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**TWO-DIMENSIONAL MESH MOVEMENT
FOR RECTANGULAR REGIONS AND OTHER
NATURAL COORDINATE SYSTEMS**

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13. ABSTRACT (Maximum 200 words) A neutrally stable two-dimensional mesh moving procedure is proposed. This procedure can be easily implemented in a two-dimensional problem domain for which a natural set of coordinates exist. Theoretical issues such as existence, uniqueness, stability, etc., are discussed. Examples are presented illustrating the two-dimensional mesh movement's attributes of stability and controllability.		
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INTRODUCTION

To most accurately approximate any physical phenomenon, a mathematical model should be posed in full three-dimensional space and time. Even with the advent of supercomputers, however, obtaining a full three-dimensional solution numerically is not feasible for many investigators. Access to a supercomputing facility is not easily obtainable by most. When access is achieved, time and/or memory constraints are sometimes imposed. In addition, software advancements are still lagging behind the great strides being made in hardware design. This makes it difficult to take full advantage of these new computer architectures.

Appropriately reducing the dimensionality of the mathematical model still seems necessary. The best scenario would be a reduction to one spatial dimension. It is rare, however, when a one-dimensional problem can be substituted for a three-dimensional one without an unacceptable loss in physical meaning. As a result, mathematical models in two spatial dimensions are still predominant in most scientific fields.

Solving two-dimensional problems is not accomplished without some level of computational difficulty. Many technological situations, modeled in two spatial dimensions and time, involve the rapid formation, propagation, and/or disintegration of small-scale structures. Some examples are shock waves in compressible flows, shear layers in laminar and turbulent flows, phase boundaries in nonequilibrium processes, combustion fronts, and classical boundary layers.

Such phenomena often arise in gun-related problems as well. For example, shock waves occur in the modeling of gas dynamics and internal/external ballistics while swift transients are found when investigating dynamic effects in gun tubes (refs 1,2).

Such structures pose numerical complications since standard methods require locally fine meshes (in space and/or time) for adequate approximation purposes. Meanwhile, the location of these phenomena is generally unknown a priori. The typical response is a globally fine mesh resulting in unnecessary computational effort for most of the problem domain.

Adaptive numerical methods were developed to address the difficulties associated with problems that would otherwise require dense meshes everywhere. An adaptive method is one which automatically adjusts its solution technique so that an answer is obtained in an optimal manner. Three basic adaptive procedures (*h*-refinement, *r*-refinement, and *p*-refinement) have evolved. All have proven to be fairly successful at optimizing their solution processes, although they accomplish this task through different means (ref 3).

To the uninitiated, *h*-refinement (also known as local mesh refinement) and adaptive methods are synonymous. An *h*-refinement approach adjusts the spatial and/or temporal discretization by adding and/or deleting mesh points so that finer meshes are used in neighborhoods of local disturbances while coarser meshes are used elsewhere (ref 4). The letter "*h*" is commonly used to designate the distance between mesh points, hence the name.

An r -refinement procedure adjusts the discretization differently than an h -refinement scheme. Instead of creating new points, existing points are moved to regions of high activity from regions of low activity as time progresses (ref 5). As a result, r -refinement is also known as mesh movement for time-dependent problems. In static problems, mesh points are said to be redistributed, as opposed to moved, consequently the letter " r " is used to describe these types of algorithms.

The basic philosophy behind p -refinement is different from the other two methods. Algorithms based on h -refinement or r -refinement attempt to modify the level of discretization in some way. The identical numerical solver is used throughout the entire spatial domain and for all time.

In p -refinement schemes, the idea is to utilize numerical methods with varying orders of accuracy in different subdomains, as opposed to directly adjusting mesh points and/or time steps. For example, a finite element method with p -refinement capabilities would automatically decide what degree polynomial (linear ($p = 1$), quadratic ($p = 2$), cubic ($p = 3$), etc.) would be appropriate to use as basis functions in various regions of the problem. In general, low order methods would be used in regions where the solution is changing very little (i.e., where solution gradients are small). The higher order methods would be employed when and where solution characteristics change dramatically, as in the cases described above (ref 6).

All three refinement schemes have their individual strengths and weaknesses (ref 3). For example, for a fixed level of discretization and a fixed order of accuracy, an r -refinement procedure will perform optimally. However, once this optimization is achieved, a finer discretization and/or a higher order method must be employed in order to increase accuracy. As a result, these adaptive strategies are often combined.

An adaptive finite element method that performed both h -refinement and r -refinement was developed for time-dependent problems in one spatial dimension (ref 7). This code has made contributions in various areas related to cannon development such as gun dynamics, impact and penetration, and rail gun technology (ref 8). However, the dimensionality restriction has proven to be too severe for this method to evolve into a practical tool for gun problems.

The goal has become to extend this work to two spatial dimensions. In particular, a two-dimensional mesh moving technique for rectangular domains was suggested by the one-dimensional results (ref 9). The purpose of this report is to examine the properties and capabilities of this two-dimensional moving scheme. An extension to other geometries is implied by the two-dimensional results, but for simplicity, this report will deal with only rectangular regions. Reports concerned with more general geometries will be forthcoming.

Ultimately, other adaptive techniques will be paired with r -refinement in order to produce a more robust computational tool. Since h -refinement and p -refinement normally involve solving a given problem more than once at selected times, they are more costly than

r-refinement. Proper mesh movement can postpone the necessity for additional adaptivity to take place and hence reduce the computational costs.

Significant accomplishments were achieved in the development of the aforementioned one-dimensional *r*-refinement procedure (ref 9). First and foremost was the mesh moving algorithm itself. This scheme has proven to be robust, reliable, stable, easily controllable, and a natural partner with other refinement techniques. In addition, a stability criterion was derived demonstrating why some other schemes become unstable and illustrating how to construct different types of schemes with various stability properties.

The two-dimensional *r*-refinement procedure analyzed here is a relatively simple extension of the one-dimensional algorithm. Due to the nature of this extension, all of the "nice" properties of the one-dimensional movement (reliability, stability, controllability, etc.) are inherited by the two-dimensional scheme.

In the next section, theoretical aspects of the two-dimensional mesh moving algorithm are presented. First, the one-dimensional algorithm is reviewed. Then, the extension to two-dimensional rectangular domains is detailed. Concepts such as existence, uniqueness, initial condition dependency, and stability are discussed and the manner in which the two-dimensional scheme inherits these properties from the one-dimensional scheme is outlined. An example is provided which illustrates the stability properties of this method as compared to another more obvious extension to two dimensions.

The third section entitled, "Control," discusses a more practical aspect of the two-dimensional mesh moving scheme. Ultimately, the scheme must be coupled with a numerical partial differential equation (PDE) solver (ref 10). In one dimension it was discovered that mesh movement could degrade the PDE solver's performance by moving points too dramatically. Space-time elements could become too distorted for the numerical solver to resolve adequately. Some control over mesh movement had to be exercised in order to avoid these situations (ref 7). The same problem is anticipated in two dimensions, and this section describes how to exercise control over the two-dimensional mesh moving algorithm in a relatively simple manner. First, space-time distortion is addressed, followed by a discussion on the resolution of space-space distortion. Examples illustrating the utility of this control methodology are presented.

The fourth section, "Discussion and Conclusions," reviews the results of the previous sections and draws conclusions. Weaknesses as well as strengths of the two-dimensional mesh moving scheme are discussed and future plans detailed. This section is followed by a list of references and a sequence of figures, which are referred to within the text of this report.

THEORY

The one-dimensional mesh moving algorithm is based on the notion of equidistribution (ref 11). In this context, a one-dimensional equidistribution problem is the determination of a dynamic partition

$$\Pi_I(t) = \{ a = x_0 < x_1(t) < x_2(t) < \dots < x_{I-1}(t) < x_I = b \} \quad (1)$$

of (a,b) into I elements such that

$$\int_{x_{i-1}(t)}^{x_i(t)} w_1(x,t) dx = \kappa(t) = \frac{1}{I} \int_a^b w_1(x,t) dx, \quad i = 1, 2, \dots, I, \quad t \geq 0, \quad (2a)$$

or equivalently

$$\int_a^{x_i(t)} w_1(x,t) dx = i \kappa(t) = \frac{i}{I} \int_a^b w_1(x,t) dx, \quad i = 1, 2, \dots, I, \quad t \geq 0, \quad (2b)$$

where the weight function $w_1(x,t) > 0$, $x \in [a,b]$, $t \geq 0$, is usually dependent on the solution of the underlying PDE. For example, w_1 has been chosen to be proportional to the solution's gradient, curvature, and local spatial discretization error (refs 12-20).

Reducing the global discretization error is the ultimate goal, and towards that end equidistributing the local discretization error would be the best procedure to follow. However, estimates of the total discretization error are not easy to make, even locally, and this has led to these other choices for the weight function. Since for most numerical methods the discretization error is proportional to some order of derivative of the solution, this has become a popular choice (refs 13,14,15,16,17,20). It has also been argued that for certain problems, an appropriate physical characteristic such as density is a suitable weight function (ref 18). When a method of lines approach is employed, the local spatial discretization error can be used since the ordinary differential equation solver can reduce the temporal error component to a relatively insignificant level (refs 12,19).

It was discovered that Eq. (2) was subject to unstable behavior as written (ref 21). In order to construct a stable mesh moving scheme, it was necessary to define

$$\Phi(x,t) = \frac{\int_a^x w_1(\xi,t) d\xi}{\int_a^b w_1(\xi,t) d\xi}, \quad (3)$$

rewrite Eq. (2b) as

$$\Phi(x_i(t),t) - \frac{i}{I} = 0, \quad i = 0, 1, 2, \dots, I, \quad (4)$$

and differentiate Eq. (4) with respect to time to obtain

$$\frac{d}{dt}[\Phi(x_i(t),t)] = 0, \quad i = 0, 1, 2, \dots, I, \quad (5a)$$

$$\Phi(x_i(0),0) = \Phi_i^0, \quad i = 0, 1, 2, \dots, I. \quad (5b)$$

With emphasis placed on the stability of the functions $\Phi(x_i(t),t)$, $I = 1, \dots, I-1$, it is seen that Eq. (5) is neutrally stable (ref 9). Initial perturbations of Φ_i (through x_i), $I = 1, \dots, I-1$ will neither grow nor decay.

Another advantage to the formulation of Eq. (5), is that as a result of the differentiation, an initially equidistributed mesh is no longer necessary. Any initial partition determines Eq. (5b) uniquely, and Eq. (5a) guarantees that the resulting movement will be stable and optimal, although the resulting mesh will not necessarily be equidistributed. Equidistribution is guaranteed for all time, however, if the initial mesh is an equidistributed one.

As mentioned previously, extending the neutrally stable mesh moving scheme (cf., Eq. (5)) to two-dimensional space can be accomplished with all of the stability properties left intact (ref 9). Consider a rectangular two-dimensional domain

$$\Omega = \{(x,y) \mid a \leq x \leq b, \quad c \leq y \leq d\} \quad (6)$$

and the extension of Eq. (5) to Ω . To that end, define

$$\Psi(x,y,t) = \frac{\int_a^x \int_c^y w_2(\xi,\eta,t) d\eta d\xi}{\int_a^b \int_c^d w_2(\xi,\eta,t) d\eta d\xi} \quad (7)$$

where $w_2(x,y,t)$ is a positive weight function on Ω such that $\Psi(x,y,t)$ is a well-defined function with the necessary continuity properties. Furthermore, let

$$\Pi_{IJ}^2(t) = \{(x_i(t),y_j(t)) \mid i = 0, 1, \dots, I, \quad j = 0, 1, \dots, J\} \quad (8)$$

denote a partition of Ω into $I \times J$ subrectangles at any time t such that

$$a = x_0 < x_1(t) < \dots < x_{I-1}(t) < x_I = b , \quad (9a)$$

$$c = y_0 < y_1(t) < \dots < y_{J-1}(t) < y_J = d . \quad (9b)$$

In order for $\Pi_{IJ}^2(t)$ to move in a neutrally stable manner for any initial partition $\Pi_{IJ}^2(0)$, demand that it obey the following equations for $t > 0$:

$$\frac{d}{dt} [\Psi(x_i(t), d, t)] = 0 , \quad i = 1, 2, \dots, I-1 , \quad (10a)$$

$$\frac{d}{dt} [\Psi(b, y_j(t), t)] = 0 , \quad j = 1, 2, \dots, J-1 , \quad (10b)$$

$$\Psi(x_i(0), d, 0) = X_i , \quad i = 1, 2, \dots, I-1 , \quad (10c)$$

$$\Psi(b, y_j(0), 0) = Y_j , \quad j = 1, 2, \dots, J-1 . \quad (10d)$$

The existence, uniqueness, and stability of the two-dimensional movement is guaranteed by the one-dimensional theory. By substituting

$$w_1(x, t) = \int_c^d w_2(x, \eta, t) d\eta \quad (11a)$$

into Eq. (5), it can be seen that Eq. (10a) is equivalent to neutrally stable, one-dimensional movement in the x -direction. Similarly, by replacing the variable x with the variable y and the interval (a, b) with the interval (c, d) in Eq. (5), and substituting

$$w_1(y, t) = \int_a^b w_2(\xi, y, t) d\xi \quad (11b)$$

it can be seen that Eq. (10b) is equivalent to neutrally stable, one-dimensional movement in the y -direction.

As in the one-dimensional case (ref 9), the normalization procedure (cf., Eq. (7)) is crucial for stability in the construction of the two-dimensional moving scheme, Eq. (10). If such a normalization is not performed the resulting equations are

$$\frac{d}{dt}[\Theta(x_i(t),d,t)] = \frac{i}{I} \frac{d}{dt}[\Theta(b,d,t)] , i = 1, 2, \dots, I-1 , \quad (12a)$$

$$\frac{d}{dt}[\Theta(b,y_j(t),t)] = \frac{j}{J} \frac{d}{dt}[\Theta(b,d,t)] , j = 1, 2, \dots, J-1 , \quad (12b)$$

$$\Theta(x_i(0),d,0) = \frac{i}{I} \Theta(b,d,0) , i = 1, 2, \dots, I-1 , \quad (12c)$$

$$\Theta(b,y_j(0),0) = \frac{j}{J} \Theta(b,d,0) , j = 1, 2, \dots, J-1 , \quad (12d)$$

where

$$\Theta(x,y,t) = \int_a^x \int_c^y w_2(\xi,\eta,t) d\eta d\xi . \quad (12e)$$

Such a system is not stable when $w_2(x,y,t)$ is a decreasing function of time as illustrated by the following example.

Example 1

Consider the two-dimensional heat equation

$$u_t = \frac{1}{8} (u_{xx} + u_{yy}), \quad 0 < x < 1, \quad 0 < y < 1, \quad t > 0 \quad (13a)$$

subject to the initial condition

$$u(x,y,0) = \sin\pi x \sin\pi y, \quad 0 < x < 1, \quad 0 < y < 1 \quad (13b)$$

and boundary conditions

$$\begin{aligned} u(0,y,t) &= u(1,y,t) = 0, \quad 0 \leq y \leq 1, \quad t \geq 0 \\ u(x,0,t) &= u(x,1,t) = 0, \quad 0 \leq x \leq 1, \quad t \geq 0 \end{aligned} \quad (13c,d,e,f)$$

The exact solution of this problem is

$$u(x,y,t) = \sin(\pi x) \sin(\pi y) \exp\left(-\frac{\pi^2}{4}t\right), \quad 0 < x < 1, \quad 0 < y < 1 \quad (14)$$

For the purposes of mesh movement take

$$w_2(x,y,t) = u(x,y,t). \quad (15)$$

Since this problem and $w_2(x,y,t)$ are separable, the correct strategy is to generate an equidistributed mesh at time $t = 0$ and use it for all time. However, $w_2(x,y,t)$ is a decreasing function of time and, thus, the solutions of Eq. (12) are expected to be unstable (ref 21).

For the given w_2 , exact solutions to Eqs. (10) and (12) can be found. The exact solution for Eq. (10) is

$$x_i(t) = x_i(0), \quad y_j(t) = y_j(0), \quad i = 1, 2, \dots, I-1, \quad j = 1, 2, \dots, J-1 \quad (16a,b)$$

The exact solution for Eq. (12a) is

$$x_i(t) = \frac{1}{\pi} \arccos[\alpha_i - (\alpha_i - \cos\pi x_i^0) \exp(\frac{\pi^2}{4}t)], \quad i = 1, 2, \dots, I-1 \quad (17a)$$

where

$$\alpha_i = 1 - 2\frac{i}{I} \quad \text{and} \quad x_i^0 = x_i(0), \quad i = 1, 2, \dots, I-1. \quad (17b,c)$$

The solution to Eq. (12b) can be found by substituting y for x , j for I , and J for I in Eq. (17). The trajectories for the meshes produced by Eqs. (16) and (17) are displayed in Figures 1 and 2, respectively, and the unstable behavior of Eq. (17) is clearly visible. A mesh with values of $I = J = 3$ was initially equidistributed, and initial perturbations of 0.01 for $I = j = 1$ and of -0.01 for $I = j = 2$ were introduced.

It should be noted that Eq. (10) could easily be adapted to nonrectangular regions if another natural set of coordinates existed for the given problem domain, e.g., polar coordinates for circular regions or boundary-fitted coordinates as used for flow around air foils (ref 22). This natural coordinate system could be used for the mesh equations (cf., Eq. (10)) as well. All that is required is to transform Eq. (10) from rectangular coordinates to the more natural system. (This is not the extension to nonrectangular geometries alluded to in the "Introduction," but rather an extension to nonrectangular coordinate systems for appropriately defined regions.)

CONTROL

In the one-dimensional situation, a positive constant can be added to w_1 to ensure the existence of a unique solution. It should be noted that adding a positive constant to w_2 will ensure the existence of a unique solution to Eq. (10) as well. It was discovered that the inclusion of this constant also provided a means in which to control the one-dimensional movement (ref 8).

Control was deemed necessary since mesh movement, as defined by Eq. (5), could be too dramatic and amplify the discretization error (especially the temporal component) of a numerical PDE solver by overly distorting the mesh (ref 7). In the one-dimensional case, there was only space-time distortion to be concerned about. Now in two dimensions there exists the possibility of space-space distortion as well. (Since in both one- and two-dimensional space there is the well-recognized problem of properly transitioning from a dense mesh to a sparse one, it will not be discussed here.) External control is even more of a necessity in two dimensions, and the addition of a positive constant to w_2 appears to be a simple method of accomplishing the task.

For one-dimensional mesh movement, it was discovered that the solution to Eq. (5), with $\alpha > 0$ added to w_1 , could be written as a linear combination of two other solutions to Eq. (5) with appropriate weight functions (ref 8). This is also the case in two dimensions since Eq. (10), as previously stated, represents one-dimensional movement in the respective coordinate directions. In order to demonstrate this fact, consider the x -component of the two-dimensional moving scheme (cf., Eq. (10a)). The y -component behaves similarly.

Let $x_i(t)$, $q_i(t)$, and u_i , $i = 0, 1, \dots, I$, denote mesh trajectories for $t \geq 0$ such that

$$\int_a^{x_i(t)} \int_c^d [w_2(\xi, \eta, t) + \alpha] d\eta d\xi = \frac{i}{I} \int_a^b \int_c^d [w_2(\xi, \eta, t) + \alpha] d\eta d\xi, \quad i = 0, 1, \dots, I, \quad (19a)$$

$$\int_a^{q_i(t)} \int_c^d w_2(\xi, \eta, t) d\eta d\xi = \frac{i}{I} \int_a^b \int_c^d w_2(\xi, \eta, t) d\eta d\xi, \quad i = 0, 1, \dots, I, \quad (19b)$$

and

$$\int_a^{u_i} \int_c^d \alpha d\eta d\xi = \frac{i}{I} \int_a^b \int_c^d \alpha d\eta d\xi, \quad i = 0, 1, \dots, I, \quad (19c)$$

Note that the trajectories, $q_i(t)$, $i = 0, 1, \dots, I$, are solutions to the equidistribution problem as originally defined and u_i , $i = 0, 1, \dots, I$, are constant trajectories defining a uniform mesh.

The solutions, $x_i(t)$, $i = 0, 1, \dots, I$, are actually linear combinations of the equidistributed trajectories, $q_i(t)$, $i = 0, 1, \dots, I$, and the constant trajectories, u_i , $i = 0, 1, \dots, I$, according to the

one-dimensional theory. Furthermore, the dominant component is determined by the following relationship (ref 8):

$$|x_i(t) - u_i| \leq \frac{\max_{a \leq x \leq b, t \geq 0} \int_c^d w_2(x, \eta, t) d\eta}{\alpha (d - c)} |q_i(t) - u_i|, \quad i = 0, 1, \dots, I. \quad (20)$$

For large α , as compared to the maximum value of w_2 , the solutions, $x_i(t)$, $I = 0, 1, \dots, I$, are very close to being constant trajectories. Conversely, for small α , the solutions, $x_i(t)$, $I = 0, 1, \dots, I$, are very close to the equidistributed positions. (N.B., for $\alpha = 0$, Eq. (19a) and Eq. (19b) are identical.) Appropriate choices of α determine the amount of movement deemed necessary as illustrated by the following example.

Example 2

Consider the first-order wave equation

$$u_t + \rho (u_x + u_y) = 0, \quad 0 < x < 1, \quad 0 < y < 1, \quad t > 0, \quad (21a)$$

with initial and boundary conditions and the constant, ρ , selected so that the exact solution is

$$u(x, y, t) = \frac{1}{2} [1 - \tanh(10x + 10y - 20t)], \quad (21b)$$

which is an oblique wave front (slope = -1) that is initially centered on the line

$$10x + 10y = 0 \quad (21c)$$

and subsequently moves across the domain, $\Omega = [0,1] \times [0,1]$, from left to right, i.e., in the direction of increasing x . If one wished to solve this problem numerically, incorporating mesh movement, a reasonable choice for the weight function, w_2 , would be:

$$w_2(x, y, t) = \text{sech}^2(10x + 10y - 20t) + \alpha \propto u_t(x, y, t) + \alpha. \quad (22)$$

Eq. (10) was solved using the differential-algebraic solver, DASSL (ref 23), with the above choice, Eq. (22), for the weight function, w_2 , and three different values for the constant, α . In all instances, the initial condition was a uniform partition with $I = J = 7$. The purpose was to illustrate the effect the parameter α has on mesh moving dynamics, as well as the amount of control available to the user.

Meshes at time, $t = 0.25$, for values of $\alpha = 0.02, 0.1, \text{ and } 0.5$ are displayed in Figures 3, 4, and 5, respectively. The gray area of the figures represents the region of the domain behind the front, while the region ahead of the front remains white. As predicted, movement from the

initially uniform positions is more dramatic for $\alpha = 0.02$, while for $\alpha = 0.5$ mesh points have barely moved at all.

The intermediate value of $\alpha = 0.1$ would seem to be optimal. Mesh points have moved significantly so as to address the issue of the moving wave front adequately. On the other hand, the movement is not so dramatic so as to possibly disrupt the ability of a PDE solver by causing excessive space-time distortion. In general, one would expect intermediate values of α to be best with specific problems and solvers demanding more extreme values on occasion.

In the above example, spatial distortion of the computational cells was not discussed. Since the moving front intersected both the x - and y -axes at 45 degrees, the aspect ratio of the cells for $\alpha = 0.1$ and 0.5 did not change appreciatively. For $\alpha = 0.02$, mesh movement tended to elongate the cells in one direction or the other. It is conceivable that this space-space distortion could become great enough so as to affect the accuracy of a PDE solver.

In order to address this problem, it should be noted that there is no requirement that α be the same value in Eq. (10a) as it is in Eq. (10b). (For that matter, there is no requirement that the entire weight function be the same in the two equations.) Different values for the constant, α , can be used to deal with spatial mesh distortion, if necessary. This fact is illustrated by the following example.

Example 3

Once again, consider the first-order wave equation

$$u_t + \rho (u_x + u_y) = 0, \quad 0 < x < 1, \quad 0 < y < 1, \quad t > 0, \quad (23a)$$

but with initial and boundary conditions and the constant, ρ , selected so that the exact solution is

$$u(x,y,t) = \frac{1}{2} [1 - \tanh(10x + 2y - 10t)], \quad (23b)$$

which is an oblique wave front (slope = -5) that is initially centered on the line

$$10x + 2y = 0 \quad (23c)$$

and subsequently moves across the domain, $\Omega = [0,1] \times [0,1]$, from left to right, i.e., in the direction of increasing x . If one wished to solve this problem numerically, incorporating mesh movement, a reasonable choice for the weight function, w_2 , would be:

$$w_2(x,y,t) = \text{sech}^2(10x + 2y - 10t) + \alpha \propto u_t(x,y,t) + \alpha. \quad (24)$$

Eq. (10) was solved using the differential-algebraic solver, DASSL (ref 23), with the above choice, Eq. (24), for the weight function, w_2 , with $\alpha = 0.1$ (in both equations), and for an initially uniform mesh with $I = J = 7$. The resulting mesh at time, $t = 0.5$, is displayed in Figure 6. The gray area of the figure represents the region of the domain behind the front, while the region ahead of the front remains white. The elongation of the computational cells is clearly evident.

Eq. (10) was then solved using Eq. (24) and an initially uniform mesh with $I = J = 7$, but with $\alpha = 1.0$ in Eq. (10a) and $\alpha = 0.1$ in Eq. (10b). The resulting mesh at time, $t = 0.5$, is displayed in Figure 7. The gray area in the figure represents the region of the domain behind the front, while the region ahead of the front remains white. The aspect ratio of the computational cells has not been altered significantly from the initial aspect ratio of 1.

DISCUSSION AND CONCLUSIONS

A neutrally stable two-dimensional mesh moving procedure was proposed (cf., Eq. (10)). Based on a successful one-dimensional algorithm (cf., Eq.(5)), this procedure can be easily implemented in a two-dimensional problem domain for which a natural set of coordinates exist, e.g., rectangular domains and Cartesian coordinates. Theoretical issues (existence, uniqueness, etc.) were discussed and it was demonstrated how these properties are inherited from the one-dimensional formulation. Examples were presented illustrating the two-dimensional mesh movement's attributes of stability and controllability. A more general extension to other geometries, regardless of the coordinate system, has been formulated but this report concentrated on the extension to rectangular regions in order to introduce the basic concepts.

A mesh moving scheme based on Eq. (10) would have various strengths and weaknesses just like any numerical method, adaptive or otherwise. For example, such a scheme would be extremely stable and could be put under user control with minimal interface requirements. Any initial mesh would be a valid initial condition and mesh points would be guaranteed not to coalesce. Unacceptable mesh distortion could occur, but both space-time and space-space distortion could easily be addressed by the user.

Individual point movement would not be independent in the strictest sense of the word, however. A routine examination of Eq. (10) reveals that Eq. (10a) is not dependent on y , while Eq. (10b) has no x dependency. This implies that all points on the same coordinate line will move in unison in the opposite coordinate direction (e.g., all points with identical x -component values will generate velocity and displacement vectors with identical y -component values, and vice versa). In a sense then, Eq. (10) delineates a line-moving scheme rather than a point-moving one. This makes it difficult for such a method to align points properly when encountering structures oblique to the coordinate axes. Orienting the initial mesh so as to align properly with the geometry of some known structure can overcome this weakness, but it is rare that such detailed information is initially available.

A mesh moving scheme with individual and independent point movement has been formulated. This independent scheme is also neutrally stable, but more susceptible to mesh distortion. A thorough analysis of this independent scheme and a comparison with the algorithm described in this report is forthcoming. Included with this analysis will be the development of a two-dimensional stability criterion analogous to the one-dimensional stability criterion for mesh movement based on equidistribution. The stability analysis presented here is specific to the particular two-dimensional moving scheme described herein.

Issues such as how best to evaluate the weight function, w_2 , and how to properly couple this moving scheme with a PDE solver have not been addressed in this report. Answers to such questions are dependent on the properties and characteristics of the PDE solver involved. It is best to delay dealing with those issues until after a particular PDE solver has been chosen. In the future, it is planned to combine this mesh moving scheme with some existing PDE solver and these issues will be reported on then, as well as the relative success or failure of such a coupling.

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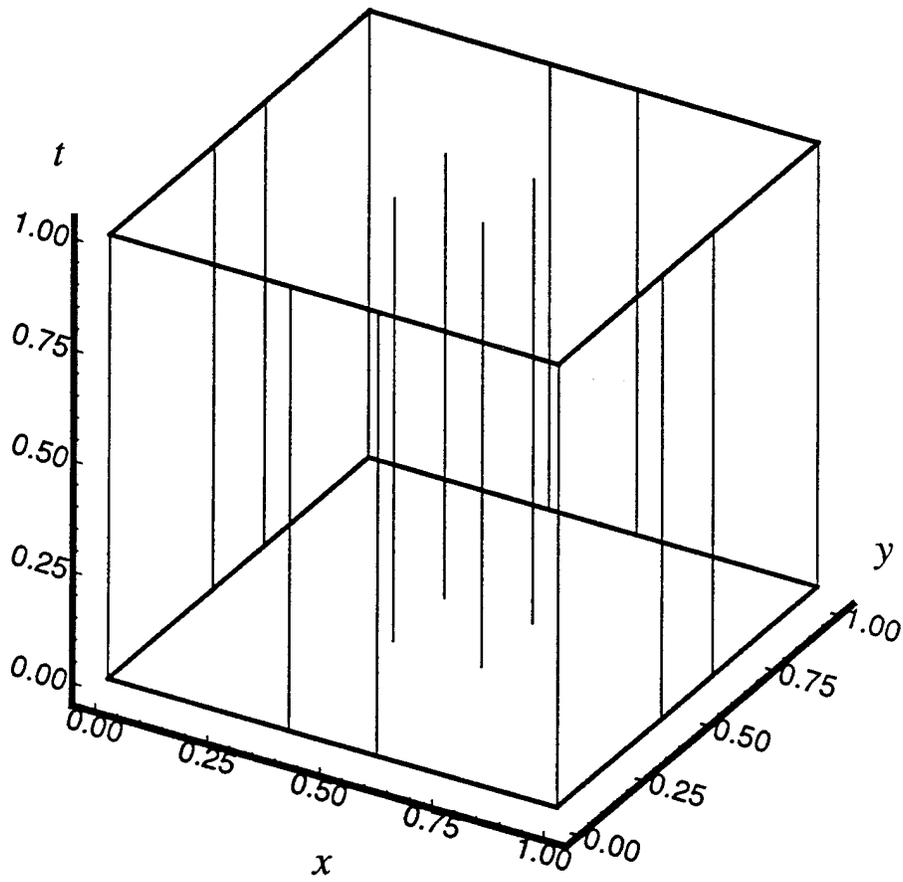


Figure 1 Stable mesh trajectories for Example 1

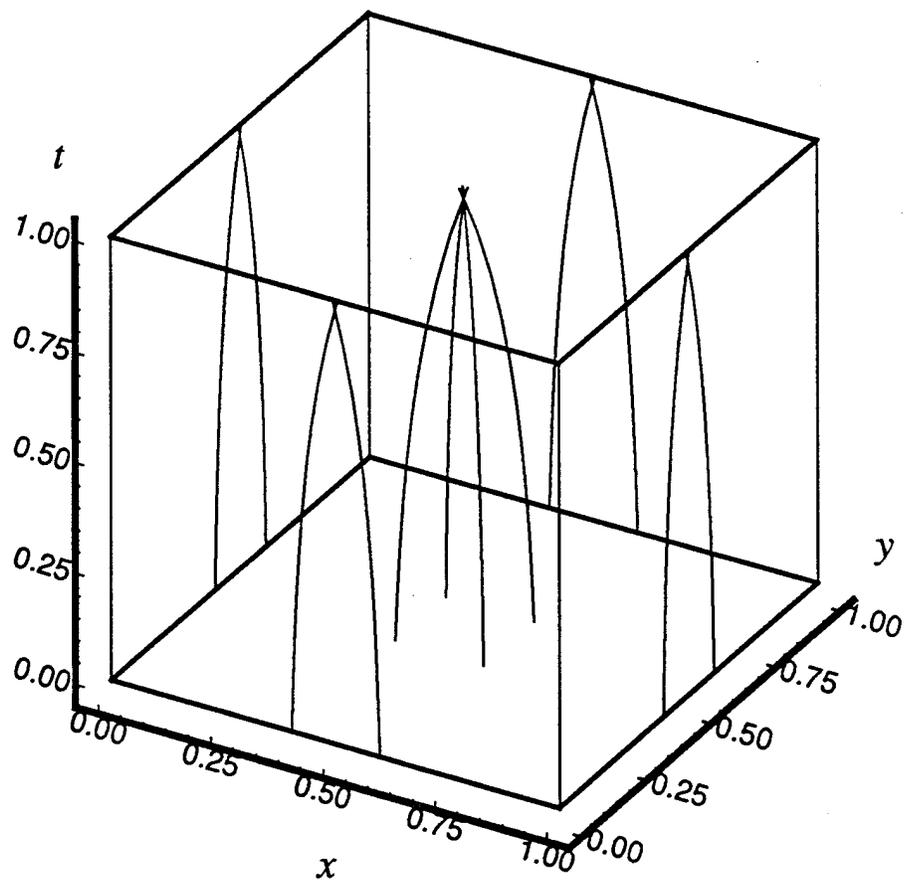


Figure 2 Unstable mesh trajectories for Example 1

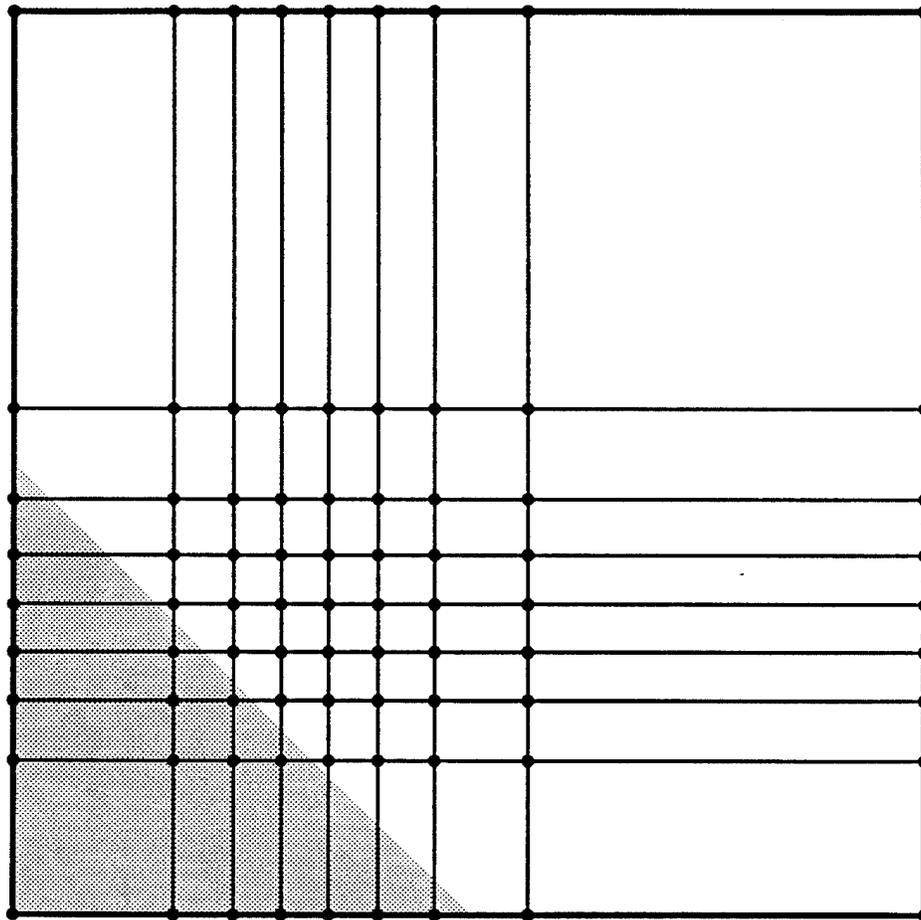


Figure 3 Mesh for Example 2 at $t = 0.25$ and $\alpha = 0.02$

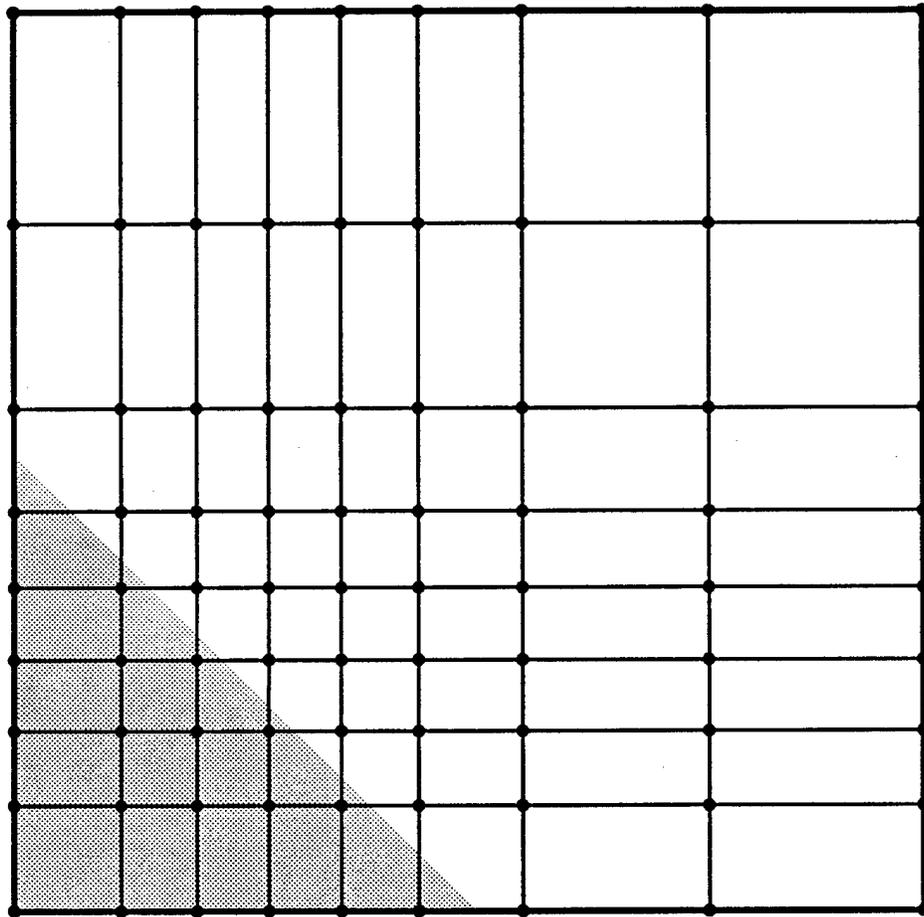


Figure 4 Mesh for Example 2 at $t = 0.25$ and $\alpha = 0.1$

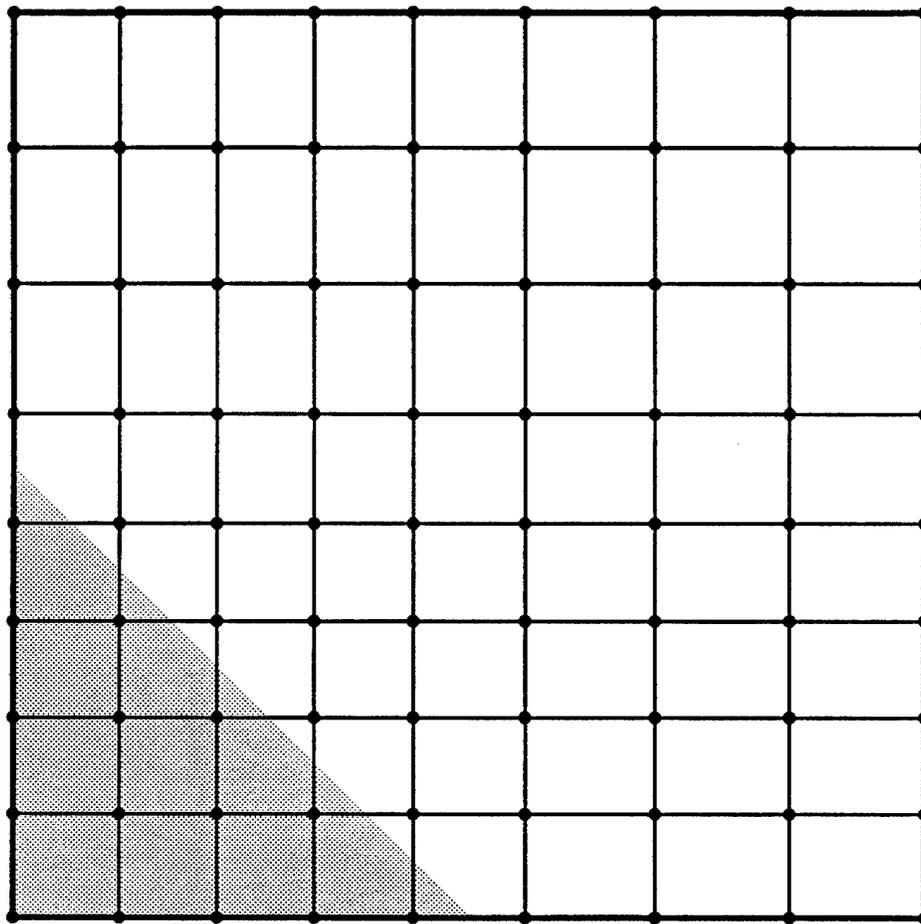


Figure 5 Mesh for Example 2 at $t = 0.25$ and $\alpha = 0.5$

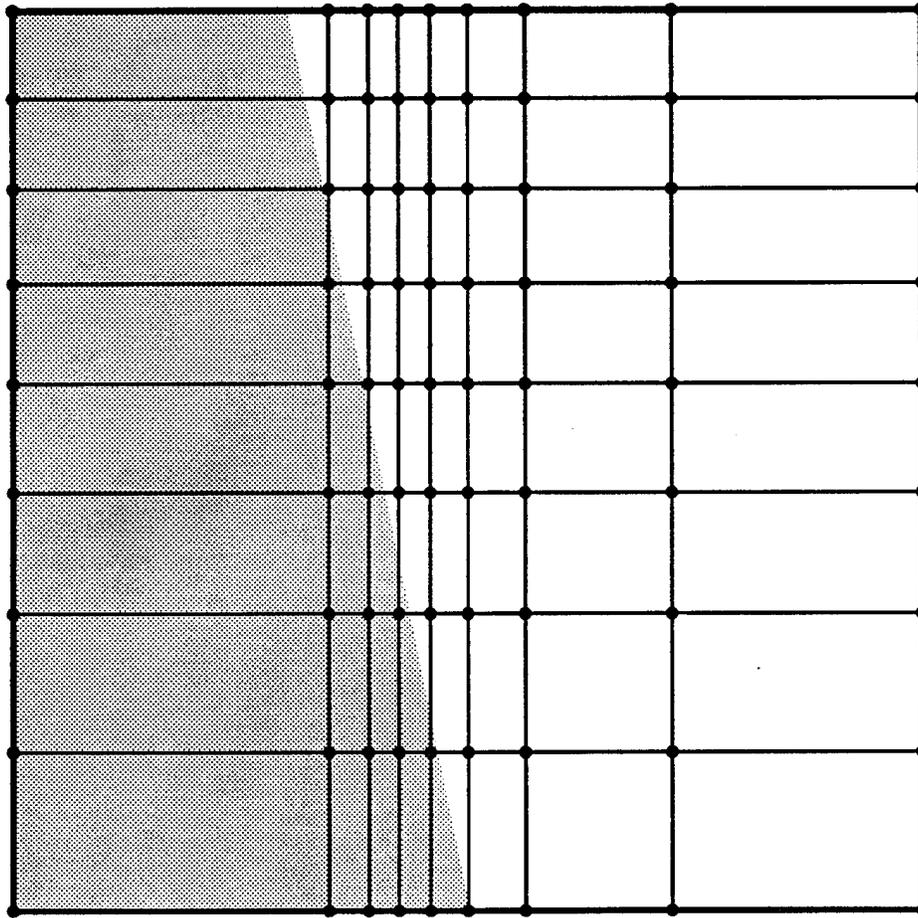


Figure 6 Mesh for Example 3 at $t = 0.5$ and $\alpha = 0.1$ in both Eqs. (10a,b)

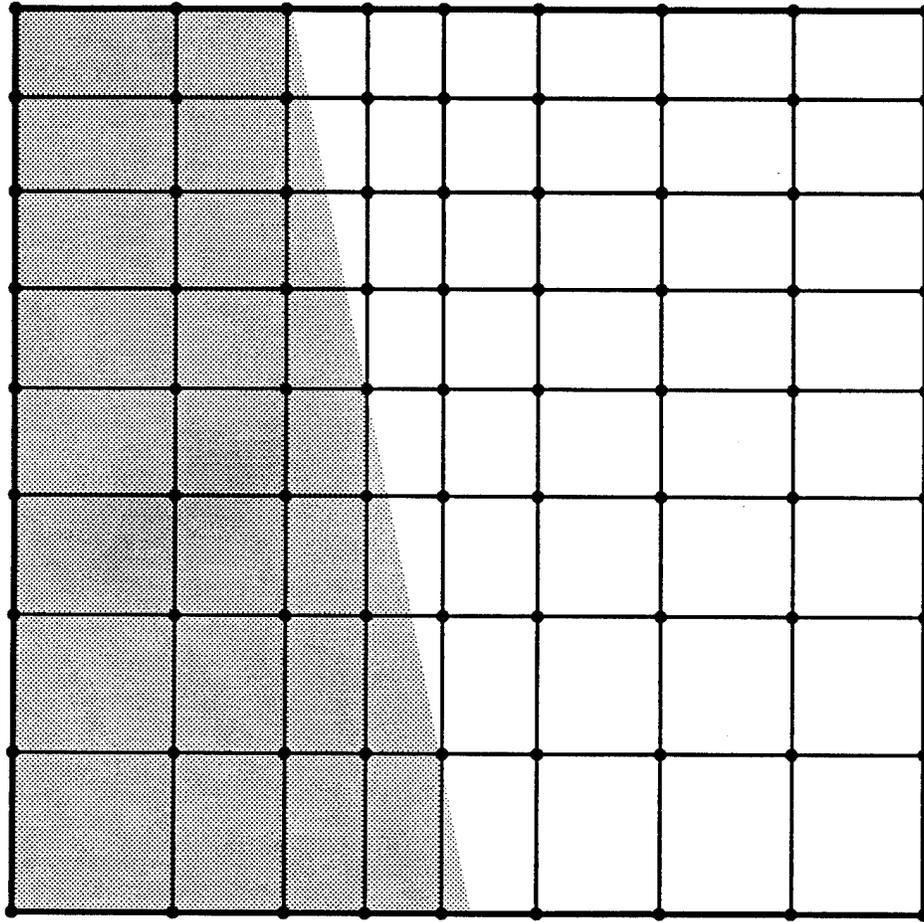


Figure 7 Mesh for Example 3 at $t = 0.5$ and $\alpha = 1.0$ in Eq. (10a) and $\alpha = 0.1$ in Eq. (10b)

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