}_b$ (for which $\sigma_\theta < \sigma_r < \sigma_z$ at the edge of the hole) which gives

$$\frac{2\nu(p_a + \sigma_u)}{2(1-N\nu)} < p_a < \frac{p_a + \sigma_u}{2(1-N\nu)} \tag{2.22}$$

Equation 2.22 can be rearranged to give

$$\frac{\nu\sigma_u}{1-(N+1)\nu} < p_a < \frac{\sigma_u}{1-2N\nu} \text{ when } 0 < N\nu < \frac{1}{2} \tag{2.23a}$$

$$\frac{\nu\sigma_u}{1-(N+1)\nu} < p_a \text{ when } \frac{1}{2} < N\nu < (N+1)\nu < 1 \tag{2.23b}$$

Equation 2.23b provides the restriction on p_a for $2N\nu > 1$ alluded to in the discussion of Equation 2.15.

In summary, the conditions under which initial yielding is governed by the yield condition Equation 2.17 constitute Case I. The conditions under which initial yielding is governed by the yield condition of Equation 2.20 constitute Case II. Thus the physical difference between Case I and Case II is the stress ordering at the edge of the hole when yielding begins. The Case I and Case II conditions are summarized in Table 2-1 (adapted from Table I of (Florence, 1978)).

2.4 FREE-FIELD YIELDING CONDITIONS.

Equilibrium in the free field dictates that

$$\sigma_r = \sigma_\theta = -p_b \tag{2.24}$$

i.e., the in-plane principal stresses are equal to the negative of the externally applied pressure. The negative sign comes from the convention that stresses are taken positive in tension, while the applied pressures (p_a and p_b) are taken positive in compression. For elastic conditions, the out-of-plane stress σ_z may be obtained from Hooke's law, the plane strain condition, and Equation 2.24.

$$E\varepsilon_z = \sigma_z - \nu(\sigma_r + \sigma_\theta) = 0 \tag{2.25a}$$

$$\sigma_z = -2\nu p_b \tag{2.23b}$$

where E is Young's modulus and ε_z is the out-of-plane strain (zero by definition).

For compressible materials $(\nu < 0.5)$, the order of the principal stresses is

$$\sigma_r = \sigma_\theta < \sigma_z \tag{2.26}$$

which leads to failure (yielding) being governed by the intersection of the following two failure surfaces.

$$f_1 = \sigma_r - N\sigma_z + \sigma_u = 0 \tag{2.27a}$$

$$f_2 = \sigma_\theta - N\sigma_z + \sigma_u = 0 \tag{2.27b}$$

The magnitude of the applied pressure at initial yield in the free field $(\hat{p}_b)$ is determined by substituting Equations 2.24 and 2.26 into either of Equations 2.27a or 2.27b,

$$-\hat{p}_b + 2N\nu\hat{p}_b + \sigma_u = 0 \tag{2.28a}$$

or

Table 2-1. Case I and Case II conditions.

Property Relations	Internal Pressure	Case
$N\nu < (N+1)\nu < 1$	$p_a < \dfrac{\nu\sigma_u}{1-(N+1)\nu}$	I
$1 < N\nu < (N+1)\nu$ or $N\nu < 1 < (N+1)\nu$	$p_a > 0$	I
$\dfrac{1}{2} < N\nu < (N+1)\nu < 1$	$p_a > \dfrac{\nu\sigma_u}{1-(N+1)\nu}$	II
$0 < N\nu < \dfrac{1}{2}$	$\dfrac{\nu\sigma_u}{1-(N+1)\nu} < p_a < \dfrac{\sigma_u}{1-2N\nu}$	II

$$\hat{p}_b = \frac{\sigma_u}{1-2N\nu} \tag{2.28b}$$

Note that if $1-2N\nu < 0$, then free-field yielding can only occur for tensile values of p_b. As discussed in Section 2.1, tensile values of p_b and p_a are precluded. Therefore, for $1-2N\nu < 0$, the earlier solution (Wintergerst, 1991) is complete.

SECTION 3
CASE I SOLUTION

In this section, the arbitrary dilatancy solution for Case I is extended to include the effects of free-field yielding. It was shown in Section 2.2 that yielding of the free field occurs when

$$p_b \geq \frac{\sigma_u}{1-2N\nu} = \hat{p}_b \quad \text{for} \quad 1-2N\nu > 0 \tag{3.1}$$

Although the earlier solution (Wintergerst, 1991) is valid for $p_b < \hat{p}_b$, the entire Case I solution is presented below for completeness.

As stated in Section 2.1, p_b is assumed equal to p_a (the internally applied pressure) until p_a reaches its final value. The external pressure, p_b, is then increased monotonically while p_a is held constant. The initial material response is assumed to be elastic (Figure 3-1). As discussed in Section 2.3, initial yielding of the material begins at the edge of the hole (r equals a) when $p_b = \left[(N+1)p_a + \sigma_u\right]/2$, with stress conditions $\sigma_\theta < \sigma_z < \sigma_r$, and is governed by the failure surface

$$f = \sigma_\theta - N\sigma_r + \sigma_u = 0 \tag{3.2}$$

After initial yielding, an inner plastic zone forms with outer boundary radius R. Outside of R, the material is elastic (as shown in Figure 3-2). For certain combinations of material properties, σ_z is always the intermediate principal stress (throughout the medium and regardless of the value of p_b); thus, no additional plastic zones will form and the free field will not yield. In this report, the conditions under which σ_z is always the intermediate principal stress will be termed Case Ia. For the case where the plastic flow rule is associative (M equal to N), the Case Ia solution is the Hendron and Aiyer solution (Hendron, 1972).

The Case Ia solution can be considered to consist of two phases:

<u>Phase 1</u>. Elastic solution
<u>Phase 2</u>. One plastic zone with an outer elastic zone

For other combinations of material properties, σ_z is not always the intermediate principal stress throughout the medium. As p_b increases, additional plastic zones can form where σ_z is the maximum principal stress. As p_b increases further, the

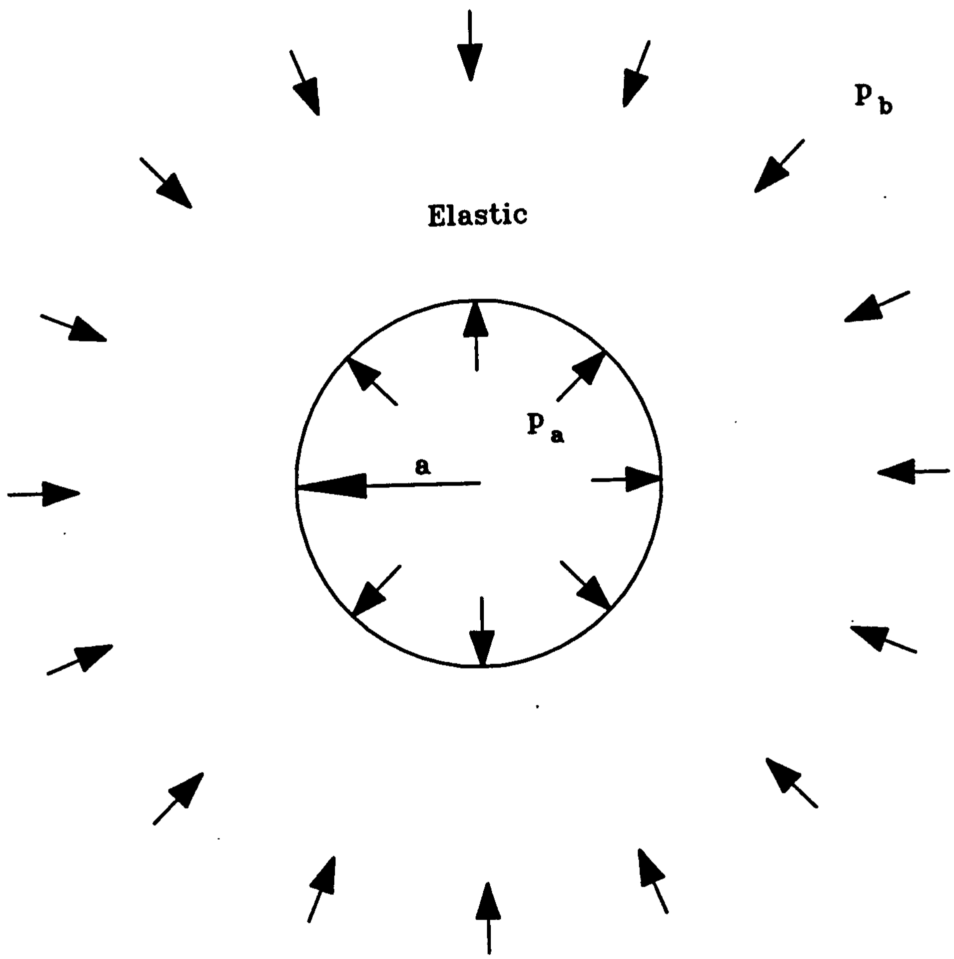


Figure 3-1. Phase 1 conditions for Cases Ia and Ib.

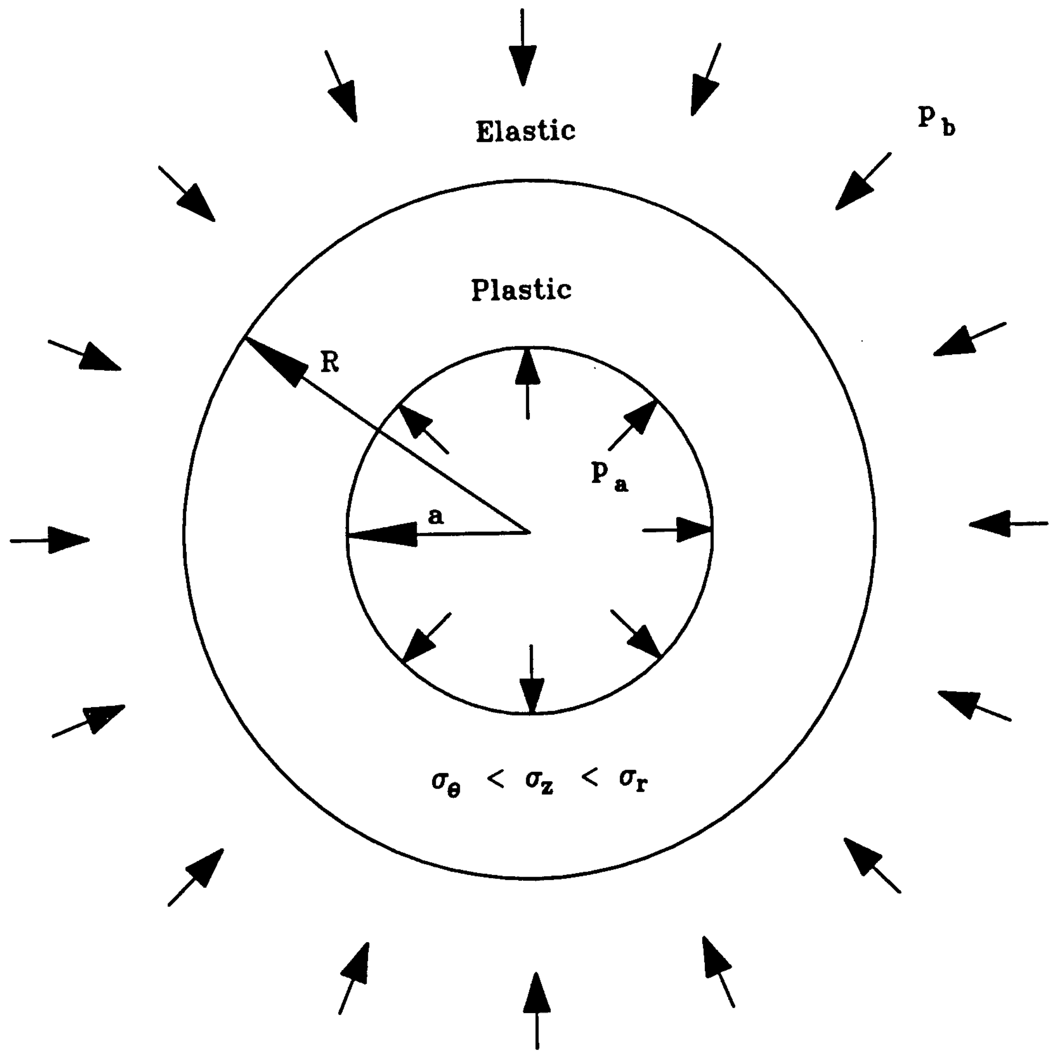


Figure 3-2. Phase 2 conditions for Cases Ia and Ib.

free field may yield if $1-2N\nu > 0$. The conditions where σ_z is not always the intermediate principal stress constitute Case Ib. The demarcation between Case Ia and Case Ib conditions is discussed in Section 3.1.

For Case Ib, the initial inner plastic zone increases in size as loading continues, until its boundary reaches a limiting value $\tilde{R}$, where $\sigma_r(\tilde{R}) = \sigma_z(\tilde{R})$. This inner plastic zone boundary, defined by $r = \tilde{R}$, remains fixed as p_b is increased. Continued loading causes the formation of two more plastic zones. In the middle plastic zone, the stress condition is $\sigma_\theta < \sigma_z = \sigma_r$ and yielding is governed by

$$f_1 = \sigma_\theta - N\sigma_r + \sigma_u = 0 \quad (3.3a)$$

$$f_2 = \sigma_\theta - N\sigma_z + \sigma_u = 0 \quad (3.3b)$$

In the outer plastic zone, the stress condition is $\sigma_\theta < \sigma_r < \sigma_z$ and yielding is governed by the single failure surface

$$f = \sigma_\theta - N\sigma_z + \sigma_u = 0 \quad (3.4)$$

The boundary between the middle and outer plastic zones, defined by $r = \overline{R}$, where $\sigma_r(\overline{R}) = \sigma_z(\overline{R})$. Outside of the plastic zones, the material response was assumed in the earlier solutions to be elastic. The boundary between the outer plastic zone and the elastic zone is defined by r = R. The three plastic zones and the elastic zone, with their boundaries and stress conditions are shown in Figure 3-3.

With further loading, the free field may yield if $1-2N\nu > 0$. If the free field yields, a fourth plastic zone (Figure 3-4) is postulated to exist. In subsequent discussions, the fourth zone is designated the "far outer plastic zone". This term is used throughout this report to indicate free-field yielding, even though there may be as few as two plastic zones. The stress condition in this zone is $\sigma_r = \sigma_\theta < \sigma_z$ and the yield conditions are

$$f_1 = \sigma_r - N\sigma_z + \sigma_u = 0 \quad (3.5a)$$

$$f_2 = \sigma_\theta - N\sigma_z + \sigma_u = 0 \quad (3.5b)$$

Based on the progression of the formation of plastic zones, the solution for Case Ib can be considered to consist of four phases:

<u>Phase 1</u>. Elastic solution

<u>Phase 2</u>. One plastic zone with an outer elastic zone

<u>Phase 3</u>. Three plastic zones with an outer elastic zone

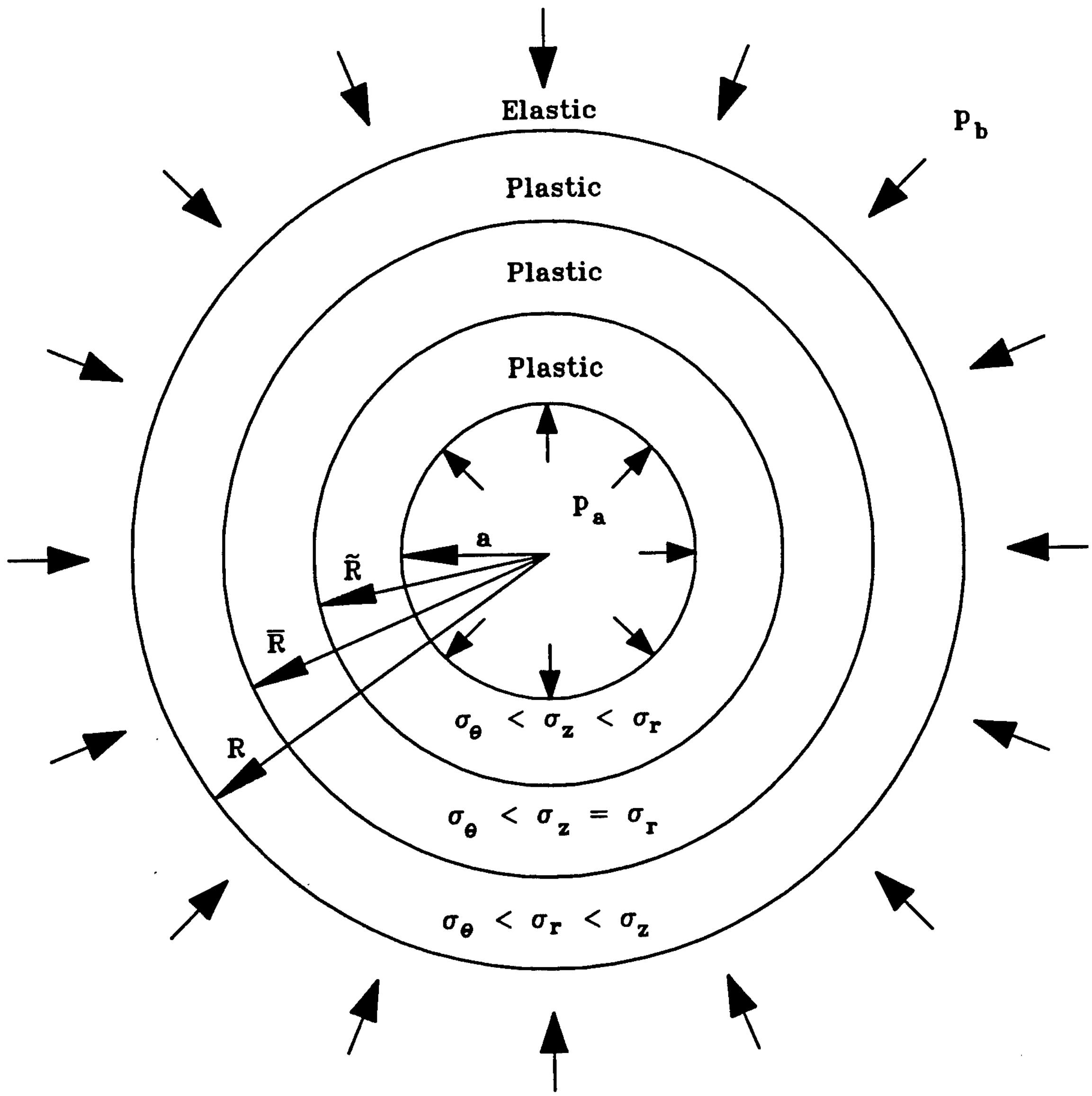


Figure 3-3. Phase 3 conditions for Case Ib.

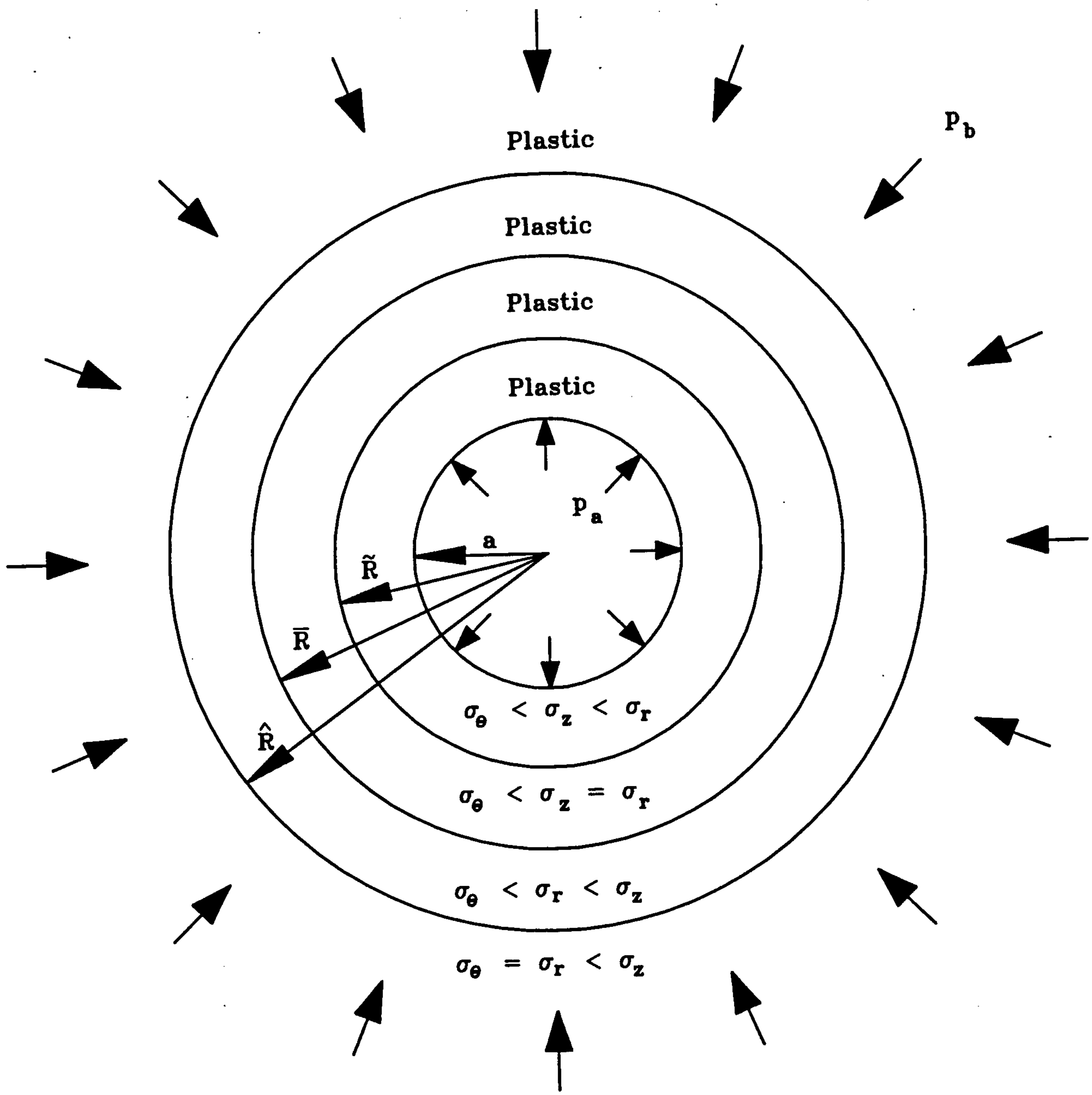


Figure 3-4. Phase 4 conditions for Case Ib.

Phase 4. Four plastic zones (free field yielding)
Note that Phase 1 and Phase 2 are identical for Cases Ia and Ib.

The elastic solution was presented in Section 2.3. The solutions for the phases with plastic zones are presented in Sections 3.1 through 3.3.

3.1 CASE Ia AND Ib, PHASE 2 SOLUTION.

In this section, we develop the solution for Phase 2 of Case I. As discussed previously, in Phase 2 there is a single (inner) plastic zone with an outer elastic zone (Figure 3-2). The derivation presented below closely follows those contained in (Florence, 1978) and (Wintergerst, 1991). The demarcation between Cases Ia and Ib is also defined.

Initial yielding of the material occurs at the edge of the hole (r = a) when p_b is equal to $\overline{p}_b$ (derived below). As discussed previously, for Case I conditions $\sigma_\theta < \sigma_z < \sigma_r$ and the yield condition is

$$f = \sigma_\theta - N\sigma_r + \sigma_u = 0 \tag{3.2}$$

Substitution of the elastic solution for σ_r and σ_θ (presented in Section 2.3) evaluated at r equals a into the yield condition of Equation 3.2 leads to the following expression for the initial yield pressure, $\overline{p}_b$.

$$\overline{p}_b = \frac{1}{2}\left[(N+1)p_a + \sigma_u\right] \tag{3.6}$$

The equation of equilibrium is given by

$$r\frac{d\sigma_r}{dr} + \sigma_r - \sigma_\theta = 0 \tag{3.7}$$

and is valid throughout the material. Using the yield condition of Equation 3.2, σ_θ can be written in terms of σ_r,

$$\sigma_\theta = N\sigma_r - \sigma_u \tag{3.8}$$

so the equilibrium Equation 3.7 can be rewritten as

$$r\frac{d\sigma_r}{dr} - \sigma_r(N-1) + \sigma_u = 0 \tag{3.9}$$

The solution to Equation 3.9, with the boundary condition $\sigma_r(a) = -p_a$, is given by

$$\sigma_r = -\left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{r}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \tag{3.10}$$

The strain rates are determined by the nonassociative flow rule, discussed in Section 2.1. For the yield condition of Equation 3.2, the plastic strain rate components are given by

$$\dot{\varepsilon}_r^{(p)} = \lambda\frac{\partial g}{\partial \sigma_r} = -\lambda M \tag{3.11a}$$

$$\dot{\varepsilon}_\theta^{(p)} = \lambda\frac{\partial g}{\partial \sigma_\theta} = \lambda \tag{3.11b}$$

$$\dot{\varepsilon}_z^{(p)} = \lambda\frac{\partial g}{\partial \sigma_z} = 0 \tag{3.11c}$$

where

$$g = \sigma_\theta - M\sigma_r + \kappa \tag{3.12}$$

and λ is the plastic multiplier. Since the loading is assumed to be monotonic and the material law is rate independent, the overdot in Equations 3.11 may be interpreted to represent either differentiation with respect to time or differentiation with respect to p_b. From the flow rule of Equations 3.11 it follows that

$$\dot{\varepsilon}_r^{(p)} + M\dot{\varepsilon}_\theta^{(p)} = 0 \tag{3.13a}$$

$$\dot{\varepsilon}_z^{(p)} = 0 \tag{3.13b}$$

which, when integrated with respect to time, give

$$\varepsilon_r^{(p)} + M\varepsilon_\theta^{(p)} = h_1(r) \tag{3.14a}$$

$$\varepsilon_z^{(p)} = h_2(r) \tag{3.14b}$$

The strain expressions in Equations 3.14, evaluated at a fixed radius r, remain constant as loading proceeds (i.e., they are independent of time, or equivalently, of load p_b). When the elastic-plastic radius R is initially at this radius r, the plastic strains are zero, which implies that $h_1(r) = 0$ and $h_2(r) = 0$, therefore

$$\varepsilon_r^{(p)} + M\varepsilon_\theta^{(p)} = 0 \tag{3.15a}$$

$$\varepsilon_z^{(p)} = 0 \tag{3.15b}$$

The strain rates are assumed to be decomposed into elastic and plastic parts

$$\dot{\varepsilon}_r = \dot{\varepsilon}_r^{(e)} + \dot{\varepsilon}_r^{(p)} \tag{3.16a}$$

$$\dot{\varepsilon}_\theta = \dot{\varepsilon}_\theta^{(e)} + \dot{\varepsilon}_\theta^{(p)} \tag{3.16b}$$

$$\dot{\varepsilon}_z = \dot{\varepsilon}_z^{(e)} + \dot{\varepsilon}_z^{(p)} \tag{3.16c}$$

Integrating Equation 3.16a with respect to time gives $\varepsilon_r - \left(\varepsilon_r^{(e)} + \varepsilon_r^{(p)}\right) = h(r)$. When r equals R, the strain is entirely elastic so $\varepsilon_r = \varepsilon_r^{(e)}$ and $\varepsilon_r^{(p)} = 0$ giving $h(r) = 0$. Using the same argument for the other two strain components leads to

$$\varepsilon_r = \varepsilon_r^{(e)} + \varepsilon_r^{(p)} \tag{3.17a}$$

$$\varepsilon_\theta = \varepsilon_\theta^{(e)} + \varepsilon_\theta^{(p)} \tag{3.17b}$$

$$\varepsilon_z = \varepsilon_z^{(e)} + \varepsilon_z^{(p)} \tag{3.17c}$$

The plane strain condition requires that $\varepsilon_z = 0$, and the flow rule led to $\varepsilon_z^{(p)} = 0$, therefore $\varepsilon_z^{(e)} = 0$.

The elastic strains are related to the stresses by Hooke's law.

$$E\varepsilon_r^{(e)} = \sigma_r - \nu\left(\sigma_\theta + \sigma_z\right) \tag{3.18a}$$

$$E\varepsilon_\theta^{(e)} = \sigma_\theta - \nu\left(\sigma_r + \sigma_z\right) \tag{3.18b}$$

$$E\varepsilon_z^{(e)} = \sigma_z - \nu\left(\sigma_r + \sigma_\theta\right) \tag{3.18c}$$

From Equation 3.18c and the condition $\varepsilon_z^{(e)} = 0$

$$\sigma_z = \nu\left(\sigma_r + \sigma_\theta\right) \tag{3.19}$$

Substituting Equation 3.19 for σ_z, and Equation 3.8 for σ_θ into Equations 3.18a and 3.18b leads to

$$2G\varepsilon_r^{(e)} = \left[1 - (N+1)\nu\right]\sigma_r + \nu\sigma_u \tag{3.20a}$$

$$2G\varepsilon_\theta^{(e)} = \left[N - (N+1)\nu\right]\sigma_r - (1-\nu)\sigma_u \tag{3.20b}$$

where σ_r is given by Equation 3.10.

The strain-displacement relations (for small strains) are given by

$$\varepsilon_r = \frac{du}{dr} \tag{3.21a}$$

$$\varepsilon_\theta = \frac{u}{r} \tag{3.21b}$$

where u represents the radial displacement. The compatibility equation is derived by eliminating u from Equations 3.21.

$$r\frac{d\varepsilon_\theta}{dr} + \varepsilon_\theta - \varepsilon_r = 0 \tag{3.22}$$

Using the strain decomposition of Equation 3.17, Equation 3.15a to eliminate $\varepsilon_r^{(p)}$, Equations 3.20 for the elastic strain components, and Equation 3.10 for σ_r, the compatibility Equation 3.22 can be rewritten as

$$r\frac{d\varepsilon_\theta^{(p)}}{dr} + (N+1)\varepsilon_\theta^{(p)} = \frac{1-\nu}{2G}(N+1)\left[(N-1)p_a + \sigma_u\right]\left(\frac{r}{a}\right)^{N-1} \tag{3.23}$$

The solution to Equation 3.23, with the boundary condition $\varepsilon_\theta^{(p)}(R) = 0$ is

$$\varepsilon_\theta^{(p)} = -\frac{1-\nu}{2G}\frac{N+1}{M+N}\left[(N-1)p_a + \sigma_u\right]\left(\frac{R}{a}\right)^{N-1}\left[\left(\frac{R}{r}\right)^{M+1} - \left(\frac{r}{R}\right)^{N-1}\right] \tag{3.24}$$

The remaining unknown is the elastic-plastic boundary radius R. Using R and p in place of a and p_a, respectively, in the elastic solution given in Section 2.3 (Equations 2.8 through 2.10) gives

$$\sigma_r(R) = -p \tag{3.25a}$$

$$\sigma_\theta(R) = -2p_b + p \tag{3.25b}$$

$$\sigma_z(R) = -2\nu p_b \tag{3.25c}$$

This notation is used throughout the report. That is, when stress or strain components are immediately followed by a single variable in parentheses, that variable denotes the spatial location where the stress or strain component is being evaluated. For example, the notation $\sigma_r(R)$ denotes the stress function σ_r evaluated at r equal to R. This rule will also apply to the radial displacement function, u.

The yield condition of Equation 3.2 still holds at r equals R, so

$$p_b = \frac{1}{2}\left[(N+1)p + \sigma_u\right] \tag{3.26}$$

$$\sigma_r(R) = -\frac{2p_b - \sigma_u}{N+1} = -p \tag{3.27}$$

Setting r equal to R in Equation 3.10 and equating the result to Equation 3.27 gives

$$-\left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{R}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} = -\frac{2p_b - \sigma_u}{N+1} \tag{3.28a}$$

or

$$\left(\frac{R}{a}\right)^{N-1} = \frac{2}{N+1} \cdot \frac{(N-1)p_b + \sigma_u}{(N-1)p_a + \sigma_u} \tag{3.28b}$$

The solution presented above is valid with continued loading as long as $\sigma_z < \sigma_r$ in the plastic zone. From Equations 3.19 and 3.2

$$\sigma_r - \sigma_z = [1-(N+1)\nu]\sigma_r + \nu\sigma_u \tag{3.29}$$

In Equation 3.29, $\sigma_z < \sigma_r$ when the right hand side of the equation, $[1-(N+1)\nu]\sigma_r + \nu\sigma_u$, is greater than zero. If $(N+1)\nu > 1$, then $[1-(N+1)\nu]$ is negative. Since σ_r is compressive (negative) and $\nu\sigma_u$ is positive, then $[1-(N+1)\nu]\sigma_r + \nu\sigma_u$ is positive and thus $\sigma_z < \sigma_r$. If $(N+1)\nu < 1$, then $[1-(N+1)\nu]\sigma_r + \nu\sigma_u$ is positive for $\sigma_r < -\frac{\nu\sigma_u}{1-(N+1)\nu}$. In summary,

(1) $\sigma_z < \sigma_r$ when $(N+1)\nu > 1$

(2) $\sigma_z < \sigma_r$ when $(N+1)\nu < 1$ and $\sigma_r < -\frac{\nu\sigma_u}{1-(N+1)\nu}$

When $(N+1)\nu > 1$, the solution presented above is complete since no more plastic zones will form and the free field will not yield. This case will be referred to as Case Ia. In the second case, when $(N+1)\nu < 1$, the minimum value of σ_r occurs at r equal to R, so that as loading proceeds the elastic-plastic boundary radius R attains a limiting value, denoted $\tilde{R}$, where

$$\sigma_r(\tilde{R}) = \sigma_z(\tilde{R}) = -\frac{\nu\sigma_u}{1-(N+1)\nu} \tag{3.30}$$

and

$$\sigma_\theta\left(\tilde{R}\right) = -\frac{(1-\nu)\sigma_u}{1-(N+1)\nu} \tag{3.31}$$

The magnitude of the loading when R equals $\tilde{R}$ is denoted as $\tilde{p}_b$ which is given by Equations 3.25, 3.26, and 3.30

$$\tilde{p}_b = \frac{\sigma_u}{2[1-(N+1)\nu]} \tag{3.32}$$

and the radius of the elastic-plastic interface can be determined from Equations 3.28b and 3.32.

$$\left(\frac{\tilde{R}}{a}\right)^{N-1} = \frac{(1-2\nu)\sigma_u}{[1-(N+1)\nu][(N-1)p_a + \sigma_u]} \tag{3.33}$$

The relationship for $\tilde{R}$ is independent of p_b, as is the stress field, and therefore both will remain constant with increased loading.

3.2 CASE Ib, PHASE 3 SOLUTION.

The solution for Phase 3 of Case Ib is developed in this section. As discussed previously, there are three plastic zones and an outer elastic zone in Phase 3 (Figure 3-3). Again, the derivation presented below closely follows those contained in (Florence, 1978) and (Wintergerst, 1991).

Florence and Schwer postulated that with increased loading $(p_b > \tilde{p}_b)$, two additional plastic zones are formed outside of the initial, inner plastic zone. In the outer plastic zone $(\overline{R} < r < R)$ the stress condition is $\sigma_\theta < \sigma_r < \sigma_z$, therefore the yield condition is

$$f = \sigma_\theta - N\sigma_z + \sigma_u = 0 \tag{3.4}$$

The plastic strain rates are obtained from the nonassociative flow rule.

$$\dot{\varepsilon}_r^{(p)} = 0 \tag{3.34a}$$

$$\dot{\varepsilon}_\theta^{(p)} = \lambda \tag{3.34b}$$

$$\dot{\varepsilon}_z^{(p)} = -M\lambda \tag{3.34c}$$

Using similar arguments for integrating the plastic strain rates as presented in Section 3.1 leads to

$$\varepsilon_z^{(p)} + M\varepsilon_\theta^{(p)} = 0 \tag{3.35a}$$

$$\varepsilon_r^{(p)} = 0 \tag{3.35b}$$

Using the strain decomposition of Equation 3.17, the plane strain condition, and Equations 3.35 leads to the following strain expressions.

$$\varepsilon_r = \varepsilon_r^{(e)} \tag{3.36a}$$

$$\varepsilon_\theta = \varepsilon_\theta^{(e)} + \frac{1}{M}\varepsilon_z^{(e)} \tag{3.36b}$$

$$\varepsilon_z = 0 \tag{3.36c}$$

Using Hooke's law (Equations 3.18), the strain Equations 3.36, and the yield condition of Equation 3.4 gives the following expressions for the radial and tangential stress components in terms of the radial and tangential strain components.

$$[MN+1-(M+1)(N+1)\nu]\sigma_r = [MN+1-(M+N)\nu]2G\varepsilon_r \\ +M(N+1)\nu 2G\varepsilon_\theta + (M-1)\nu\sigma_u \tag{3.37a}$$

$$[MN+1-(M+1)(N+1)\nu]\sigma_\theta = N(M+1)\nu 2G\varepsilon_r \\ +MN2G\varepsilon_\theta - [1-(M+1)\nu]\sigma_u \tag{3.37b}$$

Substitution of the stress Equations 3.37 into the equilibrium Equation 3.7, and using the strain-displacement Equations 3.21, leads to the following ordinary differential equation in terms of the radial displacement u,

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr}\left[1+\frac{(M-N)\nu}{MN}\beta^2\right] - \beta^2 u = -\frac{\beta^2}{MN}\frac{(1-2\nu)\sigma_u}{2G}r \tag{3.38}$$

where

$$\beta^2 = \frac{MN}{MN+1-(M+N)\nu} \tag{3.39}$$

The solution to Equation 3.38 is

$$2Gu = A_1 r^{\gamma_1} + A_2 r^{-\gamma_2} + Ar \tag{3.40}$$

where

$$A = -\frac{(1-2\nu)}{1-2N\nu}\sigma_u = -(1-2\nu)\hat{p}_b \tag{3.41}$$

$$\gamma_1 = \beta^2\frac{(N-M)\nu}{2MN} + \beta^2\sqrt{\frac{(N-M)^2\nu^2}{(2MN)^2} + \frac{1}{\beta^2}} \tag{3.42}$$

$$\gamma_2 = \beta^2\frac{(M-N)\nu}{2MN} + \beta^2\sqrt{\frac{(M-N)^2\nu^2}{(2MN)^2} + \frac{1}{\beta^2}} \tag{3.43}$$

and A_1 and A_2 are constants that are yet to be determined. In Equation 3.41, the definition of $\hat{p}_b$ from Equation 2.28b is used. The strains can be computed from the strain-displacement relations and the displacement Equation 3.40.

$$2G\varepsilon_r = \gamma_1 A_1 r^{\gamma_1 - 1} - \gamma_2 A_2 r^{-\gamma_2 - 1} + A \tag{3.44a}$$

$$2G\varepsilon_\theta = A_1 r^{\gamma_1 - 1} + A_2 r^{-\gamma_2 - 1} + A \tag{3.44b}$$

The strain Equations 3.44 can be substituted into the stress Equations 3.37 to yield

$$\sigma_r = C_{r1}A_1 r^{\gamma_1 - 1} + C_{r2}A_2 r^{-\gamma_2 - 1} - \hat{p}_b \tag{3.45a}$$

$$\sigma_\theta = C_{\theta 1}A_1 r^{\gamma_1 - 1} + C_{\theta 2}A_2 r^{-\gamma_2 - 1} - \hat{p}_b \tag{3.45b}$$

where

$$C_{r1} = \frac{1}{C}\left\{[MN + 1 - (M+N)\nu]\gamma_1 + M(N+1)\nu\right\} \tag{3.46a}$$

$$C_{r2} = \frac{1}{C}\left\{-[MN + 1 - (M+N)\nu]\gamma_2 + M(N+1)\nu\right\} \tag{3.46b}$$

$$C_{\theta 1} = \frac{1}{C}[N(M+1)\nu\gamma_1 + MN] \tag{3.46c}$$

$$C_{\theta 2} = \frac{1}{C}[-N(M+1)\nu\gamma_2 + MN] \tag{3.46d}$$

$$C = MN + 1 - (M+1)(N+1)\nu \tag{3.46e}$$

At this point the unknowns are the two constants A_1 and A_2 and the plastic zone boundary radii R and $\overline{R}$. Two of the conditions used to evaluate the unknowns are provided by enforcing continuity of radial stress and radial displacement (or, equivalently, circumferential strain) at the boundary with the outer elastic zone (r equals R). The elastic zone stresses evaluated at r equals R can be substituted into the yield condition of Equation 3.4 to give

$$\sigma_r(R) = -2(1 - N\nu)p_b + \sigma_u \tag{3.47}$$

The circumferential strain at r equals R is given by

$$2G\varepsilon_\theta(R) = -2(N-1)\nu p_b - \sigma_u \tag{3.48}$$

Continuity of radial stress at r equals R leads to

$$C_{r1}A_1 R^{\gamma_1 - 1} + C_{r2}A_2 R^{-\gamma_2 - 1} = 2(1 - N\nu)\Delta p_b \tag{3.49a}$$

while continuity of circumferential strain at r equals R leads to

$$A_1 R^{\gamma_1 - 1} + A_2 R^{-\gamma_2 - 1} = 2(N-1)\nu\Delta p_b \tag{3.49b}$$

where

$$\Delta p_b = \hat{p}_b - p_b \tag{3.50}$$

Equations 3.49 constitute a system of two equations in the two unknowns $A_1 R^{\gamma_1 - 1}$ and $A_2 R^{-\gamma_2 - 1}$,

$$\begin{bmatrix} C_{r1} & C_{r2} \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} A_1 R^{\gamma_1 - 1} \\ A_2 R^{-\gamma_2 - 1} \end{Bmatrix} = \begin{Bmatrix} 2(1 - N\nu)\Delta p_b \\ 2(N-1)\nu\Delta p_b \end{Bmatrix} \tag{3.51}$$

which can be inverted to give

$$\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} A_1 R^{\gamma_1 - 1} \\ A_2 R^{-\gamma_2 - 1} \end{Bmatrix} = \frac{1}{C_{r1} - C_{r2}} \begin{bmatrix} 1 & -C_{r2} \\ -1 & C_{r1} \end{bmatrix} \begin{Bmatrix} 2(1 - N\nu)\Delta p_b \\ 2(N-1)\nu \Delta p_b \end{Bmatrix} \tag{3.52}$$

where B_1 and B_2 are constants.

Florence and Schwer postulated that at the inner radius $\overline{R}$ of the outer plastic zone, $\sigma_r(\overline{R}) = \sigma_z(\overline{R})$, so that the yield condition of Equation 3.4 may be written as

$$N\sigma_r(\overline{R}) - \sigma_\theta(\overline{R}) = \sigma_u \tag{3.53}$$

Using the stress expressions in Equation 3.45 evaluated at $\overline{R}$ and substituted into Equation 3.53 leads to

$$(NC_{r1} - C_{\theta 1})A_1 \overline{R}^{\gamma_1 - 1} + (NC_{r2} - C_{\theta 2})A_2 \overline{R}^{-\gamma_2 - 1} - N(1 - 2\nu)\hat{p}_b = 0 \tag{3.54}$$

Substituting the results of Equation 3.52 into Equation 3.54 leads to the following nonlinear equation

$$(NC_{r1} - C_{\theta 1})B_1 \left(\frac{\overline{R}}{R} \right)^{\gamma_1 - 1} + (NC_{r2} - C_{\theta 2})B_2 \left(\frac{\overline{R}}{R} \right)^{-\gamma_2 - 1} - N(1 - 2\nu)\hat{p}_b = 0 \tag{3.55}$$

which can be solved for the radius ratio $(\overline{R}/R)$. Determination of the individual values of $\overline{R}$ and R requires a description of the state in the middle plastic zone.

As discussed earlier, Florence and Schwer postulated the existence of a middle plastic zone $(\tilde{R} < r < \overline{R})$ where $\sigma_\theta < \sigma_r = \sigma_z$, so that yielding is governed by

$$f_1 = \sigma_\theta - N\sigma_r + \sigma_u = 0 \tag{3.3a}$$

$$f_2 = \sigma_\theta - N\sigma_z + \sigma_u = 0 \tag{3.3b}$$

Solving for σ_θ in terms of σ_r using Equation 3.3a, the equilibrium Equation 3.7 has the solution

$$\sigma_r = \sigma_r(\tilde{R}) \left(\frac{r}{\tilde{R}} \right)^{N-1} + \left[1 - \left(\frac{r}{\tilde{R}} \right)^{N-1} \right] \frac{\sigma_u}{N-1} \tag{3.56}$$

Substituting $\tilde{R}$ and $\sigma_r(\tilde{R})$ from Equations 3.30 and 3.33, respectively, into Equation 3.56 gives

$$\sigma_r = -\left[p_a + \frac{\sigma_u}{N-1} \right] \left(\frac{r}{a} \right)^{N-1} + \frac{\sigma_u}{N-1} \tag{3.57}$$

Note that Equation 3.57 is identical to Equation 3.10, so the radial stress is represented by the same formula throughout the combined inner and middle plastic zones, $a < r < \overline{R}$. Providing continuity of radial stress at the interface r equals $\overline{R}$ leads to

$$C_{r1}B_1\left(\frac{\overline{R}}{R}\right)^{\gamma_1-1} + C_{r2}B_2\left(\frac{\overline{R}}{R}\right)^{-\gamma_2-1} - \hat{p}_b = -\left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{\overline{R}}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \qquad (3.58a)$$

or

$$\left(\frac{\overline{R}}{a}\right)^{N-1} = \frac{N(1-2\nu)\hat{p}_b - (N-1)C_{r1}B_1\left(\frac{\overline{R}}{R}\right)^{\gamma_1-1} + (N-1)C_{r2}B_2\left(\frac{\overline{R}}{R}\right)^{-\gamma_2-1}}{(N-1)p_a + \sigma_u} \qquad (3.58b)$$

Using Equation 3.58b, $\overline{R}$ can be determined since we have previously solved for the ratio $\overline{R}/R$ from Equation 3.55 and the constants B_1 and B_2 from Equation 3.52. After solving for $\overline{R}$, R can be computed from the ratio $\overline{R}/R$ and A_1 and A_2 can be computed using Equation 3.52.

The plastic strain rates are given by the nonassociative flow rule.

$$\dot{\varepsilon}_r^{(p)} = \lambda_1 \frac{\partial g_1}{\partial \sigma_r} + \lambda_2 \frac{\partial g_2}{\partial \sigma_r} = -M\lambda_1 \qquad (3.59a)$$

$$\dot{\varepsilon}_\theta^{(p)} = \lambda_1 \frac{\partial g_1}{\partial \sigma_\theta} + \lambda_2 \frac{\partial g_2}{\partial \sigma_\theta} = \lambda_1 + \lambda_2 \qquad (3.59b)$$

$$\dot{\varepsilon}_z^{(p)} = \lambda_1 \frac{\partial g_1}{\partial \sigma_z} + \lambda_2 \frac{\partial g_2}{\partial \sigma_z} = -M\lambda_2 \qquad (3.59c)$$

so that

$$\left[\dot{\varepsilon}_r^{(p)} + M\dot{\varepsilon}_\theta^{(p)} + \dot{\varepsilon}_z^{(p)}\right] = 0 \qquad (3.60)$$

which, when integrated with respect to time, gives

$$\left[\varepsilon_r^{(p)} + M\varepsilon_\theta^{(p)} + \varepsilon_z^{(p)}\right] = h(r) \qquad (3.61)$$

We designate the plastic strains at r equals $\overline{R}$ as $\overline{\varepsilon}_r^{(p)}$, $\overline{\varepsilon}_\theta^{(p)}$, and $\overline{\varepsilon}_z^{(p)}$, so that

$$\left[\overline{\varepsilon}_r^{(p)} + M\overline{\varepsilon}_\theta^{(p)} + \overline{\varepsilon}_z^{(p)}\right] = h(\overline{R}) \qquad (3.62)$$

But in the outer plastic zone (and therefore at r equals $\overline{R}$), the plastic strain relations of Equations 3.35 hold so $h(\overline{R})$ must be zero. Since every point in the middle plastic zone will at some time be at $\overline{R}$, then $h(\overline{R})$ is equal to zero. Therefore, Equation 3.61 becomes

$$\left[\varepsilon_r^{(p)} + M\varepsilon_\theta^{(p)} + \varepsilon_z^{(p)}\right] = 0 \tag{3.63}$$

The plane strain condition, $\varepsilon_z = \varepsilon_z^{(e)} + \varepsilon_z^{(p)} = 0$, combined with Equation 3.63 gives

$$\varepsilon_r = -M\varepsilon_\theta + M\varepsilon_\theta^{(e)} + \varepsilon_r^{(e)} + \varepsilon_z^{(e)} \tag{3.64}$$

With σ_r equal to σ_z, and σ_θ eliminated via the yield condition of Equation 3.3a, Hooke's law can be reduced to

$$E\varepsilon_r^{(e)} = E\varepsilon_z^{(e)} = \left[1-(N+1)\nu\right]\sigma_r + \nu\sigma_u \tag{3.65}$$

$$E\varepsilon_\theta^{(e)} = (N-2\nu)\sigma_r - \sigma_u \tag{3.66}$$

where σ_r is given by Equation 3.57. Substitution of the elastic strains from Equations 3.65 and 3.66 into the radial strain expression of Equation 3.64, and using the resulting expression for ε_r in the compatibility Equation 3.22 leads to the following differential equation.

$$2G\frac{d}{dr}\left(r^{M+1}\varepsilon_\theta\right) = -\frac{MN+2-2(M+N+1)\nu}{(N-1)(1+\nu)}\left[(N-1)p_a + \sigma_u\right]\frac{r^{M+N-1}}{a^{N-1}} + \frac{(M+2)(1-2\nu)\sigma_u}{(N-1)(1+\nu)}r^M \tag{3.67}$$

This differential equation is directly integrable. Integration over the limits from r to $\overline{R}$ results in the following expression for ε_θ.

$$2G\varepsilon_\theta = 2G\varepsilon_\theta\left(\overline{R}\right)\left(\frac{\overline{R}}{r}\right)^{M+1} + \frac{MN+2-2(M+N+1)\nu}{(M+N)(N-1)(1+\nu)}\left[(N-1)p_a + \sigma_u\right]\left[\left(\frac{\overline{R}}{r}\right)^{M+N} - 1\right]\left(\frac{r}{a}\right)^{N-1} - \frac{(M+2)(1-2\nu)\sigma_u}{(N-1)(M+1)(1+\nu)}\left[\left(\frac{\overline{R}}{r}\right)^{M+1} - 1\right] \tag{3.68}$$

The circumferential strain $\varepsilon_\theta\left(\overline{R}\right)$ is obtained from Equation 3.44b. The radial strain is obtained from Equations 3.64 through 3.66 giving

$$2G\varepsilon_r = -2GM\varepsilon_\theta + \frac{MN+2-2(M+N+1)\nu}{1+\nu}\sigma_r - \frac{M-2\nu}{1+\nu}\sigma_u \tag{3.69}$$

For applied pressures greater than $\tilde{p}_b$, the stresses in the inner plastic zone are still given by Equation 3.10 for σ_r, Equation 3.8 for σ_θ, and Equation 3.19 for σ_z. The differential Equation 3.23 for the plastic strain $\varepsilon_\theta^{(p)}$ still holds with increased loading, but now must be integrated from r to $\tilde{R}$, instead of from r to R, which results in

$$2G\varepsilon_\theta^{(p)} = 2G\varepsilon_\theta^{(p)}\left(\tilde{R}\right)\left(\frac{\tilde{R}}{r}\right)^{M+1}$$

$$-\frac{(N+1)(1-\nu)}{M+N}\left[(N-1)p_a+\sigma_u\right]\left(\frac{\tilde{R}}{a}\right)^{N-1}\left[\left(\frac{\tilde{R}}{r}\right)^{M+1}-\left(\frac{r}{\tilde{R}}\right)^{N-1}\right] \tag{3.70}$$

The other plastic strain components are still given by Equations 3.15. In Equation 3.70, $\varepsilon_\theta^{(p)}\left(\tilde{R}\right)$ is found by setting r equal to $\tilde{R}$ in Equation 3.68 to obtain $\varepsilon_\theta\left(\tilde{R}\right)$; then setting r equal to $\tilde{R}$ in Equation 3.66, and using Equation 3.30 for $\sigma_r\left(\tilde{R}\right)$, to obtain $\varepsilon_\theta^{(e)}\left(\tilde{R}\right)$; and then using the strain decomposition given in Equation 3.16b.

The solution developed above breaks down when the external pressure, p_b, reaches the free-field yield pressure, $\hat{p}_b$. We demonstrate this as follows. From Equation 3.50, as p_b approaches $\hat{p}_b$, Δp_b approaches zero.

$$\lim_{p_b\to\hat{p}_b} \Delta p_b = \lim_{p_b\to\hat{p}_b} \left(\hat{p}_b - p_b\right) = 0 \tag{3.71}$$

Substituting Equation 3.71 into Equation 3.52 leads to

$$\lim_{p_b\to\hat{p}_b} A_1 R^{\gamma_1-1} = 0 \tag{3.72a}$$

$$\lim_{p_b\to\hat{p}_b} A_2 R^{-\gamma_2-1} = 0 \tag{3.72b}$$

It can be shown that both γ_1 and γ_2 are greater than zero. From Equation 3.72b, as p_b approaches $\hat{p}_b$, either A_2 goes to zero or R goes to infinity, or both. Physically, one can argue that as the free-field pressure approaches $\hat{p}_b$, the outer elastic zone vanishes, with the result that R must go to infinity.

As p_b approaches $\hat{p}_b$, the circumferential strain at r equals R is computed from Equation 3.48 as

$$\lim_{p_b\to\hat{p}_b} 2G\varepsilon_\theta(R) = -2(N-1)\nu\hat{p}_b - \sigma_u = A \tag{3.73}$$

where $A = -(1-2\nu)\hat{p}_b$ as defined in Equation (3.41).

From the strain-displacement relations, it follows that

$$\lim_{p_b \to \hat{p}_b} 2Gu(R) = AR \tag{3.74}$$

Equation 3.40 for radial displacement in the outer plastic zone, evaluated at R, gives

$$2Gu(R) = A_1 R^{\gamma_1} + A_2 R^{-\gamma_2} + AR \tag{3.75}$$

From Equations 3.74 and 3.75 it follows that as p_b approaches $\hat{p}_b$, then

$$\lim_{p_b \to \hat{p}_b} \left[A_1 R^{\gamma_1} + A_2 R^{-\gamma_2}\right] = 0 \tag{3.76}$$

Using Equation 3.72b, we showed that either A_2 goes to zero or R goes to infinity, or both. This implies that Equation 3.76 may be decomposed as follows

$$\lim_{p_b \to \hat{p}_b} \left[A_2 R^{-\gamma_2}\right] = 0 \tag{3.77a}$$

$$\lim_{p_b \to \hat{p}_b} \left[A_1 R^{\gamma_1}\right] = 0 \tag{3.77b}$$

Equation 3.77b implies that A_1 must be zero since γ_1 is greater than zero.

For p_b equal to $\hat{p}_b$ (and A_1 equals zero), Equation 3.45 can be rewritten as

$$\sigma_r = C_{r2} A_2 r^{-\gamma_2 - 1} - \hat{p}_b \tag{3.78a}$$

$$\sigma_\theta = C_{\theta 2} A_2 r^{-\gamma_2 - 1} - \hat{p}_b \tag{3.78b}$$

Again, we postulate that at the inner radius $\overline{R}$ of the outer plastic zone, $\sigma_r(\overline{R}) = \sigma_z(\overline{R})$, so Equation 3.53 still holds. Using the stress expressions in Equation 3.78 evaluated at $\overline{R}$ and substituted into Equation 3.53 leads to

$$(NC_{r2} - C_{\theta 2}) A_2 \overline{R}^{-\gamma_2 - 1} - (N-1)\hat{p}_b = \sigma_u \tag{3.79}$$

which can be solved for

$$B_2 = A_2 \overline{R}^{-\gamma_2 - 1} = \frac{N(1-2\nu)\hat{p}_b}{NC_{r2} - C_{\theta 2}} \tag{3.80}$$

Continuity of radial stress at the boundary with the middle plastic zone (r equals $\overline{R}$) using Equations 3.57, 3.78a and 3.80 gives

$$C_{r2} B_2 - \hat{p}_b = -\left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{\overline{R}}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \tag{3.81a}$$

or

$$\left(\frac{\overline{R}}{a}\right)^{N-1} = \frac{N(1-2\nu)\hat{p}_b - (N-1)C_{r2}B_2}{p_a(N-1) + \sigma_u} \tag{3.81b}$$

After solving for $\overline{R}$, A_2 can be recovered from Equation 3.80. The remainder of the solution developed for $p_b < \hat{p}_b$ still applies for p_b equal to $\hat{p}_b$.

3.3 CASE Ib, PHASE 4 SOLUTION.

The solution for Phase 4, which contains free-field yielding, is presented in this section. Note that for $N\nu > \frac{1}{2}$, the free field will not yield and therefore this phase of the solution is not applicable.

After the free field yields $(p_b > \hat{p}_b)$, we postulate that a far outer plastic zone $\left(\hat{R} < r < \infty\right)$ exists with conditions

$$\sigma_\theta = \sigma_r = -p_b \tag{3.82}$$

as shown in Figure 3-4. This assumption was based on the results of numerical studies using this same Mohr-Coulomb material model. The stress condition in this far outer plastic zone is $\sigma_r = \sigma_\theta < \sigma_z$, so the yield condition is given by

$$f_1 = \sigma_r - N\sigma_z + \sigma_u = 0 \tag{3.5a}$$

$$f_2 = \sigma_\theta - N\sigma_z + \sigma_u = 0 \tag{3.5b}$$

The expressions for stress in the outer plastic zone, discussed in Section (3.3) are repeated below

$$\sigma_r = C_{r1}A_1r^{\gamma_1-1} + C_{r2}A_2r^{-\gamma_2-1} - \hat{p}_b \tag{3.45a}$$

$$\sigma_\theta = C_{\theta1}A_1r^{\gamma_1-1} + C_{\theta2}A_2r^{-\gamma_2-1} - \hat{p}_b \tag{3.45b}$$

At this point our unknowns are A_1, A_2, $\overline{R}$, and $\hat{R}$. As discussed earlier, $\overline{R}$ is the radius to the boundary between the middle and outer plastic zones. At r equal to $\hat{R}$, it is reasonable to postulate that $\sigma_\theta\left(\hat{R}\right) = -p_b$ in the outer plastic zone. This condition, along with continuity of σ_r at $\hat{R}$, results in two equations for two unknowns, $A_1\hat{R}^{\gamma_1-1}$ and $A_2\hat{R}^{-\gamma_2-1}$,

$$\begin{bmatrix} C_{r1} & C_{r2} \\ C_{\theta1} & C_{\theta2} \end{bmatrix} \begin{Bmatrix} A_1\hat{R}^{\gamma_1-1} \\ A_2\hat{R}^{-\gamma_2-1} \end{Bmatrix} = \begin{Bmatrix} \Delta p_b \\ \Delta p_b \end{Bmatrix} \tag{3.83}$$

which can be inverted to give

$$\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} A_1 \hat{R}^{\gamma_1 - 1} \\ A_2 \hat{R}^{-\gamma_2 - 1} \end{Bmatrix} = \frac{1}{C_{r1}C_{\theta 2} - C_{r2}C_{\theta 1}} \begin{bmatrix} C_{\theta 2} & -C_{r2} \\ -C_{\theta 1} & C_{r1} \end{bmatrix} \begin{Bmatrix} \Delta p_b \\ \Delta p_b \end{Bmatrix} \tag{3.84}$$

where B_1 and B_2 are constants. After solving for B_1 and B_2, the stress expressions of Equation 3.45 can be written as

$$\sigma_r = C_{r1}B_1\left(\frac{r}{\hat{R}}\right)^{\gamma_1 - 1} + C_{r2}B_2\left(\frac{r}{\hat{R}}\right)^{-\gamma_2 - 1} - \hat{p}_b \tag{3.85a}$$

$$\sigma_\theta = C_{\theta 1}B_1\left(\frac{r}{\hat{R}}\right)^{\gamma_1 - 1} + C_{\theta 2}B_2\left(\frac{r}{\hat{R}}\right)^{-\gamma_2 - 1} - \hat{p}_b \tag{3.85b}$$

As discussed in Section 3.2, at the boundary between the middle plastic zone and outer plastic zone we have

$$N\sigma_r(\bar{R}) - \sigma_\theta(\bar{R}) = \sigma_u \tag{3.53}$$

Substitution of the stress expressions from Equations 3.85 into Equation 3.53 results in the following nonlinear equation

$$(NC_{r1} - C_{\theta 1})B_1\left(\frac{\bar{R}}{\hat{R}}\right)^{\gamma_1 - 1} + (NC_{r2} - C_{\theta 2})B_2\left(\frac{\bar{R}}{\hat{R}}\right)^{-\gamma_2 - 1} - N\hat{p}_b(1 - 2\nu) = 0 \tag{3.86}$$

which can be solved for the radius ratio $\bar{R}/\hat{R}$.

As developed in Section 3.3, the radial stress in the middle plastic zone is

$$\sigma_r = -\left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{r}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \tag{3.57}$$

Continuity of radial stress at the boundary $\bar{R}$ provides

$$C_{r1}B_1\left(\frac{\bar{R}}{\hat{R}}\right)^{\gamma_1 - 1} + C_{r2}B_2\left(\frac{\bar{R}}{\hat{R}}\right)^{-\gamma_2 - 1} - \hat{p}_b = -\left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{\bar{R}}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \tag{3.87a}$$

or

$$\left(\frac{\bar{R}}{a}\right)^{N-1} = \frac{N(1-2\nu)\hat{p}_b - (N-1)C_{r1}B_1\left(\frac{\bar{R}}{\hat{R}}\right)^{\gamma_1 - 1} - (N-1)C_{r2}B_2\left(\frac{\bar{R}}{\hat{R}}\right)^{-\gamma_2 - 1}}{(N-1)p_a + \sigma_u} \tag{3.87b}$$

from which we can solve for $\bar{R}$, since B_1, B_2, and $\bar{R}/\hat{R}$ have been determined previously. Once B_1, B_2, and $\bar{R}$ are determined, the solution presented in Section 3.2 is valid for the inner, middle, and outer plastic zones.

In the far outer plastic zone, Hooke's law can be rewritten using Equation 3.82 to obtain

$$E\varepsilon_r^{(e)} = -p_b(1-\nu) - \nu\sigma_z \tag{3.88a}$$

$$E\varepsilon_\theta^{(e)} = -p_b(1-\nu) - \nu\sigma_z \tag{3.88b}$$

$$E\varepsilon_z^{(e)} = \sigma_z - 2\nu p_b \tag{3.88c}$$

The plastic strain rates are obtained from the nonassociative flow rule.

$$\dot{\varepsilon}_r^{(p)} = \lambda_1 \frac{\partial g_1}{\partial \sigma_r} + \lambda_2 \frac{\partial g_2}{\partial \sigma_r} = \lambda_1 \tag{3.89a}$$

$$\dot{\varepsilon}_\theta^{(p)} = \lambda_1 \frac{\partial g_1}{\partial \sigma_\theta} + \lambda_2 \frac{\partial g_2}{\partial \sigma_\theta} = \lambda_2 \tag{3.89b}$$

$$\dot{\varepsilon}_z^{(p)} = \lambda_1 \frac{\partial g_1}{\partial \sigma_z} + \lambda_2 \frac{\partial g_2}{\partial \sigma_z} = -M(\lambda_1 + \lambda_2) \tag{3.89c}$$

which implies that

$$M\left(\dot{\varepsilon}_r^{(p)} + \dot{\varepsilon}_\theta^{(p)}\right) + \dot{\varepsilon}_z^{(p)} = 0 \tag{3.90}$$

which, when integrated with respect to time, gives

$$M\left(\varepsilon_r^{(p)} + \varepsilon_\theta^{(p)}\right) + \varepsilon_z^{(p)} = h(r) \tag{3.91}$$

However, the plastic strain relations of Equation 3.35 still hold in the outer plastic zone; hence at r equals $\hat{R}$, h(r) must be zero. Therefore, Equation 3.91 can be rewritten as

$$\varepsilon_z^{(p)} = -M\left(\varepsilon_r^{(p)} + \varepsilon_\theta^{(p)}\right) \tag{3.92}$$

From the yield condition of Equation 3.5, the out-of-plane stress is given by

$$\sigma_z = \frac{1}{N}(-p_b + \sigma_u) \tag{3.93}$$

From the plane strain condition, strain decomposition, Equations 3.92, 3.93 and 3.88c

$$E\varepsilon_z^{(e)} = -E\varepsilon_z^{(p)} = EM\left(\varepsilon_r^{(p)} + \varepsilon_\theta^{(p)}\right) = -\frac{p_b}{N} + \frac{\sigma_u}{N} + 2\nu p_b \tag{3.94}$$

Therefore, the sum of the radial and tangential plastic strains, $\varepsilon_r^{(p)} + \varepsilon_\theta^{(p)}$, is a constant throughout the far outer plastic zone, i.e., there is no spatial variation in that sum. We will denote that constant as $\hat{C}$.

$$\varepsilon_r^{(p)} + \varepsilon_\theta^{(p)} = \frac{1}{EM}\left\{-\frac{p_b}{N} + \frac{\sigma_u}{N} + 2\nu p_b\right\} = \frac{\sigma_u - (1-2N\nu)p_b}{2GMN(1+\nu)} = \hat{C} \tag{3.95}$$

The compatibility Equation 3.22, using strain decomposition, can be written as

$$r\left(\frac{d\varepsilon_\theta^{(e)}}{dr} + \frac{d\varepsilon_\theta^{(p)}}{dr}\right) + \varepsilon_\theta^{(e)} + \varepsilon_\theta^{(p)} - \varepsilon_r^{(e)} - \varepsilon_r^{(p)} = 0 \tag{3.96}$$

From Equations 3.88a and 3.93, $\frac{d\varepsilon_\theta^{(e)}}{dr}$ is equal to zero , and $\varepsilon_\theta^{(e)}$ is equal to $\varepsilon_r^{(e)}$, so the compatibility equation reduces to

$$r\frac{d\varepsilon_\theta^{(p)}}{dr} + \varepsilon_\theta^{(p)} - \varepsilon_r^{(p)} = 0 \tag{3.97}$$

Using Equation 3.95 to eliminate $\varepsilon_r^{(p)}$ leads to

$$r\frac{d\varepsilon_\theta^{(p)}}{dr} + 2\varepsilon_\theta^{(p)} - \hat{C} = 0 \tag{3.98}$$

which can be solved to give

$$\varepsilon_\theta^{(p)} = \frac{\hat{C}}{2} + A_3 r^{-2} \tag{3.99}$$

where A_3 is a constant. The postulation that $\sigma_\theta(\hat{R}) = -p_b$ in the outer plastic zone and the yield conditions of Equations 3.4 and 3.5 lead to all three normal-stress components being continuous across the plastic-plastic boundary at r = $\hat{R}$. From Hooke's law, this implies that $\varepsilon_\theta^{(e)}$ is also continuous at r = $\hat{R}$. Continuity of tangential strain at r = $\hat{R}$ then implies that $\varepsilon_\theta^{(p)}$ must also be continuous at r = $\hat{R}$. We invoke this condition to determine the constant A_3.

$$\varepsilon_\theta^{(p)}(\hat{R}) = \frac{\hat{C}}{2} + A_3\hat{R}^{-2} \tag{3.100a}$$

or

$$A_3 = \hat{R}^2\left\{\varepsilon_\theta^{(p)}(\hat{R}) - \frac{\hat{C}}{2}\right\} \tag{3.100b}$$

where $\varepsilon_\theta^{(p)}(\hat{R})$ is computed from the solution in the outer plastic zone (Section 3.2). Equation 3.44, evaluated at $\hat{R}$, gives the total strain $\varepsilon_\theta(\hat{R})$. Equations 3.45 and 3.4 can be used to compute $\sigma_r(\hat{R})$, $\sigma_\theta(\hat{R})$ and $\sigma_z(\hat{R})$. After computing the stresses, Hooke's law can be used to compute the elastic strain $\varepsilon_\theta^{(e)}(\hat{R})$. Finally, the plastic strain $\varepsilon_\theta^{(p)}(\hat{R})$ is computed by subtracting $\varepsilon_\theta^{(e)}(\hat{R})$ from $\varepsilon_\theta(\hat{R})$.

SECTION 4
CASE II SOLUTION

In this section the arbitrary dilatancy solution for Case II is extended to include the effects of free-field yielding. Just as in the Case I solution, the earlier solution (Wintergerst, 1991) is valid only if the material response in the free field is elastic. Again, yielding of the free-field occurs when

$$p_b \geq \frac{\sigma_u}{1-2N\nu} = \hat{p}_b \quad \text{when} \quad (1-2N\nu) > 0 \tag{4.1}$$

As with Case I, the earlier solution is valid for $p_b < \hat{p}_b$ but in the interest of completeness, the entire solution is presented herein.

As derived in Section 2.3, initial yielding for Case II begins at the edge of the hole (r equals a) with stress condition $\sigma_\theta < \sigma_r < \sigma_z$ and is governed by the failure surface

$$f = \sigma_\theta - N\sigma_z + \sigma_u = 0 \tag{4.2}$$

This initial plastic zone increases in size as loading continues until σ_r is equal to σ_z at the edge of the hole. At this point a second plastic zone begins to form. This second plastic zone also begins at the edge of the hole, and like the first plastic zone, increases in size as loading proceeds. The initial (now outer) plastic zone is still governed by Equation 4.2. The second (inner) plastic zone is governed by the following yield conditions

$$f_1 = \sigma_\theta - N\sigma_r + \sigma_u = 0 \tag{4.3a}$$

$$f_2 = \sigma_\theta - N\sigma_z + \sigma_u = 0 \tag{4.3b}$$

The boundary between the inner and outer plastic zones, denoted as $\overline{R}$, is defined by

$$\sigma_r(\overline{R}) = \sigma_z(\overline{R}) \tag{4.4}$$

Outside of the plastic zones, the material response was assumed in earlier solutions to be elastic. The earlier solutions can therefore be considered to consist of three phases (in order of occurrence):

<u>Phase 1</u>. Elastic solution.

<u>Phase 2</u>. Elastic-plastic solution with one plastic zone and an outer elastic zone.

<u>Phase 3</u>. Elastic-plastic solution with two plastic zones and an outer elastic zone.

In the extended solution, free-field yielding can occur either before or after the formation of the second plastic zone. However, it is assumed never to occur prior to the formation of the first plastic zone. This assumption is consistent with the assumption that the application of the internal pressure, p_a (with p_a equal to p_b) does not cause material yielding. The case where free-field yielding occurs after the formation of the second plastic zone will be denoted as Case IIa, while the case where free-field yielding occurs before the formation of the second plastic zone will be denoted as Case IIb. The demarcation between Cases IIa and IIb will be discussed in detail in Section 4.1.

When the free field yields, a far outer plastic zone is postulated to exist. The stress condition in this zone is $\sigma_r = \sigma_\theta < \sigma_z$ and the yield conditions are

$$f_1 = \sigma_r - N\sigma_z + \sigma_u = 0 \tag{4.5a}$$

$$f_2 = \sigma_\theta - N\sigma_z + \sigma_u = 0 \tag{4.5b}$$

This outer plastic zone occupies the domain $\hat{R} < r < \infty$, with the boundary between the initial plastic zone and the far outer plastic zone given by

$$\sigma_r\left(\hat{R}\right) = \sigma_\theta\left(\hat{R}\right) \tag{4.6}$$

Based on the progression of formation of the plastic zones, the solution for Case IIa can be considered to consist of the following four phases:

<u>Phase 1</u>. Elastic solution.

<u>Phase 2</u>. Elastic-plastic solution with one plastic zone and an outer elastic zone.

<u>Phase 3</u>. Elastic-plastic solution with two plastic zones and an outer elastic zone.

<u>Phase 4</u>. Plastic solution with three plastic zones. The elastic zone is eliminated due to free-field yielding (i.e., it is replaced by a far outer plastic zone).

For Case IIb, the solution process consists of the following four phases:

<u>Phase 1</u>. Elastic solution.

<u>Phase 2</u>. Elastic-plastic solution with one plastic zone and an outer elastic zone.

Phase 3. Plastic solution with two plastic zones. The outer elastic zone of Phase 2 is eliminated due to free-field yielding (i.e., it is replaced by a far outer plastic zone).

Phase 4. Plastic solution with three plastic zones.

Phases 1, 2, and 4 are each identical for Cases IIa and IIb; only the conditions in Phase 3 differ. The conditions for each of the phases are depicted in Figures 4-1 through 4-5.

The elastic solution (Phase 1 for both Cases IIa and IIb) was presented in Section 2.3. The Phase 2 solution is identical for Cases IIa and IIb, and is presented in Section 4.1. The Phase 3 solution for Case IIa is presented in Section 4.2. The Phase 3 solution for Case IIb is presented in Section 4.3. Finally, the Phase 4 solution (identical for Cases IIa and IIb) is presented in Section 4.4.

4.1 CASE IIa AND IIb, PHASE 2 SOLUTION.

In this section, the solution is presented for Phase 2 (initial yielding with one plastic zone and an outer elastic zone as shown in Figure 4-2). Following that, the conditions are derived for the formation of the second plastic zone. As will be shown, these conditions also define Cases IIa and IIb. Finally, the solution for when p_b is equal to $\hat{p}_b$ for Case IIb is presented. The solution presented closely follows both those of (Wintergerst, 1991) and (Florence, undated), but with slightly different notation.

Initial yielding of the material occurs at the edge of the hole when p_b is equal to $\overline{p}_b$ (determined below). As discussed earlier, for Case II conditions, $\sigma_\theta < \sigma_r < \sigma_z$ at the edge of the hole, and therefore the yield condition is

$$f = \sigma_\theta - N\sigma_z + \sigma_u = 0 \tag{4.2}$$

Substitution of the elastic solution for σ_r and σ_z (presented in Section 2.3) evaluated at r = a into the yield condition of Equation 4.2 leads to the following expression for the initial yield pressure, $\overline{p}_b$:

$$\overline{p}_b = \frac{p_a + \sigma_u}{2(1 - N\nu)} \tag{4.7}$$

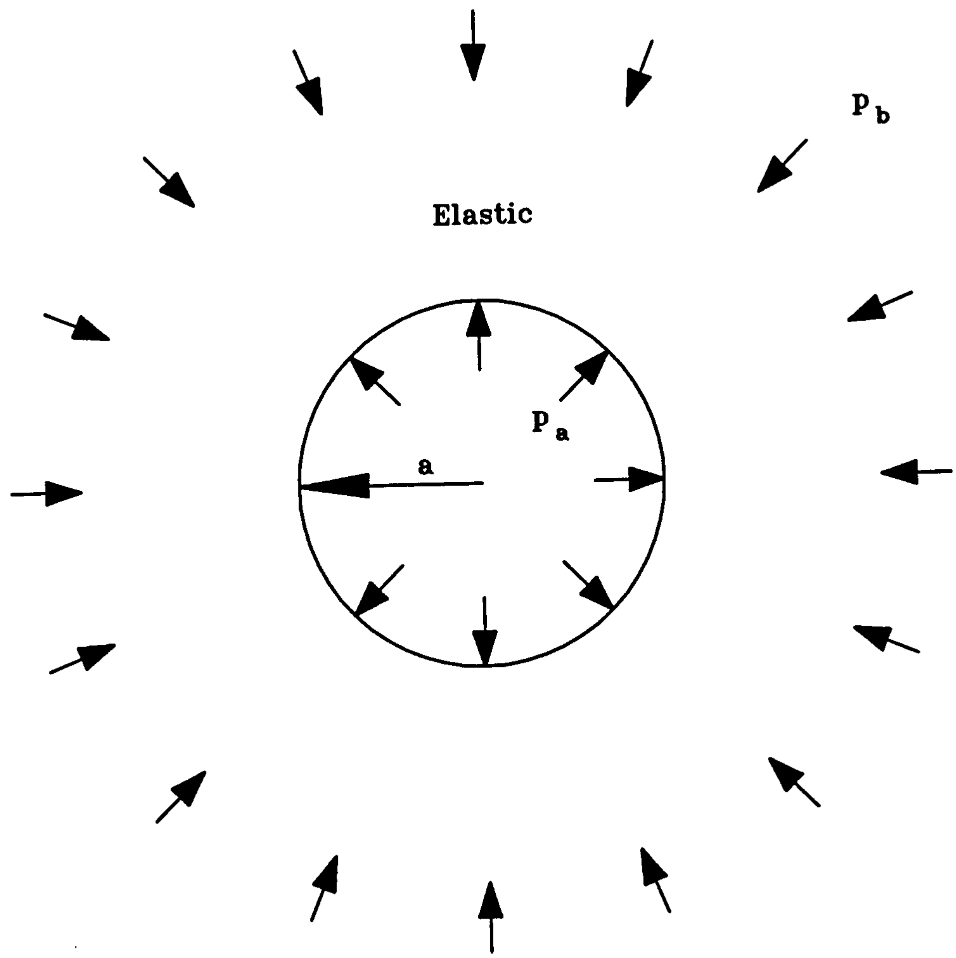


Figure 4-1. Phase 1 conditions for Case II (a and b).

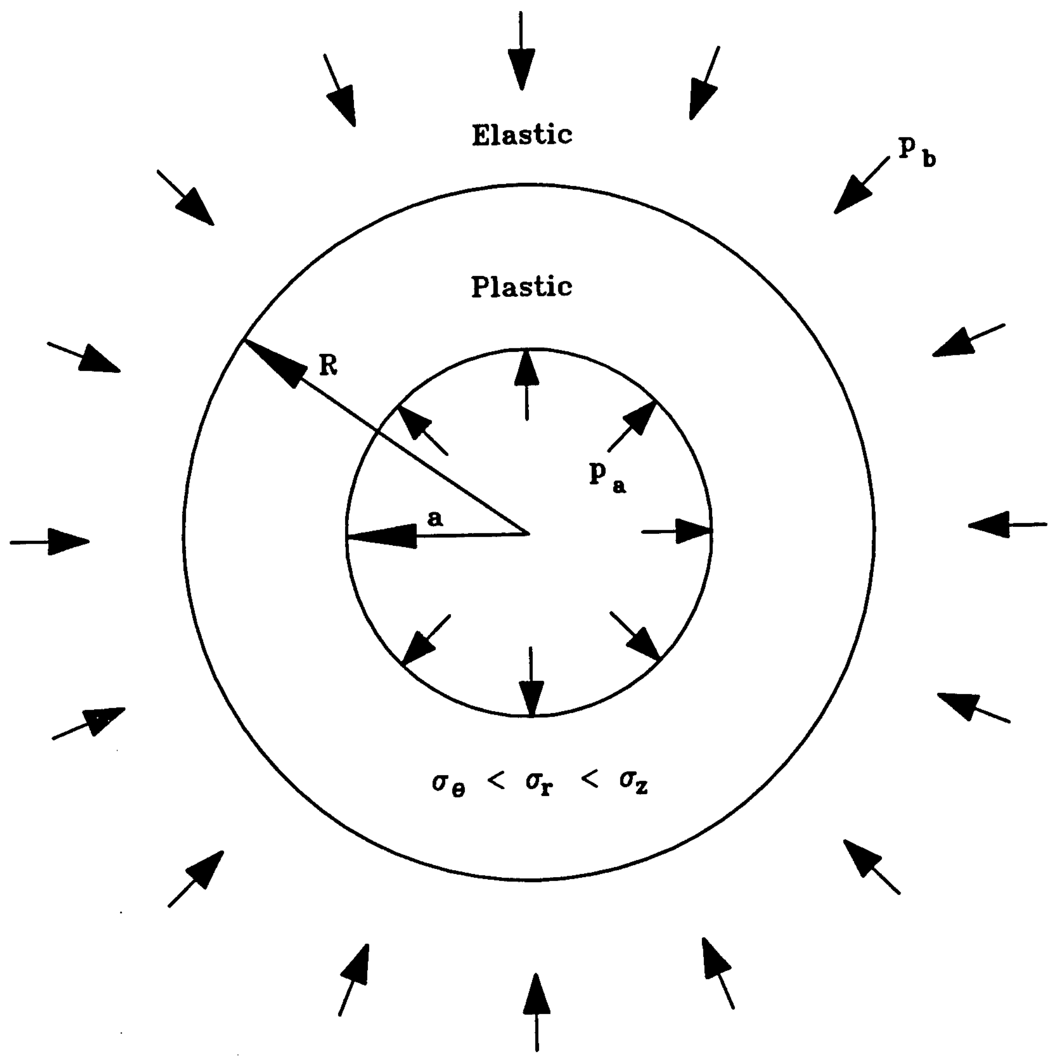


Figure 4-2. Phase 2 conditions for Case II (a and b).

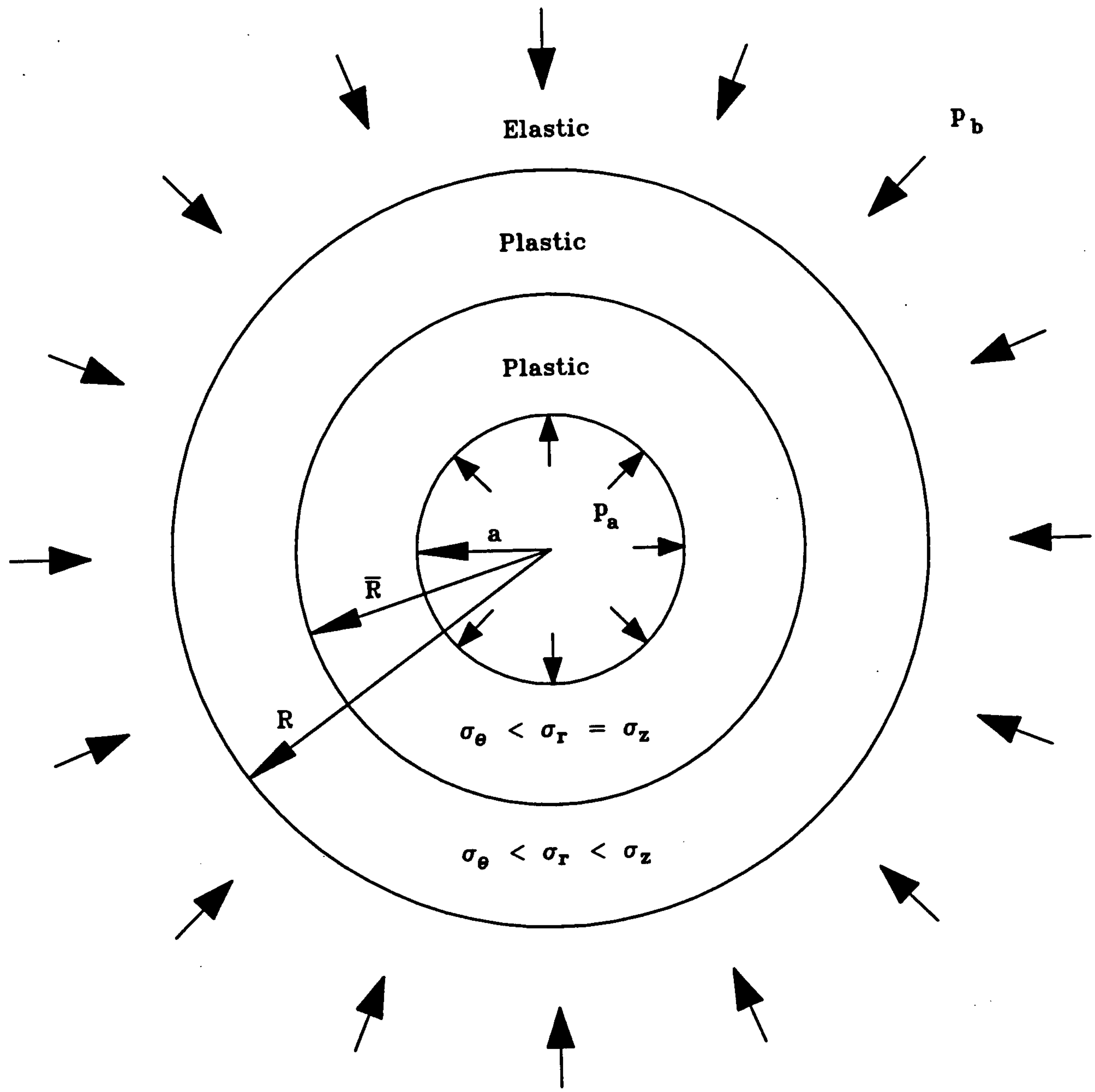


Figure 4-3. Phase 3 conditions for Case IIa.

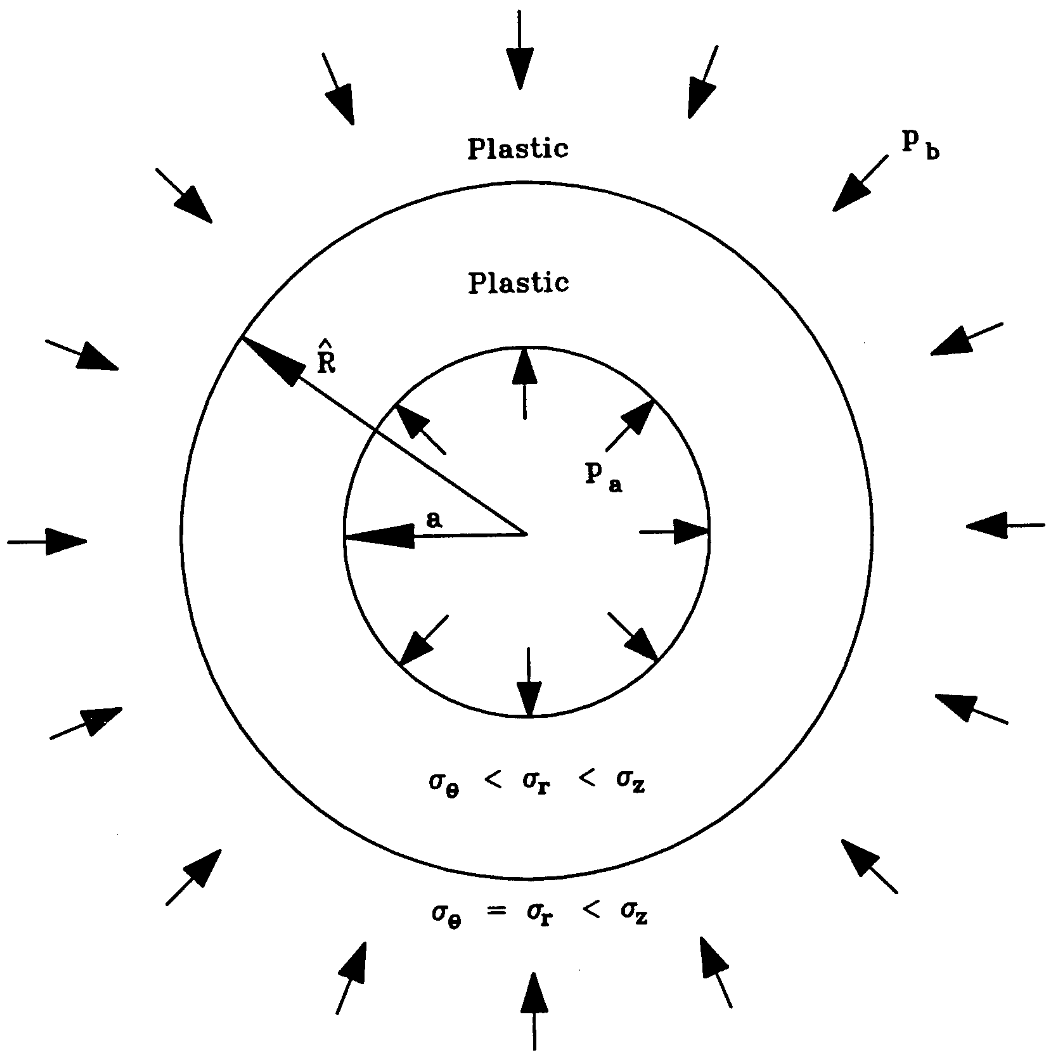


Figure 4-4. Phase 3 conditions for Case IIb.

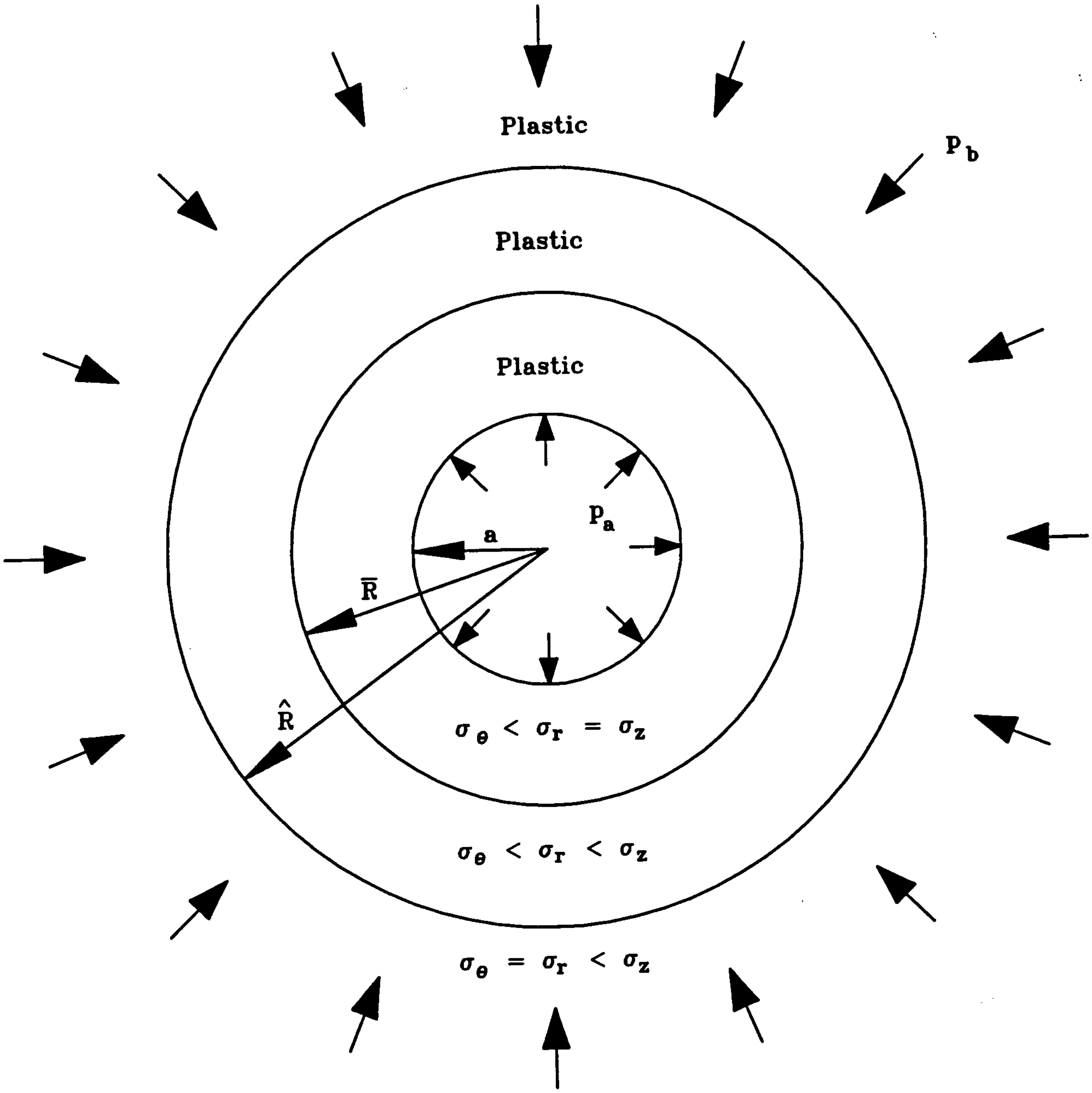


Figure 4-5. Phase 4 conditions for Case II(a and b).

The plastic strain rates are given by the nonassociative flow rule (discussed in Section 2.1)

$$\dot{\varepsilon}_r^{(p)} = \lambda \frac{\partial g}{\partial \sigma_r} = 0 \tag{4.8a}$$

$$\dot{\varepsilon}_\theta^{(p)} = \lambda \frac{\partial g}{\partial \sigma_\theta} = \lambda \tag{4.8b}$$

$$\dot{\varepsilon}_z^{(p)} = \lambda \frac{\partial g}{\partial \sigma_z} = -M\lambda \tag{4.8c}$$

where

$$g = \sigma_\theta - M\sigma_z + \kappa \tag{4.8d}$$

and λ is the plastic multiplier. As discussed in Section 3.1, the overdots in Equations 4.8 may be interpreted as either differentiation with respect to time or differentiation with respect to p_b since monotonic loading is assumed. From the flow rule of Equations 4.8 it follows that

$$\dot{\varepsilon}_z^{(p)} + M\dot{\varepsilon}_\theta^{(p)} = 0 \tag{4.9a}$$

$$\dot{\varepsilon}_r^{(p)} = 0 \tag{4.9b}$$

which, when integrated with respect to time, give

$$\varepsilon_z^{(p)} + M\varepsilon_\theta^{(p)} = h_1(r) \tag{4.10a}$$

$$\varepsilon_r^{(p)} = h_2(r) \tag{4.10b}$$

The strain expressions in Equations 4.10 evaluated at a fixed radius r remain constant as loading proceeds (i.e., they are independent of load p_b). When the elastic-plastic radius, R, is initially at this radius r, the plastic strains are zero, which implies that $h_1(r) = 0$ and $h_2(r) = 0$, and Equations 4.10 reduce to

$$\varepsilon_z^{(p)} + M\varepsilon_\theta^{(p)} = 0 \tag{4.11a}$$

$$\varepsilon_r^{(p)} = 0 \tag{4.11b}$$

The strain rates are decomposed into elastic and plastic parts:

$$\dot{\varepsilon}_r = \dot{\varepsilon}_r^{(e)} + \dot{\varepsilon}_r^{(p)} \tag{4.12a}$$

$$\dot{\varepsilon}_\theta = \dot{\varepsilon}_\theta^{(e)} + \dot{\varepsilon}_\theta^{(p)} \tag{4.12b}$$

$$\dot{\varepsilon}_z = \dot{\varepsilon}_z^{(e)} + \dot{\varepsilon}_z^{(p)} \tag{4.12c}$$

Integrating Equation 4.12a with respect to time (or p_b) gives $\varepsilon_r - \left(\varepsilon_r^{(e)} + \varepsilon_r^{(p)}\right) = h(r)$, but when r equals R, the strain is entirely elastic so $\varepsilon_r = \varepsilon_r^{(e)}$ and $\varepsilon_r^{(p)} = 0$ giving $h(r) = 0$. Using the same argument for the other strain components leads to

$$\varepsilon_r = \varepsilon_r^{(e)} + \varepsilon_r^{(p)} \tag{4.13a}$$

$$\varepsilon_\theta = \varepsilon_\theta^{(e)} + \varepsilon_\theta^{(p)} \tag{4.13b}$$

$$\varepsilon_z = \varepsilon_z^{(e)} + \varepsilon_z^{(p)} \tag{4.13c}$$

The plane strain condition dictates that $\varepsilon_z = 0$, so using Equations 4.11 and 4.13, the plastic and total strains are given by

$$\varepsilon_r^{(p)} = 0 \tag{4.14a}$$

$$M\varepsilon_\theta^{(p)} = \varepsilon_z^{(e)} \tag{4.14b}$$

$$\varepsilon_z^{(p)} = -\varepsilon_z^{(e)} \tag{4.14c}$$

and

$$\varepsilon_r = \varepsilon_r^{(e)} \tag{4.15a}$$

$$\varepsilon_\theta = \varepsilon_\theta^{(e)} + \frac{1}{M}\varepsilon_z^{(e)} \tag{4.15b}$$

$$\varepsilon_z = 0 \tag{4.15c}$$

Hooke's law relating the elastic strains to the stresses is given by

$$E\varepsilon_r^{(e)} = \sigma_r - \nu\left(\sigma_\theta + \sigma_z\right) \tag{4.16a}$$

$$E\varepsilon_\theta^{(e)} = \sigma_\theta - \nu\left(\sigma_z + \sigma_r\right) \tag{4.16b}$$

$$E\varepsilon_z^{(e)} = \sigma_z - \nu\left(\sigma_r + \sigma_\theta\right) \tag{4.16c}$$

Eliminating σ_z from Equations 4.16 via the yield condition, and substitution of the resulting elastic strains into Equations 4.15 gives the following expressions for the radial and circumferential stress components in terms of the radial and circumferential strain components.

$$\left[MN + 1 - (M+1)(N+1)\nu\right]\sigma_r = \left[MN + 1 - (M+N)\nu\right]2G\varepsilon_r + M(N+1)\nu 2G\varepsilon_\theta + (M-1)\nu\sigma_u \tag{4.17a}$$

$$[MN+1-(M+1)(N+1)\nu]\sigma_\theta = N(M+1)\nu 2G\varepsilon_r + MN2G\varepsilon_\theta - [1-(M+1)\nu]\sigma_u \tag{4.17b}$$

As discussed in Section 3, the equation of equilibrium is given by

$$r\frac{d\sigma_r}{dr} + \sigma_r - \sigma_\theta = 0 \tag{4.18}$$

and the strain-displacement equations are

$$\varepsilon_r = \frac{du}{dr} \tag{4.19a}$$

$$\varepsilon_\theta = \frac{u}{r} \tag{4.19b}$$

Substitution of the stress Equations 4.17 into the equilibrium Equation 4.18 and using the strain-displacement Equations 4.19, leads to the following differential equation in terms of the radial displacement u,

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr}\left[1+\frac{(M-N)\nu}{MN}\beta^2\right] - \beta^2 u = -\frac{\beta^2}{MN}\cdot\frac{(1-2\nu)\sigma_u}{2G}r \tag{4.20}$$

where

$$\beta^2 = \frac{MN}{MN+1-(M+N)\nu} \tag{4.21}$$

The solution to Equation 4.20 is

$$2Gu = A_1 r^{\gamma_1} + A_2 r^{-\gamma_2} + Ar \tag{4.22}$$

where

$$A = -\frac{(1-2\nu)}{1-2N\nu}\sigma_u = -(1-2\nu)\hat{p}_b \tag{4.23}$$

$$\gamma_1 = \beta^2\frac{(N-M)\nu}{2MN} + \beta^2\sqrt{\frac{(N-M)^2\nu^2}{(2MN)^2} + \frac{1}{\beta^2}} \tag{4.24}$$

$$\gamma_2 = \beta^2\frac{(M-N)\nu}{2MN} + \beta^2\sqrt{\frac{(M-N)^2\nu^2}{(2MN)^2} + \frac{1}{\beta^2}} \tag{4.25}$$

The strains can be computed from the displacement Equation 4.22 and the strain-displacement Equations 4.19

$$2G\varepsilon_r = \gamma_1 A_1 r^{\gamma_1 - 1} - \gamma_2 A_2 r^{-\gamma_2 - 1} + A \tag{4.26a}$$

$$2G\varepsilon_\theta = A_1 r^{\gamma_1 - 1} + A_2 r^{-\gamma_2 - 1} + A \tag{4.26b}$$

Substituting the strain Equations 4.26 into the stress Equations 4.17 yields

$$\sigma_r = C_{r1} A_1 r^{\gamma_1 - 1} + C_{r2} A_2 r^{-\gamma_2 - 1} - \hat{p}_b \tag{4.27a}$$

$$\sigma_\theta = C_{\theta 1} A_1 r^{\gamma_1 - 1} + C_{\theta 2} A_2 r^{-\gamma_2 - 1} - \hat{p}_b \tag{4.27b}$$

where

$$C_{r1} = \frac{1}{C}\left\{\left[MN + 1 - (M+N)\nu\right]\gamma_1 + M(N+1)\nu\right\} \tag{4.28a}$$

$$C_{r2} = \frac{1}{C}\left\{-\left[MN + 1 - (M+N)\nu\right]\gamma_2 + M(N+1)\nu\right\} \tag{4.28b}$$

$$C_{\theta 1} = \frac{1}{C}\left[N(M+1)\nu\gamma_1 + MN\right] \tag{4.28c}$$

$$C_{\theta 2} = \frac{1}{C}\left[-N(M+1)\nu\gamma_2 + MN\right] \tag{4.28d}$$

$$C = MN + 1 - (M+1)(N+1)\nu \tag{4.28e}$$

At this point, the unknowns are the two constants, A_1 and A_2, and the elastic-plastic interface radius R. Two of the conditions used to evaluate the unknowns are provided by enforcing continuity of radial stress and radial displacement (or, equivalently, circumferential strain) at R. The elastic zone stresses are given in the elastic solution presented in Section 2.3, with the substitutions a = R and $p_a = -\sigma_r(R)$. Applying the yield condition of Equation 4.2 at r equals R gives

$$\sigma_r(R) = -2(1 - N\nu)p_b + \sigma_u \tag{4.29}$$

The circumferential strain at r equals R is given by

$$2G\varepsilon_\theta(R) = -2(N-1)\nu p_b - \sigma_u \tag{4.30}$$

Using Equations 4.26, 4.27, 4.29 and 4.30, the continuity conditions are then given by

$$C_{r1} A_1 R^{\gamma_1 - 1} + C_{r2} A_2 R^{-\gamma_2 - 1} = 2(1 - N\nu)\Delta p_b \tag{4.31a}$$

$$A_1 R^{\gamma_1 - 1} + A_2 R^{\gamma_2 - 1} = 2(N-1)\nu \Delta p_b \tag{4.31b}$$

where

$$\Delta p_b = \hat{p}_b - p_b \tag{4.32}$$

Equations 4.31 constitute a system of two equations in terms of two unknowns $A_1R^{\gamma_1-1}$ and $A_2R^{-\gamma_2-1}$.

$$\begin{bmatrix} C_{r1} & C_{r2} \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} A_1R^{\gamma_1-1} \\ A_2R^{-\gamma_2-1} \end{Bmatrix} = \begin{Bmatrix} 2(1-N\nu)\Delta p_b \\ 2(N-1)\nu\Delta p_b \end{Bmatrix} \tag{4.33}$$

which can be inverted to give

$$\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} A_1R^{\gamma_1-1} \\ A_2R^{-\gamma_2-1} \end{Bmatrix} = \frac{1}{C_{r1}-C_{r2}} \begin{bmatrix} 1 & -C_{r2} \\ -1 & C_{r1} \end{bmatrix} \begin{Bmatrix} 2(1-N\nu)\Delta p_b \\ 2(N-1)\nu\Delta p_b \end{Bmatrix} \tag{4.34}$$

where B_1 and B_2 are constants.

The elastic-plastic interface radius R is found by enforcing the pressure boundary condition at the edge of the hole

$$\sigma_r(a) = -p_a \tag{4.35}$$

which leads to the nonlinear equation

$$C_{r1}B_1\left(\frac{a}{R}\right)^{\gamma_1-1} + C_{r2}B_2\left(\frac{a}{R}\right)^{-\gamma_2-1} - \hat{p}_b = -p_a \tag{4.36}$$

which can be solved for the radius ratio (a/R), which in turn gives R. Once R is determined, the constants A_1 and A_2 can be recovered from B_1 and B_2, respectively.

The solution presented above is valid with continued loading as long as $\sigma_r < \sigma_z$ in the plastic zone. Florence and Schwer performed numerical studies of the stress difference $\sigma_z - \sigma_r$ and found that $\sigma_z - \sigma_r$ decreases as p_b increases, and becomes zero first at the edge of the hole. We now derive the applied pressure, p_b, and elastic-plastic radius, R, associated with zero stress difference at the edge of the hole.

Substituting Equation 4.34 into Equations 4.27 gives

$$\sigma_r = C_{r1}B_1\left(\frac{r}{R}\right)^{\gamma_1-1} + C_{r2}B_2\left(\frac{r}{R}\right)^{\gamma_2-1} - \hat{p}_b \tag{4.37a}$$

$$\sigma_\theta = C_{\theta1}B_1\left(\frac{r}{R}\right)^{\gamma_1-1} + C_{\theta2}B_2\left(\frac{r}{R}\right)^{-\gamma_2-1} - \hat{p}_b \tag{4.37b}$$

The out-of-plane stress, σ_z, can be determined from the yield condition and Equations 4.37.

$$\sigma_z = \frac{C_{\theta 1}}{N} B_1 \left(\frac{r}{R}\right)^{\gamma_1 - 1} + \frac{C_{\theta 2}}{N} B_2 \left(\frac{r}{R}\right)^{-\gamma_2 - 1} - 2\nu \hat{p}_b \tag{4.38}$$

We now introduce two new constants, $\hat{B}_1$ and $\hat{B}_2$, defined by

$$B_1 = \hat{B}_1 \Delta p_b \tag{4.39a}$$

$$B_2 = \hat{B}_2 \Delta p_b \tag{4.39b}$$

or from Equation 4.34

$$\begin{Bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{Bmatrix} = \frac{1}{C_{r1} - C_{r2}} \begin{bmatrix} 1 & -C_{r2} \\ -1 & C_{r1} \end{bmatrix} \begin{Bmatrix} 2(1 - N\nu) \\ 2(N-1)\nu \end{Bmatrix} \tag{4.40}$$

Substituting Equations 4.39 into Equations 4.37a and 4.38 gives

$$\sigma_r = C_{r1} \hat{B}_1 \Delta p_b \left(\frac{r}{R}\right)^{\gamma_1 - 1} + C_{r2} \hat{B}_2 \Delta p_b \left(\frac{r}{R}\right)^{-\gamma_2 - 1} - \hat{p}_b \tag{4.41a}$$

$$\sigma_z = \frac{C_{\theta 1}}{N} \hat{B}_1 \Delta p_b \left(\frac{r}{R}\right)^{\gamma_1 - 1} + \frac{C_{\theta 1}}{N} \hat{B}_2 \Delta p_b \left(\frac{r}{R}\right)^{-\gamma_2 - 1} - 2\nu \hat{p}_b \tag{4.41b}$$

We denote as p_b' and R', the applied pressure and radius to the elastic-plastic interface, respectively, when $\sigma_r(a) = \sigma_z(a)$. Using Equations 4.41, and setting $\sigma_r(a) = \sigma_z(a) = -p_a$ gives

$$C_{r1} \hat{B}_1 \Delta p_b' \left(\frac{a}{R'}\right)^{\gamma_1 - 1} + C_{r2} \hat{B}_2 \Delta p_b' \left(\frac{a}{R'}\right)^{-\gamma_2 - 1} = \hat{p}_b - p_a \tag{4.42a}$$

$$\frac{C_{\theta 1}}{N} \hat{B}_1 \Delta p_b' \left(\frac{a}{R'}\right)^{\gamma_1 - 1} + \frac{C_{\theta 2}}{N} \hat{B}_2 \Delta p_b' \left(\frac{a}{R'}\right)^{-\gamma_2 - 1} = 2\nu \hat{p}_b - p_a \tag{4.42b}$$

where

$$\Delta p_b' = \hat{p}_b - p_b' \tag{4.42c}$$

Equations 4.42 constitute a system of two equations in terms of unknowns $\Delta p_b' \left(\frac{a}{R'}\right)^{\gamma_1 - 1}$ and $\Delta p_b' \left(\frac{a}{R'}\right)^{-\gamma_2 - 1}$ which can be inverted to give

$$\begin{Bmatrix} \Delta p_b' \left(\frac{a}{R'}\right)^{\gamma_1 - 1} \\ \Delta p_b' \left(\frac{a}{R'}\right)^{-\gamma_2 - 1} \end{Bmatrix} = \frac{1}{D} \begin{bmatrix} C_{\theta 2} \hat{B}_2 & -N C_{r2} \hat{B}_2 \\ -C_{\theta 1} \hat{B}_1 & N C_{r1} \hat{B}_1 \end{bmatrix} \begin{Bmatrix} \hat{p}_b - p_a \\ 2\nu \hat{p}_b - p_a \end{Bmatrix} \tag{4.43a}$$

where

$$D = \hat{B}_1 \hat{B}_2 (C_{r1} C_{\theta 2} - C_{r2} C_{\theta 1}) \tag{4.43b}$$

Equation 4.43a can be rearranged to give

$$\frac{\Delta p_b'\left(\frac{a}{R'}\right)^{-\gamma_2-1}}{\Delta p_b'\left(\frac{a}{R'}\right)^{\gamma_1-1}} = -\frac{\hat{B}_1\left[C_{\theta 1}(\hat{p}_b - p_a) - NC_{r1}(2\nu\hat{p}_b - p_a)\right]}{\hat{B}_2\left[C_{\theta 2}(\hat{p}_b - p_a) - NC_{r2}(2\nu\hat{p}_b - p_a)\right]} \tag{4.44a}$$

or

$$\left(\frac{R'}{a}\right)^{\gamma_1+\gamma_2} = -\frac{\hat{B}_1\left[C_{\theta 1}(\hat{p}_b - p_a) - NC_{r1}(2\nu\hat{p}_b - p_a)\right]}{\hat{B}_2\left[C_{\theta 2}(\hat{p}_b - p_a) - NC_{r2}(2\nu\hat{p}_b - p_a)\right]} \tag{4.44b}$$

Once $\frac{R'}{a}$ is determined, $\Delta p_b'$ can be obtained from Equation 4.43a.

$$\Delta p_b' = \frac{(C_{\theta 2} - 2\nu NC_{r2})\hat{p}_b - (C_{\theta 2} - NC_{r2})p_a}{\hat{B}_1(C_{r1}C_{\theta 2} - C_{r2}C_{\theta 1})}\left(\frac{R'}{a}\right)^{\gamma_1-1} \tag{4.45}$$

Using Equation 4.42c, p_b' can be determined from $\Delta p_b'$.

When the free field yields, R' goes to infinity implying that the denominator of the right side of Equation 4.44b must approach zero. Note that for $N\nu < \frac{1}{2}$, the free field will not yield. Numerical studies indicated that the term $\hat{B}_2$ is nonzero for Case II material parameters. This implies that the term in the denominator of the right side of Equation 4.44b which can go to zero (for $N\nu < \frac{1}{2}$) is

$$C_{\theta 2}(\hat{p}_b - p_a) - NC_{r2}(2\nu\hat{p}_b - p_a) = 0 \tag{4.46}$$

Further numerical studies indicated that when

$$p_a < \hat{p}_b \frac{2N\nu C_{r2} - C_{\theta 2}}{NC_{r2} - C_{\theta 2}} = \hat{p}_b^* \tag{4.47}$$

a solution for R' using Equation 4.44b is possible; thus yielding of the free field occurs after the formation of the second plastic zone. Therefore, the condition $p_a < \hat{p}_b^*$ defines Case IIa. Conversely, when $p_a \geq \hat{p}_b^*$, a solution for R' using Equation 4.44b is not possible, indicating that yielding occurs prior to the formation of the second plastic zone. Therefore, the condition $p_a \geq \hat{p}_b^*$ defines Case IIb.

For Case IIb, the solution presented above can be modified to include the case of p_b equal to $\hat{p}_b$. Arguments similar to those presented in Section 3.4 may be used to show that as p_b approaches $\hat{p}_b$, the constant A_1 in Equations 4.22, 4.26, and 4.27 must go to zero and that R must approach infinity.

Thus when p_b equals $\hat{p}_b$, the radial stress is given by

$$\sigma_r = C_{r2}A_2 r^{-\gamma_2 - 1} - \hat{p}_b \tag{4.48}$$

The constant A_2 can be determined from the pressure boundary condition at the edge of the hole, $\sigma_r(a) = -p_a$

$$C_{r2}A_2 a^{-\gamma_2 - 1} - \hat{p}_b = -p_a \tag{4.49a}$$

or

$$A_2 = \frac{\hat{p}_b - p_a}{C_{r2}a^{-\gamma_2 - 1}} \tag{4.49b}$$

The remainder of the solution presented above for the plastic zone is valid for p_b equals $\hat{p}_b$.

4.2 CASE IIa, PHASE 3 SOLUTION.

In this section, we present the solution for Phase 3 of Case IIa which contains two plastic zones and an outer elastic zone (Figure 4-3). We also examine the case where p_b is exactly equal to $\hat{p}_b$ and the elastic-plastic boundary goes to infinity.

Florence and Schwer postulated that with increased loading $(p_b > p_b')$ a radius $\overline{R} > a$ exists where

$$\sigma_r(\overline{R}) - \sigma_z(\overline{R}) = 0 \tag{4.50}$$

In the outer plastic zone, we still have $\sigma_\theta < \sigma_r < \sigma_z$, the same condition as in the Phase 2 solution presented in Section 4.1. In fact, the stress and strain fields developed in Section 4.1 still apply. For convenience, they are repeated below (Equations 4.26 and 4.27).

$$2G\varepsilon_r = \gamma_1 A_1 r^{\gamma_1 - 1} - \gamma_2 A_2 r^{-\gamma_2 - 1} + A \tag{4.26a}$$

$$2G\varepsilon_\theta = A_1 r^{\gamma_1 - 1} + A_2 r^{-\gamma_2 - 1} + A \tag{4.26b}$$

$$\sigma_r = C_{r1}A_1 r^{\gamma_1 - 1} + C_{r2}A_2 r^{-\gamma_2 - 1} - \hat{p}_b \tag{4.27a}$$

$$\sigma_\theta = C_{\theta 1}A_1 r^{\gamma_1 - 1} + C_{\theta 2}A_2 r^{-\gamma_2 - 1} - \hat{p}_b \tag{4.27b}$$

The continuity conditions at the elastic-plastic interface R developed in Section 4.1 still apply. These conditions led to

$$\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} A_1 R^{\gamma_1 - 1} \\ A_2 R^{-\gamma_2 - 1} \end{Bmatrix} = \frac{1}{C_{r1} - C_{r2}} \begin{bmatrix} 1 & -C_{r2} \\ -1 & C_{r1} \end{bmatrix} \begin{Bmatrix} 2(1-N\nu)\Delta p_b \\ 2(N-1)\nu\Delta p_b \end{Bmatrix} \tag{4.34}$$

However, the elastic-plastic interface R is no longer found by enforcing the pressure boundary condition at the edge of the hole (Equations 4.35 and 4.36). Instead, we use the condition of Equation 4.50. The out-of-plane stress σ_z is determined (in terms of σ_θ and σ_u) from the yield condition of Equation 4.2, and Equation 4.27b is substituted for σ_θ to yield the following nonlinear equation

$$\left(C_{r1} - \frac{C_{\theta 1}}{N}\right) B_1 \left(\frac{\overline{R}}{R}\right)^{\gamma_1 - 1} + \left(C_{r2} - \frac{C_{\theta 2}}{N}\right) B_2 \left(\frac{\overline{R}}{R}\right)^{-\gamma_2 - 1} - (1-2\nu)\hat{p}_b = 0 \tag{4.51}$$

which can be solved for the radius ratio $\left(\overline{R}/R\right)$. The solution in the inner plastic zone is required to define the individual values of $\overline{R}$ and R.

In the inner plastic zone, Florence and Schwer postulated that the stress condition is $\sigma_\theta < \sigma_r = \sigma_z$ so that yielding is governed by

$$f_1 = \sigma_\theta - N\sigma_r + \sigma_u = 0 \tag{4.3a}$$

$$f_2 = \sigma_\theta - N\sigma_z + \sigma_u = 0 \tag{4.3b}$$

From Equation 4.3a,

$$\sigma_\theta = N\sigma_r - \sigma_u \tag{4.52}$$

so the equilibrium equation (Equation 4.18) can be written as

$$r\frac{d\sigma_r}{dr} - (N-1)\sigma_r + \sigma_u = 0 \tag{4.53}$$

The solution to Equation 4.53, with $\sigma_r(a) = -p_a$, is given by

$$\sigma_r = -\left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{r}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \tag{4.54}$$

Continuity of the radial stress at the plastic-plastic interface (r equal to $\overline{R}$) leads to

$$C_{r1}B_1\left(\frac{\overline{R}}{R}\right)^{\gamma_1 - 1} + C_{r2}B_2\left(\frac{\overline{R}}{R}\right)^{-\gamma_2 - 1} - \hat{p}_b = -\left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{\overline{R}}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \tag{4.55a}$$

or

$$\left(\frac{\overline{R}}{a}\right)^{N-1} = \frac{N(1-2\nu)\hat{p}_b - (N-1)C_{r1}B_1\left(\frac{\overline{R}}{R}\right)^{\gamma_1-1} - (N-1)C_{r2}B_2\left(\frac{\overline{R}}{R}\right)^{-\gamma_2-1}}{(N-1)p_a + \sigma_u} \quad (4.55b)$$

which can be solved for $\overline{R}$ since B_1, B_2, and $(\overline{R}/R)$ have been determined previously. Once $\overline{R}$ is determined, R is computed from $(\overline{R}/R)$ and the constants A_1 and A_2 can be recovered from B_1 and B_2, respectively. Thus the solution in the outer plastic zone is complete.

In the inner plastic zone, the strain rates are given by

$$\dot{\varepsilon}_r^{(p)} = \lambda_1 \frac{\partial g_1}{\partial \sigma_r} + \lambda_2 \frac{\partial g_2}{\partial \sigma_r} = -M\lambda_1 \quad (4.56a)$$

$$\dot{\varepsilon}_\theta^{(p)} = \lambda_1 \frac{\partial g_1}{\partial \sigma_\theta} + \lambda_2 \frac{\partial g_2}{\partial \sigma_\theta} = \lambda_1 + \lambda_2 \quad (4.56b)$$

$$\dot{\varepsilon}_z^{(p)} = \lambda_1 \frac{\partial g_1}{\partial \sigma_z} + \lambda_2 \frac{\partial g_2}{\partial \sigma_z} = -M\lambda_2 \quad (4.56c)$$

From the flow rule of Equation 4.56, it follows that

$$\dot{\varepsilon}_r^{(p)} + M\dot{\varepsilon}_\theta^{(p)} + \dot{\varepsilon}_z^{(p)} = 0 \quad (4.57)$$

which, when integrated with respect to time, gives

$$\varepsilon_r^{(p)} + M\varepsilon_\theta^{(p)} + \varepsilon_z^{(p)} = h(r) \quad (4.58)$$

Let the plastic strains be $\overline{\varepsilon}_r^{(p)}$, $\overline{\varepsilon}_\theta^{(p)}$, and $\overline{\varepsilon}_z^{(p)}$ at radius r when the load p_b has placed the plastic-plastic radius at r. Then, when $\overline{R}$ equals r,

$$\overline{\varepsilon}_r^{(p)} + M\overline{\varepsilon}_\theta^{(p)} + \overline{\varepsilon}_z^{(p)} = h(r) \quad (4.59)$$

In the outer plastic zone, the plastic strain relations of Equation 4.11 still hold so $h(r)$ must be zero. Therefore Equation 4.58 becomes

$$\varepsilon_r^{(p)} + M\varepsilon_\theta^{(p)} + \varepsilon_z^{(p)} = 0 \quad (4.60)$$

Substituting Equation 4.60 into the plane strain condition $(\varepsilon_z = \varepsilon_z^{(e)} + \varepsilon_z^{(p)} = 0)$ gives

$$\varepsilon_r = -M\varepsilon_\theta + M\varepsilon_\theta^{(e)} + \varepsilon_r^{(e)} + \varepsilon_z^{(e)} \quad (4.61)$$

Using Equation 4.52, and $\sigma_r = \sigma_z$, reduces Hooke's law to

$$E\varepsilon_r^{(e)} = E\varepsilon_z^{(e)} = [1-(N+1)\nu]\sigma_r + \nu\sigma_u \quad (4.62a)$$

$$E\varepsilon_\theta^{(e)} = (N-2\nu)\sigma_r - \sigma_u \quad (4.62b)$$

where σ_r is given by Equation 4.54. Substitution of the elastic strains of Equation 4.62 into Equation 4.61, followed by the substitution of the resulting ε_r into the compatibility equation

$$r\frac{d\varepsilon_\theta}{dr}+\varepsilon_\theta-\varepsilon_r=0 \tag{4.63}$$

leads to the equation

$$2G\frac{d}{dr}\left(r^{M+1}\varepsilon_\theta\right)=-\frac{MN+2-2(M+N+1)\nu}{(N-1)(1+\nu)}\left[(N-1)p_a+\sigma_u\right]\frac{r^{M+N-1}}{a^{N-1}}$$
$$+\frac{(M+2)(1-2\nu)\sigma_u}{(N-1)(1+\nu)}r^M \tag{4.64}$$

The differential Equation 4.64 is directly integrable. Integration from r to $\overline{R}$ gives

$$2G\varepsilon_\theta=2G\varepsilon_\theta\left(\overline{R}\right)\left(\frac{\overline{R}}{r}\right)^{M+1}$$
$$+\frac{MN+2-2(M+N+1)\nu}{(N-1)(1+\nu)}\left[(N-1)p_a+\sigma_u\right]\left[\left(\frac{\overline{R}}{r}\right)^{M+N}-1\right]\left(\frac{r}{a}\right)^{N-1}$$
$$-\frac{(M+2)(1-2\nu)\sigma_u}{(N-1)(M+1)(1+\nu)}\left[\left(\frac{\overline{R}}{r}\right)^{M+1}-1\right] \tag{4.65}$$

where $2G\varepsilon_\theta\left(\overline{R}\right)$ can be determined from Equation 4.26a. Substitution of the elastic strains from Equation 4.62 into Equation 4.61 results in

$$2G\varepsilon_r=-2GM\varepsilon_\theta+\frac{MN+2-2(M+N+1)\nu}{1+\nu}\sigma_r-\frac{M-2\nu}{1+\nu}\sigma_u \tag{4.66}$$

where σ_r is given by Equation 4.54 and $2G\varepsilon_\theta$ is given by Equation 4.65.

When the increasing p_b approaches $\hat{p}_b$ (the load which initiates free-field yielding), we can use the same arguments as presented in Section 3.4 to show that the constant A_1 in Equations 4.22, 4.26, and 4.27 must go to zero, and that R must approach infinity.

Thus, when p_b approaches $\hat{p}_b$, our unknowns are reduced to A_2 and $\overline{R}$. The condition of Equation 4.50, with σ_z eliminated via the yield condition, and Equation 4.27b substituted for σ_θ (with A_1 equal to zero), gives

$$\left(C_{r2} - \frac{C_{\theta 2}}{N}\right)A_2\overline{R}^{-\gamma_2-1} - (1-2\nu)\hat{p}_b = 0 \tag{4.67}$$

which can be solved for $A_2\overline{R}^{-\gamma_2-1}$. Continuity of radial stress at $\overline{R}$ leads to

$$C_{r2}A_2\overline{R}^{-\gamma_2-1} - \hat{p}_b = -\left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{\overline{R}}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \tag{4.68}$$

which can be solved for $\overline{R}$ since $A_2\overline{R}^{-\gamma_2-1}$ was determined previously. Once the unknowns are determined, the stress and strain expressions developed for $p_b < \hat{p}_b$ are still valid for $p_b = \hat{p}_b$.

4.3 CASE IIb, PHASE 3 SOLUTION.

The solution for Phase 3 of Case IIb is presented in this section. As shown in Figure 4-4, in this phase we have an inner plastic zone with conditions $\sigma_\theta < \sigma_r < \sigma_z$ governed by the yield condition

$$f = \sigma_\theta - N\sigma_z + \sigma_u = 0 \tag{4.2}$$

and a far outer plastic zone with conditions $\sigma_r = \sigma_\theta < \sigma_z$ governed by the yield conditions

$$f_1 = \sigma_r - N\sigma_z + \sigma_u = 0 \tag{4.5a}$$

$$f_2 = \sigma_\theta - N\sigma_z + \sigma_u = 0 \tag{4.5b}$$

The conditions in the far outer plastic zone were postulated to exist based on numerical studies. From the external pressure boundary condition and the yield condition of Equation 4.5, we have

$$\sigma_r = \sigma_\theta = -p_b \tag{4.69}$$

In the inner plastic zone, the stress and strain fields developed in Section 4.1 still apply. The equations representing these fields (Equations 4.26 and 4.27) are again repeated for convenience.

$$2G\varepsilon_r = \gamma_1 A_1 r^{\gamma_1-1} - \gamma_2 A_2 r^{-\gamma_2-1} + A \tag{4.26a}$$

$$2G\varepsilon_\theta = A_1 r^{\gamma_1-1} + A_2 r^{-\gamma_2-1} + A \tag{4.26b}$$

$$\sigma_r = C_{r1}A_1 r^{\gamma_1-1} + C_{r2}A_2 r^{-\gamma_2-1} - \hat{p}_b \tag{4.27a}$$

$$\sigma_\theta = C_{\theta 1}A_1 r^{\gamma_1-1} + C_{\theta 2}A_2 r^{-\gamma_2-1} - \hat{p}_b \tag{4.27b}$$

where $A = -(1-2\nu)\hat{p}_b$.

At this point, the unknowns in the inner plastic zone are the constants A_1 and A_2, and the radius of the boundary between the inner and far outer plastic zones, denoted as $\hat{R}$. Based on the yield conditions of Equations 4.2 and 4.5, along with Equation 4.67, it is reasonable to postulate that $\sigma_\theta(\hat{R}) = -p_b$ in the inner plastic zone. This condition, along with continuity of σ_r at $\hat{R}$, results in two equations in the two unknowns $A_1\hat{R}^{\gamma_1-1}$ and $A_2\hat{R}^{-\gamma_2-1}$, which can be solved to yield

$$\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} A_1\hat{R}^{\gamma_1-1} \\ A_2\hat{R}^{-\gamma_2-1} \end{Bmatrix} = \frac{1}{C_{r1}C_{\theta 2} - C_{r2}C_{\theta 1}} \begin{bmatrix} C_{\theta 2} & -C_{r2} \\ -C_{\theta 1} & C_{r1} \end{bmatrix} \begin{Bmatrix} \Delta p_b \\ \Delta p_b \end{Bmatrix} \tag{4.70}$$

where B_1 and B_2 are constants. After solving for B_1 and B_2, the stress expressions of Equation 4.27 can be rewritten as

$$\sigma_r = C_{r1}B_1\left(\frac{r}{\hat{R}}\right)^{\gamma_1-1} + C_{r2}B_2\left(\frac{r}{\hat{R}}\right)^{-\gamma_2-1} - \hat{p}_b \tag{4.71a}$$

$$\sigma_\theta = C_{\theta 1}B_1\left(\frac{r}{\hat{R}}\right)^{\gamma_1-1} + C_{\theta 2}B_2\left(\frac{r}{\hat{R}}\right)^{-\gamma_2-1} - \hat{p}_b \tag{4.71b}$$

At the edge of the hole, we have the pressure boundary condition $\sigma_r(a) = -p_a$ which, when substituted into Equation 4.71a, leads to

$$\sigma_r(a) = C_{r1}B_1\left(\frac{a}{\hat{R}}\right)^{\gamma_1-1} + C_{r2}B_2\left(\frac{a}{\hat{R}}\right)^{-\gamma_2-1} - \hat{p}_b = -p_a \tag{4.72}$$

Equation 4.72 is a nonlinear equation which can be solved for $\left(\frac{a}{\hat{R}}\right)$, which in turn yields $\hat{R}$. Once $\hat{R}$ is determined, the constants A_1 and A_2 can be recovered from B_1 and B_2. The stress and strain fields in the inner plastic zone are now completely determined.

In the far outer plastic zone, the stress and strain fields developed in Section 3.4 apply. The stress fields are given by

$$\sigma_r = -p_b \tag{4.73a}$$

$$\sigma_\theta = -p_b \tag{4.73b}$$

$$\sigma_z = \frac{1}{N}(\sigma_u - p_b) \tag{4.73c}$$

It was shown in Section 3.4 that the sum of the radial and circumferential plastic strains was a constant, $\hat{C}$, given by

$$\varepsilon_r^{(p)} + \varepsilon_\theta^{(p)} = \frac{1}{EM}\left[-\frac{p_b}{N} + \frac{\sigma_u}{N} + 2\nu p_b\right] = \frac{\sigma_u - (1-2N\nu)p_b}{2GMN(1+\nu)} = \hat{C} \tag{4.74}$$

The equation of compatibility, strain decomposition, and Hooke's law lead to the following differential equation for $\varepsilon_\theta^{(p)}$

$$r\frac{d\varepsilon_\theta^{(p)}}{dr} + \varepsilon_\theta^{(p)} - \varepsilon_r^{(p)} = 0 \tag{4.75}$$

which, using Equation 4.74, can be solved to give

$$\varepsilon_\theta^{(p)} = \frac{\hat{C}}{2} + A_3 r^{-2} \tag{4.76}$$

where A_3 is a constant. It was also shown in Section 3.4 that the postulation $\sigma_\theta(\hat{R}) = -p_b$, Hooke's law, and continuity of total circumferential strain require that $\varepsilon_\theta^{(p)}$ must be continuous across $\hat{R}$. This condition is used to determine A_3.

$$\varepsilon_\theta^{(p)}(\hat{R}) = \frac{\hat{C}}{2} + A_3\hat{R}^{-2} \tag{4.77a}$$

or

$$A_3 = \hat{R}^2\left[\varepsilon_\theta^{(p)} - \frac{\hat{C}}{2}\right] \tag{4.77b}$$

where $\varepsilon_\theta^{(p)}(\hat{R})$ is computed from the solution in the inner plastic zone. Equation 4.26b, evaluated at $\hat{R}$, gives the total strain $\varepsilon_\theta(\hat{R})$. Equations 4.27 and 4.2 can be used to compute the stresses at $\hat{R}$. Hooke's law can then be applied to compute the elastic strain $\varepsilon_\theta^{(e)}(\hat{R})$. Finally, the plastic strain $\varepsilon_\theta^{(p)}(\hat{R})$ is computed by subtracting $\varepsilon_\theta^{(e)}(\hat{R})$ from $\varepsilon_\theta(\hat{R})$.

The solution developed above is valid until σ_z equals σ_r at r equals a. When this occurs, a third plastic zone is formed which begins at the edge of the hole and migrates outward. When the third plastic zone is present, the Phase 4 solution (which will be presented in Section 4.4) applies. However, since the Phase 4 solution applies to both Cases IIa and IIb, we present here the derivation for the load p_b which will cause the formation of a third plastic zone for Case IIb. The load p_b and plastic-plastic boundary radius $\hat{R}$ when the third plastic zone begins to form will be denoted as p_b'' and $\hat{R}''$, respectively, and are derived below. The derivation is similar to that given in Section 4.1 for p_b' and R'.

As before, we introduce two constants $\hat{B}_1$ and $\hat{B}_2$, defined by

$$B_1 = \hat{B}_1 \Delta p_b \tag{4.78a}$$

$$B_2 = \hat{B}_2 \Delta p_b \tag{4.78b}$$

From Equations 4.70 and 4.78,

$$\begin{Bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{Bmatrix} = \frac{1}{C_{r1}C_{\theta 2} - C_{r2}C_{\theta 1}} \begin{Bmatrix} C_{\theta 2} - C_{r2} \\ -C_{\theta 1} + C_{r1} \end{Bmatrix} \tag{4.79}$$

The out-of-plane stress, σ_z, can be determined from the yield condition and Equation 4.71

$$\sigma_z = \frac{C_{\theta 1}}{N} B_1 \left(\frac{r}{\hat{R}}\right)^{\gamma_1 - 1} + \frac{C_{\theta 2}}{N} B_2 \left(\frac{r}{\hat{R}}\right)^{-\gamma_2 - 1} - 2\nu \hat{p}_b \tag{4.80}$$

Substituting Equation 4.79 into Equations 4.71a and 4.80 gives

$$\sigma_r = C_{r1}\hat{B}_1 \Delta p_b \left(\frac{r}{\hat{R}}\right)^{\gamma_1 - 1} + C_{r2}\hat{B}_2 \Delta p_b \left(\frac{r}{\hat{R}}\right)^{-\gamma_2 - 1} - \hat{p}_b \tag{4.81a}$$

$$\sigma_z = \frac{C_{\theta 1}}{N} \hat{B}_1 \Delta p_b \left(\frac{r}{\hat{R}}\right)^{\gamma_1 - 1} + \frac{C_{\theta 2}}{N} \hat{B}_2 \Delta p_b \left(\frac{r}{\hat{R}}\right)^{-\gamma_2 - 1} - 2\nu \hat{p}_b \tag{4.81b}$$

Using Equations 4.81 and setting $\sigma_r(a) = \sigma_z(a) = -p_a$ gives

$$C_{r1}\hat{B}_1 \Delta p_b'' \left(\frac{a}{\hat{R}''}\right)^{\gamma_1 - 1} + C_{r2}\hat{B}_2 \Delta p_b'' \left(\frac{a}{\hat{R}''}\right)^{-\gamma_2 - 1} = \hat{p}_b - p_a \tag{4.82a}$$

$$\frac{C_{\theta 1}}{N} \hat{B}_1 \Delta p_b'' \left(\frac{a}{\hat{R}''}\right)^{\gamma_1 - 1} + \frac{C_{\theta 2}}{N} \hat{B}_2 \Delta p_b'' \left(\frac{a}{\hat{R}''}\right)^{-\gamma_2 - 1} = 2\nu \hat{p}_b - p_a \tag{4.82b}$$

where

$$\Delta p_b'' = \hat{p}_b - p_b'' \tag{4.82c}$$

Equations 4.82 constitute a system of two equations in terms of unknowns $\Delta p_b'' \left(\frac{a}{\hat{R}''}\right)^{\gamma_1 - 1}$ and $\Delta p_b'' \left(\frac{a}{\hat{R}''}\right)^{-\gamma_2 - 1}$ which can inverted to give

$$\begin{Bmatrix} \Delta p_b'' \left(\frac{a}{\hat{R}''}\right)^{\gamma_1 - 1} \\ \Delta p_b'' \left(\frac{a}{\hat{R}''}\right)^{-\gamma_2 - 1} \end{Bmatrix} = \frac{1}{D} \begin{bmatrix} C_{\theta 2}\hat{B}_2 & -NC_{r2}\hat{B}_2 \\ -C_{\theta 1}\hat{B}_1 & NC_{r1}\hat{B}_1 \end{bmatrix} \begin{Bmatrix} \hat{p}_b - p_a \\ 2\nu \hat{p}_b - p_a \end{Bmatrix} \tag{4.83a}$$

where

$$D = \hat{B}_1\hat{B}_2\left(C_{r1}C_{\theta 2} - C_{r2}C_{\theta 1}\right) \tag{4.83b}$$

Equation 4.83a can be rearranged to give

$$\frac{\Delta p_b''\left(\dfrac{a}{\hat{R}''}\right)^{-\gamma_2-1}}{\Delta p_b''\left(\dfrac{a}{\hat{R}''}\right)^{\gamma_1-1}} = -\frac{\hat{B}_1\left[C_{\theta 1}\left(\hat{p}_b - p_a\right) - NC_{r1}\left(2\nu\hat{p}_b - p_a\right)\right]}{\hat{B}_2\left[C_{\theta 2}\left(\hat{p}_b - p_a\right) - NC_{r2}\left(2\nu\hat{p}_b - p_a\right)\right]} \tag{4.84a}$$

or

$$\left(\frac{\hat{R}''}{a}\right)^{\gamma_1+\gamma_2} = -\frac{\hat{B}_1\left[C_{\theta 1}\left(\hat{p}_b - p_a\right) - NC_{r1}\left(2\nu\hat{p}_b - p_a\right)\right]}{\hat{B}_2\left[C_{\theta 2}\left(\hat{p}_b - p_a\right) - NC_{r2}\left(2\nu\hat{p}_b - p_a\right)\right]} \tag{4.84b}$$

Once $\left(\dfrac{\hat{R}''}{a}\right)$ is determined, $\Delta p_b''$ can be obtained from Equation 4.83a

$$\Delta p_b'' = \frac{\left(C_{\theta 2} - 2\nu NC_{r2}\right)\hat{p}_b - \left(C_{\theta 2} - NC_{r2}\right)p_a}{\hat{B}_1\left(C_{r1}C_{\theta 2} - C_{r2}C_{\theta 1}\right)}\left(\frac{\hat{R}''}{a}\right)^{\gamma_1-1} \tag{4.85}$$

and p_b'' can be determined from Equation 4.82c.

4.4 CASE IIa AND IIb, PHASE 4 SOLUTION.

In this section we present the solution for Phase 4, which contains three plastic zones (Figure 4-5). Again, this solution is only applicable for $N\nu < \frac{1}{2}$. As discussed earlier, yielding begins in Case IIa at the edge of the hole with an inner plastic zone. With increased loading, a second plastic zone forms which also migrates outward from the edge of the hole. The initial plastic zone is now the outer plastic zone, while the outermost region is elastic. In Phase 4, the outermost region becomes plastic when p_b becomes large enough to cause the free field to yield.

In Case IIb, yielding begins at the edge of the hole with an inner plastic zone, with the outermost region being elastic. This is followed by free-field yielding. In Phase 4, a third plastic zone is formed which begins at the edge of the hole and migrates outward. So, although the path taken to get to Phase 4 differs in Cases IIa and IIb, the number of plastic zones and the conditions in each zone are identical, and thus the solution is identical for both.

The stress condition in the inner plastic zone $\left(a < r < \overline{R}\right)$ is $\sigma_\theta < \sigma_r = \sigma_z$ with yielding governed by

$$f_1 = \sigma_\theta - N\sigma_r + \sigma_u = 0 \tag{4.3a}$$

$$f_2 = \sigma_\theta - N\sigma_z + \sigma_u = 0 \tag{4.3b}$$

The stress condition in the outer plastic zone $\left(\overline{R} < r < \hat{R}\right)$ is $\sigma_\theta < \sigma_r < \sigma_z$ so the yield condition is

$$f = \sigma_\theta - N\sigma_z + \sigma_u = 0 \tag{4.2}$$

Finally, in the far outer plastic zone $\left(\overline{R} < r < \infty\right)$, we postulate that the stress condition is $\sigma_\theta < \sigma_r = \sigma_z$ so yielding is given by

$$f_1 = \sigma_r - N\sigma_z + \sigma_u = 0 \tag{4.5a}$$

$$f_2 = \sigma_\theta - N\sigma_z + \sigma_u = 0 \tag{4.5b}$$

The external pressure boundary condition and the yield condition of Equation 4.5 lead to

$$\sigma_r = \sigma_\theta = -p_b \tag{4.69}$$

As discussed in sections 4.1 and 4.2, the stress and strain fields in the outer plastic zone are give by

$$2G\varepsilon_r = \gamma_1 A_1 r^{\gamma_1 - 1} - \gamma_2 A_2 r^{-\gamma_2 - 1} + A \tag{4.26a}$$

$$2G\varepsilon_\theta = A_1 r^{\gamma_1 - 1} + A_2 r^{-\gamma_2 - 1} + A \tag{4.26b}$$

$$\sigma_r = C_{r1} A_1 r^{\gamma_1 - 1} + C_{r2} A_2 r^{-\gamma_2 - 1} - \hat{p}_b \tag{4.27a}$$

$$\sigma_\theta = C_{\theta 1} A_1 r^{\gamma_1 - 1} + C_{\theta 2} A_2 r^{-\gamma_2 - 1} - \hat{p}_b \tag{4.27b}$$

where $A = -(1-2\nu)\hat{p}_b$.

At this point the unknowns are the constants A_1 and A_2 and the plastic zone boundary radii are $\overline{R}$ and $\hat{R}$. Just as in Sections 3.4 and 4.3, we postulate that $\sigma_\theta\left(\hat{R}\right) = -p_b$ in the outer plastic zone. This condition, along with continuity of σ_r at $\hat{R}$, results in two equations for two unknowns $A_1\hat{R}^{\gamma_1 - 1}$ and $A_2\hat{R}^{\gamma_2 - 1}$, which can be solved to yield

$$\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} A_1\hat{R}^{\gamma_1 - 1} \\ A_2\hat{R}^{-\gamma_2 - 1} \end{Bmatrix} = \frac{1}{C_{r1}C_{\theta 2} - C_{r2}C_{\theta 1}} \begin{bmatrix} C_{\theta 2} & -C_{r2} \\ -C_{\theta 1} & C_{r1} \end{bmatrix} \begin{Bmatrix} \Delta p_b \\ \Delta p_b \end{Bmatrix} \tag{4.70}$$

where B_1 and B_2 are constants. After solving for B_1 and B_2, the stress expressions of Equation 4.27 can be rewritten as

$$\sigma_r = C_{r1}B_1\left(\frac{r}{\hat{R}}\right)^{\gamma_1-1} + C_{r2}B_2\left(\frac{r}{\hat{R}}\right)^{-\gamma_2-1} - \hat{p}_b \tag{4.71a}$$

$$\sigma_\theta = C_{\theta1}B_1\left(\frac{r}{\hat{R}}\right)^{\gamma_1-1} + C_{\theta2}B_2\left(\frac{r}{\hat{R}}\right)^{-\gamma_2-1} - \hat{p}_b \tag{4.71b}$$

In the inner plastic zone, the stress and strain fields developed in Section 4.2 apply. The radial stress in that zone is given by

$$\sigma_r = -\left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{r}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \tag{4.54}$$

In Section 4.2, we also postulated that at the inner boundary of the outer plastic zone $(\overline{R})$

$$\sigma_r(\overline{R}) - \sigma_z(\overline{R}) = 0 \tag{4.50}$$

Using Equation 4.50 and the yield condition Equation 4.2 gives

$$N\sigma_r(\overline{R}) - \sigma_\theta(\overline{R}) - \sigma_u = 0 \tag{4.86}$$

Substituting Equation 4.71 into Equation 4.86 gives the following nonlinear expression

$$(NC_{r1} - C_{\theta1})B_1\left(\frac{\overline{R}}{\hat{R}}\right)^{\gamma_1-1} + (NC_{r2} - C_{\theta2})B_2\left(\frac{\overline{R}}{\hat{R}}\right)^{-\gamma_2-1} - N(1-2\nu)\hat{p}_b = 0 \tag{4.87}$$

which can be solved for the radius ratio $(\overline{R}/\hat{R})$. Using Equations 4.54 and 4.71a to enforce continuity of σ_r at $\overline{R}$ gives

$$C_{r1}B_1\left(\frac{\overline{R}}{\hat{R}}\right)^{\gamma_1-1} + C_{r2}B_2\left(\frac{\overline{R}}{\hat{R}}\right)^{-\gamma_2-1} - \hat{p}_b = \left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{\overline{R}}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \tag{4.88a}$$

or

$$\left(\frac{\overline{R}}{a}\right)^{N-1} = \frac{N(1-2\nu)\hat{p}_b - (N-1)C_{r1}B_1\left(\frac{\overline{R}}{\hat{R}}\right)^{\gamma_1-1} - (N-1)C_{r2}B_2\left(\frac{\overline{R}}{\hat{R}}\right)^{-\gamma_2-1}}{(N-1)p_a + \sigma_u} \tag{4.88b}$$

which can be solved for $\overline{R}$ since B_1, B_2, and $\frac{\overline{R}}{\hat{R}}$ have been determined previously. After solving for $\overline{R}$, $\hat{R}$ can be recovered from $\frac{\overline{R}}{\hat{R}}$ and A_1 and A_2 can be recovered from B_1 and B_2 respectively.

The solution in the far outer plastic zone is identical to that given in Section 4.3, Equations 4.74 through 4.77.

SECTION 5

PROCEDURES FOR APPLYING EXTENDED SOLUTIONS

In this section, the procedures for applying the extended solutions are summarized in 13 tables. These tables are described briefly in the following paragraphs. Because of the large number of multi-page tables and the relatively small amount of accompanying text, all tables are placed at the end of the section.

The first step in the solution process is to define the problem. The material and load parameters that must be known prior to solving the problem are listed in Table 5-1. Restrictions on the known parameters (discussed in Section 2) are also included in that table.

The second step is to determine which subcase (Case Ia, Ib, IIa or IIb) applies. The Case I and Case II definitions were developed in Section 2. The definitions of Cases Ia and Ib were developed in Section 3.1, and the definitions of Cases IIa and IIb were developed in Section 4.1. These definitions are summarized in Table 5-2. Using only the parameters listed in Table 5-1, one can use Table 5-2 to determine whether a particular problem falls into Case Ia, Case Ib, or Case II. However, an additional parameter ($\hat{p}_b^*$) is required to distinguish Case IIa from Case IIb.

A list of "computed parameters" (which can be calculated from the known parameters listed in Table 5-1 and are commonly used in the various solutions) is provided in Table 5-3. Equation numbers from the earlier derivations are listed. Where the same equation was used in Sections 3 and 4, both equation numbers are shown. Some of the parameters contained in Table 5-3 pertain to all plastic solutions; others pertain only to Case II (e.g., $\hat{p}_b^*$); while still others pertain only to either Case IIa of IIb. We have found it useful to compute and save these parameters at this point in the solution process.

Once the appropriate subcase (Case Ia, Ib, IIa, or IIb) is determined, the next step is to determine which phase of the solution is applicable. As discussed at the beginning of Section 3, Case Ia consists of two phases, while Case Ib consists of up to four phases. Both Cases IIa and IIb consist of up to four phases (Section 4). We use the terminology "up to four phases" since in the cases where $1-2Nn < 0$, the free field will not yield and only the first three of the four phases are applicable. For all cases, the applicable phase is simply a function of the value of the external pressure p_b.

The process for determining the proper case and subcase is summarized in the flowchart shown in Figure 5-1. Each phase of each subcase constitutes a branch on the flow chart. The branch symbols used in the flow chart are defined in Table 5-4. Also contained in Table 5-4 are references to figures in Sections 3 and 4 that show the stress conditions in each region for each branch.

Once the subcase and the value of the load p_b are known, one can use Table 5-5 to determine which branch of the flowchart (i.e., which phase of the solution) is applicable. Tables 5-4 and 5-5 also contain references to a "solution table" for each branch. The solution tables (Tables 5-6 through 5-13) contain step-by-step solution procedures for each of the branches. Thus, once the branch is determined, the solution procedure may be found in the applicable solution table.

The format of the solution tables is as follows. First, unknown coefficients and plastic boundary radii are computed where required. Next, stress and strain response quantities are given for each region in the solution. Finally, an expression for hole closure (referred to in the tables as "tunnel" closure) is given. If the only response quantity of interest is closure, one may skip the middle part of each solution table. That is, one can compute the required coefficients and boundary radii and then proceed directly to the closure calculation. Equation numbers are again provided for reference. Primed equation numbers are used to indicate that the form of the equation has been modified from that shown in the derivation. In the solution tables, an equation denoted with multiple numbers is an algebraic combination of the equations bearing those numbers.

Many of the solutions require solving a nonlinear equation of the form

$$C_1 x^{\gamma_1 - 1} + C_2 x^{-\gamma_2 - 1} + C_o = 0 \qquad (5.1)$$

where C_o, C_1, C_2, γ_1 and γ_2 are constants and x is the unknown. We have successfully used both linear iteration and Newton-Raphson techniques to solve nonlinear equations of this form. The linear iteration technique is summarized below.

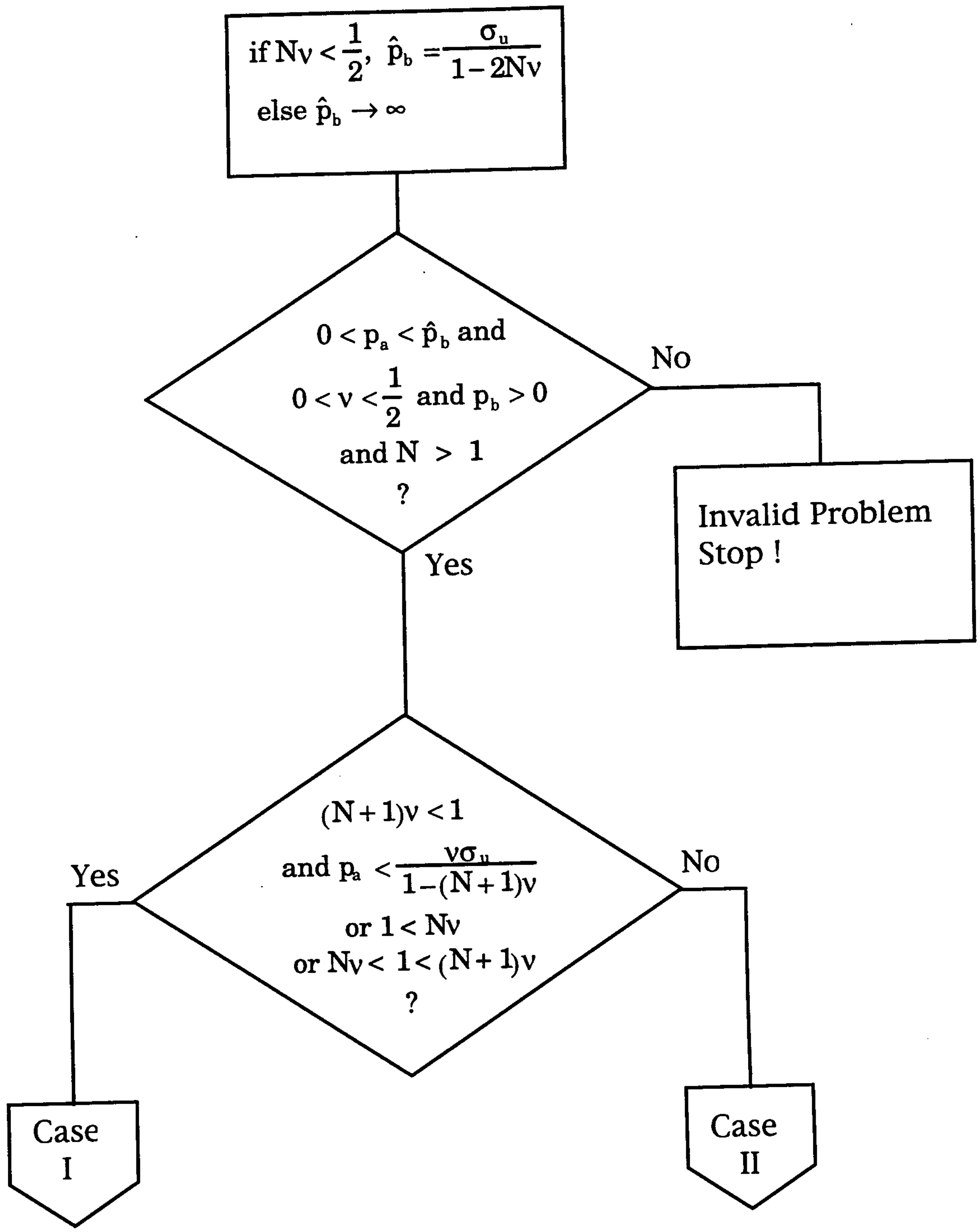


Figure 5-1. Flowchart for determining solution case.

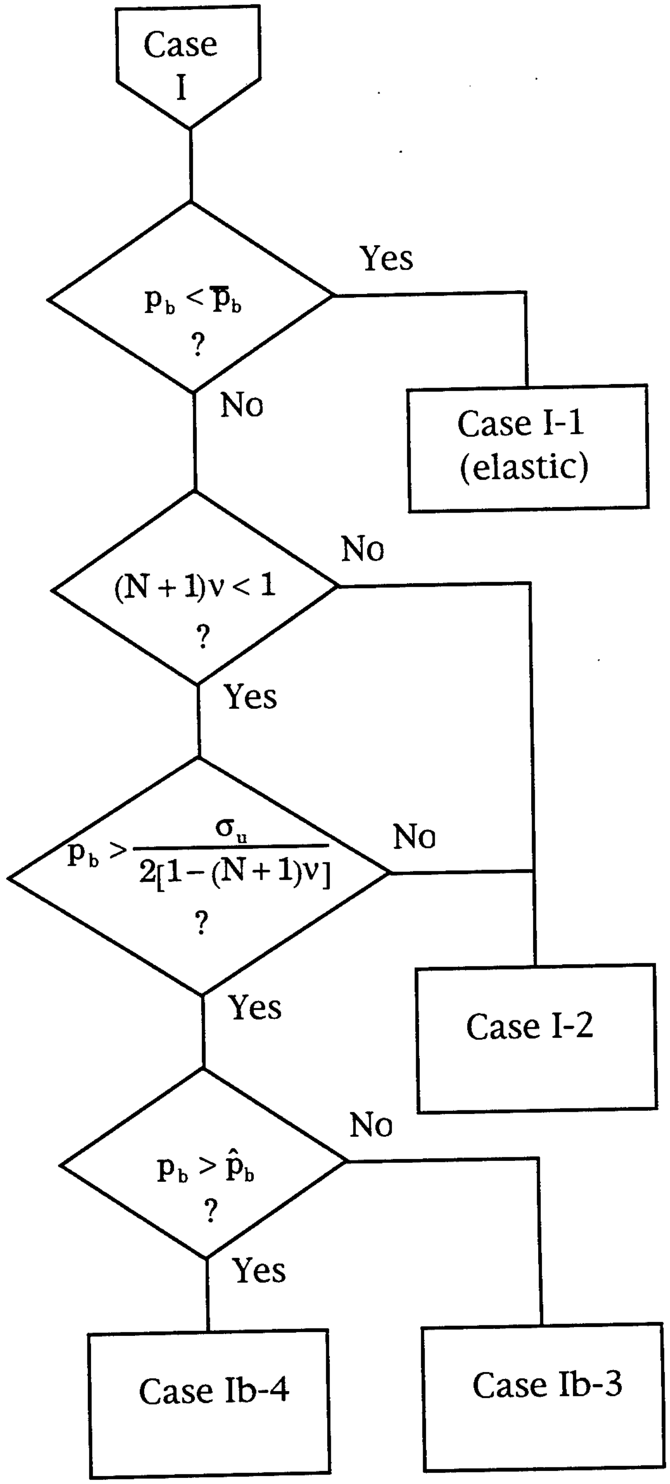


Figure 5-1. Flowchart for determining solution case (Continued).

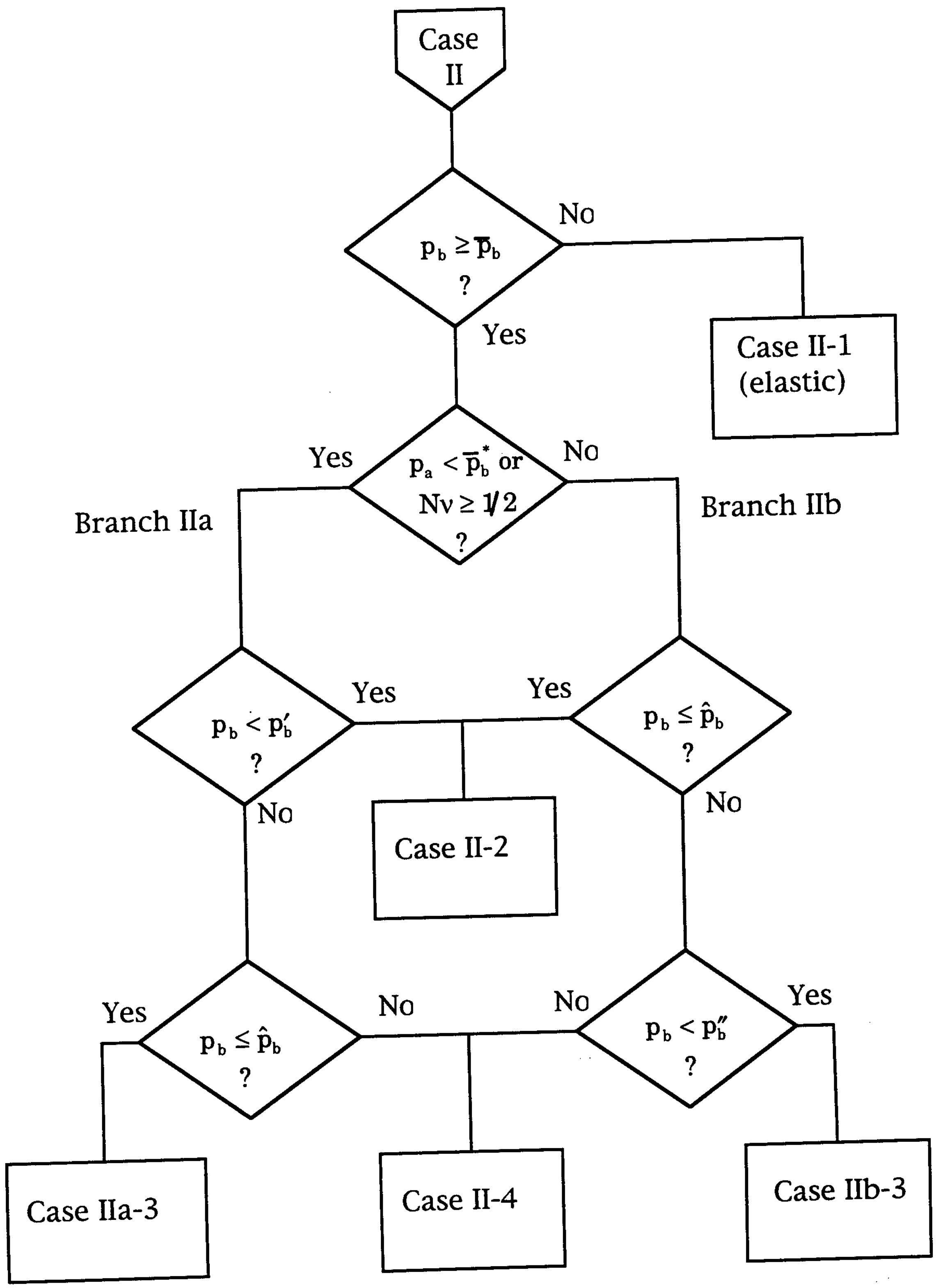


Figure 5-1. Flowchart for determining solution case (Continued).

An initial guess for the unknown x is required. For most problems γ_1 and γ_2 are approximately one, so a reasonable initial guess for x can be obtained from

$$C_1\left({}^{(o)}x\right)^{o} + C_2\left({}^{(o)}x\right)^{-2} + C_o = 0 \tag{5.2a}$$

or

$${}^{(o)}x = \sqrt{\frac{-C_2}{C_1 + C_o}} \tag{5.2b}$$

where the left superscript on x denotes the value of the iteration counter k. We then successively solve for ${}^{(k+1)}x$ using

$${}^{(k+1)}x = \left[\frac{-C_o - C_1\,{}^{(k)}x^{\gamma_1 - 1}}{C_2}\right]^{\frac{1}{-\gamma_2 - 1}} \tag{5.3}$$

until acceptable convergence is achieved. Convergence is measured by

$$\left|\frac{{}^{(k+1)}x - {}^{(k)}x}{{}^{(k+1)}x}\right| < \delta \tag{5.4}$$

where δ is a suitably small number (e.g., 10^{-4}).

Table 5-1. Known parameters.

Elastic parameters:

Poisson's ratio n $\left(0 < \nu < \frac{1}{2}\right)$

Shear modulus G (or $E = 2G(1+\nu)$)

Strength parameters

Unconfined compressive strength σ_u $(\sigma_u > 0)$

Friction angle ϕ $(\phi > 0)$, or $N = \frac{1+\sin\phi}{1-\sin\phi}$ $(N > 1)$

Dilatancy angle ω $(0 \le \omega \le \phi)$, or $M = \frac{1+\sin\omega}{1-\sin\omega}$ $(1 \le M \le N)$

Loading:

Internal pressure, p_a $\left(p_a > 0 \text{ for } N\nu \ge \frac{1}{2}, \text{ or } 0 < p_a < \frac{\sigma_u}{1-2N\nu} \text{ for } N\nu < \frac{1}{2}\right)$

External pressure, p_b $(p_b > 0)$

Geometry

Interior radius a (a=1 to normalize results with respect to interior radius).

Table 5-2. Case Ia, Ib, IIa, and IIb conditions.

Property Relations	Internal Pressure	Case
$N\nu < (N+1)\nu < 1$	$p_a < \frac{\nu\sigma_u}{1-(N+1)\nu}$	Ib
$1 < N\nu < (N+1)\nu$	$p_a > 0$	Ib
$N\nu < 1 < (N+1)\nu$	$p_a > 0$	Ia
$\frac{1}{2} < N\nu < (N+1)\nu < 1$	$p_a > \frac{\nu\sigma_u}{1-(N+1)\nu}$	IIa
$0 < N\nu < \frac{1}{2}$	$\frac{\nu\sigma_u}{1-(N+1)\nu} < p_a < \frac{\sigma_u}{1-2N\nu}$	IIb
$0 < N\nu < \frac{1}{2}$	$\frac{\nu\sigma_u}{1-(N+1)\nu} < p_a < \hat{p}_b^*$	IIa

Note: $\hat{p}_b^*$ is a function of several variables. See Equation 4.47. This equation and other contributing equations are summarized in Table 5-3.

Table 5-3. Computed parameters.

$$\beta^2 = \frac{MN}{MN + 1 - (M+N)\nu} \qquad (3.39), (4.21)$$

$$\gamma_1 = \beta^2 \frac{(N-M)\nu}{2MN} + \beta^2 \sqrt{\left(\frac{N-M}{2MN}\nu\right)^2 + \frac{1}{\beta^2}} \qquad (3.42), (4.24)$$

$$\gamma_2 = \beta^2 \frac{(M-N)\nu}{2MN} + \beta^2 \sqrt{\left(\frac{M-N}{2MN}\nu\right)^2 + \frac{1}{\beta^2}} \qquad (3.43), (4.25)$$

$$\hat{p}_b = \frac{\sigma_u}{1-2N\nu} \text{ if } N\nu < \frac{1}{2} \qquad (2.28b)$$

$$\hat{p}_b = \infty \text{ if } N\nu \geq \frac{1}{2}$$

$$A = -\frac{1-2\nu}{1-2N\nu}\sigma_u = -(1-2\nu)\hat{p}_b \qquad (3.41), (4.23)$$

$$C_{r1} = \frac{1}{C}\{[MN + 1 - (M+N)\nu]\gamma_1 + M(N+1)\nu\} \qquad (3.46a), (4.28a)$$

$$C_{r2} = \frac{1}{C}\{-[MN + 1 - (M+N)\nu]\gamma_2 + M(N+1)\nu\} \qquad (3.46b), (4.28b)$$

$$C_{\theta 1} = \frac{1}{C}[N(M+1)\nu\gamma_1 + MN] \qquad (3.46c), (4.28c)$$

$$C_{\theta 2} = \frac{1}{C}[-N(M+1)\nu\gamma_2 + MN] \qquad (3.46d), (4.28d)$$

$$C = MN + 1 - (M+1)(N+1)\nu \qquad (3.46e), (4.28e)$$

$$\hat{C} = \frac{\sigma_u - p_b(1-2N\nu)}{EMN} \qquad (3.95), (4.74)$$

$$\Delta p_b = \hat{p}_b - p_b \qquad (3.50), (4.32)$$

Case II only:

$$\hat{p}_b^* = \hat{p}_b \frac{2N\nu C_{r2} - C_{\theta 2}}{NC_{r2} - C_{\theta 2}} \qquad (4.47)$$

Table 5-3. Computed parameters (Continued).

Case IIa only:

$$\begin{Bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{Bmatrix} = \frac{1}{C_{r1} - C_{r2}} \begin{bmatrix} 1 & -C_{r2} \\ -1 & C_{r1} \end{bmatrix} \begin{Bmatrix} 2(1-N\nu) \\ 2(N-1)\nu \end{Bmatrix} \tag{4.40}$$

$$\left(\frac{R'}{a}\right)^{\gamma_1+\gamma_2} = -\frac{\hat{B}_1\left[C_{\theta 1}(\hat{p}_b - p_a) - NC_{r1}(2\nu\hat{p}_b - p_a)\right]}{\hat{B}_2\left[C_{\theta 2}(\hat{p}_b - p_a) - NC_{r2}(2\nu\hat{p}_b - p_a)\right]} \tag{4.44b}$$

$$\Delta p_b' = \frac{(C_{\theta 2} - 2\nu NC_{r2})\hat{p}_b - (C_{\theta 2} - NC_{r2})p_a}{\hat{B}_1(C_{r1}C_{\theta 2})}\left(\frac{R'}{a}\right)^{\gamma_1 - 1} \tag{4.45}$$

$$p_b' = \hat{p}_b - \Delta p_b' \tag{4.42c}$$

Case IIb only:

$$\begin{Bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{Bmatrix} = \frac{1}{C_{r1}C_{\theta 2} - C_{r2}C_{\theta 1}} \begin{Bmatrix} C_{\theta 2} - C_{r2} \\ -C_{\theta 1} + C_{r1} \end{Bmatrix} \tag{4.79}$$

$$\left(\frac{\hat{R}''}{a}\right)^{\gamma_1+\gamma_2} = -\frac{\hat{B}_1\left[C_{\theta 1}(\hat{p}_b - p_a) - NC_{r1}(2\nu\hat{p}_b - p_a)\right]}{\hat{B}_2\left[C_{\theta 2}(\hat{p}_b - p_a) - NC_{r2}(2\nu\hat{p}_b - p_a)\right]} \tag{4.84b}$$

$$\Delta p_b'' = \frac{(C_{\theta 2} - 2\nu NC_{r2})\hat{p}_b - (C_{\theta 2} - NC_{r2})p_a}{\hat{B}_1(C_{r1}C_{\theta 2})}\left(\frac{R''}{a}\right)^{\gamma_1 - 1} \tag{4.85}$$

$$p_b'' = \hat{p}_b - \Delta p_b'' \tag{4.82c}$$

Table 5-4. Identification of flow chart solution branches.

Case	Phase	Branch Symbol	Solution Table	Solution Conditions	Figure Number
I (a or b)	1	I-1	5-6	Elastic throughout	3-1
I (a or b)	2	I-2	5-7	One plastic zone, free field elastic	3-2
Ib	3	Ib-3	5-8	Three plastic zones, free field elastic	3-3
Ib	4	Ib-4	5-9	Four plastic zones, (free field yielded)	3-4
II (a or b)	1	II-1	5-6	Elastic throughout	4-1
II (a or b)	2	II-2	5-10	One plastic zone, free field elastic	4-2
IIa	3	IIa-3	5-11	Two plastic zones, free field elastic	4-3
IIb	3	IIb-3	5-12	Two plastic zones, (free field yielded)	4-4
II (a or b)	4	II-4	5-13	Three plastic zones, (free field yielded)	4-5

Table 5-5. Procedure for determining applicable solution branch.

Case Ia

External Pressure, p_b	Phase	Branch	Solution Table
$p_b \le \frac{1}{2}[(N+1)p_a + \sigma_u]$	1	I-1	Table 5-6
$p_b > \frac{1}{2}[(N+1)p_a + \sigma_u]$	2	I-2	Table 5-7

Case Ib

External Pressure, p_b	Phase	Branch	Solution Table
$p_b \le \frac{1}{2}[(N+1)p_a + \sigma_u]$	1	I-1	Table 5-6
$\frac{1}{2}[(N+1)p_a + \sigma_u] < p_b \le \frac{\sigma_u}{2[1-(N+1)\nu]}$	2	I-2	Table 5-7
$\frac{\sigma_u}{2[1-(N+1)\nu]} < p_b < \hat{p}_b$	3	Ib-3	Table 5-8
$p_b \ge \hat{p}_b$	4	Ib-4	Table 5-9

Case IIa

External Pressure, p_b	Phase	Branch	Solution Table
$p_b \le \frac{p_a + \sigma_u}{2(1-N\nu)}$	1	II-1	Table 5-6
$\frac{p_a + \sigma_u}{2(1-N\nu)} < p_b \le p'_b$	2	II-2	Table 5-10
$p'_b < p_b < \hat{p}_b$	3	IIa-3	Table 5-11
$p_b \ge \hat{p}_b$	4	II-4	Table 5-13

Table 5-5. Procedure for determining applicable solution branch (Continued).

Case IIb

External Pressure, p_b	Phase	Branch	Solution Table
$p_b \le \frac{p_a + \sigma_u}{2(1-N\nu)}$	1	II-1	Table 5-6
$\frac{p_a + \sigma_u}{2(1-N\nu)} < p_b < \hat{p}_b$	2	II-2	Table 5-10
$\hat{p}_b \le p_b < p_b''$	3	IIb-3	Table 5-12
$p_b \ge p_b''$	4	II-4	Table 5-13

Table 5-6. Procedure for elastic conditions (all cases).

The strain and stress fields are given by

$$2G\varepsilon_r = -(1-2\nu)p_b + (p_b - p_a)\frac{a^2}{r^2} \tag{2.11}$$

$$2G\varepsilon_\theta = -(1-2\nu)p_b - (p_b - p_a)\frac{a^2}{r^2} = -2(1-2\nu)p_b - 2G\varepsilon_r \tag{2.12}$$

$$\sigma_r = -p_b + (p_b - p_a)\frac{a^2}{r^2} \tag{2.8}$$

$$\sigma_\theta = -p_b - (p_b - p_a)\frac{a^2}{r^2} = -2p_b - \sigma_r \tag{2.9}$$

$$\sigma_z = -2\nu p_b \tag{2.10}$$

<u>Tunnel Closure</u>

$$\frac{\Delta D}{D} = -\varepsilon_\theta(a) = -\frac{1}{2G}\left[p_a - 2(1-\nu)p_b\right] \tag{2.12'}$$

Table 5-7. Procedure for Case I (a or b), Phase 2 solution.

Compute R from

$$R = a\left[\frac{2}{N+1}\cdot\frac{(N-1)p_b+\sigma_u}{(N-1)p_a+\sigma_u}\right]^{\frac{1}{N-1}} \qquad (3.28b')$$

<u>Plastic Zone Response</u> $(a \le r \le R)$

$$\sigma_r = -\left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{r}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \qquad (3.10)$$

$$\sigma_\theta = N\sigma_r - \sigma_u \qquad (3.8)$$

$$\sigma_z = \nu(\sigma_r + \sigma_\theta) \qquad (3.19)$$

$$2G\varepsilon_\theta^{(e)} = [N-(N+1)\nu]\sigma_r - (1-\nu)\sigma_u \qquad (3.20b)$$

$$2G\varepsilon_\theta^{(p)} = -\frac{(N+1)(1-\nu)}{M+N}[(N-1)p_a+\sigma_u]\left(\frac{R}{a}\right)^{N-1}\left[\left(\frac{R}{r}\right)^{M+1} - \left(\frac{r}{R}\right)^{N-1}\right] \qquad (3.24')$$

$$2G\varepsilon_\theta = 2G\varepsilon_\theta^{(e)} + 2G\varepsilon_\theta^{(p)} \qquad (3.17b)$$

$$2G\varepsilon_r = [1-(N+1)\nu]\sigma_r + \nu\sigma_u - M2G\varepsilon_\theta^{(p)} \qquad (3.15a), (3.20a)$$

<u>Elastic Zone Response</u> $(R \le r \le \infty)$

$$p_a^* = -\sigma_r(R) = \left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{R}{a}\right)^{N-1} - \frac{\sigma_u}{N-1} \qquad (3.10')$$

$$2G\varepsilon_r = -(1-2\nu)p_b + (p_b - p_a^*)\frac{R^2}{r^2} \qquad (2.11')$$

$$2G\varepsilon_\theta = -(1-2\nu)p_b - (p_b - p_a^*)\frac{R^2}{r^2} = -2(1-2\nu)p_b - 2G\varepsilon_r \qquad (2.12')$$

$$\sigma_r = -p_b + (p_b - p_a^*)\frac{R^2}{r^2} \qquad (2.8')$$

$$\sigma_\theta = -p_b - (p_b - p_a^*)\frac{R^2}{r^2} = -2p_b - \sigma_r \qquad (2.9')$$

$$\sigma_z = -2\nu p_b \qquad (2.10')$$

Table 5-7. Procedure for Case I (a or b), Phase 2 solution (Continued).

Tunnel Closure

$$\frac{\Delta D}{D} = -\varepsilon_\theta(a) = -\frac{1}{2G}\left[\{N-(N+1)\nu\}p_a - (1-\nu)\sigma_u\right]$$
$$+\frac{1}{2G}\left[\frac{(N+1)(1-\nu)}{M+N}\{(N-1)p_a + \sigma_u\}\right]\left[\left(\frac{R}{a}\right)^{M+N} - 1\right] \text{ (3.20b), (3.24), (3.17b)}$$

Table 5-8. Procedure for Case Ib, Phase 3 solution.

Determine boundary radii $\tilde{R}$, $\overline{R}$, and R, and coefficients A_1 and A_2.

Compute $\tilde{R}$ from

$$\tilde{R} = a\left[\frac{(1-2\nu)\sigma_u}{\{1-(N+1)\nu\}\{(N-1)p_a+\sigma_u\}}\right]^{\frac{1}{N-1}} \quad (3.33')$$

For $p_b < \hat{p}_b$, compute

$$\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = \frac{1}{C_{r1}-C_{r2}}\begin{bmatrix} 1 & -C_{r2} \\ -1 & C_{r1} \end{bmatrix}\begin{Bmatrix} 2(1-N\nu)\Delta p_b \\ 2(N-1)\nu\Delta p_b \end{Bmatrix} \quad (3.52)$$

and solve the following nonlinear equation for the ratio $\frac{\overline{R}}{R}$.

$$(NC_{r1}-C_{\theta 1})B_1\left(\frac{\overline{R}}{R}\right)^{\gamma_1-1} + (NC_{r2}-C_{\theta 2})B_2\left(\frac{\overline{R}}{R}\right)^{-\gamma_2-1} - N(1-2\nu)\hat{p}_b = 0 \quad (3.55)$$

Compute $\frac{\overline{R}}{a}$ from

$$\left(\frac{\overline{R}}{a}\right)^{N-1} = \frac{N(1-2\nu)\hat{p}_b - (N-1)C_{r1}B_1\left(\frac{\overline{R}}{R}\right)^{\gamma_1-1} - (N-1)C_{r2}B_2\left(\frac{\overline{R}}{R}\right)^{-\gamma_2-1}}{(N-1)p_a+\sigma_u} \quad (3.58b)$$

then $\overline{R} = a\left[\left(\frac{\overline{R}}{a}\right)^{N-1}\right]^{\frac{1}{N-1}}$ and $R = \dfrac{\overline{R}}{\left(\frac{\overline{R}}{R}\right)}$

Compute coefficients from

$$A_1 = \frac{B_1}{R^{\gamma_1-1}} \quad (3.52')$$

$$A_2 = \frac{B_2}{R^{-\gamma_2-1}} \quad (3.52')$$

Table 5-8. Procedure for Case Ib, Phase 3 solution (Continued).

For the case where $p_b = \hat{p}_b$, $R \to \infty$ and

$$B_2 = \frac{N(1-2\nu)}{NC_{r2} - C_{\theta 2}} \hat{p}_b \tag{3.80}$$

$$\left(\frac{\bar{R}}{a}\right)^{N-1} = \frac{N(1-2\nu)\hat{p}_b - (N-1)C_{r2}B_2}{(N-1)p_a + \sigma_u} \tag{3.81b}$$

$$A_1 = 0$$

$$A_2 = \frac{B_2}{R^{-\gamma_2 - 1}} \tag{3.80'}$$

If $p_b < \hat{p}_b$, the pressure at the internal edge of the elastic zone, $p_a^* = -\sigma_r(R)$, is required.

$$\sigma_r(R) = C_{r1}A_1R^{\gamma_1 - 1} + C_{r2}A_2R^{-\gamma_2 - 1} - \hat{p}_b \tag{3.45a'}$$

$$p_a^* = -\sigma_r(R)$$

Inner plastic zone strains depend on circumferential strains at $r = \bar{R}$ and $r = \tilde{R}$ so those are computed next.

$$2G\varepsilon_\theta(\bar{R}) = A_1\bar{R}^{\gamma_1 - 1} + A_2\bar{R}^{-\gamma_2 - 1} + A \tag{3.44a'}$$

$$2G\varepsilon_\theta(\tilde{R}) = \frac{MN + 2 - 2(M+N+1)\nu}{(M+N)(N-1)(1+\nu)}\left[(N-1)p_a + \sigma_u\right]\left[\left(\frac{\bar{R}}{\tilde{R}}\right)^{M+N} - 1\right]\left(\frac{\tilde{R}}{a}\right)^{N-1}$$
$$- \frac{(M+2)(1-2\nu)\sigma_u}{(N-1)(M+1)(1+\nu)}\left[\left(\frac{\bar{R}}{\tilde{R}}\right)^{M+1} - 1\right] + 2G\varepsilon_\theta(\bar{R})\cdot\left(\frac{\bar{R}}{\tilde{R}}\right)^{M+1} \tag{3.68'}$$

The plastic component of $\varepsilon_\theta(\tilde{R})$ is also required in the inner plastic zone. It may be determined as follows

$$2G\varepsilon_\theta^{(e)}(\tilde{R}) = -\frac{1-2\nu}{1-(N+1)\nu}\sigma_u \qquad \text{(3.66'), (3.30')}$$

$$2G\varepsilon_\theta^{(p)}(\tilde{R}) = 2G\varepsilon_\theta(\tilde{R}) - 2G\varepsilon_\theta^{(e)}(\tilde{R}) \tag{3.17b'}$$

With the information developed as described above, stresses and strains at any point may be readily determined. For a point at any radius $r \geq a$, compare r to

Table 5-8. Procedure for Case Ib, Phase 3 solution (Continued).

$\tilde{R}$, $\overline{R}$, and R to determine its location; then proceed to use the appropriate formulas below.

Inner Plastic Zone Response $\left(a \le r \le \tilde{R}\right)$

$$\sigma_r = -\left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{r}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \tag{3.10}$$

$$\sigma_\theta = N\sigma_r - \sigma_u \tag{3.8}$$

$$\sigma_z = \nu(\sigma_r + \sigma_\theta) \tag{3.19}$$

$$2G\varepsilon_r^{(e)} = [1-(N+1)\nu]\sigma_r + \nu\sigma_u \tag{3.15a}$$

$$2G\varepsilon_\theta^{(e)} = [N-(N+1)\nu]\sigma_r - (1-\nu)\sigma_u \tag{3.20b}$$

$$2G\varepsilon_\theta^{(p)} = 2G\varepsilon_\theta^{(p)}\left(\tilde{R}\right)\left(\frac{\tilde{R}}{r}\right)^{M+1}$$

$$-\frac{(N+1)(1-\nu)}{M+N}[(N-1)p_a + \sigma_u]\left(\frac{\tilde{R}}{a}\right)^{N-1}\left[\left(\frac{\tilde{R}}{r}\right)^{M+1} - \left(\frac{r}{\tilde{R}}\right)^{N-1}\right] \tag{3.70}$$

$$2G\varepsilon_\theta = 2G\varepsilon_\theta^{(e)} + 2G\varepsilon_\theta^{(p)} \tag{3.17b}$$

$$2G\varepsilon_r^{(p)} = -M2G\varepsilon_\theta^{(p)} \tag{3.15a'}$$

$$2G\varepsilon_r = 2G\varepsilon_r^{(e)} + 2G\varepsilon_r^{(p)} \tag{3.17a}$$

Middle Plastic Zone Response $\left(\tilde{R} \le r \le \overline{R}\right)$

$$2G\varepsilon_\theta = \frac{MN+2-2(M+N+1)\nu}{(M+N)(N-1)(1+\nu)}[(N-1)p_a + \sigma_u]\left[\left(\frac{\overline{R}}{r}\right)^{M+N} - 1\right]\left(\frac{r}{a}\right)^{N-1}$$

$$-\frac{(M+2)(1-2\nu)\sigma_u}{(N-1)(M+1)(1+\nu)}\left[\left(\frac{\overline{R}}{r}\right)^{M+1} - 1\right] + 2G\varepsilon_\theta\left(\overline{R}\right)\cdot\left(\frac{\overline{R}}{r}\right)^{M+1} \tag{3.68}$$

$$2G\varepsilon_r = -M2G\varepsilon_\theta + \frac{MN+2-2(M+N+1)\nu}{1+\nu}\sigma_r - \frac{M-2\nu}{1+\nu}\sigma_u \tag{3.69}$$

$$\sigma_r = -\left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{r}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \tag{3.57}$$

Table 5-8. Procedure for Case Ib, Phase 3 solution (Continued).

$$\sigma_\theta = N\sigma_r - \sigma_u \qquad (3.3a')$$

$$\sigma_z = \sigma_r \qquad (3.3a), (3.3b)$$

<u>Outer Plastic Zone Response</u> $\left(\overline{R} \le r \le R\right)$

$$2G\varepsilon_r = \gamma_1 A_1 r^{\gamma_1 - 1} - \gamma_2 A_2 r^{-\gamma_2 - 1} + A \qquad (3.44a)$$

$$2G\varepsilon_\theta = A_1 r^{\gamma_1 - 1} + A_2 r^{-\gamma_2 - 1} + A \qquad (3.44b)$$

$$\sigma_r = C_{r1} A_1 r^{\gamma_1 - 1} + C_{r2} A_2 r^{-\gamma_2 - 1} - \hat{p}_b \qquad (3.45a)$$

$$\sigma_\theta = C_{\theta 1} A_1 r^{\gamma_1 - 1} + C_{\theta 2} A_2 r^{-\gamma_2 - 1} - \hat{p}_b \qquad (3.45b)$$

$$\sigma_z = \frac{1}{N}\left(\sigma_\theta + \sigma_u\right) \qquad (3.4')$$

<u>Elastic Zone Response</u> $(R \le r \le \infty)$

$$2G\varepsilon_r = -(1-2\nu)p_b + \left(p_b - p_a^*\right)\frac{R^2}{r^2} \qquad (2.11')$$

$$2G\varepsilon_\theta = -(1-2\nu)p_b - \left(p_b - p_a^*\right)\frac{R^2}{r^2} = -2(1-2\nu)p_b - 2G\varepsilon_r \qquad (2.12')$$

$$\sigma_r = -p_b + \left(p_b - p_a^*\right)\frac{R^2}{r^2} \qquad (2.8')$$

$$\sigma_\theta = -p_b - \left(p_b - p_a^*\right)\frac{R^2}{r^2} = -2p_b - \sigma_r \qquad (2.9')$$

$$\sigma_z = -2\nu p_b \qquad (2.10')$$

Table 5-8. Procedure for Case Ib, Phase 3 solution (Continued).

Tunnel Closure

Compute $\tilde{R}$, $\bar{R}$, and R, followed by A_1 and A_2 as shown at the beginning of the table.

Now compute circumferential strain components as follows.

$$2G\varepsilon_\theta(\bar{R}) = A_1\bar{R}^{\gamma_1-1} + A_2\bar{R}^{-\gamma_2-1} + A \quad (3.44a')$$

$$2G\varepsilon_\theta(\tilde{R}) = \frac{MN+2-2(M+N+1)\nu}{(M+N)(N-1)(1+\nu)}[(N-1)p_a+\sigma_u]\left[\left(\frac{\bar{R}}{\tilde{R}}\right)^{M+N}-1\right]\left(\frac{\tilde{R}}{a}\right)^{N-1}$$
$$-\frac{(M+2)(1-2\nu)\sigma_u}{(N-1)(M+1)(1+\nu)}\left[\left(\frac{\bar{R}}{\tilde{R}}\right)^{M+1}-1\right]+2G\varepsilon_\theta(\bar{R})\cdot\left(\frac{\bar{R}}{\tilde{R}}\right)^{M+1} \quad (3.68')$$

$$2G\varepsilon_\theta^{(e)}(\tilde{R}) = -\frac{1-2\nu}{1-(N+1)\nu}\sigma_u \quad (3.66'), (3.30')$$

$$2G\varepsilon_\theta^{(p)}(\tilde{R}) = 2G\varepsilon_\theta(\tilde{R}) - 2G\varepsilon_\theta^{(e)}(\tilde{R}) \quad (3.17b')$$

$$2G\varepsilon_\theta^{(e)}(a) = -[N-(N+1)\nu]p_a - (1-\nu)\sigma_u \quad (3.20b')$$

$$2G\varepsilon_\theta^{(p)}(a) = 2G\varepsilon_\theta^{(p)}(\tilde{R})\left(\frac{\tilde{R}}{a}\right)^{M+1} - \frac{(N+1)(1-\nu)}{M+N}[(N-1)p_a+\sigma_u]\left[\left(\frac{\tilde{R}}{a}\right)^{M+N}-1\right] \quad (3.70')$$

Then

$$\frac{\Delta D}{D} = -\varepsilon_\theta(a) = -\frac{1}{2G}\left[2G\varepsilon_\theta^{(e)}(a) + 2G\varepsilon_\theta^{(p)}(a)\right] \quad (3.17b), (3.66'), (3.30'), (3.70')$$

Table 5-9. Procedure for Case Ib, Phase 4 solution.

Determine boundary radii $\tilde{R}$, $\overline{R}$, and $\hat{R}$, and coefficients A_1 and A_2.

Compute $\tilde{R}$ from

$$\tilde{R} = a\left[\frac{(1-2\nu)\sigma_u}{\{1-(N+1)\nu\}\{(N-1)p_a+\sigma_u\}}\right]^{\frac{1}{N-1}} \qquad (3.33')$$

Compute

$$\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = \frac{1}{C_{r1}C_{\theta 2} - C_{r2}C_{\theta 1}} \begin{bmatrix} C_{\theta 2} & -C_{r2} \\ -C_{\theta 1} & C_{r1} \end{bmatrix} \begin{Bmatrix} \Delta p_b \\ \Delta p_b \end{Bmatrix} \qquad (3.84)$$

and solve the following nonlinear equation for the ratio $\frac{\overline{R}}{\hat{R}}$.

$$(NC_{r1} - C_{\theta 1})B_1\left(\frac{\overline{R}}{\hat{R}}\right)^{\gamma_1 - 1} + (NC_{r2} - C_{\theta 2})B_2\left(\frac{\overline{R}}{\hat{R}}\right)^{-\gamma_2 - 1} - N(1-2\nu)\hat{p}_b = 0 \qquad (3.86)$$

Compute $\frac{\overline{R}}{a}$ from

$$\left(\frac{\overline{R}}{a}\right)^{N-1} = \frac{N(1-2\nu)\hat{p}_b - (N-1)C_{r1}B_1\left(\frac{\overline{R}}{\hat{R}}\right)^{\gamma_1-1} - (N-1)C_{r2}B_2\left(\frac{\overline{R}}{\hat{R}}\right)^{-\gamma_2-1}}{(N-1)p_a + \sigma_u} \qquad (3.87b)$$

then $\overline{R} = a\left[\left(\frac{\overline{R}}{a}\right)^{N-1}\right]^{\frac{1}{N-1}}$ and $\hat{R} = \dfrac{\overline{R}}{\left(\frac{\overline{R}}{\hat{R}}\right)}$

Compute coefficients from

$$A_1 = \frac{B_1}{\hat{R}^{\gamma_1 - 1}} \qquad (3.84')$$

$$A_2 = \frac{B_2}{\hat{R}^{-\gamma_2 - 1}} \qquad (3.84')$$

Table 5-9. Procedure for Case Ib, Phase 4 solution (Continued).

Inner plastic zone strains depend on circumferential strains at $r = \overline{R}$ and $r = \tilde{R}$, while far outer plastic zone strains depend on strains at $r = \hat{R}$, so those are computed next.

$$2G\varepsilon_\theta\left(\hat{R}\right) = A_1\hat{R}^{\gamma_1 - 1} + A_2\hat{R}^{-\gamma_2 - 1} + A \qquad (3.44a')$$

$$2G\varepsilon_\theta^{(e)}\left(\hat{R}\right) = \frac{\left[(N+1)\nu - N\right]p_b - \nu\sigma_u}{N(1+\nu)} \qquad (3.88a'), (3.5a')$$

$$2G\varepsilon_\theta^{(p)}\left(\hat{R}\right) = 2G\varepsilon_\theta\left(\hat{R}\right) - 2G\varepsilon_\theta^{(e)}\left(\hat{R}\right) \qquad (3.17b')$$

$$2G\varepsilon_\theta\left(\overline{R}\right) = A_1\overline{R}^{\gamma_1 - 1} + A_2\overline{R}^{-\gamma_2 - 1} + A \qquad (3.44a')$$

$$2G\varepsilon_\theta\left(\tilde{R}\right) = \frac{MN + 2 - 2(M+N+1)\nu}{(M+N)(N-1)(1+\nu)}\left[(N-1)p_a + \sigma_u\right]\left[\left(\frac{\overline{R}}{\tilde{R}}\right)^{M+N} - 1\right]\left(\frac{\tilde{R}}{a}\right)^{N-1}$$
$$- \frac{(M+2)(1-2\nu)\sigma_u}{(N-1)(M+1)(1+\nu)}\left[\left(\frac{\overline{R}}{\tilde{R}}\right)^{M+1} - 1\right] + 2G\varepsilon_\theta\left(\overline{R}\right)\cdot\left(\frac{\overline{R}}{\tilde{R}}\right)^{M+1} \qquad (3.68')$$

The plastic component of $\varepsilon_\theta\left(\tilde{R}\right)$ is also required in the inner plastic zone. It may be determined as follows

$$2G\varepsilon_\theta^{(e)}\left(\tilde{R}\right) = -\frac{1-2\nu}{1-(N+1)\nu}\sigma_u \qquad (3.66'), (3.30')$$

$$2G\varepsilon_\theta^{(p)}\left(\tilde{R}\right) = 2G\varepsilon_\theta\left(\tilde{R}\right) - 2G\varepsilon_\theta^{(e)}\left(\tilde{R}\right) \qquad (3.17b')$$

With the information developed as described above, stresses and strains at any point may be readily determined. For a point at any radius $r \geq a$, compare r to $\tilde{R}$, $\overline{R}$, and R to determine its location; then proceed to use the appropriate formulas below.

Table 5-9. Procedure for Case Ib, Phase 4 solution (Continued).

Inner Plastic Zone Response $\left(a \le r \le \tilde{R}\right)$

$$\sigma_r = -\left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{r}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \quad (3.10)$$

$$\sigma_\theta = N\sigma_r - \sigma_u \quad (3.8)$$

$$\sigma_z = \nu(\sigma_r + \sigma_\theta) \quad (3.19)$$

$$2G\varepsilon_r^{(e)} = [1-(N+1)\nu]\sigma_r + \nu\sigma_u \quad (3.15a)$$

$$2G\varepsilon_\theta^{(e)} = [N-(N+1)\nu]\sigma_r - (1-\nu)\sigma_u \quad (3.20b)$$

$$2G\varepsilon_\theta^{(p)} = 2G\varepsilon_\theta^{(p)}\left(\tilde{R}\right)\left(\frac{\tilde{R}}{r}\right)^{M+1}$$

$$-\frac{(N+1)(1-\nu)}{M+N}[(N-1)p_a + \sigma_u]\left(\frac{\tilde{R}}{a}\right)^{N-1}\left[\left(\frac{\tilde{R}}{r}\right)^{M+1} - \left(\frac{r}{\tilde{R}}\right)^{N-1}\right] \quad (3.70)$$

$$2G\varepsilon_\theta = 2G\varepsilon_\theta^{(e)} + 2G\varepsilon_\theta^{(p)} \quad (3.17b)$$

$$2G\varepsilon_r^{(p)} = -M2G\varepsilon_\theta^{(p)} \quad (3.15a')$$

$$2G\varepsilon_r = 2G\varepsilon_r^{(e)} + 2G\varepsilon_r^{(p)} \quad (3.17a)$$

Middle Plastic Zone Response $\left(\tilde{R} \le r \le \overline{R}\right)$

$$2G\varepsilon_\theta = \frac{MN+2-2(M+N+1)\nu}{(M+N)(N-1)(1+\nu)}[(N-1)p_a + \sigma_u]\left[\left(\frac{\overline{R}}{r}\right)^{M+N} - 1\right]\left(\frac{r}{a}\right)^{N-1}$$

$$-\frac{(M+2)(1-2\nu)\sigma_u}{(N-1)(M+1)(1+\nu)}\left[\left(\frac{\overline{R}}{r}\right)^{M+1} - 1\right] + 2G\varepsilon_\theta\left(\overline{R}\right)\cdot\left(\frac{\overline{R}}{r}\right)^{M+1} \quad (3.68)$$

$$2G\varepsilon_r = -M2G\varepsilon_\theta + \frac{MN+2-2(M+N+1)\nu}{1+\nu}\sigma_r - \frac{M-2\nu}{1+\nu}\sigma_u \quad (3.69)$$

$$\sigma_r = -\left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{r}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \quad (3.57)$$

$$\sigma_\theta = N\sigma_r - \sigma_u \quad (3.3a')$$

$$\sigma_z = \sigma_r \quad (3.3a), (3.3b)$$

Table 5-9. Procedure for Case Ib, Phase 4 solution (Continued).

Outer Plastic Zone Response $(\overline{R} \le r \le R)$

$$2G\varepsilon_r = \gamma_1 A_1 r^{\gamma_1 - 1} - \gamma_2 A_2 r^{-\gamma_2 - 1} + A \qquad (3.44a)$$

$$2G\varepsilon_\theta = A_1 r^{\gamma_1 - 1} + A_2 r^{-\gamma_2 - 1} + A \qquad (3.44b)$$

$$\sigma_r = C_{r1} A_1 r^{\gamma_1 - 1} + C_{r2} A_2 r^{-\gamma_2 - 1} - \hat{p}_b \qquad (3.45a)$$

$$\sigma_\theta = C_{\theta 1} A_1 r^{\gamma_1 - 1} + C_{\theta 2} A_2 r^{-\gamma_2 - 1} - \hat{p}_b \qquad (3.45b)$$

$$\sigma_z = \frac{1}{N}(\sigma_\theta + \sigma_u) \qquad (3.4')$$

Far Outer Plastic Zone Response $(\hat{R} \le r \le \infty)$

$$\sigma_r = -p_b \qquad (3.82)$$

$$\sigma_\theta = -p_b \qquad (3.82)$$

$$\sigma_z = \frac{1}{N}(-p_b + \sigma_u) \qquad (3.5a), (3.5b)$$

$$2G\varepsilon_r^{(e)} = \frac{[(N+1)\nu - N]p_b - \nu\sigma_u}{(1+\nu)N} \qquad (3.88a'), (3.5a')$$

$$2G\varepsilon_\theta^{(e)} = \frac{[(N+1)\nu - N]p_b - \nu\sigma_u}{(1+\nu)N} \qquad (3.88b'), (3.5a')$$

$$2G\varepsilon_\theta^{(p)} = 2G\varepsilon_\theta^{(p)}(\hat{R})\left(\frac{\hat{R}}{r}\right)^2 + G\hat{C}\left[1-\left(\frac{\hat{R}}{r}\right)^2\right] \qquad (3.100a'), (3.100b)$$

$$2G\varepsilon_r^{(p)} = 2G\hat{C} - 2G\varepsilon_\theta^{(p)} = -2G\varepsilon_\theta^{(p)}(\hat{R})\left(\frac{\hat{R}}{r}\right)^2 + G\hat{C}\left[1+\left(\frac{\hat{R}}{r}\right)^2\right] \qquad (3.95), (3.100a'), (3.100b)$$

$$2G\varepsilon_\theta = 2G\varepsilon_\theta{}^{(e)} + 2G\varepsilon_\theta{}^{(p)} \qquad (3.17b')$$

$$2G\varepsilon_r = 2G\varepsilon_r{}^{(e)} + 2G\varepsilon_\theta{}^{(p)} \qquad (3.17a')$$

Table 5-9. Procedure for Case Ib, Phase 4 solution (Continued).

<u>Tunnel Closure</u>

Compute $\tilde{R}$, $\overline{R}$, and $\hat{R}$, followed by A_1 and A_2 as shown at the beginning of the table.

Now compute circumferential strain components as follows.

$$2G\varepsilon_\theta(\overline{R}) = A_1\overline{R}^{\gamma_1-1} + A_2\overline{R}^{-\gamma_2-1} + A \qquad (3.44a')$$

$$2G\varepsilon_\theta(\tilde{R}) = \frac{MN+2-2(M+N+1)\nu}{(M+N)(N-1)(1+\nu)}[(N-1)p_a+\sigma_u]\left[\left(\frac{\overline{R}}{\tilde{R}}\right)^{M+N}-1\right]\left(\frac{\tilde{R}}{a}\right)^{N-1}$$

$$-\frac{(M+2)(1-2\nu)\sigma_u}{(N-1)(M+1)(1+\nu)}\left[\left(\frac{\overline{R}}{\tilde{R}}\right)^{M+1}-1\right]+2G\varepsilon_\theta(\overline{R})\cdot\left(\frac{\overline{R}}{\tilde{R}}\right)^{M+1} \qquad (3.68')$$

$$2G\varepsilon_\theta^{(e)}(\tilde{R}) = -\frac{1-2\nu}{1-(N+1)\nu}\sigma_u \qquad (3.66'),\ (3.30')$$

$$2G\varepsilon_\theta^{(p)}(\tilde{R}) = 2G\varepsilon_\theta(\tilde{R}) - 2G\varepsilon_\theta^{(e)}(\tilde{R}) \qquad (3.17b')$$

$$2G\varepsilon_\theta^{(e)}(a) = -[N-(N+1)\nu]p_a - (1-\nu)\sigma_u \qquad (3.20b')$$

$$2G\varepsilon_\theta^{(p)}(a) = 2G\varepsilon_\theta^{(p)}(\tilde{R})\left(\frac{\tilde{R}}{a}\right)^{M+1} - \frac{(N+1)(1-\nu)}{(M+N)}[(N-1)p_a+\sigma_u]\left[\left(\frac{\tilde{R}}{a}\right)^{M+N}-1\right] \qquad (3.70')$$

Then

$$\frac{\Delta D}{D} = -\varepsilon_\theta(a) = -\frac{1}{2G}\left[2G\varepsilon_\theta^{(e)}(a) + 2G\varepsilon_\theta^{(p)}(a)\right] \qquad (3.17b),\ (3.66'),\ (3.30'),\ (3.70')$$

Table 5-10. Procedure for Case II, Phase 2 solution.

For $p_b < \hat{p}_b$ compute

$$\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = \frac{1}{C_{r1} - C_{r2}} \begin{bmatrix} 1 & -C_{r2} \\ -1 & C_{r1} \end{bmatrix} \begin{Bmatrix} 2(1-N\nu)\Delta p_b \\ 2(N-1)\nu\Delta p_b \end{Bmatrix} \tag{4.34}$$

Determine plastic boundary radius R by solving the following nonlinear equation for the ratio (a/R).

$$C_{r1}B_1\left(\frac{a}{R}\right)^{\gamma_1 - 1} + C_{r2}B_2\left(\frac{a}{R}\right)^{-\gamma_2 - 1} = \hat{p}_b - p_a \tag{4.36}$$

Then R = a/(a/R).

Compute constants

$$A_1 = \frac{B_1}{R^{\gamma_1 - 1}} \tag{4.34'}$$

$$A_2 = \frac{B_2}{R^{-\gamma_2 - 1}} \tag{4.34'}$$

For Case IIb and p_b equals $\hat{p}_b$,

$$R \to \infty$$
$$A_1 = 0 \tag{4.49b}$$
$$A_2 = \frac{\hat{p}_b - p_a}{C_{r2}a^{-\gamma_2 - 1}}$$

With the information developed as described above, stresses and strains at any point may be readily determined. For a point at any radius $r \geq a$, compare r to R to determine its location; then proceed to use the appropriate formulas below.

Table 5-10. Procedure for Case II, Phase 2 solution (Continued).

Plastic Zone Response $(a \le r \le R)$

$$2G\varepsilon_r = \gamma_1 A_1 r^{\gamma_1 - 1} - \gamma_2 A_2 r^{-\gamma_2 - 1} + A \tag{4.26a}$$

$$2G\varepsilon_\theta = A_1 r^{\gamma_1 - 1} + A_2 r^{-\gamma_2 - 1} + A \tag{4.26b}$$

$$\sigma_r = C_{r1} A_1 r^{\gamma_1 - 1} + C_{r2} A_2 r^{-\gamma_2 - 1} - \hat{p}_b \tag{4.27a}$$

$$\sigma_\theta = C_{\theta 1} A_1 r^{\gamma_1 - 1} + C_{\theta 2} A_2 r^{-\gamma_2 - 1} - \hat{p}_b \tag{4.27b}$$

$$\sigma_z = \frac{1}{N}\left(\sigma_\theta + \sigma_u\right) \tag{4.5b}$$

Elastic Zone Response $(R \le r \le \infty)$

$$p_a^* = -\sigma_r(R) = 2(1 - N\nu)p_b + \sigma_u \tag{4.29}$$

$$2G\varepsilon_r = -(1 - 2\nu)p_b + \left(p_b - p_a^*\right)\frac{R^2}{r^2} \tag{2.11'}$$

$$2G\varepsilon_\theta = -(1 - 2\nu)p_b - \left(p_b - p_a^*\right)\frac{R^2}{r^2} = -2(1 - 2\nu)p_b - 2G\varepsilon_r \tag{2.12'}$$

$$\sigma_r = -p_b + \left(p_b - p_a^*\right)\frac{R^2}{r^2} \tag{2.8'}$$

$$\sigma_\theta = -p_b - \left(p_b - p_a^*\right)\frac{R^2}{r^2} = -2p_b - \sigma_r \tag{2.9'}$$

$$\sigma_z = -2\nu p_b \tag{2.10'}$$

Tunnel Closure

$$\frac{\Delta D}{D} = -\varepsilon_\theta(a) = -\frac{1}{2G}\left(A_1 a^{\gamma_1 - 1} + A_2 a^{-\gamma_2 - 1} + A\right) \tag{4.26b'}$$

Table 5-11. Procedure for Case IIa, Phase 3 solution.

Determine plastic boundary radii $\overline{R}$ and R and coefficients A_1 and A_2. For $p_b < \hat{p}_b$, compute

$$\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = \frac{1}{C_{r1} - C_{r2}} \begin{bmatrix} 1 & -C_{r2} \\ -1 & C_{r1} \end{bmatrix} \begin{Bmatrix} 2(1-N\nu)\Delta p_b \\ 2(N-1)\nu\Delta p_b \end{Bmatrix} \tag{4.34}$$

and then solve the following nonlinear equation for the ratio $\frac{\overline{R}}{R}$.

$$(NC_{r1} - C_{\theta 1})B_1\left(\frac{\overline{R}}{R}\right)^{\gamma_1 - 1} + (NC_{r2} - C_{\theta 2})B_2\left(\frac{\overline{R}}{R}\right)^{-\gamma_2 - 1} - N(1-2\nu)\hat{p}_b = 0 \tag{4.51'}$$

Then solve for the ratio $\overline{R}/a$ from

$$\left(\frac{\overline{R}}{a}\right)^{N-1} = \frac{N(1-2\nu)\hat{p}_b - (N-1)C_{r1}B_1\left(\frac{\overline{R}}{R}\right)^{\gamma_1 - 1} - (N-1)C_{r2}B_2\left(\frac{\overline{R}}{R}\right)^{-\gamma_2 - 1}}{(N-1)p_a + \sigma_u} \tag{4.55b}$$

Then

$$\overline{R} = a\left[\left(\frac{\overline{R}}{a}\right)^{N-1}\right]^{\frac{1}{N-1}} \text{ and } R = \frac{\overline{R}}{(\overline{R}/R)}$$

Compute constants

$$A_1 = \frac{B_1}{R^{\gamma_1 - 1}} \tag{4.34'}$$

$$A_2 = \frac{B_2}{R^{-\gamma_2 - 1}} \tag{4.34'}$$

For $p_b = \hat{p}_b$, then $R \rightarrow \infty$; first compute

$$B_2 = \frac{N(1-2\nu)\hat{p}_b}{NC_{r2} - C_{\theta 2}} \tag{4.67'}$$

and compute $\overline{R}$ from

$$\left(\frac{\overline{R}}{a}\right)^{N-1} = \frac{N(1-2\nu)\hat{p}_b - (N-1)C_{r2}B_2}{(N-1)p_a + \sigma_u} \tag{4.68'}$$

$$A_1 = 0 \text{ and } A_2 = \frac{B_2}{\overline{R}^{-\gamma_2 - 1}}$$

Table 5-11. Procedure for Case IIa, Phase 3 solution (Continued).

The pressure at the internal edge of the elastic zone, p_a^* is required when $p_b < p_b'$. Therefore we compute the stress at r = R using the solution from the outer plastic zone.

$$p_a^* = -\sigma_r(R) = -\left(C_{r1}A_1R^{\gamma_1-1} + C_{r2}A_2R^{-\gamma_2-1} - \hat{p}_b\right) \tag{4.27a'}$$

Inner Plastic Zone Response $\left(a \le r \le \overline{R}\right)$

$$2G\varepsilon_\theta\left(\overline{R}\right) = A_1\overline{R}^{\gamma_1-1} + A_2\overline{R}^{-\gamma_2-1} + A \tag{4.26a'}$$

$$2G\varepsilon_\theta = \frac{MN+2-2(M+N+1)\nu}{(M+N)(N-1)(1+\nu)}\left[(N-1)p_a+\sigma_u\right]\left[\left(\frac{\overline{R}}{r}\right)^{M+N}-1\right]\left(\frac{r}{a}\right)^{N-1}$$

$$-\frac{(M+2)(1-2\nu)\sigma_u}{(N-1)(M+1)(1+\nu)}\left[\left(\frac{\overline{R}}{r}\right)^{M+1}-1\right]+2G\varepsilon_\theta\left(\overline{R}\right)\cdot\left(\frac{\overline{R}}{r}\right)^{M+1} \tag{4.65'}$$

$$2G\varepsilon_r = -M2G\varepsilon_\theta + \frac{MN+2-2(M+N+1)\nu}{1+\nu}\sigma_r - \frac{M-2\nu}{1+\nu}\sigma_u \tag{4.66}$$

$$\sigma_r = -\left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{r}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \tag{4.54}$$

$$\sigma_\theta = N\sigma_r - \sigma_u \tag{4.3a}$$

$$\sigma_z = \sigma_r \tag{4.3a), (4.3b}$$

Outer Plastic Zone Response $\left(\overline{R} \le r \le R\right)$

$$2G\varepsilon_r = \gamma_1A_1r^{\gamma_1-1} - \gamma_2A_2r^{-\gamma_2-1} + A \tag{4.26a}$$

$$2G\varepsilon_\theta = A_1r^{\gamma_1-1} + A_2r^{-\gamma_2-1} + A \tag{4.26b}$$

$$\sigma_r = C_{r1}A_1r^{\gamma_1-1} + C_{r2}A_2r^{-\gamma_2-1} - \hat{p}_b \tag{4.27a}$$

$$\sigma_\theta = C_{\theta1}A_1r^{\gamma_1-1} + C_{\theta2}A_2r^{-\gamma_2-1} - \hat{p}_b \tag{4.27b}$$

$$\sigma_z = \frac{1}{N}\left(\sigma_\theta + \sigma_u\right) \tag{4.5b}$$

Table 5-11. Procedure for Case IIa, Phase 3 solution (Continued).

<u>Elastic Zone Response</u> $(R \le r \le \infty)$

$$2G\varepsilon_r = -(1-2\nu)p_b + \left(p_b - p_a^*\right)\frac{R^2}{r^2} \tag{2.11'}$$

$$2G\varepsilon_\theta = -(1-2\nu)p_b - \left(p_b - p_a^*\right)\frac{R^2}{r^2} = -2(1-2\nu)p_b - 2G\varepsilon_r \tag{2.12'}$$

$$\sigma_r = -p_b + \left(p_b - p_a^*\right)\frac{R^2}{r^2} \tag{2.8'}$$

$$\sigma_\theta = -p_b - \left(p_b - p_a^*\right)\frac{R^2}{r^2} = -2p_b - \sigma_r \tag{2.9'}$$

$$\sigma_z = -2\nu p_b \tag{2.10'}$$

<u>Tunnel Closure</u>

Compute boundary radii $\overline{R}$ and R, and coefficients A_1 and A_2 as shown at the beginning of the table.

Then

$$2G\varepsilon_\theta\left(\overline{R}\right) = A_1\overline{R}^{\gamma_1-1} + A_2\overline{R}^{-\gamma_2-1} + A \tag{4.26a}$$

and

$$\frac{\Delta D}{D} = -\varepsilon_\theta(a) = -\frac{MN+2-2(M+N+1)\nu}{2G(M+N)(N-1)(1+\nu)}\left[(N-1)p_a + \sigma_u\right]\left[\left(\frac{\overline{R}}{a}\right)^{M+N} - 1\right]$$
$$+\frac{(M+2)(1-2\nu)\sigma_u}{2G(N-1)(M+1)(1+\nu)}\left[\left(\frac{\overline{R}}{a}\right)^{M+1} - 1\right] - \varepsilon_\theta\left(\overline{R}\right)\cdot\left(\frac{\overline{R}}{a}\right)^{M+1} \tag{4.65'}$$

Table 5-12. Procedure for Case IIb, Phase 3 solution.

Determine plastic boundary radius $\hat{R}$, and coefficients A_1 and A_2. Begin by computing

$$\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = \frac{1}{C_{r1}C_{\theta 2} - C_{r2}C_{\theta 1}} \begin{bmatrix} C_{\theta 2} & -C_{r2} \\ -C_{\theta 1} & C_{r1} \end{bmatrix} \begin{Bmatrix} \Delta p_b \\ \Delta p_b \end{Bmatrix} \tag{4.70}$$

and then solving the following nonlinear equation for the ratio $\left(\frac{a}{\hat{R}}\right)$

$$C_{r1}B_1\left(\frac{a}{\hat{R}}\right)^{\gamma_1-1} + C_{r2}B_2\left(\frac{a}{\hat{R}}\right)^{-\gamma_2-1} - \hat{p}_b + p_a = 0 \tag{4.72}$$

then $\hat{R} = \dfrac{a}{\left(\frac{a}{\hat{R}}\right)}$

and

$$A_1 = \frac{B_1}{\hat{R}^{\gamma_1-1}} \tag{4.70'}$$

$$A_2 = \frac{B_2}{\hat{R}^{-\gamma_2-1}} \tag{4.70'}$$

Far outer plastic zone strains depend on the circumferential plastic strain at $r = \hat{R}$.

$$2G\varepsilon_\theta\left(\hat{R}\right) = A_1\hat{R}^{\gamma_1-1} + A_2\hat{R}^{-\gamma_2-1} + A \tag{4.26b'}$$

$$2G\varepsilon_\theta^{(e)}\left(\hat{R}\right) = \frac{\left[(N+1)\nu - N\right]p_b - \nu\sigma_u}{N(1+\nu)} \tag{4.73), (4.16b}$$

$$2G\varepsilon_\theta^{(p)}\left(\hat{R}\right) = 2G\varepsilon_\theta\left(\hat{R}\right) - 2G\varepsilon_\theta^{(e)}\left(\hat{R}\right) \tag{4.13b}$$

With the information developed as described above, stresses and strains at any point may be readily determined. For a point at any radius $r \geq a$, compare r to $\hat{R}$ to determine its location; then proceed to use the appropriate formulas below.

Table 5-12. Procedure for Case IIb, Phase 3 solution (Continued).

Inner Plastic Zone Response $\left(a \le r \le \hat{R}\right)$

$$2G\varepsilon_r = \gamma_1 A_1 r^{\gamma_1 - 1} - \gamma_2 A_2 r^{-\gamma_2 - 1} + A \quad (4.26a)$$

$$2G\varepsilon_\theta = A_1 r^{\gamma_1 - 1} + A_2 r^{-\gamma_2 - 1} + A \quad (4.26b)$$

$$\sigma_r = C_{r1} A_1 r^{\gamma_1 - 1} + C_{r2} A_2 r^{-\gamma_2 - 1} - \hat{p}_b \quad (4.27a)$$

$$\sigma_\theta = C_{\theta 1} A_1 r^{\gamma_1 - 1} + C_{\theta 2} A_2 r^{-\gamma_2 - 1} - \hat{p}_b \quad (4.27b)$$

$$\sigma_z = \frac{1}{N}\left(\sigma_\theta + \sigma_u\right) \quad (4.5b)$$

Far Outer Plastic Zone Response $\left(\hat{R} \le r \le \infty\right)$

$$\sigma_r = -p_b \quad (4.73a)$$

$$\sigma_\theta = -p_b \quad (4.73b)$$

$$\sigma_z = \frac{1}{N}\left(-p_b + \sigma_u\right) \quad (4.73c)$$

$$2G\varepsilon_r^{(e)} = \frac{\left[(N+1)\nu - N\right]p_b - \nu\sigma_u}{(1+\nu)N} \quad (4.73), (4.16a)$$

$$2G\varepsilon_\theta^{(e)} = \frac{\left[(N+1)\nu - N\right]p_b - \nu\sigma_u}{(1+\nu)N} \quad (4.73), (4.16b)$$

$$2G\varepsilon_\theta^{(p)} = 2G\varepsilon_\theta^{(p)}\left(\hat{R}\right)\left(\frac{\hat{R}}{r}\right)^2 + G\hat{C}\left[1 - \left(\frac{\hat{R}}{r}\right)^2\right] \quad (4.76'), (4.77b)$$

$$2G\varepsilon_r^{(p)} = 2G\hat{C} - 2G\varepsilon_\theta^{(p)} = -2G\varepsilon_\theta^{(p)}\left(\hat{R}\right)\left(\frac{\hat{R}}{r}\right)^2 + G\hat{C}\left[1 + \left(\frac{\hat{R}}{r}\right)^2\right] \quad (4.74), (4.76'), (4.77b)$$

$$2G\varepsilon_\theta = 2G\varepsilon_\theta{}^{(e)} + 2G\varepsilon_\theta{}^{(p)} \quad (4.13b')$$

$$2G\varepsilon_r = 2G\varepsilon_r{}^{(e)} + 2G\varepsilon_\theta{}^{(p)} \quad (4.13a')$$

Table 5-12. Procedure for Case IIb, Phase 3 solution (Continued).

<u>Tunnel Closure</u>

Compute $\hat{R}$, A_1, and A_2 as shown at the beginning of the table.

$$\frac{\Delta D}{D} = -\varepsilon_\theta(a) = -\frac{1}{2G}\left[A_1 a^{\gamma_1 - 1} + A_2 a^{-\gamma_2 - 1} + A\right] \qquad (4.26b')$$

Table 5-13. Procedure for Case II, Phase 4 solution.

Determine plastic boundary radii $\overline{R}$ and R. Begin by computing

$$\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = \frac{1}{C_{r1}C_{\theta 2} - C_{r2}C_{\theta 1}} \begin{bmatrix} C_{\theta 2} & -C_{r2} \\ -C_{\theta 1} & C_{r1} \end{bmatrix} \begin{Bmatrix} \Delta p_b \\ \Delta p_b \end{Bmatrix} \tag{4.70}$$

and then solving the following nonlinear equation for the ratio $\frac{\overline{R}}{\hat{R}}$.

$$(NC_{r1} - C_{\theta 1})B_1\left(\frac{\overline{R}}{\hat{R}}\right)^{\gamma_1 - 1} + (NC_{r2} - C_{\theta 2})B_2\left(\frac{\overline{R}}{\hat{R}}\right)^{-\gamma_2 - 1} - N(1-2\nu)\hat{p}_b = 0 \tag{4.87}$$

Then solve for the ratio $\overline{R}/a$ from

$$\left(\frac{\overline{R}}{a}\right)^{N-1} = \frac{N(1-2\nu)\hat{p}_b - (N-1)C_{r1}B_1\left(\frac{\overline{R}}{\hat{R}}\right)^{\gamma_1 - 1} - (N-1)C_{r2}B_2\left(\frac{\overline{R}}{\hat{R}}\right)^{-\gamma_2 - 1}}{(N-1)p_a + \sigma_u} \tag{4.88b}$$

Then

$$\overline{R} = a\left[\left(\frac{\overline{R}}{a}\right)^{N-1}\right]^{\frac{1}{N-1}} \text{ and } \hat{R} = \frac{\overline{R}}{\left(\overline{R}/\hat{R}\right)}$$

Compute constants

$$A_1 = \frac{B_1}{\hat{R}^{\gamma_1 - 1}} \tag{4.70}$$

$$A_2 = \frac{B_2}{\hat{R}^{-\gamma_2 - 1}} \tag{4.70}$$

Far outer plastic zone strains depend on the circumferential plastic strain at $r = \hat{R}$.

$$2G\varepsilon_\theta\left(\hat{R}\right) = A_1\hat{R}^{\gamma_1 - 1} + A_2\hat{R}^{-\gamma_2 - 1} + A \tag{4.26b'}$$

$$2G\varepsilon_\theta^{(e)}\left(\hat{R}\right) = \frac{\left[(N+1)\nu - N\right]p_b - \nu\sigma_u}{N(1+\nu)} \tag{4.73), (4.16b}$$

$$2G\varepsilon_\theta^{(p)}\left(\hat{R}\right) = 2G\varepsilon_\theta\left(\hat{R}\right) - 2G\varepsilon_\theta^{(e)}\left(\hat{R}\right) \tag{4.13b}$$

With the information developed as described above, stresses and strains at any point may be readily determined. For a point at any radius $r \geq a$, compare r to $\hat{R}$ to determine its location; then proceed to use the appropriate formulas below.

Table 5-13. Procedure for Case II, Phase 4 solution (Continued).

Inner Plastic Zone Response $\left(a \le r \le \overline{R}\right)$

$$2G\varepsilon_\theta\left(\overline{R}\right) = A_1\overline{R}^{\gamma_1-1} + A_2\overline{R}^{-\gamma_2-1} + A \qquad (4.26a')$$

$$2G\varepsilon_\theta = \frac{MN+2-2(M+N+1)\nu}{(M+N)(N-1)(1+\nu)}\left[(N-1)p_a + \sigma_u\right]\left[\left(\frac{\overline{R}}{r}\right)^{M+N} - 1\right]\left(\frac{r}{a}\right)^{N-1}$$

$$-\frac{(M+2)(1-2\nu)\sigma_u}{(N-1)(M+1)(1+\nu)}\left[\left(\frac{\overline{R}}{r}\right)^{M+1} - 1\right] + 2G\varepsilon_\theta\left(\overline{R}\right)\cdot\left(\frac{\overline{R}}{r}\right)^{M+1} \qquad (4.65')$$

$$2G\varepsilon_r = -M2G\varepsilon_\theta + \frac{MN+2-2(M+N+1)\nu}{1+\nu}\sigma_r - \frac{M-2\nu}{1+\nu}\sigma_u \qquad (4.66)$$

$$\sigma_r = -\left[p_a + \frac{\sigma_u}{N-1}\right]\left(\frac{r}{a}\right)^{N-1} + \frac{\sigma_u}{N-1} \qquad (4.54)$$

$$\sigma_\theta = N\sigma_r - \sigma_u \qquad (4.3a)$$

$$\sigma_z = \sigma_r \qquad (4.3a), (4.3b)$$

Outer Plastic Zone Response $\left(\overline{R} \le r \le \hat{R}\right)$

$$2G\varepsilon_r = \gamma_1 A_1 r^{\gamma_1-1} - \gamma_2 A_2 r^{-\gamma_2-1} + A \qquad (4.26a)$$

$$2G\varepsilon_\theta = A_1 r^{\gamma_1-1} + A_2 r^{-\gamma_2-1} + A \qquad (4.26b)$$

$$\sigma_r = C_{r1}A_1 r^{\gamma_1-1} + C_{r2}A_2 r^{-\gamma_2-1} - \hat{p}_b \qquad (4.27a)$$

$$\sigma_\theta = C_{\theta 1}A_1 r^{\gamma_1-1} + C_{\theta 2}A_2 r^{-\gamma_2-1} - \hat{p}_b \qquad (4.27b)$$

$$\sigma_z = \frac{1}{N}\left(\sigma_\theta + \sigma_u\right) \qquad (4.5b)$$

Table 5-13. Procedure for Case II, Phase 4 solution (Continued).

Far Outer Plastic Zone Response $\left(\hat{R} \le r \le \infty\right)$

$$\sigma_r = -p_b \qquad (4.73a)$$

$$\sigma_\theta = -p_b \qquad (4.73b)$$

$$\sigma_z = \frac{1}{N}\left(-p_b + \sigma_u\right) \qquad (4.73c)$$

$$2G\varepsilon_r^{(e)} = \frac{\left[(N+1)\nu - N\right]p_b - \nu\sigma_u}{(1+\nu)N} \qquad (4.73), (4.16a)$$

$$2G\varepsilon_\theta^{(e)} = \frac{\left[(N+1)\nu - N\right]p_b - \nu\sigma_u}{(1+\nu)N} \qquad (4.73), (4.16b)$$

$$2G\varepsilon_\theta^{(p)} = 2G\varepsilon_\theta^{(p)}\left(\hat{R}\right)\left(\frac{\hat{R}}{r}\right)^2 + G\hat{C}\left[1 - \left(\frac{\hat{R}}{r}\right)^2\right] \qquad (4.76'), (4.77b)$$

$$2G\varepsilon_r^{(p)} = 2G\hat{C} - 2G\varepsilon_\theta^{(p)} = -2G\varepsilon_\theta^{(p)}\left(\hat{R}\right)\left(\frac{\hat{R}}{r}\right)^2 + G\hat{C}\left[1 + \left(\frac{\hat{R}}{r}\right)^2\right] \qquad (4.74), (4.76'), (4.77b)$$

$$2G\varepsilon_\theta = 2G\varepsilon_\theta{}^{(e)} + 2G\varepsilon_\theta{}^{(p)} \qquad (4.13b')$$

$$2G\varepsilon_r = 2G\varepsilon_r{}^{(e)} + 2G\varepsilon_\theta{}^{(p)} \qquad (4.13a')$$

Tunnel Closure

Compute boundary radii $\overline{R}$ and R, and coefficients A_1 and A_2 as shown at the beginning of the table. Then

$$2G\varepsilon_\theta\left(\overline{R}\right) = A_1\overline{R}^{\gamma_1 - 1} + A_2\overline{R}^{-\gamma_2 - 1} + A \qquad (4.26a)$$

and

$$\frac{\Delta D}{D} = -\varepsilon_\theta(a) = -\frac{MN + 2 - 2(M+N+1)\nu}{2G(M+N)(N-1)(1+\nu)}\left[(N-1)p_a + \sigma_u\right]\left[\left(\frac{\overline{R}}{a}\right)^{M+N} - 1\right]$$
$$+ \frac{(M+2)(1-2\nu)\sigma_u}{2G(N-1)(M+1)(1+\nu)}\left[\left(\frac{\overline{R}}{a}\right)^{M+1} - 1\right] - \varepsilon_\theta\left(\overline{R}\right)\cdot\left(\frac{\overline{R}}{a}\right)^{M+1} \qquad (4.65')$$

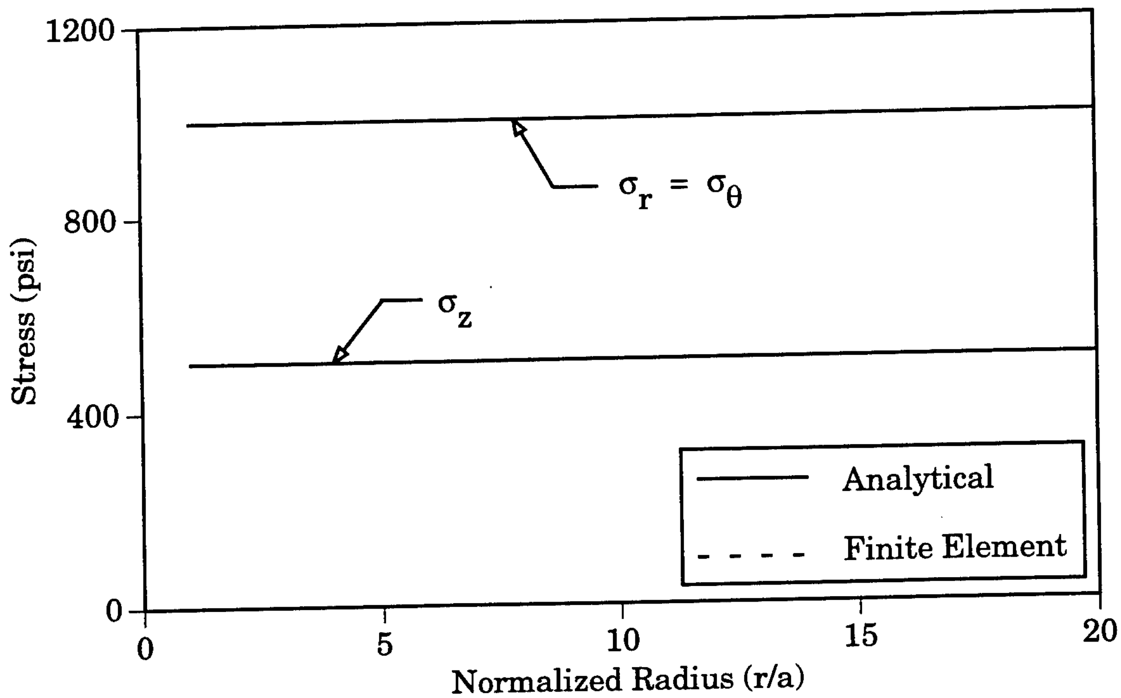
SECTION 6
NUMERICAL EXAMPLES

Numerical comparisons of the results from finite element calculations with the results from the closed-form solutions discussed in the preceding sections are presented in this section. Two examples for each of the four cases (Cases Ia, Ib, IIa, and IIb) are presented: one example with M equal to N (associative flow rule) and one example with M equal to one (nonassociative flow rule). Results using the two solution techniques are presented in terms of: (1) a table of differences in computed closure for a given load level; (2) plots of external pressure, p_b, versus closure; and (3) plots of stress and displacement versus normalized radius (r/a) at given load levels. The material properties and loads used for each example are provided in Table 6-1. The material properties were chosen to correspond to a soft rock. For each example, the radius of the hole was taken as unity and loading was applied until closure of approximately five percent was achieved. Except for the Case Ia examples, where free-field yielding cannot occur, the load levels were high enough to cause the free field to yield.

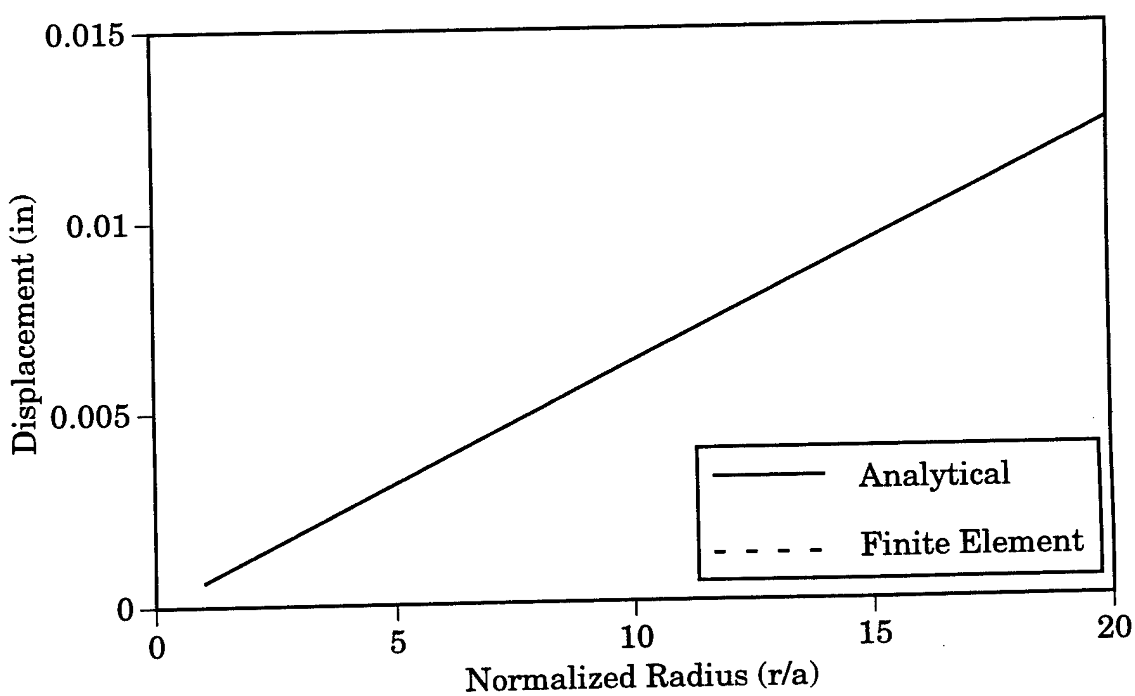
Prior to performing the nonlinear finite element calculations, a mesh discretization study was performed using a linear material. The objective of that study was to determine: (1) the size of the mesh required to approximate an infinite medium; and (2) the number of elements required to achieve accurate results. For the linear problem, we used Young's modulus of one million psi, Poisson's ratio of 0.25, and a hole radius of one inch. Loading was applied in two steps. In the first step, the internal and external pressures were both equal to one thousand psi. In the second step, the internal pressure was held at one thousand psi, while the external pressure was increased to two thousand psi. A mesh containing four hundred elements with an external radius of two hundred inches gave displacement and stress results accurate to four decimal places. This level of mesh refinement may be overly conservative, but was selected to eliminate the possibility of a coarse mesh being a significant source of error in the nonlinear finite element calculations. The linear analysis results are presented in Figure 6-1. Note that the finite element results (dashed lines) are not distinguishable from the closed-form solution (solid lines). In Figure 6-1 and all subsequent figures, compression is shown positive.

Table 6-1. Material and load parameters for example problems.

Example Number	Case	G (psi)	n	s_u (psi)	N	M	p_a(psi)	p_b (psi)
1	Case Ia	45,000	0.3	200	3.0	3.0	100	1,100
2	Case Ib	45,000	0.1	200	3.0	3.0	30	675
3	Case IIa	45,000	0.1	200	3.0	3.0	50	750
4	Case IIb	45,000	0.1	200	3.0	3.0	200	1,200
5	Case Ia	45,000	0.3	200	3.0	1.0	100	1,500
6	Case Ib	45,000	0.1	200	3.0	1.0	30	930
7	Case IIa	45,000	0.1	200	3.0	1.0	50	1,000
8	Case IIb	45,000	0.1	200	3.0	1.0	200	1,425

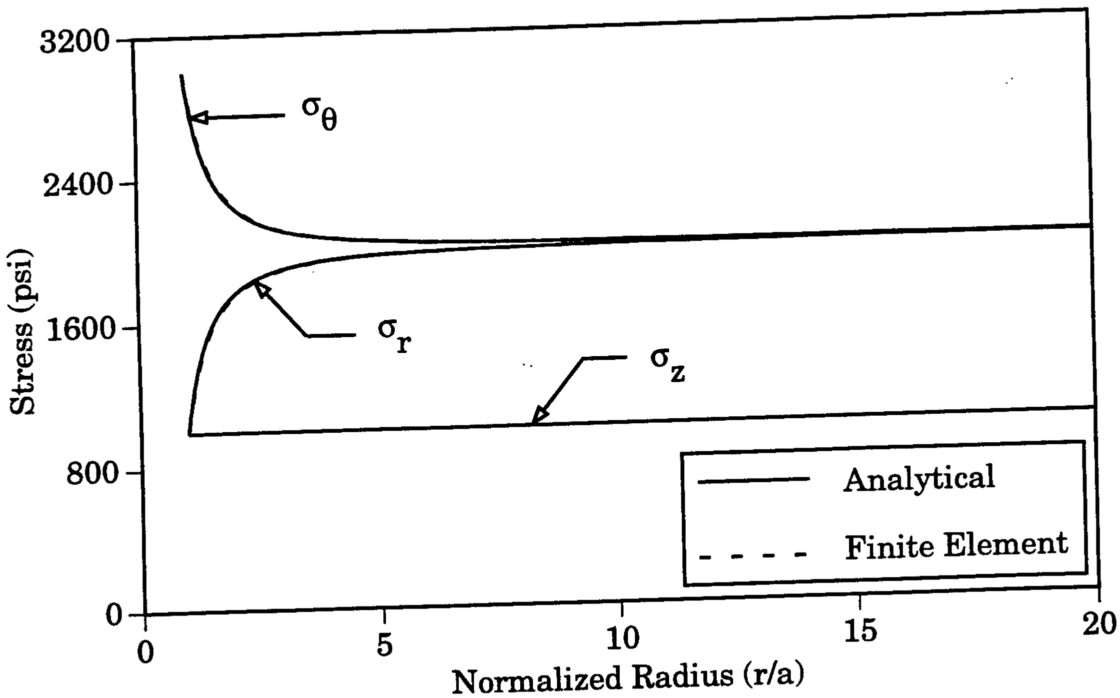


a. Step 1 stress profiles.

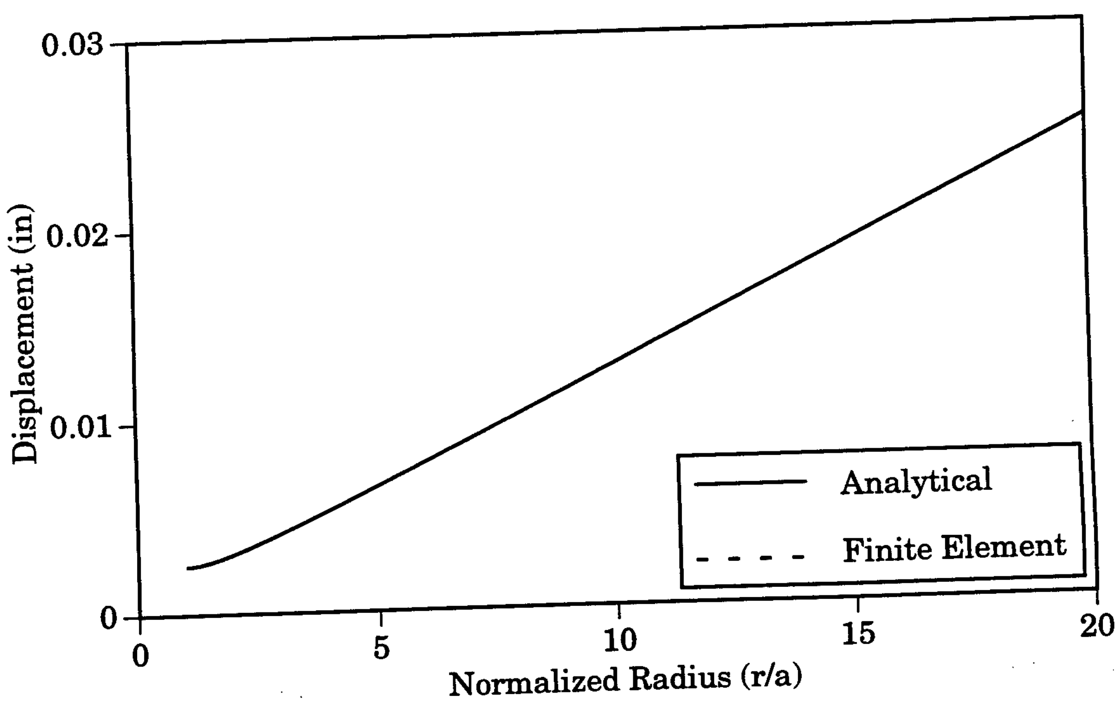


b. Step 1 displacement profile.

Figure 6-1. Comparisons of analytical and finite element analyses for linear material response.



c. Step 2 stress profiles.



d. Step 2 displacement profile.

Figure 6-1. Comparisons of analytical and finite element analyses for linear material response (Continued).

Closure results at the final load level for each of the eight nonlinear cases are presented in Table 6-2. The maximum difference in closure between the finite element and the closed-form solutions was 0.337 percent. In performing these analyses, we found that the differences in computed closures could be lessened by reducing the load increment and tightening the convergence tolerances in the nonlinear finite element analyses. This was particularly true in the nonassociative flow examples, where the elastic-plastic constitutive matrix (required for implicit finite element analysis utilizing Newton-Raphson techniques for solving nonlinear equations) becomes nonsymmetric. Our finite element program utilized a symmetric equation solver, and thus an approximation to the true elastic-plastic constitutive matrix had to be used. Although we may have been able to achieve even closer matches between analytical and numerical results, the differences presented in Table 6-2 were judged to be small enough to convince us that the hypotheses used in developing the analytical solutions were justified.

Plots of external pressure, p_b, versus closure for the eight example problems are shown in Figures 6-2 through 6-9. There is no discernible difference between the closed-form and finite element results.

Plots of stress (σ_r, σ_θ, and σ_z) versus normalized radius (r/a) and radial displacement versus normalized radius (r/a) for each of the eight cases are presented in Figures 6-10 through 6-17. Again, there is no discernible difference between the analytical and finite element results.

Table 6-2. Closure results.

Example Number	Applied External Pressure, p_b (psi)	Closure (%) (Finite Element)	Closure (%) (Analytical)	Difference (%) *
1	1,100	5.7390	5.7481	-0.158
2	675	5.0254	5.0347	-0.185
3	750	5.0573	5.0744	-0.337
4	1,200	4.9428	4.9479	-0.103
5	1,500	5.0083	5.0222	-0.277
6	930	4.9811	4.9976	-0.330
7	1,000	4.9895	5.0021	-0.252
8	1,425	5.1112	5.1180	-0.117

* Difference (in percent) is computed by subtracting the finite element solution from the analytical solution, then dividing by the analytical solution, and multiplying by 100.

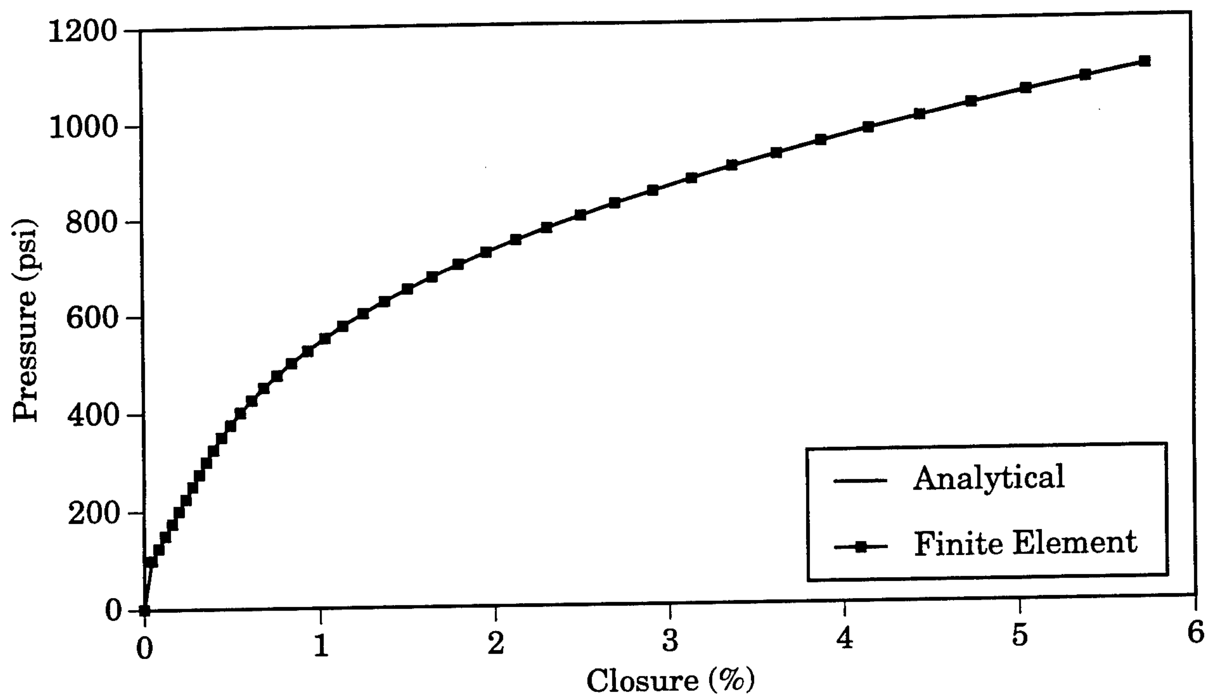


Figure 6-2. Pressure versus closure (Example 1).

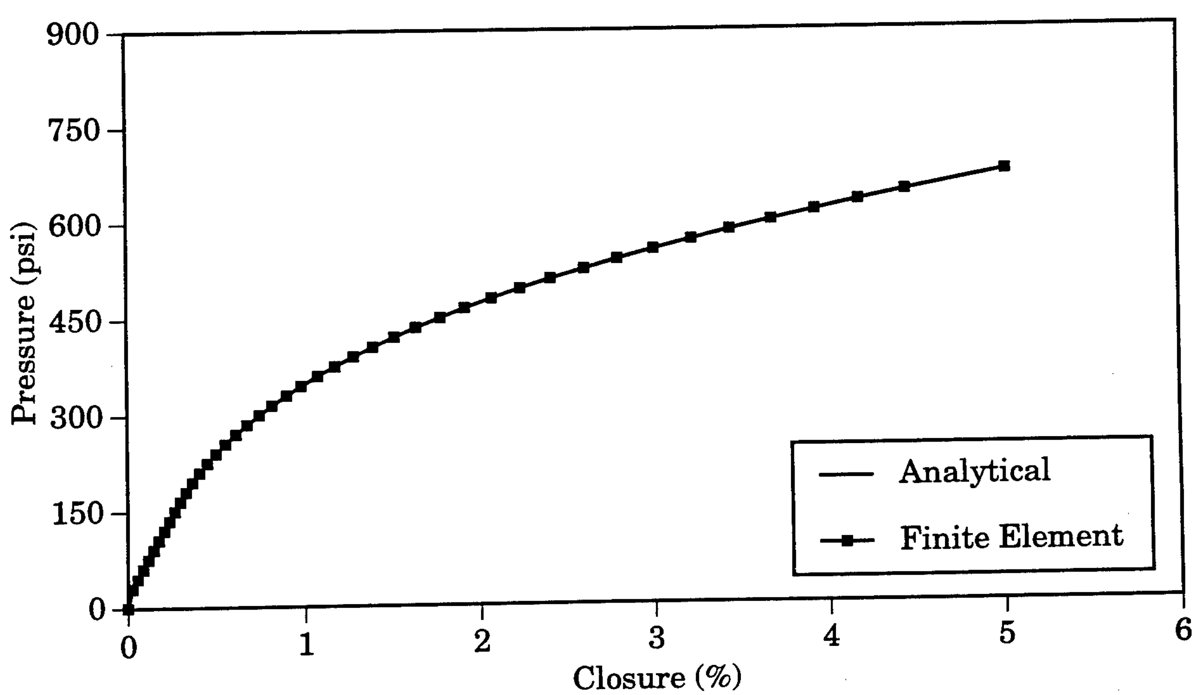


Figure 6-3. Pressure versus closure (Example 2).

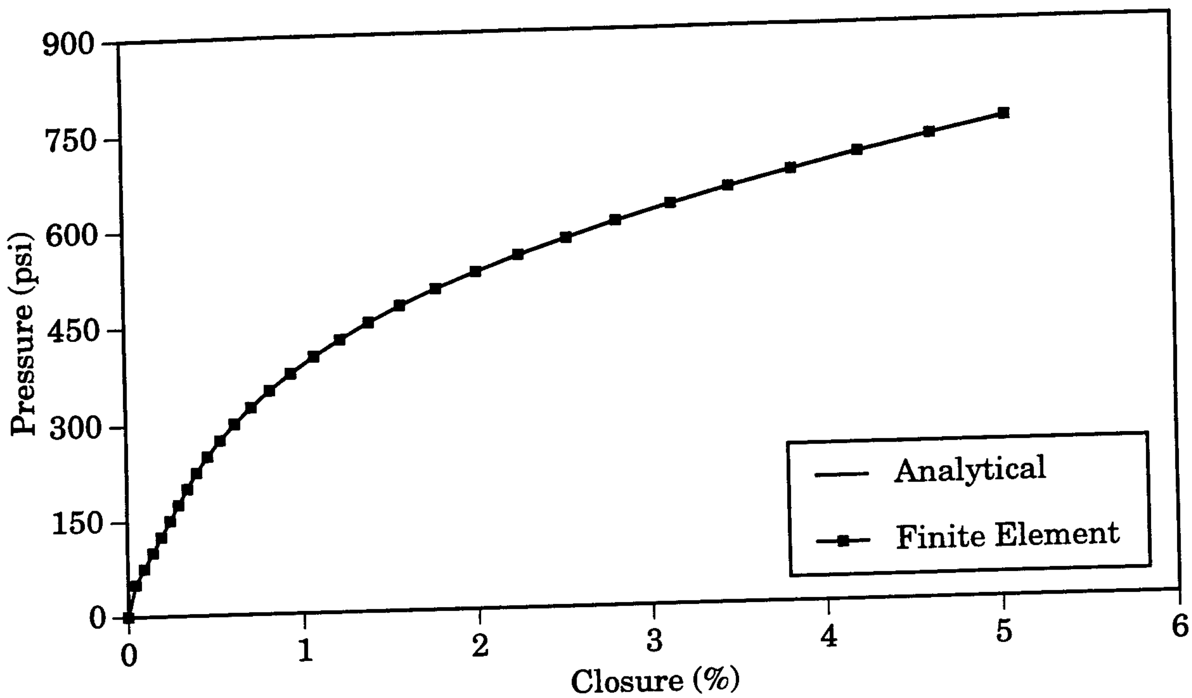


Figure 6-4. Pressure versus closure (Example 3).

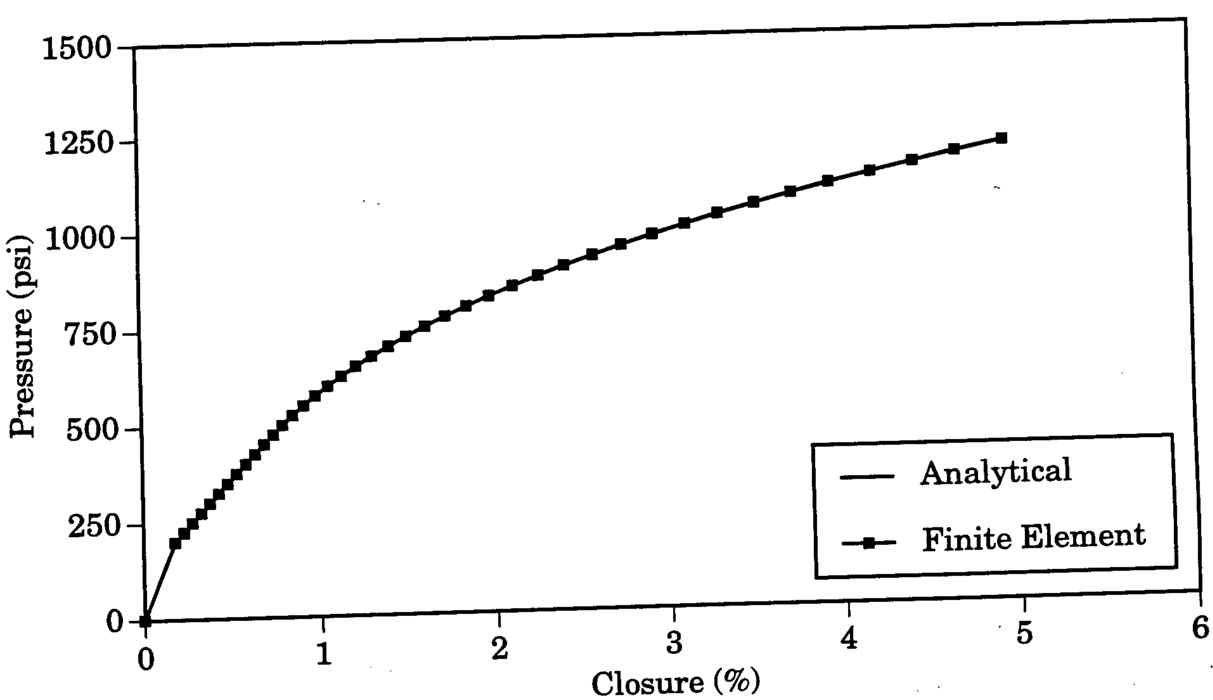


Figure 6-5. Pressure versus closure (Example 4).

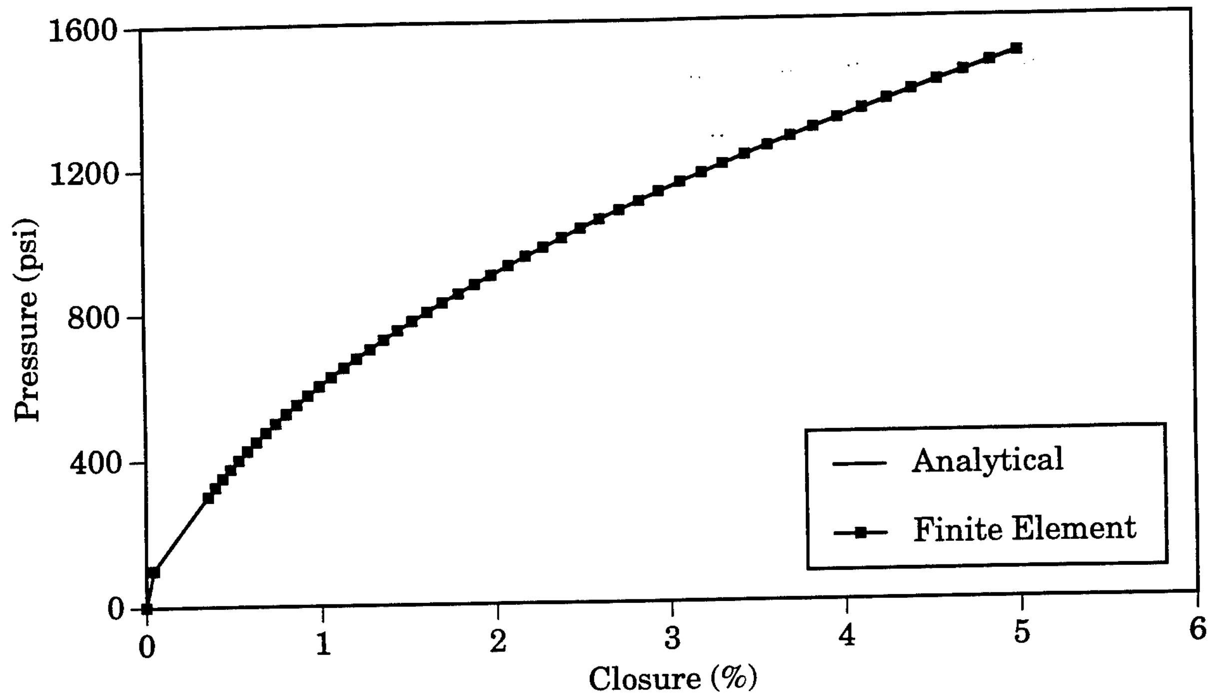


Figure 6-6. Pressure versus closure (Example 5).

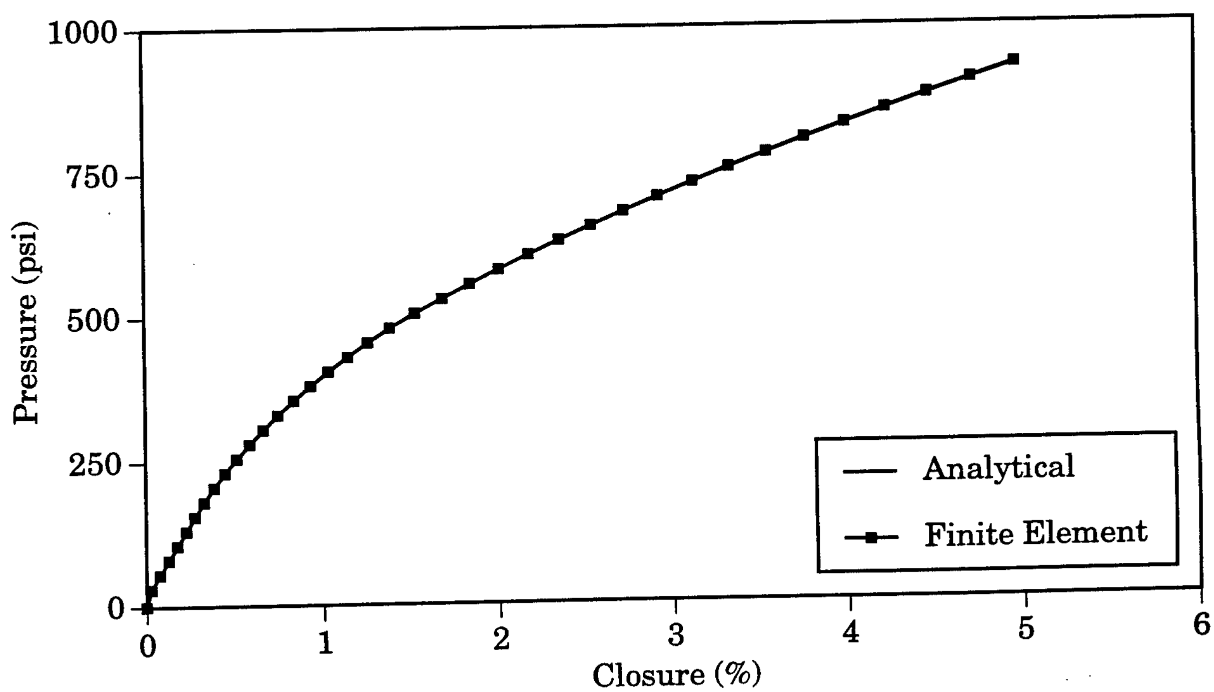


Figure 6-7. Pressure versus closure (Example 6).

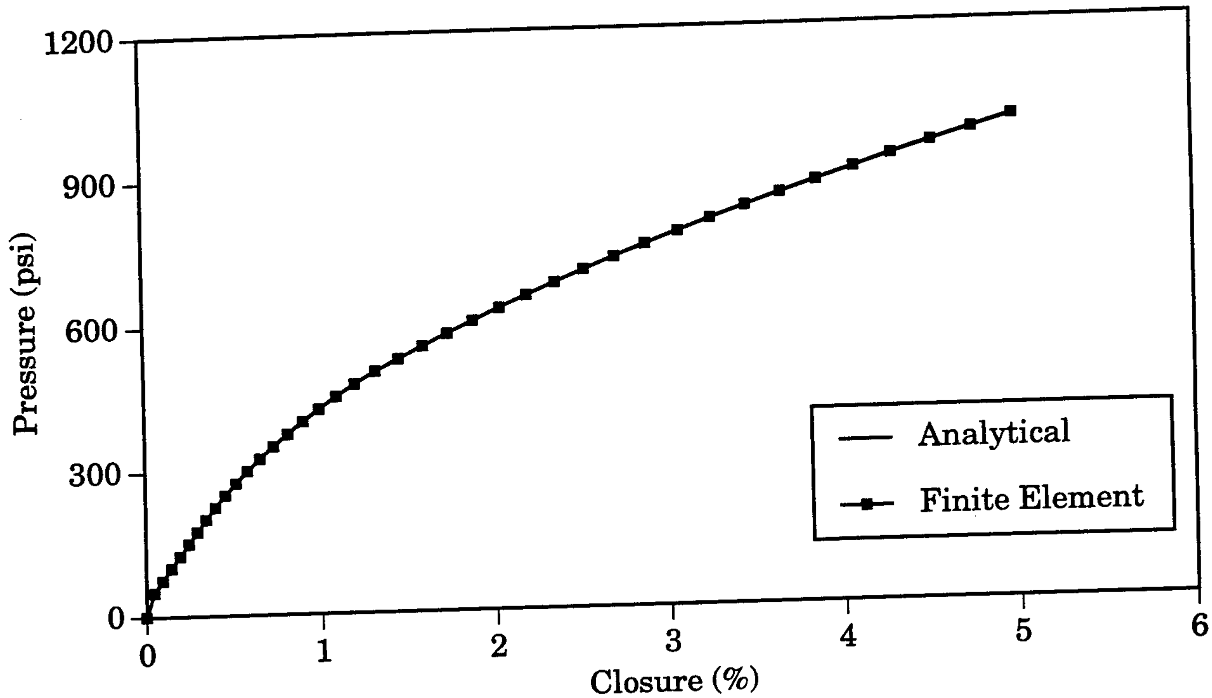


Figure 6-8. Pressure versus closure (Example 7).

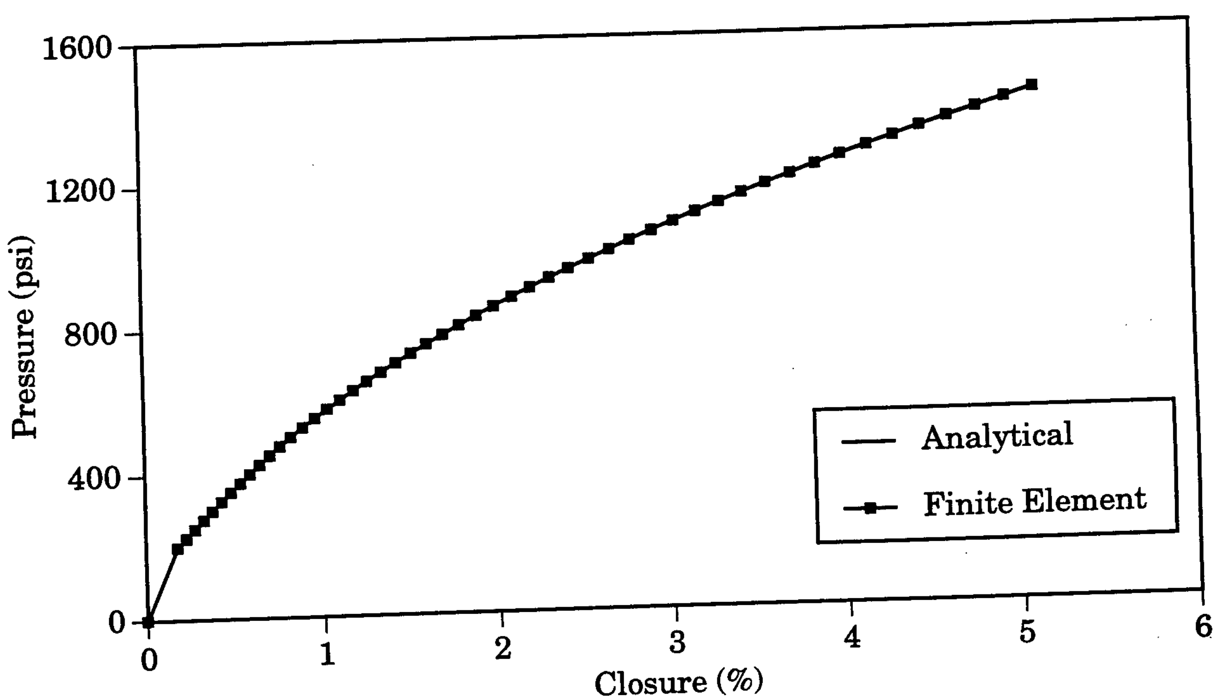
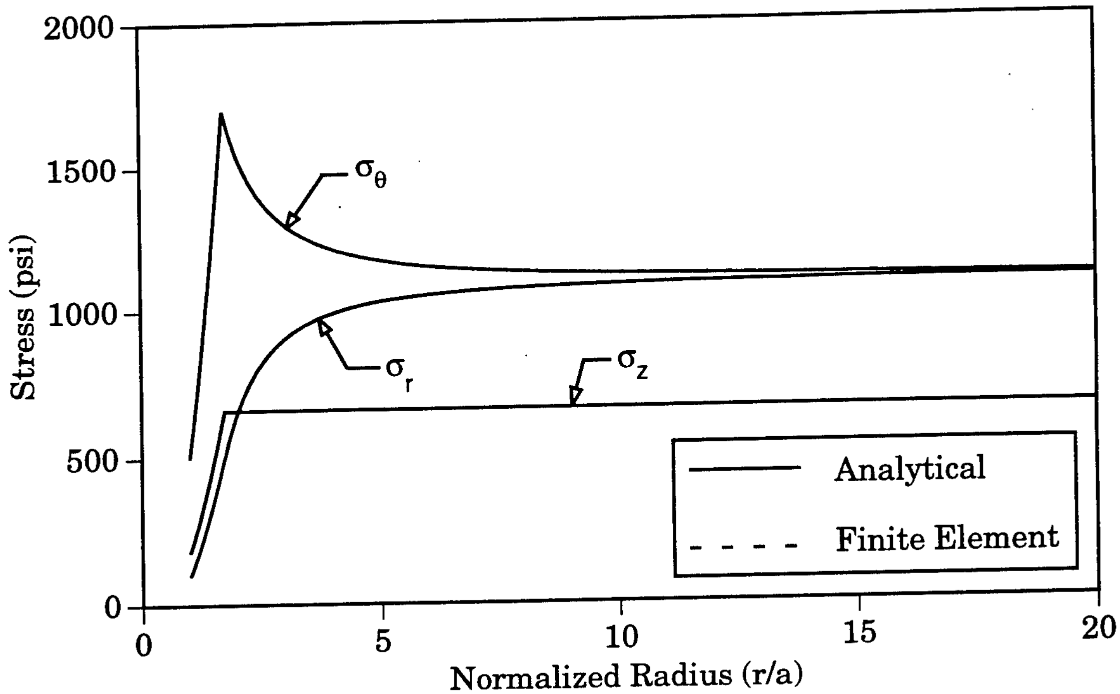
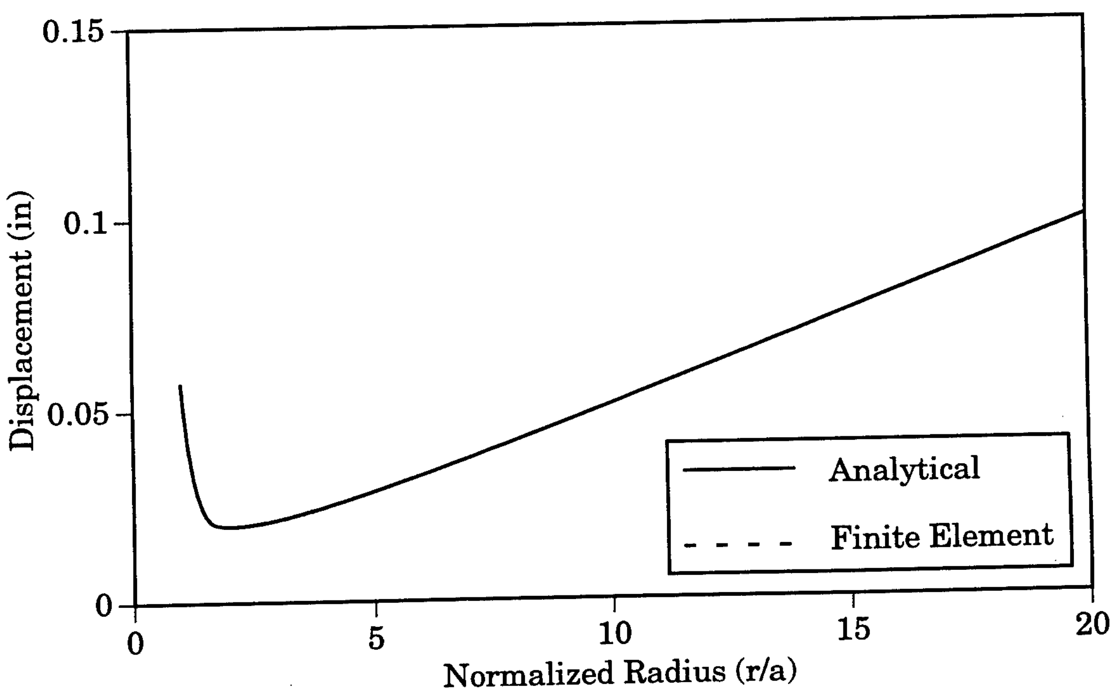


Figure 6-9. Pressure versus closure (Example 8).

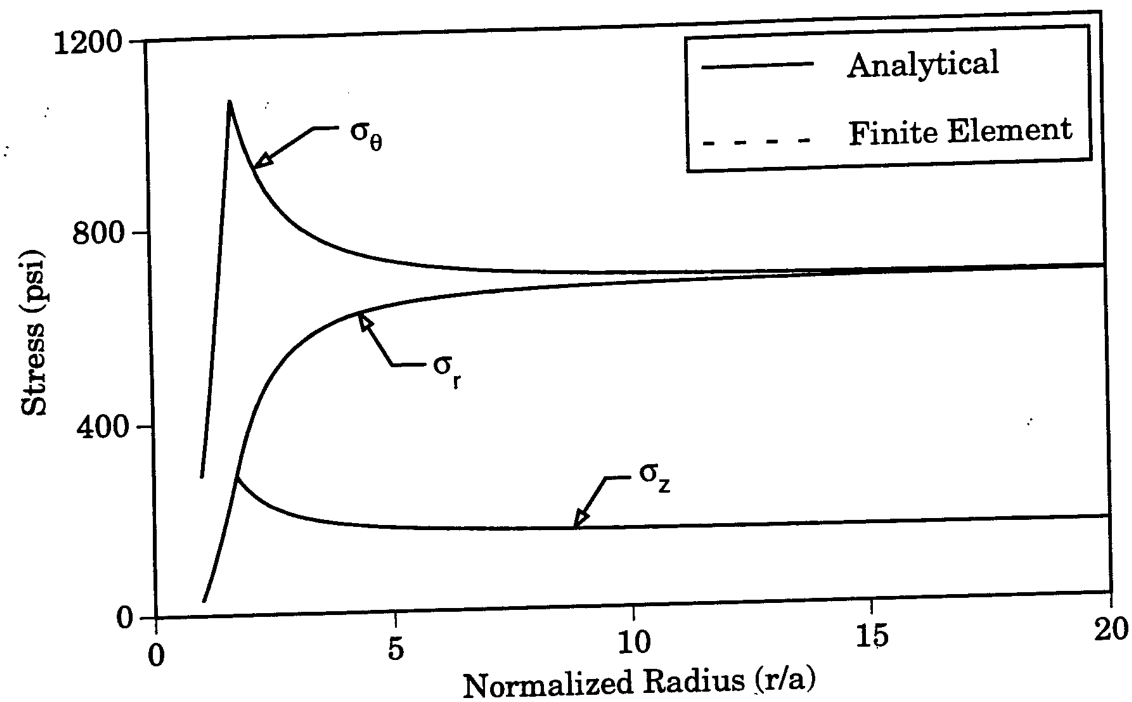


a. Stress profile.

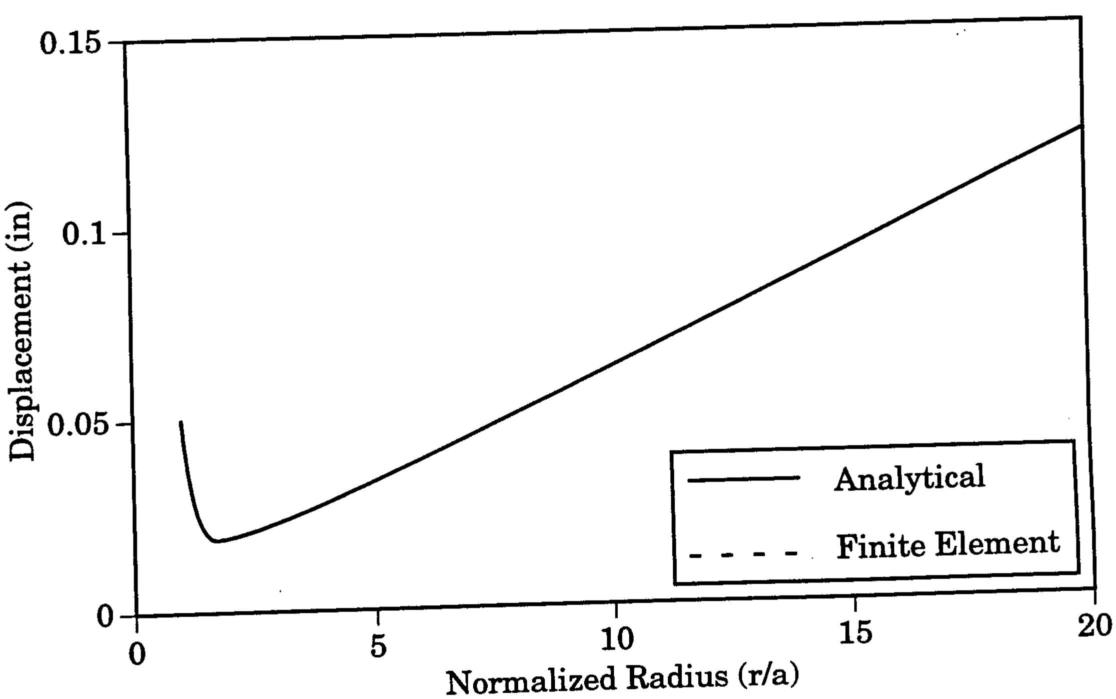


b. Displacement profile.

Figure 6-10. Example 1 response profiles.

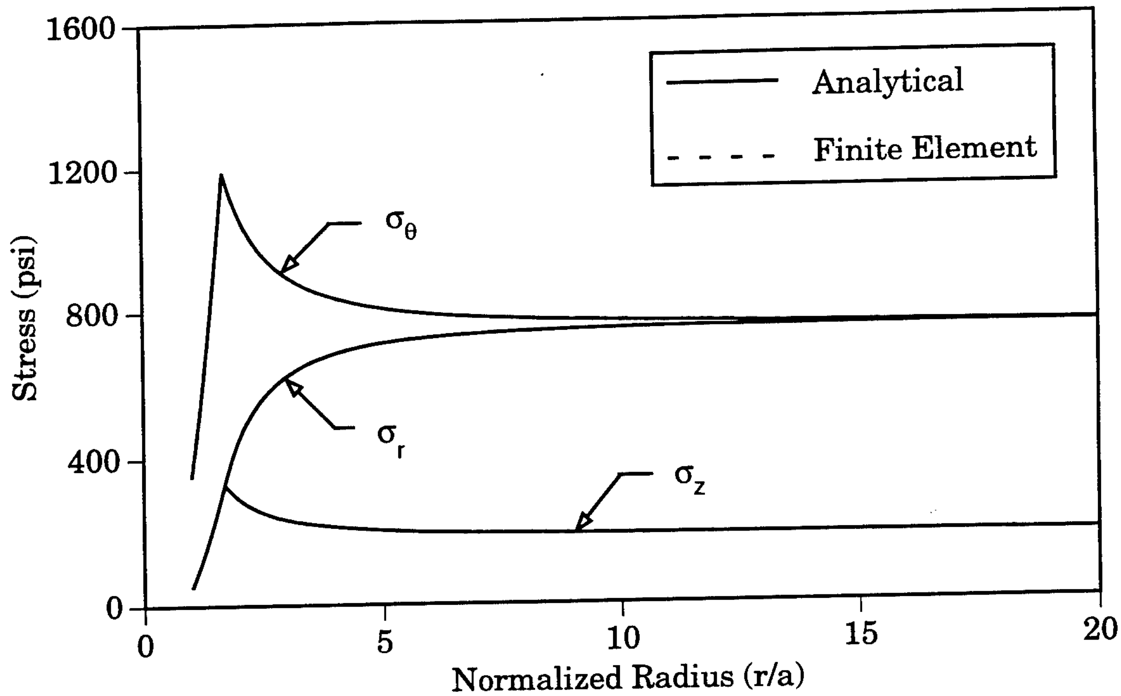


a. Stress profile.

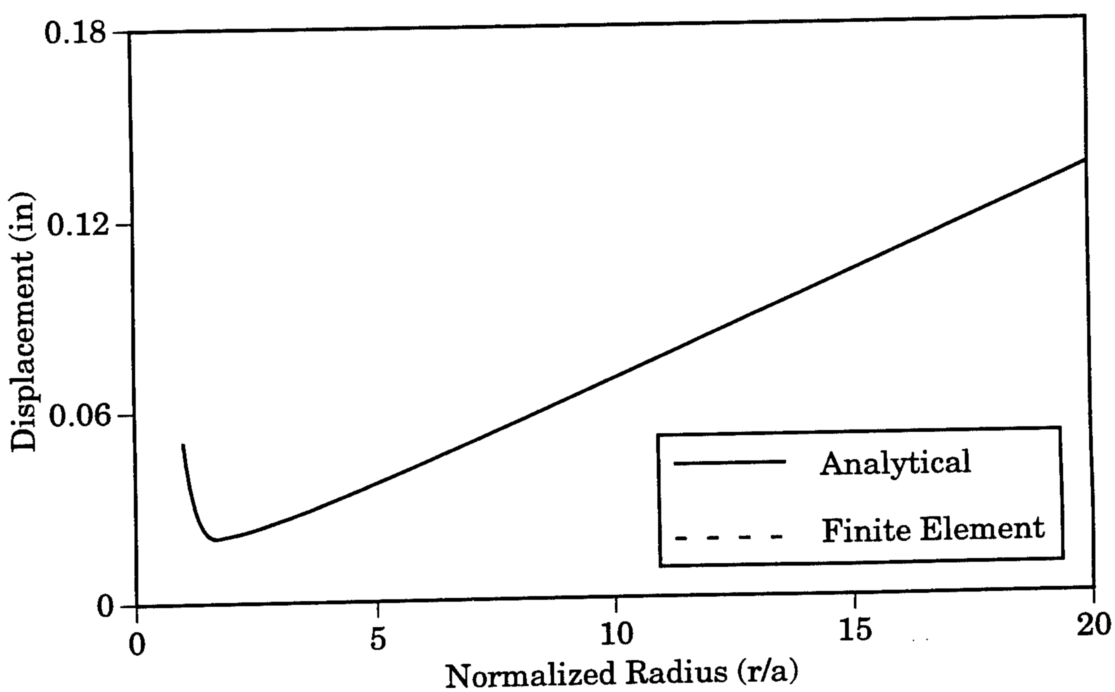


b. Displacement profile.

Figure 6-11. Example 2 response profiles.

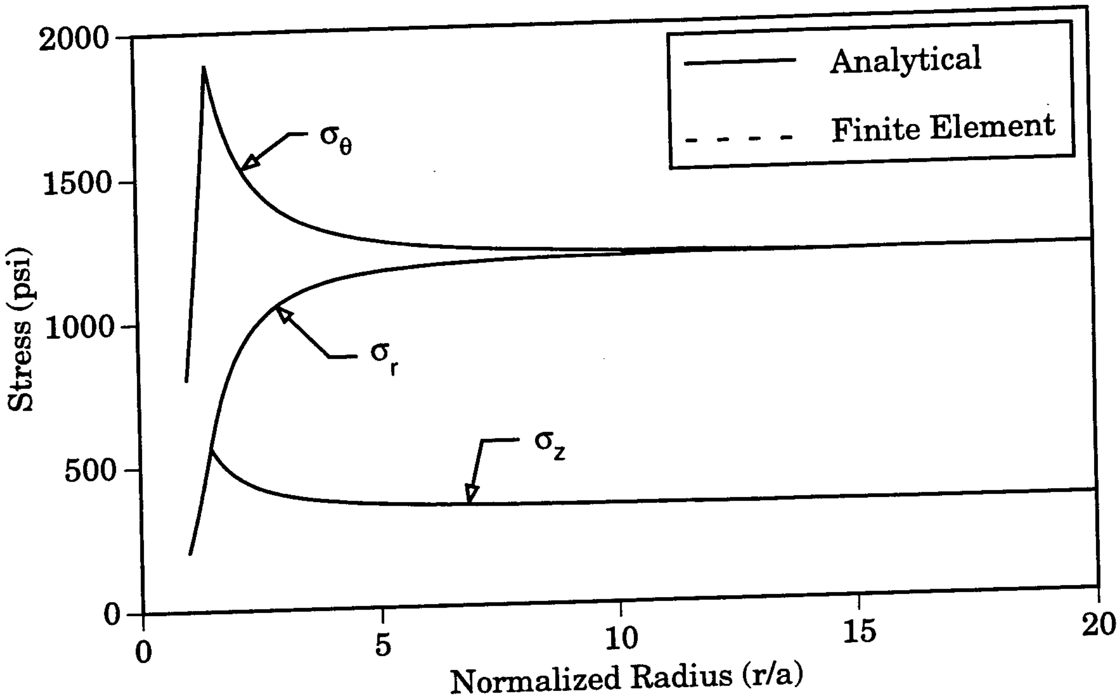


a. Stress profile.

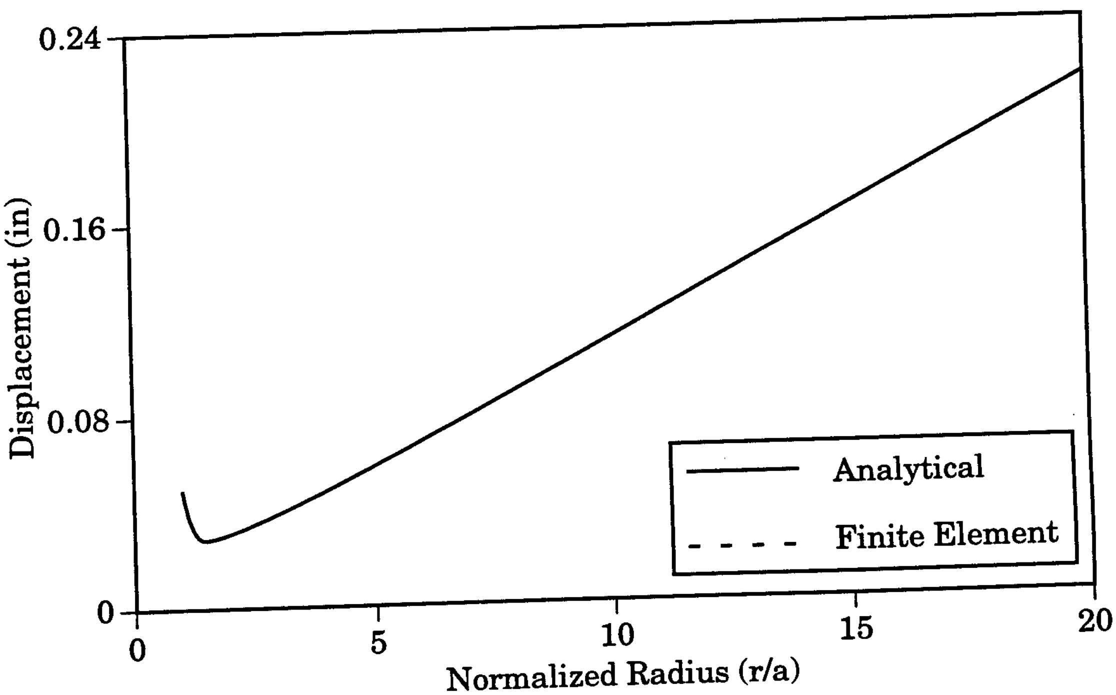


b. Displacement profile.

Figure 6-12. Example 3 response profiles.

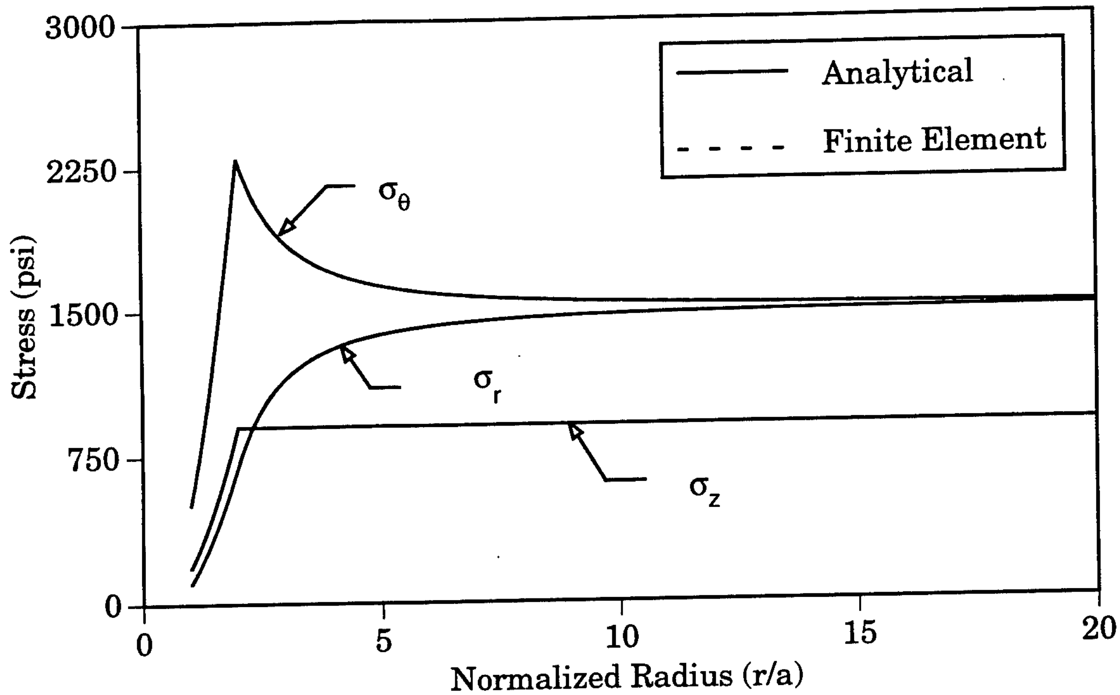


a. Stress profile.

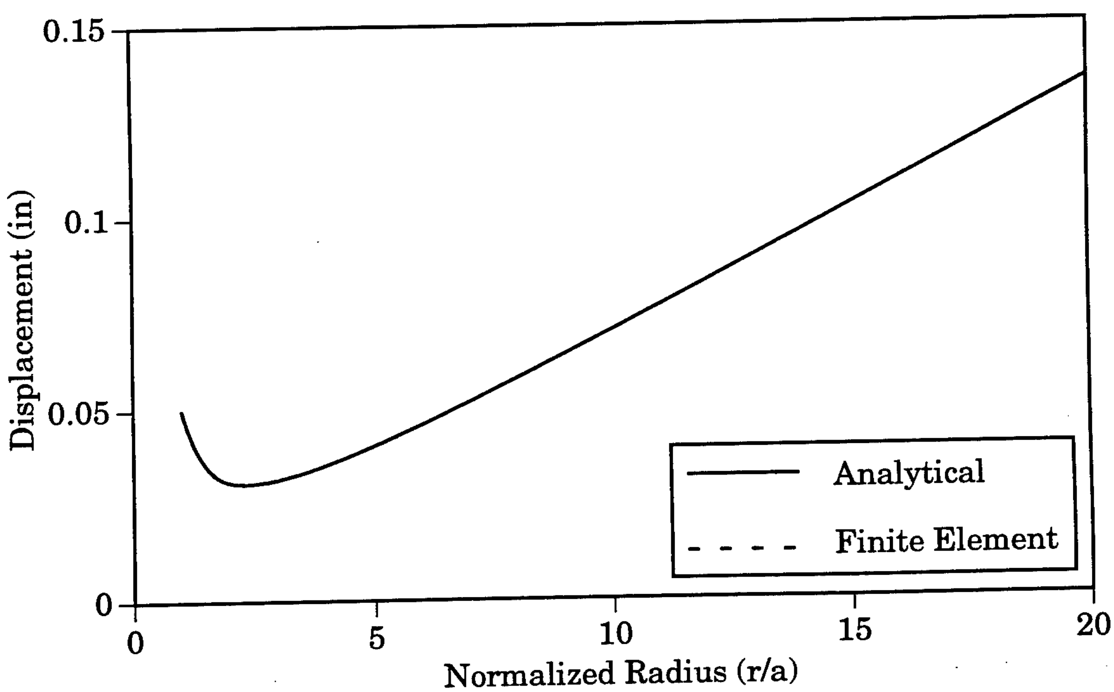


b. Displacement profile.

Figure 6-13. Example 4 response profiles.

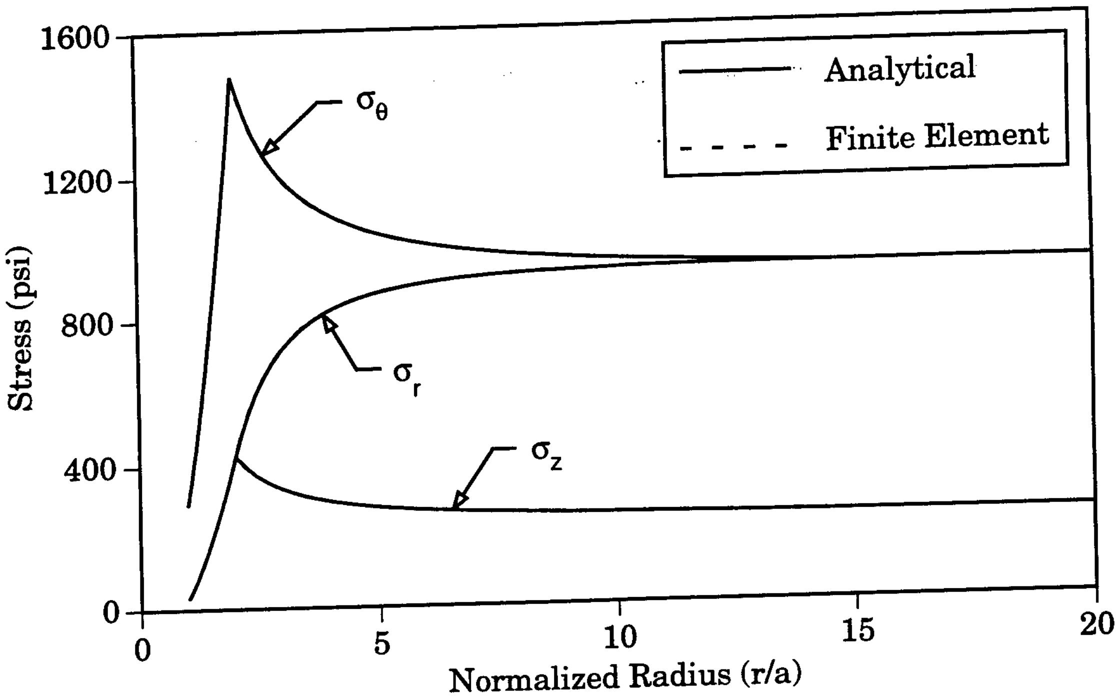


a. Stress profile.

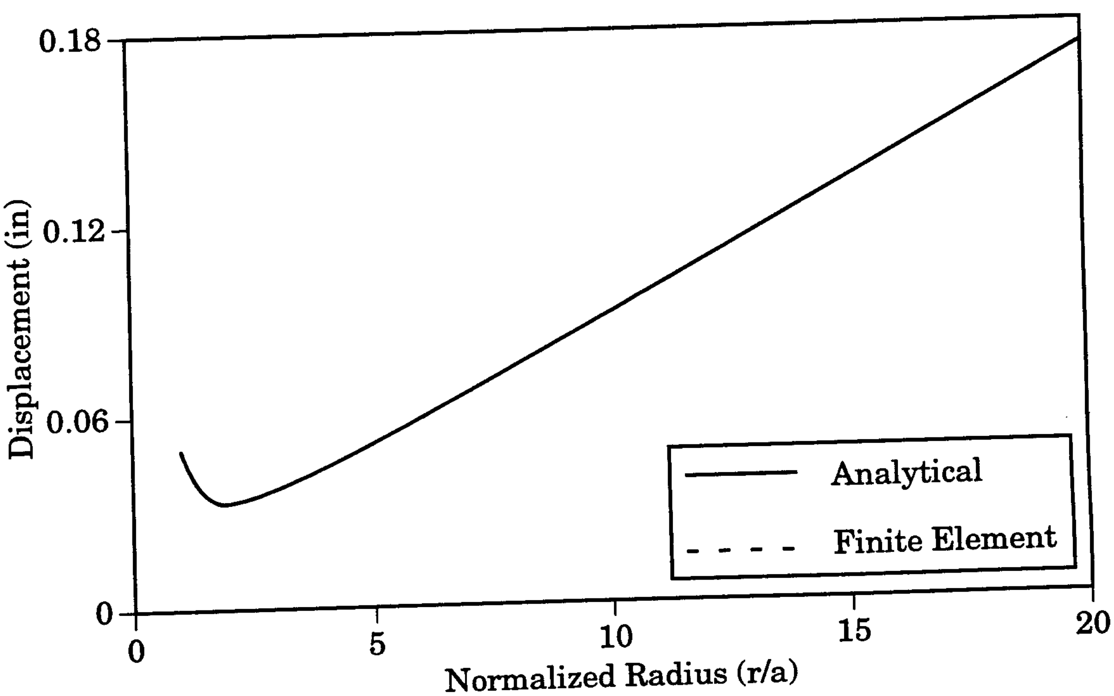


b. Displacement profile.

Figure 6-14. Example 5 response profiles.

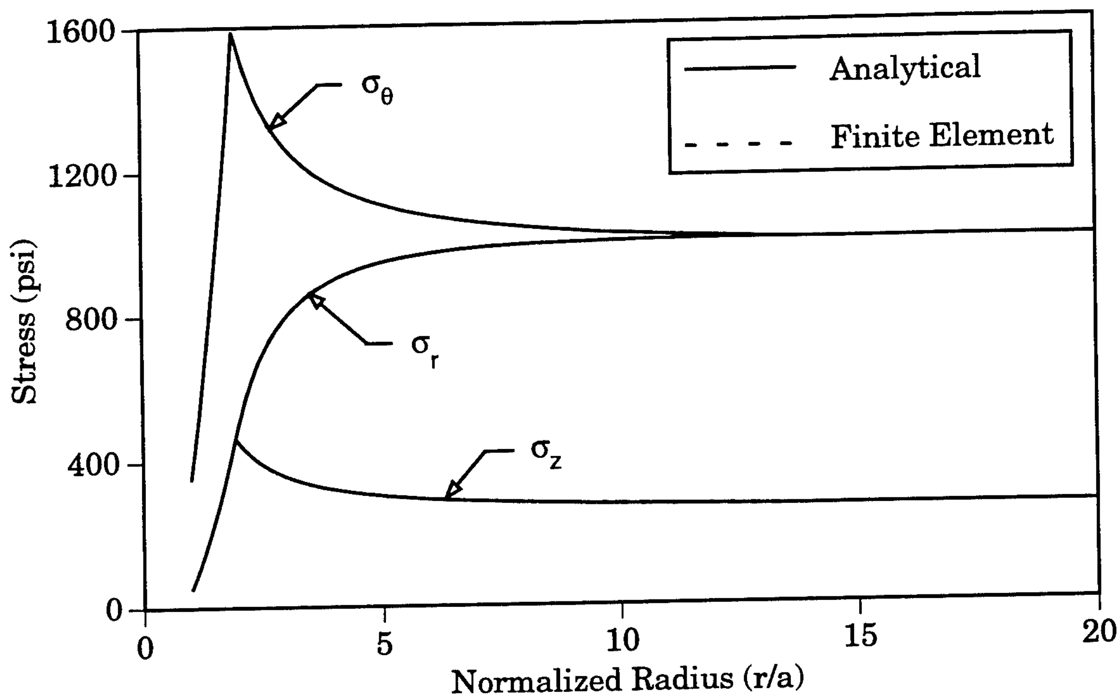


a. Stress profile.

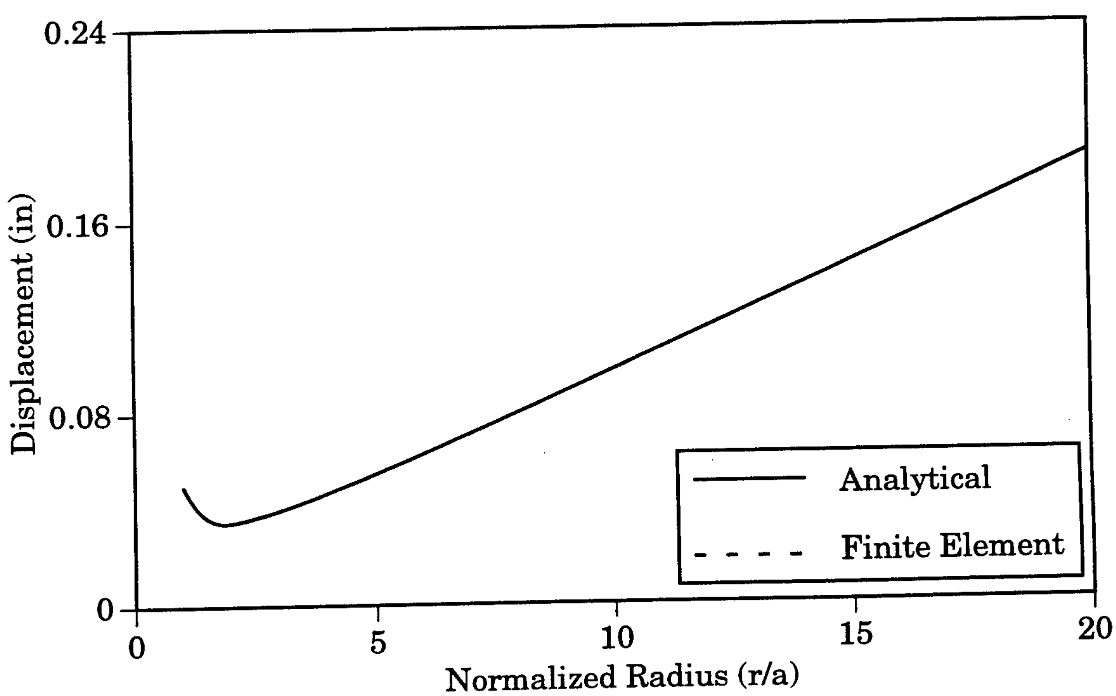


b. Displacement profile.

Figure 6-15. Example 6 response profiles.

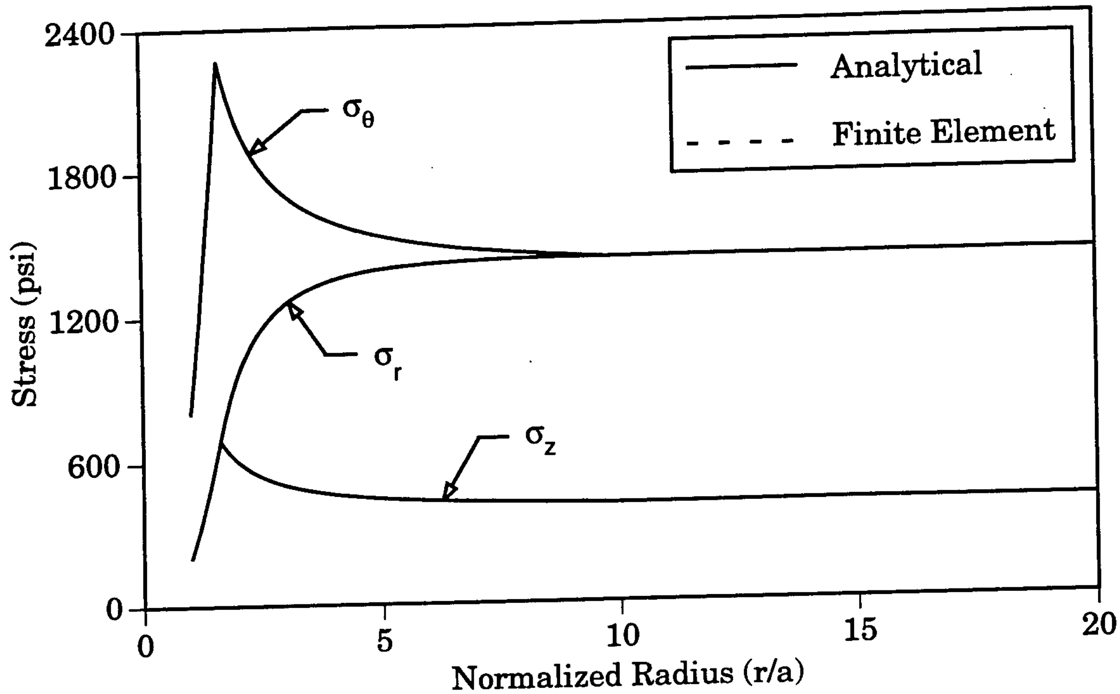


a. Stress Profile.

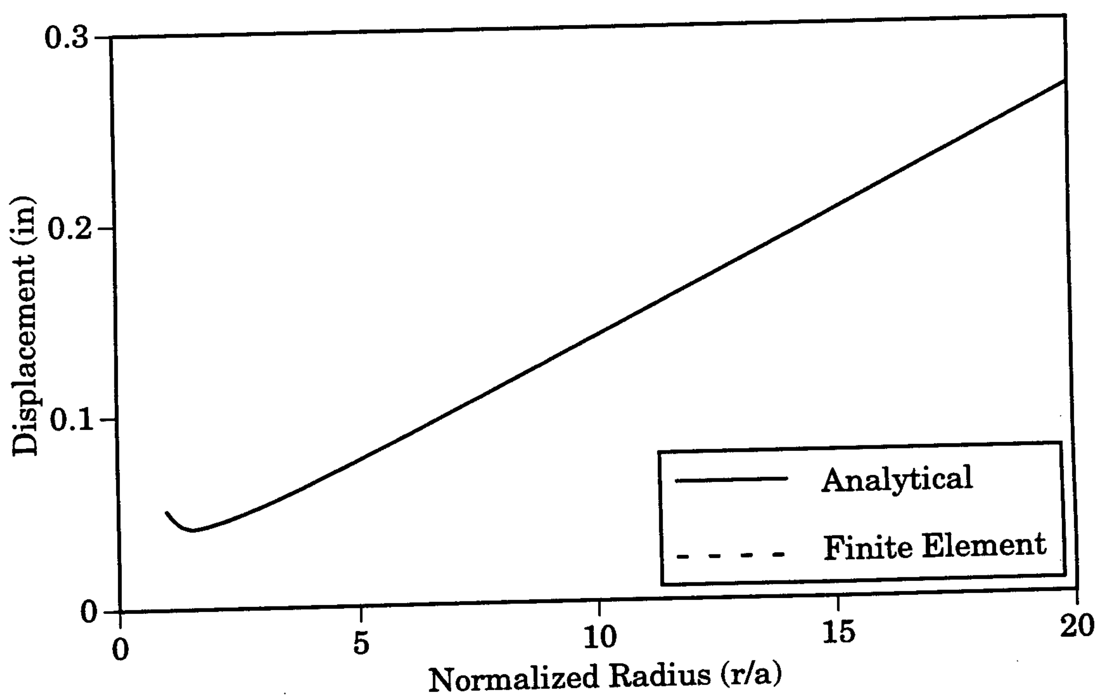


b. Displacement profile.

Figure 6-16. Example 7 response profiles.



a. Stress profile.



b. Displacement profile.

Figure 6-17. Example 8 response profiles.

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