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THE PICARD GROUP OF AN INCIDENCE RING

OF A FINITE PREORDERED SET OVER A FIELD

by

TRAE DOUGLAS HOLCOMB

B.S., Southwest Texas State University, 1991

A thesis submitted to the

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DEDICATION

I dedicate this thesis to my wife, Stephanie, who was the single most important factor in the completion of this project in a fairly short period of time. I commend her ability to manage our household while also finding ways to creatively occupy our daughter, Kaitlyn, even when I was working just down the hall. She is a true friend and a valued partner.

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CHAPTER I

INTRODUCTION

Although the history of algebra dates back almost three thousand years, much of modern algebra grew out of a desire to solve many number theoretical questions posed as recently as the sixteenth and seventeenth centuries. Many of these questions, including Fermat's Last Theorem, fascinated mathematicians such as Kummer and Dedekind, prompting them to formalize modern algebra by defining and studying the classic algebraic constructs. Dedekind, for example, developed the notion of an ideal to generalize the ideal numbers Kummer investigated. Commutative ring theorists in turn developed the ideal class group to measure the distance between a Dedekind domain and a principal ideal domain. In this thesis, we are particularly interested in studying yet a further generalization of the ideal class group--the Picard group.

To further connect this paper to the work of number theorists, we also consider the incidence algebra, which Rota introduced to generalize the Möbius inversion formula. We combine the study of these two classical constructs from number theory, and we completely investigate the Picard group of an incidence algebra of a finite preordered set over a field. In the process, we are able to formulate a structure theorem for the automorphism group of such an algebra and to solve a question pertaining to invariance under Morita equivalence. With that said, the remainder of this chapter is dedicated to making the statements of these problems and the associated terminology specific.

Throughout this thesis, we assume all rings are unital. We write group and ring homomorphisms on the right and module homomorphisms opposite the scalars. Recall that a reflexive and transitive binary relation is a **preorder**. If it is also antisymmetric, then it is a **partial order**. We let K be a field and $P=(V(P),\rho(P))$ be an arbitrary finite preordered set with vertex set $V=V(P)$ and relation set $\rho(P)$. We employ the notation $(i,j)\in\rho(P)$ to mean $i,j\in V$ and $i\leq j$ with respect to the ordering on P.

Letting $\overline{\rho}(P)=\{(i,j)|(i,j),(j,i)\in\rho(P)\}$, we define an equivalence relation $\sim$ on V via $i\sim j$ if and only if $(i,j)\in\overline{\rho}(P)$. Let $[i]$ denote the equivalence class of i for each $i\in V$, and let $\tilde{V}$ be a set of class representatives. Set $\tilde{\rho}(P)=[\tilde{V}\times\tilde{V}]\cap\rho(P)$. It is easy to verify that $\tilde{P}=(\tilde{V},\tilde{\rho}(P))$ is a poset which we call the **underlying poset** of P. By convention, $V(\tilde{P})=\tilde{V}$ and $\rho(\tilde{P})=\tilde{\rho}(P)$.

The **incidence algebra** $I(P)$ of P over K is the algebra of all functions from the set of pairs $(i,j)\in\rho(P)$ into K with addition given pointwise and multiplication given by $fg(i,j)=\sum_{i\leq k\leq j}f(i,k)g(k,j)$. The **structure matrix ring** $M(P)$ of P over K is the subring of the $|V|\times|V|$ matrix ring over K such that $M\in M(P)$ if and only if $M_{i,j}=0$ provided $(i,j)\notin\rho(P)$. Since $I(P)$ is ring isomorphic to $M(P)$ as in [21, Proposition 1.2.4], we henceforth associate $S=I(P)$ with $M(P)$. That is, we represent an element in S by the corresponding element in the structure matrix ring.

There is ample interest in understanding the structure of *Aut*(*S*), the multiplicative group of automorphisms of *S*, with special attention given to the subgroup of automorphisms of *S* which fix *K* as viewed inside *S*, which we denote by $Aut_K(S)$. See [4], [6], [7], [10], [15], [20], and [22]. We therefore state our first problem as follows:

Problem A. Represent *Aut*(*S*) as a semidirect product of several subgroups.

In [6], Coelho was able to represent $Aut_K(S)$ as the semidirect product of three particular subgroups. In Chapter III, we extend this representation to one for *Aut*(*S*) as a semidirect product of four subgroups.

The **outer automorphism group of S** is defined to be the factor group $Out(S) = Aut(S) / Inn(S)$ where $Inn(S)$ is the normal subgroup of $Aut(S)$ formed by the inner automorphisms of *S*. The decomposition of $Aut(S)$ arising from Problem A and the strong connection it allows us to demonstrate between $Out(S)$ and $Out(\tilde{S})$, where $\tilde{S}$ is the basic ring of *S*, become the primary tools we use to solve the remainder of the problems posed in this paper.

The **Picard group** $Pic(S)$ of a ring *S* is the set of invertible *S*-bimodule isomorphism classes with the group operation being the tensor product, which we denote by ⊗. The Picard group is of particular interest since it is a special subset of the projective right (and left) *S*-modules, which are instrumental in the study of module decomposition.

For $\phi \in Aut(S)$, we define ${}_\phi S = S$ as a set and as a right *S*-module. We define left multiplication by elements of *S* via $s * x = (s)^\phi x$ for $x \in {}_\phi S$ and $s \in S$. By [8], ${}_\phi S$ is an invertible S-bimodule with inverse ${}_{\phi^{-1}}S$ and therefore represents an element in $Pic(S)$. However, not every element of $Pic(S)$ is necessarily of the form $[{}_{\phi}S]$ for some $\phi \in Aut(S)$. This leads to the statement of

> **Problem B.** For $[X] \in Pic(S)$, determine the structure of X in terms of automorphisms of S and automorphisms of the underlying poset of P.

which we discuss in Chapter V. Specifically, we give a construction in Definition 5.5 generalizing the ${}_{\phi}S$ construction.

The mathematical literature demonstrates interest in knowing which ring properties are invariant under Morita equivalence. For example, the property of being artinian is a Morita invariant and is possessed by incidence rings. However, the group of outer automorphisms of an incidence ring is not an invariant. For example, $\tilde{S} \cong Q \times Q$ is Morita equivalent to $S \cong M_2(Q) \times Q$ since it is the basic ring for S. However, given the machinery of Problem A, the reader may verify that $Out(S) = 1 \not\cong \mathbf{Z}_2 = Out(\tilde{S})$. This motivates the statement of

> **Problem C.** Determine necessary and sufficient conditions so that $Out(S)$ is naturally invariant for the Morita equivalence class of S (relative to the collection of incidence K-algebras).

We provide the solution and the details of "natural invariance" in Chapter V.

CHAPTER II

PRELIMINARIES

Throughout this chapter, let R and R' be rings. Recall that $Pic(R)$ is the group formed by the set of invertible R-bimodule isomorphism classes under the tensor product. Also recall that for $\phi \in Aut(R)$, we have the invertible R-bimodule ${}_{\phi}R$, where left multiplication by elements of R is defined via $r * x = (r)^{\phi} x$ for $x \in_{\phi} R$ and $r \in R$.

The facts and results presented in this chapter will lay the foundation for the work in subsequent chapters. Specifically, for an incidence ring S with basic ring eSe, we construct a commutative diagram

$$\begin{array}{ccc} Out(S) & \longrightarrow & Pic(S) \\ \downarrow & & \downarrow \\ Out(eSe) & \longrightarrow & Pic(eSe) \end{array}$$

in Theorem 2.10 which we use in Chapters IV and V to explore the connection between $Out(S)$ and $Out(eSe)$ and to ultimately solve Problems B and C.

Definition. We say that ${}_R M$ is a **progenerator** for the category $R-mod$ if M is a finitely-generated, left R-module such that R is isomorphic to a direct summand of $M^{(n)}$ as left R-modules for some integer n. Consequently, M generates R-mod, the category of left R-modules.

Let M be an (R,R')-bimodule. As in [8], we say ${}_R M_{R'}$ is **invertible** if ${}_R M$ is a progenerator for $R-mod$ and $End({}_R M) \cong R'$ as rings. Equivalently, ${}_R M_{R'}$ is invertible if there exists an (R',R)-bimodule N and bimodule isomorphisms $\theta: M \otimes_{R'} N \to R$ and $\tau: N \otimes_R M \to R'$ making the diagrams

$$\begin{array}{ccc} M \otimes N \otimes M & \xrightarrow{\theta \otimes 1_M} & R \otimes M \\ \downarrow^{1_M \otimes \tau} & & \downarrow^{\mu} \\ M \otimes R' & \xrightarrow{\mu} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} N \otimes M \otimes N & \xrightarrow{\tau \otimes 1_N} & R' \otimes N \\ \downarrow^{1_N \otimes \theta} & & \downarrow^{\mu} \\ N \otimes R & \xrightarrow{\mu} & N \end{array}$$

commute where the μ are the standard multiplication map isomorphisms.

We now work toward establishing a connection between $Aut(R)$ and $Pic(R)$.

Lemma 2.1. Let $\phi \in Aut(R)$.

(i) ${}_\phi R \cong {}_R R$ as left R-modules.

(ii) ${}_\phi R \cong R_{\phi^{-1}}$ as R-bimodules.

(iii) ${}_\phi R \otimes_\tau R \cong {}_{\phi\tau} R$ as R-bimodules for all $\tau \in Aut(R)$.

(iv) The map $\phi \mapsto [{}_\phi R]$ is a group homomorphism from $Aut(R) \to Pic(R)$ with kernel $Inn(R)$. Consequently, $Inn(R)$ is a normal subgroup of $Aut(R)$, and ${}_\phi R \cong R$ as R-bimodules if and only if $\phi \in Inn(R)$.

Proof. We have the abelian group isomorphism $\phi: R \to {}_\phi R$ via $r \mapsto (r)^\phi$. For $r, s \in R$, $(rs)^\phi = (r)^\phi (s)^\phi = r * (s)^\phi$, so ϕ is a left R-module isomorphism showing (i). The remaining statements are straightforward adaptations of [8, Proposition 55.10 and Theorem 55.11], given that we are writing our maps on the right instead of on the left.

Notation 2.2. Recall that $Out(R) = Aut(R) / Inn(R)$. By the First Isomorphism Theorem for groups and Lemma 2.1, there is a monomorphism $\lambda: Out(R) \to Pic(R)$ such that $Inn(R)\phi \mapsto [{}_{\phi}R]$. We can therefore identify $Out(R)$ as a subgroup of $Pic(R)$ via λ. When R is an incidence ring of a preordered set P over a field K, we use λ_P to denote λ.

We observe that for an invertible R-bimodule X, $R \cong End({}_RX)$ as rings via $r \mapsto \rho_r$ where $(x)^{\rho_r} = xr$ for all $x \in X$. We use this to prove the subsequent lemma which states a necessary and sufficient condition for $[X] \in Pic(R)$ to be in the image of $\lambda: Out(R) \to Pic(R)$.

Lemma 2.3. X is an invertible R-bimodule such that either ${}_RX \cong_R R$ or $X_R \cong R_R$ if and only if there exists $\phi \in Aut(R)$ such that $X \cong {}_{\phi}R$ as R-bimodules.

Proof. We include a proof here even though this is really just [8, Theorem 55.12].

($\Leftarrow$): This holds by Lemma 2.1.

($\Rightarrow$): We suppose that ${}_RX \overset{\Psi}{\cong} {}_RR$. By the above observation, there is a ring isomorphism $\rho^X: R \to End({}_RX)$ defined via $r \mapsto \rho_r^X$ where $(x)^{\rho_r^X} = xr$ for all $x \in X$. We also have the ring isomorphism $\rho: R \to End({}_RR)$ defined via $r \mapsto \rho_r$ where $(s)^{\rho_r} = sr$ for all $s \in R$. It is easy to verify that $\alpha: End({}_RR) \to End({}_RX)$ where $(f)^{\alpha} = \Psi f \Psi^{-1}$ for $f \in End({}_RR)$ is a ring isomorphism. Notice that $R \overset{\rho\alpha}{\cong} End({}_RX)$. It is clear that for each $r \in R$, $\Psi^{-1}\rho_r^X\Psi \in End(_R R)$. Since $R \overset{\rho}{\cong} End(_R R)$, then $\Psi^{-1}\rho_r^X\Psi = \rho_{r'}$ for some unique $r' \in R$. Since $\theta = \rho^X\alpha^{-1}\rho^{-1}$ is the composition of ring isomorphisms, then $\theta \in Aut(R)$. Furthermore, $r^\theta = (\rho_r^X)^{\alpha^{-1}\rho^{-1}} = (\Psi^{-1}\rho_r^X\Psi)^{\rho^{-1}}$ implies that $\rho_{r^\theta} = (r^\theta)^\rho = \Psi^{-1}\rho_r^X\Psi$.

To show $X \overset{\Psi}{\cong} R_\theta$ as R-bimodules, we only need to show that Ψ is a right R-module homomorphism from X into R_θ. However, this holds since for $r \in R$ and $x \in X$, $(xr)^\Psi = (x)^{\rho_r^X\Psi} = (x)^{\Psi\Psi^{-1}\rho_r^X\Psi} = (x)^{\Psi\rho_{r^\theta}} = (x)^\Psi r^\theta = (x)^\Psi * r$. By Lemma 2.1, $X \cong {}_{\theta^{-1}}R$. The proof is similar if we have $X_R \cong R_R$.

Definition. Following [2], a ring R is **local** if its only idempotents are 0 and 1. A set of idempotents in a ring is said to be **orthogonal** if the idempotents comprising the set are pairwise orthogonal. A ring R is called **semiperfect** if it contains a complete set $\{e_1, e_2, \ldots, e_n\}$ of orthogonal idempotents such that for each i, e_iRe_i is a local ring. A non-zero idempotent in a ring R is **primitive** if it cannot be written as the sum of two non-zero idempotents in R. An R-module is **primitive** if it is of the form Re for some primitive idempotent e in R. A set $\{e_1, e_2, \ldots, e_n\} \subseteq R$ of primitive, orthogonal idempotents is called **basic** if $Re_1, Re_2, \ldots, Re_n$ is a complete irredundant list of representatives of the primitive left R-modules. An idempotent of a semiperfect ring R is **basic** if it the sum of a basic set of idempotents of R. A ring is a **basic ring** for R if it is isomorphic to eRe for some basic idempotent e in R.

Remark. Per [2], for a semiperfect ring R we can actually find a complete set of *primitive*, orthogonal idempotents such that for each i, e_iRe_i is local.

Lemma 2.4. If R is basic semiperfect, then the group monomorphism $\lambda: Out(R) \to Pic(R)$ in Notation 2.2 is an isomorphism.

Proof. Let $\{e_1,\ldots,e_n\}$ be a complete, basic set of idempotents of R. Then $R = Re_1 \oplus \ldots \oplus Re_n$ and the Re_i are pairwise non-isomorphic, indecomposable submodules of ${}_RR$. Let $[X] \in Pic(R)$. Then ${}_RX$ is a progenerator for R-mod and $R \cong End({}_RX)$. To show λ is surjective, we need to show there exists $\phi \in Aut(R)$ such that ${}_\phi R \cong X$ as R-bimodules. By [2, Theorem 27.11], ${}_RX \cong Re_1{}^{(A_1)} \oplus \ldots \oplus Re_n{}^{(A_n)}$ for some index sets $A_1,\ldots,A_n$ unique up to cardinality. Since ${}_RX$ is finitely-generated, then $A_1,\ldots,A_n$ can be viewed as non-negative integers.

To complete the proof, we show that ${}_RX \cong {}_RR$ by showing that each $A_i = 1$ and then appeal to Lemma 2.3. Since X is a generator, there is a positive integer k for which there is an epimorphism $X^k \to R$ which by projection yields an epimorphism $X^k \to Re_i$ for each i. Since the above decomposition of X is unique, then each Re_i is a direct summand of X and each $A_i \geq 1$. We show by contradiction that $A_i = 1$ for each i. Without loss of generality, we assume $A_1 > 1$. Since $R \cong End({}_RX)$, then $End({}_RX)$ is basic semiperfect. Let $m = \sum_{1 \leq i \leq n} |A_i|$ and let $\{f_1,\ldots,f_m\}$ be a set of idempotents of R defined such that:

$$f_1, f_2, \ldots, f_{|A_1|} = e_1;\ f_{|A_1|+1}, \ldots, f_{|A_1|+|A_2|} = e_2; \ldots;\ f_{|A_1|+\ldots+|A_{n-1}|+1}, \ldots, f_m = e_n.$$

Since ${}_RX \cong Re_1^{(A_1)} \oplus \ldots \oplus Re_n^{(A_n)} \cong \oplus_{i=1}^{m} Rf_i$, then writing our maps on the right (opposite the scalars),

$$End({}_RX) \cong \begin{pmatrix} Hom(Rf_1, Rf_1) & Hom(Rf_1, Rf_2) & \ldots & Hom(Rf_1, Rf_m) \\ Hom(Rf_2, Rf_1) & & & \\ \vdots & & & \\ Hom(Rf_m, Rf_1) & Hom(Rf_m, Rf_2) & \cdots & Hom(Rf_m, Rf_m) \end{pmatrix}.$$

Let h_i be the standard matrix idempotents for the above matrix ring. Since $f_1 = f_2$, then $End({}_RX)h_1 \cong End({}_RX)h_2$, which is impossible since $End({}_RX)$ is basic semiperfect.

Definition. Two rings R and R' are **Morita equivalent** if there are functors $F: R-mod \to R'-mod$ and $G: R'-mod \to R-mod$ generating an equivalence between the categories $R-mod$ and $R'-mod$. By [13, Morita II], there exist invertible bimodules ${}_{R'}P'_R$ and ${}_RP''_{R'}$ such that the functors F and G are naturally isomorphic to the functors $P' \otimes_R -$ and $P'' \otimes_{R'} -$, respectively. Furthermore, the functors $P' \otimes_R - \otimes_R P''$ and $P'' \otimes_{R'} - \otimes_{R'} P'$ yield isomorphisms between $Pic(R)$ and $Pic(R')$ via $[X] \mapsto [P' \otimes_R X \otimes_R P'']$ and $[Y] \mapsto [P'' \otimes_{R'} Y \otimes_{R'} P']$ for $[X] \in Pic(R)$ and $[Y] \in Pic(R')$. It is clear that Morita equivalence determines an equivalence relation and that isomorphic rings are Morita equivalent.

Lemma 2.5. Suppose R is semiperfect with complete set of primitive, orthogonal idempotents $\{e_1, \ldots, e_n\}$ which has basic subset $\{f_1, \ldots, f_m\}$. Let $e = \sum_{i=1}^{m} f_i$.

(i) R and eRe are Morita equivalent.

(ii) The ring eRe is basic semiperfect.

Proof. (i) This is [2, Proposition 27.14].

(ii) This follows from (i) and [2, Corollary 27.8].

We are now ready to define the map relating $Pic(R)$ to $Pic(eRe)$ depicted in the diagram at the beginning of the chapter.

Lemma 2.6. If R is semiperfect with basic idempotent e, then the map $e(-)e\colon Pic(R) \to Pic(eRe)$ defined via $[X] \mapsto [eXe]$ is a group isomorphism.

Proof. We have that $e = \sum_{i=1}^{m} f_i$, where $\{f_1, \ldots, f_m\}$ is a basic subset of some complete set of primitive, orthogonal idempotents $\{e_1, \ldots, e_n\} \subseteq R$.

Clearly, $ReR \subseteq R$. To show that $ReR = R$, we need only show that the $e_i \in ReR$ for then $1_R \in ReR$. Since e is basic, then for each e_i there is an f_j such that $Re_i \cong Rf_j$ as left R-modules. By [2, Exercise 7.2], there exist $r, r' \in R$ such that $e_i = e_i r f_j f_j r' e_i$. Since $f_j e = f_j$, then $e_i = e_i r f_j e r' e_i \in ReR$.

It is now clear that $Re \otimes_{eRe} eR \cong R$ as R-bimodules, and for a ring R' and bimodules ${}_{R'}Y_R$ and ${}_R Z_{R'}$, $Y \otimes_R Re \cong Ye$ as (R', eRe)-bimodules and $eR \otimes_R Z \cong eZ$ as (eRe, R')-bimodules.

Since $End(eR_R) \cong eRe \cong End({}_R Re)$ and since eR_R and ${}_R Re$ are clearly progenerators for $mod-R$ and $R-mod$, respectively, then the functor $eR \otimes_R - \otimes_R Re$ effects a group isomorphism between $Pic(R)$ and $Pic(eRe)$ via

$$[X] \mapsto [eR \otimes_R X \otimes_R Re] = [eXe].$$

We are now ready to apply these results to the specific case when we have an incidence ring of a preordered set over a field.

Notation 2.7. Recall that we let $P = (V = V(P), \rho(P))$ be a finite preordered set and $\tilde{P} = (\tilde{V} = V(\tilde{P}), \rho(\tilde{P}))$ be the underlying poset of P. Recall that we also make the association $S = I(P)$. Letting $e_i \in S$ be the matrix idempotent with 1 in the i,i position an 0's elsewhere for each $i \in V$, then $\{e_i | i \in V\}$ is a complete set of primitive, orthogonal idempotents in S. We henceforth make the associations $e = \sum_{i \in \tilde{V}} e_i$ and $\tilde{S} = eSe$.

Lemma 2.8. For $i, j \in V$, $Se_i \cong Se_j$ if and only if $[i] = [j]$.

Proof. ($\Rightarrow$): If $Se_i \overset{f}{\cong} Se_j$ for some $i, j \in V$, then $f \in Hom(Se_i, Se_j) \cong e_i S e_j$ and $f^{-1} \in Hom(Se_j, Se_i) \cong e_j S e_i$. If $[i] \neq [j]$, then either $(i,j) \notin \rho(P)$ or $(j,i) \notin \rho(P)$. If $(i,j) \notin \rho(P)$, then $e_i S e_j = 0$. Hence, $f = 0$, which contradicts that it is an isomorphism. Since we have a similar contradiction if $(j,i) \notin \rho(P)$, then $[i] = [j]$.

($\Leftarrow$): If $[i] = [j]$, then $(i,j),(j,i) \in \rho(P)$, and so $e_{ij}, e_{ji} \in S$. By transitivity, $(k,i) \in \rho(P)$ if and only if $(k,j) \in \rho(P)$ for each $k \in V$. Thus, $e_k S e_i = 0$ if and only if $e_k S e_j = 0$. Hence, Se_i and Se_j are isomorphic via the map $se_i \mapsto se_{ij} = se_{ij}e_j \in Se_j$.

Lemma 2.9.

(i) S is semiperfect with basic idempotent e; hence, eSe is the basic ring for S.

(ii) If P is a poset, S is basic semiperfect.

Proof. (i) To show S is semiperfect, it suffices to show the e_iSe_i are local rings; however, this is clear since $e_iSe_i \cong K$ is a field. By Lemma 2.8, $e = \sum_{i\in\tilde{V}} e_i$ is basic.

(ii) This holds since $\{e_i | i \in V\}$ is basic by the antisymmetry of P and Lemma 2.8.

All of the pieces are now in place to construct the diagram mentioned at the beginning of the chapter.

Theorem 2.10.

(i) The map $e(-)e\colon Pic(S) \to Pic(eSe)$ defined via $[X] \mapsto [eXe]$ is a group isomorphism.

(ii) $eSe \cong I(\tilde{P})$ and so we make the association $\tilde{S} = eSe = I(\tilde{P})$. In particular, $I(P)$ and $I(\tilde{P})$ are Morita equivalent.

(iii) The map $\lambda_{\tilde{P}}\colon Out(eSe) \to Pic(eSe)$ as in Notation 2.2 is an isomorphism; hence, the diagram

$$\begin{array}{ccc} Out(S) & \xrightarrow{\lambda_P} & Pic(S) \\ \downarrow{\scriptstyle \phi_S} & & \downarrow{\scriptstyle e(-)e} \\ Out(eSe) & \xrightarrow{\lambda_{\tilde{P}}} & Pic(eSe) \end{array}$$

commutes where $\phi_S = \lambda_P[e(-)e]\lambda_{\tilde{P}}^{-1}$. Furthermore, λ_P is an isomorphism if and only if ϕ_S is an isomorphism.

Proof. (i) This follows from Lemmas 2.9 and 2.6.

(ii) For $\alpha \in I(\tilde{P})$, define $\alpha^* \in eSe$ via $\alpha^*{}_{t_i,t_j} = \alpha_{t_i,t_j}$ for $t_i, t_j \in \tilde{V}$ and $\alpha^*{}_{t_i,t_j} = 0$ otherwise. It is easy to verify that $(-)^*: I(\tilde{P}) \to eSe$ via $\alpha \mapsto \alpha^*$ is a ring isomorphism. The proof is complete since eSe is Morita equivalent to S by Lemma 2.5.

(iii) This follows directly from (ii) and Lemmas 2.5 and 2.4.

CHAPTER III

A CHARACTERIZATION OF *AUT(I(P))*

Recall that $P = (V = V(P), \rho(P))$ is a finite preordered set with underlying poset $\tilde{P} = (\tilde{V} = V(\tilde{P}), \rho(\tilde{P}))$. Also recall that for $i \in V$, $[i] = \{j \in V | (i,j),(j,i) \in \rho(P)\}$ and that $\tilde{V}$ is a set of representatives of these classes. We make the associations $S = I(P)$ and $\tilde{S} = eSe = I(\tilde{P})$ where $e = \sum_{i \in \tilde{V}} e_i$ is a basic idempotent for S.

Our goal in this chapter is to characterize *Aut*(S), the multiplicative group of automorphisms of S. We embed our field K in S in the natural manner, and we let $Aut_K(S)$ denote the subgroup of automorphisms of S which fix K. We extend Coelho's description of $Aut_K(S)$ in [6] as the semidirect product of three of its subgroups to a characterization of *Aut*(S) as a four-subgroup semidirect product. This yields a representation of *Out*(S) (and $Out(\tilde{S})$) as a semidirect product which we use to explore the connection between *Out*(S) and $Out(\tilde{S})$ in Chapter IV and to solve Problems B and C in Chapter V.

Definition. Let $\sigma \in S_V$; that is, σ is a permutation on V. We say σ is an **automorphism** of P if $(i^\sigma, j^\sigma) \in \rho(P)$ if and only if $(i,j) \in \rho(P)$. Let *Aut*(P) denote the multiplicative group of automorphisms of P. For $\sigma \in Aut(P)$, we define $\hat{\sigma}: S \to S$ by setting $(e_{i,j})^{\hat{\sigma}} = e_{i^\sigma, j^\sigma}$ and extending linearly. That is, for $(a_{i,j}) \in S$, the $\{i, j\}$-entry of $(a_{i,j})^{\hat{\sigma}}$ is $a_{i^{\sigma^{-1}}, j^{\sigma^{-1}}}$. In [6], Coelho points out that $\hat{\sigma} \in Aut_K(S)$.

Lemma 3.1.

(i) The map $(\hat{\ }): Aut(P) \to Aut(S)$ is a group monomorphism.

(ii) For $\sigma \in Aut(P)$, $([i])^\sigma = [i^\sigma]$ for each $i \in V$.

(iii) Let α denote the composition $Aut(P) \xrightarrow{(\hat{\ })} Aut(S) \xrightarrow{\ \nu\ } Out(S)$ where ν is the canonical projection. Set $L = \{\sigma \in Aut(P) | ([j])^\sigma = [j] \text{ for each } j \in V\}$; i.e., L consists of those σ leaving each class invariant. Then $L = Ker(\alpha)$.

(iv) If P is a poset, α is injective.

Proof. (i) This follows directly from the comments in [21, pg. 273].

(ii) This is clear since for $i \in V$,

$$\begin{aligned} j \in [i] &\Leftrightarrow (i, j), (j, i) \in \rho(P) \\ &\Leftrightarrow (i^\sigma, j^\sigma), (j^\sigma, i^\sigma) \in \rho(P) \\ &\Leftrightarrow j^\sigma \in [i^\sigma]. \end{aligned}$$

(iii) Choose class representatives $\{v_1, \ldots v_t\}$.

($\subseteq$): Let $\sigma \in L$. It suffices to show that $\hat{\sigma} \in Inn(S)$. Since $[x^\sigma] = [x]$ for each $x \in V$, then $[x^{\sigma^{-1}}] = ([x])^{\sigma^{-1}} = ([x^\sigma])^{\sigma^{-1}} = [x]$ by (ii). Hence, $u(\sigma) = \sum_{i=1}^{t} \sum_{x \in [v_i]} e_{x x^\sigma}$ and

$u(\sigma^{-1}) = \sum_{i=1}^{t} \sum_{x \in [v_i]} e_{x x^{\sigma^{-1}}}$ are in S. We show that $u(\sigma)$ and $u(\sigma^{-1})$ are inverse elements.

Notice that $u(\sigma)$ and $u(\sigma^{-1})$ have only one non-zero entry, namely a 1, in each row and each column since $x^{\sigma} \neq y^{\sigma}$ and $x^{\sigma^{-1}} \neq y^{\sigma^{-1}}$ for $x \neq y$. For $i, j \in V$, $[u(\sigma)u(\sigma^{-1})]_{i,j}$ $= \sum_{k \in V} u(\sigma)_{i,k} u(\sigma^{-1})_{k,j} = u(\sigma^{-1})_{i^{\sigma},j}$. If $i = j$, then $i^{\sigma} = j^{\sigma}$, so that $u(\sigma^{-1})_{i^{\sigma},j}$ $= u(\sigma^{-1})_{j^{\sigma},j^{\sigma\sigma^{-1}}} = 1$. If $i \neq j$, then $i^{\sigma} \neq j^{\sigma}$ so that $u(\sigma^{-1})_{i^{\sigma},j} = 0$. Therefore, $u(\sigma)u(\sigma^{-1}) = 1_S$. Similarly, $u(\sigma^{-1})u(\sigma) = 1_S$, and $u(\sigma)$ is a unit in S with inverse $u(\sigma^{-1})$. Since for $A \in S$, we have

$$
\begin{aligned}
[u(\sigma^{-1})Au(\sigma)]_{i,j} &= \sum_{k \in V} u(\sigma^{-1})_{i,k} [Au(\sigma)]_{k,j} \\
&= [Au(\sigma)]_{i^{\sigma^{-1}},j} \\
&= \sum_{m \in V} A_{i^{\sigma^{-1}},m}\, u(\sigma)_{m,j} \\
&= A_{i^{\sigma^{-1}},j^{\sigma^{-1}}} \\
&= (A)^{\hat{\sigma}}{}_{i,j},
\end{aligned}
$$

then $(A)^{\hat{\sigma}} = u(\sigma^{-1})Au(\sigma)$ and $\hat{\sigma} \in Inn(S)$.

($\supseteq$): Let $\sigma \in Ker(\alpha)$. Then $\hat{\sigma} \in Inn(S)$, and there is a unit U in S such that $A^{\hat{\sigma}} = UAU^{-1}$ for every A in S. We want to show that σ leaves each equivalence class invariant. Let $X = [v_i]$ for some $1 \leq i \leq t$, and let $e_X = \sum_{x \in X} e_{xx}$. Notice that $(e_X)^{\hat{\sigma}}$

$= \sum_{x \in X} (e_{xx})^{\hat{\sigma}} = \sum_{x \in X} e_{x^{\sigma} x^{\sigma}} = e_{X^{\sigma}}$ where $X^{\sigma} = (X)^{\sigma}$ is an equivalence class. It remains to show $e_{X^{\sigma}} = e_X$, since then $X^{\sigma} = X$.

For any equivalence class Y and $A, B \in S$, $e_Y ABe_Y = (e_Y Ae_Y)(e_Y Be_Y) = e_Y Ae_Y Be_Y$. To see this, let $i, j \in Y$ so that

$$[e_Y ABe_Y]_{i,j} = \sum_{k \in V} [e_Y A]_{i,k} [Be_Y]_{k,j}$$

$$= \sum_{k \in Y} [e_Y A]_{i,k} [Be_Y]_{k,j} + \sum_{k \notin Y} [e_Y A]_{i,k} [Be_Y]_{k,j} .$$

For $z \notin Y$, either (i, z) (and (j, z)) or (z, i) (and (z, j)) $\notin \rho(P)$; hence,

$$[e_Y ABe_Y]_{i,j} = \sum_{k \in Y} [e_Y A]_{i,k} [Be_Y]_{k,j}$$

$$= \sum_{k \in Y} [e_Y Ae_Y]_{i,k} [Be_Y]_{k,j}$$

$$= [e_Y Ae_Y Be_Y]_{i,j} .$$

We now show that $e_X = e_{X^{\sigma}}$ as follows:

$$e_{X^{\sigma}} = e_{X^{\sigma}} e_{X^{\sigma}} e_{X^{\sigma}} \text{ since } e_{X^{\sigma}} \text{ is idempotent}$$

$$= e_{X^{\sigma}} U e_X U^{-1} e_{X^{\sigma}} \text{ since } e_{X^{\sigma}} = (e_X)^{\hat{\sigma}} = U e_X U^{-1}$$

$$= e_{X^{\sigma}} U e_{X^{\sigma}} e_X e_{X^{\sigma}} U^{-1} e_{X^{\sigma}} \text{ by the above observation.}$$

If $e_X \neq e_{X^{\sigma}}$, then $e_{X^{\sigma}} e_X e_{X^{\sigma}} = 0$ which yields the impossibility

$$0 = e_{X^{\sigma}} U e_{X^{\sigma}} e_X e_{X^{\sigma}} U^{-1} e_{X^{\sigma}} = e_{X^{\sigma}} .$$

(iv) If P is a poset with $V = \{v_1,\dots,v_t\}$, then the equivalence classes are the singleton sets $\{v_1\},\dots,\{v_t\}$. By (iii), $\sigma \in Ker(\alpha)$ implies that $v_i^{\sigma} = v_i$ for each i; thus, σ is trivial.

Set $Out(P) = Aut(P) / L$. By the First Isomorphism Theorem for groups, there is a monomorphism $Out(P) \to Out(S)$ such that $L\sigma \mapsto Inn(S)\hat{\sigma}$, and so we identify $Out(P)$ as a subgroup of $Out(S)$ via this monomorphism.

Notation 3.2. Since Coelho's description of $Aut_K(S)$ requires that $V = \{1,2,\dots,n_P\}$ and since we wish to extend this to a description of $Aut(S)$, then we make this stipulation on V throughout the remainder of this paper. We set $\mathcal{P}(P) = \{\, \hat{\sigma} \in Aut(S) \mid \sigma \in Aut(P)$ such that $i^{\sigma} < j^{\sigma}$ as integers whenever $(i,j) \in \overline{\rho}(P)$ and $i < j$ as integers$\}$. As in [6], $\mathcal{P}(P)$ is a subgroup of $Aut_K(S)$

Recall that a group G is said to be the __**semidirect product**__ of subgroups N and H if N is normal in G and $N \cap H = 1$. We use the symbol $\oslash$ to denote the semidirect product, and we write $G = N \oslash H$. For $h \in H$, $h^{-1}nh \in N$ for each $n \in N$ since N is normal. We can view $h \in H$ as a map from N into N via the action $n^h = h^{-1}nh$, and we say that __***H* acts on *N* by conjugation**__.

If $n, n' \in N$ and $h, h' \in H$ such that $nh = n'h'$, then $n'^{-1}n = h'h^{-1} \in N \cap H = 1$. Hence, $n = n'$ and $h = h'$. By induction, we see that we have uniqueness of representation for a semidirect product of any number of factors.

The next result connects the subgroup L mentioned in Lemma 3.1 with $\mathcal{P}(P)$.

Proposition 3.3. *Aut*(*P*) is the semidirect product of L by $\mathcal{P} = \{\, \sigma \in Aut(P) \mid i^{\sigma} < j^{\sigma}$ as integers whenever $(i, j) \in \overline{\rho}(P)$ and $i < j$ as integers$\}$; in particular, $\mathcal{P}(P) \cong \mathcal{P} \cong Out(P)$.

Proof. We first show $\mathcal{P} \cap L = 1$ by contradiction. Assume $\sigma \in \mathcal{P} \cap L$ such that $\sigma \neq 1$. By Lemma 3.1, σ leaves each equivalence class invariant, so that $(i, i^{\sigma}) \in \overline{\rho}(P)$ for each $i \in V$. Since $\sigma \neq 1$, there exists $i \in V$ such that either $i < i^{\sigma}$ or $i^{\sigma} < i$ as integers (where the inequality is strict). Since $\sigma \in \mathcal{P}$, there exists either a chain $i < i^{\sigma} < i^{\sigma^2} < \ldots$ or a chain $\ldots < i^{\sigma^2} < i^{\sigma} < i$. This is impossible since V is finite; hence, $\mathcal{P} \cap L = 1$.

To show $Aut(P) = L \cdot \mathcal{P}$, let $\sigma \in Aut(P)$. Suppose P has m equivalence classes, and let $V = \{x_{1,1}, x_{1,2}, \ldots, x_{1,k_1}, x_{2,1}, x_{2,2}, \ldots, x_{2,k_2}, \ldots, x_{m,1}, x_{m,2}, \ldots, x_{m,k_m}\}$ such that for each $1 \leq i \leq m$, the vertices $x_{i,1}, x_{i,2}, \ldots, x_{i,k_i}$ comprise the *i*-th equivalence class and $x_{i,1} < x_{i,2} < \ldots < x_{i,k_i}$ as integers. For each $1 \leq i \leq m$, let $1 \leq i_1, \ldots, i_{k_i} \leq k_i$ such that $(x_{i,i_1})^{\sigma} < (x_{i,i_2})^{\sigma} < \ldots < (x_{i,i_{k_i}})^{\sigma}$ as integers, and define a map τ_i such that $x_{i,j} \mapsto x_{i,i_j}$ for each $1 \leq j \leq k_i$ and all other vertices remain fixed. The $\tau_i \in L$ since they leave the equivalence classes invariant; hence, $\tau = \prod_{1 \leq i \leq m} \tau_i \in L$. To show that $\tau\sigma \in \mathcal{P}$, take $x_{i,j}, x_{i,k}$ such that $x_{i,j} < x_{i,k}$ as integers; that is $1 \leq j < k \leq k_i$. Then

$$(x_{i,j})^{\tau\sigma} = (x_{i,j})^{\tau_i\sigma} = (x_{i,i_j})^{\sigma} < (x_{i,i_k})^{\sigma} = (x_{i,k})^{\tau_i\sigma} = (x_{i,k})^{\tau\sigma},$$

so that $\tau\sigma \in \mathcal{P}$ and $\sigma = 1\sigma \in L\sigma = L\tau\sigma \subseteq L\mathcal{P}$.

Finally, L is normal since it is the kernel of a group homomorphism.

We now establish the notation necessary to define the third and final subgroup in Coelho's decomposition of $Aut_K(S)$.

Notation 3.4. Let $\Delta(P)$ denote the graph associated with the relation $\rho \setminus \overline{\rho}$ as defined in [6]. Specifically, the vertices of $\Delta(P)$ are the $i \in V$ such that either (i, j) or (j, i) $\in \rho(P) \setminus \overline{\rho}(P)$ for some $j \in V$, and the edges of $\Delta(P)$ are the unordered pairs $\{i, j\}$ such that (i, j) or $(j, i) \in \rho(P) \setminus \overline{\rho}(P)$. Let $T(P)_m$ denote a spanning tree of the m-th component of $\Delta(P)$; i.e., $T(P)_m$ contains every vertex in the component. Let $\overline{\overline{\rho}}(P)$ be the subset of $\rho(P)$ such that $(i, j) \in \overline{\overline{\rho}}(P)$ if and only if $(i, j) \in \rho(P)$ and $\{i, j\}$ is an edge in the union of all the trees of all the components of $\Delta(P)$. Let $J(P)$ denote the vertices of P that are not in the union of the trees, and let $\overline{\overline{\overline{\rho}}}(P) = \overline{\rho}(P) \cap [J(P) \times J(P)]$.

As in [6], a function $g: \rho(P) \to K^*$ is **transitive** if $(i, j)^g (j,k)^g = (i,k)^g$ for all $(i, j), (j,k) \in \rho(P)$. Every transitive function $g: \rho(P) \to K^*$ gives rise to an automorphism g^* of S by defining $(e_{ij})^{g^*} = (i, j)^g e_{ij}$ and extending linearly. Per [6], $\mathcal{G}(P) = \{ g^* \in Aut(S) \mid g: \rho(P) \to K^*$ is transitive and $(i, j)^g = 1$ for all $(i, j) \in \overline{\overline{\rho}}(P) \cup \overline{\overline{\overline{\rho}}}(P) \}$ forms a subgroup of $Aut_K(S)$ under composition; that is, $g^* \circ h^* = (gh)^*$ where $gh: \rho(P) \to K^*$ is defined via $(i, j)^{gh} = (i, j)^g (i, j)^h$.

We caution the reader that $\mathcal{G}(P)$ depends on the choice of trees; different trees yield a different $\mathcal{G}(P)$.

Example. Given the strict constraints imposed in the definition of $\mathcal{G}(P)$, one might wonder if $\mathcal{G}(P)$ is almost always trivial. In [6, Example 2], Coelho constructs an example of a preordered set P for which $\mathcal{G}(P)$ is nontrivial. As we will see in Theorem 4.5, $\mathcal{G}(P) \cong \mathcal{G}(\tilde{P})$, so $\mathcal{G}(\tilde{P})$ is also nontrivial. To see this, note that $V(\tilde{P}) = \{1,2,4,5\}$ and $\rho(\tilde{P}) = \{(1,1),(1,4),(1,5),(2,2),(2,4),(2,5),(4,4),(5,5)\}$, so that $V(\Delta(\tilde{P})) = \{1,2,4,5\}$ and $\rho(\Delta(\tilde{P})) = \{\{1,4\},\{1,5\},\{2,4\},\{2,5\}\}$. We let $T(\tilde{P})$ be a tree for the single component of $\Delta(\tilde{P})$ with edges $\{\{1,5\},\{2,4\},\{2,5\}\}$. For any $\alpha \in K^*$ such that $\alpha \neq 1$, the transitive function $g: \rho(\tilde{P}) \to K^*$ via $(1,4)^g = \alpha$ and $(i,j)^g = 1$ otherwise is in $\mathcal{G}(\tilde{P})$ since $\overline{\overline{\rho}}(\tilde{P}) = \{(1,5),(2,4),(2,5)\}$ and $\overline{\overline{\overline{\rho}}}(\tilde{P}) = \varnothing$. Thus, in this case, $\mathcal{G}(P) \cong \mathcal{G}(\tilde{P}) \cong K^*$.

Set $Out_K(S) = Aut_K(S) / Inn(S)$. By [6], $Aut_K(S) = (Inn(S) \oslash \mathcal{G}(P)) \oslash \mathcal{P}(P)$; thus, the subgroups $\mathcal{G}(P)$, $\mathcal{P}(P)$, and $Inn(S)$ are pairwise disjoint. As a result, the compositions $\mathcal{G}(P) \xrightarrow{\subseteq} Aut_K(S) \to Out_K(S)$ and $\mathcal{P}(P) \xrightarrow{\subseteq} Aut_K(S) \to Out_K(S)$ are monic. We shall identify $\mathcal{G}(P)$ and $\mathcal{P}(P)$ with their images in $Out_K(S)$ under these compositions.

As we mentioned at the beginning of the chapter, our description of $Aut(S)$ requires one more subgroup. Before giving its definition, we need a suitable description of the center of S.

Proposition 3.5. Suppose P has n components $P_1, \ldots, P_n$.

(i) $S \cong A_1 \oplus \ldots \oplus A_n$ where A_i is the incidence ring of P_i over K.

(ii) The center C of S is isomorphic to $K^{(n)}$.

(iii) $Aut_K(C) \cong S_n$.

Proof. (i) We represent the matrices of S in "block form". That is, we group the vertices based upon their inclusion in the components of P and index the matrices of S beginning with the vertices in $V(P_1)$, followed by those in $V(P_2)$, and ending with those in $V(P_n)$. The matrices are thus of the form

$$\begin{bmatrix} [A_1] & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & [A_n] \end{bmatrix}.$$

It is clear that $f_i S f_i \cong A_i$ as rings where for $1 \le i \le n$, $f_i = \sum_{j \in V(P_i)} e_j$. So, S $= f_1 S f_1 \oplus \ldots \oplus f_n S f_n \cong A_1 \oplus \ldots \oplus A_n$. In subsequent discussion, we identify A_i with $f_i S f_i$, and we may write $S = A_1 \oplus \ldots \oplus A_n$.

(ii) Since $e_{i\,i} \in S$, then for α in C, $\alpha e_{i\,i} = e_{i\,i}\alpha$. However, this only holds if α has non-zero entries only on its diagonal. We show that the entries on the diagonal of α corresponding to vertices in the same component must have the same value. For $\beta \in S$, we have $\beta_{i,j}\alpha_{j,j} = \sum_{k \in V} \beta_{i,k}\alpha_{k,j} = (\beta\alpha)_{i,j} = (\alpha\beta)_{i,j} = \sum_{k \in V} \alpha_{i,k}\beta_{k,j} = \alpha_{i,i}\beta_{i,j}$. So, if $(i, j) \in \rho(P)$, then we must have $\alpha_{i,i} = \alpha_{j,j}$ Finally, it is clear that $Cf_i \cong K$ where the f_i are the central idempotents in (i); hence, $C \cong K^{(n)}$.

(iii) Let $\sigma \in Aut_K(C)$. Consider the central idempotents f_i in (i). Since $Cf_i \cong K$ is indecomposable for each $1 \le i \le n$, the f_i are primitive in *C*. We claim the f_i are precisely the primitive central idempotents of *C*. Viewing *C* as $K^{(n)}$, we see that any idempotent of *C* must have either a 0 or 1 in each coordinate. Since the f_i correspond to the idempotents with 1 in the *i*-th coordinate and 0's elsewhere, then any other idempotent *f* of *C* must have a 1 in at least two positions. But, then $Cf \cong K^m$ for some $m \ge 2$, which is not indecomposable.

Since any automorphism must take primitive central idempotents to primitive central idempotents, there exists $\tau \in S_n$ such that $(f_i)^\sigma = f_{i^\tau}$. Since σ fixes *K*, $(af_i)^\sigma = af_{i^\tau}$ for all $a \in K$. It is easy to verify the map $\sigma \mapsto \tau$ is a group monomorphism. Finally, let $\tau \in S_n$. Since $C \cong K^{(n)}$ is a free *K*-module with the f_i forming a free basis for *C*, then by [2, Exercise 8.11], there exists $\sigma \in Aut(C)$ such that $(f_i)^\sigma = f_{i^\tau}$ for each *i*. Hence, the map is an isomorphism.

We are now ready to define the final subgroup necessary for our representation of $Aut(S)$ as a semidirect product.

Notation 3.6. As in Proposition 3.5, $S = A_1 \oplus \ldots \oplus A_n$ where *n* is the number of components of *P*. If $\alpha \in Aut(K)$, we can define $\breve{\alpha} \in Aut(A_k)$ entrywise; that is, $(a_{ij})^{\breve{\alpha}} = (a_{ij}{}^{\alpha})$ where $(a_{ij}) \in A_k$. We then have a group monomorphism $Aut(K) \to Aut(A_k)$, which yields a group monomorphism $\prod^n Aut(K) \to Aut(S)$ via

$(\alpha_i) \mapsto (\breve{\alpha}_i)$ where $\breve{\alpha}_i \in Aut(A_i)$. Denote the image of this map by $Aut(K)^n$. Since automorphisms of S take the center C to itself, the restriction map sends $Aut(K)^n$ to a subgroup of $Aut(C)$ which we also denote by $Aut(K)^n$.

Proposition 3.7. $Aut(C) = Aut(K)^n \oslash Aut_K(C)$.

Proof. Let $\sigma \in Aut(K)^n \cap Aut_K(C)$. Viewing C as $K^{(n)}$ as in Proposition 3.5, then $k \in K$ corresponds to the element $(k,k,\ldots,k)$ inside C. Since $\sigma \in Aut(K)^n$, then $\sigma = (\sigma_i)$ where the σ_i are automorphisms on the coordinates. Since $\sigma \in Aut_K(C)$, then $(k,k,\ldots,k) = (k,k,\ldots,k)^\sigma = (k^{\sigma 1}, k^{\sigma 2}, \ldots, k^{\sigma n})$. It follows that the σ_i are the identity automorphism on K, and σ is the identity on C.

We now show $Aut(C) = Aut(K)^n \cdot Aut_K(C)$. Let $\sigma \in Aut(C)$. As in Proposition 3.5, there exists $\tau \in S_n$ such that $(f_i)^\sigma = f_{i^\tau}$. Define $\sigma' \in Aut_K(C)$ via $(\sum_{1 \le i \le n} a_i f_i)^{\sigma'} = \sum_{1 \le i \le n} a_i f_{i^\tau}$. We claim that $\theta = \sigma(\sigma')^{-1} \in Aut(K)^n$. It suffices to show θ fixes each f_i since then θ leaves each Cf_i invariant. But, $f_i^\theta = f_i^{\sigma(\sigma')^{-1}} = f_{i^\tau}^{(\sigma')^{-1}} = f_i$. Thus, $\sigma = \theta\sigma' \in Aut(K)^n \cdot Aut_K(C)$.

We finally show that $Aut(K)^n$ is normal in $Aut(C)$. Let $\beta = (\beta_i) \in Aut(K)^n$ and $\sigma \in Aut_K(C)$. Since $Aut_K(C) \cong S_n$, we may identify σ with the associated permutation; that is, we may assume $(xf_i)^\sigma = xf_{i^\sigma}$ for $x \in K$. For $(x_i) \in C$,

$$((x_i))^{\sigma\beta\sigma^{-1}} = ((x_{i\sigma^{-1}}))^{\beta\sigma^{-1}}$$

$$= ((x_{i\sigma^{-1}}{}^{\beta_i}))^{\sigma^{-1}}$$

$$= (x_i{}^{(\beta_{i\sigma})}),$$

and so $\sigma\beta\sigma^{-1} = (\beta_{i\sigma}) \in Aut(K)^n$.

We are almost ready to state the main theorem of this chapter which presents $Aut(S)$ as a semidirect product of the four previously described subgroups. Its proof will follow from the following two lemmas.

Lemma 3.8. $Aut(S) = Aut(K)^n \cdot Aut_K(S)$ and $Aut(K)^n \cap Aut_K(S) = 1$.

Proof. To show that $Aut(K)^n \cap Aut_K(S) = 1$, let $\sigma \in Aut(K)^n \cap Aut_K(S)$. We know $k \in K$ corresponds to the scalar matrix $kI \in S$ where I is the identity. But, $kI = (kI_1, \ldots, kI_n)$ where the I_j are the identity elements of the A_j. Since $\sigma \in Aut(K)^n$, then $\sigma = (\sigma_i)$ where the σ_i are automorphisms defined entrywise on the A_i. Since $\sigma \in Aut_K(S)$, then $(kI_1, kI_2, \ldots, kI_n) = kI = (kI)^\sigma = ((kI_1)^{\sigma_1}, (kI_2)^{\sigma_2}, \ldots, (kI_n)^{\sigma_n})$. It follows that σ is the identity on S.

Let $\sigma \in Aut(S)$. Then $\sigma|_C \in Aut(C)$, and by Proposition 3.7, we may write $\sigma|_C = \beta\gamma$ where $\beta \in Aut(K)^n \subseteq Aut(C)$ and $\gamma \in Aut_K(C)$. We extend β to an automorphism of S in the obvious manner, and we denote this automorphism also by β. If $x \in K \subseteq C \subseteq S$, then

$$
\begin{aligned}
(x)^{\beta^{-1}\sigma} &= (x)^{(\beta^{-1}\sigma)|_C} \\
&= (x)^{\beta^{-1}|_C \sigma|_C} \\
&= (x)^{\gamma} \\
&= x
\end{aligned}
$$

which shows $\beta^{-1}\sigma \in Aut_K(S)$. Thus, $\sigma \in Aut(K)^n \cdot Aut_K(S)$.

Lemma 3.9. Suppose P has n components $P_1, P_2, \ldots, P_n$.

(i) $Aut(K)^n$ normalizes both $Inn(S)$ and $\mathcal{G}(P)$.

(ii) $\mathcal{P}(P)$ normalizes $Aut(K)^n$, and if $n = 1$, then $Aut(K)^n$ normalizes $\mathcal{P}(P)$.

Proof. Let $\breve{\beta} = (\breve{\beta}_i) \in Aut(K)^n$.

(i) Since $Inn(S)$ is a normal subgroup of $Aut(S)$, $Aut(K)^n$ normalizes $Inn(S)$. To show $Aut(K)^n$ normalizes $\mathcal{G}(P)$, we let $g^* \in \mathcal{G}(P) = \{\, g^* \in Aut(S) \mid g\colon \rho(P) \to K^*$ is transitive and $(i,j)^g = 1$ for all $(i,j) \in \overline{\overline{\rho}}(P) \cup \overline{\overline{\overline{\rho}}}(P)\,\}$. Recall that g^* is defined by setting $(e_{ij})^{g^*} = (i,j)^g e_{ij}$ and extending linearly. Since $\rho(P) = \rho(P_1) \cup \ldots \cup \rho(P_n)$, then $g = (g_k)$ where $g_k\colon \rho(P_k) \to K^*$ is a transitive function for P_k. Since the $\beta_k \in Aut(K)$, we have $\beta_k{}^{-1}\colon K^* \to K^*$, and the compositions $g_k\beta_k{}^{-1}\colon \rho(P_k) \to K^*$ are transitive. If $(i,j) \in \rho(P)$, then $(i,j) \in \rho(P_k)$ for exactly one k, and we define $(i,j)^h = (i,j)^{g_k\beta_k{}^{-1}}$. Clearly, $h\colon \rho(P) \to K^*$ is transitive. Since $(1)^{\beta_k{}^{-1}} = 1$, then $(i,j)^h = 1$ for all $(i,j) \in \overline{\overline{\rho}}(P) \cup \overline{\overline{\overline{\rho}}}(P)$ and $h^* \in \mathcal{G}(P)$. Let e_{kij} denote a matrix basis element belonging to A_k. If $x \in K$, then

$$(xe_{kij})^{\breve{\beta} g^* \breve{\beta}^{-1}} = (x^{\beta_k} e_{kij})^{g^* \breve{\beta}^{-1}}$$

$$= (x^{\beta_k} (i,j)^{g_k} e_{kij})^{\breve{\beta}^{-1}}$$

$$= x(i,j)^{g_k \beta_k{}^{-1}} e_{kij}$$

$$= (xe_{kij})^{h^*},$$

so that $\breve{\beta} g^* \breve{\beta}^{-1} = h^* \in \mathcal{G}(P)$.

(ii) Let $\hat{\sigma} \in \mathcal{P}(P)$. We know σ sends components of P to components of P. For a component P_k, we let P_{k^σ} denote the component $(P_k)^\sigma$. Let $a = (a_k) \in S$ where the a_k are in A_k, and write $a_k = (a_{kij})$. Then $\hat{\sigma}\breve{\beta}\hat{\sigma}^{-1} = (\beta_{k^\sigma}) \in Aut(K)^n$ since

$$(a)^{\hat{\sigma}\breve{\beta}\hat{\sigma}^{-1}} = ((a_{kij}))^{\hat{\sigma}\breve{\beta}\hat{\sigma}^{-1}}$$

$$= ((a_{k^{\sigma^{-1}} i^{\sigma^{-1}} j^{\sigma^{-1}}}))^{\breve{\beta}\hat{\sigma}^{-1}}$$

$$= ((a^{\beta_k}_{k^{\sigma^{-1}} i^{\sigma^{-1}} j^{\sigma^{-1}}}))^{\hat{\sigma}^{-1}}$$

$$= ((a_{kij}^{\beta_{k^\sigma}}))$$

$$= a^{(\breve{\beta}_{k^\sigma})}.$$

For $n = 1$, let $\breve{\beta} \in Aut(K)$ and $\hat{\sigma} \in \mathcal{P}(P)$. Since for $k \in K$, $(ke_{ij})^{\hat{\sigma}\breve{\beta}} = k^{\beta} e_{i^\sigma j^\sigma}$ $= (ke_{ij})^{\breve{\beta}\hat{\sigma}}$, then by extension, $\hat{\sigma}\breve{\beta} = \breve{\beta}\hat{\sigma}$. Consequently, $\breve{\beta}\hat{\sigma}\breve{\beta}^{-1} = \hat{\sigma} \in \mathcal{P}(P)$.

Theorem 3.10. $Aut(S) = [(Inn(S) \oslash \mathcal{G}(P)) \oslash Aut(K)^n] \oslash \mathcal{P}(P)$.

Proof. By [6], $Aut_K(S) = (Inn(S) \oslash \mathcal{G}(P)) \oslash \mathcal{P}(P)$. By Lemma 3.8, $Aut(K)^n \cap Aut_K(S) = 1$ and $Aut(S) = Aut(K)^n \cdot Aut_K(S)$. Since $Aut(K)^n$ normalizes $Inn(S) \oslash \mathcal{G}(P)$ by Lemma 3.9, then

$$Aut(S) = Aut(K)^n \cdot (Inn(S) \oslash \mathcal{G}(P)) \cdot \mathcal{P}(P)$$

$$= [(Inn(S) \oslash \mathcal{G}(P)) \oslash Aut(K)^n] \cdot \mathcal{P}(P).$$

Since $\mathcal{P}(P)$ normalizes $Aut(K)^n$ by Lemma 3.9, and since $\mathcal{P}(P)$ normalizes $Inn(S) \oslash \mathcal{G}(P)$, then $\mathcal{P}(P)$ normalizes $(Inn(S) \oslash \mathcal{G}(P)) \oslash Aut(K)^n$. Thus,

$$Aut(S) = [(Inn(S) \oslash \mathcal{G}(P)) \oslash Aut(K)^n] \oslash \mathcal{P}(P).$$

Corollary 3.11. If P is connected, then $Aut(S) = Aut_K(S) \oslash Aut(K)$.

Proof. By Lemma 3.8, $Aut(S) = Aut(K) \cdot Aut_K(S)$ and $Aut(K) \cap Aut_K(S) = 1$. By Lemma 3.9, $Aut(K)$ normalizes $Aut_K(S) = (Inn(S) \oslash \mathcal{G}(P)) \oslash \mathcal{P}(P)$ which means $Aut(S) = Aut_K(S) \oslash Aut(K)$.

Corollary 3.12. $Out(S) = [\mathcal{G}(P) \oslash Aut(K)^n] \oslash \mathcal{P}(P)$; if P is connected, then $Out(S) = Out_K(S) \oslash Aut(K)$.

Proof. By Lemma 3.9, $Aut(K)^n$ normalizes $\mathcal{G}(P)$, so that $\mathcal{G}(P) \oslash Aut(K)^n$ is defined. Define $\pi: Out(S) \to \mathcal{P}(P)$ via $Inn(S) g^* \breve{\beta} \hat{\sigma} \mapsto \hat{\sigma}$. To show π is well-defined, suppose $Inn(S) g^* \breve{\beta} \hat{\sigma} = Inn(S) h^* \breve{\delta} \hat{\tau}$. Then $g^* \breve{\beta} \hat{\sigma} = \alpha h^* \breve{\delta} \hat{\tau}$ for some $\alpha \in Inn(S)$. By the uniqueness of representation of the semidirect product, then $g^* = h^*$, $\breve{\beta} = \breve{\delta}$, and $\hat{\sigma} = \hat{\tau}$. It is easy to see that π is a group epimorphism.

Consider the sequence $\mathcal{G}(P) \oslash Aut(K)^n \xrightarrow{\nu} Out(S) \xrightarrow{\pi} \mathcal{P}(P) \to 1$ where ν is the canonical map. By

Theorem 3.10, $Aut(S)$ contains the disjoint subgroups $Inn(S)$ and $\mathcal{G}(P) \oslash Aut(K)^n$; hence, ν is monic. Clearly, $Im(\nu) \subseteq Ker(\pi)$. Suppose $Inn(S)g^*\breve{\beta}\hat{\sigma}$ $\in Ker(\pi)$. Then $\hat{\sigma} = \hat{1}$, so that $Inn(S)g^*\breve{\beta}\hat{\sigma} = Inn(S)g^*\breve{\beta} = (g * \breve{\beta})^\nu$. The sequence splits since the monomorphism $\mathcal{P}(P) \xrightarrow{\subseteq} Aut_K(S) \to Out_K(S)$ is a backmap for $\pi{:}\, Out(S) \to \mathcal{P}(P)$.

If $n = 1$, then by Corollary 3.11, $Aut(S) = Aut_K(S) \oslash Aut(K)$. We define $\pi{:}\, Out(S) \to Aut(K)$ by $Inn(S)g * \hat{\sigma}\breve{\beta} \mapsto \breve{\beta}$. As above, the sequence $1 \to Out_K(S) \xrightarrow{\subseteq} Out(S) \xrightarrow{\pi} Aut(K) \to 1$ is split exact with the monomorphism $Aut(K) \xrightarrow{\subseteq} Aut(S) \to Out(S)$ a backmap for $\pi{:}\, Out(S) \to Aut(K)$.

CHAPTER IV

THE CONNECTION BETWEEN $OUT(I(P))$ AND $OUT(I(\tilde{P}))$

Recall that $P = (V = V(P), \rho(P))$ is a finite preordered set with underlying poset $\tilde{P} = (\tilde{V} = V(\tilde{P}), \rho(\tilde{P}))$. Also recall that for $i \in V$, $[i] = \{j \in V | (i,j),(j,i) \in \rho(P)\}$ and that $\tilde{V}$ is a set of representatives of these classes. We make the associations $S = I(P)$ and $\tilde{S} = eSe = I(\tilde{P})$ where $e = \sum_{i \in \tilde{V}} e_i$ is a basic idempotent for S.

Our goal in the next chapter is to solve Problems B and C. Working to that end, we explore the connection between $Out(S)$ and $Out(eSe)$. We have the maps $\lambda_P: Out(S) \to Pic(S)$ and $\lambda_{\tilde{P}}: Out(eSe) \to Pic(eSe)$ as defined in Notation 2.2. Recall that by Theorem 2.10, $\lambda_{\tilde{P}}$ and $e(-)e: Pic(S) \to Pic(eSe)$ are group isomorphisms so that the diagram

$$\begin{array}{ccc} Out(S) & \xrightarrow{\lambda_P} & Pic(S) \\ \downarrow{\scriptstyle \phi_S} & & \downarrow{\scriptstyle e(-)e} \\ Out(eSe) & \xrightarrow{\lambda_{\tilde{P}}} & Pic(eSe) \end{array}$$

commutes, where $\phi_S = \lambda_P[e(-)e]\lambda_{\tilde{P}}^{-1}$.

The philosophy of this paper is that with a sufficient understanding of $Out(eSe)$ and the isomorphisms $\lambda_{\tilde{P}}$ and $e(-)e$, a reasonable description of the elements of $Pic(S)$ is within grasp as we will see in Chapter V. However, the description is particularly nice if λ_P is an isomorphism or, equivalently, if ϕ_S is an isomorphism. Since we would like to use knowledge about the simpler group $Out(eSe)$ to glean information about the structure of $Pic(S)$, then the goal of this chapter is to take a large step in that direction by discovering necessary and sufficient conditions under which ϕ_S is an isomorphism.

We begin with the connection between $\mathcal{G}(P)$ and $\mathcal{G}(\tilde{P})$. Specifically, we address the connection between $\overline{\overline{\rho}}(\tilde{P})$ and $\overline{\overline{\rho}}(P)$ and the connection between $\overline{\overline{\overline{\rho}}}(\tilde{P})$ and $\overline{\overline{\overline{\rho}}}(P)$. We start with the following lemma.

Lemma 4.1.

(i) $\overline{\rho}(\tilde{P}) = \{(i,i) | i \in \tilde{V}\} = \overline{\rho}(P) \cap [\tilde{V} \times \tilde{V}]$.

(ii) $\Delta(\tilde{P}) = \Delta(P) \cap [\tilde{V} \times \tilde{V}]$. Moreover, there is a bijection between the set of components of $\Delta(P)$ and the set of components of $\Delta(\tilde{P})$ via $C \mapsto C \cap \Delta(\tilde{P})$.

Proof. (i) This is clear since $\tilde{P}$ is a poset and therefore antisymmetric.

(ii) We first show that $V(\Delta(\tilde{P})) = V(\Delta(P) \cap [\tilde{V} \times \tilde{V}])$. Note that the proofs of the inclusions below are similar if we use (j,i) instead of (i,j).

($\subseteq$): If $i \in V(\Delta(\tilde{P}))$, then there exists $j \in \tilde{V}$ such that $(i,j) \in \rho(\tilde{P}) \setminus \overline{\rho}(\tilde{P})$, so that $(i,j) \in \rho(P) \setminus \overline{\rho}(P)$ and $i \in V(\Delta(P) \cap [\tilde{V} \times \tilde{V}])$.

($\supseteq$): If $i \in V(\Delta(P) \cap [\tilde{V} \times \tilde{V}])$, then there exists $j \in V(\Delta(P))$ such that $(i,j) \in \rho(P) \setminus \overline{\rho}(P)$. Let $m \in \tilde{V}$ such that $m \in [j]$. Then $(i,m) \in \rho(\tilde{P})$. However, $(i,m) \notin \overline{\rho}(\tilde{P})$, for otherwise, $i = m \in [j]$, which contradicts that $(i,j) \notin \overline{\rho}(P)$. Thus, $(i,m) \in \rho(\tilde{P}) \setminus \overline{\rho}(\tilde{P})$ and $i \in V(\Delta(\tilde{P}))$.

The edges also agree since for $i, j \in \tilde{V}$,

$$\{i,j\} \in \rho(\Delta(\tilde{P})) \Leftrightarrow (i,j) \in \rho(\tilde{P}) \setminus \overline{\rho}(\tilde{P}) \text{ (or } (j,i) \in \rho(\tilde{P}) \setminus \overline{\rho}(\tilde{P}))$$

$$\Leftrightarrow (i,j) \in \rho(P) \setminus \overline{\rho}(P) \text{ (or } (j,i) \in \rho(P) \setminus \overline{\rho}(P))$$

$$\Leftrightarrow \{i,j\} \in \rho(\Delta(P) \cap [\tilde{V} \times \tilde{V}]).$$

Clearly, each component of $\Delta(\tilde{P})$ is contained in some component of $\Delta(P)$. Suppose C_1 and C_2 are distinct components of $\Delta(\tilde{P})$ contained in some component C of $\Delta(P)$. Take $i \in V(C_1) \subseteq \tilde{V}$ and $j \in V(C_2) \subseteq \tilde{V}$. Since C is connected, there exist edges $\{i,x_1\},\{x_1,x_2\},\ldots,\{x_k,j\}$ in $\rho(\Delta(P))$. Let x_i' be the unique element in $\tilde{V} \cap [x_i]$ for each $1 \leq i \leq k$. It is then clear that $\{i,x_1'\},\{x_1',x_2'\},\ldots,\{x_k',j\}$ are edges in $\rho(\Delta(P) \cap [\tilde{V} \times \tilde{V}]) = \rho(\Delta(\tilde{P}))$ which implies that $j \in V(C_1)$. This contradicts that C_1 and C_2 are distinct, thereby showing the correspondence.

Recall that the definitions of $\mathcal{G}(P)$ and $\mathcal{G}(\tilde{P})$ depend on the choice of spanning trees for the components of P and $\tilde{P}$, respectively. Since we wish to connect $\mathcal{G}(P)$ and $\mathcal{G}(\tilde{P})$, we make a connection between spanning trees for components of $\Delta(\tilde{P})$ and spanning trees for components of $\Delta(P)$.

Definition. Let $\tilde{C}$ be a component of $\Delta(\tilde{P})$ with associated component C of $\Delta(P)$. Let $T(\tilde{P})$ be a spanning tree for $\tilde{C}$. If $T(P)$ is a spanning tree for a subgraph G of C containing $\tilde{C}$ such that $T(\tilde{P}) = T(P) \cap \Delta(\tilde{P})$ and each edge in $T(P)$ is incident on at least one vertex in $\tilde{V}$, we say that $T(P)$ **extends** $T(\tilde{P})$ and $T(\tilde{P})$ **lifts** to $T(P)$.

Lemma 4.2. If $T(\tilde{P})$ is a spanning tree for a component $\tilde{C}$ of $\Delta(\tilde{P})$, then there exists a spanning tree $T(P)$ for the corresponding component C of $\Delta(P)$ that extends $T(\tilde{P})$ as in the above definition.

Proof. We show that we can extend $T(\tilde{P})$ to such a spanning tree for any subgraph of C containing $\tilde{C}$ by inducting on the number of vertices in such a subgraph.

Suppose we have a subgraph G of C such that k is the only vertex in $V(G) \setminus V(\tilde{C})$. Let $j \in \tilde{V}$ such that $j \in [k]$. Since $k \in V(C)$, then $j \in V(C)$ as well. Thus, $j \in V(\tilde{C})$ and there exists an $m \in \tilde{V}$ such that $\{j,m\}$ is in $T(\tilde{P})$. So, either (j,m) (or (m,j)) is in $\rho(\tilde{P}) \setminus \overline{\rho}(\tilde{P}) = \rho(\tilde{P}) \setminus (\overline{\rho}(P) \cap [\tilde{V} \times \tilde{V}])$. Thus, (j,m) (or (m,j)) $\in \rho(P) \setminus \overline{\rho}(P)$. Since $j \in [k]$, then (k,m) (or (m,k)) $\in \rho(P) \setminus \overline{\rho}(P)$ and $\{k,m\} \in \rho(\Delta(P))$. The graph $T(\tilde{P}) \cup \{k,m\}$ formed by adding the vertex k and the edge $\{k,m\}$ to $T(\tilde{P})$ contains no circuits since $\{k,m\}$ connects a vertex in $\tilde{V}$ to a vertex not in $\tilde{V}$. Hence, $T(\tilde{P}) \cup \{k,m\}$ is a spanning tree for G that extends $T(\tilde{P})$.

We now suppose that we can extend $T(\tilde{P})$ to a spanning tree for any subgraph of C containing n more vertices than $\tilde{C}$ such that each edge in the tree is incident on at least one vertex of $\tilde{V}$, and we further suppose that we have a subgraph G of C such that $V(G) \setminus V(\tilde{C}) = \{v_1, \ldots, v_n, v_{n+1}\}$. By hypothesis, we can extend $T(\tilde{P})$ to an appropriate spanning tree for the graph formed by deleting from G the vertex v_{n+1} and any edges incident on it. This tree in turn can be extended appropriately to include v_{n+1} as described in the induction basis paragraph.

Remark. To connect $\mathcal{G}(P)$ and $\mathcal{G}(\tilde{P})$, we use Lemma 4.2 to extend trees for $\Delta(\tilde{P})$ to trees for $\Delta(P)$. Consequently, our choice of trees for $\Delta(P)$ depends on the trees chosen for $\Delta(\tilde{P})$. This, in turn, determines $\mathcal{G}(P)$. We impose this assumption throughout the rest of this paper.

Lemma 4.3. If the trees for the components of $\Delta(P)$ extend the trees for the components of $\Delta(\tilde{P})$, then

(i) $\overline{\overline{\rho}}(\tilde{P}) = \overline{\overline{\rho}}(P) \cap \rho(\tilde{P})$

(ii) $J(\tilde{P}) = J(P) \cap \tilde{V}$

(iii) $\overline{\overline{\overline{\rho}}}(\tilde{P}) = \overline{\overline{\overline{\rho}}}(P) \cap \rho(\tilde{P})$

Proof. (i) Suppose $(i,j) \in \overline{\overline{\rho}}(\tilde{P})$. Then $(i,j) \in \rho(\tilde{P})$ and $\{i,j\}$ is an edge in some tree $T(\tilde{P})$ for a component of $\Delta(\tilde{P})$. Since $\{i,j\}$ is also an edge in the tree $T(P)$ which extends $T(\tilde{P})$, then $(i,j) \in \overline{\overline{\rho}}(P) \cap \rho(\tilde{P})$.

Conversely, suppose $(i,j) \in \overline{\overline{\rho}}(P) \cap \rho(\tilde{P})$. Then, $\{i,j\}$ is an edge in some tree $T(P)$ for a component of $\Delta(P)$. Since this tree extends a tree $T(\tilde{P})$ in $\Delta(\tilde{P})$ and since $i,j \in \tilde{V}$, then $\{i,j\}$ must be an edge in $T(\tilde{P})$. So, $(i,j) \in \overline{\overline{\rho}}(\tilde{P})$.

(ii) Suppose $i \in J(\tilde{P})$. Then i is not a vertex in any tree of $\Delta(\tilde{P})$. Since $i \in \tilde{V}$, then it is also not a vertex in any tree of $\Delta(P)$. Thus, $i \in J(P) \cap \tilde{V}$.

Conversely, suppose $i \in J(P) \cap \tilde{V}$. Then i is not in any tree of $\Delta(P)$. Since the trees of $\Delta(P)$ contain those of $\Delta(\tilde{P})$, then i is not in any tree of $\Delta(\tilde{P})$ and $i \in J(\tilde{P})$.

(iii) By Lemma 4.1 and (ii),

$$(i,j) \in \overline{\overline{\overline{\rho}}}(\widetilde{P}) = \overline{\rho}(\widetilde{P}) \cap [J(\widetilde{P}) \times J(\widetilde{P})]$$

$$\Leftrightarrow (i,j) \in (\overline{\rho}(P) \cap [\widetilde{V} \times \widetilde{V}]) \cap [(J(P) \cap \widetilde{V}) \times (J(P) \cap \widetilde{V})]$$

$$\Leftrightarrow (i,j) \in (\overline{\rho}(P) \cap [J(P) \times J(P)]) \cap [\widetilde{V} \times \widetilde{V}]$$

$$\Leftrightarrow (i,j) \in \overline{\overline{\overline{\rho}}}(P) \cap \rho(\widetilde{P}).$$

Before stating the theorem establishing the connection between $\mathcal{G}(P)$ and $\mathcal{G}(\widetilde{P})$, we need the following lemma which we also use in many of the remaining proofs presented in this paper.

Lemma 4.4. If $\sigma \in Aut(S)$ such that $e^{\sigma} = e$, then $e({}_{\sigma}S)e =_{\sigma|_{eSe}} (eSe)$ as eSe-bimodules.

Proof. We make $e({}_{\sigma}S)e$ into an eSe-bimodule with right multiplication defined by $ete*ese = (ete)(ese) = e(tes)e$ and left multiplication $ese * ete = (ese)^{\sigma}(ete) = e(s^{\sigma}et)e$. Since this is simply the action of S on ${}_{\sigma}S$ restricted to eSe, $e({}_{\sigma}S)e$ is a well-defined eSe-bimodule.

We show that $e({}_{\sigma}S)e =_{\sigma|_{eSe}} (eSe)$ as eSe-bimodules by showing the identity map $I: e({}_{\sigma}S)e \rightarrow_{\sigma|_{eSe}} (eSe)$ is an eSe-bimodule isomorphism. Let $ete \in e({}_{\sigma}S)e$, $ese \in eSe$, and let $\bullet$ denote the left and right eSe-multiplication in ${}_{\sigma|_{eSe}}(eSe)$. Then $((ete)*(ese))^{I}$ $= (ete)(ese) = (ete)^{I} \bullet (ese)$. Also, $((ese)*(ete))^{I} = (ese)^{\sigma}(ete) = (ese) \bullet (ete)^{I}$. So, *1* is an *eSe*-bimodule monomorphism. Since both $e({}_{\sigma}S)e$ and ${}_{\sigma|_{eSe}}(eSe)$ equal *eSe* as sets, *1* is an isomorphism.

Theorem 4.5.

(i) If $g:\rho(P)\to K^*$ is a transitive function, then $g|_{\rho(\tilde{P})}:\rho(\tilde{P})\to K^*$ is a transitive function.

(ii) If $g^*\in\mathcal{G}(P)$, then $g^*|_{eSe}=(g|_{\rho(\tilde{P})})^*\in\mathcal{G}(\tilde{P})$.

(iii) The map $(-)|_{eSe}:\mathcal{G}(P)\to\mathcal{G}(\tilde{P})$ via $g^*\mapsto g^*|_{eSe}$ is a group isomorphism making the diagram

$$\begin{array}{ccccccc} \mathcal{G}(P) & \xrightarrow{\subseteq} & Aut(S) & \xrightarrow{\nu} & Out(S) & \xrightarrow{\lambda_P} & Pic(S) \\ \downarrow{\scriptstyle (-)|_{eSe}} & & & & \downarrow{\scriptstyle \phi_S} & & \downarrow{\scriptstyle e(-)e} \\ \mathcal{G}(\tilde{P}) & \xrightarrow{\subseteq} & Aut(eSe) & \xrightarrow{\nu} & Out(eSe) & \xrightarrow{\lambda_{\tilde{P}}} & Pic(eSe) \end{array} \tag{4.6}$$

commute where the ν are the canonical projections.

Proof. (i) This follows since $\rho(\tilde{P})\subseteq\rho(P)$.

(ii) Suppose $g^*\in\mathcal{G}(P)$. Then g is transitive and $(i,j)^g=1$ for all (i,j) $\in\overline{\overline{\rho}}(P)\cup\overline{\overline{\overline{\rho}}}(P)$. Since $\overline{\overline{\rho}}(\tilde{P})\subseteq\overline{\overline{\rho}}(P)$ and $\overline{\overline{\overline{\rho}}}(\tilde{P})\subseteq\overline{\overline{\overline{\rho}}}(P)$ by Lemma 4.3, then $\overline{\overline{\rho}}(\tilde{P})\cup\overline{\overline{\overline{\rho}}}(\tilde{P})\subseteq\overline{\overline{\rho}}(P)\cup\overline{\overline{\overline{\rho}}}(P)$. Thus, $(k,m)^{g|_{\rho(\tilde{P})}}=1$ for all $(k,m)\in\overline{\overline{\rho}}(\tilde{P})\cup\overline{\overline{\overline{\rho}}}(\tilde{P})$ and $(g|_{\rho(\tilde{P})})^*\in\mathcal{G}(\tilde{P})$.

Since $(i,i)^g = (i,i)^g(i,i)^g$ inside K^*, then $(i,i)^g = 1$ for all $i \in V$. Thus, $e^{g^*} = e$ and $(eSe)^{g^*} = e^{g^*}S^{g^*}e^{g^*} = eS^{g^*}e$. Since $S^{g^*} = S$, then $(eSe)^{g^*} = eSe$ and $g^*|_{eSe} \in Aut(eSe) = Aut(I(\tilde{P}))$.

If $eAe = \sum_{(i,j)\in\rho(\tilde{P})} \alpha_{i,j}e_{ij}$ is an element of eSe, then

$$\begin{aligned}(eAe)^{(g|_{\rho(\tilde{P})})^*} &= \sum_{(i,j)\in\rho(\tilde{P})} \alpha_{i,j}(e_{ij})^{(g|_{\rho(\tilde{P})})^*} \\ &= \sum_{(i,j)\in\rho(\tilde{P})} \alpha_{i,j}(i,j)^g(e_{ij}) \\ &= \sum_{(i,j)\in\rho(\tilde{P})} \alpha_{i,j}(e_{ij})^{g^*|_{eSe}} \\ &= (eAe)^{g^*|_{eSe}}.\end{aligned}$$

(iii) By (ii), $(-)|_{eSe}$ is well-defined. Since $g^*|_{eSe} \in Aut(eSe)$, then $(gh)^*|_{eSe}$ $= (g^* \circ h^*)|_{eSe} = (g^*|_{eSe})(h^*|_{eSe})$, and $(-)|_{eSe}$ is a group homomorphism. To show that it is epic, we let $h^* \in \mathcal{G}(\tilde{P})$ and construct $g^* \in \mathcal{G}(P)$ such that $g^*|_{eSe} = h^*$ as follows:

For $i, j \in V$, there exist $i', j' \in \tilde{V}$ such that $i' \in [i]$ and $j' \in [j]$. We define $g: \rho(P) \to K^*$ via $(i,j)^g = (i',j')^h$. Since $(i,j) \in \rho(P)$ implies $(i',j') \in \rho(\tilde{P})$, g is well-defined. Notice that $(i,j)^g = (i,j)^h$ for all $(i,j) \in \rho(\tilde{P})$ and $(i,j)^g = 1$ if $[i] = [j]$.

We start by showing g is transitive. Let $(i,l), (l,j) \in \rho(P)$, and let $i', j', l' \in \tilde{V}$ such that $i' \in [i]$, $j' \in [j]$, and $l' \in [l]$. By the transitivity of h, $(i,j)^g = (i',j')^h$ $= (i',l')^h(l',j')^h = (i,l)^g(l,j)^g$.

To show $g^* \in \mathcal{G}(P)$, we must show $(i,j)^g = 1$ for all $(i,j) \in \overline{\overline{\rho}}(P) \cup \overline{\overline{\overline{\rho}}}(P)$. If $(i,j) \in \overline{\overline{\overline{\rho}}}(P)$, then $[i] = [j]$ and $(i,j)^g = 1$. If $(i,j) \in \overline{\overline{\rho}}(P)$, then $(i,j) \in \rho(P)$ and $\{i,j\}$ is an edge in a tree *T*(*P*) for some component of *Δ*(*P*). By assumption, *T*(*P*) extends the tree $T(\tilde{P})$ for the associated component of $\Delta(\tilde{P})$ as described in Lemma 4.2. By definition, $(i,j)^g = (i',j')^h$ where $i', j' \in \tilde{V}$, $i' \in [i]$, and $j' \in [j]$. Since, by construction, each edge in *T*(*P*) is incident on at least one vertex in $\tilde{V}$, we must consider the following cases:

Case 1: If $i, j \in \tilde{V}$, then $\{i,j\}$ is an edge in $T(\tilde{P})$, $(i,j) \in \overline{\overline{\rho}}(\tilde{P})$, and $(i,j)^g = (i,j)^h = 1$.

Case 2: Suppose $i \notin \tilde{V}$ and $j \in \tilde{V}$. The proof is similar if we suppose $i \in \tilde{V}$ and $j \notin \tilde{V}$. By the construction of *T*(*P*), we know $\{i,j\}$ was added to $T(\tilde{P})$ to "graft in" the vertex *i* and $\{i',j\}$ is an edge in $T(\tilde{P})$. Since $(i,j) \in \rho(P)$, then $(i',j) \in \rho(P)$ so that $(i',j) \in \overline{\overline{\rho}}(\tilde{P})$. Hence, $(i,j)^g = (i',j)^h = 1$.

Since $(i,j)^g = (i,j)^h$ for all $(i,j) \in \rho(\tilde{P})$, then clearly $g^*|_{eSe} = h^*$ and $(-)|_{eSe}$ is surjective.

By [6, Theorem C], *Aut*(*S*) contains the disjoint subgroups $\mathcal{G}(P)$ and *Inn*(*S*); hence, the top row of diagram (4.6) is monic. Since the bottom row is similarly monic and since $e(-)e$ is monic, then we will have shown $(-)|_{eSe}$ is injective once we show the large rectangle commutes. However, this will also show the diagram commutes since the right square commutes by definition.

The image inside *Pic*(*S*) of $g^* \in \mathcal{G}(P)$ under the composition of the top row is $[{}_{g^*}S]$. After applying $e(-)e$, we have the element $[e({}_{g^*}S)e]$ in *Pic*(*eSe*). By part (ii), $e^{g^*} = e$. Hence, $e({}_{g^*}S)e = {}_{g^*|_{eSe}}(eSe)$ as *eSe*-bimodules by Lemma 4.4. The diagram commutes since $[{}_{(g^*|_{eSe})}(eSe)]$ is the image of $g^*|_{eSe}$ in *Pic*(*eSe*) under the composition of the bottom row.

We now turn our attention to connecting $\mathcal{P}(P)$ and $\mathcal{P}(\widetilde{P})$.

Definition. Define $(\widetilde{=}): Aut(P) \to Aut(\widetilde{P})$ via $\sigma \mapsto \widetilde{\sigma}$ where, for $i \in \widetilde{V}$, $i^{\widetilde{\sigma}}$ is the unique element in $\widetilde{V} \cap [i^{\sigma}]$. We show that $\widetilde{\sigma} \in Aut(\widetilde{P})$ below.

Lemma 4.7. Let $\sigma \in Aut(P)$.

(i) $\widetilde{\sigma} \in Aut(\widetilde{P})$.

(ii) $(\widetilde{=})$ is a group homomorphism with kernel *L*. Recall that *L* is the set $\{\sigma \in Aut(P) | ([j])^{\sigma} = [j] \text{ for each } j \in V\}$; i.e., *L* consists of those σ leaving each class invariant.

Proof. (i) We must show $\widetilde{\sigma} \in S_{\widetilde{V}}$ and $(i^{\widetilde{\sigma}}, j^{\widetilde{\sigma}}) \in \rho(\widetilde{P})$ whenever $(i, j) \in \rho(\widetilde{P})$. We first show $\widetilde{\sigma}$ is a bijection of $\widetilde{V}$.

Suppose $i, j \in \widetilde{V}$ such that $i^{\widetilde{\sigma}} = j^{\widetilde{\sigma}}$. Then $[i^{\sigma}] = [j^{\sigma}]$, so that $[i] = [j]$. Since $i, j \in \widetilde{V}$, which is a set of equivalence class representatives, then $i = j$ and $\widetilde{\sigma}$ is injective.

Let $j \in \tilde{V}$. Since $\sigma \in Aut(P)$, there exists $k \in V$ such that $k^{\sigma} = j$. Let $i \in \tilde{V}$ such that $i \in [k]$. Then $[i^{\sigma}] = [k^{\sigma}] = [j]$, $i^{\tilde{\sigma}} = j$, and $\tilde{\sigma}$ is a bijection.

For $i, j \in \tilde{V}$, $i^{\tilde{\sigma}} \in [i^{\sigma}]$ implies that $(i^{\tilde{\sigma}}, i^{\sigma}), (i^{\sigma}, i^{\tilde{\sigma}}) \in \rho(P)$, and $j^{\tilde{\sigma}} \in [j^{\sigma}]$ implies that $(j^{\tilde{\sigma}}, j^{\sigma}), (j^{\sigma}, j^{\tilde{\sigma}}) \in \rho(P)$. Thus, by transitivity,

$$\begin{aligned}
(i, j) \in \rho(\tilde{P}) &\Leftrightarrow (i, j) \in \rho(P) \\
&\Leftrightarrow (i^{\sigma}, j^{\sigma}) \in \rho(P) \\
&\Leftrightarrow (i^{\tilde{\sigma}}, j^{\tilde{\sigma}}) \in \rho(P) \\
&\Leftrightarrow (i^{\tilde{\sigma}}, j^{\tilde{\sigma}}) \in \rho(\tilde{P}).
\end{aligned}$$

(ii) Let $\sigma, \tau \in Aut(P)$. Then $i^{\widetilde{(\sigma\tau)}} = m$ where $m \in \tilde{V} \cap [i^{\sigma\tau}]$. On the other hand, $i^{\tilde{\sigma}} = j$ where $j \in \tilde{V} \cap [i^{\sigma}]$, which implies that $i^{\tilde{\sigma}\tilde{\tau}} = j^{\tilde{\tau}} = k$ where $k \in \tilde{V} \cap [j^{\tau}]$. Since $[k] = [j^{\tau}] = [i^{\sigma\tau}] = [m]$ and $k, m \in \tilde{V}$, then $k = m$. So, $i^{\widetilde{(\sigma\tau)}} = m = k = i^{\tilde{\sigma}\tilde{\tau}}$, and $(\simeq)$ is a group homomorphism.

Finally, we have

$$\begin{aligned}
\sigma \in Ker(\simeq) &\Leftrightarrow \tilde{\sigma} = 1 \text{ in } Aut(\tilde{P}) \\
&\Leftrightarrow i^{\tilde{\sigma}} = i \text{ for all } i \in \tilde{V} \\
&\Leftrightarrow [i^{\sigma}] = [i] \text{ for all } i \in \tilde{V} \\
&\Leftrightarrow [i^{\sigma}] = [i] \text{ for all } i \in V \\
&\Leftrightarrow \sigma \in L.
\end{aligned}$$

Theorem 4.8. Let the representatives comprising $\widetilde{V}$ be chosen such that $i \in \widetilde{V}$ implies $i \leq j$ as integers for all $j \in [i]$.

(i) ($\simeq$) induces a monomorphism $(-)|_{eSe} : \mathcal{P}(P) \to \mathcal{P}(\widetilde{P})$ via $\hat{\sigma} \mapsto \hat{\sigma}|_{eSe} = \widehat{\tilde{\sigma}}$.

(ii) The group homomorphism ϕ_S is given by

$$(Inn(S)g^*\breve{\beta}\hat{\sigma})^{\phi_S} = Inn(\widetilde{S})(g^*\breve{\beta}\hat{\sigma})|_{eSe} = Inn(\widetilde{S})(g^*|_{eSe})(\breve{\beta}|_{eSe})\widehat{\tilde{\sigma}}$$

for $g^* \in \mathcal{G}(P)$, $\breve{\beta} \in Aut(K)^n$, and $\hat{\sigma} \in \mathcal{P}(P)$.

Proof. (i) Suppose $\hat{\sigma} \in \mathcal{P}(P)$. Because of the way we chose the elements in $\widetilde{V}$, $\sigma|_{\widetilde{V}} \in Aut(\widetilde{P})$ so that $\sigma|_{\widetilde{V}} = \tilde{\sigma}$. Hence, $\hat{\sigma}$ fixes e and $\hat{\sigma}|_{eSe} \in Aut(eSe)$. For an element $eAe = \sum_{(i,j)\in\rho(\widetilde{P})} \alpha_{i,j}e_{ij}$ of eSe,

$$(eAe)^{\widehat{\tilde{\sigma}}} = \sum_{(i,j)\in\rho(\widetilde{P})} \alpha_{i,j}(e_{i^{\tilde{\sigma}}j^{\tilde{\sigma}}}) = \sum_{(i,j)\in\rho(\widetilde{P})} \alpha_{i,j}(e_{i^{\sigma}j^{\sigma}}) = (eAe)^{\hat{\sigma}},$$

so that $\hat{\sigma}|_{eSe} = \widehat{\tilde{\sigma}}$.

Since $\widehat{\sigma\tau}|_{eSe} = \hat{\sigma}\hat{\tau}|_{eSe} = \hat{\sigma}|_{eSe}\,\hat{\tau}|_{eSe}$ for $\hat{\tau} \in \mathcal{P}(P)$, we have a group homomorphism. If $\widehat{\tilde{\sigma}} = \widehat{\tilde{\tau}}$, then $\tilde{\sigma} = \tilde{\tau}$ since $(\hat{-}): Aut(\widetilde{P}) \to Aut(\widetilde{S})$ is monic. Since $\sigma, \tau \in \mathcal{P}(P)$, they are completely determined by $\tilde{\sigma}$ and $\tilde{\tau}$, respectively. Thus, $\sigma = \tau$ and $\hat{\sigma} = \hat{\tau}$.

(ii) Since $\widetilde{P}$ also has n components, $Aut(K)^n$ embeds inside $Aut(\widetilde{S})$. We abuse notation slightly and identify $Aut(K)^n$ as residing in both $Aut(S)$ and $Aut(\widetilde{S})$.

Since $\breve{\beta} \in Aut(K)^n$ fixes e, it leaves eSe invariant. In particular, $\breve{\beta}|_{eSe} \in Aut(eSe)$. Since $\breve{\beta}$ merely acts on the entries of the matrices of S and eSe, then

$\breve{\beta}|_{eSe} \in Aut(K)^n$ as contained in $Aut(eSe)$. It is easy to verify that the restriction map induces a group isomorphism between $Aut(K)^n \subseteq S$ and $Aut(K)^n \subseteq Aut(eSe)$.

By Corollary 3.12, $Out(S) = [\mathcal{G}(P) \oslash Aut(K)^n] \oslash \mathcal{P}(P)$. Furthermore, arbitrary elements in $Out(S)$ are uniquely of the form $Inn(S)g^*\breve{\beta}\hat{\sigma}$ where $g^* \in \mathcal{G}(P)$, $\breve{\beta} \in Aut(K)^n$, and $\hat{\sigma} \in \mathcal{P}(P)$. By (i), $e^{\hat{\sigma}} = e$. Since $e^{\breve{\beta}} = e$ and since $e^{g^*} = e$ by Theorem 4.5, then $e({}_{g^*\breve{\beta}\hat{\sigma}}S)e =_{(g^*\breve{\beta}\hat{\sigma})|_{eSe}} (eSe)$ as eSe-bimodules by Lemma 4.4. Since $g^*|_{eSe}, \breve{\beta}|_{eSe} \in Aut(eSe)$, then $(g^*\breve{\beta}\hat{\sigma})|_{eSe} = (g^*|_{eSe})(\breve{\beta}|_{eSe})(\hat{\sigma}|_{eSe})$. By (i), $\hat{\sigma}|_{eSe} = \hat{\tilde{\sigma}}$. Thus,

$$
\begin{aligned}
(Inn(S)g^*\breve{\beta}\hat{\sigma})^{\phi_S} &= (Inn(S)g^*\breve{\beta}\hat{\sigma})^{\lambda_P[e(-)e]\lambda_{\tilde{P}}^{-1}} \\
&= ([e({}_{g^*\breve{\beta}\hat{\sigma}}S)e])^{\lambda_{\tilde{P}}^{-1}} \\
&= ([{}_{(g^*\breve{\beta}\hat{\sigma})|_{eSe}}(eSe)])^{\lambda_{\tilde{P}}^{-1}} \\
&= Inn(eSe)(g * \breve{\beta}\hat{\sigma})|_{eSe} \\
&= Inn(\tilde{S})(g^*|_{eSe})(\breve{\beta}|_{eSe})(\hat{\tilde{\sigma}}).
\end{aligned}
$$

We are about to accomplish the stated goal for this chapter; that is, we will list necessary and sufficient conditions under which ϕ_S is an isomorphism. To do so, we need one more definition.

Definition. Following Bolla [5], we say S has the ***<u>Aut-Pic property</u>*** if $\lambda_P : Out(S) \to Pic(S)$ is an isomorphism.

Corollary 4.9. The following statements are equivalent.

(i) The group monomorphism $\phi_S : Out(S) \to Out(\tilde{S})$ is an isomorphism.

(ii) The group monomorphism $(-)|_{eSe} : \mathcal{P}(P) \to \mathcal{P}(\tilde{P})$ is an isomorphism.

(iii) The group homomorphism $(\approx) : Aut(P) \to Aut(\tilde{P})$ is surjective.

(iv) S has the *Aut-Pic* property.

Proof. (i)⇔(ii) Consider the following diagram where the rows are as in Corollary 3.12.

$$\begin{array}{ccccccccc} 1 \to & \mathcal{G}(P) \oslash Aut(K)^n & \to & Out(S) & \to & \mathcal{P}(P) & \to 1 \\ & \downarrow (-)|_{eSe} & & \downarrow \phi_S & & \downarrow (-)|_{eSe} & \\ 1 \to & \mathcal{G}(\tilde{P}) \oslash Aut(K)^n & \to & Out(\tilde{S}) & \to & \mathcal{P}(\tilde{P}) & \to 1 \end{array}$$

For $g^*\breve{\beta}, g_1{}^*\breve{\beta}_1 \in \mathcal{G}(P) \oslash Aut(K)^n$, $(g^*\breve{\beta} g_1{}^*\breve{\beta}_1)|_{eSe} = (g^*\breve{\beta})|_{eSe}\,(g_1{}^*\breve{\beta}_1)|_{eSe}$ since $g^*\breve{\beta}|_{eSe} \in Aut(eSe)$. Hence, the leftmost vertical map $(-)|_{eSe}$ is a group homomorphism. Since $(-)|_{eSe} : \mathcal{G}(P) \to \mathcal{G}(\tilde{P})$ is an isomorphism by Theorem 4.5, we have an isomorphism.

By Theorem 4.8, both the left and right squares clearly commute. Hence, the entire diagram commutes.

If ϕ_S is surjective, then $(-)|_{eSe} : \mathcal{P}(P) \to \mathcal{P}(\tilde{P})$ is surjective by the commutativity of the right square. Surjectivity of $(-)|_{eSe} : \mathcal{P}(P) \to \mathcal{P}(\tilde{P})$ implies surjectivity of ϕ_S since the leftmost $(-)|_{eSe}$ is an isomorphism.

(ii)$\Rightarrow$(iii) Take $\alpha \in Aut(\tilde{P})$. Then $\hat{\alpha} \in \mathcal{P}(\tilde{P})$ so that there exists $\hat{\sigma} \in \mathcal{P}(P)$ such that $\hat{\tilde{\sigma}} = \hat{\sigma}|_{eSe} = \hat{\alpha}$. Since $(\hat{\ }) : Aut(\tilde{P}) \to Aut(\tilde{S})$ is monic, then $\tilde{\sigma} = \alpha$.

(iii)$\Rightarrow$(ii) Take $\hat{\alpha} \in \mathcal{P}(\tilde{P})$. Then $\alpha \in Aut(\tilde{P})$ so that there exists $\beta \in Aut(P)$ such that $\tilde{\beta} = \alpha$. By Proposition 3.3, $\beta = \tau\sigma$ for some $\tau \in L$ and some $\sigma \in Aut(P)$ such that $\hat{\sigma} \in \mathcal{P}(P)$. Since $L = Ker(\tilde{\ })$, then $\alpha = \tilde{\beta} = \tilde{\tau}\tilde{\sigma} = \tilde{\sigma}$ and $\hat{\sigma}|_{eSe} = \hat{\tilde{\sigma}} = \hat{\alpha}$.

(i)$\Leftrightarrow$(iv) This is trivial.

CHAPTER V

THE SOLUTION TO PROBLEMS B AND C

Recall that $P = (V = V(P), \rho(P))$ is a finite preordered set with underlying poset $\tilde{P} = (\tilde{V} = V(\tilde{P}), \rho(\tilde{P}))$. Recall that for $i \in V$, $[i] = \{j \in V | (i,j), (j,i) \in \rho(P)\}$. We suppose $\tilde{V} = \{v_1, \ldots v_t\}$ is the set of minimal representatives of these classes (as integers) since we wish to apply Theorem 4.8. Also recall that we make the associations $S = I(P)$ and $\tilde{S} = eSe = I(\tilde{P})$ where $e = \sum_{i \in \tilde{V}} e_i$ is a basic idempotent for S.

Since our first goal is to explicitly describe the elements of $Pic(S)$, we revisit the following commutative diagram from Theorem 2.10 where $\phi_S = \lambda_P[e(-)e]\lambda_{\tilde{P}}^{-1}$.

$$\begin{array}{ccc} Out(S) & \xrightarrow{\lambda_P} & Pic(S) \\ \downarrow^{\phi_S} & & \downarrow^{e(-)e} \\ Out(eSe) & \xrightarrow{\lambda_{\tilde{P}}} & Pic(eSe) \end{array}$$

By Lemma 2.3, if X is an invertible S-bimodule such that either ${}_S X \cong_S S$ or $X_S \cong S_S$, then there exists $\varphi \in Aut(S)$ such that $X \cong {}_\varphi S$ as S-bimodules. If this is the case for each invertible S-bimodule, then λ_P is an isomorphism, which makes ϕ_S an isomorphism. Since we are assuming that we have a sufficient understanding of $Out(eSe)$, then, in this case, a nice description of $Pic(S)$ is easily within grasp as we see next.

Suppose λ_P and ϕ_S are isomorphisms. Then if $[X] \in Pic(S)$, there exists $\theta \in Aut(S)$ such that $X \cong_\theta S$. However, θ can be chosen as a lifting of an automorphism of $\tilde{S}$. We know from Corollary 3.12 that each element in $Out(\tilde{S})$ is uniquely of the form $Inn(\tilde{S})g^*\breve{\beta}\hat{\sigma}$ for some $g^* \in \mathcal{G}(\tilde{P})$, $\breve{\beta} \in Aut(K)^n$, and $\hat{\sigma} \in \mathcal{P}(\tilde{P})$. Since ϕ_S is epic, then by Theorem 4.8, $Inn(\tilde{S})g^*\breve{\beta}\hat{\sigma} = (Inn(S)g_1^*\breve{\beta}_1\hat{\sigma}_1)^{\phi_S}$ where $g_1^* \in \mathcal{G}(P)$ such that $g_1^*|_{eSe} = g^*$, $\breve{\beta}_1 \in Aut(K)^n \subseteq Aut(S)$ such that $\breve{\beta}_1|_{eSe} = \breve{\beta}$, and $\hat{\sigma}_1 \in \mathcal{P}(P)$ such that $\hat{\sigma}_1|_{eSe} = \widehat{(\tilde{\sigma}_1)} = \hat{\sigma}$. Since $\phi_S{}^{-1}\lambda_P$ is an isomorphism, $[{}_{g_1^*\breve{\beta}_1\hat{\sigma}_1}S] = (Inn(\tilde{S})g^*\breve{\beta}\hat{\sigma})^{\phi_S{}^{-1}\lambda_P}$ $\in Pic(S)$. Hence, if we understand the structure of the elements of $Out(\tilde{S})$, then since we know how to form the image under $\phi_S{}^{-1}$ of these elements, we can construct a representative of each element in $Pic(S)$.

However, not all elements in $Pic(S)$ are necessarily of the form ${}_\varphi S$ for some $\varphi \in Aut(S)$. In fact, λ_P is an isomorphism if and only if $(\approx): Aut(P) \to Aut(\tilde{P})$ is surjective by Corollary 4.9. Hence, describing the elements of $Pic(S)$ is interesting when $(\approx)$ is not surjective. However, since $\lambda_{\tilde{P}}(e(-)e)^{-1}$ is an isomorphism, there is still hope to characterize $Pic(S)$ based upon the structure of elements in $Out(\tilde{S})$.

Definition 5.1. For $\sigma \in Aut(\tilde{P})$, we set

$$\rho(\sigma) = \{(i,j) | ((i')^\sigma, j) \in \rho(P) \text{ where } i' \in \tilde{V} \text{ such that } i \in [i']\}.$$

We define $Z(\sigma)$ to be the subset of the $|V|\times|V|$ matrix ring over K such that $z \in Z(\sigma)$ if and only if $z_{i,j} = 0$ provided $(i,j) \notin \rho(\sigma)$. We call $Z(\sigma)$ the **S-bimodule induced by $\rho(\sigma)$** or the **incidence S-bimodule induced by σ**. We justify this name below.

Lemma 5.2. Let $\sigma \in Aut(\tilde{P})$.

(i) $Z(\sigma)$ is an S-bimodule.

(ii) $Z(\sigma)_S \cong (e_{v_1}S)^{(b_1)} \oplus \ldots \oplus (e_{v_t}S)^{(b_t)}$ where for $1 \le i \le t$, $b_i = \left|[(v_i)^{\sigma^{-1}}]\right|$.

Proof. (i) To show $Z(\sigma)$ is a left S-module we need only show $sz \in Z(\sigma)$ for all $z \in Z(\sigma)$ and $s \in S$. Let $z \in Z(\sigma)$. By linearity, we may consider only the $e_{k\,l}$ in S such that $(k,l) \in \rho(P)$. To show $e_{k\,l}z \in Z(\sigma)$, it suffices to show that for $i, j \in V$, $(e_{k\,l}z)_{i,j} \neq 0$ implies $(i,j) \in \rho(\sigma)$. However, $(e_{k\,l}z)_{i,j} = 0$ unless $i = k$. So, we may suppose $i = k$. Then, $(e_{k\,l}z)_{i,j} = \sum_{m \in V} (e_{k\,l})_{k,m} z_{m,j} = (e_{k\,l})_{k,l} z_{l,j} = z_{l,j}$. If $z_{l,j} \neq 0$, then $(l,j) \in \rho(\sigma)$ and $((l')^{\sigma}, j) \in \rho(P)$ where $l' \in \tilde{V}$ such that $l \in [l']$. Since $e_{i\,l} = e_{k\,l} \in S$, then $(i,l) \in \rho(P)$, so that $((i')^{\sigma}, (l')^{\sigma}) \in \rho(P)$ where $i' \in \tilde{V}$ such that $i \in [i']$. By transitivity, $((i')^{\sigma}, j) \in \rho(P)$. Hence, $(i,j) \in \rho(\sigma)$, and $Z(\sigma)$ is a left S-module. $Z(\sigma)$ is similarly a right S-module. Since S and $Z(\sigma)$ are subsets of $M_n(K)$ where $n = |V(P)|$, the multiplication is associative and $Z(\sigma)$ is an S-bimodule.

(ii) For $i \in [(v_j)^{\sigma^{-1}}]$, $e_i(Z(\sigma)) \cong e_{(v_j)^{\sigma^{-1}\sigma}} S = e_{v_j}S$ by the definition of $Z(\sigma)$. So, there are $b_j = \left|[(v_j)^{\sigma^{-1}}]\right|$ copies of $e_{v_j}S$ in $Z(\sigma)$.

We next show that $Z(\sigma) \in Pic(S)$ and $e(Z(\sigma))e \cong_{\hat{\sigma}} (eSe)$ as eSe-bimodules.

This will allow us to identify $(Inn(\tilde{S})g^{*}\breve{\beta}\hat{\sigma})^{\lambda_{\tilde{P}}(e(-)e)^{-1}} \in Pic(S)$ as $[\,_{g_1^{*}\breve{\beta}_1} Z(\sigma)]$ where

$g_1^{*} \in \mathcal{G}(P)$ extends g^{*} and $\breve{\beta}_1 \in Aut(K)^n \subseteq Aut(S)$ is the obvious extension of $\breve{\beta}$.

Lemma 5.3. Let $\sigma \in Aut(\tilde{P})$.

(i) $Z(\sigma) \in Pic(S)$.

(ii) $e(Z(\sigma))e \cong_{\hat{\sigma}} (eSe)$ as eSe-bimodules.

Proof. Recall that by Lemma 5.2, $Z(\sigma)_S \cong (e_{v_1}S)^{(b_1)} \oplus \ldots \oplus (e_{v_t}S)^{(b_t)}$ where for $1 \leq i \leq t$, $b_i = \left|[(v_i)^{\sigma^{-1}}]\right|$.

(i) $Z(\sigma)_S$ is a finite sum of finitely-generated, projective right S-modules; therefore, $Z(\sigma)_S$ is a finitely-generated, projective right S-module. Since each $b_i \geq 1$, $Z(\sigma)_S$ is easily seen to be a generator for *mod-S*.

Since $Pic(S) = \{[\,_S Z_S] \mid Z_S$ is a progenerator for *mod-S* and $End(Z_S) \cong S$ as rings$\}$, it remains to show $End(Z(\sigma)_S) \cong S$. We again rely on our basic set of idempotents $\{e_{v_1}, \ldots, e_{v_t}\}$. Since $Z(\sigma)_S \cong (e_{v_1}S)^{(b_1)} \oplus \ldots \oplus (e_{v_t}S)^{(b_t)}$ as right S-modules, then writing our maps on the left (opposite the scalars), $End(Z(\sigma)_S) \cong (h_{i,j})$ where $h_{i,j} = Hom((e_{v_j}S)^{(b_j)}, (e_{v_i}S)^{(b_i)})$. But, $Hom((e_{v_j}S)^{(b_j)}, (e_{v_i}S)^{(b_i)})$ is isomorphic to the $b_i \times b_j$ matrix ring over $Hom(e_{v_j}S, e_{v_i}S)$, which in turn is isomorphic to $e_{v_i}Se_{v_j}$ as abelian groups. Consequently, $Hom((e_{v_j}S)^{(b_j)}, (e_{v_i}S)^{(b_i)}) \cong M_{b_i \times b_j}(e_{v_i}Se_{v_j})$, and

$$End(Z(\sigma)_S) \cong \begin{pmatrix} M_{b_1\times b_1}(e_{v_1}Se_{v_1}) & M_{b_1\times b_2}(e_{v_1}Se_{v_2}) & \cdots & M_{b_1\times b_t}(e_{v_1}Se_{v_t}) \\ M_{b_2\times b_1}(e_{v_2}Se_{v_1}) & & & \\ \vdots & & & \\ M_{b_t\times b_1}(e_{v_t}Se_{v_1}) & M_{b_t\times b_2}(e_{v_t}Se_{v_2}) & \cdots & M_{b_t\times b_t}(e_{v_t}Se_{v_t}) \end{pmatrix}$$

Throughout the rest of the proof, we associate $End(Z(\sigma)_S)$ with the above matrix ring where the ring multiplication is standard matrix multiplication. Since $\sigma \in Aut(\tilde{P})$, then by the definition of the b_i, $|V| = b_1 + \ldots + b_t$.

We will be done if we can show that $End(Z(\sigma)_S)$ is a structure matrix ring for *P* over *K* since it will then be isomorphic to $I(P) = S$. We have that $e_{v_i}Se_{v_j} \cong K$ or $e_{v_i}Se_{v_j} = 0$ for each $i, j \in V$. To show we have a structure matrix ring, we label the vertices in *V* $v_1^1, v_1^2, \ldots, v_1^{|v_1|}, v_2^1, \ldots, v_2^{|v_2|}, \ldots, v_t^1, \ldots, v_t^{|v_t|}$ such that for each $1 \le i \le t$, $v_i^1 = v_i$ and for each $1 \le k \le |[v_i]|$, $v_i^k \in [v_i]$. Since $\sigma^{-1} \in Aut(\tilde{P})$, it induces a permutation on $\tilde{V} = \{v_1, \ldots, v_t\}$ which can be viewed as a permutation on $\{1, \ldots, t\}$. We therefore identify $v_{i^{\sigma^{-1}}}$ with $(v_i)^{\sigma^{-1}}$ for each *i*. We index the rows and columns of $End(Z(\sigma)_S)$ with the vertices in the following order: $v^1_{1^{\sigma^{-1}}}, v^2_{1^{\sigma^{-1}}}, \ldots, v^{b_1}_{1^{\sigma^{-1}}}, v^1_{2^{\sigma^{-1}}}, \ldots, v^{b_2}_{2^{\sigma^{-1}}}, \ldots, v^1_{t^{\sigma^{-1}}}, \ldots, v^{b_t}_{t^{\sigma^{-1}}}$.

To complete the proof that $End(Z(\sigma)_S)$ is a structure matrix ring for *P* over *K*, it suffices to show that if $a \in End(Z(\sigma)_S)$ and $a_{v^k_{i^{\sigma^{-1}}}, v^l_{j^{\sigma^{-1}}}} \neq 0$, then $(v^k_{i^{\sigma^{-1}}}, v^l_{j^{\sigma^{-1}}}) \in \rho(P)$.

Since $a_{v^k_{i^{\sigma^{-1}}}, v^l_{j^{\sigma^{-1}}}}$ is a non-zero entry in $M_{b_i \times b_j}(e_{v_i}Se_{v_j})$, then $(v_i, v_j) \in \rho(P)$. This implies that $(v_{i^{\sigma^{-1}}}, v_{j^{\sigma^{-1}}}) \in \rho(P)$ which in turn implies that $(v^k_{i^{\sigma^{-1}}}, v^l_{j^{\sigma^{-1}}}) \in \rho(P)$.

(ii) For $k \in \widetilde{V}$, $e_k(Z(\sigma)) \cong e_{k^\sigma} S$ as right S-modules by the definition of $Z(\sigma)$.

Thus, for $(a_{i,j}) = \sum_{k,l \in \widetilde{V}} a_{k,l} e_{k\,l} \in e(Z(\sigma))e$, we can define $(a_{i,j})' = \sum_{k,l \in \widetilde{V}} a_{k,l} e_{k^\sigma\,l} \in_{\hat{\sigma}} (eSe)$.

We define the map $(-)' : e(Z(\sigma))e \rightarrow {}_{\hat{\sigma}}(eSe)$ via $(a_{i,j}) \mapsto (a_{i,j})'$. We clearly have an additive group isomorphism since $\sigma \in Aut(\widetilde{P})$. We examine its action on the elements of the form $ae_{k\,l}$ where $a \in K$ and $k, l \in \widetilde{V}$ to show it is a bimodule homomorphism. Let $s \in eSe$ so that $s \cdot ae_{k\,l} = \sum_{i \in \widetilde{V}} s_{i,k} ae_{i\,l}$ and

$$
\begin{aligned}
(s \cdot ae_{k\,l})' &= \sum_{i \in \widetilde{V}} s_{i,k} ae_{i^\sigma\,l} \\
&= \sum_{i^{\sigma^{-1}} \in \widetilde{V}} s_{i^{\sigma^{-1}},k} ae_{i\,l} \text{ by substituting } i^{\sigma^{-1}} \text{ for } i \\
&= \sum_{i \in \widetilde{V}} s_{i^{\sigma^{-1}},k} ae_{i\,l} \text{ since } \sigma \text{ is a bijection on } \widetilde{V} \\
&= \sum_{i \in \widetilde{V}} (s^{\hat{\sigma}})_{i,k^\sigma} ae_{i\,l} \\
&= (s^{\hat{\sigma}})(ae_{k^\sigma\,l}) \\
&= s \bullet (ae_{k^\sigma\,l}) \text{ where } \bullet \text{ is the left } eSe\text{-} \\
&\qquad\qquad \text{multiplication in } {}_{\hat{\sigma}}(eSe) \\
&= s \bullet (ae_{k\,l})'.
\end{aligned}
$$

Since $ae_{k\,l} \cdot s = \sum_{j \in \widetilde{V}} as_{l,j} e_{k\,j}$, then $(ae_{k\,l} \cdot s)' = \sum_{j \in \widetilde{V}} as_{l,j} e_{k^\sigma\,j} = (ae_{k^\sigma\,l})s = (ae_{k\,l})' s$.

Lemma 5.4. Let S be a ring, $\theta \in Aut(S)$, and X be an invertible S-bimodule. Then ${}_{\theta}S \otimes X \cong {}_{\theta}X$ as S-bimodules.

Proof. The map ${}_{\theta}S \otimes X \to {}_{\theta}X$ via $t \otimes x \mapsto tx$ is clearly an S-bimodule isomorphism.

Definition 5.5. Let $Inn(\tilde{S})\varphi \in Out(\tilde{S})$. By Theorem 3.10, $\varphi = \alpha g^{*}\breve{\beta}\hat{\sigma}$ for some $\alpha \in Inn(\tilde{S})$, $g^{*} \in \mathcal{G}(\tilde{P})$, $\breve{\beta} \in Aut(K)^{n}$, and $\hat{\sigma} \in \mathcal{P}(\tilde{P})$. As in Chapter IV, there is $g_1^{*} \in \mathcal{G}(P)$ such that $g_1^{*}|_{eSe} = g^{*}$ and $\breve{\beta}_1 \in Aut(K)^{n} \subseteq Aut(S)$ such that $\breve{\beta}_1|_{eSe} = \breve{\beta}$. We define $Z(\varphi) = {}_{g_1^{*}\breve{\beta}_1} Z(\sigma)$ where $Z(\sigma)$ is as in Definition 5.1. If $Inn(\tilde{S})\alpha g^{*}\breve{\beta}\hat{\sigma}$ $= Inn(\tilde{S})\gamma h^{*}\breve{\delta}\hat{\tau}$, there exists $\alpha' \in Inn(\tilde{S})$ such that $\alpha'\alpha g^{*}\breve{\beta}\hat{\sigma} = \gamma h^{*}\breve{\delta}\hat{\tau}$. By uniqueness of factorization of the semidirect product, we have $g^{*} = h^{*}$, $\breve{\beta} = \breve{\delta}$, and $\hat{\sigma} = \hat{\tau}$, which shows $Z(\varphi)$ is well-defined.

Notice that by Lemma 5.4, Lemma 4.4, and Lemma 5.3, we have $[eZ(\varphi)e]$ $= [e({}_{g_1^{*}\breve{\beta}_1} Z(\sigma))e] = [e({}_{g_1^{*}\breve{\beta}_1} S)e] \otimes [eZ(\sigma)e] = [{}_{g^{*}\breve{\beta}}(eSe)] \otimes [{}_{\hat{\sigma}}(eSe)] = [{}_{\varphi}(eSe)]$. Also observe that since $e(-)e$ is a group isomorphism, $[X] = [Y]$ in $Pic(S)$ if and only if $[eXe] = [eYe]$ in $Pic(\tilde{S})$. Our structure theorem for elements of $Pic(S)$ appears next.

Theorem 5.6. Let $[X] \in Pic(S)$.

(i) $X \cong Z(\varphi)$ for some $Inn(\widetilde{S})\varphi \in Out(\widetilde{S})$.

(ii) If $X \cong_{\xi} S$ for some $\xi \in Aut(S)$, then ${}_{\theta}S \cong X \cong Z(\theta|_{eSe})$ for some $\theta \in Aut(S)$.

(iii) The arithmetic in *Pic*(*S*) is defined by $[Z(\varphi)] \otimes [Z(\chi)] = [Z(\varphi\chi)]$; that is, $Z(\varphi) \otimes Z(\chi) \cong Z(\varphi\chi)$.

Proof. (i) Since $[eXe] \in Pic(\widetilde{S})$, there exists $\varphi \in Aut(\widetilde{S})$ such that $[eXe]$ $= (Inn(\widetilde{S})\varphi)^{\lambda_{\widetilde{P}}} = [{}_{\varphi}(eSe)] = [eZ(\varphi)e]$, which completes the proof by the above observations.

(ii) Suppose $X \cong_{\xi} S$ for some $\xi \in Aut(S)$. By Theorem 3.10, $\xi = \alpha g^{*}\breve{\beta}\hat{\sigma}$ for some $\alpha \in Inn(S)$, $g^{*} \in \mathcal{G}(P)$, $\breve{\beta} \in Aut(K)^{n}$, and $\hat{\sigma} \in \mathcal{P}(P)$. Letting $\theta = g^{*}\breve{\beta}\hat{\sigma}$, then $[X] = [{}_{\xi}S] = (Inn(S)\xi)^{\lambda_P} = (Inn(S)\theta)^{\lambda_P} = [{}_{\theta}S]$. Since $\lambda_P(e(-)e) = \phi\lambda_{\widetilde{P}}$, then,

$$
\begin{aligned}
[eXe] &= [e({}_{\theta}S)e] \\
&= (Inn(S)\theta)^{\lambda_{P(e(-)e)}} \\
&= (Inn(S)\theta)^{\phi\lambda_{\widetilde{P}}} \\
&= (Inn(S)\theta|_{eSe})^{\lambda_{\widetilde{P}}} \quad \text{by Theorem 4.8} \\
&= [{}_{\theta|_{eSe}}(eSe)] \\
&= [eZ(\theta|_{eSe})e]
\end{aligned}
$$

so that $X \cong Z(\theta|_{eSe})$ as *S*-bimodules by the observations preceding this theorem.

(iii) This is clear since

$$\begin{aligned}[e(Z(\varphi)\otimes Z(\chi))e] &= [eZ(\varphi)e]\otimes[eZ(\chi)e] \\ &= [{}_{\varphi}(eSe)]\otimes[{}_{\chi}(eSe)] \\ &= [{}_{\varphi\chi}(eSe)] \\ &= [eZ(\varphi\chi)e].\end{aligned}$$

This completes the solution of Problem B. We now pursue the solution to Problem C which we now restate.

Problem C. Determine necessary and sufficient conditions so that $Out(S)$ is naturally invariant for the Morita equivalence class of S (relative to the collection of incidence K-algebras).

Let $\mathcal{P}$ denote the set of finite preordered sets. We define an equivalence relation $\approx$ on $\mathcal{P}$ via $P' \approx P''$ if and only if $\tilde{P}' \cong \tilde{P}''$ where $\tilde{P}'$ and $\tilde{P}''$ are the underlying posets of P' and P'', respectively. For each $P' \in \mathcal{P}$, let $[P']$ be the equivalence class of P'.

Lemma 5.7. Let $P_1, P_2 \in \mathcal{P}$; let $S_1 = I(P_1)$ and $S_2 = I(P_2)$; and let $\tilde{S}_1$ and $\tilde{S}_2$ be the basic rings for S_1 and S_2, respectively. The following are equivalent:

(i) $[P_1] = [P_2]$

(ii) $\tilde{S}_1 \cong \tilde{S}_2$

(iii) S_1 and S_2 are Morita equivalent.

Proof. Throughout this proof, we let e_{ij} and f_{kl} denote the standard matrix basis elements for $i, j \in V_1$ and $k, l \in V_2$. We let e_i denote e_{ii} and f_k denote f_{kk}. By Lemma 2.9, $e = \sum_{i \in \tilde{V}_1} e_i$ and $f = \sum_{k \in \tilde{V}_2} f_k$ are basic idempotents for S_1 and S_2, respectively. Hence, we may assume that $\tilde{S}_1 = eS_1e$ and $\tilde{S}_2 = fS_2f$.

(i)⇒(ii) Suppose $[P_1] = [P_2]$, so that $\tilde{P}_1 \overset{\sigma}{\cong} \tilde{P}_2$ for some σ. Thus, $f = \sum_{i \in \tilde{V}_1} f_{i^\sigma}$, and the map $\sum_{(i,j) \in \rho(\tilde{P}_1)} a_{i,j} e_{ij} \mapsto \sum_{(i,j) \in \rho(\tilde{P}_1)} a_{i,j} f_{i^\sigma j^\sigma}$ is easily seen to be a ring isomorphism between $\tilde{S}_1$ and $\tilde{S}_2$.

(ii)⇒(i) This is [22, Theorem 1].

(ii)⇔(iii) This is [2, Proposition 27.14].

Since Morita equivalence determines an equivalence relation on the collection of incidence algebras over *K*, we determine necessary and sufficient conditions on *P* so that the equivalence class of *S* has the property that $Out(S)$ is a natural invariant. Specifically, suppose *S* and S' are Morita equivalent via the functors $- \otimes_S Q'_{S'}: Mod - S \to Mod - S'$ and ${}_{S'}Q_S \otimes -: S - Mod \to S' - Mod$. We say that $Out(S)$ and $Out(S')$ are **naturally isomorphic** if the functor ${}_{S'}Q_S \otimes - \otimes_S Q'_{S'}: Pic(S) \to Pic(S')$ restricts to a group isomorphism $Q \otimes - \otimes Q'|_{Out(S)} : Out(S) \to Out(S')$ where we identify $Out(S)$ with its image in $Pic(S)$ under λ_P. That is, $Out(S)$ is naturally isomorphic to $Out(S')$ if and only if the vertical maps in the following commutative diagram

$$\begin{array}{ccc} Out(S) & \xrightarrow{\lambda_P} & Pic(S) \\ \downarrow{\scriptstyle \lambda_P(Q\otimes - \otimes Q')\lambda_{P'}^{-1}} & & \downarrow{\scriptstyle Q\otimes - \otimes Q'} \\ Out(S') & \xrightarrow{\lambda_{P'}} & Pic(S') \end{array}$$

are isomorphisms. Note that this definition depends on our choice of functors.

Let [*S*] denote the Morita equivalence class of *S*. We say $Out(S)$ is a **natural invariant** for [*S*] if $Out(S)$ and $Out(S')$ are naturally isomorphic for all $S' = I(P') \in [S]$.

Recall that we say *S* has the ***Aut-Pic* property** if $\lambda_P : Out(S) \to Pic(S)$ is an isomorphism.

Lemma 5.8. The following statements are equivalent.

(i) ϕ_S is an isomorphism.

(ii) $(\simeq): Aut(P) \to Aut(\tilde{P})$ is surjective.

(iii) *S* has the *Aut-Pic* property.

(iv) $Out(S)$ and $Out(\tilde{S})$ are naturally isomorphic.

Proof. (i)⇔(ii)⇔(iii) This is Corollary 4.9.

(i)⇒(iv) This is clear since the isomorphism $e(-)e: Pic(S) \to Pic(eSe)$ in the commutative diagram

$$\begin{array}{ccc} Out(S) & \xrightarrow{\lambda_P} & Pic(S) \\ \downarrow{\scriptstyle \phi_S} & & \downarrow{\scriptstyle e(-)e} \\ Out(eSe) & \xrightarrow{\lambda_{\tilde{P}}} & Pic(eSe) \end{array}$$

is really the functor $eS \otimes - \otimes Se$ as in Lemma 2.6 and since $\phi_S = \lambda_P[e(-)e]\lambda_{\tilde{P}}^{-1}$.

(iv)⇒(iii) Since $Out(S)$ and $Out(\tilde{S})$ are naturally isomorphic, then λ_P is an isomorphism if and only if $\lambda_{\tilde{P}}$ is as well. But, $\lambda_{\tilde{P}}$ is an isomorphism by Theorem 2.10.

We close this paper with our solution to Problem C.

Theorem 5.9. The following statements are equivalent.

(i) $Out(S)$ is a natural invariant for [*S*].

(ii) $Aut(\tilde{P}) = 1$.

(iii) Every element in [*S*] has the *Aut-Pic* property.

Proof. We first note that Lemma 5.7 permits us to assume that $\tilde{P}$ is the underlying poset of P' for any $S' = I(P') \in [S]$ by relabeling vertices if necessary.

(i)⇒(ii) For $S' = I(P') \in [S]$, we know that $\tilde{S}' \in [S]$, and so $Out(S)$ and $Out(S')$ are naturally isomorphic as are $Out(S)$ and $Out(\tilde{S}')$. Since the composition of functors is again a functor, then $Out(S')$ and $Out(\tilde{S}')$ are naturally isomorphic. Thus, by Lemma 5.8, $(\approx)\colon Aut(P') \to Aut(\tilde{P})$ is surjective for every $S' \in [S]$.

By contradiction, assume there exists $1 \neq \sigma \in Aut(\tilde{P})$. Then there exist $x, y \in \tilde{V}$ such that $x \neq y$ and $x^{\sigma} = y$. We show there is a preordered set $P' \in [P]$ for which σ is not in the image of $(\approx)\colon Aut(P') \to Aut(\tilde{P})$. Let *z* be a symbol not in $\tilde{V}$, let $V' = \tilde{V} \cup \{z\}$, and let $\rho(P')$ be the transitive closure of $\rho(\tilde{P}) \cup \{(z,z),(z,x),(x,z)\}$. Since $\tilde{P}$ is reflexive, so is P'. P' is preordered since it is transitive by definition. Since $[x] = \{x, z\}$ and since for $i \in V$ such that $i \notin [x]$ we have $[i] = \{i\}$, then $\tilde{P}$ is the underlying poset of P'. Since $\|[x]\| = 2 \neq 1 = \|[y]\|$, there is no $\theta \in Aut(P')$ such that $\tilde{\theta} = \sigma$, contradicting the surjectivity of $(\tilde{=}): Aut(P') \to Aut(\tilde{P})$. Thus, $Aut(\tilde{P}) = 1$.

(ii)⇒(iii) If $Aut(\tilde{P}) = 1$, then for any $P' \in [P]$, $(\tilde{=}): Aut(P') \to Aut(\tilde{P})$ is surjective. By Lemma 5.8, then each $S' = I(P') \in [S]$ has the *Aut-Pic* property.

(iii)⇒(i) If $\lambda_{P'}: Out(S') \to Pic(S')$ is an isomorphism for all $S' = I(P') \in [S]$, then $Out(S)$ is clearly a natural invariant for [*S*].

Example. Let *P* be the preordered set such that $V(P) = \{v_1, v_2, v_3\}$ and $\rho(P) = \{(v_1,v_1),(v_1,v_2),(v_1,v_3),(v_2,v_1),(v_2,v_2),(v_2,v_3),(v_3,v_3)\}$. Since $\tilde{P}$ is such that $V(\tilde{P}) = \{v_1, v_3\}$ and $\rho(\tilde{P}) = \{(v_1,v_1),(v_1,v_3),(v_3,v_3)\}$, then $Aut(\tilde{P})$ is clearly trivial. Consequently, the incidence algebra

$$S = \begin{pmatrix} K & K & K \\ K & K & K \\ 0 & 0 & K \end{pmatrix}$$

for the preordered set *P* has the *Aut-Pic* property and *Out*(*S*) is a natural invariant for the Morita equivalence class [*S*].

On the other hand, the incidence algebra

$$S = \begin{pmatrix} K & K & 0 \\ K & K & 0 \\ 0 & 0 & K \end{pmatrix}$$

does not have the *Aut-Pic* property as shown in [17]. This holds since $Aut(\tilde{P}) \cong \mathbf{Z}_2$.

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