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PHASE NOISE CANCELLATION IN A MIXER CIRCUIT: ANALYSIS USING A RANDOM PHASE FUNCTION

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A stochastic signal model is used to study the phase noise suppression in an ideal mixer circuit. The phase noise in a harmonic oscillator is modeled by a sinewave containing a random phase term. The random phase term is a random walk stochastic process. The problem is to calculate the phase noise reduction (i.e., the decrease in sideband power) which occurrs when the oscillator signal is coherently mixed witha time delay replica of itself. The results confirm the validity of the well know engineering formula for the phase noise reduction as a function of the time delay. An improved formula is also obtained. Although the results are derived for a pure sinusoidal oscillator, the same analysis can also be applied to modulated signals such as FMCW, etc.

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CONTENTS

Introduction

It is well known that the phase noise of a sinusoidal oscillator is reduced by the time-delayed mixing scheme shown in figure 1a. This phenomenon, called the *correlation effect* (ref 1) is important in the phase noise analysis of radar and communications systems. It is usually explained using FM modulation theory, (refs 1 through 4) where it is shown that the reduction in the sideband power at the output relative to the sideband power of the source is given by the well known formula

$$K(\omega) = 2\big[1 - \cos(\omega\tau_0)\big] = 4\sin^2(\omega\tau_0/2) \tag{1}$$

where τ_0 is the time delay and ω is the radian frequency offset from the carrier. A precise definition of the function $K(\omega)$ is given below. In this report, the correlation effect is analyzed using the theory of stochastic processes. The signal model consists of a fixed frequency sinusoid that is phase modulated by a random-walk phase function, i.e., a random-walk phase noise. Other signal models are possible but shall not be considered in this report. It is shown that an analysis based on this random signal model leads to a similar, though somewhat more detailed formula than equation (1). In one respect, the analysis presented here places equation (1) on a more rigorous mathematical foundation. In light of this analysis, it is fair to conclude that equation (1) is a good approximation which is generally valid except for very small frequency offsets.

The problem considered here is illustrated in figure 1a. This is a common signal processing operation which can be thought of as the front end of a coherent CW or FMCW radar receiver. The signal $v(t)$ is a sinewave containing a random phase function, i.e.,

$$v(t) = \cos[\omega_0 t + \varphi(t) + \varphi_0] \tag{2}$$

where $\varphi(t)$ is a random process.[1] The phase noise originating in the source oscillator $v(t)$ propagates through the mixer and low-pass filter (LP) and manifests itself in the output $x(t)$. The problem is to compare the phase noise spectrum of the output signal $x(t)$ to the phase noise spectrum of the source $v(t)$. The term *phase noise spectrum* is defined in the section Spectral Purity. In order to simplify the comparison of these power spectra, it is helpful to frequency-shift the power spectrum of the source oscillator $v(t)$ from the carrier frequency ω_0 down to D.C. This is accomplished very easily by means of the circuit shown in figure 1b. In figure 1b, the local oscillator is an ideal reference oscillator

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[1] For an FMCW radar, an FM modulation term should be included in the argument of the cosine. However, this modulation term does not effect the analysis and is omitted for simplicity.
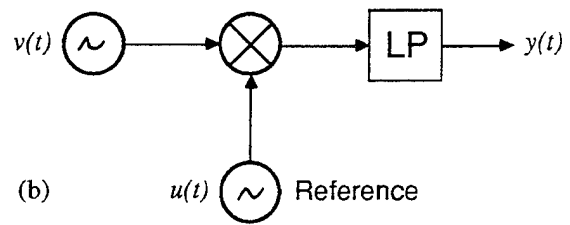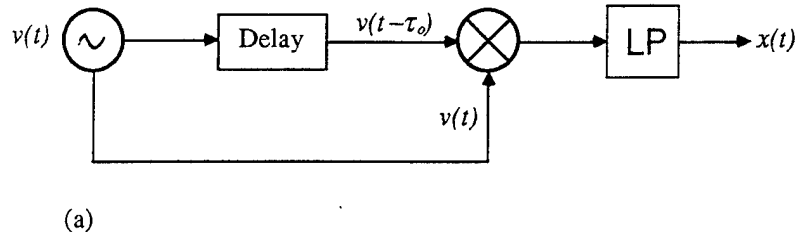
(a)

(b)

Figure 1. Circuit Block diagram.

$u(t) = \cos \omega_0 t$ which has zero phase noise. Thus, the power spectrum of the output $y(t)$ is almost identical to the power spectrum of the source $v(t)$, except that the entire spectrum is downshifted to D.C. Observe that the power spectrum of $v(t)$ can be easily measured in this case by connecting the output $y(t)$ to a baseband spectrum analyzer. This is the basic principle behind various methods of phase noise measurement (ref 5).

The main objective of this study is to compare the power spectral density of the mixer output $x(t)$ to the power spectral density of the source oscillator, i.e., $y(t)$. It will be shown that there is a reduction, i.e., a decrease in the phase noise spectrum of the signal $x(t)$ relative to the phase noise spectrum of $y(t)$. The reduction in phase noise power is characterized by the ratio

$$K(\omega) = \frac{S_x(\omega)}{S_y(\omega)}, \tag{3}$$

where $S_x(\omega)$ and $S_y(\omega)$ are the power spectral densities of $x(t)$ and $y(t)$, respectively. The quantity $K(\omega)$ will be called the *reduction factor* or the *correlation factor*. (In ref 4 it is called the *delay function*). Using dB notation, the reduction factor can be expressed in the form

$$K(\omega) = 10 \log_{10} \frac{S_x(\omega)}{S_y(\omega)} \quad \text{dB}. \tag{4}$$

The main goal of this report is to calculate the reduction factor for the random signal model given by equation (2).

Spectral Purity

The *spectral purity* of a sinusoidal oscillator describes how closely the frequency spectrum of the oscillator represents that of an ideal sinewave. The term *phase noise* refers to the short term (< 1 second) frequency fluctuations resulting from random phase modulation of the sinusoidal signal. Phase noise is also called FM noise. In addition to phase noise, an oscillator is also subject to amplitude fluctuations or AM noise. However, for an amplitude stabilized oscillator, the AM noise is usually much less than the FM noise.

The power spectrum of a good oscillator is characterized by a narrow spectral peak at $f = f_0$ (and one at $-f_0$). A simple measure of spectral purity is given by the linewidth of the power spectrum $S(f)$ as shown in figure 2. Note that the power spectrum as used here is defined for both positive and negative frequencies. Moreover, it is an even function of frequency. The oscillator linewidth is characterized by either the full-width at half-maximum (FWHM)
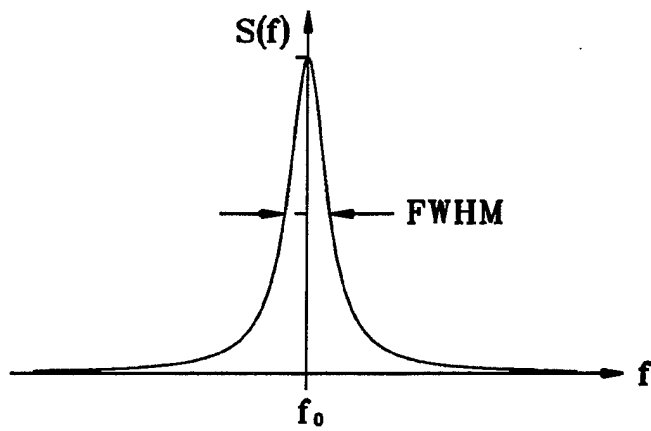
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Figure 2. The linewidth of the power spectrum.

or the half-width at half-maximum (HWHM). Thus, an elementary measure of spectral purity is given by the ratio

$$\eta = \frac{f_{3dB}}{f_0}, \tag{5}$$

where f_{3dB} is equal to the HWHM. The parameter η will be called the linewidth parameter.

A much more precise measure of spectral purity is given by the quantity $\mathcal{L}(f)$, called script–L–F. It seems that $\mathcal{L}(f)$ was first introduced into the engineering literature by V. Van Duzer in 1964 (ref 6). Another early appearance of $\mathcal{L}(f)$ in the literature was due to D.J. Glaze in 1970 (ref 7). A formal definition of $\mathcal{L}(f)$ was given by D.W. Allen, J.H. Shoaf, and D. Halford in 1974 (ref 8). As originally defined in reference 8:

$$\mathcal{L}(f) = \frac{P(f)}{P_0}, \tag{6}$$

where $P(f)$ is the average signal power in a 1 Hz bandwidth and at a frequency offset f from the carrier, and P_0 is the total average signal power. To be more precise, $P(f)$ is defined to be the *power spectral density* at a frequency offset f from the carrier, where f lies in the range $-f_0 < f < \infty$. Since it is sometimes a point of confusion, it is important to stress that P_0 is not the power spectral density at f_0, but the *total average signal power*. It is important to make this distinction. An immediate consequence of the above definition is that $\mathcal{L}(f)$ satisfies the normalization condition

$$\int_{-f_0}^{\infty} \mathcal{L}(f) \, df = 1. \tag{7}$$

Therefore, $\mathcal{L}(f)$ will be called the normalized power spectral density. As one might expect, $\mathcal{L}(f)$ is directly related to the power spectrum $S(f)$ of the oscillator. By the definition of $S(f)$, the total average signal power is given by

$$P_0 = \int_{-\infty}^{\infty} S(f) \, df. \tag{8}$$

Using the fact that $S(f)$ is an even function of f, that is, $S(-f) = S(f)$, this can be written as an integral over positive frequencies in the form

$$P_0 = \int_{0}^{\infty} 2S(f) \, df. \tag{9}$$

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The average power contained in the frequency band from f_1 to f_2 ($0 \leq f_1 \leq f_2$) is given by

$$P_{12} = \int_{f_1}^{f_2} 2S(f) \, df \tag{10}$$

and the fraction of the total power which is contained in that frequency band is

$$\frac{P_{12}}{P_0} = \frac{\int_{f_1}^{f_2} S(f) \, df}{\int_0^\infty S(f) \, df}. \tag{11}$$

The spectral density $P(f)$ can be obtained from equation (9) or (10) by inspection: at a frequency offset f from the carrier $P(f) = 2S(f_0 + f)$. Therefore, one obtains the relation

$$\mathcal{L}(f) = \frac{2S(f_0 + f)}{P_0}. \tag{12}$$

Consider now the units (physical dimensions) of $\mathcal{L}(f)$. Because $P(f)$ is a power spectral density and P_0 is a power, the units of $\mathcal{L}(f)$ are Hz^{-1}. Using dB notation, $\mathcal{L}(f)$ can be written in the form

$$\mathcal{L}(f) = 10 \log_{10} \frac{P(f)}{P_0}. \tag{13}$$

The units in this case are dBc/Hz which are commonly abbreviated as dBc.

As defined by equation (6), the normalized power spectrum $\mathcal{L}(f)$ is the standard way to characterize oscillator spectral purity. In the measurement community, $\mathcal{L}(f)$ is called the single-sideband (SSB) phase noise. Measurements of oscillator spectral purity seek to measure $\mathcal{L}(f)$ as a function of f. Assuming that the spectral profile is symmetric about f_0, measurements are generally made only for positive (or only for negative) values of the frequency offset f. For the system designer, acceptable limits of phase noise are usually specified by quoting an upper bound for $\mathcal{L}(f)$ at a specified offset f from the carrier. For a more detailed specification, an upper bound on $\mathcal{L}(f)$ might be specified at several discrete frequency offsets, $f_1, f_2, \ldots, f_n$; or over a continuous range of frequencies, $f_1 < f < f_2$.

Stochastic Phase Modulation Theory

The analysis given here follows Papoulis (ref 9, 2nd ed., p. 321-324). It is assumed that the signal source is amplitude stabilized so that the amplitude fluctuations of the oscillator output are negligible. The phase fluctuations of the oscillator give rise to a stochastic process of the form

$$v(t) = \cos[\omega_0 t + \varphi(t) + \varphi_0], \tag{14}$$

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where the phase function $\varphi(t)$ is a stochastic process and φ_0 is a random variable that is uniformly distributed on the interval $[0, 2\pi]$. In addition, it is assumed that $\varphi(t)$ and φ_0 are statistically independent for all t. There are two ways to specify the process $\varphi(t)$: Either $\varphi(t)$ is a given stochastic process (phase modulation) with known statistics, or it is defined indirectly by the integral

$$\varphi(t) = \int_0^t w(s)\, ds, \tag{15}$$

where $w(t)$ is a given stochastic process (frequency modulation) with known statistics. Technically, equation (15) is a stochastic integral which must be treated with special mathematical care; however, the representation (15) shall not be used here. For the purposes of analysis, it is desirable that the oscillator signal $v(t)$ be at least wide-sense stationary so that the process mean and autocorrelation function are invariant with respect to a shift of the time origin.

To simplify the mathematics, the real valued process (14) can be represented as the real part of the complex process $V(t)$ defined by

$$V(t) = e^{i[\omega_0 t + \varphi(t) + \varphi_0]}. \tag{16}$$

The statistics of $v(t)$ are closely related to the statistics of the process

$$\Phi(t) = e^{i[\varphi(t) + \varphi_0]}, \tag{17}$$

since $V(t) = \Phi(t)e^{i\omega_0 t}$. The objective now is to find the power spectrum of the oscillator signal $v(t)$. First of all, the process $v(t)$ has zero mean since

$$E\left[e^{i\varphi_0}\right] = E[\cos\varphi_0] + iE[\sin\varphi_0] = 0,$$

and therefore, using the fact that the $\varphi(t)$ and φ_0 are statistically independent,

$$\begin{cases} E[V(t)] = E\left(e^{i[\omega_0 t + \varphi(t)]}\right) E\left[e^{i\varphi_0}\right] = 0 \\ E[V^*(t)] = E\left(e^{-i[\omega_0 t + \varphi(t)]}\right) E\left[e^{-i\varphi_0}\right] = 0, \end{cases}$$

which imply

$$E[v(t)] = E\left[\frac{V(t) + V^*(t)}{2}\right] = 0.$$

Next, the autocorrelation function of the process $v(t)$ can be written in the form

$$R_v(t_2, t_1) = E\left[v(t_2)v(t_1)\right] = E\left\{\left[\frac{V(t_2) + V^*(t_2)}{2}\right]\left[\frac{V(t_1) + V^*(t_2)}{2}\right]\right\}$$

$$= \frac{1}{4}\left\{E\left[V(t_2)V(t_1)\right] + E\left[V(t_2)V^*(t_1)\right] + E\left[V^*(t_2)V(t_1)\right]\right.$$

$$\left. + E\left[V^*(t_2)V^*(t_1)\right]\right\}. \tag{18}$$

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Evaluating the first term yields

$$E[V(t_2)V(t_1)] = E[e^{i\omega_0(t_2+t_1)}e^{i[\varphi(t_2)+\varphi(t_1)]}] E[e^{i2\varphi_0}] = 0. \qquad (19)$$

The second term in equation (18) can be written in the form

$$E[V(t_2)V^*(t_1)] = R_\Phi(t_2,t_1)e^{i\omega_0(t_2-t_1)}, \qquad (20)$$

or equivalently,

$$R_V(t_2,t_1) = R_\Phi(t_2,t_1)e^{i\omega_0(t_2-t_1)}, \qquad (21)$$

where

$$R_\Phi(t_2,t_1) = E[\Phi(t_2)\Phi^*(t_1)] = E\left(e^{i[\varphi(t_2)-\varphi(t_1)]}\right). \qquad (22)$$

The second and third terms in equation (18) are obtained from the first and second terms by interchanging t_1 and t_2, or equivalently, by complex conjugation. Therefore, substituting equations (19) and (20) into equation (18), one obtains

$$R_v(t_2,t_1) = \frac{1}{4}\left[R_\Phi(t_2,t_1)e^{i\omega_0\tau} + R_\Phi^*(t_2,t_1)e^{-i\omega_0\tau}\right], \qquad (23)$$

where $\tau = t_2 - t_1$. This equation expresses the autocorrelation function of $v(t)$ in terms of the autocorrelation function of $\Phi(t)$. As stated previously, it is desirable that the random process $v(t)$ be wide-sense stationary. If R_Φ is only a function of $\tau = t_2 - t_1$, then so is R_v. Thus, a sufficient condition for stationarity is easy to obtain. If the first order p.d.f. of the increment $\varphi(t_2) - \varphi(t_1)$ appearing in equation (22) depends only on $t_2 - t_1$, then $R_\Phi(t_2,t_1)$ is only a function of τ. Consequently, equation (23) takes the form

$$R_v(\tau) = \frac{1}{4}\left[R_\Phi(\tau)e^{i\omega_0\tau} + R_\Phi^*(\tau)e^{-i\omega_0\tau}\right]. \qquad (24)$$

Therefore, the autocorrelation function of the process $v(t)$ only depends on the time difference τ, and since $E[v(t)] = 0$, this implies that $v(t)$ is wide-sense stationary. In general, for a Gaussian phase modulation, a necessary and sufficient condition for the process $v(t)$ to be wide-sense stationary is given in appendix A.

Having obtained the autocorrelation function of $v(t)$, it is now possible to obtain the power spectrum by simply taking the Fourier transform of $R_v(\tau)$. Using the well known frequency-shift theorem, the Fourier transform of equation (24) is given by

$$S_v(\omega) = \frac{1}{4}\left[S_\Phi(\omega-\omega_0) + S_\Phi(\omega+\omega_0)\right], \qquad (25)$$

where $S_\Phi(\omega)$ is the power spectrum of $\Phi(t)$. Of course, $S_\Phi(\omega)$ is real since by equation (22), $R_\Phi(-\tau) = R_\Phi^*(\tau)$. Equation (25) gives the power spectrum of the oscillator signal $v(t)$ in terms of the power spectrum of $\Phi(t)$. In order to proceed with the analysis, it is necessary to assume some particular form for the process $\varphi(t)$.

Random-Walk Phase Modulation

It is known that due to thermal and shot noise, the phase $\varphi(t)$ in an electrical oscillator executes a continuous random walk (refs 10 and 11). Therefore, it will be assumed that $\varphi(t)$ is a zero mean Wiener process with parameter α. The first order p.d.f. of $\varphi(t)$ is given by

$$f(\varphi, t) = \frac{1}{\sqrt{2\pi\alpha t}}\, e^{-\varphi^2/2\alpha t}, \tag{26}$$

where $t > 0$. According to Edson (ref 10), the coherence time τ_c is defined to be the time interval required for the standard deviation (variance) of the phase to increase by 1 radian. For the Wiener process (26), this condition takes the form

$$\mathrm{Var}\left[\varphi(t)\right] = E\left[\varphi^2(t)\right] = \alpha t = 1, \tag{27}$$

and therefore, the coherence time is given by

$$\tau_c = 1/\alpha. \tag{28}$$

One way to define the Wiener process is by the integral

$$\varphi(t) = \int_0^t w(s)\, ds,$$

where $w(t)$ is a zero mean, stationary, white noise Gaussian process with autocorrelation function $R_w(\tau) = \alpha\delta(\tau)$. Thus, for a random-walk phase modulation the instantaneous frequency deviation $\omega(t) - \omega_0 = w(t)$ is a zero mean, stationary, white noise Gaussian process where the rms frequency fluctuations are given by

$$\sigma_w^2 = \mathrm{Var}\left[\omega(t) - \omega_0\right] = \mathrm{Var}\left[w(t)\right] = \alpha. \tag{29}$$

The variance σ_w^2 is related to the linewidth of the power spectrum. Combining equations (28) and (29) shows that the rms frequency fluctuations are related to the coherence time by the relation

$$\sigma_w^2 = \frac{1}{\tau_c}. \tag{30}$$

To avoid the use of the stochastic integral, the Wiener process can also be defined by the following properties:

1. $\varphi(0) = 0$.

2. $f\big(\varphi(t_2) = \varphi_2 \,|\, \varphi(t_1) = \varphi_1\big) =$

$$\frac{1}{\sqrt{2\pi\alpha(t_2 - t_1)}} \exp\left[-\frac{(\varphi_2 - \varphi_1)^2}{2\alpha(t_2 - t_1)}\right], \quad t_2 > t_1 \geq 0.$$

3. The increments $\varphi(t_2)-\varphi(t_1)$ and $\varphi(t_4)-\varphi(t_3)$ are statistically independent for every set of points $0 \leq t_1 < t_2 \leq t_3 < t_4$.

4. The sample functions $\varphi(t)$ are continuous.

These properties have the following simple consequences:

5. The first order p.d.f. of the increment $\varphi(t + \tau) - \varphi(t)$ is independent of t.

6. The joint p.d.f. of $\varphi(t_1), \varphi(t_2), \ldots, \varphi(t_n)$, is Gaussian for every sequence $0 \leq t_1 < t_2 < \cdots < t_n$.

It is well known (ref 9) that the autocorrelation function of $\varphi(t)$ is given by

$$R_\varphi(t_2, t_1) = \alpha \min(t_1, t_2). \tag{31}$$

This shows that $\varphi(t)$ *is not a stationary process.* Nevertheless, the process $v(t)$ turns out to be stationary.

The properties of the Wiener process will now be used to compute the power spectrum of the oscillator signal $v(t)$. Since a linear combination of Gaussian random variables is itself a Gaussian random variable, the increment $\varphi(t_2) - \varphi(t_1)$ has a Gaussian p.d.f. with mean value zero and variance $\alpha|t_2 - t_1|$. Thus, the autocorrelation function

$$R_\Phi(t_2, t_1) = E\big(e^{i[\varphi(t_2)-\varphi(t_1)]}\big) \tag{32}$$

is essentially the characteristic function of the Gaussian random variable $\varphi(t_2) - \varphi(t_1)$. From probability theory, if x is a Gaussian random variable with mean μ and variance σ^2, then the characteristic function of x is

$$E\big[e^{ixz}\big] = e^{i\mu z - \sigma^2 z^2/2}. \tag{33}$$

Using this result together with property 2 of the Wiener process, it follows that

$$R_\Phi(t_2, t_1) = e^{-\alpha|t_2-t_1|/2}, \tag{34}$$

or more simply,

$$R_\Phi(\tau) = e^{\alpha|\tau|/2}. \tag{35}$$

Taking the Fourier transform of this function yields the power spectrum

$$S_\Phi(\omega) = \frac{\alpha}{(\alpha/2)^2 + \omega^2}. \tag{36}$$

This is the well known Lorentzian spectrum. The half-width at half-maximum of the spectral peak is given by

$$\omega_{3dB} = \frac{\alpha}{2} = \frac{1}{2\tau_c}. \tag{37}$$

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This relates the coherence time to the the linewidth of the power spectrum. Using equations (36) and (25), the power spectrum of the oscillator signal $v(t)$ is given by

$$S_v(\omega) = \frac{1}{4}\left[\frac{\alpha}{(\alpha/2)^2 + (\omega - \omega_0)^2} + \frac{\alpha}{(\alpha/2)^2 + (\omega + \omega_0)^2}\right]. \tag{38}$$

This is the desired result. The PSD (38) consists of two Lorentzians symmetrically located at $\omega = \pm\omega_0$.

Phase Noise of the Source Oscillator

The phase noise of the source oscillator can be described by either the RF power spectrum $S_v(\omega)$ or by the normalized phase-noise spectrum $\mathcal{L}(f)$. These two spectral representations are related through equation (12). The average power in the signal $v(t)$, the total signal power, is given by

$$P_0 = E\big[v^2(t)\big] = R_v(0) = \frac{1}{2}, \tag{39}$$

where the value 1/2 is obtained from equations (24) and (35). Substituting equations (39) and (38) into equation (12), the SSB phase noise of the source oscillator is given by

$$\mathcal{L}(f) = \left[\frac{\alpha}{(\alpha/2)^2 + \omega^2} + \frac{\alpha}{(\alpha/2)^2 + (\omega + 2\omega_0)^2}\right]_{\omega = 2\pi f}. \tag{40}$$

For a frequency offset ω such that $|\omega| \ll \omega_0$, the magnitude of the first term is much greater than the magnitude of the second term. Hence, it is a good approximation to simply write

$$\mathcal{L}(f) = \left[\frac{\alpha}{(\alpha/2)^2 + \omega^2}\right]_{\omega = 2\pi f}. \tag{41}$$

Thus, the normalized SSB phase-noise spectrum has the same Lorentzian spectral profile as the RF power spectrum.

This result can also be obtained by means of the mixer circuit in figure 1b. In this circuit, the noisy source oscillator $v(t)$ is mixed with a phase-noise-free reference oscillator $u(t) = \cos(\omega_0 t)$. The input to the low-pass filter is

$$v(t)u(t) = \cos[\omega_0 t + \varphi(t) + \varphi_0]\cos(\omega_0 t). \tag{42}$$

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Using the cosine identity

$$\cos A \cos B = \frac{1}{2}\big[\cos(A - B) + \cos(A + B)\big], \tag{43}$$

together with the fact that $\varphi(t)$ is a lowpass process, the filtered mixer output is given by

$$y(t) = \frac{1}{2}\cos\big[\varphi(t) + \varphi_0\big]. \tag{44}$$

Since $\varphi(t)$ is not strictly a low-pass process, this equality is not exact. A random process $x(t)$ is a strictly low-pass process if $S_x(\omega) = 0$ for $|\omega| > B$, where $B > 0$ is the cutoff frequency. Nevertheless, for an oscillator with good spectral purity, the process $\varphi(t)$, and therefore the process $\exp[i\varphi(t)]$, is very nearly a low-pass process so that equation (44) is a reasonable approximation. To continue with the analysis, let $Y(t)$ be the complex signal

$$Y(t) = \frac{1}{2}e^{i[\varphi(t) + \varphi_0]}. \tag{45}$$

Note that $2Y(t) = \Phi(t)$ as defined by equation (17). Proceeding as in section 2, it is clear that $E[Y(t)] = E[Y^*(t)] = 0$, and therefore $E[y(t)] = 0$. The autocorrelation function of $y(t)$ can be written in the form

$$
\begin{aligned}
R_y(t_2, t_1) = E\big[y(t_2)y(t_1)\big] &= E\left\{\left[\frac{Y(t_2) + Y^*(t_2)}{2}\right]\left[\frac{Y(t_1) + Y^*(t_1)}{2}\right]\right\} \\
&= \frac{1}{4}\Big\{E\big[Y(t_2)Y(t_1)\big] + E\big[Y(t_2)Y^*(t_1)\big] + E\big[Y^*(t_2)Y(t_1)\big] \\
&\qquad + E\big[Y^*(t_2)Y^*(t_1)\big]\Big\}.
\end{aligned}
\tag{46}
$$

Using the results

$$E\big[Y(t_2)Y(t_1)\big] = E\big[Y^*(t_2)Y^*(t_1)\big] = 0,$$

and

$$E\big[Y(t_2)Y^*(t_1)\big] = \frac{1}{4}E\left[e^{i[\varphi(t_2) - \varphi(t_1)]}\right] = \frac{1}{4}R_\Phi(\tau),$$

equation (46) implies

$$R_y(\tau) = \frac{1}{8}R_\Phi(\tau), \tag{47}$$

where R_Φ is defined by equation (22) and $\tau = t_2 - t_1$. By taking the Fourier transform, this immediately yields the power spectrum

$$S_y(\omega) = \frac{1}{8}S_\Phi(\omega). \tag{48}$$

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Substituting the result (36) for a random-walk phase modulation, one obtains

$$S_y(\omega) = \frac{1}{8}\left[\frac{\alpha}{(\alpha/2)^2 + \omega^2}\right]. \tag{49}$$

This is the desired power spectrum. Equation (12) will now be used to calculate the normalized power spectrum $\mathcal{L}(f)$. From equation (47), the total average power in the signal $y(t)$ is found to be $P_0 = 1/8$. Combining this with equation (49), the normalized PSD of $y(t)$ is given by

$$\mathcal{L}(f) = \left[\frac{2\alpha}{(\alpha/2)^2 + \omega^2}\right]_{\omega=2\pi f}. \tag{50}$$

Except for an additional factor of two, this is identical to equation (41). Consequently, the normalized phase noise spectrum of $y(t)$ is two times the normalized phase noise spectrum of the source oscillator $v(t)$.

For the purpose of making calculations, the phase-noise spectrum (41) can be written in the more convenient form

$$\mathcal{L}(f) = \frac{1}{\pi f_{3dB}}\left[\frac{1}{1 + (f/f_{3dB})^2}\right], \tag{51}$$

where the HWHM is given by

$$f_{3dB} = \frac{\alpha}{4\pi}. \tag{52}$$

Alternatively, this can be expressed in dB notation in the form

$$\mathcal{L}(f) = 10\log_{10}\left\{\frac{1}{\pi f_{3dB}}\left[\frac{1}{1 + (f/f_{3dB})^2}\right]\right\} \quad \text{dBc/Hz.} \tag{53}$$

For a 36 GHz oscillator with a linewidth parameter $\eta = 10^{-7}$, the normalized phase-noise spectrum of $v(t)$, equation (53), is plotted in figure 3. The phase noise of the signal $y(t)$ can be obtained by simply adding 3 dB. The spectrum in figure 3 is the well known Lorentzian. It is flat from $f = 0$ to $f_{3dB} = 3600$ Hz and then rolls off with a slope of 20 dB per decade. In the low frequency limit, as $f \to 0$, $\mathcal{L}(f) \to -10\log(\pi f_{3dB}) = -40.5$ dBc/Hz.

Autocorrelation Function of the Mixer Output

With reference to figure 1a, the two signals entering the mixer are the oscillator signal $v(t)$ and the received signal $\varepsilon v(t - \tau_0)$, where $\tau_0 > 0$ is the time delay
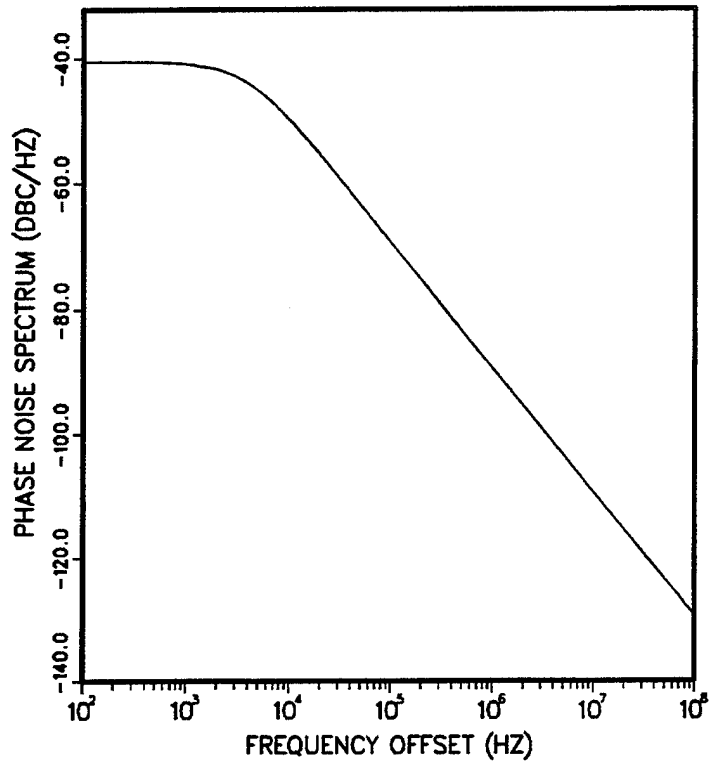
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Figure 3.   Normalized power spectrum for a 36 GHz
oscillator with a HWHM (3 dB frequency) of 3600 Hz.

of the received signal and $\varepsilon > 0$ is its magnitude. The input to the low-pass filter is the product

$$\varepsilon v(t)v(t - \tau_0) = \varepsilon \cos[\omega_0 t + \varphi(t) + \varphi_0] \cos[\omega_0(t - \tau_0) + \varphi(t - \tau_0) + \varphi_0].$$

Using the trigonometric identity

$$\cos A \cos B = \frac{1}{2}\big[\cos(A - B) + \cos(A + B)\big],$$

together with the assumption that $\varphi(t)$ is a low-pass process, the filtered mixer output is given by

$$x(t) = \frac{\varepsilon}{2} \cos\big[\omega_0 \tau_0 + \varphi(t) - \varphi(t - \tau_0)\big], \tag{54}$$

where $t \geq \tau_0$. Since $\varphi(t)$ is not strictly a low-pass process this equality is not exact. However, for an oscillator with good spectral purity $\varphi(t)$ is very nearly a low-pass process so that equation (54) is a good approximation. Proceeding as before, define the complex signal

$$X(t) = e^{i[\omega_0 \tau_0 + \varphi(t) - \varphi(t - \tau_0)]}, \tag{55}$$

where $t \geq \tau_0$. Using definition (22) together with the fact that $R_\Phi(\tau)$ is real, it is clear that

$$\begin{cases} E\big[X(t)\big] = e^{i\omega_0 \tau} R_\Phi(\tau) \\ E\big[X^*(t)\big] = e^{-i\omega_0 \tau} R_\Phi(\tau). \end{cases}$$

Therefore, the mean of the process $x(t)$ is given by

$$E\big[x(t)\big] = \frac{\varepsilon}{2} E\left[\frac{X(t) + X^*(t)}{2}\right] = \frac{\varepsilon}{2} \cos(\omega_0 \tau_0) R_\Phi(\tau_0). \tag{56}$$

Substituting the result (35) for the Wiener process, one obtains

$$E\big[x(t)\big] = \frac{\varepsilon}{2} \cos(\omega_0 \tau_0) e^{-\alpha \tau_0/2}. \tag{57}$$

Due to the phase noise of the source oscillator, the expected D.C. output level $(\varepsilon/2)\cos(\omega_0 \tau_0)$ is reduced in magnitude by the factor $e^{-\alpha \tau_0/2}$. However, for the small delays usually encountered in practice, the inequality $\alpha \tau_0 \ll 1$ is well satisfyied so that $e^{-\alpha \tau_0/2} \simeq 1$. In this case, this factor is neglidgible and equation (57) reduces to the usual result obtained in the absence of any phase noise. For large time delays such that $\alpha \tau_0 \gg 1$, the two mixer inputs become uncorrelated and $E[x(t)] \to 0$.

The rest of this section will be devoted to a calculation of the autocorrelation function of $x(t)$. Using the complex signal $X(t)$, this autocorrelation function may be written in the form

$$R_x(t_2, t_1) = E\big[x(t_2)x(t_1)\big] = \frac{\varepsilon^2}{4}E\left\{\left[\frac{X(t_2) + X^*(t_2)}{2}\right]\left[\frac{X(t_1) + X^*(t_1)}{2}\right]\right\}$$

$$= \frac{\varepsilon^2}{16}\Big\{E\big[X(t_2)X(t_1)\big] + E\big[X(t_2)X^*(t_1)\big]$$

$$+ E\big[X^*(t_2)X(t_1)\big] + E\big[X^*(t_2)X^*(t_1)\big]\Big\}. \tag{58}$$

It is convenient to define the shorthand notation

$$\begin{cases} \Delta\varphi_1 = \varphi(t_1) - \varphi(t_1 - \tau_0) \\ \Delta\varphi_2 = \varphi(t_2) - \varphi(t_2 - \tau_0). \end{cases}$$

Thus, the first and second terms in equation (58) may be written in the form

$$E\big[X(t_2)X(t_1)\big] = e^{i2\omega_0\tau_0}E\big[e^{i(\Delta\varphi_2 + \Delta\varphi_1)}\big], \tag{59}$$

$$E\big[X(t_2)X^*(t_1)\big] = \quad E\big[e^{i(\Delta\varphi_2 - \Delta\varphi_1)}\big]. \tag{60}$$

These expectations may be evaluated as follows: assume that $t_2 \geq t_1 \geq \tau_0$. There are two cases to consider (figure 4): case 1 where $t_1 \leq t_2 - \tau_0$, and case 2 where $t_2 - \tau_0 \leq t_1$. In case 1, $\Delta\varphi_1$ and $\Delta\varphi_2$ are independent and identically distributed Gaussian random variables with mean zero and variance $\alpha\tau_0$. Therefore, using the known characteristic function for a single Gaussian random variable, one obtains

$$E\big[e^{i(\Delta\varphi_2 + \Delta\varphi_1)}\big] = E\big[e^{i\Delta\varphi_2}\big]E\big[e^{i\Delta\varphi_1}\big] = e^{-\alpha\tau_0}, \quad |t_2 - t_1| \geq \tau_0. \tag{61}$$

Similarly, the random variable $(-\Delta\varphi_1)$ is also Gaussian with mean zero and variance $\alpha\tau_0$, which implies

$$E\big[e^{i(\Delta\varphi_2 - \Delta\varphi_1)}\big] = E\big[e^{i\Delta\varphi_2}\big]E\big[e^{-i\Delta\varphi_1}\big] = e^{-\alpha\tau_0}, \quad |t_2 - t_1| \geq \tau_0. \tag{62}$$

In case 2, $\Delta\varphi_1$ and $\Delta\varphi_2$ are no longer independent. In this case, let

$$\begin{cases} x_1 = \varphi(t_2 - \tau_0) - \varphi(t_1 - \tau_0), \\ x_2 = \varphi(t_1) - \varphi(t_2 - \tau_0), \\ x_3 = \varphi(t_2) - \varphi(t_1). \end{cases}$$
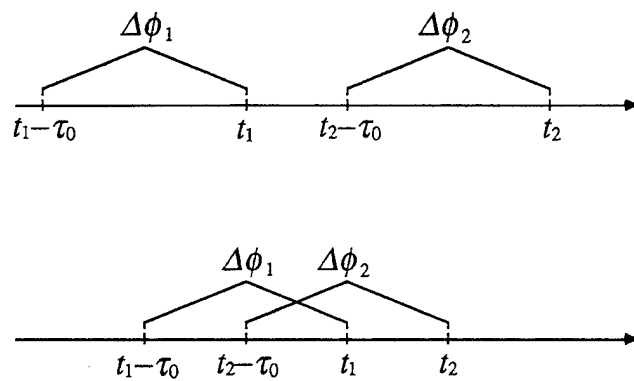
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Figure 4. Illustration of overlap between the phase increments at times 1 and 2.

Because $\varphi(t)$ is a Wiener process, the quantities x_1, x_2, and x_3 are mutually independent Gaussian random variables with mean values $E[x_1] = E[x_2] = E[x_3] = 0$, and variances

$$\begin{cases} \sigma_1^2 = \alpha |t_2 - t_1|, \\ \sigma_2^2 = \alpha \big(\tau_0 - |t_2 - t_1|\big), \\ \sigma_3^2 = \alpha |t_2 - t_1|. \end{cases}$$

Therefore, by virtue of the relations

$$\begin{cases} \Delta\varphi_1 = x_1 + x_2 \\ \Delta\varphi_2 = x_2 + x_3, \end{cases}$$

it follows that

$$\begin{cases} \Delta\varphi_2 + \Delta\varphi_1 = x_1 + 2x_2 + x_3 \\ \Delta\varphi_2 - \Delta\varphi_1 = x_3 - x_1. \end{cases}$$

Thus, in case 2, one finds

$$\begin{aligned} E\big[e^{i(\Delta\varphi_2 + \Delta\varphi_1)}\big] &= E\big[e^{ix_1}\big] E\big[e^{i2x_2}\big] E\big[e^{ix_3}\big] \\ &= \big(e^{-\sigma_1^2/2}\big)\big(e^{-\sigma_2^2 2^2/2}\big)\big(e^{-\sigma_3^2/2}\big) \\ &= e^{-\alpha(2\tau_0 - |\tau|)}, \qquad |\tau| \le \tau_0, \end{aligned} \tag{63}$$

where $\tau = t_2 - t_1$. And similarly,

$$\begin{aligned} E\big[e^{i(\Delta\varphi_2 - \Delta\varphi_1)}\big] &= E\big[e^{ix_3}\big] E\big[e^{-ix_1}\big] \\ &= \big(e^{-\sigma_3^2/2}\big)\big(e^{-\sigma_1^2/2}\big) \\ &= e^{-\alpha|\tau|}, \qquad |\tau| \le \tau_0. \end{aligned} \tag{64}$$

Substituting equations (63) and (64) into equations (59) and (60), one finds

$$E\big[X(t_2)X(t_1)\big] = e^{i2\omega_0\tau_0} \begin{cases} e^{-\alpha(2\tau_0 - |\tau|)} & \text{if} \quad |\tau| \le \tau_0, \\ e^{-\alpha\tau_0} & \text{if} \quad |\tau| \ge \tau_0; \end{cases} \tag{65}$$

and

$$E\big[X(t_2)X^*(t_1)\big] = \begin{cases} e^{-\alpha|\tau|} & \text{if} \quad |\tau| \le \tau_0, \\ e^{-\alpha\tau_0} & \text{if} \quad |\tau| \ge \tau_0. \end{cases} \tag{66}$$

These equations can be written more compactly in the form

$$E\big[X(t_2)X(t_1)\big] = e^{-\alpha\tau_0} e^{i2\omega_0\tau_0} f(\tau), \tag{67}$$

$$E\big[X(t_2)X^*(t_1)\big] = e^{-\alpha\tau_0} g(\tau), \tag{68}$$

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where

$$f(\tau) = \begin{cases} e^{-\alpha(\tau_0 - |\tau|)} & \text{if} \quad |\tau| \leq \tau_0, \\ 1 & \text{if} \quad |\tau| \geq \tau_0; \end{cases} \quad (69)$$

and

$$g(\tau) = \begin{cases} e^{\alpha(\tau_0 - |\tau|)} & \text{if} \quad |\tau| \leq \tau_0, \\ 1 & \text{if} \quad |\tau| \geq \tau_0. \end{cases} \quad (70)$$

Note that $f(-\tau) = f(\tau)$ and $g(\tau) = 1/f(\tau)$. Substituting equations (67) and (68) into equation (58), the autocorrelation function of the filtered mixer output $x(t)$ is found to be

$$R_x(t_2, t_1) = \frac{\varepsilon^2}{8} e^{-\alpha\tau_0} \left[\cos(2\omega_0\tau_0) f(\tau) + g(\tau) \right]. \quad (71)$$

Since this function only depends on the time difference $\tau = t_2 - t_1$, the process $x(t)$ is wide-sense stationary as required.

Power Spectrum of the Mixer Output

Taking the Fourier transform of the autocorrelation function (71), the power spectrum of $x(t)$ is given by

$$S_x(\omega) = \frac{\varepsilon^2}{8} e^{-\alpha\tau_0} \left[\cos(2\omega_0\tau_0) F(\omega) + G(\omega) \right], \quad (72)$$

where $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(\tau)$ and $g(\tau)$, respectively. In order to compute these transforms, let

$$\begin{cases} \widetilde{f}(\tau) = f(\tau) - 1 \\ \widetilde{g}(\tau) = g(\tau) - 1, \end{cases} \quad (73)$$

so that

$$\begin{cases} \widetilde{F}(\omega) = F(\omega) - \delta(\omega) \\ \widetilde{G}(\omega) = G(\omega) - \delta(\omega). \end{cases} \quad (74)$$

Using equations (69) and (70), a straightforward evaluation of these transforms yields

$$\widetilde{F}(\omega) = \frac{2\alpha}{\alpha^2 + \omega^2} \left[\cos(\omega\tau_0) - (\alpha\tau_0)\frac{\sin(\omega\tau_0)}{(\omega\tau_0)} - e^{-\alpha\tau_0} \right], \quad (75)$$

and

$$\widetilde{G}(\omega) = \frac{-2\alpha}{\alpha^2 + \omega^2} \left[\cos(\omega\tau_0) + (\alpha\tau_0)\frac{\sin(\omega\tau_0)}{(\omega\tau_0)} - e^{+\alpha\tau_0} \right]. \quad (76)$$

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Note that $\widetilde{G}(\omega)$ is obtained from $\widetilde{F}(\omega)$ by simply reversing the sign of α. Substituting equations (74), (75), and (76) into equation (72) one obtains

$$S_x(\omega) = \frac{\varepsilon^2}{8} e^{-\alpha\tau_0} \left\{ [\cos(2\omega_0\tau_0) + 1]\delta(\omega) + \left(\frac{2\alpha}{\alpha^2 + \omega^2}\right) A(\omega) \right\}, \qquad (77)$$

where

$$A(\omega) = \cos(\omega\tau_0)\big[\cos(2\omega_0\tau_0) - 1\big]$$

$$- (\alpha\tau_0)\frac{\sin(\omega\tau_0)}{(\omega\tau_0)}\big[\cos(2\omega_0\tau_0) + 1\big]$$

$$+ e^{+\alpha\tau_0} - e^{-\alpha\tau_0}\cos(2\omega_0\tau_0). \qquad (78)$$

Using the identities

$$\begin{cases} e^x = \cosh(x) + \sinh(x) \\ e^{-x} = \cosh(x) - \sinh(x), \end{cases}$$

and

$$\begin{cases} 1 - \cos(2x) = 2\sin^2 x \\ 1 + \cos(2x) = 2\cos^2 x, \end{cases}$$

the function $A(\omega)$ can be written in the more compact form

$$A(\omega) = \big[\cosh(\alpha\tau_0) - \cos(\omega\tau_0)\big] 2\sin^2(\omega_0\tau_0)$$

$$+ \left[\sinh(\alpha\tau_0) - (\alpha\tau_0)\frac{\sin(\omega\tau_0)}{(\omega\tau_0)}\right] 2\cos^2(\omega_0\tau_0). \qquad (79)$$

Discussion: The power spectrum (77) is the sum of two terms where the first term is a delta function and the second term is a continuous function. The delta function in equation (77) cooresponds to the power contained in the average D.C. signal

$$E\big[x(t)\big] = \mu = \frac{\varepsilon}{2} e^{-\alpha\tau_0/2} \cos(\omega_0\tau_0). \qquad (80)$$

For the purpose of studying the phase noise, the delta function part of the power spectrum will be separated from the continuous part. This is achieved by separately considering the power spectra of the two signals $x(t) - \mu$ and $\mu = $ const. Mathematically, this amounts to using the autocovariance of the process $x(t)$ instead of the autocorrelation. The autocovariance of the process $x(t)$ is defined by

$$C(t_2, t_1) = E\Big\{ \big[x(t_2) - \mu(t_2)\big] \big[x(t_1) - \mu(t_1)\big] \Big\}$$

$$= R(t_2, t_1) - \mu(t_2)\mu(t_1), \qquad (81)$$

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where $\mu(t) = E[x(t)]$. For the wide-sense stationary process $x(t)$, the mean is constant and $C_x(\tau) = R_x(\tau) - \mu^2$ so that by taking a Fourier transform,

$$C_x(\omega) = S_x(\omega) - \mu^2 \delta(\omega). \tag{82}$$

Hence, the delta function part of the power spectrum has been subtracted out and $C_x(\omega)$ is simply the continuous part of the power spectrum. Obviously, if the process mean is zero, then $C(\omega) = S(\omega)$. For the mixer output $x(t)$, the continuous part of the power spectrum is given, using equation (77), by

$$C_x(\omega) = \frac{\varepsilon^2}{8} e^{-\alpha\tau_0} \left(\frac{2\alpha}{\alpha^2 + \omega^2} \right) A(\omega). \tag{83}$$

This equation is one of the main results of this report. For a fixed time delay τ_0, the first two factors are positive constants. The frequency dependence of the power spectrum is contained in the last two factors. The third factor is the well known Lorentzian spectrum with a HWHM given by

$$\omega_{3dB} = \alpha \doteq \frac{1}{\tau_c}, \tag{84}$$

where τ_c is the correlation time of the oscillator signal $v(t)$. Comparing this with equation (37), the HWHM of the mixer output $x(t)$ is twice the HWHM of the source oscillator $v(t)$. Therefore, one effect of the mixing process is a broadening of the Lorentzian part of the power spectrum by a factor of 2. In addition, and more importantly, the amplitude of the spectrum is modulated by the factor $A(\omega)$. It is this modulation factor, $A(\omega)$, which is responsible for the correlation effect, i.e., the reduction in phase-noise of the mixer output. The behavior of the function $A(\omega)$ is investigated in appendix B.

Phase Noise of the Mixer Output

The normalized power spectrum of the mixer output $x(t)$ can be calculated from the PSD $S_x(\omega)$. Using equation (71), the total average power in the signal $x(t)$ is given by

$$P_0 = E[x^2(t)] = R_x(0) = \frac{\varepsilon^2}{8} e^{-\alpha\tau_0} \left[e^{-\alpha\tau_0} \cos(2\omega_0\tau_0) + e^{+\alpha\tau_0} \right]. \tag{85}$$

Applying various trigonometric and exponential identities, this may be put into the form

$$P_0 = \frac{\varepsilon^2}{4} e^{-\alpha\tau_0} \left[\cosh(\alpha\tau_0) \cos^2(\omega_0\tau_0) + \sinh(\alpha\tau_0) \sin^2(\omega_0\tau_0) \right]. \tag{86}$$

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For the small time delays met in practice, it is almost always true that $\alpha\tau_0 \ll 1$. Therefore, making the approximation $e^{+\alpha\tau_0} \simeq e^{-\alpha\tau_0} \simeq 1$, equation (85) takes the simplified form

$$P_0 \simeq \frac{\varepsilon^2}{8}\left[\cos(2\omega_0\tau_0) + 1\right] = \frac{\varepsilon^2}{4}\cos^2(\omega_0\tau_0). \tag{87}$$

This is a well known result which says that the output power is proportional to $\cos^2(\omega_0\tau_0)$, essentially the cosine squared of the delay (or range). Of course, this result is exact when the phase noise of the source is zero. But, even in the presence of phase noise, this result is a very good approximation as long as the phase noise of the source oscillator is low and the time delay is small, i.e., $\alpha\tau_0 \ll 1$.

In order to calculate the phase-noise spectrum, the delta function component of the PSD (77) must be omitted. This is because the average D.C. component of the signal $x(t)$ does not contribute to the SSB phase noise $\mathcal{L}(f)$. Using the continuous part of the PSD, equation (83), together with equation (12), the normalized phase-noise spectrum is given by

$$\mathcal{L}(f) = \left.\frac{2C_x(\omega)}{P_0}\right|_{\omega=2\pi f} = \left(\frac{4\alpha}{\alpha^2 + \omega^2}\right)\left[\frac{A(\omega)}{B}\right]_{\omega=2\pi f}, \tag{88}$$

where

$$B = \cosh(\alpha\tau_0)\cos^2(\omega_0\tau_0) + \sinh(\alpha\tau_0)\sin^2(\omega_0\tau_0). \tag{89}$$

The factor of two on the right side of equation (88) is due to the fact that both positive and negative frequency components contribute equally to the measured noise power. The function $A(\omega)$ may be written in the compact form

$$A(\omega) = \left\{\left[1 - \cos(\omega\tau_0)\right] + a\right\}2\sin^2(\omega_0\tau_0)$$
$$+ \left\{(\alpha\tau_0)\left[1 - \frac{\sin(\omega\tau_0)}{(\omega\tau_0)}\right] + b\right\}2\cos^2(\omega_0\tau_0), \tag{90}$$

where

$$\begin{cases} a = \cosh(\alpha\tau_0) - 1 \\ b = \sinh(\alpha\tau_0) - (\alpha\tau_0). \end{cases} \tag{91}$$

The same comments made for equation (83) are applicable to equation (88). To reiterate, equation (88) shows that the phase-noise spectrum of the mixer output is the product of two terms. The first term is just the Lorentzian spectral function. If equation (88) is rewritten in the form

$$\mathcal{L}(f) = \left(\frac{2\alpha}{\alpha^2 + \omega^2}\right)\left[\frac{2A(\omega)}{B}\right]_{\omega=2\pi f}, \tag{92}$$

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then by comparison with equation (41) it is clear that the linewidth of the Lorentzian term is two times the linewidth of the source oscillator. This is one effect of the mixing process. A more important effect arises from the second term $2A(\omega)/B$ which imposes a modulation onto the Lorenzian. It is this modulation factor which is responsible for the coherence effect, i.e., phase-noise reduction.

Phase Noise Reduction Factor

As stated in the introduction, the main objective is to compare the phase noise of the mixer output in figure 1a to the phase noise of the source. The ratio of the power spectral density of the filtered mixer output $x(t)$ to that of the source oscillator $y(t)$ is given by

$$K(\omega) = \frac{S_x(\omega)}{S_y(\omega)}. \tag{93}$$

This will be called the *reduction factor*. Substituting the power spectra (49) and (83) into (93), and assuming there is no attenuation of the delayed signal ($\varepsilon = 1$), the reduction factor is found to be

$$K(\omega) = e^{-\alpha\tau_0} \left[\frac{(\alpha/2)^2 + \omega^2}{\alpha^2 + \omega^2} \right] 2A(\omega), \tag{94}$$

where $A(\omega)$ is given by equation (7.6). This equation is the main result of this report. With minor modifications, equation (8.2) may be written in the form

$$K(\omega) = e^{-\alpha\tau_0} \left[\frac{(\alpha\tau_0/2)^2 + (\omega\tau_0)^2}{(\alpha\tau_0)^2 + (\omega\tau_0)^2} \right] 2A(\omega). \tag{95}$$

For the time delays encountered in practice, it is generally true that $\alpha\tau_0 \ll 1$. Consequently, the first and second factors in equation (95) are approximately unity, and it is a good approximation to write

$$K(\omega) \simeq 2A(\omega). \tag{96}$$

Therefore, the frequency response of $K(\omega)$ is contained in the function $A(\omega)$ which again is given by:

$$A(\omega) = \left\{ \left[1 - \cos(\omega\tau_0) \right] + a \right\} 2\sin^2(\omega_0\tau_0)$$

$$+ \left\{ (\alpha\tau_0) \left[1 - \frac{\sin(\omega\tau_0)}{(\omega\tau_0)} \right] + b \right\} 2\cos^2(\omega_0\tau_0). \tag{97}$$

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The general behavior of this function is studied in appendix B. The first thing to note is that $A(\omega)$ is a rapidly varying function of the delay τ_0. Due to the factors $\sin^2(\omega_0\tau_0)$ and $\cos^2(\omega_0\tau_0)$, the first and second terms in equation (97) oscillate in and out of phase as τ_0 cycles through a period of $1/f_0$. For example, if $f_0 = 33$ GHz, then $1/f_0 = 30$ ps (picoseconds) which shows that these terms are oscillating very rapidly indeed. Therefore, it is necessary to consider three separate cases:

$$\text{Case (1):} \quad \omega_0\tau_0 = n\pi, \quad n = 1,2,3,\ldots;$$

$$\text{Case (2):} \quad \omega_0\tau_0 = (2n+1)(\pi/2), \quad n = 0,1,2,\ldots;$$

$$\text{Case (3):} \quad \omega_0\tau_0 = (2n+1)(\pi/4), \quad n = 0,1,2,\ldots.$$

Equivalently, these cases are characterized by

$$\text{Case (1):} \quad \cos^2(\omega_0\tau_0) = 1,$$

$$\text{Case (2):} \quad \sin^2(\omega_0\tau_0) = 1,$$

$$\text{Case (3):} \quad \sin^2(\omega_0\tau_0) = \cos^2(\omega_0\tau_0) = 1/2.$$

The behavior of $A(\omega)$ in each of these special cases is discussed in appendix B.

To illustrate the above results, the reduction factor was calculated for a 36 GHz oscillator with a linewidth parameter $\eta = 10^{-7}$. For the purpose of making calculations, it is convenient to introduce the variables

$$\begin{cases} x = \alpha\tau_0 \\ y = \omega\tau_0. \end{cases} \tag{98}$$

Making these substitutions, the reduction factor (95) takes the form

$$K(\omega) = e^{-x}\left[\frac{(x/2)^2 + y^2}{x^2 + y^2}\right] 2A(\omega), \tag{99}$$

where

$$A(\omega) = \left\{\left[1 - \cos(y)\right] + a\right\} 2\sin^2(\omega_0\tau_0)$$

$$+ \left\{(x)\left[1 - \frac{\sin(y)}{y}\right] + b\right\} 2\cos^2(\omega_0\tau_0), \tag{100}$$

and

$$\begin{cases} a = \cosh(x) - 1 \\ b = \sinh(x) - x. \end{cases} \tag{101}$$

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In the low frequency limit, as $\omega \to 0$, equation (107) shows that $K(\omega) \to (\alpha\tau_0)^2/2$ where the additional factor of $1/4$ comes from the second term in equation (94). To locate the frequency of transition from region 1 to region 2, expand the cosine in equation (107) to obtain

$$K(\omega) \approx 2\big[(\omega\tau_0)^2 + (\alpha\tau_0)^2\big], \tag{108}$$

which is valid for $\omega\tau_0$ small. The transition occurs when $\omega\tau_0 \simeq \alpha\tau_0$ or $\omega = \alpha$. To obtain the frequency of transition from region 2 to region 3, note that $A(\omega)$ reaches its maximum at $\omega\tau_0 = \pi$, i.e., $\omega = \pi/\tau_0$, where the maximum value is $A(\omega\tau_0 = \pi) = 8 + 2(\alpha\tau_0)^2$. These results may be summarized as follows.

Region 1: $\quad 0 < \omega < \alpha, \qquad K(\omega) \simeq (\alpha\tau_0)^2/2 \equiv K_0;$

Region 2: $\quad \alpha < \omega < \pi/\tau_0, \quad K(\omega) \simeq K_0 + 2(\omega\tau_0)^2;$

Region 3: $\quad \omega > \pi/\tau_0, \qquad K(\omega) \simeq 4.$

The value of 4 in region 3 is one-half the maximum value of 8.

Also plotted in figure 5 is the approximate formula

$$K(\omega) \approx 2\big[1 - \cos(\omega\tau_0)\big] = 4\sin^2(\omega\tau_0/2) \tag{109}$$

derived from FM modulation theory (refs 1 through 3). A different and much simpler derivation of equation (109) is given in appendix C. It is clear from figure 5 that equation (109) is a good approximation in regions 2 and 3 where it approaches the exact solution for case 3. However, (109) fails in the low frequency limit, that is, in region 1. The breakdown of this formula can be remedied by simply adding a constant. Thus, based on the previous analysis, a more accurate formula is

$$K(\omega) \approx 4\big[1 - \cos(\omega\tau_0)\big] + (\alpha\tau_0)^2/2, \tag{110}$$

where $\alpha = 2\omega_{3dB} = 2\eta\omega_0$ and ω_{3dB} is the HWHM of the source oscillator. This formula represents a slight improvement over equation (109).

So far, the discussion of the solution has focused on the three special cases: cases 1, 2, and 3. With this done, it is necessary to discuss what happens when the delay τ_0 lies somewhere in the range between these cases. It is possible to write equation (97) in the form

$$A(\omega) = A_1 \cos^2(\omega_0\tau_0) + A_2 \sin^2(\omega_0\tau_0), \tag{111}$$

where

$$A_1 = A_1(\omega) = 2\left\{(\alpha\tau_0)\left[1 - \frac{\sin(\omega\tau_0)}{(\omega\tau_0)}\right] + b\right\}, \tag{112}$$

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and

$$A_2 = A_2(\omega) = 2\Big\{\big[1 - \cos(\omega\tau_0)\big] + a\Big\}. \qquad (113)$$

The function $A(\omega)$ reduces to $A_1(\omega)$ in case 1 and $A_2(\omega)$ in case 2. By virtue of the identity

$$\sin^2(\theta) + \cos^2(\theta) = 1, \qquad (114)$$

equation (111) shows that the response $A(\omega)$ lies between $A_1(\omega)$ and $A_2(\omega)$ for all values of τ_0 between case 1 and 2.

There is another prominent feature of figure 5, namely, that the separation between the response curves for cases 1 and 2 is independent of frequency. This is, in fact, a general feature of the solution. Ignoring the constants a and b in equations (112) and (113), the ratio of the response in case 1 to that in case 2 is given by

$$\frac{A_1(\omega)}{A_2(\omega)} \simeq (\alpha\tau_0)\frac{1 - \sin(y)/y}{1 - \cos(y)}. \qquad (115)$$

It is interesting that the quotient appearing in equation (115) is very nearly constant on the interval from $y = 0$ to $y = \pi$. While this fact is difficult to demonstrate analytically, it is easy to show graphically in figure 6 where the value of this quotient is seen to be approximately $1/3$. Therefore, the ratio of $A_1(\omega)$ to $A_2(\omega)$ is approximately constant and

$$\frac{A_1(\omega)}{A_2(\omega)} \simeq \frac{\alpha\tau_0}{3}, \qquad (116)$$

where $\alpha\tau_0 = 2\eta\omega_0\tau_0$. Thus, for the delay $\tau_0 = 10^{-9}$ seconds in figure 5, the separation between the response curves for cases 1 and 2 is $\alpha\tau_0/3 = -48.2$ dB.

As discussed in connection with equation (97), the reduction factor is a very sensitive function of the delay τ_0. More specifically, as τ_0 cycles through a period of $1/f_0 = 28$ picoseconds the reduction factor goes from case 1 to case 2 and then back to 1 again. Of course, in the middle is case 3. By inspection of figure 5 the difference between case 2 and case 3 is 3 dB. Analytically, in case 3 one has, using equations (111) and (116),

$$A(\omega) = \frac{1}{2}\big[A_1(\omega) + A_2(\omega)\big] \simeq \frac{1}{2}A_2(\omega), \qquad (117)$$

since $\alpha\tau_0 \ll 1$. This accounts for the 3 dB difference. Note that the response lies between cases 2 and 3 for one-half of the period $1/f_0$ due to equation (111), i.e., case 3 occurs when $\theta = \pi/2$, case 2 occurs when $\theta = \pi$, and case 3 occurs again when $\theta = 3\pi/2$. The question arises: for what percent of the period $1/f_0$ is the response within 3 dB of case 3? The answer is 67 % of the time. To see
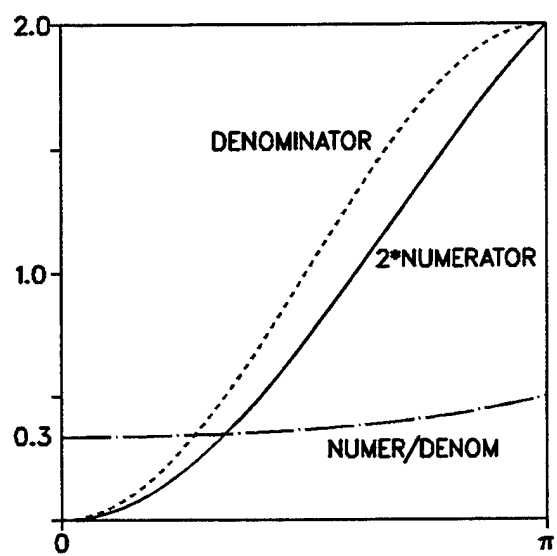
28

Figure 6. The quotient (numer/denom) in equation
(115) is approximately constant from zero to pi.

this, note that if $\theta = 0$ corresponds to case 1 and $\theta = \pi/2$ corresponds to case 3, then when $\theta = \pi/3$,

$$A(\omega) = \frac{3}{4}A_1(\omega) + \frac{1}{4}A_2(\omega) \simeq \frac{1}{4}A_2(\omega), \qquad (118)$$

which corresponds to case 3 minus 3 dB. Hence, the reduction factor is within 3 dB of the solution in case 3 throughout 67 % of the period $1/f_0$.

In practice, these rapid oscillations of the reduction factor can be ignored since in a radar system, for example, the radar return is "smeared out" in time. That is, the return signal is the sum of many components with different delays. This results in an averaging effect whereby the reduction factor is averaged over one period of the variable $\theta = \omega_0 \tau_0$. Since the average of $\sin^2(\theta)$ and $\cos^2(\theta)$ are both equal to $1/2$, the averaging effect yields

$$A(\omega) = \frac{1}{2}\big[A_1(\omega) + A_2(\omega)\big] \simeq \frac{1}{2}A_2(\omega), \qquad (119)$$

Therefore, using equation (96), the reduction factor in the presence of the averaging effect is given by

$$K(\omega) \simeq A_2(\omega) = 2\big[1 - \cos(\omega\tau_0)\big] + (\alpha\tau_0)^2, \qquad (120)$$

or, equivalently,

$$K(\omega) \simeq 4\sin^2(\omega\tau_0/2) + (\alpha\tau_0)^2. \qquad (121)$$

Conclusions

The phase noise of an amplitude stabilized oscillator was modeled as a sinewave with a random-walk phase function. The propagation of phase noise through a mixing circuit was studied to determine the reduction of phase noise due to mixing the time delayed signal with itself. The normalized (baseband) power spectrum of the source oscillator is given in equation (41) and the spectrum of the filtered mixer output is given in equation (83) or (88). The main result of the report is the reduction factor (correlation factor) which is given in equation (95). The results show that the well known formula (1) for the reduction factor is a good approximation except in the low frequency limit ($\omega \to 0$). An improved formula was given, either equation (110) or (121), which is valid for any frequency offset.

It is important to realize that the theoretical results obtained in this study are only applicable to the extent that the given mathematical model is valid. In

general, an oscillator will exhibit both AM and FM noise components; whereas, in this study it was assumed that the amplitude noise is zero. Nevertheless, for an amplitude stabilized oscillator employing current state-of-the-art technology, the AM noise is usually neglidgible compared to the FM noise. Another limitation of the model employed here is its restriction to a random walk phase function. This is not always the case for electrical oscillators. A microwave oscillator, for example, does not exhibit a perfect random-walk (f^{-2}) phase noise spectrum. In practice, real oscillators have phase noise spectra which exhibit combinations of different slopes, i.e., f^{-4}, f^{-3}, f^{-2}, f^{-1}, f^0, and, more generally, noninteger slopes $f^{-\alpha}$ where $\alpha > 0$ is a real parameter. Nevertheless, the random walk f^{-2} case is one of the most important since it is mathematically tractable and since it is characteristic of a resonantly tuned oscillator operating near its thermal performance limit (ref 12).

References

[1] W. K. Saunders, in *Radar Handbook*, 2nd ed., M. I. Skolnik, ed., McGraw-Hill, New York, 1990, chapter 14.

[2] W. P. Robins, *Phase Noise in Signal Sources*, Peter Peregrinus, London, 1982.

[3] V. Manassewitsch, *Frequency Synthesizers: Theory and Design*, 3rd ed., Wiley, New York, 1987.

[4] S. J. Goldman, *Phase Noise Analysis in Radar Systems*, Wiley, New York, 1989.

[5] *Characterization of Clocks and Oscillators*, National Institute of Standards and Technology (NIST) Technical Note 1337, D. B. Sullivan, D. W. Allen, D. A. Howe, and F. L. Walls, eds., U.S. Government Printing Office, Washington D.C., 1990.

[6] V. Van Duzer, in *IEEE–NASA Symposium on Short-Term Frequency Stability*, Proceedings of the IEEE–NASA Symposium on Definition and Measurement of Short-Term Frequency Stability held at the Goddard Space Flight Center, Greenbelt, Maryland, 1964, NASA Special Publication SP-80, 1964, p. 269–272.

[7] D. J. Glaze, "Improvements in Atomic Cesium Beam Frequency Standards at the National Bureau of Standards," *IEEE Trans. on Instrumentation and Measurement*, Vol. 19, p. 156, 1970.

[8] D.W. Allen, J.H. Shoaf, and D. Halford, "Statistics of Time and Frequency Data Analysis," in *Time and Frequency: Theory and Fundamentals*, NBS Monograph 140, B.E. Blair, ed., U.S. Department of Commerce, 1974, p. 153–204.

[9] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, 2nd ed., McGraw-Hill, New York, 1984.

[10] W.A. Edson, "Noise in Oscillators," *Proc. Inst. Radio Eng.*, Vol. 48, 1960, p. 1454–1466. Reprinted in [13].

[11] L.S. Cutler and C.L. Searle, "Some Aspects of the Theory and Measurement of Frequency Fluctuations in Frequency Standards," *Proc. IEEE*, Vol 54, 1966, p. 136–154. Reprinted in [13].

[12] J. Rutman, "Very-Short-Term Frequency Instability of an Oscillator Disturbed by Additive Internal Thermal Noise," *Proc. IEEE*, Vol 58, 1970, p. 958–959.

[13] V. F. Kroupa, ed., *Frequency Stability: Fundamentals and Measurement*, IEEE Press, New York, 1983.

Appendix A

Necessary and Sufficient Conditions
for Stationarity

For a zero mean Gaussian phase modulation $\varphi(t)$ (not necessarily stationary) it is easy to obtain a necessary and sufficient condition for the stationarity of the process $v(t)$. Since the sum and difference of Gaussian random variables is Gaussian the increment $\varphi(t_2) - \varphi(t_1)$ is a Gaussian random variable with mean zero and variance

$$
\begin{aligned}
\sigma^2(t_2, t_1) &= \text{Var}\left[\varphi(t_2) - \varphi(t_1)\right] \\
&= E\left(\left[\varphi(t_2) - \varphi(t_1)\right]^2\right) = \sigma_\varphi^2(t_2) - 2R_\varphi(t_2, t_1) + \sigma_\varphi^2(t_1).
\end{aligned}
\tag{A-1}
$$

By using the characteristic function for a Gaussian random variable, it follows that

$$
R_\Phi(t_2, t_1) = E\left(e^{i[\varphi(t_2) - \varphi(t_1)]}\right) = \exp\left[-\sigma^2(t_2, t_1)/2\right].
\tag{A-2}
$$

Note that this is a *real* function. Consequently, using equation (24) the autocorrelation function of $v(t)$ takes the form

$$
R_v(t_2, t_1) = \frac{1}{2} R_\Phi(t_2, t_1) \cos(\omega_0 \tau).
\tag{A-3}
$$

Since $E[v(t)] = 0$, this equation implies that $v(t)$ is wide-sense stationary if and only if the function $R_\Phi(t_2, t_1)$ depends only on the time difference $\tau = t_2 - t_1$. From the previous considerations, it follows that if $\sigma^2(t_2, t_1)$ depends only on τ, then $R_\Phi(t_2, t_1) = R_\Phi(\tau)$ is strictly a function of τ. Conversely, if $R_\Phi(t_2, t_1) = R_\Phi(\tau)$ is strictly a function of τ, then the variance $\sigma^2(t_2, t_1)$ is also a function of τ. This proves the following theorem.

Theorem. *If $\varphi(t)$ is a zero mean Gaussian process, then the stochastic process $v(t) = \cos[\omega_0 t + \varphi(t) + \varphi_0]$ is wide-sense stationary if and only if $\text{Var}[\varphi(t_2) - \varphi(t_1)]$ is a function of τ only, that is:*

$$
\text{Var}\left[\varphi(t_2) - \varphi(t_1)\right] = \sigma_\varphi^2(t_2) - 2R_\varphi(t_2, t_1) + \sigma_\varphi^2(t_1) = \sigma^2(\tau).
\tag{A-4}
$$

In this case the autocorrelation function of $v(t)$ is given by

$$
R_v(\tau) = \frac{1}{2} R_\Phi(\tau) \cos(\omega_0 \tau),
\tag{A-5}
$$

where

$$
R_\Phi(\tau) = \exp\left[-\tfrac{1}{2}\sigma^2(\tau)\right].
\tag{A-6}
$$

In the special case when $\varphi(t)$ is a stationary process, (A-1) is only a function of τ. Therefore, one has the following corollary.

Corollary. *If $\varphi(t)$ is a stationary zero mean Gaussian process, then the stochastic process $v(t)$ is wide-sense stationary with autocorrelation function given by (A-5) and (A-6).*

Appendix B

Analysis of the Amplitude Factor

The functional dependence of the reduction factor $K(\omega)$ is primarily contained in the function $A(\omega)$. Therefore, a study of $A(\omega)$ will shed light on the general behavior of $K(\omega)$. Since $A(\omega)$ also depends on the time delay τ_0 it will sometimes be denoted by $A(\omega, \tau_0)$, where

$$A(\omega, \tau_0) = \left[\cosh(\alpha\tau_0) - \cos(\omega\tau_0)\right] 2\sin^2(\omega_0\tau_0)$$
$$+ \left[\sinh(\alpha\tau_0) - (\alpha\tau_0)\frac{\sin(\omega\tau_0)}{(\omega\tau_0)}\right] 2\cos^2(\omega_0\tau_0). \qquad \text{(B-1)}$$

It is clear that $A(\omega, \tau_0) > 0$ for all $\tau_0 > 0$ and for all ω real. Also, $A(-\omega) = A(\omega)$, so that $A(\omega)$ is an even function of ω. One important observation should be made at this point: Due to the factors $\sin^2(\omega_0\tau_0)$ and $\cos^2(\omega_0\tau_0)$ the relative contributions of the first and second terms alternate as τ_0 cycles through a period of length $T_0 = 1/f_0$. For frequencies in the microwave range, T_0 is on the order of 100 picoseconds and therefore $A(\omega)$ is a very sensitive function of τ_0. That is, the first and second terms are very rapidly going in and out of phase. On the other hand, in a radar system, for example, the radar return is "smeared out" in time so that on average each of these terms will contribute equally to the mixer output. This averaging effect is discussed further in section 8. In equation (B-1), as τ_0 increases from 0^+ there are three cases to consider:

Case (1): $\omega_0\tau_0 = n\pi, \quad n = 1, 2, 3, \ldots$;

Case (2): $\omega_0\tau_0 = (2n+1)(\pi/2), \quad n = 0, 1, 2, \ldots$;

Case (3): $\omega_0\tau_0 = (2n+1)(\pi/4), \quad n = 0, 1, 2, \ldots$.

Equivalently, these cases are characterized by

Case (1): $\cos^2(\omega_0\tau_0) = 1$,

Case (2): $\sin^2(\omega_0\tau_0) = 1$,

Case (3): $\sin^2(\omega_0\tau_0) = \cos^2(\omega_0\tau_0) = 1/2$.

Because the mixer output $x(t)$ is proportional to $\cos^2(\omega_0\tau_0)$, case (1) might be expected to have the dominant effect on the output phase noise. However, it will be shown that the coefficient of the $\sin^2$ term in (B-1) is on the order of unity, while the coefficient of the $\cos^2$ term is on the order of $\alpha\tau_0$ where, in general, $\alpha\tau_0 \ll 1$. Therefore, cases (2) and (3) are also important.

CASE 1: $\omega_0\tau_0 = n\pi$.

In this case, $A(\omega)$ is given by

$$A(\omega) = 2\left[\sinh(\alpha\tau_0) - (\alpha\tau_0)\frac{\sin(\omega\tau_0)}{(\omega\tau_0)}\right]. \qquad \text{(B-2)}$$

For the delays usually encountered in practice, it is generally true that $\alpha\tau_0 \ll 1$. Making the approximation $\sinh(x) \simeq x + x^3/3!$, this takes the form

$$A(\omega) \simeq 2(\alpha\tau_0)\left[1 - \frac{\sin(\omega\tau_0)}{(\omega\tau_0)}\right] + 2b, \qquad \text{(B-3)}$$

where $b = (\alpha\tau_0)^3/3!$. The magnitude of this term is determined by the factor $\alpha\tau_0$. This function is sketched for the case $n = 1$ in Figure B-1. Note the linear scale. It is apparent that the function $A(\omega)$ will cause a reduction or nulling out of the phase noise spectrum for frequencies in the neighborhood of zero. For frequencies from zero to $\omega\tau_0 = \pi$, $A(\omega)$ increases monotonically from the minimum value $2b$ to the value $2(\alpha\tau_0 + b)$. For frequencies greater than $\omega\tau_0 = \pi$, $A(\omega)$ is nearly flat as it approaches the asymptotic value $2(\alpha\tau_0 + b)$. Consequently, the nulling effect is greatest from $\omega = 0$ to $\omega\tau_0 = \pi$. To gain some feeling for the overall magnitude of the nulling effect suppose that $\alpha\tau_0 = 10^{-4}$. Then $2b \simeq 10^{-16}$ and $2(\alpha\tau_0 + b) \approx 10^{-4}$. The nulling effect caused by $A(\omega)$ is quite strong in case 1.

Consider now the behavior of $A(\omega, \tau_0)$ as a function of τ_0. For a fixed time delay $\tau_0 = n\pi/\omega_0$ (n fixed), the zeros of $\sin(\omega\tau_0)$ occur at $\omega\tau_0 = m\pi$, or

$$\frac{\omega}{\omega_0} = \frac{m}{n}, \qquad \text{(B-4)}$$

where $m = 0, 1, 2, \ldots$. The behavior of $A(\omega)$ for different values of the delay n is illustrated in Figure B-2. In general, $A(\omega)$ increases monotonically from the minimum value $2b$ at $\omega = 0$ to the value $2(\alpha\tau_0 + b)$ at $\omega = \omega_0/n$. For $\omega \geq \omega_0/n$, $A(\omega)$ is nearly constant.

CASE 2: $\omega_0\tau_0 = (2n + 1)(\pi/2)$.

In this case, $A(\omega)$ reduces to the form

$$A(\omega) = 2\big[\cosh(\alpha\tau_0) - \cos(\omega\tau_0)\big]. \qquad \text{(B-5)}$$

For the small delays encountered in applications it is almost always true that $\alpha\tau_0 \ll 1$. Therefore, making the approximation $\cosh(x) \simeq 1 + x^2/2$, equation (B-5) takes the form

$$A(\omega) = 2\big[1 - \cos(\omega\tau_0)\big] + 2a, \qquad \text{(B-6)}$$

where $a = (\alpha\tau_0)^2/2$. This function is plotted on a linear scale and displayed in Figure B-3. This function exibits large amplitude swings as a function of ω. As ω increases from zero to $\omega\tau_0 = \pi$ the function $A(\omega)$ increases monotonically from its minimum value $2a$ to its maximum value $2 + 2a$. For example, if
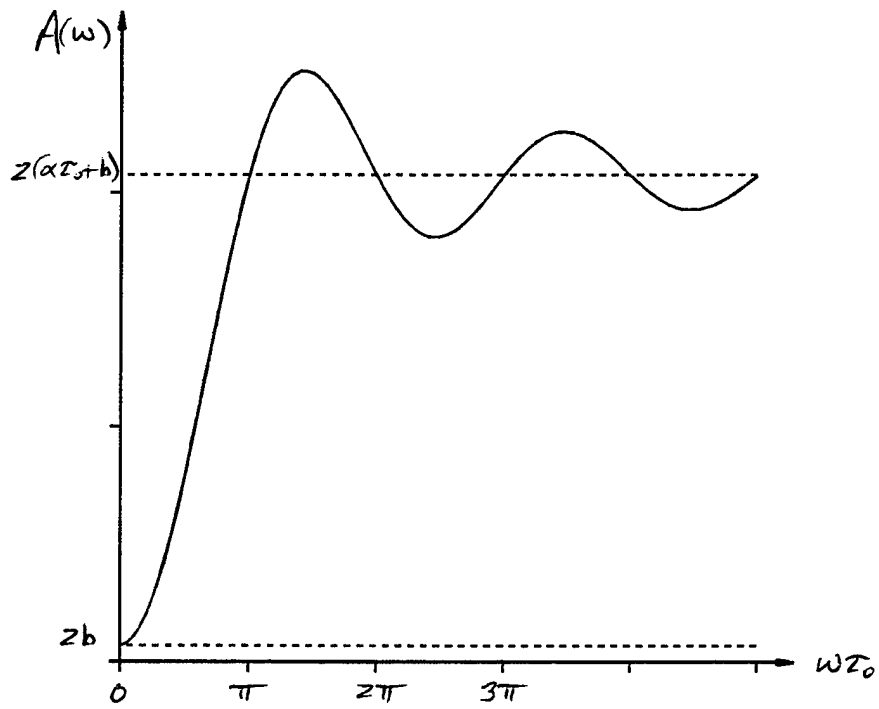
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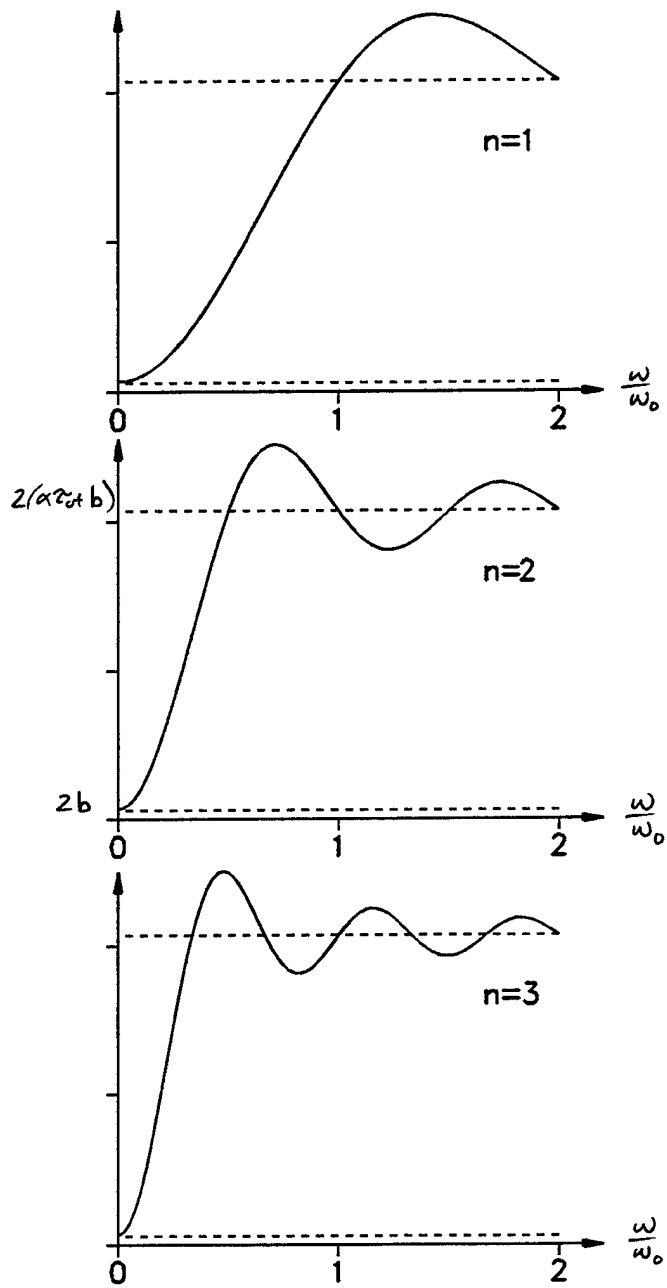Figure B-1. Linear plot of the amplitude response versus frequency in case 1.

Figure B-2. Amplitude versus frequency in case 2
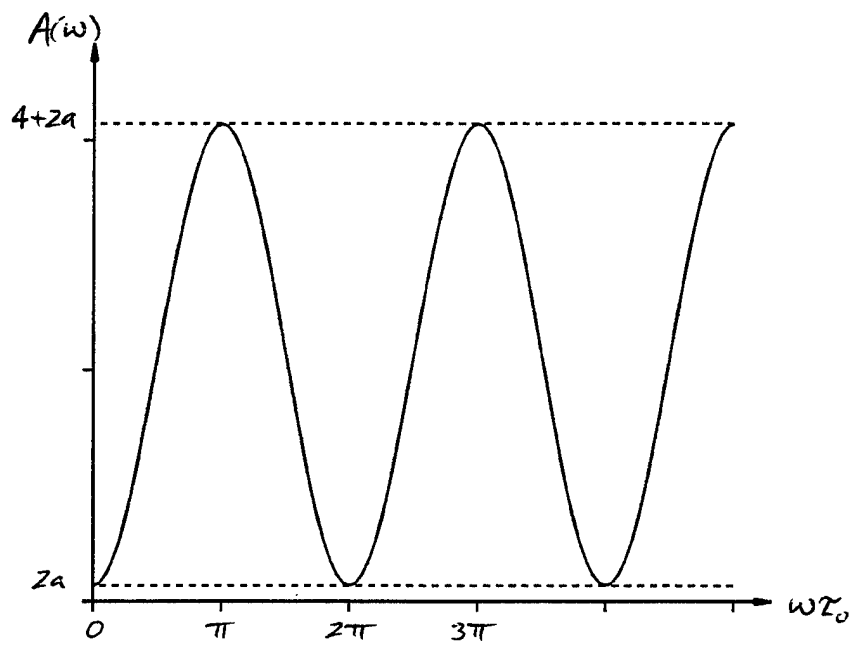for different values of the delay n.

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Figure B-3. Amplitude versus frequency in case 2.

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$\alpha\tau_0 = 10^{-4}$, then $2a = 10^{-8}$ and $2 + 2a \approx 2.0$. Therefore, in this example, $A(\omega)$ spans eight orders of magnitude between $\omega = 0$ and $\omega\tau_0 = \pi$. Since the phase noise spectrum is located in the neighborhood of $\omega = 0$, i.e., $\omega \ll \omega_0$, this shows that $A(\omega)$ causes a significant reduction or nulling out of the phase noise. Of course, since the mixer output goes as $\cos^2(\omega_0\tau_0)$, the mixer output is zero in case (2) and the nulling effect is irrelevant. Nevertheless, the above analysis is relevant for a study of case (3).

Next, consider the dependence of $A(\omega, \tau_0)$ on the time delay. In case (2), the time delay is given by $\tau_0 = (2n + 1)\pi/2\omega_0$. The extrema of $A(\omega)$ occur when $\omega\tau_0 = m\pi$, $m = 1, 2, \ldots$. For a fixed value of τ_0 (fixed n), the extrema occur for those frequencies which satisfy

$$\frac{\omega}{\omega_0} = \frac{2m}{2n + 1}, \tag{B-7}$$

where $m = 1, 2, \ldots$, n fixed. The behavior of $A(\omega)$ for different values of n are shown in Figure B-4. As the time delay n increases, the bandwidth from $\omega = 0$ to the first maximum $\omega/\omega_0 = 2/(2n + 1)$ decreases. Consequently, the bandwidth of the nulling effect also decreases as shown in Figure B-4. Nonetheless, the nulling effect is significant except for very large values of n (large delays) since the phase noise is concentrated in a narrow frequency band close to the carrier frequency.

CASE 3: $\omega_0\tau_0 = (2n + 1)\pi/4$.

This case is a combination of cases 1 and 2. Making the same approximations for sinh and cosh as in equations (B-3) and (B-6), one obtains

$$A(\omega) = (\alpha\tau_0)\left[1 - \frac{\sin(\omega\tau_0)}{(\omega\tau_0)}\right] + \left[1 - \cos(\omega\tau_0)\right] + (a + b), \tag{B-8}$$

where

$$a = (\alpha\tau_0)^2/2, \qquad b = (\alpha\tau_0)^3/3!. \tag{B-9}$$

Since $\alpha\tau_0 \ll 1$, the first term in (B-8) is much less than the second term and the overall response is very similar to that of case 2 shown in Figures B-3 and B-4.

Appendix C

Derivation of the Approximate Formula

In this appendix an elementary derivation is given of the approximate formula (1). The phase of the output signal is $\phi(t) = \varphi(t) - \varphi(t - \tau_0)$, where $\varphi(t)$ is the phase of the source. Therefore, the autocorrelation function of the output *phase* is given by

$$
\begin{aligned}
S_\phi(\tau) &= E\big[\phi(t_2)\phi(t_1)\big] \\
&= E\Big\{ \big[\varphi(t_2) - \varphi(t_2 - \tau_0)\big]\big[\varphi(t_1) - \varphi(t_1 - \tau_0)\big] \Big\} \\
&= 2R_\varphi(\tau) - R_\varphi(\tau + \tau_0) - R_\varphi(\tau - \tau_0).
\end{aligned} \tag{C-1}
$$

Taking the Fourier transform and using the time shift property, this yields the result

$$
S_\phi(\omega) = 2\big[1 - \cos(\omega\tau_0)\big] S_\varphi(\omega). \tag{C-2}
$$

Therefore, the ratio of the power spectrum of the output phase $\phi(t)$ to the power spectrum of the input phase $\varphi(t)$ is given by

$$
\frac{S_\phi(\omega)}{S_\varphi(\omega)} = 2\big[1 - \cos(\omega\tau_0)\big] = 4\sin^2(\omega\tau_0/2). \tag{C-3}
$$

This is just formula (1). The problem with this result is that the power spectrum of the *oscillator signal* $v(t)$ is not equal to the power spectrum of the *phase function* $\varphi(t)$, and the PSD of $x(t)$ is not equal to the PSD of $\phi(t)$. In the case of a random-walk phase function, a more careful analysis leads to the correct result (94).

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