WEAPON SYSTEMS EFFECTIVENESS AND MINIMUM COST FOR BALLISTIC MISSILE DEFENSE ALTERNATIVES

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1. Statement of Problem

The use of SCUD missiles by the Iraqis in the 1991 Gulf War signaled the emergence of a new threat against which current U.S. defenses are limited. One message from the Gulf War is that defending ports, strategic off-load air fields, marshaling areas, and population centers against Theater Ballistic Missiles (TBM) will be of mounting concern in future conflicts. The Ballistic Missile Defense Organization (BMDO), in conjunction with the military services, is currently evaluating various Theater Ballistic Missile Defense (TBMD) systems to defend critical friendly assets (called targets in this paper) against current and projected short range, medium range, and long range TBMs. The cost for defending these assets depends on the number and type of threat TBMs that emerge, and the mix of defensive missiles arrayed against them.

There are several problems associated with designing the most cost-effective mix of TBMD systems. First, the time needed to deploy all systems relative to the warning time available is uncertain. Second, even if enemy TBM inventories are known (and there is some uncertainty here, especially in projected forces), the targeting strategy may be unknown. We don't know, for example, precisely what mix of short, medium, and long range TBMs will be used against a particular target. How then, in the face of such uncertainties, can we plan to acquire reasonable defenses in the most cost-effective way?

In this paper we develop the first steps of an approach to determine the TBMD architecture that minimizes cost or maximizes effectiveness. This approach is based on a statistical theory of enemy attack and TBMD strategies, and employs the general formalism used earlier by one of us (Kohlberg 1980) [1] for a very different problem in nuclear C^3 survivability. Addressing the issue of TBMD cost-effectiveness is particularly relevant since near-term decisions must be made regarding the choice of different systems against the threat(s).

It is useful to consider the nature of the threat. A summary of the threat is shown in Table I. Each of these missile types may have various types of warheads. In the simplest case, these warheads would be unitary high explosives (HE), while in more sophisticated cases some could use submunitions, and weapons of mass destruction (WMD): nuclear, chemical, or biological weapons. The importance of destroying a missile will be a function of its warhead type. Thus, the formalism to be described in this paper permits assigning values to TBMs according to the likelihood of the use of WMDs.

Table I. Illustrative TBM Threat

Type	Apogee (km)	Range (km)
short - range	below 35	less than 150 km
medium - range	between 35 and 125	between 150 and 500 km
long - range	above 125	over 500 km

To counter the threat array shown in Table I the Army and Navy each have proposed a set of lower tier and upper tier defensive systems. The Army's lower tier system is called PAC-3 and its upper tier system is called Theater High Altitude Area Defense (THAAD); the Navy's lower and upper tier systems are called Navy Area Defense (NAD) and Navy Theater-Wide (NTW) Defense, respectively. Table II shows the types of threat missiles the different systems are optimized to interdict.

The TBMD systems are designed for different purposes. The PAC-3 and NAD systems defend the region within a few tens of kilometers around a single target against short- and medium-range missiles, while the THAAD or NTW could protect multiple assets over a much larger region from medium- or long-range TBMs.

Table II. Engagement Capabilities of Defensive Systems Compared with Threat

TBMD System	Threat Range
PAC-3	short, medium
NAD	short, medium
THAAD	medium, long
NTW	medium, long

There are an enormous number of possible scenarios that could be proposed when we take into account: the mixture of TBMs against a selected set of targets, warning time for TBMD deployment, time for the TBMs to reach the intended targets, detection points and times for incoming TBMs, firing rates of missiles (offensive and defense), battle management concepts of operations, probability of kill for each pair of TBM/TBMD, etc. Each possible scenario will lead to a different outcome in effectiveness.

Cost is usually measured as the life cycle cost: the remaining research and development (R&D) and other start-up costs, the costs for procuring all the missiles in the inventory, and costs associated with 20 years of operations and maintenance. Table III shows the approximate inventory objectives of the four systems just mentioned and the approximate 20-year life cycle cost and average cost per missile.

Table III. Illustrative Inventory and Life Cycle Cost

TBMD System	Objective Inventory	20-year Life Cycle Cost (Constant FY 96 $B)	Per Missile Cost ($ M)
PAC-3	1200	6	2
NAD	1500	6	2
THAAD	1300	15	6
NTW	600	7	6

To formulate the minimum cost algorithm in Section 2, we need the analytical relationship between the life cycle cost for each system and the number of missiles produced. If "i" denotes a member of the set of 4 defensive systems (i = 1, 2, 3, 4), the cost, C_i, will be a function of the total number, N_i, of missiles of that type produced;

$$C_i = C_i(N_i). \qquad (1.1)$$

C_i increases as N_i increases, but we expect the cost per missile,

$$\alpha_i = \partial C_i / \partial N_i \quad , \qquad (1.2)$$

to decrease along a production learning curve as N_i increases.

2. Theoretical Approach for TBMD

Starting with scenario-generated results of the outcome of engagement analysis, we show: (1) how to minimize the system cost to achieve a desired outcome, or (2) how to maximize the effectiveness of a complex of missile systems if the overall cost is fixed. The treatment in this paper is general and follows the theoretical methodology [1] used in other optimization analyses. It is beyond our purpose to recommend at this time the most desirable mixture of TBMD systems for specific situations without additional computations based on the theoretical approach developed here.

The approach places no *a prori* restrictions on the outcome of individual engagements, and can automatically decide whether a specific defensive system should be completely deleted from the inventory (that is, it can answer the question as to whether any one or more of the four systems identified in Table II should not be built).

As indicated in Section 1, there are a large number of variables in this problem which can all be incorporated into the mathematical framework. However, for the purpose of explaining the approach we shall limit the explicit notation to the threat, targets, and defensive systems. The following variables are now identified:

$n_i^{(j)}$ = number of defensive missiles of type "i" allocated to defend target j (where i = 1 to 4; for example, PAC-3 corresponds to i = 1),

$m_k^{(j)}$ = number of threat missiles of type "k" impinging on target j (k = 1 to 3; for example, k = 1 corresponds to short range missiles),

L = total number of targets (e.g., defended assets),

$N_i = \sum_{j=1}^{L} n_i^{(j)}$ = total number of type i defensive missiles required for defense of L targets.

Using these definitions it is possible to construct a scenario for target j consisting of $m_1^{(j)}$, $m_2^{(j)}$, $m_3^{(j)}$ threat missiles and $n_1^{(j)}$, $n_2^{(j)}$, $n_3^{(j)}$, $n_4^{(j)}$ defensive missiles. For brevity, we now introduce the set notation

$$\tilde{m}^{(j)} = \left\{ m_1^{(j)}, m_2^{(j)}, m_3^{(j)} \right\} \quad , (2.1)$$

$$\tilde{n}^{(j)} = \left\{ n_1^{(j)}, n_2^{(j)}, n_3^{(j)}, n_4^{(j)} \right\} \quad . (2.2)$$

The total number of TBMs fired at the jth target is

$$M^{(j)} = \sum_{k=1}^{3} m_k^{(j)}. \qquad (2.3)$$

Once the details of the engagement are defined (this will include the specifics of the incoming trajectory, probabilities of identification/discrimination, probabilities of kill for each offensive/defensive pair, firing rates, etc.) we can compute the number of offensive missiles getting through. For simplicity in this paper we make no distinction in effectiveness between short, intermediate, or long range TBMs. Accordingly we define

$$P^{(j)}\left(\tilde{m}^{(j)}, \tilde{n}^{(j)}\right)$$

as the scenario kill probability for $M^{(j)}$ missiles fired at the jth target in a composite of $\tilde{m}^{(j)}$, $\tilde{n}^{(j)}$ sets of offensive and defensive missiles. The number of missiles that get through at the jth target is

$$M_T^{(j)} = M^{(j)}\left(1 - P^{(j)}\left(\tilde{m}^{(j)}, \tilde{n}^{(j)}\right)\right). \quad (2.4)$$

It is readily appreciated that the determination of $P^{(j)}\left(\tilde{m}^{(j)}, \tilde{n}^{(j)}\right)$ in a realistic scenario – that is, one that involves numerous incoming missiles and where intercepts may occur at different altitudes using different defensive systems, is extremely difficult to determine analytically, and invariably involves real-time simulation modeling. The explicit dependence of $P^{(j)}\left(\tilde{m}^{(j)}, \tilde{n}^{(j)}\right)$ on the index "j" indicates that the results may depend on the detailed nature of the target site itself. Otherwise $P^{(j)}\left(\tilde{m}^{(j)}, \tilde{n}^{(j)}\right)$ would be independent of j, and $M_T^{(j)}$ would be given by the equation

$$M_{TS}^{(j)} = M^{(j)}\left(1 - P\left(\tilde{m}, \tilde{n}\right)\right), (2.5)$$

where $P(\tilde{m},\tilde{n})$ is now the site-independent kill probability for the sets $\tilde{m},\tilde{n}$.

$P^{(j)}\left(\tilde{m}^{(j)}, \tilde{n}^{(j)}\right)$ has the properties

$$P^{(j)}\left(\phi, \tilde{n}^{(j)}\right) = 1 \quad , \qquad (a)$$

$$P^{(j)}\left(\tilde{m}^{(j)}, \phi\right) = 0 \quad , \qquad (b) \qquad (2.6)$$

where ϕ is the null set. Beyond Eq. (2.6) it may not be especially meaningful to render bounding statements since they will depend quite strongly on the firing doctrine employed in the scenario. For example, in one case, we may find that a maximum of only two anti-missiles can be fired against a single threat missile, while in cases involving targeting by independent batteries more than two defensive missiles can be used against a single incoming threat. Essentially, we must rely on the information supplied by scenario calculation to provide the function $P^{(j)}\left(\tilde{m}^{(j)}, \tilde{n}^{(j)}\right)$.

For the purposes of this discussion it is not necessary to become involved in the detailed description of $P^{(j)}\left(\tilde{m}^{(j)}, \tilde{n}^{(j)}\right)$, but simply to recognize that such a function can be computed for arbitrary $\tilde{m}^{(j)},\tilde{n}^{(j)}$. The practical aspects of the problem enter when one realizes that not all mathematical possibilities can be realized in the solution due to inherent constraints on both the availability of large numbers of TBMD missiles distributed over a large number of targets, and the intrinsic probability of TBMD/TBM kill.

For example, if the intrinsic probability of kill, p, were the same for all TBMD/TBM engagements, and if no more than two attempts could be made against an individual threat missile, the effective kill probability would be

$$P_k = p + (1-p)p = p(2-p) \quad . \qquad (2.7)$$

Using Eq. (2.7) and assuming statistical independence of targeting, the probability of knocking out all $M^{(j)}$ threat missiles would be

$$P_{M^{(j)}} = \left(p(2-p)\right)^{M^{(j)}} \quad . \qquad (2.8)$$

Equation (2.8) shows that no amount of TBMD missiles allocated to protect target j could destroy all $M^{(j)}$ incoming missiles with a probability exceeding $P_{M^{(j)}}$.

For the example just given, the maximum number of TBMD missiles that could be used in the scenario is $2M^{(j)}$, while the average number that would be used is

$$N^{(j)} = M^{(j)} n_k \quad , \qquad (2.9)$$

where n_k is the average number of defense missiles fired at a TBM; it is

$$n_k = 1 + (1-p) = (2-p) \quad . \qquad (2.10)$$

By recognizing the inherent limitations of certain situations one can avoid seeking optimum solutions with unrealistic constraints. Eventually, the user of the optimization algorithm to be described would come to the realization of what is feasible, but simple calculations of the type just rendered limit the search space required at the outset.

Suppose that we have conducted analyses for a sufficient number of scenarios so that we have obtained a suitable analytical representation of $P^{(j)}\left(\tilde{m}^{(j)}, \tilde{n}^{(j)}\right)$. (We assume in some of the formalism to follow that the elements of the set $\tilde{n}^{(j)}$ are continuous, even though they actually are discrete integers. However, this is a detail that can be rectified in a numerical calculation.) The remaining tasks are to concurrently establish what we want to maximize or minimize within the constraints on the problem.

Initially, we fix the total cost C_T. We thus require:

$$C_T = \sum_{i=1}^{4} C_i(N_i) = C_o \quad , \qquad (2.11)$$

where C_o is a constant, and

$$N_i = \sum_{i=1}^{L} n_i^{(j)} \quad . \qquad (2.12)$$

Given the constraint of Eq. (2.11) we seek to optimize an objective function of the engagement. Let's suppose, for example, that each threat missile that gets through is equally important, and that the objective then becomes the minimization of M_T, the sum of all incoming missiles that get through (on the other hand, a different weighting scheme could be introduced here to preferentially deny penetration by WMDs or by any other type TBM). M_T is given by

$$M_T = \sum_{j=1}^{L} M_T^{(j)} = \sum_{j=1}^{L} M^{(j)}\left(1 - P^{(j)}\left(\tilde{m}^{(j)}, \tilde{n}^{(j)}\right)\right). \qquad (2.13)$$

Using the method of Lagrange multipliers [3], we minimize the functional

$$F = M_T - \lambda C_T \quad , \qquad (2.14)$$

where λ is a new variable, the Lagrange multiplier, to be determined by the conditions of the problem. Taking the variation δF gives

$$\delta F = -\sum_{j=1}^{L} M^{(j)} \delta P^{(j)} - \lambda \sum_{i=1}^{4} \delta C_i(N_i) \, , \qquad (2.15)$$

where

$$\delta P^{(j)} = \sum_{i=1}^{4} \frac{\partial P^{(j)}}{\partial n_i^{(j)}} \delta n_i^{(j)} \quad , \qquad (2.16)$$

$$\delta C_i(N_i) = \frac{\partial C_i}{\partial N_i} \delta N_i = \frac{\partial C_i}{\partial N_i} \sum_{j=1}^{L} \delta n_i^{(j)} \, . \qquad (2.17)$$

Substituting Eqs. (2.16) and (2.17) into Eq. (2.15), and setting $\delta F = 0$ for the minimum gives

$$\delta F = \sum_{j=1}^{L} \sum_{i=1}^{4} \left(M^{(j)} \frac{\partial P^{(j)}}{\partial n_i^{(j)}} + \lambda \frac{\partial C_i}{\partial N_i} \right) \delta n_i^{(j)} = 0 \, , \qquad (2.18)$$

Following standard Lagrange multiplier methods, the solution of Eq. (2.18) is achieved when the conditions

$$M^{(j)} \frac{\partial P^{(j)}}{\partial n_i^{(j)}} + \lambda \frac{\partial C_i}{\partial N_i} = 0 \qquad (2.19)$$

are satisfied for all i , j.

If there are four possible TBDM systems (see Table II) and, say, 20 targets, Eq. (2.19) provides the solution of 80 of the unknowns, the $n_i^{(j)}$, in terms of the 81st unknown, λ. That is, the solution of Eq. (2.19) is of the general form

$$n_i^{(j)} = f_i^{(j)}(\lambda), \qquad (2.20)$$

where $f_i^{(j)}$ is an analytic function of λ.

Substituting Eq. (2.20) into Eq. (2.12) gives

$$N_i = \sum_{j=1}^{L} n_i^{(j)} = \sum_{j=1}^{L} f_i^{(j)}(\lambda) = F_i(\lambda) \ , \qquad (2.21)$$

where $F_i(\lambda)$ is another known function of λ determined from the summation in Eq. (2.21). Inserting Eq. (2.21) into Eq. (2.11) gives

$$C_i(N_i) = C_i(F_i(\lambda)) \ , \qquad (2.22)$$

and ultimately

$$C_T = \sum_{i=1}^{4} C_i(F_i(\lambda)) = C_T(\lambda) \ . \qquad (2.23)$$

Since the cost, C_T, is assumed fixed, Eq. (2.23) can be inverted to provide

$$C_o = C_T(\lambda^*) \ , \qquad (a)$$
$$\lambda^* = \lambda^*(C_o) \ , \qquad (b) \qquad (2.24)$$

where λ^* is the value of λ that satisfies the equalities in Eq. (2.24).

Using Eq. (2.24) in Eq. (3.20) gives the unique set of values, $\hat{n}_i^{(j)}$, that corresponds to the fixed cost, C_o

$$\hat{n}_i^{(j)} = f_i^{(j)}\left(\lambda^*(C_o)\right). \qquad (2.25)$$

When the mathematical calculations are carried out correctly we will be guaranteed to have the minimum number of penetrating missiles achieved at the designated cost. This is given by

$$M_{T,\min} = \sum_{j=1}^{L} M^{(j)}\left(1 - P^{(j)}\left(\tilde{m}^{(j)}, \ \tilde{\hat{n}}^{(j)}\right)\right) \qquad (2.26)$$

where $\tilde{\hat{n}}$ is the set of $\hat{n}_i^{(j)}$.

The analysis presented between Eqs. (2.14) to (2.26) is a straightforward application of Lagrange multiplier theory. The solution represents the most unrestricted case since it only involves a single constraint, namely that given by the fixed cost, although we introduce restrictions in the next paragraph. When applied to the TBMD domain it also allows for an arrangement of $\hat{n}^{(j)}$ without explicit consideration of whether the defensive missiles will have any difficulty in being available to defend the targets. All targets have implicitly been considered to be equally important since no *a priori* consideration has been given to selected threats.

Constrained solutions are the more relevant ones in TBMD. Realistic applications place restrictions or limitations on the availability of defensive missiles, because of logistics, deployment constraints, limitations on the location of defensive batteries in the theater, and other factors. For example, one may envision a constraint that limits the maximum number of missiles of a certain type that may be allocated to any site. This would lead to the mathematical restrictions,

$$n_i^{(j)} \le N_{i,\max}^{(j)} \ . \qquad (2.27)$$

Another possibility of the same type would be the constraint

$$\sum_{i=1}^{4} n_i^{(j)} \le N_{\max}^{(j)} \ , \qquad (2.28)$$

a constraint that restricts the total number of defensive systems available to defend a particular target. Other availability constraints can be invoked.

In addition to the constraints of Eqs. (2.27) and (2.28) one may attach relative importance to selected targets, such as population centers, or critical military facilities. Those would be expressed in the form

$$M^{(j)} \le M_{\max}^{(j)}. \quad (2.29)$$

Since the Lagrange multiplier solution for the unrestricted case is the optimum unconstrained defense, additional constraints are guaranteed to lead to a larger number of missiles that get through, when summed over all targets. Solution of any of these constrained inequality problems requires an optimization routine [e.g. **2**].

There is one additional consideration that should be addressed; all the results for the $\tilde{n}^{(j)}$ and the related quantities that depend on $\tilde{n}^{(j)}$ were computed for a specified threat laydown. That is, the solution of $\tilde{n}^{(j)}$ determined from Eq. (2.19) contains the elements of $\tilde{m}^{(j)}$. If the threat specific specification should change, the method of solution would be the same but the $\tilde{n}^{(j)}$ would be different, although also derived from a minimum cost principle. Thus, the final results for any analysis of this type should be averaged over the probability distribution for the threat.

In contrast to the foregoing analysis, which demonstrates one particular way that a TBMD system can be deduced for a fixed cost, we can also use the formalism for finding the minimum cost, $C_{\min}$, to achieve a required level of effectiveness. For illustrative purposes let us consider the "flip side" of the previous problem in which we impose the fixed system requirement

$$M_T = \sum_{j=1}^{L} M^{(j)}\left(1 - P^{(j)}\left(\tilde{m}^{(j)}, \tilde{n}^{(j)}\right)\right) = M_o\ , \quad (2.30)$$

and seek to minimize the cost

$$C_T = \sum_{i=1}^{4} C_i(N_i) \quad (2.31)$$

associated with the constraint M_o.

The function to be minimized is now

$$G = C_T - \Omega M_T \quad , \quad (2.32)$$

where Ω is the new Lagrange multiplier. Taking the variational of Eq. (2.32) gives

$$\delta G = \sum_{j=1}^{L} \sum_{i=1}^{4} \left(\frac{\partial C_i}{\partial N_i} + \Omega M^{(j)} \frac{\partial P^{(j)}}{\partial n_i^{(j)}} \right) \delta n_i^{(j)} = 0 . \quad (2.33)$$

The foregoing equation is identical to Eq. (2.18) with the replacement of λ by $(1/\Omega)$. Hence the solution for this problem is of the form

$$n_i^{(j)} = f_i^{(j)}\left(\frac{1}{\Omega}\right) \quad , \quad (2.34)$$

where the parameter Ω is determined from the equation

$$\sum_{j=1}^{M} M^{(j)}\left(1 - P^{(j)}\left(\tilde{m}^{(j)}, \tilde{n}^{(j)}(1/\Omega)\right)\right) = M_o . \quad (2.35)$$

Once Ω is determined, the particular values of $n_i^{(j)}$ are determined from Eq. (2.34), and the corresponding minimum cost is computed from Eqs. (2.11) and (2.12), analogous to the previous procedure.

The two similar examples just rendered have been presented to demonstrate the concept. In practice, we would most likely anticipate the introduction of several inequality constraints.

3. Sample Calculations

We have not yet applied the formalism in this paper to an actual problem. The purpose of this paper is to establish the foundations for such applications. For insight into some issues concerning the use of the methodology developed, we have analyzed three situations, each of which has a different result.

For simplicity, assume that there are only two TBMD systems under consideration, called A and B. These might be the PAC-3 and NAD systems. Moreover, we assume that these two systems have the same probability-of-kill against all TBMs that would be aimed at the targets they protect.

Where they differ is: (1) availability in the theater at the time attacks begin, (2) cost, and (3) access to and coverage of the region immediately surrounding some of the targets. Availability in the theater reflects the possibility that some types of TBMD systems are more readily transported to a given theater than another. In the absence of prepositioning or early deployments, land-based TBMD defenses might not be as available in the numbers needed for full protection of targets as would ship-based TBMD systems. Cost differences in acquisition and O&S are to be expected between two different TBMD systems. And, despite availability in the theater, one of the TBMD systems might be limited by location from covering targets the other could protect. For example, short-range ship-board TBMD systems may not be able to protect targets deep inland.

We examine three different situations. For illustration purposes, we use the linear relationship between cost and number of missiles:

$$C_i = C_{oi} + \alpha_i N_i \quad , \qquad (3.1)$$

where C_{oi} is a fixed cost and α_i is a constant. For example, application of Eq. (3.1) to Table III gives C_o = \$3.6 B and α = \$2.0 M for PAC-3. For NAD, C_o = \$3.0B and α = \$2.0 M.

These three different cases we examine are summarized in Table IV, and characterized more completely in the related text. In all cases a total of M TBMs are fired in defense of the complex of theater targets.

Table IV. Illustrative Cases for TBMD Solutions

	Case I	Case II	Case III
Avail-ability	A, B both unconstrained	A, B both unconstrained	A constrained to N_A*; B unconstrained
Cost	$C_{oA} > C_{oB}$; $\alpha_A = \alpha_B = \alpha$	$C_{oA} > C_{oB}$; $\alpha_A = \alpha_B = \alpha$	$C_{oA} > C_{oB}$; $\alpha_A = \alpha_B = \alpha$
Coverage of Targets	A, B can defend all targets	A can defend all; B can defend 50%	A can defend all; B can defend 50%

For simplicity, we assume that the defenses fire 2 TBMD missiles against each incoming TBM. Greater complexity can be added, as in Eq. (2.7), but at the expense of clarity. Thus, a total of $2M$ TBMD missiles must be in the theater. The question is: how many of

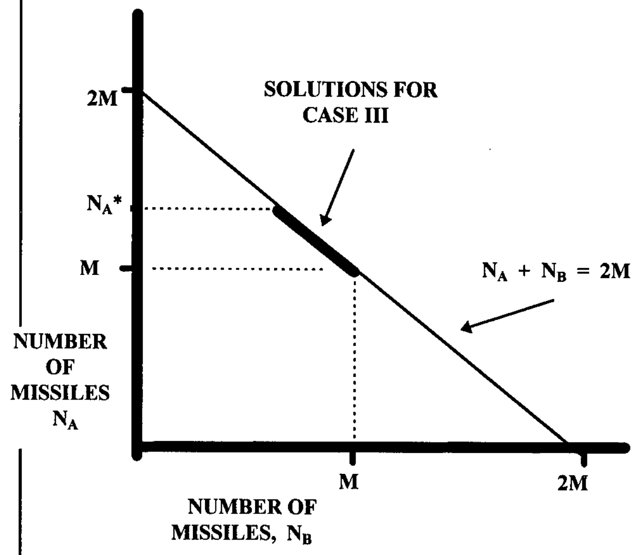

Figure 1. Constraint Relationships for N_A and N_B.

these $2M$ missiles are A-type missiles and how many are B-type missiles? The examples are chosen so that there are different answers to the same question, depending on the constraints.

An understanding of the solution is facilitated and summarized with the aid of Figure 1. The ordinate is the number of missiles of type A, and the abscissa is the number of type B missiles.

The negatively sloped line connecting the ordinate and the abscissa in Figure 1 is the locus of all points for which $N_A + N_B = 2M$. The solution to any of the three cases lies somewhere along this line (or to the right), the exact point or points being determined by the specific constraints.

There are three general solutions to the cases summarized in Table IV, subject to the equality constraint

$$N_A + N_B = 2M. \qquad (3.2)$$

They are

$$C_T = C_{0A} + \alpha N_A = C_{0A} + \alpha(2M) \quad , \qquad (3.3a)$$
$$C_T = C_{0B} + \alpha N_B = C_{0B} + \alpha(2M) \quad , \qquad (3.3b)$$

$$C_T = C_{0A} + C_{0B} + \alpha(N_A + N_B) = C_{0A} + C_{0B} + \alpha(2M) \quad . \qquad (3.3c)$$

Case I. Reference to Table IV shows that this corresponds to a totally unconstrained solution to the TBMD question. Both systems A and B can be in the theater to the extent required. Both are equally effective, but the costs are different. And both systems can be located anywhere within the theater to protect all targets. Thus either alone could perform to meet defense requirements. *The least cost solution in Case I is to buy only one system, the lower cost system B.* This solution corresponds to Eq. (3.3b).

Case II. One of the conditions in Case I is constrained in this case. See Table IV for the details. Here both A and B are available in any numbers required (as in Case I), except that B can be positioned, no matter how numerous in the theater, to protect only 50 percent of the targets. Thus the solution in Case I, while the least costly, fails to meet the requirement that all targets be protected with 2M missiles. The possible solutions are reduced to either Eq. (3.3a) or Eq.(3.3c). An examination of the costs in Table III shows that the least-cost solution here is to buy only one system, but this time system A. Even though A is more costly than B to develop and bring to the point of production (i.e. $C_{oA} > C_{oB}$) , the cost of a single system that protects all targets is lower than a mix of A and B, due to the relatively large R&D costs associated with acquiring a second system. *Thus, the architecture that meets requirements at lowest cost in Case II is the single system A, even though its cost is higher than that of B.* Its cost is given by Eq. (3.3a).

Case III. This is a doubly constrained case, even more constrained than Case II. In Case III, as Table IV shows, only B can be in the theater in unconstrained numbers. System A is available up to a limit, a limit that prevents full protection of all targets to the high level required. This limit might be a force size constraint (there is a maximum number of missile batteries in the force) or to strategic deployment constraints (inadequate time and lift assets available). Thus A alone fails to meet the specified requirement. On the other hand, even though B can be available in unlimited numbers, it is constrained as in Case II to defending only 50 percent of the targets. System B alone also fails to meet the defensive requirement. Thus, both A and B are constrained in some fashion in this case. The general solution would therefore be a mixed solution and lie somewhere on the darkened portion of the line in Figure 1.

The expected solution here is more complex than for the other cases analyzed. We look separately at three different domains of N_A*: (i) $N_A^* \leq M$, (ii) $M \leq N_A^* \leq 2M$, and (iii) $2M \leq N_A^*$. The special conditions on A arise from the fact that B can defend only up to one-half the targets by expending M missiles.

(i) $N_A^* \leq M$. If there are this few A-type missiles available, Figure 1 shows that there is no solution that meets the requirements of engaging every incoming TBM with 2 missiles. *So, if the constraints are too severe, there is no solution at any cost that meets the requirements.* New requirements would be needed to proceed.

(ii) $M \leq N_A^* \leq 2M$. In this intermediate case, both A and B together can defend all targets. There exists a least cost solution that meets the requirements, and it is given by Eq. (3.3c). While the minimum cost is determined uniquely, the appropriate mix of A and B is not, since A and B missiles cost exactly the same to produce (by assumption). The architecture with equal numbers of A and B missiles (i.e. $N_A = N_B = M$) is just as effective as the one with N_A^* type-A missiles and $(2M - N_A^*)$ type-B missiles. Graphically, all the points on the dark line in Figure 1 are equal cost and equal effectiveness solutions. *So, for some conditions, there may be many solutions that are equally cost-effective.*

(iii) $2M \leq N_A^*$. Finally, we examine a case outside the boundaries. If N_A^* had actually exceeded $2M$, this case is identical with Case II (except for the number of missiles), and the single system A architecture becomes preferred. So this condition is not fundamentally different from one already analyzed. This observation makes us realize that there is a discontinuity in the solution and the cost at $N_A^* = 2M$. When N_A^* is slightly larger than $2M$, only a single system (system A) is the most cost-effective. When N_A^* is slightly less than $2M$, the combined A and B system architecture, with a considerably larger total cost, is the most cost-effective. Here we need to decide whether the requirement is more important than the cost. This points to the need for probing for discontinuity points by small excursions around the optimum solution point when solving much more complex problems by numerical means, ones in which the results cannot easily be guessed beforehand. *Points of non-analytic behavior with discontinuities in system choices and costs can occur.*

4. Conclusions

In summary, we have presented the basic theory, using Lagrange multiplier techniques, for estimating the most cost-effective TBMD architecture. We have shown by example how inequality constraints can greatly influence the solution. Lowest cost, highest performance, selective defense of designated targets, prioritized defense against probable WMD warheads, or any other constraint criterion can be introduced within its framework. This approach was adapted from one developed in another field (nuclear C^3 survivability) where it has been used with great success for a number of years.

The approach to finding the most cost-effective architecture is straightforward, although demanding in its attention to detail and ultimate extensive use of computer resources for realistic cases. Extensions of the methodology to go beyond fixed assumptions (such as target selection by enemy TBMs) and to include distributions of the critical parameters in the theory are contemplated, but are not included in the present version of the theory. We have avoided these extensions in this paper, the purpose of which is to introduce the main outlines of the theory with a minimum of embellishments, rather than to develop it in its entirety.

We have not yet applied the full theory to finding the most cost-effective mix of TBMD systems for a realistic scenario. But we have examined several simple cases in order to anticipate some of the difficulties to be expected in such an application. Despite the power of the approach outlined here, optimum solutions obtained from the formalism should be used with care. The solution found by an optimization algorithm may not be the only one of interest. Several observations on this are made next.

We have examined several simple cases that serve to illustrate in a transparent fashion how metastable the least cost solution can be. Much depends on the constraints imposed on the problem. We have selected here for illustration the availability, R&D cost, and theater coverage, because they are realistic variables that would constrain and possibly dominate any more complex treatments. These would probably continue to exert powerful influences on any TBMD acquisition problem, no matter how many other variables were introduced.

The extreme case is the situation in Case III in which there are numerous equally cost-effective solutions, only one of which will probably be found by a numerical search algorithm. The exact solution found would depend on where in its initial guess the search algorithm began its search. In a more realistic case with many more variables and constraints, there may reside a range of solutions that differ insignificantly from one another in cost and effectiveness, but which are hard to discover by reasonable numerical means. Criteria other than the ones in the model, to include subjective and political ones, could then be brought to bear on selecting the "best" of the otherwise "equal" solutions.

On the other hand, the solution found may be near one or more discontinuity points, such as the one also discussed in Case III in this paper. This type of solution, if its metastability is discovered, would probably require a re-examination of the conditions that initially constrained the solution. A small change in requirements could result in a dramatic change in costs for the least-cost solution that meets all the criteria, an area worth exploring.

Finally, the use of simplified examples, such as those in Section 3, can serve to help us choose good first guesses in starting the search algorithms for the more general and more complex constrained realistic cases. Because of the large volume of space in which the search is to take place, a good starting point is crucial to realistic computer search times and convergence criteria, as noted in the examples studied.

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