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PILE TRANSFER FUNCTIONS

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PILE TRANSFER FUNCTIONS

The purpose of this memo is to present a detailed derivation of the transfer function of a critical pile and to relate this steady-state description and the transient response to a step function change of reactivity. Reference to G.E. report JIO-1 - Revised is implied throughout this discussion.

The transfer function of a critical pile can be determined from the differential equations of the pile. The reactivity of the pile is assumed to be controlled by the delayed neutrons so that k_{eff} is always less than 1 + B For these assumptions elementary pile theory gives -

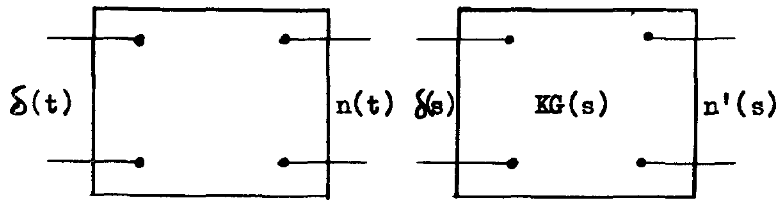
$$\frac{\ell}{B}\frac{dn}{dt} = (D - 1)n + \sum_i \lambda_i \ell\, C_i/B \qquad (1)$$

$$\frac{d\, C_i/B}{dt} = \frac{f_i}{\ell} n - \lambda_i \quad C_i/B \qquad (2)$$

where: f_i = B_i/B
n = neutron flux
ℓ = mean life of a neutron
B_i = % of group-i fission neutrons
B = % of neutrons emitted which are delayed
C_i = concentration of fission fragments which emit group-i delayed neutrons
λ_i = decay constant of group-i neutrons
t = time (seconds)
D = reactivity in dollars = $\frac{\delta k_{eff}}{B}$
δk_{eff} = $k_{eff} - 1$
k_{eff} = ratio of fission neutrons produced to those lost for an infinite pile

The discussion thus far is somewhat misleading. We have been considering a pile on the basis of elementary pile theory, when actually we are going to use an electronic pile simulator whose correspondence to a pile depends on the accuracy of the assumptions of elementary pile theory. To avoid considering these assumptions we will be more exact and say that we desire to determine the transfer function of the electronic pile simulator. The simulator obeys the same differential equations. To determine the transfer function consider the simulator as a four-terminal network. The input function is the reactivity, $D = \delta(t)$ and the output is the neutron flux. For a sinusoidal function of time input the transfer function $KG(s)$ is the ratio of the Laplace transforms of the output and input. It consists of K = a constant, or frequency invariant portion, and $G(s)$ = the frequency variant portion, or the portion dependent on time derivatives. In general, we define $f(s)$ as the Laplace transform of $f(t)$ by this integral:

$$f(s) = \int_0^t e^{-st} f(t)\, dt$$



$$\delta(s)\, KG(s) = n'(s)$$

Let the reactivity have small sinusoidal oscillations of amplitude ϵ

$$D = \delta(t) = \epsilon\, e^{jwt} \tag{3}$$

Since the pile is critical, the neutron flux will vary simusoidally $[n']$ about a steady level, n_o. The same is true for the concentrations of fission fragments. Thus:

$$n(t) = n_o + n'(t) \tag{4}$$

$$C_i(t) = C_{io} + C_{i'}(t) \tag{5}$$

By definition the transfer function relates simusoidal variations, so that

$$KG(s) = \frac{n'(s)}{\delta(s)} \tag{6}$$

To obtain this result we eliminate C_i from equations (1) and (2), take the Laplace transform and substitute to obtain an expression for n'.

From (2) $$\frac{\lambda_i C_i}{B} = \frac{f_i}{\ell} n - \frac{dC_i/B}{dt} \tag{7}$$

Substituting in (1) and using (3):

$$\frac{\ell}{B}\frac{dn}{dt} = (\delta - 1)n + \sum_i f_i \quad n - \sum_i \ell \frac{d(C_i/B}{dt} \tag{8}$$

By (4);(5):

$$\frac{\ell}{B}\frac{dn'}{dt} = n_o\delta + n'\delta - n + \sum_i f_i n - \sum_i \ell \frac{d(C_i'/B)}{dt} \tag{9}$$

$$\text{but } n = \sum_i f_i n$$

It is convenient to neglect $n'(t)\,\delta(t)$. For power levels greater than one quarter power $n_o\,\delta(t) \gg n'(t)\,\delta(t)$, so that this assumption will not introduce appreciable error. Taking the Laplace transform of (9) gives

$$\frac{\ell}{B} s\, n'(s) = n_o\,\delta(s) - \sum_i \frac{\ell s}{B}\, C_i(s) \tag{10}$$

Rewriting eq. (2) in terms of $n'(t)$ and $C_i'(t)$ we have:

$$\frac{d\ C_i'/B}{dt} = \frac{f_i}{\ell} n_o + \frac{f_i}{\ell} n' - \lambda_i \frac{C_{i_o}}{B} - \lambda_i \frac{C_i'}{B} \tag{11}$$

But the steady-state condition $\frac{dC_i/B}{dt} = 0$ shows that

$$\frac{f_i\ n_o}{\ell} = \lambda_i \frac{C_{i_o}}{B} \tag{12}$$

taking the Laplace transform and then solving for C_i':

$$\frac{s\ C_i'(s)}{B} = \frac{f_i}{\ell} n'(s) - \frac{\lambda_i\ C_i'(s)}{B}$$

$$\frac{C_i'(s)}{B} = \frac{f_i\ n'(s)}{\ell(s + \lambda_i)} \tag{13}$$

Substituting (13) in (10)

$$\left[\frac{\ell}{B} s + \sum_i \frac{s\ f_i}{s + \lambda_i}\right] n'(s) = n_o\delta(s) \tag{14}$$

$$\frac{n'(s)}{\delta(s)} = \frac{n_o}{s\left[\frac{\ell}{B} + \sum\limits_i \frac{f_i}{s + \lambda_i}\right]} = \frac{n'(s)}{\delta(s)} = \frac{B n_o}{\ell s\left[1 + \sum\limits_i \frac{B_i/\ell}{(s + \lambda_i)}\right]} \tag{15}$$

This can be rewritten:

$$= \frac{n_oB}{\ell s}\left[\frac{(s+\lambda_1)(s+\lambda_2)\dots(s+\lambda_i)\dots}{(s+\lambda_1)\dots(s+\lambda_i)\dots+\frac{B_1}{\ell}(s+\lambda_2)\dots(s+\lambda_i)\dots+\frac{B_2}{\ell}(s+\lambda_1)(s+\lambda_3)\dots+\dots}\right] \quad (16)$$

$$\frac{n'(s)}{(s)} = \frac{n_oB}{\ell s}\;\frac{\prod_i (s+\lambda_i)}{a_o+a_1s+a_2s^2+\dots+a_is^i+\dots} \quad (17)$$

$$= \frac{n_oB}{\ell}\;\frac{\prod_i (s+\lambda_i)}{s(s+s_1)(s+s_2)\dots(s+s_i)\dots} \quad (18)$$

This is the general form of the transfer function. All the λ_i's and s_i's are positive. That s_i is positive follows from the fact that all the a_i's being positive excludes positive roots. This just says that the pile transfer function can be handled by the standard techniques of servo theory. (This was implicit in the original assumptions).

This general result may now be applied to a simulator having five groups of delayed neutrons.

Half-life	Decay constant	Fraction of Fission Neutrons
$t_{1/2}$	$\lambda i = \frac{0.693}{t_{1/2}}$ sec^{-1}	B_i
0.05	$\lambda 1 = 14$	$B_1 = 0.00029$
0.43	$\lambda 2 = 1.61$	$B_2 = 0.00084$
1.52	$\lambda 3 = 0.456$	$B_3 = 0.0024$
4.51	$\lambda 4 = 0.154$	$B_4 = 0.0021$
22.0	$\lambda 5 = 0.0315$	$B_5 = 0.0017$

Assume $\ell = 5 \times 10^{-5}$ sec

Eq. (17) is now

$$\frac{n'(s)}{(s)} = \frac{n_oB}{\ell s}\;\frac{(s+\lambda_1)(s+\lambda_2)(s+\lambda_3)(s+\lambda_4)(s+\lambda_5)}{s^5+a_4s^4+a_3s^3+a_2s^2+a_1s+a_o} \quad (19)$$

The a_i's can be written in terms of λ_i, B_i and ℓ by comparing them with the corresponding coefficients of the denominator expressed in eq. (16).

The results are shown in Appendix I. Now the roots of this fifth degree equation must be determined.

$$D(s) = s^5 + 163\, s^4 + 2270\, s^3 + 3750\, s^2 + 1200\, s + 73.2 = 0 \qquad (20)$$

$$= (s + s_1)(s + s_2)(s + s_3)(s + s_4)(s + s_5) \qquad (21)$$

The results are:

$$s_1 = 148$$
$$s_2 = 13.3$$
$$s_3 = 1.45$$
$$s_4 = 0.32$$
$$s_5 = 0.08$$

These results can be used to plot the Bode diagram of Figure 1 by replacing s by jw.

The Bode diagram (Figure 1) shows the asymptotes of the steady-state amplitude gain of the pile simulator for a sinusoidal input. Replacing s by jw is an obvious operation, since s may be considered equivalent to the time derivative. Taking the time derivative of a sinusoid, exp jwt, is just multiplication by jw. The values of s_i and λ_i are the break points of the asymptotes of the function whose analytical expression is:

$$\begin{aligned} Lm\left|KG(jw)\right| = Lm\, n_o &+ Lm \frac{B}{\ell} & &- Lm\left|jw + 148\right| \qquad (22)\\ &+ Lm\left|jw + 14\right| & &- Lm\left|jw + 13.3\right| \\ &+ Lm\left|jw + 1.61\right| & &- Lm\left|jw + 1.45\right| \\ &+ Lm\left|jw + 0.456\right| & &- Lm\left|jw + 0.32\right| \\ &+ Lm\left|jw + 0.154\right| & &- Lm\left|jw + 0.08\right| \\ &+ Lm\left|jw + 0.0315\right| & &- Lm\left|jw\right| \end{aligned}$$

Transient Response

The response of a steady-state pile to a step function change of reactivity, δ, can be expressed as a sum of exponential terms. Usually

$$N = N_o \sum_{i=1}^{7} A_i e^{p_i t} \qquad (23)$$

we are interested in six groups of delayed neutrons giving seven terms in the expansion. For each amplitude, δ, there are seven values of p. Each p has an A associated with it. Figure 2 shows the calculated values of p_i and A_i for values of reactivity, δ, from - \$2.00 to + \$1.00 (-1.5% to + 0.75%). These curves also are plotted for various values of the mean life of a neutron, ℓ.

These step-function response graphs can be related to the Bode diagram of steady-state sinusoidal response.

The transfer function of the simulator KG(s) depends only on the simulator's physical characteristics. On the other hand the constants obtained from the step-function response graphs depend on the amplitude of the input. If we apply a unit impulse function (unit step-function minus another unit step-function) we can use the step-response graphs to determine the output. The values of A_i and p_i are those corresponding to $\delta = 0$, and in a sense do not depend on reactivity δ, but only on the characteristics of the simulator. Now by previous definitions -

$$n(s) = \delta(s)KG(s) \tag{24}$$

If $\delta(t)$ is an impulse function $\delta(s) = 1$ and

$$n(s)\Big|_{(t)=0} = KG(s) \tag{25}$$

The step-function response

$$n(t) = n_o \sum_{i=1}^{6} A_i \quad {}^{p_i t} \tag{26}$$

has the Laplace transform

$$n(s) = n_o \sum_{i=1}^{6} \frac{A_i}{s - p_i} \tag{27}$$

subject to certain mathematical restrictions (the physical equivalent of not allowing the pile to blow up). For the case of 6 groups of delayed neutrons the transfer function

$$KG(s) = \frac{Bn_o}{\ell} \quad \frac{(s + \lambda_1)(s + \lambda_2)(s + \lambda_3)(s + \lambda_4)(s + \lambda_5)(s + \lambda_6)}{s(s + s_1)(s + s_2)(s + s_3)(s + s_4)(s + s_5)(s + s_6)} \tag{28}$$

Equation (25) states that this equals

$$n(s)\Big|_{(t)=0} = n_o \sum_{i=1}^{6} \frac{A_i}{s - p_i}$$

$$= n_o \frac{b_6 s^6 + b_5 s^5 + b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_o}{(s-p_1)(s-p_2)(s-p_3)(s-p_4)(s-p_5)(s-p_6)(s-p_7)} \tag{29}$$

There is a correspondence between the s_i 's and the p_i 's.

$p_1 = 0$	
$p_2 = -0.015$	$s_6 = 0.015$
$p_3 = -0.08$	$s_5 = 0.08$
$p_4 = -0.33$	$s_4 = 0.32$
$p_5 = -1.45$	$s_3 = 1.45$
$p_6 = -13.3$	$s_2 = 13.3$
$p_7 =$ ---	$s_1 = 148$

The value of s_6 was obtained from the Bode diagram of G.E. report JIO-1. The values of p_1 to p_6 and s_6 to s_2 do not change appreciably for different values of neutron life ℓ, so that the above comparison is valid even though the p_i's were computed for $\ell = 10^{-5}$ and the s_i's for $\ell = 5 \times 10^{-5}$. The value of s_1 (and p_7) varies considerably for different values of ℓ, but for a given ℓ, s_1 is the negative of p_7. This correspondence between steady-state and transient response gives a convenient check and can be used to correlate results.

Appendix I

$$a_4 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \beta/\ell$$

$$\begin{aligned} a_3 = {} & \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{15} + \lambda_{23} + \lambda_{24} + \lambda_{25} + \lambda_{34} + \lambda_{35} + \lambda_{45} \\ & + \frac{\beta_1}{\ell}(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5) + \frac{\beta_2}{\ell}(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5) + \frac{\beta_3}{\ell}(\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5) \\ & + \frac{\beta_4}{\ell}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_5) + \frac{\beta_5}{\ell}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \end{aligned}$$

$$\begin{aligned} a_2 = {} & \lambda_{123} + \lambda_{124} + \lambda_{125} + \lambda_{134} + \lambda_{135} + \lambda_{145} + \lambda_{234} + \lambda_{235} + \lambda_{245} + \lambda_{345} \\ & + \frac{\beta_1}{\ell}(\lambda_{23} + \lambda_{24} + \lambda_{25} + \lambda_{34} + \lambda_{35} + \lambda_{45}) \\ & + \frac{\beta_2}{\ell}(\lambda_{13} + \lambda_{14} + \lambda_{15} + \lambda_{34} + \lambda_{35} + \lambda_{45}) + \frac{\beta_3}{\ell}(\lambda_{12} + \lambda_{14} + \lambda_{15} + \lambda_{24} + \lambda_{25} + \lambda_{45}) \\ & + \frac{\beta_4}{\ell}(\lambda_{12} + \lambda_{13} + \lambda_{15} + \lambda_{23} + \lambda_{25} + \lambda_{35}) + \frac{\beta_5}{\ell}(\lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{23} + \lambda_{24} + \lambda_{34}) \end{aligned}$$

$$\begin{aligned} a_1 = {} & \lambda_{1234} + \lambda_{1235} + \lambda_{1245} + \lambda_{1345} + \lambda_{2345} + \frac{\beta_1}{\ell}(\lambda_{234} + \lambda_{235} + \lambda_{245} + \lambda_{345}) \\ & + \frac{\beta_2}{\ell}(\lambda_{134} + \lambda_{135} + \lambda_{145} + \lambda_{345}) + \frac{\beta_3}{\ell}(\lambda_{124} + \lambda_{125} + \lambda_{145} + \lambda_{245}) \\ & + \frac{\beta_4}{\ell}(\lambda_{123} + \lambda_{125} + \lambda_{135} + \lambda_{235}) + \frac{\beta_5}{\ell}(\lambda_{123} + \lambda_{124} + \lambda_{134} + \lambda_{234}) \end{aligned}$$

$$a_0 = \lambda_{12345} + \underline{\beta}\,\lambda_{2345} + \frac{\beta_2}{\ell}\lambda_{1345} + \frac{\beta_3}{\ell}\lambda_{1245} + \frac{\beta_4}{\ell}\lambda_{1235} + \frac{\beta_5}{\ell}\lambda_{1234}$$

where $\lambda_{ijk\ldots} = \lambda_i \lambda_j \lambda_k \cdots$

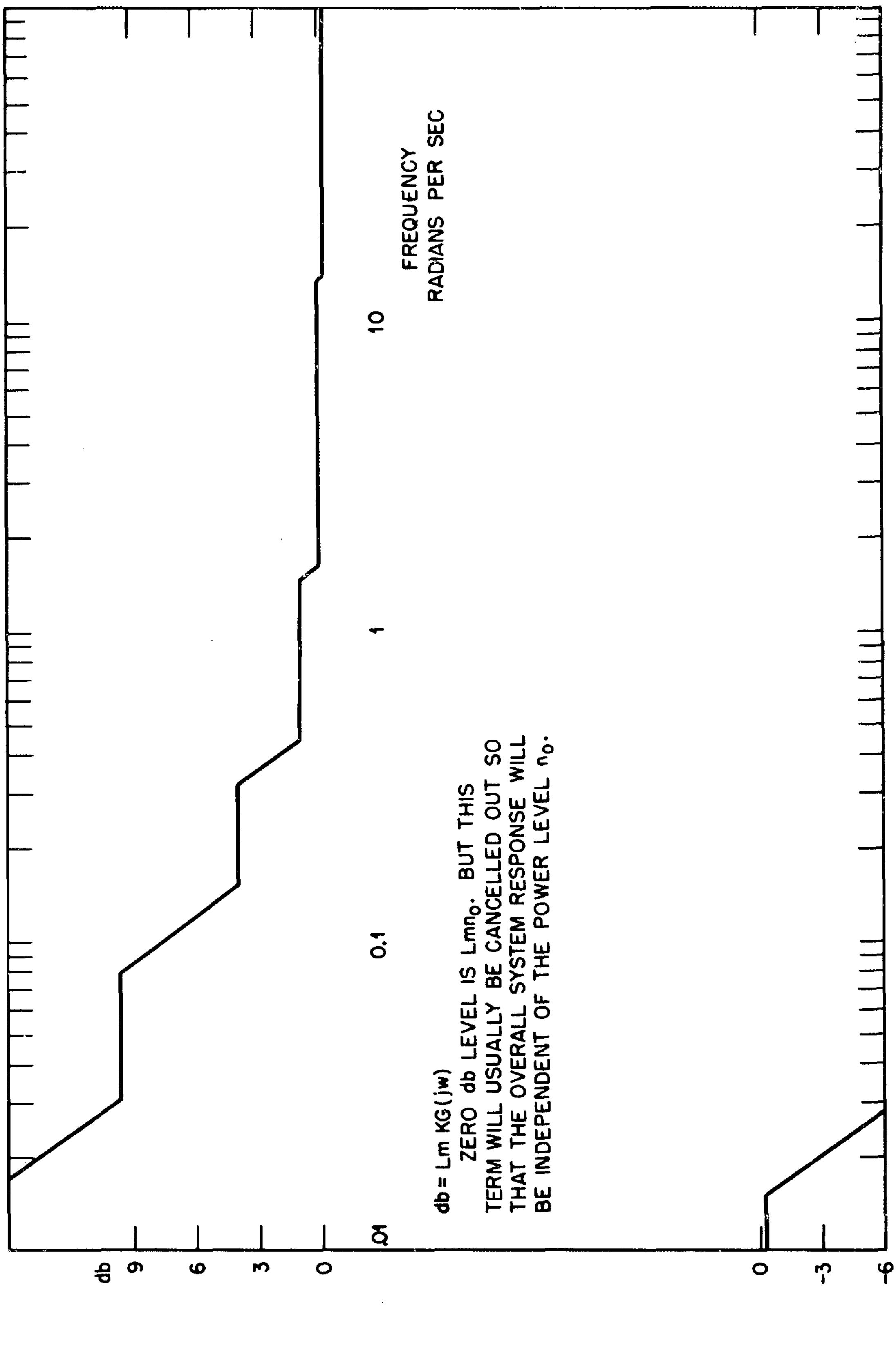


Fig. 1--Bode Diagram of Pile Simulator.
Asymptotes of db gain vs. frequency.

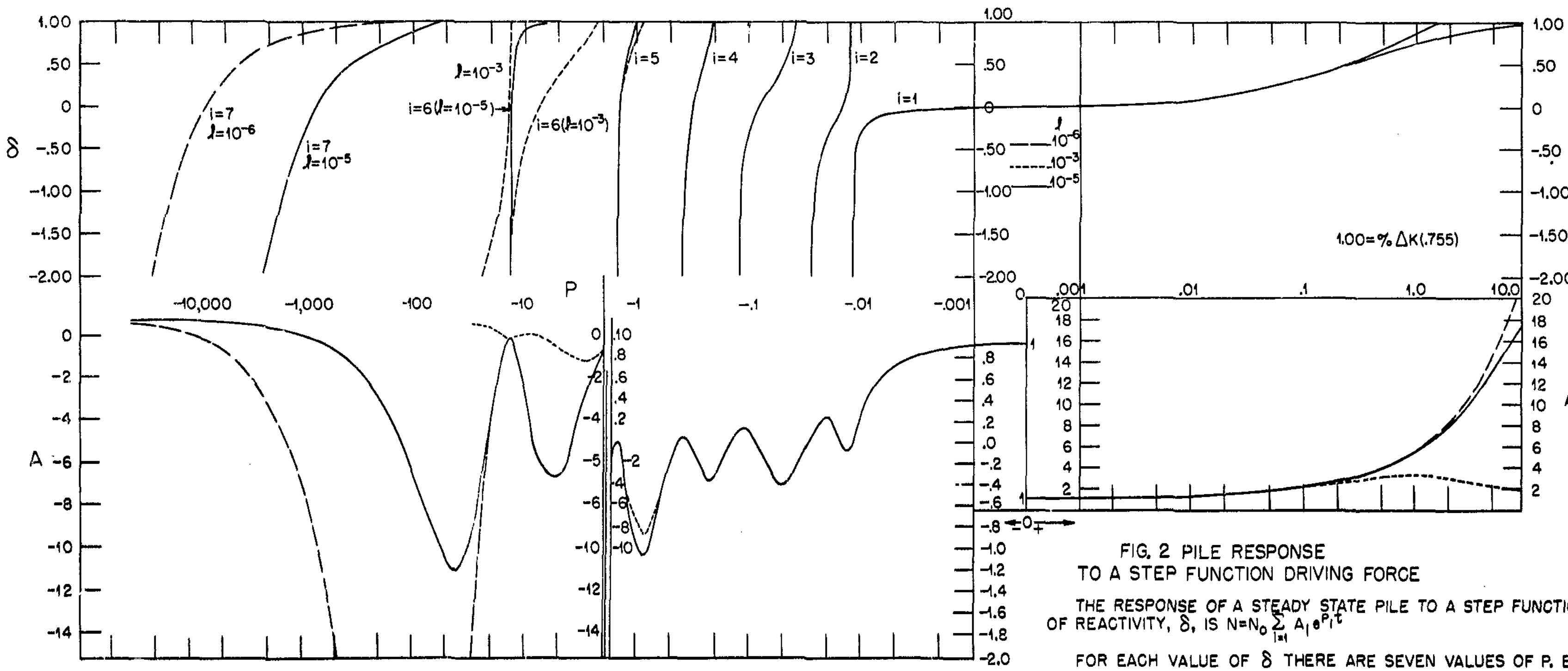


FIG. 2 PILE RESPONSE
TO A STEP FUNCTION DRIVING FORCE

THE RESPONSE OF A STEADY STATE PILE TO A STEP FUNCTION CHANGE OF REACTIVITY, δ, IS $N = N_0 \sum_{i=1} A_i e^{P_i t}$

FOR EACH VALUE OF δ THERE ARE SEVEN VALUES OF P. EACH P HAS AN A ASSOCIATED WITH IT.

THE FOLLOWING TWO EXAMPLES ILLUSTRATE THE USE OF THIS CHART.

E.G.1. FOR A \$.25 STEP FUNCTION APPLIED TO A $\ell = 10^{-5}$ PILE RUNNING AT STEADY STATE, N_0, THE RESPONSE, N, IS $N = N_0 [1.8 e^{.045 t} - .03 e^{-.0135 t} - .28 e^{-.063 t} - .1 e^{-.3 t} - .04 e^{-1.4 t} - .05 e^{-13 t} - .3 e^{-560 t}]$

E.G.2: FOR A $-1.01\%\ \Delta K$ $\left(\frac{1.01\%\ \Delta K}{.755} = \$1.35\right)$ STEP FUNCTION APPLIED TO A N_0 STEADY STATE PILE WHOSE $\ell = 10^{-3}$ $N = N_0 [.06 e^{-.012 t} + .1 e^{-.028 t} + .14 e^{-.125 t} + .08 e^{-.4 t} + .02 e^{-1.55 t} + 0 + .60 e^{-17,500 t}]$

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