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WEDGE THEORY / COMPOUND MATRICES: PROPERTIES AND APPLICATIONS

Debra L. Boutin, M.S.
Department of Mathematics
Cornell University
Ithaca, NY 14853

Ronald F. Gleeson, Ph.D.
Department of Physics
Trenton State College
Trenton, NJ 08650

Robert M. Williams, Ph.D.
Mission Avionics Technology Dept. (Code 455100R07)
Naval Air Systems Team
Warminster, PA 18974-5000

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*Debra Boutin, M.S. Robert M. Williams, Ph.D.
**Ronald F. Gleeson, Ph.D.

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*Department of Mathematics
Cornell University
Ithaca, NY 14853
**Department of Physics
Trenton State College
Trenton, NJ 08650

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The Navy utilizes matrices to analyze radar signals to determine the direction and velocity of aircraft. Matrix analysis is also useful in the sonar classification of submarines. One powerful tool for obtaining information about matrices is wedge theory. (The traditional terminology is "compound matrix theory", whereas modern texts speak of "mappings on the exterior algebra".) Wedge theory is a fundamental tool in multilinear algebra with important applications to group representations and tensor analysis. Current research indicates that it may also be useful in analyzing noisy data matrices, but this potential has not yet been fully explored. The purpose of this report is to collect details about wedge theory, in one accessible place, to facilitate future exploration of this topic. First, basic properties of the wedge operation are given along with definitions and examples. Then, an application to calculating the rank of a matrix with noise is considered. Finally, since the basic constructions can now be easily implemented on desktop computer algebra systems, the procedures for several such packages are illustrated.

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ABSTRACT

The Navy utilizes matrices to analyze radar signals to determine the direction and velocity of aircraft. Matrix analysis is also useful in the sonar classification of submarines. One powerful tool for obtaining information about matrices is wedge theory. (The traditional terminology is "compound matrix theory", whereas modern texts speak of "mappings on the exterior algebra".) Wedge theory is a fundamental tool in multilinear algebra with important applications to group representations and tensor analysis. Current research indicates that it may also be useful in analyzing noisy data matrices, but this potential has not yet been fully explored. The purpose of this report is to collect details about wedge theory, in one accessible place, to facilitate future exploration of this topic. First, basic properties of the wedge operation are given along with definitions and examples. Then, an application to calculating the rank of a matrix with noise is considered. Finally, since the basic constructions can now be easily implemented on desktop computer algebra systems, the procedures for several such packages are illustrated.

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I. INTRODUCTION

The Navy utilizes matrices to analyze radar signals to determine the direction and velocity of aircraft. Matrix analysis is also useful in the sonar classification of submarines. The role of matrices in Navy data analysis and a tool for extracting information from these matrices have been discussed in an earlier report (NAWCADWAR–96-21-TR) by Gleeson, Stiller and Williams.

Another tool for squeezing information out of matrices is wedge theory. In the older literature wedge theory is referred to as compound matrix theory. In more modern texts one speaks of mappings on the exterior algebra. Wedge theory appears to have the potential to be quite useful in analyzing noisy data matrices, but this potential has not yet been fully explored. The purpose of this report is to collect details about wedge theory, in one accessible place, to facilitate future exploration of this topic.

The wedge product of a matrix is defined in Section II. Also, certain basic facts and properties of the wedge product are discussed. In addition, the eigenvalues and characteristic equation for the wedge product, along with a method for efficiently computing the coefficients of the characteristic equation, will be considered. Finally, we examine theorems relating to different types of compound determinants.

In Section III we create a rank three 4×4 matrix with noise deliberately added. We then show that the wedge products of this matrix have a proportionally larger nullity than the original matrix. This increased nullity may afford a method for gaining greater sensitivity in determining the effective rank of the matrix.

Tools such as the wedge product have become more feasible with the development of desktop computer algebra software packages. We will show in Appendix A how to produce the wedge products using Maple, Mathematica and Fermat.

II. THEORETICAL BACKGROUND

A. DEFINITIONS

Given a matrix A, our goal is to define $\wedge^p(A)$, a matrix whose entries are the $p \times p$ minors of A. This matrix has been called the p^{th} "wedge" of A, "compound" of A or "exterior product" of A, depending on the literature. In this paper we shall refer to $\wedge^p(A)$ as the p^{th} wedge of A or as the order p wedge of A.

First a little necessary notation.

Definition: Let A be an $n \times n$ matrix whose entry in the i^{th} row and j^{th} column is denoted by $a_{i,j}$.

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}$$

Lexicographically ordered p-sets: Fix a number p between 1 and n, inclusive. Consider all possible sets of p distinct numbers between 1 and n. Order the elements of each set by increasing magnitude. Order the sets by increasing magnitude of the first elements, and when the first elements are equal, by increasing magnitude of the second elements, and when the first and second elements are equal, by increasing magnitude of the third elements, etc. That is, order the sets using a lexicographic or dictionary order. Denote these sets by $S_1, S_2, \ldots, S_{\binom{n}{p}}$ so that $S_1 < S_2 < \cdots < S_{\binom{n}{p}}$.

Example: Let $n = 4$ and $p = 2$. Then $S_1 = \{1,2\}$, $S_2 = \{1,3\}$, $S_3 = \{1,4\}$, $S_4 = \{2,3\}$, $S_5 = \{2,4\}$, $S_6 = \{3,4\}$.

Definition: Let $A_{i,j}$ be the $p \times p$ matrix formed by the intersection, in A, of the rows whose numbers are in the set S_i and the columns whose numbers are in the set S_j. That is, $A_{i,j}$ is formed by the entries $a_{h,k}$ of A where h is an element of S_i and k is an element of S_j, maintaining the relative placement of the entries.

Note: The determinant of any $p \times p$ submatrix of A is called an order p minor of A, or alternately, a $p \times p$ minor of A. This terminology will be used throughout the paper.

Example: Let A be an abstract 4×4 matrix. Let $p = 2$. Then $S_2 = \{1,3\}$ and $S_5 = \{2,4\}$, so $A_{2,5}$ has entries from the intersection of rows 1 and 3 with columns 2 and 4 of A. Thus

$$A_{2,5} = \begin{pmatrix} a_{1,2} & a_{1,4} \\ a_{3,2} & a_{3,4} \end{pmatrix}.$$

Definition: Let $\wedge^p(A)$ be the matrix of size $\binom{n}{p} \times \binom{n}{p}$ whose $(i,j)^{th}$ entry is the determinant of the matrix $A_{i,j}$.

Example: Let $A = \begin{pmatrix} -3 & -1 & 1 & 2 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. Then,

$$\wedge^2(A) = \begin{pmatrix} \begin{vmatrix} -3 & -1 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} -3 & 1 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} -3 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} -3 & -1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} -3 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -3 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} -3 & -1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} -3 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} -3 & 2 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -9 & -3 & -2 & 1 & -5 \\ 0 & -3 & -3 & -1 & -1 & -1 \\ 0 & 0 & -6 & 0 & -2 & 2 \\ 0 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & -2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Basic Facts about Wedge Products

Let A be an $n \times n$ matrix.

1) The matrix $\wedge^p(A)$ has dimension $\binom{n}{p} \times \binom{n}{p}$. Thus the dimensions of the various wedge products of A correspond to the n^{th} row of Pascal's Triangle.

2) For theoretical purposes, $\wedge^0(A)$ is defined to be the 1×1 matrix with entry 1.

3) $\wedge^1(A) = \wedge(A) = A$.

This is clear since by definition, $\wedge^1(A)$ has entries which are determinants of the 1×1 submatrices of A. That is, the entries of $\wedge(A)$ are the entries of A.

4) $\wedge^n(A) = (\det(A))$.

This is an immediate consequence of the definition of $\wedge^n(A)$. The entries of $\wedge^n(A)$ are the determinants of the $n \times n$ submatrices of A. But A is the only $n \times n$ submatrix of itself.

B. PROPERTIES OF THE WEDGE OPERATOR

The beauty and usefulness of the wedge operator is demonstrated in the following properties. A good reference for most of these properties is Determinants and Matrices by A. C. Aitken, [1]. In particular, properties 1), 4), 6), 10) and 11) can be found there. Property 2) can be found in Algebra, Volume 2 by P. M. Cohn, [2].

Theorem: Let A and B be $n \times n$ matrices. Let λ be a real number. Let I and I' be identity matrices of appropriate dimensions. Then:

Property 1) $\wedge^p(I) = I'$.

This can be seen easily from the definition of $\wedge^p(A)$.

Property 2) $\wedge^p(AB) = \wedge^p(A)\wedge^p(B)$.

Example: Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. Then $AB = \begin{pmatrix} 1 & -2 & 4 & -4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & -3 \end{pmatrix}$, so we can compute $\wedge^3(AB) = \begin{pmatrix} 2 & -3 & 5 & -6 \\ 0 & -3 & 3 & 6 \\ 0 & 0 & -6 & 12 \\ 0 & 0 & 0 & -6 \end{pmatrix}$. Further, $\wedge^3(A) = \begin{pmatrix} -2 & -1 & -1 & -2 \\ 0 & -3 & 3 & 6 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & -6 \end{pmatrix}$ and $\wedge^3(B) = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Now we can compute $\wedge^3(A)\wedge^3(B) = \begin{pmatrix} 2 & -3 & 5 & -6 \\ 0 & -3 & 3 & 6 \\ 0 & 0 & -6 & 12 \\ 0 & 0 & 0 & -6 \end{pmatrix}$ which is equal to $\wedge^3(AB)$.

Property 3) $\wedge^p(\lambda A) = \lambda^p \cdot \wedge^p(A)$.

In the case $A = I$ an easy computation yields $\wedge^p(\lambda I) = \lambda^p I'$. If $A \neq I$, we can write $\lambda A = \lambda I A$. Using 2), the wedge product becomes $\wedge^p(\lambda A) = \wedge^p(\lambda I A) = \wedge^p(\lambda I)\wedge^p(A) = \lambda^p I' \wedge^p(A) = \lambda^p \wedge^p(A)$.

Example: Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$. Then we can compute $2A = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 6 \end{pmatrix}$ and $\wedge^3(2A) = \begin{pmatrix} -16 & -8 & -8 & -16 \\ 0 & -24 & 24 & 48 \\ 0 & 0 & 48 & 48 \\ 0 & 0 & 0 & -48 \end{pmatrix}$, which can easily be seen to be $2^3\wedge^3(A) = 8\begin{pmatrix} -2 & -1 & -1 & -2 \\ 0 & -3 & 3 & 6 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & -6 \end{pmatrix}$.

Property 4) $\left(\wedge^p(A)\right)^{-1} = \wedge^p(A^{-1})$.

This is a consequence of 1) and 2). Since $AA^{-1} = I$, we have that $I' = \wedge^p(I) = \wedge^p(AA^{-1}) = \wedge^p(A)\wedge^p(A^{-1})$. Thus $\wedge^p(A^{-1})$ is the inverse of $\wedge^p(A)$.

Example: With A as above, $A^{-1} = \begin{pmatrix} 1 & 1 & -1 & -\frac{1}{3} \\ 0 & -1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}$ and $\wedge^3(A^{-1}) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{6} \\ 0 & -\frac{1}{3} & \frac{1}{6} & -\frac{1}{6} \\ 0 & 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & -\frac{1}{6} \end{pmatrix}$.

We can now compute $\wedge^3(A)\wedge^3(A^{-1}) = \begin{pmatrix} -2 & -1 & -1 & -2 \\ 0 & -3 & 3 & 6 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & -6 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{6} \\ 0 & -\frac{1}{3} & \frac{1}{6} & -\frac{1}{6} \\ 0 & 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & -\frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$. Thus $\wedge^3(A^{-1})$ is actually the inverse of $\wedge^3(A)$.

Property 5) $\left(\wedge^p(A)\right)^t = \wedge^p(A^t)$.

This is an easy consequence of the definition of $\wedge^p(A)$.

Example: With A as above, $A^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 3 \end{pmatrix}$ and $\wedge^3(A^t) = \begin{pmatrix} -2 & 0 & 0 & 0 \\ -1 & -3 & 0 & 0 \\ -1 & 3 & 6 & 0 \\ -2 & 6 & 6 & -6 \end{pmatrix}$, which can easily be seen to be the transpose of $\wedge^3(A) = \begin{pmatrix} -2 & -1 & -1 & -2 \\ 0 & -3 & 3 & 6 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & -6 \end{pmatrix}$.

Property 6) If A is a symmetric matrix, then $\wedge^p(A)$ is symmetric.

This is an immediate consequence of 5). If A is symmetric $A = A^t$. Then by 5), $\wedge^p(A) = \wedge^p(A^t) = \left(\wedge^p(A)\right)^t$, which means that $\wedge^p(A)$ is symmetric.

Example: Let $A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$, a symmetric matrix. Then $\wedge^3(A) = \begin{pmatrix} 0 & -2 & 0 & 2 \\ -2 & -4 & 2 & 0 \\ 0 & 2 & -4 & -2 \\ 2 & 0 & -2 & 0 \end{pmatrix}$ is also symmetric.

Property 7) If A is upper (lower) triangular, then $\wedge^p(A)$ is upper (lower) triangular.

Example: Note that in the example for 3) the original matrix is upper triangular and so is the wedge product. The example for 5) is a lower triangular matrix, A^t, whose wedge is also lower triangular.

Property 8) If A is a diagonal matrix, then $\wedge^p(A)$ is diagonal.

This is an immediate consequence of 7). If A is diagonal matrix, then it is both upper and lower triangular; by 7) so is $\wedge^p(A)$.

Example: Let $A = \begin{pmatrix} 1&0&0&0 \\ 0&2&0&0 \\ 0&0&3&0 \\ 0&0&0&4 \end{pmatrix}$. Then $\wedge^3(A) = \begin{pmatrix} 6&0&0&0 \\ 0&8&0&0 \\ 0&0&12&0 \\ 0&0&0&24 \end{pmatrix}$ is also diagonal.

Property 9) The complex conjugate of $\wedge^p(A)$ is the p^{th} wedge of the complex conjugate of A.

Example: Let $A = \begin{pmatrix} 1-i & 1 & 2-i & -2i \\ 0 & 1+i & -1 & 1+2i \\ 0 & 0 & i & i \\ 0 & 0 & 0 & -i \end{pmatrix}$. Then $\wedge^3(A) = \begin{pmatrix} 2i & 2i & -4i & 5-3i \\ 0 & -2i & 1+i & -1+4i \\ 0 & 0 & 1-i & 1 \\ 0 & 0 & 0 & 1+i \end{pmatrix}$ and $\bar{A} = \begin{pmatrix} 1+i & 1 & 2+i & 2i \\ 0 & 1-i & -1 & 1-2i \\ 0 & 0 & -i & -i \\ 0 & 0 & 0 & i \end{pmatrix}$ is the complex conjugate of A. We can then compute $\wedge^3(\bar{A}) = \begin{pmatrix} -2i & -2i & 4i & 5+3i \\ 0 & 2i & 1-i & -1-4i \\ 0 & 0 & 1+i & 1 \\ 0 & 0 & 0 & 1-i \end{pmatrix}$ which is the complex conjugate of $\wedge^3(A)$.

Property 10) The Hermitian conjugate of $\wedge^p(A)$ is the p^{th} wedge of the Hermitian conjugate of A.

Since the Hermitian conjugate of a matrix is the complex conjugate transposed, this is a straightforward consequence of 9) and 5).

Property 11) (Sylvester) $\det(\wedge^p(A)) = (\det(A))^{\binom{n-1}{p-1}}$.

Example: In the example for 3), $\det(A) = -6$ while $\det(\wedge^3(A)) = -216 = (-6)^{\binom{3}{2}}$.

Property 12) As a linear operator on a vector space of dimension $\binom{n}{p}$, $\wedge^p(A)$ is injective (surjective) if and only if A is injective (surjective) as a linear operator on a vector space of dimension n.

For those who are less familiar with injectivity and surjectivity, we provide the following:

A linear operator A is said to be injective if for any two vectors x and y, $Ax = Ay$ only if $x = y$. That is, no two distinct vectors have the same image under the linear operator.

A linear operator is said to be surjective if for any vector v there is some vector u so that $Au = v$. A consequence of A being surjective is that the image of the vector space under the linear operator A is again the whole vector space.

For finite dimensional vector spaces the notions of injectivity, surjectivity and non-zero determinant are equivalent. Thus Property 12) follows from Property 11).

C. EIGENVALUES OF THE WEDGE PRODUCT

Theorem: If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, then the eigenvalues of $\wedge^p(A)$ are the $\binom{n}{p}$ distinct products of the λ_i taken p at a time.

Proof:

Given a matrix A there is related matrix, called the Jordan form, which has the same eigenvalues as A, with the added advantage of being lower triangular. The Jordan form will be extremely helpful in finding the eigenvalues of $\wedge^p(A)$.

Let B be the Jordan Form of the matrix A. Then B is similar to A, which means that $B = PAP^{-1}$ for some matrix P. Since similar matrices have the same eigenvalues and since the Jordan form is lower triangular, B has the eigenvalues of A as its diagonal entries.

We can compute $\wedge^p(B) = \wedge^p(PAP^{-1}) = \wedge^p(P) \wedge^p (A) \wedge^p (P)^{-1}$ by the properties of the wedge operator.

This means that $\wedge^p(B)$ and $\wedge^p(A)$ are also similar matrices and therefore have identical eigenvalues. Thus we have that the eigenvalues of $\wedge^p(A)$ are the eigenvalues of $\wedge^p(B)$.

Note that since B is a lower triangular matrix, $\wedge^p(B)$ is lower triangular and its diagonal entries are $|B_{1,1}|, |B_{2,2}|, \cdots, |B_{\binom{n}{p},\binom{n}{p}}|$, in the notation of Section II.A. Thus the eigenvalues of $\wedge^p(B)$ (and therefore of $\wedge^p(A)$) are precisely these $|B_{i,i}|$.

Let's see what one of the $|B_{i,i}|$ looks like.

Recall the p-sets $S_1, \ldots, S_{\binom{n}{p}}$ defined earlier. Let $S_i = \{i_1, i_2, \ldots, i_p\}$. Then $|B_{i,i}|$ is the determinant of the $p \times p$ matrix formed by entries of B from rows $i_1, i_2, \ldots, i_p$ and columns $i_1, i_2, \ldots, i_p$. That is:

$$B_{i,i} = \begin{pmatrix} b_{i_1,i_1} & b_{i_1,i_2} & b_{i_1,i_3} & \cdots & b_{i_1,i_p} \\ b_{i_2,i_1} & b_{i_2,i_2} & b_{i_2,i_3} & \cdots & b_{i_2,i_p} \\ b_{i_3,i_1} & b_{i_3,i_2} & b_{i_3,i_3} & \cdots & b_{i_3,i_p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{i_p,i_1} & b_{i_p,i_2} & b_{i_p,i_3} & \cdots & b_{i_p,i_p} \end{pmatrix}.$$

Since B is a lower triangular matrix, whenever $i_j < i_k$ we know that $b_{i_j,i_k} = 0$. In $B_{i,i}$ this occurs precisely when $j < k$. Thus $B_{i,i}$ is a lower triangular matrix. Further, since the diagonal entries of B are the eigenvalues of A we know that $b_{i_j,i_j} = \lambda_{i_j}$, the ${i_j}^{th}$ eigenvalue of A. Thus,

$$B_{i,i} = \begin{pmatrix} \lambda_{i_1} & 0 & 0 & \cdots & 0 \\ b_{i_2,i_1} & \lambda_{i_2} & 0 & \cdots & 0 \\ b_{i_3,i_1} & b_{i_3,i_2} & \lambda_{i_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{i_p,i_1} & b_{i_p,i_2} & b_{i_p,i_3} & \cdots & \lambda_{i_p} \end{pmatrix}.$$

Now we can easily compute $|B_{i,i}| = \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_p}$, an eigenvalue of $\wedge^p(B)$ and therefore of $\wedge^p(A)$.

Since the eigenvalues of $\wedge^p(A)$ are precisely the $|B_{i,i}|$, which are the products of the eigenvalues of A taken p at a time, the eigenvalues of $\wedge^p(A)$ are the $\binom{n}{p}$ distinct products of the λ_i taken p at a time.

Example: If A is a 4×4 matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ then $\wedge^2(A)$ has eigenvalues $\lambda_1\lambda_2,\ \lambda_1\lambda_3,\ \lambda_1\lambda_4,\ \lambda_2\lambda_3,\ \lambda_2\lambda_4$, and $\lambda_3\lambda_4$, and $\wedge^3(A)$ has eigenvalues $\lambda_1\lambda_2\lambda_3,\ \lambda_1\lambda_2\lambda_4,\ \lambda_1\lambda_3\lambda_4$ and $\lambda_2\lambda_3\lambda_4$.

Example: Using the numeric example from Section II.A, we see that A has $-3, -1, 1$ and 2 as its eigenvalues, which implies that $\wedge^2(A)$ has eigenvalues $-3 \cdot -1 = 3$, $-3 \cdot 1 = -3$, $-3 \cdot 2 = -6$, $-1 \cdot 1 = -1$, $-1 \cdot 2 = -2$, and $1 \cdot 2 = 2$. We can see immediately that this is true by looking at the upper triangular matrix $\wedge^2(A)$ which we computed.

D. THE CHARACTERISTIC POLYNOMIAL OF THE WEDGE PRODUCT

We would like to be able to find the characteristic coefficients (that is, the coefficients of the characteristic polynomial) of $\wedge^p(A)$, directly from the characteristic coefficients of A. This turns out to be a straightforward task for two of the characteristic coefficients.

For the purposes of this paper, the characteristic polynomial of the matrix A is defined to be $\det(xI - A)$, though in some literature it is defined as $\det(A - xI)$. The polynomials resulting from these two definitions differ by a power of -1. However, we find it convenient to have the highest degree term in the characteristic polynomial be positive, and our definition ensures that.

Let c_k be the coefficient of the term of degree $n - k$ in the characteristic polynomial of A. In the characteristic polynomial of $\wedge^p(A)$, let d_k be the coefficient of the term of degree $\binom{n}{p} - k$.

More explicitly, the characteristic polynomial of A will be written as $x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n$, while if we let $m = \binom{n}{p}$, the characteristic polynomial of $\wedge^p(A)$ will be written as $x^m + d_1x^{m-1} + \cdots + d_{m-1}x + d_m$.

It is a well known fact in linear algebra that in the characteristic polynomial of an $n \times n$ matrix, the coefficient of the term of degree $n-k$ is $(-1)^k s_k$ where s_k is the k^{th} symmetric function on the eigenvalues of the matrix. That is, s_k is the sum of products of eigenvalues taken k at a time.

So in the characteristic polynomial of $\wedge^p(A)$, d_1, the coefficient of the term of degree $\binom{n}{p} - 1$, is $(-1)^1 s_1$, the negative of the sum of the eigenvalues of $\wedge^p(A)$. Since these eigenvalues are products of eigenvalues of A taken p at a time, d_1 is, up to sign, actually the p^{th} symmetric function on the eigenvalues of A. Thus:

Theorem: In the characteristic polynomial of $\wedge^p(A)$, $d_1 = (-1)^{p+1}c_p$.

We now use a similar method to find the constant term, $d_{\binom{n}{p}}$.

Recall from the properties of the wedge product that $\det(\wedge^p(A)) = \left(\det(A)\right)^{\binom{n-1}{p-1}}$ and that the constant term of the characteristic polynomial of a matrix is, up to sign, the determinant of the matrix. Thus:

Theorem: In the characteristic polynomial of $\wedge^p(A)$, the constant term, $d_{\binom{n}{p}}$, is $\left(c_n\right)^{\binom{n-1}{p-1}}$, up to sign.

The other characteristic coefficients of $\wedge^p(A)$ don't lend themselves to such simple formulae. However, they are still polynomials in the c_k's, and we can actually find them.

Method for Finding Characteristic Coefficients

Let A be an abstract $n \times n$ matrix with characteristic polynomial

$$p(x) = x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n.$$

Suppose that B is a matrix with the same characteristic polynomial as A. Then the eigenvalues of B, i.e. the roots of the characteristic polynomial, are identical to the eigenvalues of A. Thus $\wedge^p(A)$ and $\wedge^p(B)$ also have identical eigenvalues. Since the coefficients of the characteristic polynomial of a matrix are completely determined by its eigenvalues, $\wedge^p(B)$ has the same characteristic polynomial as $\wedge^p(A)$.

So what we want is a matrix B, with the same characteristic polynomial as A, whose entries are 0's, 1's and c_k's. If $\wedge^p(B)$ also has entries of 0's, 1's and c_k's, the characteristic coefficients of $\wedge^p(B)$ will be polynomials in the c_k's. Luckily, the companion matrix for $p(x)$ fits all these requirements perfectly.

$$\text{Let } B = \begin{pmatrix} 0 & 0 & \cdots & 0 & -c_n \\ 1 & 0 & \cdots & 0 & -c_{n-1} \\ 0 & 1 & \cdots & 0 & -c_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_1 \end{pmatrix}, \text{ the companion matrix for } p(x).$$

B has the same characteristic polynomial as A, and the entries of B and $\wedge^p(B)$ are 0's, 1's and c_k's, so the characteristic coefficients of $\wedge^p(B)$ (and therefore of $\wedge^p(A)$) are polynomials in the c_k's.

In particular we can use a computer algebra program such as Mathematica, Maple or Fermat to compute the characteristic polynomial of $\wedge^p(B)$. The result will yield a formula for each characteristic coefficient of $\wedge^p(A)$ in terms of the coefficients of the characteristic polynomial of the original matrix, the c_k's.

This needs only be done once for each n and each p. Then, given a matrix with a known characteristic polynomial we need only use the appropriate formulae to find the characteristic polynomial of the p^{th} wedge. Some of these formulae are listed in Appendix B.

Example: Let A be a 4×4 abstract matrix which has characteristic polynomial of the form $x^4 + c_1x^3 + c_2x^2 + c_3x + c_4$. If we let $B = \begin{pmatrix} 0 & 0 & 0 & -c_4 \\ 1 & 0 & 0 & -c_3 \\ 0 & 1 & 0 & -c_2 \\ 0 & 0 & 1 & -c_1 \end{pmatrix}$, we can then compute

$$\wedge^2(B) = \begin{pmatrix} 0 & 0 & c_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_4 \\ 1 & 0 & -c_2 & 0 & c_3 & 0 \\ 0 & 1 & -c_1 & 0 & 0 & c_3 \\ 0 & 0 & 0 & 1 & -c_1 & c_2 \end{pmatrix} \text{ and } \wedge^3(B) = \begin{pmatrix} 0 & -c_4 & 0 & 0 \\ 0 & 0 & -c_4 & 0 \\ 0 & 0 & 0 & -c_4 \\ 1 & -c_1 & c_2 & -c_3 \end{pmatrix}.$$

We can now compute the characteristic polynomial of $\wedge^2(B)$, which is $x^6 + (-c_2)x^5 + (c_3c_1 - c_4)x^4 + (2c_4c_2 - c_4c_1^2 - c_3^2)x^3 + (c_4c_3c_1 - c_4^2)x^2 + (-c_4^2c_2)x + c_4^3$, and the characteristic polynomial of $\wedge^3(B)$, which is $x^4 + c_3x^3 + c_4c_2x^2 + c_4^2c_1x + c_4^3$.

Thus we have the characteristic polynomials of $\wedge^2(A)$ and $\wedge^3(A)$ respectively, and the characteristic coefficients are polynomials in the characteristic coefficients of A.

Note that in the characteristic polynomial of $\wedge^2(A)$, $d_1 = (-1)^{2+1}c_2$ and $d_6 = (c_4)^{\binom{3}{1}}$ as promised.

Also, in the characteristic polynomial of $\wedge^3(A)$, $d_1 = (-1)^{3+1}c_3$ and $d_4 = (c_4)^{\binom{3}{2}}$, again as expected.

Example: Let A be the 4×4 numeric matrix from the example in Section II.A.

The characteristic polynomial of A is $x^4 + x^3 - 7x^2 - x + 6$, so $c_1 = 1$, $c_2 = -7$, $c_3 = -1$, and $c_4 = 6$.

Using these in the formulae above, we get that the characteristic polynomial of $\wedge^2(A)$ is

$$\begin{aligned}
&x^6 + -(-7)x^5 + ((-1)(1) - (6))x^4 + (2(6)(-7) - (6)(1)^2 - (-1)^2)x^3 \\
&\qquad + ((6)(-1)(1) - (6)^2)x^2 + (-(6)^2(-7))x + (6)^3 \\
&= x^6 + 7x^5 - 7x^4 - 91x^3 - 42x^2 + 252x + 216,
\end{aligned}$$

and the characteristic polynomial of $\wedge^3(A)$ is

$$x^4 + (-1)x^3 + (6)(-7)x^2 + (6)^2(1)x + (6)^3 = x^4 - x^3 - 42x^2 + 36x + 216.$$

E. DIFFERENT TYPES OF COMPOUND DETERMINANTS

In this section, we cover some results about minors of "compound matrices." The wedge product is one such compound matrix, but there are other compound matrix constructions different from the wedge product. An excellent reference for this section is, Determinants and Matrices by A. C. Aitken, [1].

Adjugate Wedge Products

Recall that the entries of $\wedge^p(A)$ are the order p minors of A. Here we wish to study a matrix, which we will call the p^{th} adjugate wedge of A, whose entries are, up to sign, order $n-p$ minors of A. This new matrix is not quite $\wedge^{n-p}(A)$, but it's close. Many of the signs of the entries will be different, and their positions within the matrix will be different. Specifically, we wish the entries of this new matrix to be the cofactors of the minors which comprise $\wedge^p(A)$. So first we must define what we mean by the cofactor of a minor.

Definition: Let A be an abstract $n \times n$ matrix. Let M be the minor which is the determinant of the $m \times m$ submatrix obtained from rows $r_1, \ldots, r_m$ and columns $c_1, \ldots, c_m$. Then the **complementary** minor to M is the determinant of the $(n-m) \times (n-m)$ matrix obtained by deleting rows $r_1, \dots, r_m$ and columns $c_1, \dots, c_m$. The **cofactor** of the minor M is the complementary minor times $(-1)^{r_1+\cdots+r_m+c_1+\cdots+c_m}$.

Example: Let $A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$. Let $M = \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}$. Then the minor complementary to M is $\begin{vmatrix} b_1 & b_4 \\ d_1 & d_4 \end{vmatrix}$ and the cofactor of M is $(-1)^{2+3+1+3} \begin{vmatrix} b_1 & b_4 \\ d_1 & d_4 \end{vmatrix} = (-1) \begin{vmatrix} b_1 & b_4 \\ d_1 & d_4 \end{vmatrix}$.

Definition Let the p^{th} adjugate wedge of a matrix A, denoted $\mathrm{adj}^p(A)$, be the $\binom{n}{p} \times \binom{n}{p}$ matrix whose $(i,j)^{th}$ entry is the cofactor of the order p minor which is the $(j,i)^{th}$ entry of $\wedge^p(A)$.

Example: Let $A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$.

Then

$$\wedge^3(A) = \begin{pmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} & \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} & \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix} \\ \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ d_1 & d_2 & d_4 \end{vmatrix} & \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ d_1 & d_3 & d_4 \end{vmatrix} & \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ d_2 & d_3 & d_4 \end{vmatrix} \\ \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{vmatrix} & \begin{vmatrix} a_1 & a_3 & a_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix} & \begin{vmatrix} a_2 & a_3 & a_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix} \\ \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} & \begin{vmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{vmatrix} & \begin{vmatrix} b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix} & \begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix} \end{pmatrix}, \text{ and}$$

$$\mathrm{adj}^3(A) = \begin{pmatrix} d_4 & -c_4 & b_4 & -a_4 \\ -d_3 & c_3 & -b_3 & a_3 \\ d_2 & -c_2 & b_2 & -a_2 \\ -d_1 & c_1 & -b_1 & a_1 \end{pmatrix}.$$

Note that $\mathrm{adj}^p(A)$ is "almost" $\wedge^{n-p}(A)$. Up to a factor of $(-1)^{i+j}$, the $(i,j)^{th}$ entry of $\mathrm{adj}^p(A)$ is the $(n-j+1, n-i+1)^{th}$ entry of $\wedge^{n-p}(A)$. That is, to get $\mathrm{adj}^p(A)$, we take $\wedge^{n-p}(A)$, reflect the entries over the right-leaning diagonal, and adjust the entries by $(-1)^{i+j} = (-1)^{n-j+1+n-i+1}$.

Example: Let $A = \begin{pmatrix} 2 & 1 & 0 & 2 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then we can compute $\wedge^1(A) = A$, $\wedge^2(A) =$

$$\begin{pmatrix} 4 & 2 & -2 & 1 & -5 & -2 \\ 0 & -2 & -8 & -1 & -4 & 2 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & -8 & -5 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \text{ and } \wedge^3(A) = \begin{pmatrix} -4 & -16 & -10 & -9 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

Then $\mathrm{adj}^1(A) = \begin{pmatrix} -2 & 1 & 1 & 9 \\ 0 & -2 & -2 & -10 \\ 0 & 0 & 4 & 16 \\ 0 & 0 & 0 & -4 \end{pmatrix}$ has the appropriate relationship with $\wedge^3(A)$;

$\mathrm{adj}^2(A) = \begin{pmatrix} -1 & -1 & -5 & 0 & 2 & 2 \\ 0 & 2 & 8 & 1 & 4 & -5 \\ 0 & 0 & -2 & 0 & -1 & -1 \\ 0 & 0 & 0 & 2 & 8 & -2 \\ 0 & 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$ is related in the desired way to $\wedge^{4-2}(A) = \wedge^2(A)$; and

$\mathrm{adj}^3(A) = \begin{pmatrix} 1 & 4 & -1 & -2 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ is similarly related to $\wedge^1(A) = A$.

Note: The first adjugate wedge of A, $\mathrm{adj}^1(A)$ is just called the **adjugate** of A and is often denoted simply by $\mathrm{adj}(A)$. In some literature this is called the classical adjoint of A.

Theorem: (Cauchy) $\det(\mathrm{adj}(A)) = \left(\det(A)\right)^{n-1}$.

Example: In the example above, we can easily see that $\det(A) = -4$ while $\det(\mathrm{adj}(A)) = -64 = (-4)^{4-1}$ which fits our theorem.

Theorem: $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$.

Example: With A as in the example above

$$\frac{1}{\det(A)}\,\mathrm{adj}(A) = -\frac{1}{4}\begin{pmatrix} -2 & 1 & 1 & 9 \\ 0 & -2 & -2 & -10 \\ 0 & 0 & 4 & 16 \\ 0 & 0 & 0 & -4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & -\frac{9}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix} = A^{-1}, \text{ since}$$

$$\frac{1}{\det(A)}\mathrm{adj}(A)\ A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & -\frac{9}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 2 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem: (Jacobi) Let A be a matrix. Then any minor of order r of $\mathrm{adj}(A)$ is equal to the cofactor of the corresponding minor in the transpose of A, multiplied by $\det(A)^{r-1}$.

Example: In the previous numeric example, we can see that the minor of $\mathrm{adj}(A)$ of order 3, obtained by deleting the last row and last column, is 16. The cofactor of the corresponding minor of the transpose of A is $(-1)^{1+2+3+1+2+3} \cdot 1 = 1$. When this is multiplied by $\det(A)^{3-1} = (-4)^2$ we get back the value of the original minor.

Theorem: (Franke) Any minor of order r in the p^{th} wedge of A is equal to the co-factor of corresponding minor in the transpose of the p^{th} adjugate wedge of A, times $(\det(A))^{r-\binom{n-1}{p}}$.

Example: Again let $A = \begin{pmatrix} 2 & 1 & 0 & 2 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then we have $\wedge^3(A) = \begin{pmatrix} -4 & -16 & -10 & -9 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & -2 \end{pmatrix}$ and $(\mathrm{adj}^3(A))^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ -2 & 0 & -1 & 2 \end{pmatrix}$.

Consider the order 2 minor of $\wedge^3(A)$ which is the determinant of the matrix obtained from rows 1 and 2 and columns 2 and 3. It is $\begin{vmatrix} -16 & -10 \\ 4 & 2 \end{vmatrix} = 8$. The cofactor of the corresponding minor in $(\mathrm{adj}^3(A))^t$ is $(-1)^{1+2+2+3}\begin{vmatrix} -1 & 0 \\ -2 & 2 \end{vmatrix} = (-1)^{2+3+1+2} \cdot (-2) = -2$. We can see that $\det(A)$ is -4 and $8 = (-2) \cdot (-4)^{2-\binom{3}{3}}$, as predicted.

The Bazin Hybrid

There are more ways of creating new matrices from old ones which result in relationships similar to those we saw above. In the following, we discuss Bazin compound matrices and their more general form, Reiss compound matrices, and theorems regarding both.

Definition: Let A and B be $n \times n$ matrices. The **Bazin hybrid compound** of A and B is the $n \times n$ matrix whose $(i,j)^{th}$ entry is the determinant of the matrix obtained by replacing column i of A with column j of B.

Example: Let A and B be abstract 2×2 matrices with general entries $a_{i,j}$ and $b_{i,j}$, respectively. Then the Bazin Hybrid of A and B is

$$\begin{pmatrix} \begin{vmatrix} b_{1,1} & a_{1,2} \\ b_{2,1} & a_{2,2} \end{vmatrix} & \begin{vmatrix} b_{1,2} & a_{1,2} \\ b_{2,2} & a_{2,2} \end{vmatrix} \\ \begin{vmatrix} a_{1,1} & b_{1,1} \\ a_{2,1} & b_{2,1} \end{vmatrix} & \begin{vmatrix} a_{1,1} & b_{1,2} \\ a_{2,1} & b_{2,2} \end{vmatrix} \end{pmatrix}.$$

Note: There is no reason to believe that Bazin hybrid of A and B is equal to the Bazin hybrid of B and A. In fact they are not generally equal. So we call the Bazin hybrid of B and A the **dual** Bazin hybrid of A and B.

Theorem: (Bazin) The determinant of the Bazin hybrid compound of A and B is equal to $(\det(A))^{n-1}\det(B)$.

Example: Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 1 & -1 & 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}$. Then the Bazin hybrid compound of A and B is $\begin{pmatrix} 12 & 0 & 0 & 0 \\ -6 & -6 & 0 & 0 \\ 6 & 6 & 4 & 0 \\ -12 & -3 & 1 & -3 \end{pmatrix}$, which has determinant 864. We can see that this is equal to $\left(\det(A)\right)^{n-1}\det(B) = (6)^{4-1} \cdot (4)$, as was stated above.

The dual Bazin hybrid (that is the Bazin hybrid of B and A) is $\begin{pmatrix} 2 & 0 & 0 & 0 \\ -2 & -4 & 0 & 0 \\ 0 & 6 & 6 & 0 \\ -6 & 6 & 2 & -8 \end{pmatrix}$ with determinant $384 = \left(\det(B)\right)^{n-1}\det(A) = 4^{4-1} \cdot 6$. This also fits Bazin's theorem when we reverse the roles of A and B.

Theorem: (Reiss) Any minor of order r of the Bazin hybrid compound of A and B is equal to the cofactor of the corresponding minor in the transpose of the dual hybrid, multiplied by $\left(\det(A)\right)^{r-1}\left(\det(B)\right)^{r-n+1}$.

Example: Let A and B be as above. Then again the Bazin hybrid compound of A and B is $\begin{pmatrix} 12 & 0 & 0 & 0 \\ -6 & -6 & 0 & 0 \\ 6 & 6 & 4 & 0 \\ -12 & -3 & 1 & -3 \end{pmatrix}$ and the transpose of the dual Bazin hybrid is $\begin{pmatrix} 2 & -2 & 0 & -6 \\ 0 & -4 & 6 & 6 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & -8 \end{pmatrix}$.

The minor of the Bazin hybrid obtained from rows 3 and 4 and columns 1 and 3 is $\begin{vmatrix} 6 & 4 \\ -12 & 1 \end{vmatrix} = 54$. The cofactor of the corresponding minor of the transpose of the dual hybrid is obtained from rows 1 and 2 and columns 2 and 4 and is $(-1)^{3+4+1+3}\begin{vmatrix} -2 & -6 \\ -4 & 6 \end{vmatrix} = -1 \cdot (-36) = 36$. We can compute $\left(\det(A)\right)^{r-1}\left(\det(B)\right)^{r-n+1}$ to be $(6)^1(4)^{-1} = \frac{3}{2}$ and multiplying 36 by this gives us 54, the value of the original minor.

The Reiss Hybrid

Note: This is a simple generalization of the Bazin hybrid. Similarly, the theorems are generalizations of those for the Bazin hybrid.

Definition: Let A and B be $n \times n$ matrices. Define the p^{th} **Reiss hybrid** compound of A and B to be the $\binom{n}{p} \times \binom{n}{p}$ matrix whose $(i,j)^{th}$ entry is the determinant of the matrix obtained by replacing columns $i_1, \ldots, i_p$ of A with columns $j_1, \ldots, j_p$ of B, where $S_i = \{i_1, \ldots, i_p\}$ and $S_j = \{j_1, \ldots, j_p\}$ are the ordered p-subsets of $\{1, 2, \ldots, n\}$ defined at the beginning of this paper.

Note: We can also define the p^{th} **dual** Reiss hybrid of A and B as the p^{th} Reiss hybrid compound of B and A.

Example: Let $n = 4$ and $p = 2$. Let A and B be abstract matrices with general entries $a_{i,j}$ and $b_{i,j}$. Then $S_1 = \{1,2\}$, $S_2 = \{1,3\}$, $S_3 = \{1,4\}$, $S_4 = \{2,3\}$, $S_5 = \{2,4\}$, $S_6 = \{3,4\}$.

The entry in row 1, column 5 of the second Reiss hybrid of A and B is the determinant of the matrix with columns 1 and 2 of A replaced with columns 2 and 4 of B:

$$\begin{vmatrix} b_{1,2} & b_{1,4} & a_{1,3} & a_{1,4} \\ b_{2,2} & b_{4,4} & a_{2,3} & a_{2,4} \\ b_{2,3} & b_{3,4} & a_{3,3} & a_{3,4} \\ b_{4,2} & b_{4,4} & a_{4,3} & a_{4,4} \end{vmatrix}.$$

Theorem: (Reiss, Picquet) The determinant of the p^{th} Reiss hybrid compound of A and B is equal to $\left(\det(A)\right)^{\binom{n-1}{p}}\left(\det(B)\right)^{\binom{n-1}{p-1}}$.

Notice that when $p = 1$ we have the statement of the theorem by Bazin.

Example: Let A and B be as in the previous example. Then the second Reiss hybrid compound is $\begin{pmatrix} -12 & 0 & 0 & 0 & 0 & 0 \\ 12 & 8 & 0 & 0 & 0 & 0 \\ -6 & 2 & -6 & 0 & 0 & 0 \\ 0 & -4 & 0 & -4 & 0 & 0 \\ -9 & -1 & 3 & -1 & 3 & 0 \\ 9 & 9 & -3 & 3 & -3 & -2 \end{pmatrix}$. As stated in the theorem, it's determinant is $13824 = \left(\det(A)\right)^{\binom{3}{2}}\left(\det(B)\right)^{\binom{3}{1}} = 6^3 \cdot 4^3$.

The dual Reiss hybrid compound is $\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 & 0 \\ 3 & 1 & -4 & 0 & 0 & 0 \\ -3 & -3 & 0 & -6 & 0 & 0 \\ -9 & -1 & 4 & -2 & 8 & 0 \\ 9 & 9 & 0 & -6 & -12 & -12 \end{pmatrix}$. It's determinant is also $13824 = \left(\det(B)\right)^{\binom{3}{2}}\left(\det(A)\right)^{\binom{3}{1}} = 4^3 \cdot 6^3$.

Theorem: (Reiss, Picquet) Any $r \times r$ minor of the p^{th} Reiss hybrid compound is equal to the cofactor of the corresponding minor in the transpose of the dual Reiss hybrid, multiplied by $\left(\det(A)\right)^{r-\binom{n-1}{p-1}}\left(\det(B)\right)^{r-\binom{n-1}{p}}$.

Example: Let A and B be as before. Then we need the second Reiss hybrid of A and B $\begin{pmatrix} -12 & 0 & 0 & 0 & 0 & 0 \\ 12 & 8 & 0 & 0 & 0 & 0 \\ -6 & 2 & -6 & 0 & 0 & 0 \\ 0 & -4 & 0 & -4 & 0 & 0 \\ -9 & -1 & 3 & -1 & 3 & 0 \\ 9 & 9 & -3 & 3 & -3 & -2 \end{pmatrix}$, and the transpose of the dual Reiss hybrid $\begin{pmatrix} -2 & 3 & 3 & -3 & -9 & 9 \\ 0 & 3 & 1 & -3 & -1 & 9 \\ 0 & 0 & -4 & 0 & 4 & 0 \\ 0 & 0 & 0 & -6 & -2 & -6 \\ 0 & 0 & 0 & 0 & 8 & -12 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{pmatrix}$.

Consider the minor of the second Reiss hybrid which is the determinant of the matrix obtained from rows and columns 1,2,3 and 4. It's value is -2304. The cofactor of the corresponding minor in the transpose of the dual is $(-1)^{1+2+3+4+1+2+3+4}(-96) = -96$. Computing $\left(\det(A)\right)^{4-\binom{4-1}{2-1}}\left(\det(B)\right)^{4-\binom{4-1}{2}}$ yields $(6)^1 \cdot (4)^1 = 24$ and multiplying -96 by this gives us -2304, as expected.

III. EXAMPLE OF USEFULNESS

In the earlier report (NAWCADWAR–96-21-TR) by Gleeson, Stiller and Williams it was shown that the characteristic coefficients (c_k's) could be used to predict the effective rank of a noisy matrix. These coefficients, when properly normalized, fall below predetermined threshold values for k greater than the effective rank. These normalized coefficients are called P_k's.

To illustrate how the wedge product could be used in this type of analysis, we create the following matrix:

$$A = \begin{pmatrix} 17.91 & 28.05 & 6.45 & 10.33 \\ -5.97 & -11.56 & -15.03 & -36.72 \\ 22.04 & 33.83 & 37.56 & 38.39 \\ -24.17 & -37.40 & -17.85 & -19.93 \end{pmatrix}.$$

The matrix A was generated by first creating a rank three 4×4 matrix, and then adding a small random noise contribution to each element. For readability, each element was then rounded to two decimal places. The details of the generation process are spelled out at some length in NAWCADWAR–96-21-TR.

For the matrix A, we have:

$$P_1 = 0.53, \qquad P_2 = 0.60, \qquad P_3 = 0.57, \qquad P_4 = 0.09.$$

The earlier report studied 7×7 matrices with different effective ranks and noise levels. For small noise, the threshold was found to be typically in the range of 0.2 and 0.3. Let us assume that this threshold range does not vary drastically and applies to 4×4 matrices. The above distribution of the P_k's has P_3 above the threshold and P_4 below the threshold. This is the profile of a matrix whose effective rank is three.

Now let us consider the second wedge of A. The second wedge of A is the following 6×6 matrix:

$$\wedge^2(A) = \begin{pmatrix} -39.57 & -230.62 & -595.90 & -347.01 & 910.63 & -81.47 \\ -12.43 & 530.56 & 459.88 & 835.58 & 727.55 & -140.47 \\ 8.23 & -163.86 & -107.32 & -259.65 & -172.86 & 55.88 \\ 52.85 & 107.08 & 580.25 & 74.30 & 798.60 & 802.25 \\ -56.15 & -256.63 & -768.42 & -355.64 & -1142.77 & -355.87 \\ -6.64 & 514.42 & 488.52 & 800.76 & 761.42 & -63.36 \end{pmatrix}.$$

Recall, that if the given matrix A has rank three, then only three of the four eigenvalues are nonzero. With noise added the fourth would also be nonzero, but small compared to the other three. The eigenvalues of the second wedge are equal to the products of A's eigenvalues taken two at a time, so there are $\binom{3}{2}$ or three significant eigenvalues in $\wedge^2(A)$. With three significant eigenvalues, $\wedge^2(A)$ should appear to have effective rank three.

The P_k's for the $\wedge^2(A)$ are the following:

$$P_1 = 0.60, \qquad P_2 = 0.56, \qquad P_3 = 0.57, \qquad P_4 = 0.22, \qquad P_5 = 0.13, \qquad P_6 = 0.09.$$

Assuming that the 7×7 thresholds work for 6×6 matrices, P_5 and P_6 clearly fall below the threshold range. P_4 is on the border; whereas, P_3 is higher than the threshold range. This profile indicates that $\wedge^2(A)$ is either rank three or four. Actually, the fact that P_6 is lower than the threshold, alone implies that the rank of A is very likely less than four. Moreover, when we see that P_5 is also less than the threshold, the likelihood of the rank of A being less than four is amplified. Finally, P_3 is greater than the threshold. This implies that the rank of $\wedge^2(A)$ is at least three. This, in turn, implies that the rank of A itself is three. The main point here is that $\wedge^2(A)$ has more coefficients with which to work and has greater nullity. Using the normalized characteristic coefficients of $\wedge^2(A)$ should give us a greater handle and enhanced sensitivity in determining the rank of the matrix A.

As discussed in Section II, it is not necessary to actually compute $\wedge^2(A)$ to determine its P_k's. All one needs are the characteristic coefficients of $\wedge^2(A)$, and these can be computed in terms of polynomials in the characteristic coefficients of A. A few examples of these polynomials are included in Appendix B.

IV. CONCLUSION

This report may be regarded as a primer on the theory of wedge products.

In Section II we have brought together definitions, properties and theorems relating to the wedge product. Also, we have discussed how the companion matrix can be used to compute the coefficients of the characteristic polynomials of the various wedge products. Finally, theorems which relate to other forms of compound matrices have been explained.

In Section III we have illustrated how the wedge product has the potential to be useful in the determination of the effective rank of a noisy data matrix.

Future work in this area includes determining quantitatively the extent of this potential for predicting the effective rank. That is, we should determine whether or not the normalized coefficients of the characteristic polynomial of $\wedge^2(A)$ are more successful than those of A in finding effective rank. Characteristic coefficients for higher order wedge products should also be studied. Ultimately, the order(s) of the wedge product(s) that optimize results and computation speed should be compared with existing methods for predicting effective rank.

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APPENDIX A

SAMPLE RUN SESSIONS

Here we have included sample code that illustrates the syntax for evaluating wedges of matrices on three different mathematical software packages: Maple (version 5, release 3.0), Mathematica (version 2.2), and Fermat (version 1.1). In the symbolic examples below $WA2$ is the second order wedge of matrix A.

Maple

In Maple, there is a wedge product function (&^) in the differential forms library. Unfortunately, the intent of this function was for differential forms, not for linear algebra. The wedge product command does not work on all matrices; specifically, it may fail on matrices containing zero entries. This would put limitations on the number of matrices for which the Navy could find the wedge product. In light of this, George Nakos has written a program using Maple syntax that computes the wedge product of any matrix. This file can be written in any text editor. Maple can access this file with the command 'read < filename >; '. The program is as follows:

```
with(linalg):
with(combinat,cartprod):

#This function takes a list of lists [[a,b,...],[a1,b1,...],...
#and computes the Cartesian product [a,b,...] × [a1,b1,...] × ...
 lesslists0 := proc(lis)
  local ll,car;
  ll := [];
  car := carprod(lis);
  while not car[finished] do
     ll := [op(ll),car[nextvalue]()] od:
  RETURN(ll);
  end:

# This function takes a list of numbers and returns a list of lists
#of numbers with entries ≤ to the
#corresponding entries of the original.  For example,
#Lesslists([2,4,3]); yields [1,1,1],[1,1,2],...,[2,4,3]
Lesslists := proc(lis)
  local ll,i,n;
  ll := [];
  for i from 1 to nops(lis) do
  ll := [op(ll),[$1..lis[i]]]:
  od:
```

```
  ll := lesslists0(ll);
  RETURN(ll);
  end:

#This function tests whether the length of f is n
lengthNQ := proc(f,n) evalb(nops(f) = n) end:

# From a list lis, pick the elements of length n
PickLengthN := proc(lis,n) select (lengthNQ,lis,n) end:

#This function deletes repeated elements in a list
DeleteRepeated := proc(lis)
  local ll,i;
  for i from 2 to nops(lis) do
     if evalb(not member(lis[i],ll)) then
     ll := [op(ll),lis[i]];fi:
  od:
  RETURN(ll);
  end:
```

This function takes two >0 integers n and k, n $>$ k, and returns all pairs

#of the form (i,j) with i $<$ j and i $\leq$ k and j $\leq$ n.

```
SubmatrixIndex := proc(n,k)
  local tt;
  tt:= [$(n-k+1)..n];
  LessLists(tt);
  map(convert,",set);
  PickLengthN(",k);
  DeleteRepeated(");
  tt := map(convert,",list);
  Return(tt);
  end:

# This function takes a list of numbers and returns the
#complete Cartesian product of the element lists
ListOfPairs := proc(lis)
  local ll,nn,i,j;
  nn := nops(lis);
  for i from 1 to nn do
     for j from 1 to nn do
        ll := [op(ll),[lis[i],lis[j]]];
     od:
  od:
```

```
  RETURN(ll);
  end:

# MAIN FUNCTION. Computes the k x k minors of matrix A
Minors := proc(A,k)
  local n,dimmat,kk,ll,i;
  n := vectdim(row(A,1));
  SubmatrixIndex(n,k);
  dimmat := nops(");
  kk := ListOfPairs("");
  ll := [];
  for i from 1 to nops(kk) do
     ll := [op(ll),det(submatrix(A,op(kk[i])))];
  od:
  RETURN(matrix(dimmat,dimmat,ll));
  end:
```

Example:

To compute the second order wedge for the matrix $A = \begin{pmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \\ c1 & c2 & c3 & c4 \\ d1 & d2 & d3 & d4 \end{pmatrix}$, one would read in the above file and then type:

$> A := matrix(3, 3, [a1, a2, a3, a4, b1, b2, b3, b4, c1, c2, c3, c4, d1, d2, d3, d4]);$

$> WA2 := Minors(A, 2);$

The output to this would be the following matrix:

$$WA2 := \begin{pmatrix} a1b2-a2b1 & a1b3-a3b1 & a1b4-a4b1 & a2b3-a3b2 & a2b4-a4b2 & a3b4-a4b3 \\ a1c2-a2c1 & a1c3-a3c1 & a1c4-a4c1 & a2c3-a3c2 & a2c4-a4c2 & a3c4-a4c3 \\ a1d2-a2d1 & a1d3-a3d1 & a1d4-a4d1 & a2d3-a3d2 & a2d4-a4d2 & a3d4-a4d3 \\ b1c2-b2c1 & b1c3-b3c1 & b1c4-b4c1 & b2c3-b3c2 & b2c4-b4c2 & b3c4-b4c3 \\ b1d2-b2d1 & b1d3-b3d1 & b1d4-b4d1 & b2d3-b3d2 & b2d4-b4d2 & b3d4-b4d3 \\ c1d2-c2d1 & c1d3-c3d1 & c1d4-c4d1 & c2d3-c3d2 & c2d4-c4d2 & c3d4-c4d3 \end{pmatrix}.$$

Mathematica

Mathematica has a built in command for computing the wedge product, namely "Minors[A,p]". This command takes two parameters; the first is the name of the matrix, and the second is a positive integer denoting the order of the wedge.

Example :

As in the previous example, if we wish to compute the second wedge of a 4×4 matrix in Mathematica, we would enter:

$A = \{\{a1, a2, a3, a4\}, \{a5, a6, a7, a8\}, \{a9, a10, a11, a12\}, \{a13, a14, a15, a16\}\}$
$WA2 =$ Minors$[A, 2]$

Fermat

Fermat is a mathematical software system written by Robert Lewis. While there is no built-in function for computing the wedge product, there will be soon. Michael Hirsch has written a program in the Fermat language that can be used in the absence of a wedge product function. Run times are faster than Mathematica and Maple for large symbolic matrices. Once the code is within the Fermat shell, run times using Fermat should be even shorter than they are currently (using the Hirsch program).

```
;This is the main function for the program
:Wedge(p,d,matrix2; m,n,i) =
  :cols = Cols[p];
  :rows = Deg[p]/cols;
  :n = Ç (cols,d);
  :m = Ç (rows,d);
  :ri = 0;
  :ci = 0;
  :b[m,n];
  :x[d];
  :y[d];
  :temp[d,d];
  Genr(rows,d,1,1);
  :[b] =  TM  [b];
  @[temp];
  :[matrix2] = [b];
  @[b];
  0.;

;This function generates all of the possible row combinations
;for the wedge
:Genr(m,dd,j,s;i)=
  if (dd = 0)
     then (:ri+; :ci =0; Genc(cols,d,1,1) )
  else ( for ( :i=s,m+1-dd) do
```

```
    ( :x[j] = i;
     Genr(m, dd-1, j+1, i+1 );
    )
 ).;

;This function generates all of the possible
;column combinations for the wedge
:Genc(n,dc,k,l;i)=
 if (dc = 0)
    then (:ci+; Dump(ri,ci) )
 else ( for ( :i=l,n+1-dc) do
    ( :y[k] = i;
     Genc(n, dc-1, k+1, i+1 );
    )
 ).;

;This function calculates the determinant of each
;submatrix for the entries of the wedge
:Dump(row,col; q,w)=
 :temp[d,d];
 for( :q=1, d ) do
    ( for( :w=1,d )) do
         ( :temp[q,w] = p[x[q],y[w]] )
(     );
 :b[col,row] = Det[temp];
 @[temp].;
```

After reading this program into Fermat, using the wedge function is easy. Just type the following:

>Wedge(matrix1,d,matrix2)

where matrix1 is the original matrix, d is the order of the wedge, and matrix2 is the variable that will be assigned to the wedge.

<u>Example:</u>

In Fermat to compute the second wedge of a 4×4, we would enter:

>: $a[4,4]$
>: $[a] = [[a1, a2, a3, a4,$,,
$a5, a6, a7, a8,$,,
$a9, a10, a11, a12,$,,
$a13, a14, a15, a16]]$

$>: wa2[6,6]$
$> Wedge([a],2,[wa2])$

APPENDIX B:

CHARACTERISTIC COEFFICIENT RELATIONSHIPS

This Appendix contains a few sample coefficients of the characteristic polynomial of the wedge products (d_k's) expressed as polynomials in the coefficients of the characteristic polynomial (c_k's) of the given matrix (A).

If A is a 3×3 matrix:

For $\wedge^2(A)$:

$$d_1 = -c_2$$

$$d_2 = c_1 c_3$$

$$d_3 = -c_3^2$$

For $\wedge^3(A)$:

$$d_1 = c_3$$

If A is a 4×4 matrix:

For $\wedge^2(A)$:

$$d_1 = -c_2$$

$$d_2 = c_1 c_3 - c_4$$

$$d_3 = 2c_2 c_4 - c_1^2 c_4 - c_3^2$$

$$d_4 = c_1 c_3 c_4 - c_4^2$$

$$d_5 = -c_2 c_4^2$$

$$d_6 = c_4^3$$

4×4 matrix cont.

For $\wedge^3(A)$:

$$d_1 = c_3$$

$$d_2 = c_2 c_4$$

$$d_3 = c_1 c_4^2$$

$$d_4 = c_4^3$$

For $\wedge^4(A)$:

$$d_1 = -c_4$$

If A is a 5×5 matrix:

For $\wedge^2(A)$:

$$d_1 = -c_2$$

$$d_2 = c_1 c_3 - c_4$$

$$d_3 = c_1 c_5 + 2c_2 c_4 - c_1^2 c_4 - c_3^2$$

$$d_4 = c_3 c_5 - 3c_1 c_2 c_5 + c_1^3 c_5 - c_4^2 + c_1 c_3 c_4$$

$$d_5 = -c_2 c_4^2 - c_1^2 c_3 c_5 + 2c_2 c_3 c_5 + 2c_1 c_4 c_5 - 2c_5^2$$

$$d_6 = c_4^3 + c_1 c_2 c_4 c_5 - 3c_3 c_4 c_5 - c_1^2 c_5^2 + c_2 c_5^2$$

$$d_7 = c_4 c_5^2 + 2c_1 c_3 c_5^2 - c_2^2 c_5^2 - c_1 c_4^2 c_5$$

$$d_8 = c_2 c_4 c_5^2 - c_1 c_5^3$$

$$d_9 = -c_3 c_5^3$$

$$d_{10} = c_5^4$$

5×5 matrix cont.

For $\wedge^3(A)$:

$$d_1 = c_3$$

$$d_2 = c_2c_4 - c_1c_5$$

$$d_3 = -c_4c_5 - 2c_1c_3c_5 + c_2^2c_5 + c_1c_4^2$$

$$d_4 = c_2c_5^2 - c_1^2c_5^2 - 3c_3c_4c_5 + c_1c_2c_4c_5 + c_4^3$$

$$d_5 = 2c_5^3 - 2c_1c_4c_5^2 - 2c_2c_3c_5^2 + c_1^2c_3c_5^2 + c_2c_4^2c_5$$

$$d_6 = c_1c_3c_4c_5^2 - c_4^2c_5^2 + c_1^3c_5^3 - 3c_1c_2c_5^3 + c_3c_5^3$$

$$d_7 = c_3^2c_5^3 + c_1^2c_4c_5^3 - 2c_2c_4c_5^3 - c_1c_5^4$$

$$d_8 = c_1c_3c_5^4 - c_4c_5^4$$

$$d_9 = c_2c_5^5$$

$$d_{10} = c_5^6$$

For $\wedge^4(A)$:

$$d_1 = -c_4$$

$$d_2 = c_3c_5$$

$$d_3 = -c_2c_5^2$$

$$d_4 = c_1c_5^3$$

$$d_5 = -c_5^4$$

For $\wedge^5(A)$:

$$d_1 = c_5$$

If A is a 6×6 matrix:

For $\wedge^2(A)$:

$$d_1 = -c_2$$

$$d_2 = c_1c_3 - c_4$$

$$d_3 = -c_3^2 - c_1^2c_4 + 2c_2c_4 + c_1c_5 - c_6$$

$$d_4 = c_1c_3c_4 - c_4^2 + c_1^3c_5 - 3c_1c_2c_5 + c_3c_5 - c_1^2c_6 + 2c_2c_6$$

$$\begin{aligned} d_5 = &-c_2c_4^2 - c_1^2c_3c_5 + 2c_2c_3c_5 + 2c_1c_4c_5 - 2c_5^2 \\ &-c_1^4c_6 + 4c_1^2c_2c_6 - 2c_2^2c_6 - 3c_1c_3c_6 + 2c_4c_6 \end{aligned}$$

$$\begin{aligned} d_6 = &\, c_4^3 + c_1c_2c_4c_5 - 3c_3c_4c_5 - c_1^2c_5^2 + c_2c_5^2 \\ &+c_1^3c_3c_6 - 3c_1c_2c_3c_6 + 3c_3^2c_6 - c_1^2c_4c_6 + 3c_1c_5c_6 - 2c_6^2 \end{aligned}$$

$$\begin{aligned} d_7 = &-c_1c_4^2c_5 - c_2^2c_5^2 + 2c_1c_3c_5^2 + c_4c_5^2 - c_1^2c_2c_4c_6 \\ &+2c_2^2c_4c_6 + c_1c_3c_4c_6 - c_4^2c_6 + c_1^3c_5c_6 - c_1c_2c_5c_6 \\ &-3c_3c_5c_6 - c_1^2c_6^2 \end{aligned}$$

$$\begin{aligned} d_8 = &\, c_2c_4c_5^2 - c_1c_5^3 + c_1^2c_4^2c_6 - 2c_2c_4^2c_6 + c_1c_2^2c_5c_6 \\ &-2c_1^2c_3c_5c_6 - c_2c_3c_5c_6 + c_1c_4c_5c_6 + c_5^2c_6 - c_1^2c_2c_6^2 \\ &+c_2^2c_6^2 + 3c_1c_3c_6^2 \end{aligned}$$

$$\begin{aligned} d_9 = &-c_3c_5^3 - c_1c_2c_4c_5c_6 + 3c_4c_5c_6 + c_1^2c_5^2c_6 + c_2c_5^2c_6 \\ &-c_2^3c_6^2 + 3c_1c_2c_3c_6^2 - 3c_3^2c_6^2 - c_1^2c_4c_6^2 - 3c_1c_5c_6^2 + 2c_6^3 \end{aligned}$$

$$\begin{aligned} d_{10} = &\, c_5^4 + c_1c_3c_5^2c_6 - 4c_4c_5^2c_6 + c_2^2c_4c_6^2 - 2c_1c_3c_4c_6^2 + 2c_4^2c_6^2 \\ &-2c_1c_2c_5c_6^2 + 3c_3c_5c_6^2 + 2c_1^2c_6^3 - 2c_2c_6^3 \end{aligned}$$

$$\begin{aligned} d_{11} = &-c_1c_5^3c_6 - c_2c_3c_5c_6^2 + 3c_1c_4c_5c_6^2 + c_5^2c_6^2 + c_2^2c_6^3 \\ &-c_1c_3c_6^3 - 2c_4c_6^3 \end{aligned}$$

$$d_{12} = c_2c_5^2c_6^2 + c_3^2c_6^3 - 2c_2c_4c_6^3 - c_1c_5c_6^3 + c_6^4$$

$$d_{13} = c_2c_6^4 - c_3c_5c_6^3$$

$$d_{14} = c_4c_6^4$$

6×6 matrix cont.

For $\wedge^2(A)$:

$$d_{15} = -c_6^5$$

For $\wedge^3(A)$:

$$d_1 = c_3$$

$$d_2 = c_2c_4 - c_1c_5 + c_6$$

$$d_3 = c_1c_4^2 + c_2^2c_5 - 2c_1c_3c_5 - c_4c_5 - c_1c_2c_6 + 3c_3c_6$$

$$\begin{aligned} d_4 = {} & c_4^3 + c_1c_2c_4c_5 - 3c_3c_4c_5 - c_1^2c_5^2 + c_2c_5^2 + c_2^3c_6 \\ & -3c_1c_2c_3c_6 + 3c_3^2c_6 + c_1^2c_4c_6 - c_2c_4c_6 + c_1c_5c_6 \end{aligned}$$

$$\begin{aligned} d_5 = {} & c_2c_4^2c_5 + c_1^2c_3c_5^2 - 2c_2c_3c_5^2 - 2c_1c_4c_5^2 + 2c_5^3 \\ & +c_1c_2^2c_4c_6 - 2c_1^2c_3c_4c_6 - c_2c_3c_4c_6 + 3c_1c_4^2c_6 - 2c_1^2c_2c_5c_6 \\ & +3c_2^2c_5c_6 + 2c_1c_3c_5c_6 - 4c_4c_5c_6 + 2c_1^3c_6^2 - 4c_1c_2c_6^2 \\ & +3c_3c_6^2 \end{aligned}$$

$$\begin{aligned} d_6 = {} & c_1c_3c_4c_5^2 - c_4c_5^2 + c_1^3c_5^3 - 3c_1c_2c_5^3 + c_3c_5^3 \\ & +c_2^2c_4^2c_6 - 2c_1c_3c_4^2c_6 + 2c_4^3c_6 + c_1^2c_2c_3c_5c_6 - 2c_2^2c_3c_5c_6 \\ & -c_1c_3^2c_5c_6 - 3c_1^3c_4c_5c_6 + 6c_1c_2c_4c_5c_6 - 2c_3c_4c_5c_6 + 4c_2c_5^2c_6 \\ & -c_1^2c_2^2c_6^2 + 2c_2^3c_6^2 + c_1^3c_3c_6^2 - 2c_1c_2c_3c_6^2 + 3c_3^2c_6^2 \\ & +4c_1^2c_4c_6^2 - 8c_2c_4c_6^2 - 4c_1c_5c_6^2 + 3c_6^3 \end{aligned}$$

$$\begin{aligned} d_7 = {} & c_3^2c_5^3 + c_1^2c_4c_5^3 - 2c_2c_4c_5^3 - c_1c_5^4 + c_1c_2c_3c_4c_5c_6 \\ & -3c_3^2c_4c_5c_6 - 3c_1^2c_4^2c_5c_6 + 5c_2c_4^2c_5c_6 + c_1^3c_2c_5^2c_6 - 3c_1c_2^2c_5^2c_6 \\ & -c_1^2c_3c_5^2c_6 + c_2c_3c_5^2c_6 + 4c_1c_4c_5^2c_6 + c_5^3c_6 + c_1^3c_3^2c_6^2 \\ & -3c_1c_2c_3^2c_6^2 + 3c_3^3c_6^2 - 2c_1^3c_2c_4c_6^2 + 5c_1c_2^2c_4c_6^2 + c_1^2c_3c_4c_6^2 \\ & -5c_2c_3c_4c_6^2 + c_1c_4^2c_6^2 - c_1^4c_5c_6^2 + 4c_1^2c_2c_5c_6^2 \\ & +c_2^2c_5c_6^2 - c_1c_3c_5c_6^2 - 6c_4c_5c_6^2 + c_1^3c_6^3 - 6c_1c_2c_6^3 \\ & +3c_3c_6^3 \end{aligned}$$

6 × 6 matrix cont.

For $\wedge^3(A)$:

$$
\begin{aligned}
d_8 = {} & c_1c_3c_5^4 - c_4c_5^4 + c_2c_3^2c_5^2c_6 + c_1^2c_2c_4c_5^2c_6 - 2c_2^2c_4c_5^2c_6 \\
& -4c_1c_3c_4c_5^2c_6 + 4c_4^2c_5^2c_6 - 2c_1c_2c_5^3c_6 + c_1^2c_3^2c_4c_6^2 - 2c_2c_3^3c_4c_6^2 \\
& -2c_1^2c_2c_4^2c_6^2 + 4c_2^2c_4^2c_6^2 - c_4^3c_6^2 + c_1^4c_3c_5c_6^2 - 4c_1^2c_2c_3c_5c_6^2 \\
& +4c_1c_3^2c_5c_6^2 - 2c_1^3c_4c_5c_6^2 + 11c_1c_2c_4c_5c_6^2 - 5c_3c_4c_5c_6^2 + 2c_1^2c_5^2c_6^2 \\
& +c_2c_5^2c_6^2 - c_1^4c_2c_6^3 + 4c_1^2c_2^2c_6^3 - c_2^3c_6^3 - 5c_1c_2c_3c_6^3 \\
& +3c_3^2c_6^3 + c_1^2c_4c_6^3 - 7c_2c_4c_6^3 - 5c_1c_5c_6^3 + 3c_6^4
\end{aligned}
$$

$$
\begin{aligned}
d_9 = {} & c_2c_5^5 + c_1c_2c_3c_5^3c_6 - 6c_2c_4c_5^3c_6 - c_1c_5^4c_6 + c_1c_3^3c_5c_6^2 \\
& +c_1^3c_3c_4c_5c_6^2 - 5c_1c_2c_3c_4c_5c_6^2 - c_3^2c_4c_5c_6^2 - c_1^2c_4^2c_5c_6^2 + 10c_2c_4^2c_5c_6^2 \\
& -c_1c_2^2c_5^2c_6^2 - 2c_1^2c_3c_5^2c_6^2 + 5c_2c_3c_5^2c_6^2 + 6c_1c_4c_5^2c_6^2 + c_5^3c_6^2 \\
& -c_1c_2c_3^2c_6^3 + c_3^3c_6^3 + c_1^5c_4c_6^3 - 6c_1^3c_2c_4c_6^3 \\
& +10c_1c_2^2c_4c_6^3 + 5c_1^2c_3c_4c_6^3 - 10c_2c_3c_4c_6^3 - 5c_1c_4^2c_6^3 \\
& -c_1^4c_5c_6^3 + 6c_1^2c_2c_5c_6^3 - 5c_2^2c_5c_6^3 - 3c_1c_3c_5c_6^3 - 5c_4c_5c_6^3 \\
& +c_1^3c_6^4 - 5c_1c_2c_6^4 + 6c_3c_6^4
\end{aligned}
$$

$$
\begin{aligned}
d_{10} = {} & c_5^6 + c_2^2c_5^4c_6 - 6c_4c_5^4c_6 + c_1^2c_3^2c_5^2c_6^2 - 4c_2^2c_4c_5^2c_6^2 \\
& -2c_1c_3c_4c_5^2c_6^2 + 10c_4^2c_5^2c_6^2 - 2c_1c_2c_5^3c_6^2 + 6c_3c_5^3c_6^2 + c_3^4c_6^3 \\
& -4c_2c_3^2c_4c_6^3 + c_1^4c_4^2c_6^3 - 4c_1^2c_2c_4^2c_6^3 + 6c_2^2c_4^2c_6^3 + 4c_1c_3c_4^2c_6^3 \\
& -4c_4^3c_6^3 - 2c_1^2c_2c_3c_5c_6^3 + 4c_2^2c_3c_5c_6^3 - 2c_1c_3^2c_5c_6^3 - 2c_1^3c_4c_5c_6^3 \\
& +12c_1c_2c_4c_5c_6^3 - 12c_3c_4c_5c_6^3 + 2c_1^2c_5^2c_6^3 - 4c_2c_5^2c_6^3 + c_1^6c_6^4 \\
& -6c_1^4c_2c_6^4 + 10c_1^2c_2^2c_6^4 - 4c_2^3c_6^4 + 6c_1^3c_3c_6^4 - 12c_1c_2c_3c_6^4 \\
& +6c_3^2c_6^4 - 4c_1^2c_4c_6^4 - 2c_2c_4c_6^4 + 2c_1c_5c_6^4
\end{aligned}
$$

6 × 6 matrix cont.

For $\wedge^3(A)$:

$$\begin{aligned}
d_{11} = {} & c_2c_5^5c_6 + c_1c_2c_3c_5^3c_6^2 - 6c_2c_4c_5^3c_6^2 - c_1c_5^4c_6^2 + c_1c_3^3c_5c_6^3 \\
& + c_1^3c_3c_4c_5c_6^3 - 5c_1c_2c_3c_4c_5c_6^3 - c_3^2c_4c_5c_6^3 - c_1^2c_4^2c_5c_6^3 + 10c_2c_4^2c_5c_6^3 \\
& - c_1c_2^2c_5^2c_6^3 - 2c_1^2c_3c_5^2c_6^3 + 5c_2c_3c_5^2c_6^3 + 6c_1c_4c_5^2c_6^3 \\
& + c_5^3c_6^3 - c_1c_2c_3^2c_6^4 + c_3^3c_6^4 + c_1^5c_4c_6^4 - 6c_1^3c_2c_4c_6^4 \\
& + 10c_1c_2^2c_4c_6^4 + 5c_1^2c_3c_4c_6^4 - 10c_2c_3c_4c_6^4 - 5c_1c_4^2c_6^4 - c_1^4c_5c_6^4 \\
& + 6c_1^2c_2c_5c_6^4 - 5c_2^2c_5c_6^4 - 3c_1c_3c_5c_6^4 - 5c_4c_5c_6^4 + c_1^3c_6^5 \\
& - 5c_1c_2c_6^5 + 6c_3c_6^5
\end{aligned}$$

$$\begin{aligned}
d_{12} = {} & c_1c_3c_5^4c_6^2 - c_4c_5^4c_6^2 + c_2c_3^2c_5^2c_6^3 + c_1^2c_2c_4c_5^2c_6^3 - 2c_2^2c_4c_5^2c_6^3 \\
& - 4c_1c_3c_4c_5^2c_6^3 + 4c_4^2c_5^2c_6^3 - 2c_1c_2c_5^3c_6^3 + c_1^2c_3^2c_4c_6^4 - 2c_2c_3^2c_4c_6^4 \\
& - 2c_1^2c_2c_4^2c_6^4 + 4c_2^2c_4^2c_6^4 - c_4^3c_6^4 + c_1^4c_3c_5c_6^4 - 4c_1^2c_2c_3c_5c_6^4 \\
& + 4c_1c_3^2c_5c_6^4 - 2c_1^3c_4c_5c_6^4 + 11c_1c_2c_4c_5c_6^4 - 5c_3c_4c_5c_6^4 + 2c_1^2c_5^2c_6^4 \\
& + c_2c_5^2c_6^4 - c_1^4c_2c_6^5 + 4c_1^2c_2^2c_6^5 - c_2^3c_6^5 - 5c_1c_2c_3c_6^5 \\
& + 3c_3^2c_6^5 + c_1^2c_4c_6^5 - 7c_2c_4c_6^5 - 5c_1c_5c_6^5 + 3c_6^6
\end{aligned}$$

$$\begin{aligned}
d_{13} = {} & c_3^2c_5^3c_6^3 + c_1^2c_4c_5^3c_6^3 - 2c_2c_4c_5^3c_6^3 - c_1c_5^4c_6^3 + c_1c_2c_3c_4c_5c_6^4 \\
& - 3c_3^2c_4c_5c_6^4 - 3c_1^2c_4^2c_5c_6^4 + 5c_2c_4^2c_5c_6^4 + c_1^3c_2c_5^2c_6^4 - 3c_1c_2^2c_5^2c_6^4 \\
& - c_1^2c_3c_5^2c_6^4 + c_2c_3c_5^2c_6^4 + 4c_1c_4c_5^2c_6^4 + c_5^3c_6^4 + c_1^3c_3^2c_6^5 \\
& - 3c_1c_2c_3^2c_6^5 + 3c_3^3c_6^5 - 2c_1^3c_2c_4c_6^5 + 5c_1c_2^2c_4c_6^5 + c_1^2c_3c_4c_6^5 \\
& - 5c_2c_3c_4c_6^5 + c_1c_4^2c_6^5 - c_1^4c_5c_6^5 + 4c_1^2c_2c_5c_6^5 + c_2^2c_5c_6^5 - c_1c_3c_5c_6^5 \\
& - 6c_4c_5c_6^5 + c_1^3c_6^6 - 6c_1c_2c_6^6 + 3c_3c_6^6
\end{aligned}$$

$$\begin{aligned}
d_{14} = {} & c_1c_3c_4c_5^2c_6^4 - c_4^2c_5^2c_6^4 + c_1^3c_5^3c_6^4 - 3c_1c_2c_5^3c_6^4 + c_3c_5^3c_6^4 \\
& + c_2^2c_4^2c_6^5 - 2c_1c_3c_4^2c_6^5 + 2c_4^3c_6^5 + c_1^2c_2c_3c_5c_6^5 - 2c_2^2c_3c_5c_6^5 \\
& - c_1c_3^2c_5c_6^5 - 3c_1^3c_4c_5c_6^5 + 6c_1c_2c_4c_5c_6^5 - 2c_3c_4c_5c_6^5 + 4c_2c_5^2c_6^5 \\
& - c_1^2c_2^2c_6^6 + 2c_2^3c_6^6 + c_1^3c_3c_6^6 - 2c_1c_2c_3c_6^6 + 3c_3^2c_6^6 \\
& + 4c_1^2c_4c_6^6 - 8c_2c_4c_6^6 - 4c_1c_5c_6^6 + 3c_6^7
\end{aligned}$$

6×6 matrix cont.

For $\wedge^3(A)$:

$$d_{15} = c_2c_4^2c_5c_6^5 + c_1^2c_3c_5^2c_6^5 - 2c_2c_3c_5^2c_6^5 - 2c_1c_4c_5^2c_6^5 + 2c_5^3c_6^5$$
$$+c_1c_2^2c_4c_6^6 - 2c_1^2c_3c_4c_6^6 - c_2c_3c_4c_6^6 + 3c_1c_4^2c_6^6 - 2c_1^2c_2c_5c_6^6$$
$$+3c_2^2c_5c_6^6 + 2c_1c_3c_5c_6^6 - 4c_4c_5c_6^6 + 2c_1^3c_6^7 - 4c_1c_2c_6^7$$
$$+3c_3c_6^7$$

$$d_{16} = c_4^3c_6^6 + c_1c_2c_4c_5c_6^6 - 3c_3c_4c_5c_6^6 - c_1^2c_5^2c_6^6 + c_2c_5^2c_6^6$$
$$+c_2^3c_6^7 - 3c_1c_2c_3c_6^7 + 3c_3^2c_6^7 + c_1^2c_4c_6^7 - c_2c_4c_6^7$$
$$+c_1c_5c_6^7$$

$$d_{17} = c_1c_4^2c_6^7 + c_2^2c_5c_6^7 - 2c_1c_3c_5c_6^7 - c_4c_5c_6^7 - c_1c_2c_6^8$$
$$+3c_3c_6^8$$

$$d_{18} = c_2c_4c_6^8 - c_1c_5c_6^8 + c_6^9$$

$$d_{19} = c_3c_6^9$$

$$d_{20} = c_6^{10}$$

For $\wedge^4(A)$:

$$d_1 = -c_4$$

$$d_2 = +(c_3c_5) - c_2c_6$$

$$d_3 = -c_2c_5^2 - c_3^2c_6 + 2c_2c_4c_6 + c_1c_5c_6 - c_6^2$$

$$d_4 = +(c_1c_5^3) + c_2c_3c_5c_6 - 3c_1c_4c_5c_6 - c_5^2c_6 - c_2^2c_6^2$$
$$+c_1c_3c_6^2 + 2c_4c_6^2$$

$$d_5 = -c_5^4 - c_1c_3c_5^2c_6 + 4c_4c_5^2c_6 - c_2^2c_4c_6^2 + 2c_1c_3c_4c_6^2$$
$$-2c_4^2c_6^2 + 2c_1c_2c_5c_6^2 - 3c_3c_5c_6^2 - 2c_1^2c_6^3 + 2c_2c_6^3$$

6×6 matrix cont.

For $\wedge^4(A)$:

$$d_6 = +(c_3c_5^3c_6) + c_1c_2c_4c_5c_6^2 - 3c_3c_4c_5c_6^2 - c_1^2c_5^2c_6^2 - c_2c_5^2c_6^2$$
$$+c_2^3c_6^3 - 3c_1c_2c_3c_6^3 + 3c_3^2c_6^3 + c_1^2c_4c_6^3 + 3c_1c_5c_6^3 - 2c_6^4$$

$$d_7 = -c_2c_4c_5^2c_6^2 + c_1c_5^3c_6^2 - c_1^2c_4^2c_6^3 + 2c_2c_4^2c_6^3 - c_1c_2^2c_5c_6^3$$
$$+2c_1^2c_3c_5c_6^3 + c_2c_3c_5c_6^3 - c_1c_4c_5c_6^3 - c_5^2c_6^3 + c_1^2c_2c_6^4$$
$$-c_2^2c_6^4 - 3c_1c_3c_6^4$$

$$d_8 = +(c_1c_4^2c_5c_6^3) + c_2^2c_5^2c_6^3 - 2c_1c_3c_5^2c_6^3 - c_4c_5^2c_6^3 + c_1^2c_2c_4c_6^4$$
$$-2c_2^2c_4c_6^4 - c_1c_3c_4c_6^4 + c_4^2c_6^4 - c_1^3c_5c_6^4 + c_1c_2c_5c_6^4$$
$$+3c_3c_5c_6^4 + c_1^2c_6^5$$

$$d_9 = -c_4^3c_6^4 - c_1c_2c_4c_5c_6^4 + 3c_3c_4c_5c_6^4 + c_1^2c_5^2c_6^4 - c_2c_5^2c_6^4$$
$$-c_1^3c_3c_6^5 + 3c_1c_2c_3c_6^5 - 3c_3^2c_6^5 + c_1^2c_4c_6^5 - 3c_1c_5c_6^5$$
$$+2c_6^6$$

$$d_{10} = +(c_2c_4^2c_6^5) + c_1^2c_3c_5c_6^5 - 2c_2c_3c_5c_6^5 - 2c_1c_4c_5c_6^5 + 2c_5^2c_6^5$$
$$+c_1^4c_6^6 - 4c_1^2c_2c_6^6 + 2c_2^2c_6^6 + 3c_1c_3c_6^6 - 2c_4c_6^6$$

$$d_{11} = -c_1c_3c_4c_6^6 + c_4^2c_6^6 - c_1^3c_5c_6^6 + 3c_1c_2c_5c_6^6 - c_3c_5c_6^6$$
$$+c_1^2c_6^7 - 2c_2c_6^7$$

$$d_{12} = +(c_3^2c_6^7) + c_1^2c_4c_6^7 - 2c_2c_4c_6^7 - c_1c_5c_6^7 + c_6^8$$

$$d_{13} = -c_1c_3c_6^8 + c_4c_6^8$$

$$d_{14} = +(c_2c_6^9)$$

$$d_{15} = -c_6^{10}$$

For $\wedge^5(A)$:

$$d_1 = c_5$$

$$d_2 = c_4c_6$$

$$d_3 = c_3c_6^2$$

6×6 matrix cont.

For $\wedge^5(A)$:

$$d_4 = c_2 c_6^3$$

$$d_5 = c_1 c_6^4$$

$$d_6 = c_6^5$$

For $\wedge^6(A)$:

$$d_1 = c_6$$

APPENDIX C

d_k DEPENDENCE

As mentioned in Section II D, one can find the d_k's by looking at the characteristic polynomial of the p^{th} wedge of the companion matrix for a given polynomial. Appendix B specifies how the c_k's and d_k's are related. But how are the d_k's related to each other? Although the c_k's are algebraically independent, the d_k's are not. Below is code from Mathematica which uses the Gröbner basis tool to calculate the d_k relations. The Gröbner basis tool was applied to the equations from Appendix B using the second wedge of a 4×4 matrix. Recall from Appendix B, that the second order wedge of the 4×4 matrix has six d_k's. In the ideal generated by this tool, there are five polynomials that only involve the six d_k's. By setting these polynomials equal to zero and simplifying we get five useful d_k relations.

```
GroebnerBasis[{d1 + c2,
               d2 + c4 - c1 * c3,
               d3 + c4 * c1^2 + c3^2 - 2c2 * c4,
               d4 + c4^2 - c1 * c3 * c4,
               d5 + c2 * c4^2,
               d6 - c4^3},
               {c1, c2, c3, c4, d1, d2, d3, d4, d5, d6}]
```

The output to this command is a Gröbner basis. The first five polynomials in the basis, however, depend only on the d_k's. They are as follows:

$$
\begin{aligned}
&d2^3 d6 - d4^3, \\
&d1 d4^2 - d2^2 d5, \\
&d1 d2 d6 - d4 d5, \\
&d1^2 d4 d6 - d2 d5^2, \\
&d1^3 d6^2 - d5^3.
\end{aligned}
$$

Setting each of these equations equal to zero and reducing them provides a way of examining the relations between the d_k's. In fact, for this example, it turns out that the equations reduce to the following:

$$
\begin{aligned}
d5 &= d1 * d6^{1/3} \\
d4 &= d2 * d6^{2/3}.
\end{aligned}
$$

Hence, this gives us an easier way to calculate the d_k's.

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