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ANOTHER INVERTIBLE PIECEWISE GREULING SOLUTION TO THE STRAIGHT AHEAD TRANSPORT EQUATION

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1. Introduction

We consider the straight ahead transport equation [1]

$$\frac{\partial F(x,E)}{\partial x} + \mu(E)F = \int_E^{E_0} K(t,E)F(x,t)dt + K(E_0,E)e^{-\mu(E_0)x} \qquad (1.1)$$

$$F(0,E) = 0,$$

under the hypothesis of separability, i.e.

$$K(t,E) = G(t)H(E).$$

We assume in addition that the interval $(0,E_0)$ can be partitioned into a finite number of subintervals $(E_{n+1} < E < E_n)$ $(n = 0,1,\ldots,N)$ such that on each of these intervals $K(E,E)$ may be closely approximated by some constant multiple of $d\mu/dE$, i.e.

$$K(E,E) = \alpha_n \frac{d\mu}{dE} \qquad (E_n < E < E_{n-1}).$$

In an earlier report [1] we discussed the solution of Eq. (1.1) under the following circumstances:

(i) α_n is a finite integer $(n = 1,\ldots,N+1)$.

(ii) α_n is a finite integer $(n = 1,\ldots,N+1, n\neq p)$, $\alpha_p = \infty$, $0 < \alpha_1 < \ldots < \alpha_{p-1}$, $\alpha_{p+1} < \ldots < \alpha_{N+1} < 0$.

(iii) $N = 1$, $\alpha_1 = 1$, α_2 arbitrary.

(iv) $N = 1$, $\alpha_2 = -1$, α_1 arbitrary.

(v) α_n is a finite integer $(n=1,\ldots,N)$, $\alpha_{N+1} = -\frac{1}{2}$,

$0 < \alpha_1 < \ldots < \alpha_N$.

The solutions in all these cases are expressible in terms of polynomials, exponentials, Bessel functions and confluent hypergeometric functions with second argument 2.

We shall discuss here the following case, which generalizes case (v):

(vi) α_n is a finite integer $(n=1,\ldots,N)$, $\alpha_{N+1} = -\frac{1}{2}$,

$0 < \alpha_1 < \ldots < \alpha_p$, $\alpha_{p+1} < \ldots < \alpha_N < 0$.

When $p \neq N$, the solution will involve the function

$$G(k,v) = \int_0^v e^{-u}\, I_o(ku)du \tag{1.2}$$

used and tabulated by Rice [2].

2. <u>Properties of the Function</u> G(k,v)

Since the function G(k,v) is relatively new in analysis, it is advisable to record some of its properties. Some of these properties will be useful in the next section where we actually solve Eq. (1.1).

Let us define

$$H(k,v) = k\, G(k,v), \qquad \delta = k^2 \frac{\partial}{\partial k}\,.$$

Then it is true that

$$\int_0^v u^m e^{-u}\, I_o(ku)du = \frac{1}{k^{m+1}}\left[\delta^m\, H(k,v) - ve^{-v} \sum_{p=o}^{m-1} \delta^p \left\{k^{m+1-p}\, I_o(kv)\right\} v^{m-1-p}\right]. \tag{2.1}$$

The proof of this identity can be made quite simply by a mathematical induction on m if one replaces u by t/k and multiplies by k^{m+1} before applying the operator $\mathcal{D}$.

If we differentiate Eq. (1.2) with respect to k and integrate by parts twice we can show that G(k,v) satisfies the differential equation

$$(k^2-1)\,\frac{\partial G}{\partial k} = -kG + ve^{-v}\left\{k\,I_0(kv) + I_1(kv)\right\}. \tag{2.2}$$

In particular, when k = 1 we have that

$$G(1,v) = ve^{-v}\left\{I_0(v) + I_1(v)\right\}, \tag{2.3}$$

a result first noticed by Bennett [3]. Since it is obvious that

$$G(0,v) = 1 - e^{-v}, \tag{2.4}$$

it follows from Eq. (2.2) that

$$\sqrt{1-k^2}\;G = 1 - e^{-v} - ve^{-v}\int_0^k \frac{t\,I_0(tv) + I_1(tv)}{\sqrt{1-t^2}}\,dt \tag{2.5}$$

$$= ve^{-v}\int_k^1 \frac{t\,I_0(tv) + I_1(tv)}{\sqrt{1-t^2}}\,dt.$$

If we let v approach $+\infty$, we see that [4, p. 384]

$$G(k,\infty) = 1\Big/\sqrt{1-k^2} \tag{2.6}$$

when $k^2 < 1$.

It follows from repeated differentiations of Eq. (2.2) that the integral on the left of Eq. (2.1) can be expressed in terms of

G, polynomials in v and k, e^{-v} and Bessel functions of kv when $k^2 < 1$. In particular,

$$(1-k^2) \int_0^v ue^{-u} I_o(ku)du = G(k,v) - ve^{-v}\left\{I_o(kv) + kI_1(kv)\right\} \tag{2.7}$$

$$(1-k^2)^2 \int_0^v u^2e^{-u} I_o(ku)du = (2+k^2)G(k,v) - ve^{-v}\left[\left\{2+k^2+(1-k^2)v\right\}I_o(kv) + \left\{3k + (1-k^2)kv\right\} I_1(kv)\right] . \tag{2.8}$$

The computation of G(k,v) for small values of k can be performed by expanding $I_o(ku)$ into a power series and integrating term by term. In this manner we find that [2, p. 44]

$$G(k,v) = \sum_{n=o}^{\infty} \frac{1}{(n!)^2} \left(\frac{k}{2}\right)^{2n} \int_0^v u^{2n}e^{-u}du = \sum_{n=o}^{\infty} \frac{(2n)!}{(n!)^2} \left(\frac{k}{2}\right)^{2n} A_n(v), \tag{2.9}$$

in which

$$A_n(v) = 1 - \left[1 + v + \frac{v^2}{2!} + \dots + \frac{v^{2n}}{(2n)!}\right]e^{-v} .$$

Since $0 < A_n(v) < 1$ when $0 \leqq v < \infty$, the convergence of this series is dominated, uniformly in v, by the convergence of the power series for $(1-k^2)^{-1/2}$.

When $1-k^2 \ll 1$, it is more convenient to expand $I_o(ku)$ into a power series in $1-k^2$ [4, p.142]

$$I_o(ku) = \sum_{m=o}^{\infty} \frac{(-1)^m(1-k^2)^m(\frac{1}{2}u)^m I_m(u)}{m!} ,$$

and integrate term by term. If we observe that

$$\int_0^v e^{-u} I_m(u) u^m du = \frac{e^{-v} v^{m+1}\left\{I_{m+1}(v) + I_m(v)\right\}}{2m+1},$$

(this can be proved by differentiating both sides with respect to v), we then see that

$$G(k,v) = ve^{-v} \sum_{m=0}^{\infty} \frac{\left\{(k^2-1)v/2\right\}^m \left\{I_{m+1}(v) + I_m(v)\right\}}{(2m+1)m!}. \tag{2.10}$$

If we make use of the inequalities [4, p. 49]

$$I_m(v) \leq \frac{(v/2)^m}{m!} e^v, \quad v^m e^{-v} \leq m^m e^{-m}$$

and the asymptotic behavior of the gamma function, we can show that the series for $e^{-v}G(k,v)$ converges uniformly in v and k when $|k^2-1| \leq 1 - \delta < 1$, the convergence being dominated by that of the series

$$\sum_{n=0}^{\infty} \frac{|k^2-1|^m}{m}.$$

Although the series (2.9) and (2.10) both converge uniformly in v, neither is entirely satisfactory for calculating values of G when v is large and k is near one. To derive an approximate formula for these conditions we notice that as a consequence of the asymptotic expansion for I_0 and Eq. (2.6) we have that

$$G(k,v) \sim \frac{1}{\sqrt{1-k^2}} - \int_v^{\infty} \frac{e^{-(1-k)u}}{(2\pi ku)^{1/2}} \sum_{m=0}^{\infty} \frac{(-1)^m (0,m)}{(2ku)^m} du$$

$$\sim \frac{1}{\sqrt{1-k^2}} - \sqrt{\frac{2}{\pi k(1-k)}} \int_{t_1}^{\infty} e^{-t^2} \sum_{m=0}^{\infty} \frac{(k-1)^m (0,m)}{(2kt^2)^m} dt,$$

in which

$$(0,m) = \frac{\Gamma(m+\frac{1}{2})}{m!\Gamma(\frac{1}{2}-m)}, \quad t_1 = \sqrt{(1-k)v} \quad .$$

The integral with integrand $e^{-t^2}t^{-2m}$ can be reduced after m integrations by parts to the error function integral. It turns out that sufficient accuracy for most purposes can be obtained by retaining only two terms in the sum and so we shall not carry out this reduction in detail. For the first two terms we have that

$$G(k,v) \sim \frac{1}{\sqrt{1-k^2}} - \sqrt{\frac{2}{\pi k(1-k)}} \left\{ \int_{t_1}^{\infty} e^{-t^2} dt + \left(\frac{1-k}{8k}\right) \int_{t_1}^{\infty} \frac{e^{-t^2} dt}{t^2} \right\}$$

$$\sim \frac{1}{\sqrt{1-k^2}} - \frac{1}{4\sqrt{k^3}} \left\{ \frac{(5k-1)\left[1-H(t_1)\right]}{\sqrt{2(1-k)}} + \frac{e^{-t_1^2}}{\sqrt{2\pi v}} \right\} , \tag{2.11}$$

in which $H(t_1)$ is the error function extensively tabulated elsewhere [5],

$$H(t_1) = \frac{2}{\sqrt{\pi}} \int_0^{t_1} e^{-t^2} dt.$$

3. Solution of the Transport Equation.

It is known [1] that if f(s,E) is the Laplace transform of F(x,E), then under the hypothesis of separability,

$$f(s,E) = \frac{K(E_o,E)}{(\mu_o+s)(\mu+s)} \exp \int_E^{E_o} \frac{K(E',E')dE'}{\mu(E')+s} .$$

If it is true that

$$K(E,E) = \alpha_n \frac{d\mu}{dE} \qquad (E_n < E < E_{n-1}),$$

then it is true that

$$f(s,E) = \frac{K(E_o,E)(\mu_o+s)^{\alpha_1-1}}{(\mu+s)^{\alpha_{n+1}}} \prod_{k=1}^{n-1} (\mu_k+s)^{\alpha_{k+1}-\alpha_k} \qquad (E_n < E < E_{n-1}).$$

Let us now suppose that the constants α_n are integers $(n=1,\ldots,N)$, that $\alpha_{N+1} = -\frac{1}{2}$, that $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_p$, and that $\alpha_{p+1} < \alpha_{p+2} < \ldots < \alpha_N < 0$. Then if $E > E_N$, $f(s,E)$ is a rational function of s, and so its inverse Laplace transform can be readily computed in terms of exponentials and polynomials. If $E < E_N$, then

$$f(s,E) = \frac{K(E_o,E)(\mu_o+s)^{\alpha_1-1} \prod_{k=1,\,k\neq p}^{N-1} (\mu_k+s)^{\alpha_{k+1}-\alpha_k} (\mu_N+s)^{-\alpha_N}}{(\mu+s)^{1/2}(\mu_N+s)^{1/2}(\mu_p+s)^{\alpha_p-\alpha_{p+1}}}$$

$$= \frac{K(E_o,E)P(s)}{(\mu+s)^{1/2}(\mu_N+s)^{1/2}(\mu_p+s)^{\alpha_p-\alpha_{p+1}}} \qquad (3.1)$$

in which $P(s)$ is a polynomial in s of degree $\alpha_p-\alpha_{p+1}-1$ since α_1-1, $\alpha_{k+1}-\alpha_k$ and $-\alpha_N$ are non-negative integers. The inverse Laplace transform of f can therefore be expressed in terms of the inverse Laplace transform of

$$\frac{1}{(\mu+s)^{1/2}(\mu_N+s)^{1/2}(\mu_p+s)} \tag{3.2}$$

and its derivatives with respect to μ_p. The inverse Laplace transform of (3.2) is [6, No. 219, No. 555]

$$e^{-\mu_p x}\int_0^x e^{\mu_p y}\, e^{-1/2(\mu+\mu_N)y}\, I_0\left\{\tfrac{1}{2}(\mu-\mu_N)y\right\}dy$$

$$= \frac{2e^{-\mu_p x}}{\mu+\mu_N-2\mu_p}\, G\left\{\frac{\mu-\mu_N}{\mu+\mu_N-2\mu_p},\ \tfrac{1}{2}\,(\mu+\mu_N-2\mu_p)x\right\}.$$

It follows that the solution $F(x,E)$ can be expressed in terms of the function G and its derivatives with respect to k, together with exponentials, Bessel functions and polynomials. In view of Eq. (2.2), it is not necessary to use derivatives of G with respect to k.

If $p = 0$, i.e. if $\alpha_1 < \alpha_2 < \cdots < \alpha_N < \alpha_{N+1} = -\frac{1}{2}$, the above analysis needs to be modified slightly. In this case we have, when $E < E_N$, that

$$f(s,E) = \frac{K(E_0,E)P(s)}{(\mu_0+s)^{1-\alpha_1}(\mu+s)^{1/2}(\mu_N+s)^{1/2}} ,$$

in which $P(s)$ is a polynomial in s of degree $-\alpha_1$. The formal manipulations to invert $f(s,E)$ are thus identical with those to invert (3.1) if we set $p = 0$, $\alpha_p = 1$.

We shall now apply this general analysis to a particular case in which the details are relatively simple.

Let us suppose that N = 1, p = 0. Then

$$\frac{f(s,E)}{K(E_o,E)} = \frac{(\mu_1+s)^{-\alpha_1}}{(\mu_o+s)^{1-\alpha_1}(\mu+s)^{1/2}(\mu_1+s)^{1/2}}$$

$$= \frac{1}{(\mu_o+s)(\mu+s)^{1/2}(\mu_1+s)^{1/2}} \sum_{r=o}^{-\alpha_1} \frac{(-\alpha_1)!}{r!(-\alpha_1-r)!} \left(\frac{\mu_1-\mu_o}{\mu_o+s}\right)^r$$

$$= \left\{ \sum_{r=o}^{-\alpha_1} \frac{(-\alpha_1)!(\mu_o-\mu_1)^r}{(r!)^2(-\alpha_1-r)!} \frac{\partial^r}{\partial u_o^r} \right\} \frac{1}{(\mu_o+s)(\mu+s)^{1/2}(\mu_1+s)^{1/2}} .$$

Therefore

$$\frac{F(x,E)}{K(E_o,E)} = \sum_{r=o}^{-\alpha_1} \frac{(-\alpha_1)!(\mu_o-\mu_1)^r}{(r!)^2(-\alpha_1-r)!} \frac{\partial^r}{\partial u_o^r} .$$

$$. \int_0^x e^{\mu_o(y-x)} e^{-\frac{1}{2}(\mu+\mu_1)y} I_o\left\{\frac{1}{2}(\mu-\mu_1)y\right\} dy$$

$$\frac{e^{\mu_o x}F(x,E)}{K(E_o,E)} = \sum_{r=o}^{-\alpha_1} \frac{(-\alpha_1)!(\mu_o-\mu_1)^r}{(r!)^2(-\alpha_1-r)!} .$$

$$. \int_0^x (y-x)^r e^{-\frac{1}{2}(\mu+\mu_1-2\mu_o)y} I_o\left\{\frac{1}{2}(\mu-\mu_1)y\right\} dy.$$

There is little to be gained by using Eqs. (2.1) and (2.2) to reduce this expression in general. The method can be illustrated in case $\alpha_1 = -1$. Then

$$\frac{e^{\mu_o x} F(x,E)}{K(E_o,E)} = \int_0^x e^{-\frac{1}{2}(\mu+\mu_1-2\mu_o)y} I_o\left\{\frac{1}{2}(\mu-\mu_1)y\right\}\left[1-(\mu_1-\mu_o)(y-x)\right] dy$$

$$= \frac{2\left\{1+(\mu_1-\mu_o)x\right\}}{\mu+\mu_1-2\mu_o} G\left\{\frac{\mu-\mu_1}{\mu+\mu_1-2\mu_o}, \frac{1}{2}(\mu+\mu_1-2\mu_o)x\right\}$$

$$-(\mu_1-\mu_o)\int_0^x y\, e^{-\frac{1}{2}(\mu+\mu_1-2\mu_o)y} I_o\left\{\frac{1}{2}(\mu-\mu_1)y\right\} dy.$$

If we now use Eq. (2.7) we find after a little manipulation that

$$\frac{e^{\mu_o x} F(x,E)}{K(E_o,E)} = \frac{2}{\mu+\mu_1-2\mu_o}\left\{(1 - \frac{2h}{1-k^2} + 2hv)G(k,v) + \frac{2hve^{-v}\left[I_o(kv)+kI_1(kv)\right]}{1-k^2}\right\},$$

in which

$$h = \frac{\mu_1-\mu_o}{\mu+\mu_1-2\mu_o}, \quad k = \frac{\mu-\mu_1}{\mu+\mu_1-2\mu_o}, \quad v = \frac{1}{2}(\mu+\mu_1-2\mu_o)x,$$

$$kv = \frac{1}{2}(\mu-\mu_1)x.$$

This result has already been applied to calculate the transmission of 2 Mev γ-rays in tungsten [7].

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