CCM 79 10

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AXISYMMETRIC ANALYSIS OF THE CYLINDRICALLY
ORTHOTROPIC DISK OF
VARIABLE FIBER ORIENTATION

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-- 1 OF 1

-- ***DTIC DOES NOT HAVE THIS ITEM***

-- 1 - AD NUMBER: D429746

-- 5 - CORPORATE AUTHOR: DELAWARE UNIV NEWARK CENTER FOR COMPOSITE MATERIALS

-- 6 - UNCLASSIFIED TITLE: AXISYMMETRIC ANALYSIS OF THE CYLINDRICALY ORTHOTROPIC DISK OF VARIABLE FIBER ORIENTATION,

--10 - PERSONAL AUTHORS: GILLESPIE,J. W. , JR.;PIPES,R. B. ;

--11 - REPORT DATE: APR 01, 1979

--12 - PAGINATION: 50P

--14 - REPORT NUMBER: CCM-79-10

--20 - REPORT CLASSIFICATION: UNCLASSIFIED

--22 - LIMITATIONS (ALPHA): APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED. AVAILABILITY: CENTER FOR COMPOSITE MATERIALS, COLLEGE OF ENGINEERING, UNIVERSITY OF DELAWARE, NEWARK, DELAWARE 19711.

--33 - LIMITATION CODES: 1 24

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Axisymmetric Analysis of the Cylindrically Orthotropic Disk of Variable Fiber Orientation

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Prepared for

Rogers Corporation

Lurie Research and Development Center

Rogers, Connecticut

April 1, 1979

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Abstract

Injection molding of axisymmetric bodies with fiber-reinforced molding compounds results in cylindrically orthotropic components in which the fiber orientation varies with radial position. Consequently, development of analysis techniques for determining the effect of material property variability on the response of these components is of great importance. In the present study, a numerical integration scheme for the analysis of cylindrically orthotropic annular disks with variable elastic constants is presented. The influence of material property variations, temperature changes and centrifugal forces on the response of an annular disk subjected to internal and external pressure or displacement boundary condition is included in the analysis. Correlation of numerical integration results with analytical and finite-element solutions is excellent.

Axisymmetric Analysis of the Cylindrically Orthotropic Disk of Variable Fiber Orientation

Introduction

Injection molding of axisymmetric bodies with fiber-reinforced molding compounds results in cylindrically orthotropic components. In general, the fiber orientation will not be constant throughout the body, but will be a function of radial position. It has recently been shown [1] that the flow field and mold geometry determine fiber orientation. For example, the influence of converging and diverging flow fields on fiber orientation are shown clearly in Figure 1. Consequently, injection molding of an axisymmetric mold uniformly on the inner or outer radius (axisymmetric flow) will result in fiber orientation which is dependent solely on the radial position. Material properties are determined uniquely by fiber orientation, constituent properties, fiber aspect ratio and fiber volume fraction. McCullough [1, 2] has introduced the following parameters to quantify fiber orientation:

$$f = 1/2\ [3 < \cos^2\phi > - 1]$$

$$g = 1/4\ [5 < \cos^4\phi > - 1]$$

$$<\cos^m\phi> = \int_o^{\pi/2} N(\phi)\cos^m\phi\sin\phi d\phi$$

where ϕ is the angle which a fiber makes with the longitudinal direction and $N(\phi)$ is the percentage of fibers with that direction. In figure 2, the bounds on Young's modulus as a function of the orientation parameter "f" are shown. Clearly, the variation of fiber orientation with radial position significantly influences the mechanical properties of the component.

In the present study, a numerical integration scheme is developed to determine the response of cylindrically orthotropic disks with elastic constants which vary in the radial direction. Uniform temperature variations and centrifugal body forces, as well as, pressure on displacement boundary conditions prescribed on the inner and outer radii are considered.

Formulation of the governing equation in terms of displacements yields a second order linear ordinary differential equation with non-constant coefficients (See Appendix A). In general, a closed form solution of this equation does not exist. The integration scheme requires the reduction of the governing equation to two simultaneous first order differential equations which are solved using Hamming's Predictor-Corrector Method [3]. Unfortunately, Hamming's Method requires two initial value conditions (one for each first order equation) whereas only one is known in the actual boundary value problem. Consequently, a half-interval search technique is incorporated into the program in which upper and lower bounds for the unknown initial condition are prescribed. The average value is employed in the integration and correlation of the solution with the second known boundary condition enables the interval of uncertainty to be halved. Subsequent iterations converge quickly to the solution. In fact, if Δ_1 is the length of the starting interval, then the number (N) of interval halving operations required to reduce the interval of uncertainty to Δ_N is given by

$$N = \frac{\ln(\Delta_1/\Delta_n)}{\ln 2}$$

Closed-form solutions for two special variations of elastic properties are presented in Appendices B and C to verify numerical integration results. The first solution is for uniform properties and the second assumes that the modulii vary along a radius according to a power law:

$$E_r = E_{rm}\, r^m, \; E_\theta = E_{\theta m}\, r^m$$

where m is an arbitrary real number, and the Poisson's ratios are held constant. In addition, finite element results for a linear variation of properties are compared to the numerical integration results.

Correlation of Results

Agreement of numerical integration results with analytical and finite element results was found to be excellent. In Figures 3-6, typical results for temperature variations, centrifugal body forces and internal and external radial stress tractions are presented. The solid line in all figures corresponds to the numerical integration prediction. Superimposed are analytic and finite element results shown as symbols. The excellent agreement is obtained using an integration step-size of 0.001 inches (0.025 mm). Approximately 15-20 iterations are required to determine the unknown initial condition with sufficient accuracy to satisfy the remaining boundary condition on the outer radius. Although only the correlation of stress components are presented in Figures 3-6, excellent agreement was obtained for displacement and strain component values as well.

For reference, the material properties employed in the various solutions are given below:

Constant Properties: Radial Fiber Orientation

$E_r = 2.6$ Msi(17.9GPa) $\quad \alpha_r = 4.5\text{x}10^{-6}/°F$ $(8.1\text{x}10^{-6}/°C)$

$E_\theta = 1.4$ Msi(9.65GPa) $\quad \alpha_\theta = 10.1\text{x}10^{-6}/°F$ $(18.2\text{x}10^{-6}/°C)$

$\nu_{\theta r} = 0.27$

Constant Properties: Tangential Fiber Orientation

$E_r = 1.4$ Msi(9.65GPa) $\quad \alpha_r = 10.1\text{x}10^{-6}/°F$ $(18.2\text{x}10^{-6}/°C)$

$E_\theta = 2.6$Msi(17.9GPa) $\quad \alpha_\theta = 4.5\text{x}10^{-6}/°F$ $(8.1\text{x}10^{-6}/°C)$

$\nu_{\theta r} = .501$

Power Law Variation (m specified)

$E_r = 2.6\ r^m$ Msi $(17.9r^m \text{GPa})$ $\quad \nu_{r\theta} = .501$

$E_\theta = 1.4\ r^m$ Msi $(9.65r^m \text{GPa})$ $\quad \alpha_r = (4.5\text{x}10^{-6})r^m/°F$

$(8.1\text{x}10^{-6}r^m/°C)$

$\nu_{\theta r} = .27$ $\quad \alpha_\theta = (10.1\text{x}10^{-6})r^m/°F$

$(18.2\text{x}10^{-6}r^m/°C)$

Linear Variation

$$E_r = (\frac{-1.2}{(b-a)}\ (r-a) + 2.6)\text{Msi}\ ((\frac{-8.27}{(b-a)}\ (r-a) + 17.9)\ \text{GPa})$$

$$E_\theta = (\frac{1.2}{(b-a)}\ (r-a) + 1.4)\text{Msi}\ ((\frac{8.27}{(b-a)}\ (r-a) + 9.65)\ \text{GPa})$$

$$\nu_{\theta r} = .27$$

$$\nu_{r\theta} = \nu_{\theta r}\ E_r/E_\theta$$

$$\alpha_r = (\frac{5.6(r-a)}{(b-a)} + 4.5)\text{x}10^{-6}/°F\ ((\frac{10.1(4-a)}{(b-a)} + 8.1)\text{x}10^{-6}/°C)$$

$$\alpha_\theta = (\frac{-5.6(r-a)}{(b-a)} + 10.1)\text{x}10^{-6}/°F\ ((\frac{-10.1(r-a)}{(b-a)} + 18.2)\text{x}10^{-6}/°C)$$

where

E_r - Young's Modulus in radial direction

E_θ - Young's Modulus in tangential direction

$\nu_{r\theta}$, $\nu_{\theta r}$ - Poisson ratios

α_r - Thermal coefficient of expansion in radial direction

α_θ - Thermal coefficient of expansion in tangential direction

User's Guide

Fortran computer codes have been developed for two analytical solutions and for the Hamming's Predictor-Corrector Method. Analysis details and program listings may be found in Appendices A, B and C. The programs have been written in an interactive format which necessitates execution from a terminal or similar device. In Table 1, program symbol definitions are defined. The following examples will be illustrative.

Analytic Solution: Constant Properties (VARPROP/CFD)

Table 2 indicates the line numbers in VARPROP/CFD which describe the material properties. These are the only lines that must be altered when another set of properties are to be input. In Figure 7 a sample program execution is presented where data input is requested by the program. Note that displacement boundary conditions are not possible.

Analytic Solution: Power Law Variation (VARPROP/CF2)

The anlytic solution assumes the following material property variation:

$$E_r = E_{rm} r^m \qquad \nu_{r\theta} = \nu_{\theta r} E_{rm}/E_{\theta m}$$

$$E_\theta = E_{\theta m} r^m \qquad \alpha_r = \alpha_{rm} r^m$$

$$\nu_{\theta r} = \text{constant} \qquad \alpha_\theta = \alpha_{\theta m} r^m$$

In Table 3, the line numbers in VARPROP/CF2 which describe the material properties are shown. Execution is straightforward as indicated in Figure 8. Note that centrifugal body forces and displacement boundary conditions are not included.

Numerical Integration (VARPROP/NUMD)

The material property section of VARPROP/WUMD is located in Subroutine PROP (See Appendix A for further detail). All property dependence on radial position must be input. Derivatives of these properties are also required. In Table 4, for example, the material properties employed in the correlation of numerical with analytical results (constant properties) are presented. Material property input for power law and linear variations are shown in Tables 5 and 6, respectively. Note that derivatives are input in a straightforward manner.

Execution of VARPROP/NUMD is also in an interactive mode. The program versatility allows displacement or stress boundary conditions in the inner and outer surface. As mentioned previously, upper and lower bounds on the unknown initial condition at the inner radius must be supplied. If a radial stress initial condition is specified, bounds on the tangential stress components are input as shown in Figures 9 and 10. For displacement initial conditions, bounds on the radial strain component at the inner radius are required (See Figures 11 and 12). Results from numerical integration and the analytical solution for constant properties (radial fiber orientation) are presented in Figures 7 and 9. Comparison of solutions illustrates the excellent accuracy obtained with the Hamming's Predictor -Corrector Method.

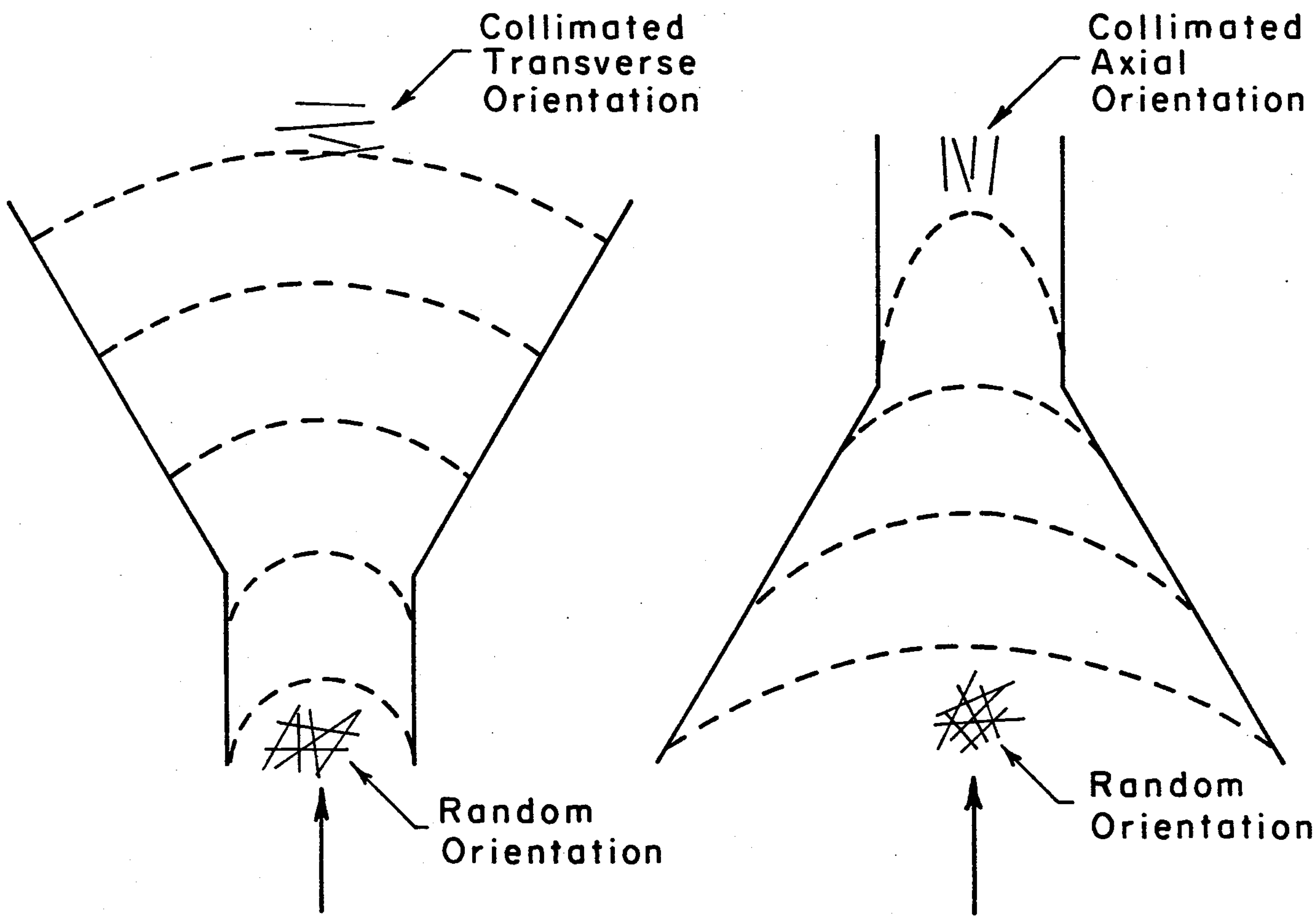

Figure 1. Fiber orientations of converging and diverging flow fields

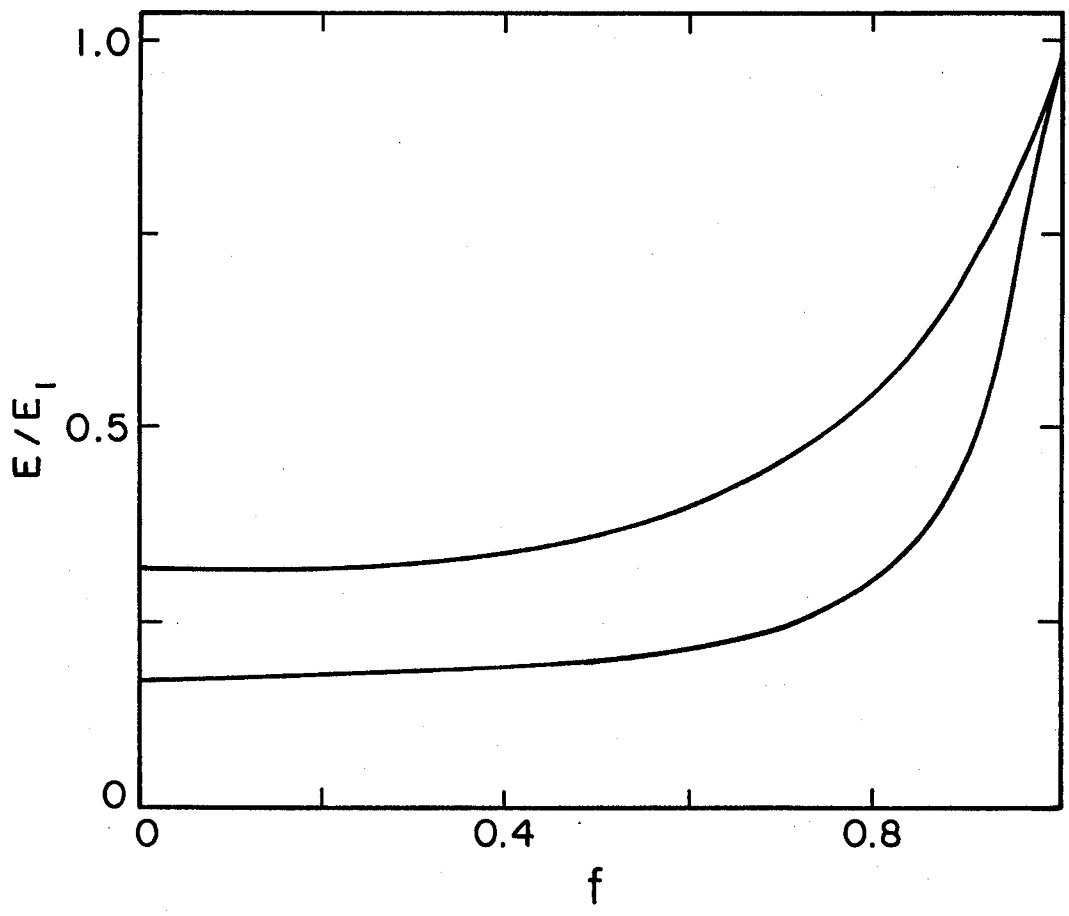

Figure 2 Dependence of Young's modulus on fiber orientation.

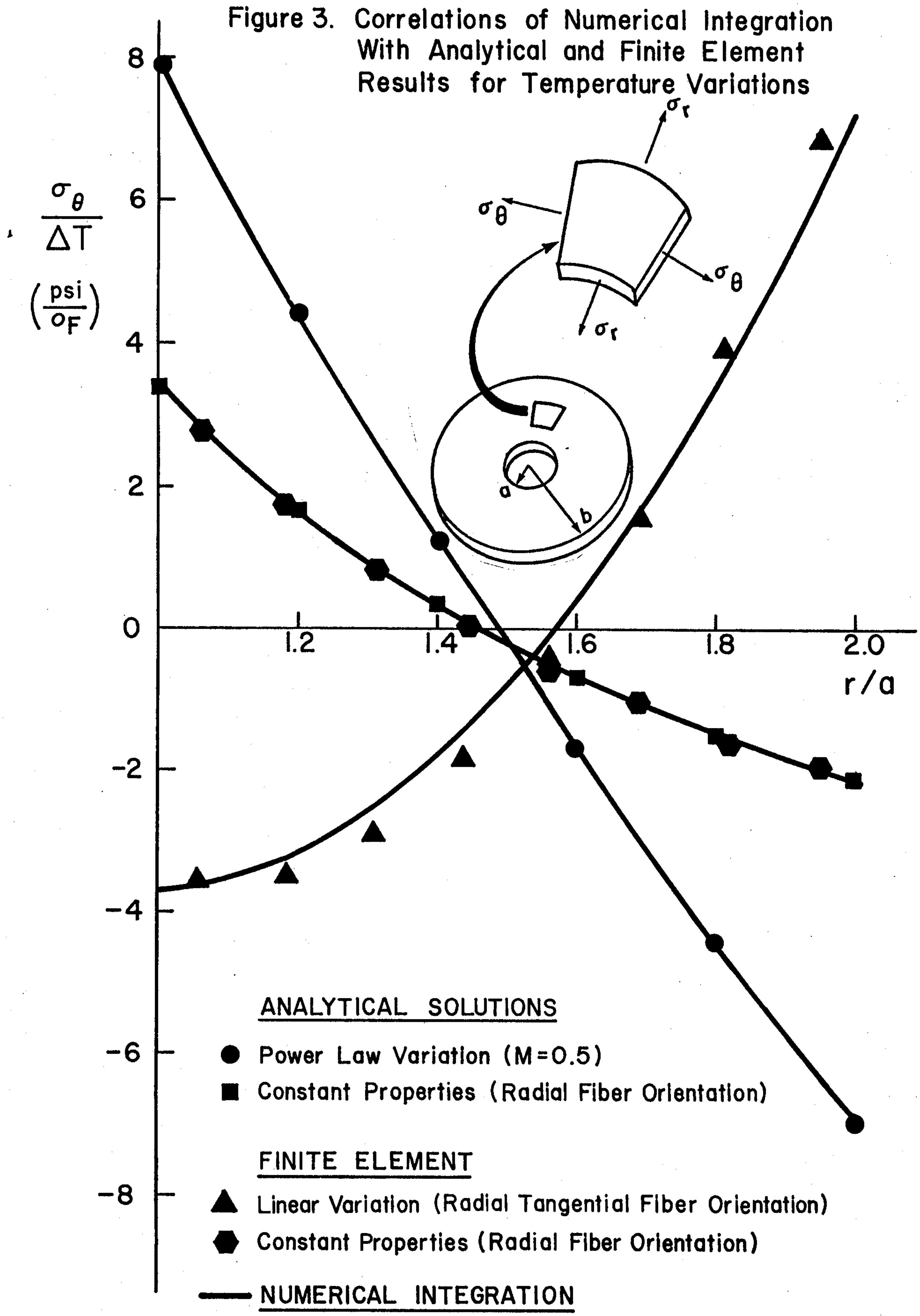

Figure 3. Correlations of Numerical Integration With Analytical and Finite Element Results for Temperature Variations
$\frac{\sigma_\theta}{\Delta T}$
$\left(\frac{psi}{^oF}\right)$
8
6
4
2
0
-2
-4
-6
-8
1.2
1.4
1.6
1.8
2.0
r/a
σ_r
σ_θ
σ_θ
σ_r
a
b
ANALYTICAL SOLUTIONS
Power Law Variation (M=0.5)
Constant Properties (Radial Fiber Orientation)
FINITE ELEMENT
Linear Variation (Radial Tangential Fiber Orientation)
Constant Properties (Radial Fiber Orientation)
NUMERICAL INTEGRATION

FIGURE 4. Correlation of Numerical Integration Results With Analytical Solution For Centrifigal Body Forces

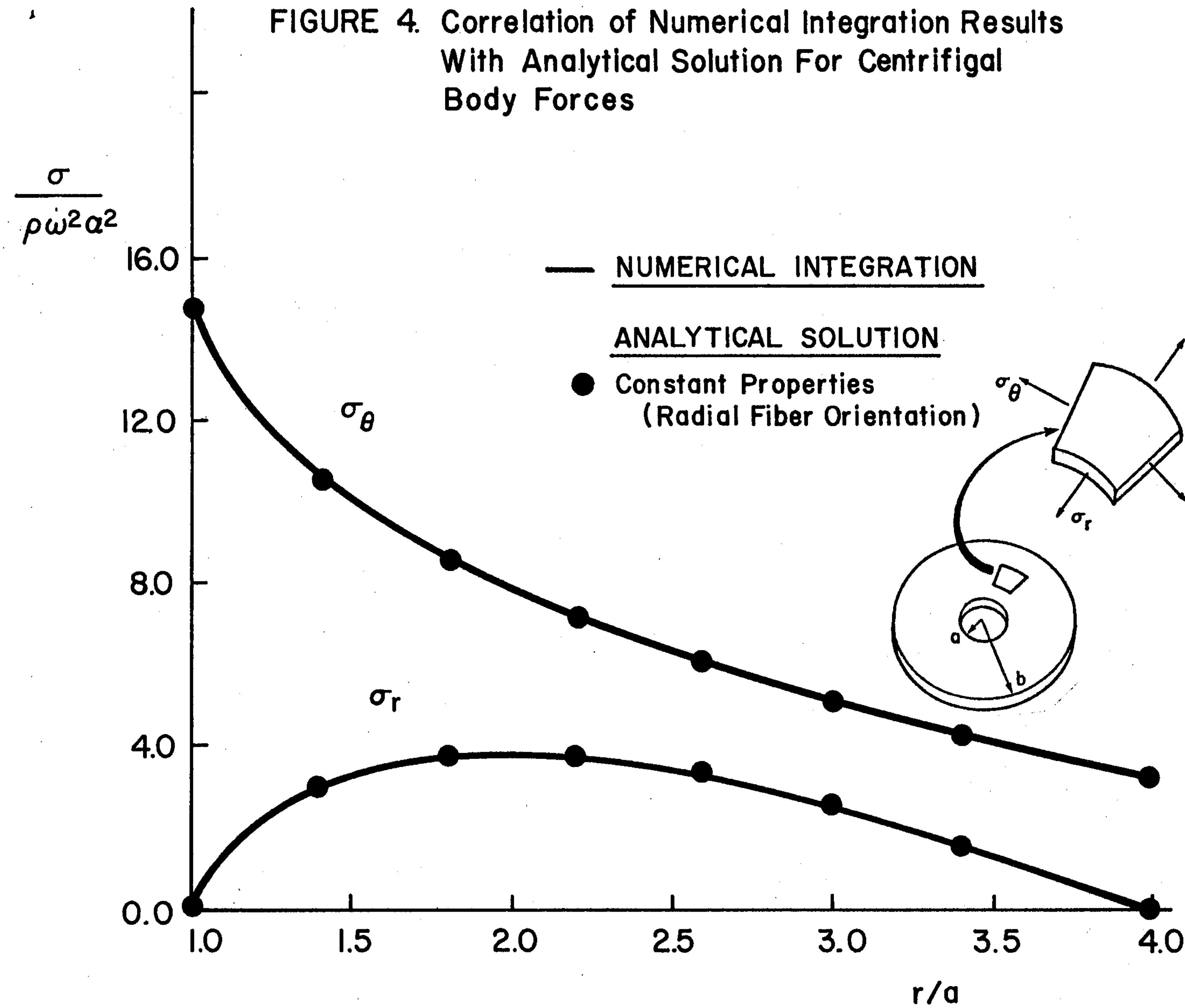

FIGURE 5. Correlation of Numerical Integration Results With Analytical Solution For Radial Traction On Inner Surface

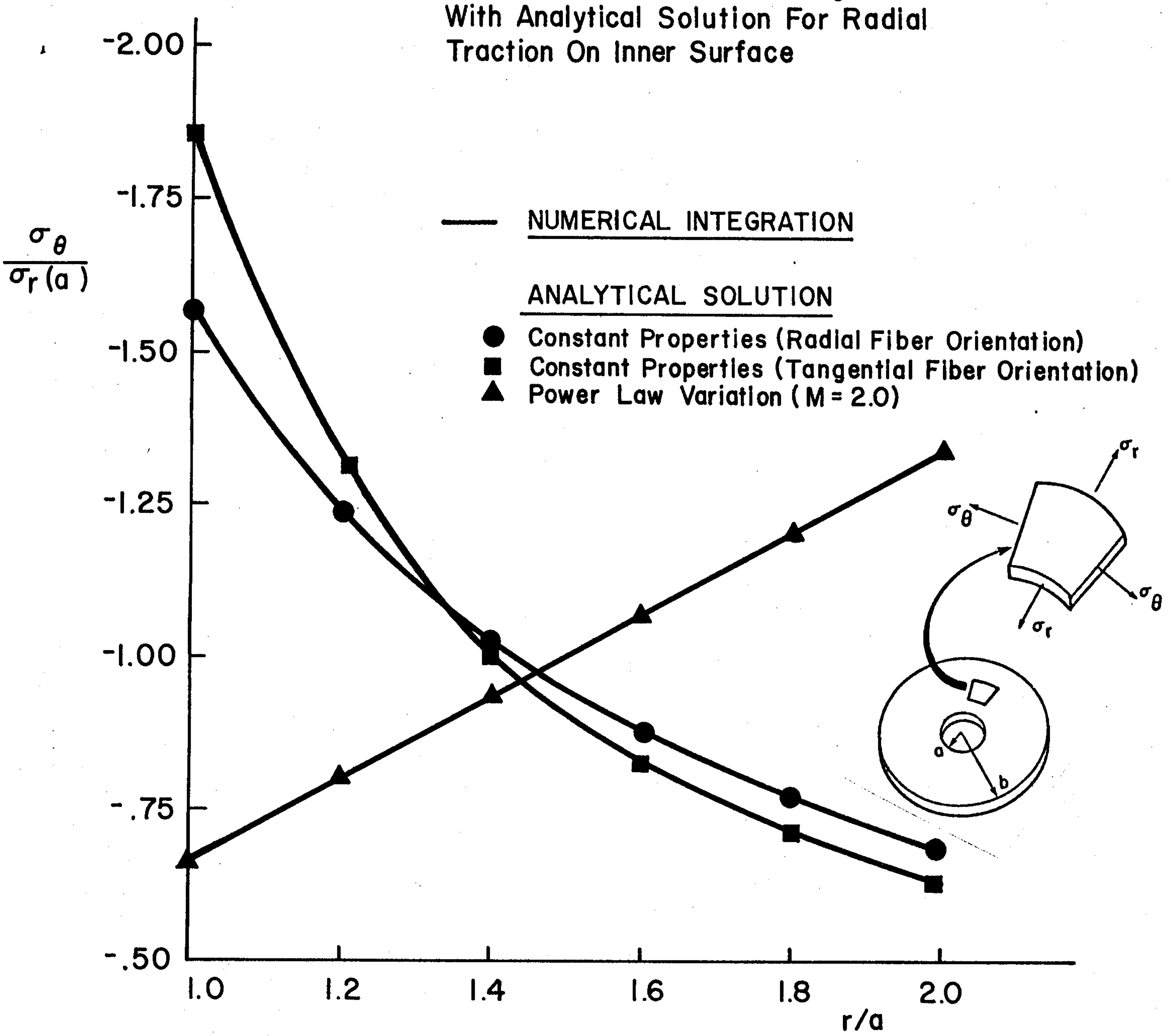

Figure 6. Correlation of Numerical Integration Results With Analytical Solution For Radial Stress Traction On Outer Surface

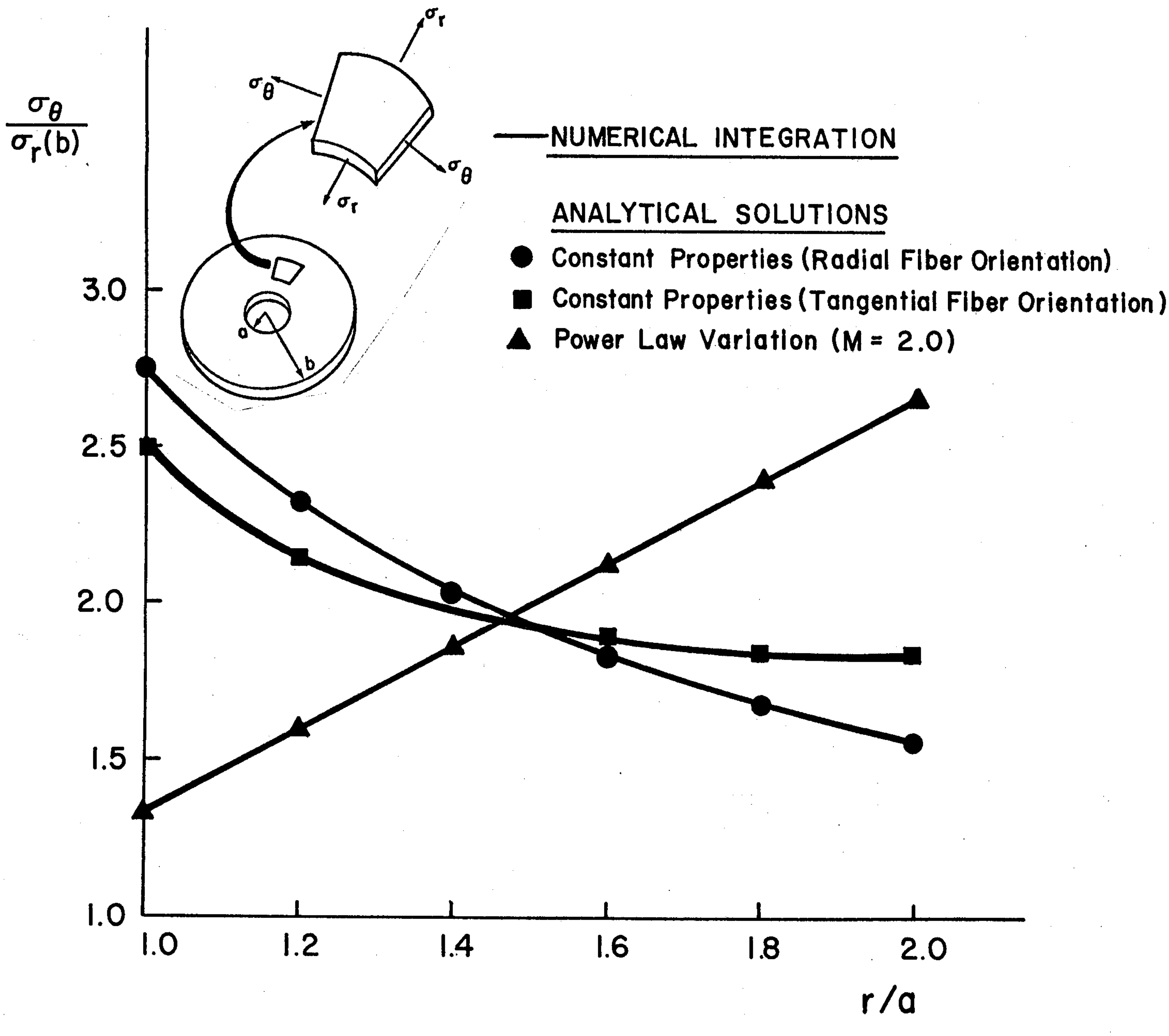

Table 1 Symbol Definitions

LIST OF SYMBOLS:

XI	INITIAL STARTING VALUE FOR INDEPENDENT VARIABLE
H	INTEGRATION STEP-SIZE
XMAX	UPPER LIMIT OF INTEGRATION
N	NUMBER OF INITIAL CONDITIONS
YR(I)	N INITIAL CONDITIONS AT X
FR(I)	DERIVATIVES IN ORIGINAL SYSTEM OF DIFFERENTIAL EQUATIONS
Y(I,J)	Y VALUE AT ITH X VALUE FOR JTH DIFFERENTIAL EQUATION
F(I,J)	DERIVATIVE AT ITH X VALUE FOR JTH DIFFERENTIAL EQUATION
TE(I)	TRUNCATION ERROR FOR ITH CORRECTOR EQUATION
INT	NUMBER OF INTEGRATIONS BETWEEN OUTPUT
NN	UPPER LIMIT ON THE NUMBER OF HALF-INTERVAL ITERATIONS
Y2LEFT	LOWER LIMIT ON UNKNOWN INITIAL VALUE
Y2RITE	UPPER LIMIT ON UNKNOWN INITIAL VALUE
NHIS	NUMBER OF HALF-INTERVAL ITERATIONS BETWEEN PRINTOUT
SGI	RADIAL STRESS B.C. ON INNER RADIUS
SGO	RADIAL STRESS B.C. ON OUTER RADIUS
SIGNL	HAS THE VALUE -1, IF Y1(XMAX)<SIGO WHEN Y2(XI)=Y2LEFT;OTHERWISE THE VALUE IS 1
DT	TEMPERATURE DIFFERENCE RELATIVE TO STRESS FREE STATE
W	ROTATIONAL SPEED(RPM)
RHO	DENSITY(#/IN**3)

Table 2 Material Property Section of VARPROP/CFD

```
C     INPUT MATERIAL PROPERTIES
C    (T: TANGENTIAL, R:RADIAL):
C
           ET=1.4E6
           VTR=.27
           ER=2.6E6
           VRT=VTR*ER/ET
           AT=10.1E-6
           AR=4.5E-6
           RHO=.1
C
C      END MATERIAL PROPERTY DESCRIPTION
C
#
```

Table 3 Material Property Section of VARPROP/CF 2

```
C
C    INPUT MATERIAL PROPERTIES
C   (T: TANGENTIAL, R:RADIAL):
C
          ETM=1.4E6
          VTR=.27
          ERM=2.6E6
          ATM=10.1E-6
          ARM=4.5E-6
C
C     END MATERIAL PROPERTY DESCRIPTION
C
```

Table 4 Material Property Section of VARPROP/NUMD
(Constant Properties)

```
C
C   INPUT MATERIAL PROPERTY DEPENDENCE ON RADIAL POSITION
C
          ET=1.4E6
          VTR=.27
          ER=2.6E6
          VTRP=0.
          AT=10.1E-6
          AR=4.5E-6
          ERP=0.
          ETP=0.
          ATP=0.
          ARP=0.
          VRT=ER*VTR/ET
          VRTP=0.
C   END MATERIAL PROPERTY INPUT
```

Table 5 Material Property Section of VARPROP/NUMD
(Power Law Variation)

```
27504 C
27505 C   INPUT MATERIAL PROPERTY DEPENDENCE ON RADIAL POSITION
27506 C
27600        ETM=1.4E6
27800        VTR=.27
27900        VTRP=0.
28000        ERM=2.6E6
28006        VRT=ERM*VTR/ETM
28008        VRTP=0.
28100        ATM=10.1E-6
28300        ARM=4.5E-6
28310        ET=ETM*X**2
28312        ER=ERM*X**2
28314        AR=ARM*X**2
28316        AT=ATM*X**2
28318        ERP=2.*ERM*X
28319        ETP=2.*ETM*X
28320        ATP=2.*ATM*X
28321        ARP=2.*ARM*X
28322 C   END MATERIAL PROPERTY INPUT
#
```

Table 6 Material Property Section of VARPROP/NUMD
(Linear Variation)

```
31100 C
31200 C   INPUT MATERIAL PROPERTY DEPENDENCE ON RADIAL POSITION
31300 C
31400        ETM=1.4E6
31500        VTR=.27
31700        ERM=2.6E6
31900        VTRP=0.
32000        ATM=10.1E-6
32100        ARM=4.5E-6
32200        ET=(ERM-ETM)*X/(XMAX-XI)+ETM-(ERM-ETM)*XI/(XMAX-XI)
32300        ER=(ETM-ERM)*X/(XMAX-XI)+ERM-(ETM-ERM)*XI/(XMAX-XI)
32400        AR=(ATM-ARM)*X/(XMAX-XI)+ARM-(ATM-ARM)*XI/(XMAX-XI)
32500        AT=(ARM-ATM)*X/(XMAX-XI)+ATM-(ARM-ATM)*XI/(XMAX-XI)
32600        ERP=(ETM-ERM)/(XMAX-XI)
32700        ETP=(ERM-ETM)/(XMAX-XI)
32800        ATP=(ARM-ATM)/(XMAX-XI)
32900        ARP=(ATM-ARM)/(XMAX-XI)
32950        VRT=ER*VTR/ET
32960        VRTP=VTR*ERP/ET-VTR*ER/ET**2*ETP
33000 C   END MATERIAL PROPERTY INPUT
#
```

Figure 7 Typical Output of VARPROP/CFD

```
RUN VARPROP/CFD
#RUNNING 5396
ENTER XI,XMAX,DX,DT,W
1,2,.2,100,10000
#?
ENTER SGI,SGO
100,50


XI     =      .100E+01
XMAX   =     .2000E+01
SGO    =     .5000E+02
SGI    =     .1000E+03
ET     =     .1400E+07
ER     =     .2600E+07
AT     =     .1010E-04
AR     =     .4500E-05
VTR    =      .270E+00
DT     =     .1000E+03
W      =     .1000E+05
RHO    =     .1000E+00

       X          SIGR            SIGT            EPR             EPT             U

    1.0000    .1000000E+03    .1316823E+04    .2345029E-03    .1931302E-02    .1931302E-02
    1.2000    .2143555E+03    .9662531E+03    .3460956E-03    .1658841E-02    .1990609E-02
    1.4000    .2343291E+03    .7153699E+03    .4021624E-03    .1475786E-02    .2066101E-02
    1.6000    .2019383E+03    .5212714E+03    .4271377E-03    .1343391E-02    .2149426E-02
    1.8000    .1370740E+03    .3618789E+03    .4329298E-03    .1242049E-02    .2235689E-02
    2.0000    .5000000E+02    .2247240E+03    .4258911E-03    .1160874E-02    .2321749E-02
#ET=56.2 PT=0.2 IO=0.2
```

Figure 8 Typical Output of VARPROP/CF 2

```
R VARPROP/CF2
#RUNNING 5339
ENTER XI,XMAX,DX,DT
#?
1,2,.2,100
ENTER SGI,SGO,M
100,50,2


XI    =      .100E+01
XMAX  =     .2000E+01
SGO   =     .5000E+02
SGI   =     .1000E+03
ETM    =     .1400E+07
ERM    =     .2600E+07
ATM    =     .1010E-04
ARM    =     .4500E-05
VTR   =      .270E+00
DT    =     .1000E+03
M       =   .2000000E+01

        X          SIGR            SIGT            EPR             EPT             U

     1.0000    .1000000E+03    .3772882E+04   -.2391657E-03    .3685630E-02    .3685630E-02
     1.2000    .6970974E+03    .3475520E+04    .3687192E-03    .3085007E-02    .3702008E-02
     1.4000    .1024690E+04    .2341979E+04    .8526347E-03    .2732265E-02    .3825171E-02
     1.6000    .1061023E+04    .7181989E+02    .1305998E-02    .2525707E-02    .4041131E-02
     1.8000    .7580607E+03   -.3681644E+04    .1767134E-02    .2415627E-02    .4348129E-02
     2.0000    .5000000E+02   -.9311337E+04    .2253747E-02    .2374851E-02    .4749701E-02
#ET=1:09.7 PT=0.2 IO=0.2
```

Figure 9 Typical Output of VARPROP/NUMD: Stress Initial Condition - Stress Outer Boundary Condition

Constant Properties (Radial Fiber Orientation)

```
RUN VARPROP/NUMD
#RUNNING 5451
ENTER XI,H,XMAX,INT,DT,W(RPM),RHO(LB/IN**3)
#?
1,.001,2,200,100,10000,.1
ENTER KNOWN I.C.,TYPE(STRESS=1,DISPL.=2)
100,1
ENTER Y2LEFT,Y2RITE
1000,1500
ENTER SECOND B.C.,TYPE(STRESS=1,DISPL.=2)
50,1
ENTER NN,NHIS,SIGNL
15,15,-1


H       =      .100E-02
XI      =       1.0000
XMAX    =       2.0000
N       =            2
INT     =          200
YR1     =     100.0000
TIC1    =       1.0000
Y2LEFT=      1000.0000
Y2RITE=      1500.0000
NN      =           15
NHIS    =           15
SIGNL   =           -1
DT      =     100.0000
RHO     =       0.100v
W       =   10000.0000

      X                          Y
                         U                 EPR                 EPT                 SIGR                 SIGT

ITERATION   1    LOWER=    1000.0000   UPPER=    1500.0000   UNKNOWN I.C.=    1250.0000
ITERATION   2    LOWER=    1250.0000   UPPER=    1500.0000   UNKNOWN I.C.=    1375.0000
ITERATION   3    LOWER=    1250.0000   UPPER=    1375.0000   UNKNOWN I.C.=    1312.5000
ITERATION   4    LOWER=    1312.5000   UPPER=    1375.0000   UNKNOWN I.C.=    1343.7500
ITERATION   5    LOWER=    1312.5000   UPPER=    1343.7500   UNKNOWN I.C.=    1328.1250
ITERATION   6    LOWER=    1312.5000   UPPER=    1328.1250   UNKNOWN I.C.=    1320.3125
ITERATION   7    LOWER=    1312.5000   UPPER=    1320.3125   UNKNOWN I.C.=    1316.4063
ITERATION   8    LOWER=    1316.4063   UPPER=    1320.3125   UNKNOWN I.C.=    1318.3594
ITERATION   9    LOWER=    1316.4063   UPPER=    1318.3594   UNKNOWN I.C.=    1317.3828
ITERATION  10    LOWER=    1316.4063   UPPER=    1317.3828   UNKNOWN I.C.=    1316.8945
ITERATION  11    LOWER=    1316.4063   UPPER=    1316.8945   UNKNOWN I.C.=    1316.6504
ITERATION  12    LOWER=    1316.6504   UPPER=    1316.8945   UNKNOWN I.C.=    1316.7725
ITERATION  13    LOWER=    1316.7725   UPPER=    1316.8945   UNKNOWN I.C.=    1316.8335
ITERATION  14    LOWER=    1316.7725   UPPER=    1316.8335   UNKNOWN I.C.=    1316.8030
ITERATION  15    LOWER=    1316.8030   UPPER=    1316.8335   UNKNOWN I.C.=    1316.8182
   1.20000         .1990606E-02      .3460961E-03      .1658838E-02      .2143548E+03      .9662493E+03
   1.40000         .2066098E-02      .4021626E-03      .1475784E-02      .2343280E+03      .7153666E+03
   1.60000         .2149423E-02      .4271377E-03      .1343390E-02      .2019369E+03      .5212684E+03
   1.80000         .2235686E-02      .4329298E-03      .1242048E-02      .1370724E+03      .3618762E+03
   2.00000         .2321746E-02      .4258910E-03      .1160873E-02      .4999837E+02      .2247214E+03
#ET=2:26.3 PT=21.3 IO=0.2
```

Figure 10 Typical Output of VARPROP/NUMD: Stress Initial Condition - Displacement Outer Boundary Conditions

Constant Properties (Radial Fiber Orientation)

```
RUN VARPROP/NUMD
#RUNNING 5609
ENTER XI,H,XMAX,INT,DT,W(RPM),RHO(LB/IN**3)
#?
1,.001,2,200,100,10000,.1
ENTER KNOWN I.C.,TYPE(STRESS=1,DISPL.=2)
100,1
ENTER Y2LEFT,Y2RITE
0,10000
ENTER SECOND B.C.,TYPE(STRESS=1,DISPL.=2)
.005,2
ENTER NN,NHIS,SIGNL
15,15,-1


H      =      .100E-02
XI     =        1.0000
XMAX   =        2.0000
N      =             2
INT    =           200
YR1    =      100.0000
TIC1   =        1.0000
Y2LEFT=         0.0000
Y2RITE=     10000.0000
NN     =            15
NHIS   =            15
SIGNL  =            -1
DT     =      100.0000
RHO    =        0.1000
W      =    10000.0000

       X                            Y
                           U                  EPR                EPT                SIGR               SIGT

ITERATION   1   LOWER=        0.0000   UPPER=  10000.0000   UNKNOWN I.C.=    5000.0000
ITERATION   2   LOWER=     5000.0000   UPPER=  10000.0000   UNKNOWN I.C.=    7500.0000
ITERATION   3   LOWER=     5000.0000   UPPER=   7500.0000   UNKNOWN I.C.=    6250.0000
ITERATION   4   LOWER=     5000.0000   UPPER=   6250.0000   UNKNOWN I.C.=    5625.0000
ITERATION   5   LOWER=     5000.0000   UPPER=   5625.0000   UNKNOWN I.C.=    5312.5000
ITERATION   6   LOWER=     5312.5000   UPPER=   5625.0000   UNKNOWN I.C.=    5468.7500
ITERATION   7   LOWER=     5312.5000   UPPER=   5468.7500   UNKNOWN I.C.=    5390.6250
ITERATION   8   LOWER=     5312.5000   UPPER=   5390.6250   UNKNOWN I.C.=    5351.5625
ITERATION   9   LOWER=     5312.5000   UPPER=   5351.5625   UNKNOWN I.C.=    5332.0313
ITERATION  10   LOWER=     5312.5000   UPPER=   5332.0313   UNKNOWN I.C.=    5322.2656
ITERATION  11   LOWER=     5312.5000   UPPER=   5322.2656   UNKNOWN I.C.=    5317.3828
ITERATION  12   LOWER=     5317.3828   UPPER=   5322.2656   UNKNOWN I.C.=    5319.8242
ITERATION  13   LOWER=     5317.3828   UPPER=   5319.8242   UNKNOWN I.C.=    5318.6035
ITERATION  14   LOWER=     5318.6035   UPPER=   5319.8242   UNKNOWN I.C.=    5319.2139
ITERATION  15   LOWER=     5318.6035   UPPER=   5319.2139   UNKNOWN I.C.=    5318.9087
    1.20000         .4733722E-02    -.6829588E-04     .3944768E-02     .8242265E+03     .4331217E+04
    1.40000         .4749964E-02     .2076776E-03     .3392831E-02     .1205982E+04     .3661579E+04
    1.60000         .4809768E-02     .3769476E-03     .3006105E-02     .1401005E+04     .3172818E+04
    1.80000         .4896502E-02     .4820016E-03     .2720279E-02     .1484847E+04     .2795299E+04
    2.00000         .4999834E-02     .5457404E-03     .2499917E-02     .1497600E+04     .2490236E+04
#ET=3:11.1 PT=21.1 IO=0.2
```

Figure 11 Typical Output for VARPROP/NUMD:
Displacement Initial Condition -
Displacement Outer Boundary Condition

Constant Properties (Radial Fiber Orientation)

```
RUN VARPROP/NUMD
#RUNNING 5500
ENTER XI,H,XMAX,INT,DT,W(RPM),RHO(LB/IN**3)
#?
1,.001,2,200,100,10000,.1
ENTER KNOWN I.C.,TYPE(STRESS=1,DISPL.=2)
.01,2
ENTER Y2LEFT,Y2RITE
-.1,.1
ENTER SECOND B.C.,TYPE(STRESS=1,DISPL.=2)
.05,2
ENTER NN,NHIS,SIGNL
15,15,-1


H      =      .100E-02
XI     =        1.0000
XMAX   =        2.0000
N      =             2
INT    =           200
YR1    =        0.0100
TIC1   =        2.0000
Y2LEFT=        -0.1000
Y2RITE=         0.1000
NN     =            15
NHIS   =            15
SIGNL  =            -1
DT     =      100.0000
RHO    =        0.1000
W      =    10000.0000

      X                          Y
                     U                    EPR                 EPT                 SIGR                 SIGT

ITERATION   1   LOWER=     -0.1000   UPPER=       0.1000   UNKNOWN I.C.=        0.0000
ITERATION   2   LOWER=      0.0000   UPPER=       0.1000   UNKNOWN I.C.=        0.0500
ITERATION   3   LOWER=      0.0500   UPPER=       0.1000   UNKNOWN I.C.=        0.0750
ITERATION   4   LOWER=      0.0500   UPPER=       0.0750   UNKNOWN I.C.=        0.0625
ITERATION   5   LOWER=      0.0500   UPPER=       0.0625   UNKNOWN I.C.=        0.0563
ITERATION   6   LOWER=      0.0500   UPPER=       0.0563   UNKNOWN I.C.=        0.0531
ITERATION   7   LOWER=      0.0531   UPPER=       0.0563   UNKNOWN I.C.=        0.0547
ITERATION   8   LOWER=      0.0531   UPPER=       0.0547   UNKNOWN I.C.=        0.0539
ITERATION   9   LOWER=      0.0531   UPPER=       0.0539   UNKNOWN I.C.=        0.0535
ITERATION  10   LOWER=      0.0531   UPPER=       0.0535   UNKNOWN I.C.=        0.0533
ITERATION  11   LOWER=      0.0533   UPPER=       0.0535   UNKNOWN I.C.=        0.0534
ITERATION  12   LOWER=      0.0534   UPPER=       0.0535   UNKNOWN I.C.=        0.0535
ITERATION  13   LOWER=      0.0535   UPPER=       0.0535   UNKNOWN I.C.=        0.0535
ITERATION  14   LOWER=      0.0535   UPPER=       0.0535   UNKNOWN I.C.=        0.0535
ITERATION  15   LOWER=      0.0535   UPPER=       0.0535   UNKNOWN I.C.=        0.0535
    1.20000           .1987381E-01      .4580194E-01      .1656151E-01      .1490054E+06      .6200357E+05
    1.40000           .2849024E-01      .4067638E-01      .2035017E-01      .1366683E+06      .6397668E+05
    1.60000           .3624159E-01      .3702574E-01      .2265099E-01      .1275585E+06      .6473819E+05
    1.80000           .4336097E-01      .3428931E-01      .2408943E-01      .1204976E+06      .6484556E+05
    2.00000           .4999722E-01      .3215548E-01      .2499861E-01      .1148191E+06      .6458522E+05
#ET=3:10.2 PT=21.1 IO=0.2
```

```
RUN VARPROP/NUMD
#RUNNING 5765
ENTER XI,H,XMAX,INT,DT,W(RPM),RHO(LB/IN**3)
#?
1,.001,2,200,100,10000,.1
ENTER KNOWN I.C.,TYPE(STRESS=1,DISPL.=2)
.005,2
ENTER Y2LEFT,Y2RITE
-.01,.01
ENTER SECOND B.C.,TYPE(STRESS=1,DISPL.=2)
100,1
ENTER NN,NHIS,SIGNL
15,15,-1
```

Figure 12 Typical Output for VARPROP/NUMD: Displacement Initial Condition - Stress Outer Boundary Condition

```
H       =       .100E-02
XI      =         1.0000
XMAX    =         2.0000
N       =              2
INT     =            200
YR1     =         0.0050
TIC1    =         2.0000
Y2LEFT=          -0.0100
Y2RITE=           0.0100
NN      =             15
NHIS    =             15
SIGNL   =             -1
DT      =       100.0000
RHO     =         0.1000
W       =     10000.0000

      X                             Y
                         U                    EFR                  EPT                    SIGR                    SIGT

ITERATION   1    LOWER=      -0.0100    UPPER=       0.0100    UNKNOWN I.C.=         0.0000
ITERATION   2    LOWER=      -0.0100    UPPER=       0.0000    UNKNOWN I.C.=        -0.0050
ITERATION   3    LOWER=      -0.0050    UPPER=       0.0000    UNKNOWN I.C.=        -0.0025
ITERATION   4    LOWER=      -0.0025    UPPER=       0.0000    UNKNOWN I.C.=        -0.0013
ITERATION   5    LOWER=      -0.0025    UPPER=      -0.0013    UNKNOWN I.C.=        -0.0019
ITERATION   6    LOWER=      -0.0019    UPPER=      -0.0013    UNKNOWN I.C.=        -0.0016
ITERATION   7    LOWER=      -0.0016    UPPER=      -0.0013    UNKNOWN I.C.=        -0.0014
ITERATION   8    LOWER=      -0.0014    UPPER=      -0.0013    UNKNOWN I.C.=        -0.0013
ITERATION   9    LOWER=      -0.0014    UPPER=      -0.0013    UNKNOWN I.C.=        -0.0014
ITERATION  10    LOWER=      -0.0014    UPPER=      -0.0013    UNKNOWN I.C.=        -0.0013
ITERATION  11    LOWER=      -0.0013    UPPER=      -0.0013    UNKNOWN I.C.=        -0.0013
ITERATION  12    LOWER=      -0.0013    UPPER=      -0.0013    UNKNOWN I.C.=        -0.0013
ITERATION  13    LOWER=      -0.0013    UPPER=      -0.0013    UNKNOWN I.C.=        -0.0013
ITERATION  14    LOWER=      -0.0013    UPPER=      -0.0013    UNKNOWN I.C.=        -0.0013
ITERATION  15    LOWER=      -0.0013    UPPER=      -0.0013    UNKNOWN I.C.=        -0.0013
   1.20000              .4797600E-02     -.7318717E-03      .3998000E-02     -.1128006E+04      .3878638E+04
   1.40000              .4691240E-02     -.3609490E-03      .3350886E-02     -.5380035E+03      .3131979E+04
   1.60000              .4644290E-02     -.1256343E-03      .2902682E-02     -.1942907E+03      .2597296E+04
   1.80000              .4635576E-02      .2775477E-04      .2575320E-02      .1176168E+01      .2191765E+04
   2.00000              .4651894E-02      .1282869E-03      .2325947E-02      .1010172E+03      .1869601E+04
#ET=2:22.8 PT=21.0 IO=0.2
```

Conclusions

Development of a numerical integration scheme for the analysis of cylindrically orthotropic annular disks with variable elastic constants has been accomplished. The integration scheme utilizes Hamming's Predictor-Corrector Method in conjunction with a half-interval search technique which rapidly converges to the exact solution. Radial stress and displacement boundary conditions may be specified. Correlation of numerical integration results with analytical and finite-element solutions was found to be excellent. This analysis capability enables the determination of the influence of prescribed material property variations, temperature changes, and centrifugal forces on the response of an annular disk. The disk may be subjected to internal and external pressure or displacement boundary conditions as well. In addition, interference fit can be approximated by specifying displacement boundary conditions on the disk inner radius.

References

1. D. G. Taggart, R. Byron Pipes and J. C. Mosko, Test Method Evaluation for Fiber-Reinforced Molding Materials, Internal Report, Center for Composite Materials, University of Delaware, Newark, Delaware, 19711, 1978.
2. R. L. McCullough, R. B. Pipes, D. Taggart and J. C. Mosko, "Influence of Fiber Orientation on the Properties of Short Fiber Composites," Composite Materials in the Automobile Industry, ASME, New York, 1978.
3. B. Carnahan et al, Applied Numerical Methods, Wiley and Sons, Inc., New York, 1969.
4. S. G. Lekhnitskii, Anisotropic Plates, translated by S. W. Tsai and T. Cheron, Gordon and Breach Science Publishers, New York, 1968.

Appendix A

Formulation of Governing Equations: Variable Properties

For an axisymmetric body rotating at a constant angular velocity ω, the equilibrium equation in the radial direction is given by:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \rho\omega^2 r = 0 \tag{1}$$

The stress - strain relations in polar coordinates for a cylindrically orthotropic material are given as follows:

$$\varepsilon_r = \frac{\sigma_r}{E_r} - \frac{\nu_{\theta r}\sigma_\theta}{E_\theta} + \alpha_r \Delta T \tag{2}$$

$$\varepsilon_\theta = \frac{\sigma_\theta}{E_\theta} - \frac{\nu_{r\theta}\sigma_r}{E_r} + \alpha_\theta \Delta T \tag{3}$$

Inverting (2) and (3) and solving in terms of the stress components yields,

$$\sigma_r = Q_{rr}(\varepsilon_r - \alpha_r \Delta T) + Q_{r\theta}(\varepsilon_\theta - \alpha_\theta \Delta T) \tag{4}$$

$$\sigma_\theta = Q_{r\theta}(\varepsilon_r - \alpha_r \Delta T) + Q_{\theta\theta}(\varepsilon_\theta - \alpha_\theta \Delta T) \tag{5}$$

where

$$Q_{rr} = \frac{E_r}{1-\nu_{\theta r}\nu_{r\theta}} \qquad Q_{\theta\theta} = \frac{E_\theta}{1-\nu_{\theta r}\nu_{r\theta}} \tag{6}$$

$$Q_{r\theta} = Q_{\theta r} = \frac{E_r \nu_{\theta r}}{1-\nu_{\theta r}\nu_{r\theta}} = \frac{E_\theta \nu_{r\theta}}{1-\nu_{\theta r}\nu_{r\theta}} \tag{7}$$

The strain - displacement relations for an axisymmetric body are

$$\varepsilon_r = \frac{du}{dr} \tag{8}$$

$$\varepsilon_\theta = \frac{u}{r} \tag{9}$$

Where u is the radial displacement. To obtain the governing equation in terms of displacement equations (4), (5), (8) and (9) are substituted into the equilibrium equation (1). Simplifying one obtains,

$$\frac{d^2u}{dr^2} + S(r)\,\frac{du}{dr} + T(r)u = F(r) \tag{10}$$

where

$$S(r) = \left(\frac{1}{Q_{rr}}\;\frac{dQ_{rr}}{dr} + \frac{1}{r}\right) \tag{11}$$

$$T(r) = \left(\frac{1}{rQ_{rr}}\;\frac{dQ_{r\theta}}{dr} - \frac{k^2}{r^2}\right. \tag{12}$$

$$F(r) = \frac{-\rho\omega^2 r}{Q_{rr}} + \left\{\left[\frac{1}{Q_{rr}}\;\frac{dQ_{rr}}{dr} + \frac{1-\nu_{\theta r}}{r}\right]\alpha_r\right. \tag{13}$$

$$+ \left[\frac{1}{Q_{rr}}\;\frac{dQ_{r\theta}}{dr} + \frac{k^2(\nu_{r\theta}-1)}{r}\right]\alpha_\theta$$

$$\left. + \frac{d\alpha_r}{dr} + \nu_{\theta r}\,\frac{d\alpha_\theta}{dr}\right\}\Delta T$$

$$k^2 = E_\theta/E_r$$

and employing equations (6) and (7),

$$\frac{1}{Q_{rr}}\frac{dQ_{rr}}{dr} = \frac{1}{E_r}\frac{dE_r}{dr} + \frac{\nu_{r\theta}\frac{d\nu_{\theta r}}{dr}}{1-\nu_{\theta r}\nu_{r\theta}} + \frac{\nu_{\theta r}\frac{d\nu_{r\theta}}{dr}}{1-\nu_{\theta r}\nu_{r\theta}} \quad (14)$$

$$\frac{1}{Q_{r\theta}}\frac{dQ_{r\theta}}{dr} = \frac{\nu_{\theta r}}{E_r}\frac{dE_r}{dr} + \frac{1}{(1-\nu_{r\theta}\nu_{\theta r})}\left[\frac{d\nu_{\theta r}}{dr} + \nu_{\theta r}^2\frac{d\nu_{r\theta}}{dr}\right] \quad (15)$$

Reduction of equation (10) to a first order system follows:

Let $u_1 = u$

$$u_2 = \frac{du}{dr}$$

then $$\frac{du_1}{dr} = \frac{du}{dr} = u_2 \quad (16)$$

$$\frac{du_2}{dr} = \frac{d^2u}{dr^2}$$

Therefore (10) becomes

$$\frac{du_1}{dr} = u_2 \quad (17)$$

$$\frac{du_2}{dr} = F - S\,u_2 - T\,u_1 \quad (18)$$

and the initial conditions required at the inner radius "a" for Hamming's Method are,

$$u_1(a) = u(a)$$

$$u_2(a) = \frac{du}{dr}(a) \quad (19)$$

For the original boundary value problem, one boundary condition (displacement or radial stress component) will be prescribed on the inner and outer radii. Consequently, the unknown initial condition will be bounded and the half-interval method will be employed to iterate to the solution. Note that Hamming's Method always requires boundary conditions at the inner radius in terms of displacement (see eq. (19)). Therefore, if u(a) is prescribed, then bounds on $\frac{du}{dr}(a)$ are established and the solution is obtained in a straightforward manner. However, if $\sigma_r(a)$ is prescribed, bounds on $\sigma_\theta(a)$ are expected. In this case, the stress components are transformed into the displacement and radial strain boundary conditions utilizing equations (2), (3), (8) and (9).

The listing of the program which performs the numerical integration scheme is given below.

```
#FILE (1345)VARPROP/NUMD ON PACK
$RESET FREE
FILE  6(KIND=REMOTE,MAXRECSIZE=22)
C
C      HAMMING'S PREDICTOR-CORRECTOR METHOD
C
C      PROGRAM SOLVES A SYSTEM OF N FIRST ORDER ORDINARY DIFFERENTIAL
C      EQUATIONS
C
C      LIST OF SYMBOLS:
C
C          XI       INITIAL STARTING VALUE FOR INDEPENDENT VARIABLE
C          H        INTEGRATION STEP-SIZE
C          XMAX     UPPER LIMIT OF INTEGRATION
C          N        NUMBER OF INITIAL CONDITIONS
C          YR(I)    N INITIAL CONDITIONS AT X
C          FR(I)    DERIVATIVES  IN ORIGINAL SYSTEM OF DIFFERENTIAL
C                   EQUATIONS
C         Y(I,J)    Y VALUE AT ITH X VALUE FOR JTH DIFFERENTIAL EQUATION
C         F(I,J)    DERIVATIVE AT ITH X VALUE FOR JTH DIFFERENTIAL
C                   EQUATION
C         TE(I)     TRUNCATION ERROR FOR ITH CORRECTOR EQUATION
C         INT       NUMBER OF INTEGRATIONS BETWEEN OUTPUT
C         NN        UPPER LIMIT ON THE NUMBER OF HALF-INTERVAL ITERATIONS
C         Y2LEFT    LOWER LIMIT ON UNKNOWN INITIAL VALUE
C         Y2RITE    UPPER LIMIT ON UNKNOWN INITIAL VALUE
C         NHIS      NUMBER OF HALF-INTERVAL ITERATIONS BETWEEN PRINTOUT
C         SGI       RADIAL STRESS B.C. ON INNER RADIUS
C         SGO       RADIAL STRESS B.C. ON OUTER RADIUS
C         SIGNL     HAS THE VALUE -1, IF Y1(XMAX)<SIGO WHEN
C                     Y2(XI)=Y2LEFT;OTHERWISE THE VALUE IS 1
C         DT        TEMPERATURE DIFFERENCE RELATIVE TO STRESS
C                   FREE STATE
C         W         ROTATIONAL SPEED(RPM)
C         RHO       DENSITY(#/IN**3)
C
C        SUBROUTINES RUNGE AND HAMMING ARE BASED ON ALGORITHM'S
C        FOUND IN "APPLIED NUMERICAL METHODS" BY CARNAHAN, LUTHER
C        AND WILKES (PAGES 367-402)
C
C        MAIN PROGRAM
C
       INTEGER COUNT,RUNGE,HAMING
       LOGICAL PRED
       DIMENSION TE(10),YR(10),FR(10),Y(4,10),F(3,10),PHI(10),SAVEY(10)
      1,YPRED(10)
       COMMON DT,W,RHO,XI,XMAX
C
C        READ INPUT DATA
C
```

```
C
      N=2
      WRITE(6,10)
      READ(5,/)XI,H,XMAX,INT,DT,W,RHO
      WRITE(6,20)
      READ(5,/)YR1,TIC1
      WRITE(6,25)
      READ(5,/)Y2LEFT,Y2RITE
      WRITE(6,26)
      READ(5,/)YR3,TIC3
      WRITE(6,27)
      READ(5,/)NN,NHIS,SIGNL
C
C       PRINT HEADING AND INPUT DATA
C
      WRITE(6,30)H,XI,XMAX,N,INT,YR1,TIC1,Y2LEFT,Y2RITE,NN,NHIS,SIGNL
     1 ,DT,RHO,W
      WRITE(6,40)
      W=W*2.*3.1415927/60.
      RHO=RHO/32.2/12.
C
C       INITIALIZE STEP COUNTER
C       SET FIRST ROW OF Y MATRIX EQUAL TO INITIAL VALUES
C       INITIALIZE TRUNCATION ERRORS TO ZERO
C
      DO 90 II=1,NN
      X=XI
      YR(1)=YR1
      Y2ZERO=(Y2LEFT+Y2RITE)/2.
      YR(2)=Y2ZERO
      WRITE(6,35)II,Y2LEFT,Y2RITE,Y2ZERO
C
C       CONVERT STRESS B.C. TO DISPLACEMENT B.C.
C
      IF(TIC1.EQ.2)GO TO 55
      CALL PROP(X,YR(1),YR(2),FF,P,Q,YR(2),EPT,YR(1),1)
   55 COUNT=0.
      M=0.
      DO 60 J=1,N
      TE(J)=0.
   60 Y(4,J)=YR(J)
C
C       CALL FOURTH-ORDER RUNGE-KUTTA FUNCTION TO INTEGRATE
C       OVER FIRST THREE STEPS
C       SUBROUTINE DERIV CALUCULATES DERIVATIVES
C
   65  IF(RUNGE(M,N,YR,FR,X,H,PHI,SAVEY).NE.1)GOTO 70
        CALL DERIV(N,1,ISUB,YR,Y,FR,F,X)
       GOTO 65
C
C       PUT APPROPRIATE INITIAL VALUES IN Y AND F MATRICES
   70 COUNT=COUNT+1
```

```
   70 COUNT=COUNT+1
      ISUB=4-COUNT
      DO 75 J=1,N
   75 Y(ISUB,J)=YR(J)
       CALL DERIV(N,2,ISUB,YR,Y,FR,F,X)
C
C      PRINT SOLUTIONS AFTER INT STEPS
C
   80 IF(COUNT/INT*INT.NE.COUNT)GO TO 85
      IF(.NOT.(II/NHIS*NHIS.EQ.II.OR.II.EQ.NN))GO TO 85
      IF(COUNT.LE.3)EPT=Y(ISUB,1)/X
      IF(COUNT.LE.3)CALL PROP(X,F(ISUB,1),EPT,FF,P,Q,SIGR,SIGT,U,3)
      IF(COUNT.GT.3)EPT=Y(1,1)/X
      IF(COUNT.GT.3)CALL PROP(X,F(1,1),EPT,FF,P,Q,SIGR,SIGT,U,3)
      IF(COUNT.LE.3)WRITE(6,50)X,Y(ISUB,1),F(ISUB,1),EPT,SIGR,SIGT
      IF(COUNT.GT.3)WRITE(6,50)X,Y(1,1),F(1,1),EPT,SIGR,SIGT
C
C      IF X > XMAX TERMINATE INTEGRATION
C
   85 CONTINUE
      IF(X.GT.XMAX-H/2)GOTO 100
C
C      CALL RUNGE OR HAMMING TO INTEGRATE NEXT STEP
C
      IF(COUNT.LT.3)GOTO 65
      PRED=.TRUE.
  105 MM=HAMING(N,Y,F,X,H,TE,PRED,YPRED)
      CALL DERIV(N,3,ISUB,YR,Y,FR,F,X)
      IF(MM.EQ.1)GOTO 105
C
C      INCREMENT STEP COUNTER AND CONTINUE INTEGRATION
      COUNT=COUNT+1
      GO TO 80
C
C     COMPARE SOLUTION TO KNOWN OUTER B.C.
C
  100 IF(TIC3.EQ.2)GO TO 102
C
C     CONVERT DISP. AND STRAIN TO STRESS
C
      EPT=Y(1,1)/X
      EPR=F(1,1)
      CALL PROP(X,EPR,EPT,FF,P,Q,SIGR,SIGT,U,3)
      IF((SIGR-YR3)*SIGNL.GT.0)GO TO 106
      Y2RITE=Y2ZERO
      GO TO 90
  106 Y2LEFT=Y2ZERO
      GO TO 90
  102 IF((Y(1,1)-YR3)*SIGNL .GT. 0)GO TO 110
      Y2RITE=Y2ZERO
      GO TO 90
```

```
  110 Y2LEFT=Y2ZERO
   90 CONTINUE
C
C       FORMAT STATEMENTS
C
   10 FORMAT(1X,'ENTER XI,H,XMAX,INT,DT,W(RPM),RHO(LB/IN**3)')
   20 FORMAT(1X,'ENTER KNOWN I.C.,TYPE(STRESS=1,DISPL.=2)')
   25 FORMAT(1X,'ENTER Y2LEFT,Y2RITE')
   26 FORMAT(1X,'ENTER SECOND B.C.,TYPE(STRESS=1,DISPL.=2)')
   27 FORMAT(1X,'ENTER NN,NHIS,SIGNL')
   30 FORMAT(///1X,'H      =',E15.3/1X,'XI     =',F15.4/
     1 1X,'XMAX   =',F15.4/1X,'N      =',I15/1X,'INT    =',I15/
     1  1X,'YR1    =',F15.4/1X,'TIC1   =',F15.4/1X,'Y2LEFT=',F15.4/
     1 1X,'Y2RITE=',F15.4/1X,'NN     =',I15/1X'NHIS   =',I15/
     1 1X,'SIGNL =',I15/1X,'DT     =',F15.4/1X,'RHO    =',F15.4/1X,
     1 'W      ='F15.4//)
   35 FORMAT(1X,'ITERATION',I5,5X,'LOWER='F15.4,5X,'UPPER=',F15.4,
     1 5X,'UNKNOWN I.C.='F15.4)
   40 FORMAT(7X,'X',30X,'Y',/30X,'U',18X,'EPR',16X,'EPT',16X
     1  ,'SIGR',16X,'SIGT'/)
   50 FORMAT(1X,F10.5,5X,5(5X,E15.7))
      CALL EXIT
      END
C**********************************************************************
      FUNCTION RUNGE(M,N,Y,F,X,H,PHI,SAVEY)
C
      INTEGER RUNGE
      DIMENSION PHI(N),SAVEY(N),Y(N),F(N)
      M=M+1
      GO TO (1,2,3,4,5), M
C
    1 RUNGE=1
      RETURN
C
    2 DO 22 J=1,N
      SAVEY(J)=Y(J)
      PHI(J)=F(J)
   22 Y(J)=SAVEY(J)+.5*H*F(J)
      X=X+.5*H
      RUNGE=1
      RETURN
C
    3 DO 33 J=1,N
      PHI(J)=PHI(J)+2.*F(J)
   33 Y(J)=SAVEY(J)+.5*H*F(J)
      RUNGE=1
      RETURN
C
    4 DO  44 J=1,N
      PHI(J)=PHI(J)+2.*F(J)
   44 Y(J)=SAVEY(J)+H*F(J)
```

```
      X=X+.5*H
      RUNGE=1
      RETURN
C
    5 DO 55 J=1,N
   55 Y(J)=SAVEY(J)+(PHI(J)+F(J))*H/6.
      H=0.
      RUNGE=0
      RETURN
      END
C*****************************************************************
C
      SUBROUTINE DERIV(N,FCOUNT,ISUB,YR,Y,FR,F,X)
C
      DIMENSION YR(N),Y(4,N),FR(N),F(3,N)
      CALL PROP(X,SIGR,SIGT,FF,P,Q,EPR,EPT,U,2)
      GO TO(1,2,3), FCOUNT
    1 FR(1)=YR(2)
      FR(2)=FF-P*YR(2)-Q*YR(1)
      RETURN
    2 F(ISUB,1)=YR(2)
      F(ISUB,2)=FF-P*YR(2)-Q*YR(1)
      RETURN
    3 F(1,1)=Y(1,2)
      F(1,2)=FF-P*Y(1,2)-Q*Y(1,1)
      RETURN
      END
C*****************************************************************
C
      FUNCTION HAMING(N,Y,F,X,H,TE,PRED,YPRED)
C
      INTEGER HAMING
      LOGICAL PRED
      DIMENSION YPRED(N),TE(N),Y(4,N),F(3,N)
C      IS CALL FOR PREDICTION OR CORRECTOR SECTION
      IF(.NOT.PRED) GOTO 4
C
C      PREDICTOR SECTION OF HAMING
C      COMPUTE PREDICTED VALUES AT NEXT POINT
      DO 1 J=1,N
    1 YPRED(J)=Y(4,J)+4.*H*(2.*F(1,J)-F(2,J)+2.*F(3,J))/3.
C
C      UPDATE THE Y AND F TABLES
      DO 2 J=1,N
      DO 2 K5=1,3
      K=5-K5
      Y(K,J)=Y(K-1,J)
      IF(K.LT.4)F(K,J)=F(K-1,J)
    2 CONTINUE
C
C      MODIFY PREDICTED Y(J) VALUES USING THE TRUNCATION ERROR
```

```
C         ESTIMATES FROM THE PREVIOUS STEP,INCREMENT X VALUE
          DO 3 J=1,N
     3    Y(1,J)=YPRED(J)+112.*TE(J)/9.
          X=X+H
C
C         SET PRED AND REQUEST UPDATED DERIVATIVE VALUES
          PRED=.FALSE.
          HAMING=1
          RETURN
C
C         CORRECTOR SECTION OF HAMING
C         COMPUTE CORRECTED AND IMPROVED VALUES OF THE Y(J) AND
C         SAVE TRUNCATION ERROR ESTIMATES FOR THE CURRENT STEP
     4    DO 5 J=1,N
          Y(1,J)=(9.*Y(2,J)-Y(4,J)+3.*H*(F(1,J)+2.*F(2,J)-F(3,J)))/8.
          TE(J)=9.*(Y(1,J)-YPRED(J))/121.
     5    Y(1,J)=Y(1,J)-TE(J)
C
C         SET PRED AND RETURN WITH SOLUTIONS FOR CURRENT STEP
          PRED=.TRUE.
          HAMING=2
          RETURN
          END
C*******************************************************************
C
          SUBROUTINE PROF(X,A1,A2,FF,P,Q,A3,A4,A5,OPT1)
C
C  INPUT MATERIAL PROPERTY DEPENDENCE ON RADIAL POSITION X
C  (T: TANGENTIAL, R:RADIAL):
C
C      ET    -TANGENTIAL MODULI
C      ETP   -DERIVATIVE OF ET
C      VTR   -POISSON RATIO
C      VTRP  -DERIVATIVE OF VTR
C      ER    -RADIAL MODULI
C      ERP   -DERIVATIVE OF ER
C      VTR   -POISSON RATIO
C      VTRP  -DERIVATIVE OF VTR
C      AT    -TANGENTIAL COEFFICIENT OF THERMAL EXPANSION
C      ATP   -DERIVATIVE OF AT
C      AR    -RADIAL COEFFICIENT OF THERMAL EXPANSION
C      DT    -TEMPERATURE CHANGE(POSITIVE VALUE CORROSPONDS TO AN INCREASE
C             RELATIVE TO THE STRESS FREE STATE)
C     EPR   -RADIAL STRAIN COMPONENT
C       EPT   -TANGENTIAL STRAIN COMPONENT
C      U     -RADIAL DISPLACEMENT
C
          REAL K
          COMMON DT,W,RHO,XI,XMAX
C
C  INPUT MATERIAL PROPERTY DEPENDENCE ON RADIAL POSITION
```

```
C
      ETM=1.4E6
      VTR=.27
      ERM=2.6E6
      VRTP=0.
      ATM=10.1E-6
      ARM=4.5E-6
      ET=(ERM-ETM)*X/(XMAX-XI)+ETM-(ERM-ETM)*XI/(XMAX-XI)
      ER=(ETM-ERM)*X/(XMAX-XI)+ERM-(ETM-ERM)*XI/(XMAX-XI)
      AR=(ATM-ARM)*X/(XMAX-XI)+ARM-(ATM-ARM)*XI/(XMAX-XI)
      AT=(ARM-ATM)*X/(XMAX-XI)+ATM-(ARM-ATM)*XI/(XMAX-XI)
      ERP=(ETM-ERM)/(XMAX-XI)
      ETP=(ERM-ETM)/(XMAX-XI)
      ATP=(ARM-ATM)/(XMAX-XI)
      ARP=(ATM-ARM)/(XMAX-XI)
      VRT=ER*VTR/ET
      VRTP=VTR*ERP/ET-VTR*ER/ET**2*ETP
C  END MATERIAL PROPERTY INPUT
      VD=1.-VRT*VTR
      QRR=ER/VD
      QTT=ET/VD
      QRT=ER*VTR/VD
      QRRP=ERP/ER+VRT*VTRP/VD+VTR*VRTP/VD
      QRTP=VTR*ERP/ER+(VTRP+VTR**2*VRTP)/VD
      K=SQRT(ET/ER)
      IF(OPT1.EQ.1)GO TO 1
      IF(OPT1.EQ.3)GO TO 3
      FF=-RHO*W**2*X/QRR+((QRRP+(1-VTR)/X)*AR
     1 +(QRTP+K**2*(VRT-1.)/X)*AT+ARP+VTR*ATP)*DT
      P=1./X+QRRP
      Q=QRTP/X-(K/X)**2
      IF(OPT1.EQ.2)GO TO 2
    1 A3=A1/ER-VTR*A2/ET+AR*DT
      A4=-VRT*A1/ER+A2/ET+AT*DT
      A5=X*A4
      GO TO 2
    3 A3=QRR*A1+QRT*A2-DT*(QRR*AR+QRT*AT)
      A4=QRT*A1+QTT*A2-DT*(QRT*AR+QTT*AT)
    2 RETURN
      END
#
```

Appendix B

Analytic Solution: Constant Properties

For the case of constant properties, equation (10) can be rewritten as follows:

$$r^2 \frac{d^2u}{dr^2} + r\,\frac{du}{dr} - k^2 u = C_1 r - C_2 r^3 \tag{20}$$

where

$$C_1 = \Delta T \quad \{\ 2_r(1-\nu_{\theta r}) + k^2 \alpha_\theta (\nu_{r\theta} - 1)\ \}$$

$$C_2 = \frac{(1-\nu_{\theta r}\nu_{r\theta})}{E_r}\ \rho\omega^2 \tag{21}$$

Noting that (20) is in the form of Euler's equations, the general solution can be expressed as follows:

$$u = Ar^k + Br^{-k} + \frac{C_1 r}{1-k^2} - \frac{C_2 r^3}{9-k^2} \quad (k \neq 1 \text{ or } 3) \tag{22}$$

where A and B are constants to be determined from the boundary conditions. The corresponding stress components are:

$$\sigma_r = \frac{A\ E_r(k+\nu_{\theta r})\,r^{k-1}}{1-\nu_{\theta r}\nu_{r\theta}} + \frac{B\ E_r(\nu_{\theta r}-k)\,r^{-k-1}}{1-\nu_{\theta r}\nu_{r\theta}} - \frac{(3+\nu_{\theta r})\rho\omega^2 r^2}{9-k^2} + \frac{\Delta T\ E_\theta}{(1-k^2)}(\alpha_r - \alpha_\theta) \tag{23}$$

$$\sigma_\theta = \frac{A\ E_\theta(1+k\nu_{r\theta})}{(1-\nu_{r\theta}\nu_{\theta r})}\ r^{k-1} + \frac{B\ E_\theta(1-k\nu_{r\theta})}{(1-\nu_{r\theta}\nu_{\theta r})}\ r^{-k-1}$$

$$- \frac{(1+3\nu_{r\theta})k^2\rho\omega^2 r^2}{9-k^2} + \frac{\Delta T E_\theta}{1-k^2}\ (\alpha_r-\alpha_\theta) \tag{24}$$

If radial stress boundary conditions are prescribed on the inner (a) and outer radius (b),

$$\sigma_r(a) = p \tag{25}$$

$$\sigma_r(b) = q$$

The unknown constants are,

$$A = \frac{(Pd^{k+1} - Q)}{b^{k-1}[d^{2k}-1]\ Q_{rr}(k+\nu_{\theta r})}$$

$$B = \frac{(-P + d^{k-1}Q)d^{k+1}}{b^{-k-1}[d^{2k}-1]\ Q_{rr}(\nu_{\theta r}-k)} \tag{26}$$

where $$P = p + \frac{(3+\nu_{\theta r})\rho\omega^2 a^2}{9-k^2} - \frac{\Delta T\ E_\theta(\alpha_r-\alpha_\theta)}{1-k^2} \tag{27}$$

$$Q = q + \frac{(3+\ \nu_{\theta r})\rho\omega^2 b^2}{9-k^2} - \frac{\Delta T\ E_\theta(\alpha_r-\alpha_\theta)}{1-k^2}$$

$$d = a/b,\ k^2 = E_\theta/E_r$$

Employing equation (26), we obtain the following relations for the stress components:

$$\sigma_r = \frac{Pd^{k+1}-Q}{d^{2k}-1}\left(\frac{r}{b}\right)^{k-1} - \frac{(P-d^{k-1}Q)d^{k+1}}{d^{2k}-1}\left(\frac{r}{b}\right)^{-k-1}$$

$$- \frac{(3+\nu_{\theta r})\rho\omega^2 r^2}{9-k^2} + \frac{\Delta T E_\theta}{(1-k^2)}(\alpha_r-\alpha_\theta) \qquad (28)$$

$$\sigma_\theta = \frac{k^2(1+k\nu_{r\theta})}{(k+\nu_{\theta r})}\frac{(Pd^{k+1}-Q)}{(d^{2k}-1)}\left(\frac{r}{b}\right)^{k-1}$$

$$- \frac{k^2(1-k\nu_{r\theta})}{(\nu_{\theta r}-k)}\frac{(P-d^{k-1}Q)}{(d^{2k}-1)}d^{k+1}\left(\frac{r}{b}\right)^{-k-1} \qquad (29)$$

$$- \frac{(1+3\ \nu_{r\theta})k^2\rho\omega^2 r^2}{9-k^2} + \frac{\Delta T E_\theta}{(1-k^2)}(\alpha_r-\alpha_\theta)$$

A program listing for the analytical solution derived for radial stress boundary conditions is given below. Stress components are determined directly from equations (28) and (29) and strain components and radial displacement are calculated from equations (2), (3) and (9), respectively.

```
#FILE (1345)VARPROP/CFD ON PACK
$RESET FREE
FILE  6(KIND=REMOTE,MAXRECSIZE=22)
       REAL K
C     LIST OF SYMBOLS:
C
C          XI      INITIAL STARTING VALUE FOR INDEPENDENT VARIABLE
C           DX      X INCREMENT
C          XMAX    UPPER LIMIT OF INTEGRATION
C         SGI      RADIAL STRESS B.C. ON INNER RADIUS
C         SGO      RADIAL STRESS B.C. ON OUTER RADIUS
C
C     ET   -TANGENTIAL MODULI
C     VTR  -POISSON RATIO
C     ER   -RADIAL MODULI
C     AT   -TANGENTIAL COEFFICIENT OF THERMAL EXPANSION
C     AR   -RADIAL COEFFICIENT OF THERMAL EXPANSION
C     DT   -TEMPERATURE CHANGE(POSITIVE VALUE CORROSPONDS TO AN INCREASE
C           RELATIVE TO THE STRESS FREE STATE)
C     EPR  -RADIAL STRAIN COMPONENT
C     EPT  -TANGENTIAL STRAIN COMPONENT
C     U    -RADIAL DISPLACEMENT
C     W    -ROTATIONAL SPEED(RPM)
C   RHO    -DENSITY(#/IN**3)
C
C   INPUT MATERIAL PROPERTIES
C  (T: TANGENTIAL, R:RADIAL):
C
        ET=2.6E6
        VTR=.501
        ER=1.4E6
        VRT=VTR*ER/ET
        AT=4.5E-6
        AR=10.1E-6
        RHO=.1
C
C    END MATERIAL PROPERTY DESCRIPTION
C
C        READ INPUT DATA
C
        WRITE(6,10)
        READ(5,/)XI,XMAX,DX,DT,W
        WRITE(6,20)
        READ(5,/)SGI,SGO
C
C        PRINT HEADING AND INPUT DATA
C
        WRITE(6,30)XI,XMAX,SGO,SGI,ET,ER,AT,AR,VTR,DT,W,RHO
        WRITE(6,40)
```

```
C
      W=W*2*3.1415927/60.
      RHO=RHO/32.2/12.
      D=XI/XMAX
      K=SQRT(ET/ER)
      DEN=D**(2*K)-1.
      P1=DT*(AR-AT)/(1.-K**2)
      P2=(3+VTR)*RHO*W**2/(9.-K**2)
      P=SGI+P2*XI**2-P1*K**2*ER
      Q=SGO+P2*XMAX**2-P1*K**2*ER
      X=XI
   80 R=X/XMAX
      SIGR=(P*D**(K+1)-Q)*R**(K-1)/DEN-(P-D**(K-1)*Q)*D**(K+1)
     1 /DEN*R**(-K-1)-P2*X**2+P1*K**2*ER
      SIGT=K**2*(1+K*VRT)/(K+VTR)*(P*D**(K+1)-Q)/DEN*R**(K-1)
     1 -K**2*(1-K*VRT)/(VTR-K)*(P-D**(K-1)*Q)/DEN*D**(K+1)
     1 *R**(-K-1)-(1+3.*VRT)*RHO*(W*K*X)**2/(9-K**2)+P1*ET
   70 EPR=SIGR/ER-VTR*SIGT/ET+AR*DT
      EPT=-VTR*SIGR/ET+SIGT/ET+AT*DT
      U=X*EPT
      WRITE(6,50)X,SIGR,SIGT,EPR,EPT,U
      IF(X.GE.XMAX-DX)GO TO 2
      X=X+DX
      GO TO 80
C
C       FORMAT STATEMENTS
C
   10 FORMAT(1X,'ENTER XI,XMAX,DX,DT,W')
   20 FORMAT(1X,'ENTER SGI,SGO')
   30 FORMAT(///1X,'XI     =',E15.3/1X,'XMAX   =',E15.4/
     1  1X,'SGO    =',E15.4/1X,'SGI    =',E15.4/1X,'ET     =',E15.4/
     1 1X,'ER     =',E15.4/1X,'AT     =',E15.4/1X'AR     =',E15.4/
     1 1X,'VTR    =',E15.3/1X,'DT     =',E15.4/1X,'W      =',E15.4/
     1 1X,'RHO    ='E15.4//)
   40 FORMAT(17X,'X',14X,'SIGR',16X,'SIGT',16X,'EPR',17X
     1 ,'EPT',18X,'U'/)
   50 FORMAT(10X,F10.4,5(5X,E15.7))
    2 RETURN
       END
#
```

Appendix C

Analytical Solution: Power Law Variation of Properties

For the power law variation of modulii, the stress function approach [4] is utilized to obtain a closed form solution. Employing the strain-displacement relations in (8) and (9) and eliminating u in (2) and (3), we obtain

$$\left(- \frac{r\nu_{\theta r}}{E_r} \sigma_r \right)' + \left(\frac{r}{E_\theta} \sigma_\theta \right)' + (r\alpha_\theta \Delta T)' \tag{30}$$

$$= - \frac{\sigma_r}{E_r} - \frac{\nu_{\theta r}}{E_\theta} \sigma_\theta + \alpha_r \Delta T$$

where prime denotes derivates with respect to radial position. For an axisymmetric body, the relations between the stress function and stress components which identically satisfy Eq. (1), (in the absence of body forces) are

$$\sigma_r = g/r, \ \sigma_\theta = g' \tag{31}$$

Substitution of (26) into (25) yields the governing differential equation,

$$g'' + \left(\frac{1}{r} - \frac{E_\theta'}{E_\theta}\right) g' + \left(\frac{\nu_{\theta r} E_\theta'}{r E_\theta} - \frac{\nu_{\theta r}'}{r} - \frac{E_\theta}{r^2 E_r}\right) g \tag{32}$$

$$= - \Delta T \left[\frac{E_\theta (\alpha_\theta - \alpha_r)}{r} + E_\theta \alpha_\theta' \right]$$

Assuming the exponential form for the modulii equation (32) may be ordered

$$E_r = E_{rm}\, r^m, \quad E_\theta = E_{\theta m}\, r^m$$

$$\nu_{\theta r} = \text{constant}, \quad \nu_{r\theta} = \nu_{\theta r}\,\frac{E_{rm}}{E_{\theta m}} \tag{33}$$

$$r^2 g'' + r(1-m)g' + (m\nu_{\theta r} - k^2)g$$

$$= -\,\Delta T\, E_{\theta m}\, r^{m+2}\,\left[\frac{(\alpha_\theta - \alpha_r)}{r} + \alpha_\theta'\right] \tag{34}$$

The general solution of equation (29) can be expressed as,

$$g = Ar^{N_1} + Br^{N_2} + g_p \tag{35}$$

where $N_{1,2} = \dfrac{m \pm \;\; m^2 + 4(k^2 - m\nu_{\theta r})}{2}$

and g_p is the particular solution.

The particular solution is determined by a variation of parameter's approach,

$$g_p = v_1 r^{N_1} + v_2 r^{N_2} \tag{36}$$

where

$$v_1' = \frac{-r^{1-N_1} C_3}{(N_2 - N_1)}$$

$$v_2' = \frac{r^{1-N_2} C_3}{(N_2 - N_1)} \tag{37}$$

and C_3 is defined as the right-hand side of equation (34)

$$C_3 = -\Delta T E_{\theta m} r^{m+2} \left[\frac{(\alpha_\theta - \alpha_r)}{r} + \alpha_\theta{}' \right] \tag{38}$$

The stress components defined in equation (31) are:

$$\sigma_r = Ar^{N_1-1} + Br^{N_2-1} + r^{N_1-1} v_1 + r^{N_2-1} v_2 \tag{39}$$

$$\sigma_\theta = AN_1 r^{N_1-1} + BN_2 r^{N_2-1} + N_1 r^{N_1-1} v_1 + N_2 r^{N_2-1} v_2 \tag{40}$$

and A and B are unknown constants to be determined from the boundary conditions. The components of displacement and strain are calculated from equations (9), (2) and (3), respectively.

If radial stress boundary conditions are prescribed in equation (25), the unknown constants are given as follows:

$$A = \frac{Pd^{1-N_2} - Q}{b^{N_1-1}(d^{N_1-N_2} - 1)} \tag{41}$$

$$B = \frac{-(P - d^{N_1-1} Q) d^{1-N_2}}{b^{N_2-1}(d^{N_1-N_2} - 1)} \tag{42}$$

where

$$P = p - a^{N_1-1} V_1(a) - a^{N_2-1} V_2(a) \tag{43}$$

$$Q = q - b^{N_1-1} V_1(b) - b^{N_2-1} V_2(b) \tag{44}$$

and the V_i (i=1,2) are obtained through integration of equation (37) and evaluation at the boundaries.

For completeness, assume the coefficients of thermal expansion also vary as a power law, i.e.,

$$\alpha_\theta = \alpha_{\theta m}\, r^m \tag{45}$$

$$\alpha_r = \alpha_{rm}\, r^m \tag{46}$$

Equation (38) reduces to

$$C_3 = C_4 r^{2m-1} \tag{47}$$

where

$$C_4 = -\Delta T E_{\theta m}[\alpha_{\theta m}(m+1) - \alpha_{rm}] \tag{48}$$

and integrating equation (37) yields

$$V_1 = \frac{-C_4 r^{2m-N_1+1}}{(N_2-N_1)\,(2m-N_1+1)} \tag{49}$$

$$V_2 = \frac{C_4 r^{2m-N_2+1}}{(N_2-N_1)\,(2m-N_2+1)} \tag{50}$$

Employing equations (41), (42), (43) and (44) we obtain the following expression for the stress components:

$$\sigma_r = \left[\frac{Pd^{1-N_2} - Q}{(d^{N_1-N_2} - 1)} \right] \left(\frac{r}{b}\right)^{N_1-1} - \left[\frac{P - d^{N_1-1} Q}{d^{N_1-N_2} - 1} \right] d^{1-N_2} \left(\frac{r}{b}\right)^{N_2-1}$$

$$+ \frac{C_4 r^2}{(2m-N_1+1)(2m-N_2+1)} \tag{51}$$

$$\sigma_\theta = \left[\frac{Pd^{1-N_2} - Q}{d^{N_1-N_2} - 1} \right] N_1 \left(\frac{r}{b}\right)^{N_1-1} - \left[\frac{P - d^{N_1-1} Q}{d^{N_1-N_2} - 1} \right] N_2 d^{(1-N_2)} \left(\frac{r}{b}\right)^{N_2-1}$$

$$+ \frac{C_4 r^{2m} (1+2m)}{(2m-N_1+1)(2m-N_2+1)} \tag{52}$$

Equations (43) and (44) simplify to the following:

$$P = p - \frac{C_4 a^{2m}}{(2m-N_1+1)(2m-N_2+1)} \tag{53}$$

$$Q = q - \frac{C_4 b^{2m}}{(2m-N_1+1)(2m-N_2+1)} \tag{54}$$

Note that for $m = 0$ ($N_1=k$, $N_2=-k$), equations (51) - (54) reduce to the solution given in Appendix B for constant material properties. The program listing for this solution is given next.

```
#FILE (1345)VARPROP/CF2 ON PACK
$RESET FREE
FILE  6(KIND=REMOTE,MAXRECSIZE=22)
       REAL K,N1,N2,M
C      LIST OF SYMBOLS:
C
C           XI      INITIAL STARTING VALUE FOR INDEPENDENT VARIABLE
C            DX      X INCREMENT
C           XMAX    UPPER LIMIT OF INTEGRATION
C          SGI      RADIAL STRESS B.C. ON INNER RADIUS
C          SGO      RADIAL STRESS B.C. ON OUTER RADIUS
C
C      ETM   -TANGENTIAL MODULI
C      VTR  -POISSON RATIO
C      ERM   -RADIAL MODULI
C      ATM   -TANGENTIAL COEFFICIENT OF THERMAL EXPANSION
C      ARM   -RADIAL COEFFICIENT OF THERMAL EXPANSION
C      DT   -TEMPERATURE CHANGE(POSITIVE VALUE CORROSPONDS TO AN INCREASE
C             RELATIVE TO THE STRESS FREE STATE)
C      EPR  -RADIAL STRAIN COMPONENT
C      EPT  -TANGENTIAL STRAIN COMPONENT
C      U    -RADIAL DISPLACEMENT
C
C    INPUT MATERIAL PROPERTIES
C   (T: TANGENTIAL, R:RADIAL):
C
       ETM=1.4E6
       VTR=.27
       ERM=2.6E6
       ATM=10.1E-6
       ARM=4.5E-6
C
C     END MATERIAL PROPERTY DESCRIPTION
C
C        READ INPUT DATA
C
       WRITE(6,10)
       READ(5,/)XI,XMAX,DX,DT
       WRITE(6,20)
       READ(5,/)SGI,SGO,M
C
C        PRINT HEADING AND INPUT DATA
C
       WRITE(6,30)XI,XMAX,SGO,SGI,ETM,ERM,ATM,ARM,VTR,DT,M
       WRITE(6,40)
```