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FINITE ELEMENT METHOD FOR SOLVING PROBLEMS WITH SINGULAR SOLUTIONS

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Finite element method for solving problems with singular solutions

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July 19, 1995

Abstract

Numerical treatment of the elliptic boundary value problem with nonsmooth solution by the finite element method is discussed. The nonsmoothness could have its origin in the unsmooth boundary or the differential equation. This paper, which is a survey of the recent results, elaborates among others on the method of auxiliary mapping, the partition of unity finite element method and the *hp* version of FEM in three-dimensions. Numerical examples illustrate mathematical results.

1 Introduction

Finite element methods face significant problems if the exact solution of the solved problem is not sufficiently smooth.

The unsmoothness could have very different character.

a) Let us consider, as the model problem, the boundary value problem for the Laplace or elasticity equation on the domain $\Omega \subset \mathbf{R}^2$. If the boundary of the domain has corner, located with the internal angle ω in the origin, then the solution u, in the neighborhood of the origin, has essentially the form $u =$

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$r^\beta \phi(\theta)$ where (r, θ) are the polar coordinates, $\beta > 0$ and $\phi(\theta)$ is an analytic function in θ. For the Laplace equation, the mixed boundary conditions and internal angle $\omega = 2\pi$, we have $\beta = 1/4$. For the elasticity equation, mixed boundary conditions and a particular internal angle, the exponent β could be very small, e.g. see [1]. The same occurs when operators with piecewise smooth coefficients are considered, e.g. see [2]. The solution which essentially has the form $u = r^\beta \phi(\theta)$ belongs to the Besov space $B^k_{2,\infty}(\Omega)$ for $1 < k \leq \beta$, but not for $k > 1 + \beta$. Also, $u \in H^k(\Omega)$ for $k < 1 + \beta$, but $u \notin H^k(\Omega)$ for $k \geq 1 + \beta$. Furthermore, the solution belongs to countably normed spaces introduced in [3], [4], and [5]. Particularly, we have for any $\alpha = (\alpha_1, \alpha_2)$

$$|D^\alpha u| \leq C\, \alpha! d^{|\alpha|} r^{\gamma - |\alpha|}, \tag{1.1}$$

with $\gamma > 0$, $|\alpha| > 1$ and C independent of α.

We have shown in [3], [4], and [5] that the solution of elliptic problems with piecewise analytic data belongs to these spaces. In [6], [7], and [8], we have proven that if the exact solution belongs to this space then the *hp* version of the finite element method converges exponentially with

$$\|e\|_E \leq Ce^{-\rho N^{1/3}} \tag{1.2}$$

where N is the number of degrees of freedom and $\rho > 0$ depends on γ in (1.1); particularly, $\rho \to 0$ as $\gamma \to 0$. In [9] we have shown that in one-dimension, $\rho \to 0$ as $\gamma \to 0$ for any *hp* version, more precisely, $\|e\|_E \geq Ce^{-\rho N^{1/2}}$, (for more see § 2.1).

If β or γ is *small*, then we will speak about a *strong singularity*, which should be distinguished from the case when β or γ is not extremely small. We mention here only the two-dimensional problem, although the situation is similar in three-dimensions, too. (See §4 of this paper).

b) The second type of unsmooth solutions has completely different character. Consider the model problem solving the equation

$$-\Delta u + Cu = f \tag{1.3}$$

with C being large (in absolute value) either positive or negative (for example, if considering Helmholtz problem) or

$$\sum_{ij} -\frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} = f \tag{1.4}$$

where $a_{ij}(x)$ are rough functions. Then also, if f is smooth, the solution of (1.3) has a boundary layer character ($C \gg 1$) or is highly oscillatory when $C \ll -1$. The solution of (1.3) or (1.4) is not smooth, but has very different character when compared with the one described in *a*.

In this paper, we will address methods for solving problems when the solution is singular in the sense mentioned above.

2 The method of auxiliary mapping

Before we address the method of the auxiliary mapping (MAM), we will discuss in detail the one-dimensional finite element method based on polynomial approximation. This will lead to an insight into difficulties when the standard finite element method is used.

2.1 The *hp* version of the finite element method in one-dimension.

Let us consider the problem

$$\begin{aligned} -u_\beta'' &= f, \quad x \in I = (0,1), \\ u_\beta(0) &= 0, \\ u_\beta(1) &= 1 \end{aligned} \tag{2.1}$$

with the exact solution

$$u_\beta = x^\beta, \quad \beta > 1/2. \tag{2.2}$$

This case is analogous to the two-dimensional case when $u = r^\alpha \phi(\theta)$ with $\alpha > 0$ and $\beta = 1/2 + \alpha$.

Consider the finite element method on the mesh Δ_n,

$$\Delta_n : 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1,$$

with $I_j = (x_{j-1}, x_j)$, $h_j = x_j - x_{j-1}$, $j = 1, \ldots, n$ and polynomial shape functions of degrees $p_j \geq 1$ on I_j, $j = 1, \ldots, n$, with $P_n = (p_1, \ldots, p_n)$. Let

$$S(\Delta_n, P_n) = \{u \in H^1(I), u(0) = 0, u(1) = 1\}.$$

Denoting $N(S(\Delta_n, P_n)) = \dim S(\Delta_n, P_n) = -1 + \sum_{i=1}^n p_i$, N is obviously the number of degrees of freedom.

Let $u^{[\beta]}_{S(\Delta_n,P_n)}$ be the classical finite element solution of (2.1), (2.2). Furthermore, let

$$e^{[\beta]}_{S(\Delta_n,P_n)} = u_\beta - u^{[\beta]}_{S(\Delta_n,P_n)} \tag{2.3}$$

and

$$E_\beta(\Delta_n, P_n) = \|e^{[\beta]}_{S(\Delta_n,P_n)}\|_E = \left(\int_0^1 \left(\frac{d}{dx} \left(u_\beta - u^{[\beta]}_{S(\Delta_n,P_n)} \right) \right)^2 dx \right)^{1/2}. \tag{2.4}$$

We will now address the question as to what can be said about $E_\beta(\Delta_n, P_n)$ as a function of (Δ_n, P_n). The first essential question is about the *lower* bound of $E_\beta(\Delta_n, P_n)$ with $\dim S(\Delta_n, P_n) = N$. In [9], we have proven among others

Theorem 2.1 *Let*

$$\varepsilon_\beta(N) = \inf_{\dim S(\Delta_n, P_n)=N} E_\beta(\Delta_n, P_n),$$

then

$$\varepsilon_\beta(N) \geq C(\beta)\frac{q_o^{\sqrt{(\beta-1/2)N}}}{\sqrt{N^{\beta-1/2}}}.$$

where

$$q_o = \left(\sqrt{2}-1\right)^2$$

and $C(\beta)$ is a constant independent of N. □

Theorem 2.1 shows that when $\beta - 1/2$ is very small, i.e., if the solution is strongly singular, then *no finite element method using polynomial shape functions can be efficient.* Let us show now that we can construct a sequence (Δ_n, P_n) so that $E_\beta(\Delta_n, P_n)$ is essentially the same as $\varepsilon_\beta(N)$. To this end, for $0 < q < 1$, define

$$\Delta_n(q) : 0 = x_0 < x_1 < \cdots < x_n$$

with

$$x_i = q^{n-i}, i = 1, \ldots, n$$

and for $0 < s$ let $P_n(s) = (p_1, \ldots, p_n)$ with

$$p_i = [1 + s(i-1)], i = 1, \ldots, n$$

where by $[a]$, we denote the closest integer to a. Denote by $S(s, q, n)$ the associated finite element space for $\Delta_n(q)$ and $P_n(s)$ and let $N(s, q, n) = \dim S(s, q, n)$. As before, we denote by $u^{[\beta]}_{S(s,q,n)}$ the finite element solution and by $e^{[\beta]}_{S(s,q,n)}$ its error. Then we have proven in [9]:

Theorem 2.2 *If*

1. *$s > s_0$, then*

$$\|e^{[\beta]}_{S(s,q,n)}\|_E \leq C(\beta, q, s) q^{(\beta-1/2)\sqrt{2N/s}}; \tag{2.5}$$

2. *$s < s_0$, then*

$$\|e^{[\beta]}_{S(s,q,n)}\|_E \leq C(\beta, q, s) r^{\sqrt{2Ns}}; \tag{2.6}$$

3. *$s = s_0$, then*

$$\|e^{[\beta]}_{S(s,q,n)}\|_E \leq C(\beta, q,) e^{-\sqrt{(\beta-1/2)N}\sqrt{2\ln q \ln r}} \tag{2.7}$$

with

$$r = \frac{1-\sqrt{q}}{1+\sqrt{q}} \tag{2.8}$$

and

$$s_0 = (\alpha - 1/2)\frac{\ln q}{\ln r}. \tag{2.9}$$

Furthermore, if

$$q_{\text{opt}} = \left(\sqrt{2}-1\right)^2 \tag{2.10}$$

$$s_{\text{opt}} = 2\alpha - 1, \tag{2.11}$$

then

$$\|e^{[\beta]}_{S(s_{\text{opt}},q_{\text{opt}},n)}\|_E \le C(\beta) q_{\text{opt}}^{\sqrt{(\beta-1/2)N}} = C(\beta) e^{-1.7626\sqrt{(\beta-1/2)N}}. \tag{2.12}$$

The constants C in (2.5), (2.6), (2.7), and (2.12) are independent of N. □

Remark 2.1 We have proven in [9] that instead of $\le$ used in Theorem 2.2, the equivalency holds, i.e., that the lower bound has the same form as the upper bound.

In adaptive procedures, often $q = 1/2$ is used. Then for optimal distribution of the degrees given by (2.9),

$$\|e^{[\beta]}_{S(s_o,1/2,n)}\|_E \le C(\beta) e^{-1.5632\sqrt{(\beta-1/2)N}}. \tag{2.13}$$

Remark 2.2 We have $q_{\text{opt}} = 0.1715$. It is always better slightly to overrefine the mesh and the value $q = 0.15$ is recommended.

Comparing (2.12) with (2.4), we see that the mesh $\Delta_n(q)$ and degrees distribution $P_n(s)$ leads essentially to the best possible error. To obtain the error which is comparable with the best error (2.4), it is essential that degrees of elements are not uniform, *they are low when the elements are small and high when the elements are large.* The question arises what is the error for uniform p. In [9], we have proven

Theorem 2.3 *Let*

$$P_n(s) = (p_1, \ldots, p_n), p_i = [sn]$$

1. *If $s > s_0$, then*

$$\|e^{[\beta]}_{S(s,q,n)}\|_E \le C(\beta,q,s) \frac{q^{(\beta-1/2)\sqrt{N/s}}}{\sqrt{sN^{2\beta-1}}}; \tag{2.14}$$

2. *If $s < s_0$, then*

$$\|e^{[\beta]}_{S(s,q,n)}\|_E \le C(\beta,q,s) \frac{r^{\sqrt{sN}}}{\sqrt{sN^{\beta}}}; \tag{2.15}$$

3. *If $s = s_0$, then*

$$\|e^{[\beta]}_{S(s,q,n)}\|_E \le C(\beta,q,s) \frac{e^{-\sqrt{(\beta-1/2)N}\sqrt{\ln q \ln r}}}{\sqrt{sN^{\sigma}}}, \tag{2.16}$$

where

$$s_0 = (\beta - 1/2)\frac{\ln q}{\ln r}$$

$$r = \frac{1-\sqrt{q}}{1+\sqrt{q}} \tag{2.17}$$

$$\sigma = \min(2\beta - 1, \beta). \tag{2.18}$$

Furthermore, if

$$q = q_{\text{opt}} = \left(\sqrt{2}-1\right)^2 \tag{2.19}$$

$$s = s_{\text{opt}} = 2\beta - 1, \tag{2.20}$$

then

$$\|e^{[\beta]}_{S(s_{\text{opt}},q_{\text{opt}},n)}\|_E \leq C(\beta)\frac{q_{\text{opt}}^{\sqrt{(\beta-1/2)N/2}}}{\sqrt{N^\sigma}} = \frac{C(\beta)}{\sqrt{N^\sigma}}e^{-1.2464\sqrt{(\beta-1/2)N}}. \tag{2.21}$$

□

Comparing Theorem 2.2 and Theorem 2.3, we clearly see that uniform degrees distribution decreases the rate of convergence.

In [9], we analyzed in great detail the one-dimensional case. From this analysis, it follows, that for the optimal *hp* version, the elemental errors should be asymptotically equal, and the degree in the first element has to be the smallest one. If it is not, then the mesh is underrefined. If the singularity is strong, the optimal mesh is so strongly refined, that implementational difficulties (round-offs) could occur.

As we stated in Theorem 2.2 and Theorem 2.3 $\approx$ instead of $\leq$ could be used.

Let us illustrate the effectiveness of the estimates in Theorem 2.3. Consider the case $\beta = 0.75$, $q = 0.15$ and $s = 2\beta - 1 = 0.5$ and uniform p. This case is analogous to the case of Laplace equation mentioned in the introduction when $\gamma = 1/4$. Based on Theorem 2.3, we have

$$\|e^{[\beta]}_{S(s,0.15,n)}\|_E \leq C\frac{1}{N^{0.375}}e^{-0.5778N^{1/2}}. \tag{2.22}$$

In Table 2.1, we report for n, p, N the relative error

$$\varepsilon\% = \|e^{[\beta]}_{S(s,0.15,n)}\|_E/\|u^{[\beta]}\| \cdot 100$$

and the constant

$$D(N) = \|e^{[\beta]}_{S(s,0.15,n)}\|_E N^{0.375}e^{0.5778N^{1/2}} \tag{2.23}$$

which illustrates the effectiveness of (2.22). Based on the theory presented in [9], $D(N)$ should be uniformly bounded from below and above. Because $s = 1/2$, we have for odd n, $p_n = \text{integer} + 1/2$ and hence, we report the errors for both p integers closest to s_n. Finally, we report the ratio of $h_{\min}/h_{\max}$. We see that in fact, the formula (2.22) is very accurate

n	p	N	$\varepsilon\%$	D	$h_{\min}/h_{\max}$
1	2	1	23.81	0.424	1.
2	1	1	23.10	0.412	0.18
2	2	3	15.10	0.620	0.18
3	1	2	17.60	0.516	0.26 (−1)
3	2	5	9.842	0.655	0.26 (−1)
4	2	7	6.786	0.650	0.40 (−2)
5	2	9	5.140	0.663	0.59 (−3)
5	3	14	3.175	0.742	0.59 (−3)
6	3	17	2.206	0.691	0.89 (−4)
7	3	20	1.688	0.688	0.13 (−4)
7	4	27	1.074	0.744	0.13 (−4)
8	4	31	0.6865	0.621	0.20 (−5)
9	4	35	0.5572	0.645	0.30 (−6)
9	5	44	0.3223	0.616	0.30 (−6)
10	5	49	0.2421	0.595	0.45 (−7)

Table 2.1. The performance of the *hp* version of the finite element method.

We have shown that Theorem 2.1, Theorem 2.2, and Theorem 2.3 completely and very precisely characterize the performance of the *hp* version. We see that for $\beta - 1/2$ very small, the *hp* version using polynomial shape function cannot be effective. Although, the lower bound is available only in the one-dimensional case, it is obvious that we can expect in two- (and three-) dimensions that when γ is very small, *any* finite element method based on polynomial shape functions will converge very slowly.

2.2 The 2-dimensional model problem

Consider the elasticity problem (isotropic material, $E = 1000$, $\nu = 0.3$) without body forces on the domain

$$\Omega = \{x_1, x_2 \mid |x_1| < 2, |x_2| < 2\} \setminus \{x_1, x_2 \mid 0 < x_1 < 2, x_2 = 0\}$$

shown in Figure 2.1 and with the following boundary conditions

$$u_n = u_t = 0 \ \text{(fixed) on } \Gamma_1 \cup \Gamma_2 \tag{2.24a}$$

$$T_n = 10, \ T_t = 2 \ \text{(prescribed tractions) on } \Gamma_5 \tag{2.24b}$$

$$T_n = T_t = 0 \ \text{(free) on } \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_5) \tag{2.24c}$$

The solution has major singularities in the neighborhood of the origin $P_1 = (0, 0)$, where $\gamma \approx 0.3$ and at the point $P_2 = (2, 2)$, where $\gamma \approx 0.7$. Hence, we have the roughly analogous case as discussed in § 2.1. In Figure 2.2(a) and Figure 2.2(b), we show the mesh I with 22 elements and mesh II with 48 elements. The geometric factor is 0.15 (figure is not in scale), so that the ratio between the size of the smallest and largest element in the mesh II is 0.843×10^{-3}.

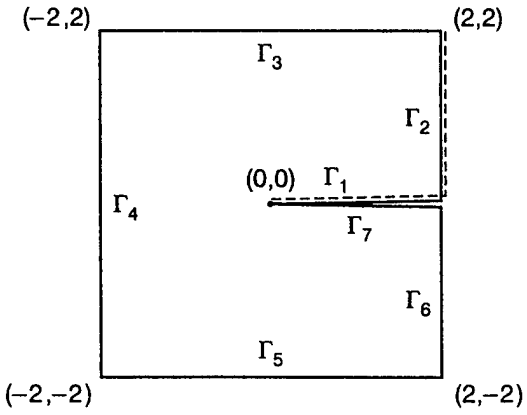

Figure 2.1: The scheme of the model problem.

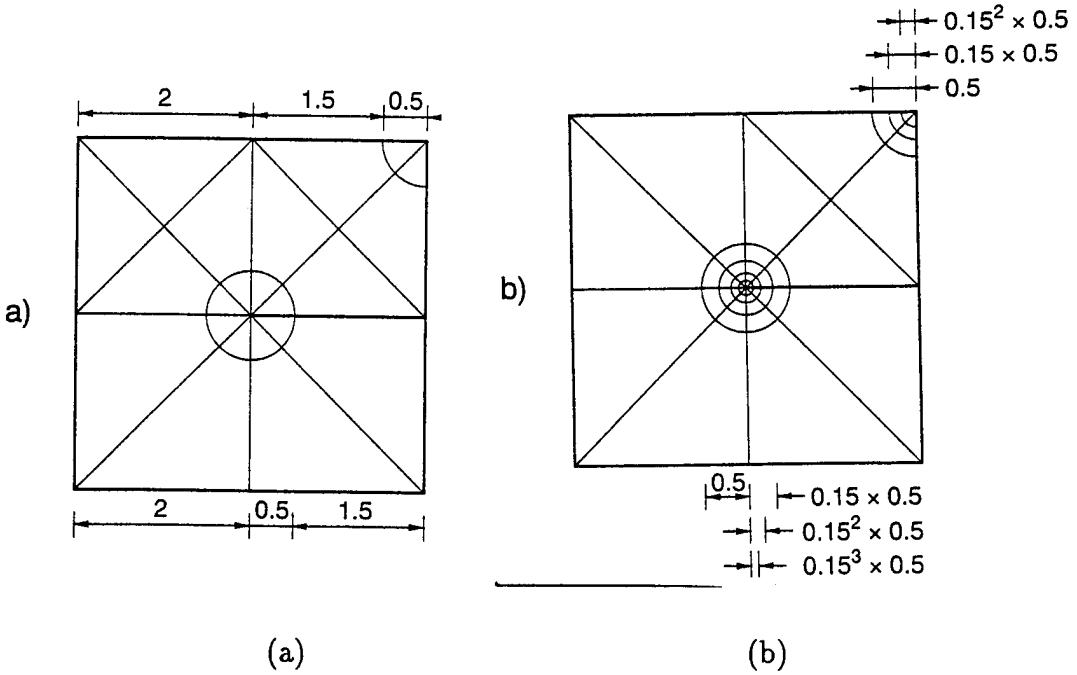

Figure 2.2: The mesh I (22 elements) and mesh II (48 elements).

In Table 2.2, we give relative error in the energy norm as a function of the degree p (uniform) of the elements. The error was computed from the strain energy of the finite element solution and the exact solution computed by extrapolation.

	Mesh I		Mesh II	
k	DOF	$\varepsilon\%$	DOF	$\varepsilon\%$
1	38	64.60	92	46.52
2	120	49.04	280	21.21
3	226	42.26	488	15.46
4	376	37.59	792	12.12
5	570	34.33	1192	10.70
6	808	31.87	1688	9.82
7	1090	29.88	2280	9.14
8	1416	28.23	2960	8.61

Table 2.2. The relative error ε in the energy norm in %.

From Table 2.2, we see an algebraic rate and that for $p \geq 5$, the mesh II is under-refined.

2.3 The method of auxiliary mapping

In the neighborhood of the singular point, for example, the origin, the elements have two straight lines and one circular arc. The shape functions are mapped polynomials when the blending mapping is used so that the mapping of the side (AB) of the standard element on the circular arc is linear in the arc length (for more, see [10]). The scheme is shown in Figure 2.2

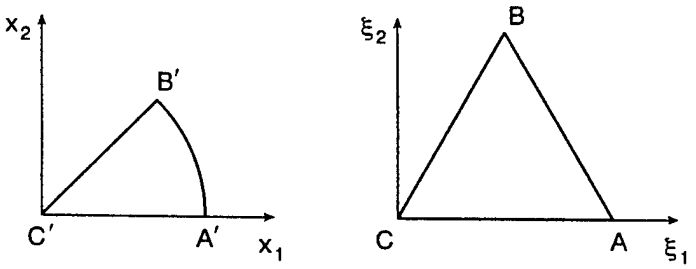

Figure 2.3: The elements.

Let $\mathcal{T}$ be the element with one circular side as shown in Figure 2.3.

Consider the conformal map $z = \zeta^\gamma$ which mapps $\tilde{\mathcal{T}}$ onto $\mathcal{T}$. Then we have the following lemma:

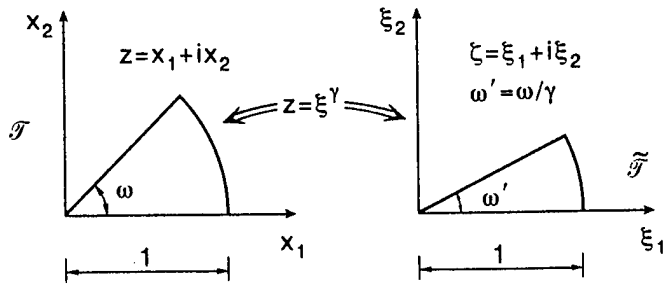

Figure 2.4: Mapped element.

Lemma 2.4 *Let u be defined on $\mathcal{T}$ and U on $\tilde{\mathcal{T}}$ and let $u(z) = U(\zeta)$. Then for $i = 1, 2$*

$$\int_{\mathcal{T}} \left(\frac{\partial u}{\partial x_i}\right)^2 dx \leq C_1 \int_{\tilde{\mathcal{T}}} \left(\left(\frac{\partial U}{\partial \zeta_1}\right)^2 + \left(\frac{\partial U}{\partial \zeta_2}\right)^2\right) d\zeta$$

$$\int_{\tilde{\mathcal{T}}} \left(\frac{\partial U}{\partial \zeta_i}\right)^2 dx \leq C_2 \int_{\mathcal{T}} \left(\left(\frac{\partial U}{\partial x_1}\right)^2 + \left(\frac{\partial U}{\partial x_2}\right)^2\right) dx$$

with C_1 and C_2 independent of u. □

Let (r, θ) be the polar coordinates in the plane x_1, x_2 and (ρ, ψ) in the plane ζ_1, ζ_2. Then, in the special case $u = r^\alpha \phi(\theta)$, we get $U = \rho^{\gamma\alpha} \phi(\gamma\psi)$ with for $\gamma > 0$. Hence U has smaller singularity than u. Using the pull-back polynomials from the standard element, we see that U on $\mathcal{T}$ can be approximated in $H^1(\tilde{\mathcal{T}})$-seminorm much better than u on $\mathcal{T}$. In general, U is smoother than u when u is singular due to the corner of the boundary. Using Lemma 2.4, we can approximate u well on $\mathcal{T}$ by pull-back polynomials if, in addition, the conformal map is used. To obtain conforming elements with common circular side, the blending mapping preserving the length as explained above, is essential.

Implementation of this method, which we call Method of Auxiliary Mapping (MAM), is very easy. This method is very effective when the solution is very singular due to corner or interface. In Table 2.3, we show the error in the energy norm for the MAM on the mesh I using $\gamma = 6$ for the elements in the origin $P_1 = (0, 0)$ and $\gamma = 2$ in for the elements in the point $P_2 = (2, 2)$.

p	DOF	$\varepsilon\%$
1	38	44.13
2	120	15.81
3	226	9.69
4	376	4.22
5	570	1.80
6	808	1.10
7	1090	0.65
8	1416	0.37

Table 2.3. The accuracy of MAM.

The MAM method on mesh I has obviously the same DOF as the standard p-method and the cost is also exactly the same. The effectiveness of the MAM method is obvious by comparing Table 2.2 and Table 2.3. It is necessary to underline that although the method is based on conformal mapping, it has nothing to do with solving Laplace equation. The mapping is used for a smoothening and it is essential that the H^1-seminorm is preserved when going from the mapped element to the original one.

The MAM was discussed and analyzed in [11], [12], [13] in the two-dimensions and in [14] in three-dimensions. It was shown that the method is not too sensitive to the selection of the smoothening parameter. The MAM is very easy to implement as p version or hp version in the frame of standard codes. It avoids the problem of slow convergence of the classical hp version when the solution is very singular due to corners of the domain or interfaces.

3 The PUFEM—The Partition of Unity Finite Element Method

3.1 Introduction

Let us first address the main idea of the proof of the convergence of the classical p version of FEM. We construct a function in the finite element space which approximates well the exact solution in the energy norm. Then the finite element solution has the error which is majorized by the approximations error of the constructed function. The construction proceeds as follows (see [7], [8], [15]):

(a) Given the partition of the domain Ω into elements $\mathcal{T}$, an approximation by polynomials of degree p on every *single* element is constructed. Because of the completeness of the polynomials, the error of approximation can be made arbitrarily small by selecting sufficiently high degree p.

(b) Using the fact that the exact solution belongs to $H^1(\Omega)$ (the energy space), the discontinuity of the approximation along the element boundary can be estimated (in $H^{1/2}(\partial\mathcal{T})$) and a correction is used so that the constructed function

is continuous. A theorem ([15], [16]) on the polynomial extension from $\partial\mathcal{T}$ to $\mathcal{T}$ is utilized.

Let us assume now that we solve, for example, the Laplace problem, i.e., we know *a-priori* that the solution is harmonic. Then realizing that the harmonic polynomials are complete in the space of harmonic functions, we can approximate well the exact solution on every element by *harmonic polynomials only*. Hence, there is no difference when compared with the step (a) above.

Nevertheless, the part (b) now creates essential difficulty so that the idea of approximation by harmonic polynomials cannot be used. This is especially important when a still more general complete set of approximation functions is used. For example, Bergmann [17] an Vekua [18] construct an analog of harmonic polynomials satisfying homogeneous equations of second order with analytic coefficients. This can be used for example, when equation (1.3) with $f = 0$ is considered.

The major step is to construct a continuous function from the piecewise continuous (on patches) functions. This will be made by a partition of unity approach. The idea was used and theoretically analyzed in [19], [20], [21], and [22]. This is the PUFEM method presented here. For detailed description of the PUFEM method, see [20], [21], and [22].

3.2 The PUFEM method

Let us describe the major ingredients of the PUFEM method. The critical notion is here the notion of (M, C_∞, C_G) partitions of unity.

Definition 3.1 *Let $\Omega \subset \mathbf{R}^n$ be an open set, $\{\Omega_i\}$ be an open cover of Ω satisfying a pointwise overlap condition:*

$$\exists M \in \mathbf{N}, \forall x \in \Omega, \operatorname{card}\{i \mid x \in \Omega_i\} \leq M.$$

Let $\{\phi_i\}$ be a Lipschitz partition of unity subordinate to the cover $\{\Omega_i\}$ satisfying

$$\begin{aligned} \operatorname{supp}\phi_i &\subset \overline{\Omega}_i, \quad \forall i, \\ \sum_i \phi_i &\equiv 1 \qquad \text{on } \Omega, \\ \|\phi_i\|_{L^\infty(\mathbf{R}^n)} &\leq C_\infty, \\ \|\nabla\phi_i\|_{L^\infty(\mathbf{R}^n)} &\leq \frac{C_G}{\operatorname{diam}\Omega_i}, \end{aligned}$$

where C_∞ and C_G are two constants. Then $\{\phi_i\}$ is called a (M, C_∞, C_G) partition of unity subordinate to cover $\{\Omega_i\}$. The partition of unity $\{\phi_i\}$ is said to be of degree $m \in \mathbf{N}_0$ if $\{\phi_i\} \subset C^m(\mathbf{R}^m)$. The covering sets $\{\Omega_i\}$ are called patches. □

Definition 3.2 *Let $\{\Omega_i\}$ be an open cover of $\Omega \subset \mathbf{R}^n$ and let $\{\phi_i\}$ be a (M, C_∞, C_G) partition of unity subordinate to the cover $\{\Omega_i\}$. Let $V_i \subset H^1(\Omega_i \cap \Omega)$ be given. Then space*

$$V := \sum_i \phi_i V_i = \left\{ \sum_i \phi_i V_i \mid v_i \in V_i \right\} \subset H^1(\Omega) \tag{3.1}$$

is called the PUFEM space. The PUFEM space V is said to be of degree $m \in \mathbf{N}_0$ if $V \subset C^m(\Omega)$. The spaces V_i are referred to as the local approximation spaces. □

Let us now mention the basic theorem [20], [21], [22]:

Theorem 3.1 *Let $\Omega \subset \mathbf{R}^n$ be given. Let $\{\Omega_i\}$, $\{\phi_i\}$ and $\{V_i\}$ be as in Definition 3.1 and Definition 3.2. Let $u \in H^1(\Omega)$ be the function to be approximated. Assume that the local approximation spaces V_i have the following approximation properties: On each patch $\Omega_i \cap \Omega$, u can be approximated by a function $v_i \in V_i$ such that*

$$\|u - v_i\|_{L^2(\Omega_i \cap \Omega)} \leq \varepsilon_1(i), \tag{3.2}$$

$$\|\nabla(u - v_i)\|_{L^2(\Omega_i \cap \Omega)} \leq \varepsilon_2(i). \tag{3.3}$$

Then the function

$$u_{ap} = \sum_i \phi_i v_i \in V \subset H^1(\Omega) \tag{3.4}$$

satisfies

$$\|u - u_{ap}\|_{L^2(\Omega)} \leq \sqrt{M} C_\infty \left(\sum_i \varepsilon_1^2(i) \right)^{1/2} \tag{3.5}$$

$$\|\nabla(u - u_{ap})\|_{L^2(\Omega)} \leq \sqrt{2M} \left(\sum_i \left(\frac{C_G}{\operatorname{diam} \Omega_i} \right)^2 \varepsilon_1^2(i) + \varepsilon_2^2(i) \right)^{1/2}. \tag{3.6}$$

□

The PUFEM can be understood as the h, p, or hp version. Consider for example the p version. Let $\{\Omega_i\}$ be the patches covering Ω and assume that the exact solution u is harmonic. Assume that the spaces V_i are the spaces of harmonic polynomials of degree p. Then

$$\varepsilon_1(i) \leq C \operatorname{diam}(\Omega_i) p^{-(k-1)} \|u\|_{H^k(\Omega_i \cap \Omega)} \tag{3.7}$$

$$\varepsilon_2(i) \leq C p^{-(k-1)} \|u\|_{H^k(\Omega_i \cap \Omega)} \tag{3.8}$$

and the error estimate from Theorem 3.1 takes the form

$$\|\nabla(u - u_{ap})\|_{L_2(\Omega)} \leq M\sqrt{2} C (C_G + C_\infty) p^{-(k-1)} \|u\|_{H^k(\Omega)}. \tag{3.9}$$

We see that the PUFEM method allows to do what was mentioned in § 3.1.

In [22], more details of PUFEM method including the *a-posterion* error estimation are given.

3.3 A numerical example

Let us consider the Helmholz problem on unit square

$$-\Delta u - k^2 u = 0 \quad \text{on } \Omega = (0,1)\times(0,1), \tag{3.10}$$

$$\frac{\partial u}{\partial n} + iku = g \quad \text{on } \partial\Omega, \tag{3.11}$$

where g is chosen such that the exact solution u is a plane wave of the form

$$u(x) = e^{ik(x_1\cos\theta + x_2\sin\theta)}, \theta = \frac{\pi}{16}. \tag{3.12}$$

The following types of local approximations spaces were analyzed in [21]. The first type are "generalized harmonic polynomials" of Bergmann-Vekua type. Then

$$V^v(p) = \text{span}\,\{e^{\pm in\theta} J_n(k,r) \mid n = 0,\ldots,p\} \tag{3.13}$$

where the functions J_n are Bessel functions of the first kind. The second type are systems of plane waves given by

$$W(p) = \text{span}\,\{e^{ik(x_1\cos\theta_j + x_2\sin\theta_j)} \mid \theta_j = \frac{2\pi}{p}j, j = 0,\ldots,p-1\} \tag{3.14}$$

with $p = 4n+2$, n an integer, so that $u(x)$ given by (3.12) does not belong to $W(p)$ for any p. It can be shown that functions from $V^v(k)$ and $W(p)$ can approximate well any function satisfying (3.10). For more, see [21]. If u is solution of (3.10), and $\overline{\Omega} \subset \Omega$, then

$$\inf_{u_p \in V^v(p)} \|u - u_p\|_{H^1(\overline{\Omega})} \leq C(\gamma,\overline{\Omega})e^{-\gamma p}$$

$$\inf_{w_p \in W(p)} \|u - w_p\|_{H^1(\overline{\Omega})} \leq C(\gamma,\overline{\Omega})e^{-\gamma p}$$

holds for any $\gamma > 0$.

Assume now that on Ω, we have a square mesh with size $h = 1/n$ and use the partition of unity created by the standard bilinear pyramid shape functions.

We now compare the effectiveness of the PUFEM method with some other ones. Let us discuss, as example, the case $k = 100$ and be interested in the error measured in the L_2 norm. We will compare the PUFEM based on the space $W(p)$ with the usual Gaberkin method (FEM), the generalized least squares finite element method (GLSFEM [23]) and the quasi-stabilized finite element method (QSFEM [24]). The FEM, GLSFEM and QSFEM are based on piecewise linear functions on uniform mesh and they differ in their choice of bilinear form. In particular, the bilinear form of QSFEM is constructed such that the "pollution" (see [24]) is minimized and it is virtually the best method available which is based on piecewise linear functions.

In Table 3.1, we show the necessary DOF to obtain relative error $\varepsilon\%$ in L_2 norm. We report in Table 3.1 also the DOF of the best approximant by bilinear functions.

$\varepsilon\%$	Best approx. by bilin. funct.	QSFEM	GLSFEM	FEM
30	2.045 (+3)	3.969 (+3)	2.016 (+4)	7.734 (+4)
10	5.041 (+3)	1.000 (+4)	6.150 (+4)	2.352 (+5)
5	3.464 (+3)	1.960 (+4)	1.274 (+5)	4.692 (+5)

Table 3.1. *DOF* necesary to obtain the accuracy $\varepsilon\%$ in L_2 norm; $k = 100$.

In Table 3.2, we show the *DOF* for PUFEM, for $n = 4$ and compare it with the other methods

p	$\varepsilon\%$	PUFEM	Best approx. by bilin. funct.	QSFEM	FEM
26	10.8	6.50 (+2)	3.80 (+3)	7.95 (+3)	2.08 (+5)
30	0.69	7.50 (+2)	5.89 (+4)	1.23 (+5)	3.23 (+6)
34	0.11	8.50 (+2)	3.45 (+5)	7.23 (+5)	1.90 (+7)

Table 3.2. *DOF* necesary to achieve various accuracies in L_2 for PUFEM with $n = 4$ and various other methods, $k = 100$.

The excellent performance of the PUFEM method shown in Table 3.2 is due to the fact that PUFEM employs the general character of the solution; while the other methods utilize only piecewise linear functions. In Table 3.3, we report the number of operations using band elimination, for PUFEM, $n = 4$.

p	$\varepsilon\%$	PUFEM	QSFEM	FEM
26	10.8	1.76 (+7)	6.3 (+7)	4.3 (+11)
30	0.69	2.71 (+7)	1.5 (+10)	1.01 (+13)
34	0.11	3.94 (+7)	5.2 (+11)	3.6 (+14)

Table 3.2. The number of operations using band elimination, $k = 100$, error in L_2, $k = 100$.

Let us underline that the construction of the stiffness matrix for the PUFEM based on $W(p)$ is relatively cheap and the cost is negligible when compared with the cost of the solver. The stiffness matrix for the PUFEM based on the space $V^v(p)$ is more expensive than for $W(p)$. Nevertheless, the space $V^v(p)$ is in some sense optimal (see [22]).

In Table 3.4 and Table 3.5, we show the results for $k = 32$ and the error measured in the H^1-seminorm. Table 3.4 shows the accuracies and number of iterations and operations for the iterative method proposed in [25]. Table 3.5 shows the operation count for PUFEM method with the band elimination, $k = 32$ and $n = 1$.

$\sqrt{DOF}$	FEM			SFEM		
	H^1 error $\varepsilon\%$	# of iterat.	NOP	H^1 error $\varepsilon\%$	# of iterat.	NOP
33	65.0	232	4.51 (+6)	30.5	272	5.29 (+6)
65	21.7	434	3.37 (+7)	14.3	492	3.82 (+7)
129	8.16	831	2.68 (+8)	7.02	953	2.96 (+8)
257	3.64	1665	2.07 (+9)	3.48	1863	2.31 (+9)
513	1.72	3263	1.62 (+10)	1.69	3762	1.86 (+10)

Table 3.4. Operation count for solving the linear system, error in H^1-seminorm, $k = 32$.

p	H^1 error $\varepsilon\%$	NOP
18	46.	1.3 (+5)
22	6.7	2.3 (+5)
26	0.38	3.8 (+5)
30	0.00025	5.9 (+5)

Table 3.5. Operation count for band elimination for PUFEM, $k = 32$, $n = 1$, error in H^1-seminorm

Let us underline once more that the cost of the stiffness matrix is negligible in comparison with the cost of the solver. The PUFEM method belongs to the family of mesh free methods, see [26], [27], nevertheless, we underline in this paper the flexibility of PUFEM which allows us to employ the properties of solved differential equations.

4 The *hp* version of FEM in three-dimensions

4.1 The *hp* version

In contrast to the two-dimensional case, the character of the singularities in the neighborhood of the boundary is much more complex. In three-dimensions, we have to distinguish between the behavior in the neighborhood of the edges far from the vertices, close to the vertices, and in the neighborhood of the vertices, and in the neighborhood of the vertices which is (conically) far from the edges. In the two-dimensional case, only one type exist in the neighborhood of the corner.

In [28], [29], [30], and [31], the regularity of the solutions in terms of countable spaces is analyzed. Based on these results, the *hp* versions converge exponentially

$$\|e\| \leq Ce^{-\gamma N^{1/5}}, \gamma > 0. \tag{4.1}$$

In contrast to the one-dimensional case, there is no proof that the exponent $1/5$ is the optimal one, but we conjecture that it is. The meshes leading to this rate have elements with large aspect ratios, which increases as $N \to \infty$. These "needle" elements are in the neighborhoods of the edges and reflect the fact that the solution

is smooth along the edges and unsmooth in the direction perpendicular to the edge. Also, the optimal upper-bound (and the lower-bound) depending on the strength of the singularities, is not known, but analogously as in the one-dimensional case, it is possible to expect that γ in (4.1) could be very small in the case when the singularity is strong. So far, we discussed the *hp* version. The *p* version was analyzed in [32] and [33]. See also, [34] for some computational experience.

4.2 Computational Examples

In this section, we show briefly an example analyzed by the code STRIPE developed at Aeronautical Research Institute of Sweden. Consider the problem on the domain shown in Figure 4.1 where the boundary conditions are

- on the faces *A-C-E-N*, *A-B-E-F*, *A-B-C-D*, and *I-J-L-M*, we have $(T_x, T_y, T_z) = 0$;
- on the face *G-H-I-J*, we have $(T_x, T_y, T_z) = (-10, 0, 0)$;
- on the face *D-K-L-M-N-C*, we have $u = 0$, $T_y = T_z = 0$;
- on the face *N-M-J-H-F-E*, we ahve $v = 1$, $T_x = T_z = 0$;
- on the face *K-G-I-L*, we ahve $v = 0$, $T_x = T_z = 0$; and
- on the face *K-D-B-F-H-G*, we ahve $w = 0$, $T_x = T_y = 0$.

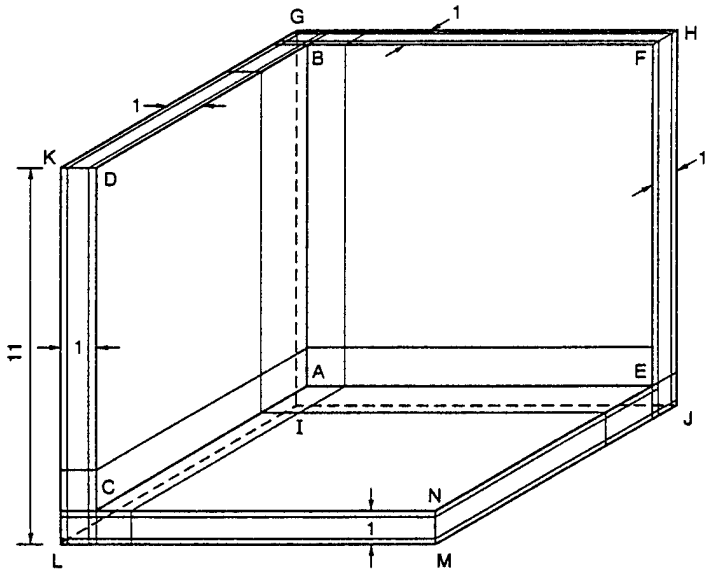

Figure 4.1: The domain

The singularity of the solution occurs in the vertex *A*, along the edges *A-B*, *A-C*, *A-E*, and the edge *I-J*. The singularity along the edge *I-J* is weaker than along

A-B, A-C, and A-E. The basic mesh is shown also in Figure 4.1. Around the edges, the mesh is refined in the analogous way as in two-dimensions, with M_1, M_2 layers in the neighborhood of the edges A-B, A-C, A-E, and I-J. Figure 4.2 shows the case with $M_1 = 1$ and $M_2 = 3$. In the neighborhood of the vertex A, the mesh is a

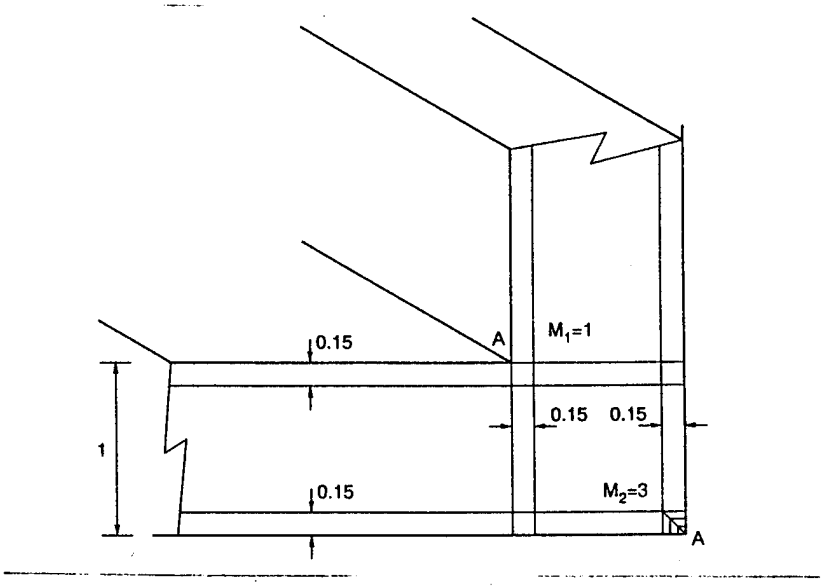

Figure 4.2: Detail of the Mesh

complex one accommodating the meshes along the edges A-B, A-C, and A-E. The hp version together with increasing N does increase the number of layers around the edges where the singularity is located. The p version fixes the mesh and increases the degree uniformly or selectively. In the Figure 4.3, we show in the scale $\log \|e\| \times N^{1/5}$, the error for various p (uniform) and M_1 with $M_2 = 1$. As predicted, we see the rate $e^{-\gamma N^{1/5}}$ if we combine properly M_1 and p. It will appear in the graph as the straight line. As seen from Figure 4.3, we have to combine in a proper way the degrees of elements and the number of layers. In the range of accuracy under consideration, $M_2 = 1$ is sufficient. In Table 4.1, we show the proper combination which leads to the exponential convergence with respect to $N = DOF$

M_1	p
0	6
2	5,6,7
4	6,7,8

Table 4.1. The combination of $M_1 \times p$ for the hp version.

Remark 4.1 The relative error in the energy norm shown in Figure 4.3 was computed from the strain energies

$$\frac{\|e\|_E}{\|u\|_E} = \left(\frac{\varepsilon_{\text{exact}} - \varepsilon_{\text{FE}}}{\varepsilon_{\text{EX}}}\right)^{1/2}$$

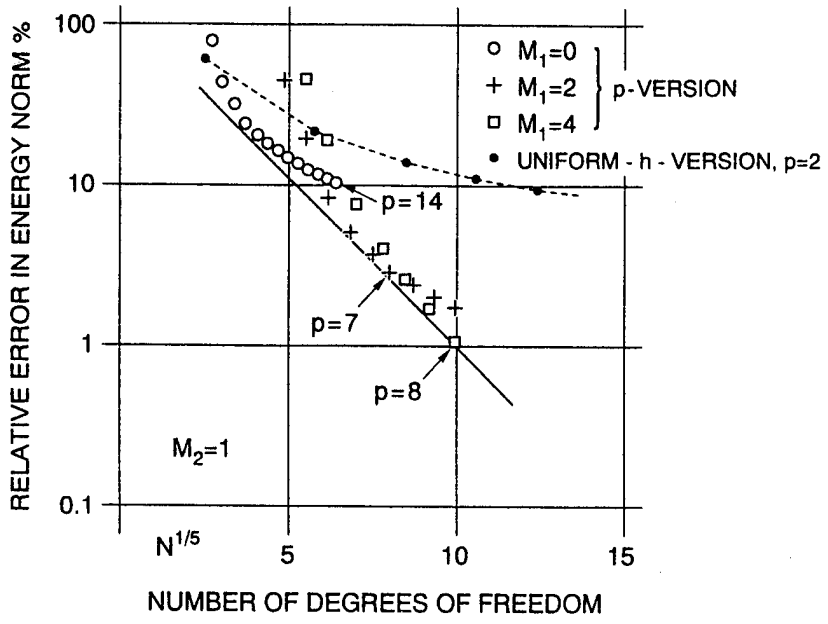

Figure 4.3: The error in the energy norm

where the energy $\varepsilon_{\text{exact}}$ was computed from $M_1 = 4$, $M_2 = 2$ and $p = 10$ and extrapolation so that the error in the given range shown in Figure 4.3 is guaranteed. □

Figure 4.3 shows also the error for the h version with uniform mesh and elements of degree 2. In Figure 4.4, we show the CPU time on one processor computation of a Silicon Graphics Challenger for the hp version and the h version. In the CPU time, the total time is included, i.e., the construction of the stiffness matrix, direct solver and the postprocessing. In Figure 4.4, we show also CPU time for the iterative solver written by J. Mandel (Solver International, Inc.) which is a special solver based on PCG method.

So far, we have addressed the case with uniform degrees. We can also use an adaptive procedure which is more effective. In the adaptive procedure, different degrees of the shape functions (but not the full space of shape functions of degree p) are used. In Figure 4.5, we show the relation between the CPU time and achieved accuracy measured in the energy norm for the adaptive approach and uniform p approach for $M_1 = 4$ and $M_2 = 1$.

We see clearly the effectiveness of the hp version in the three-dimensional examples for solutions with singularities.

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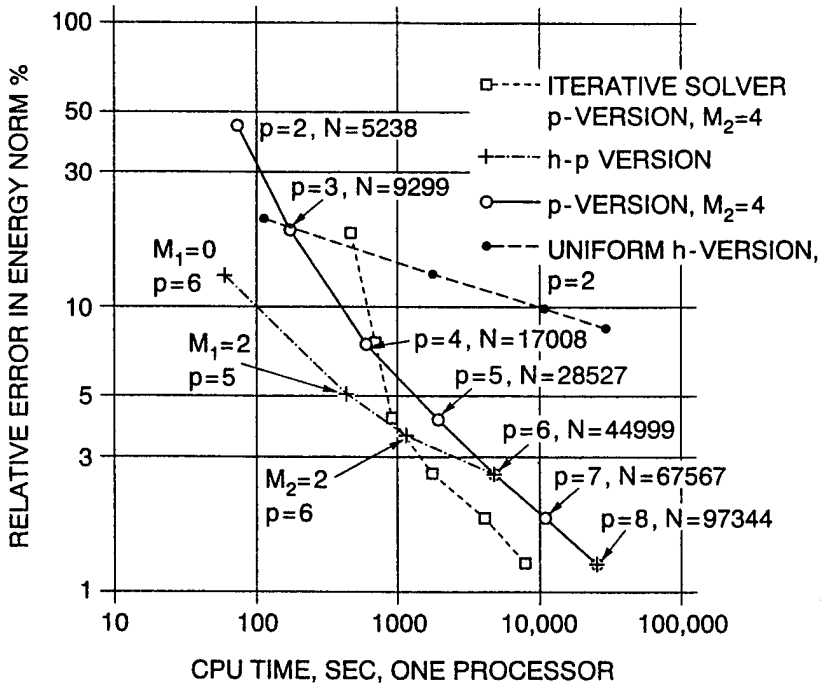

Figure 4.4: The CPU time for different versions

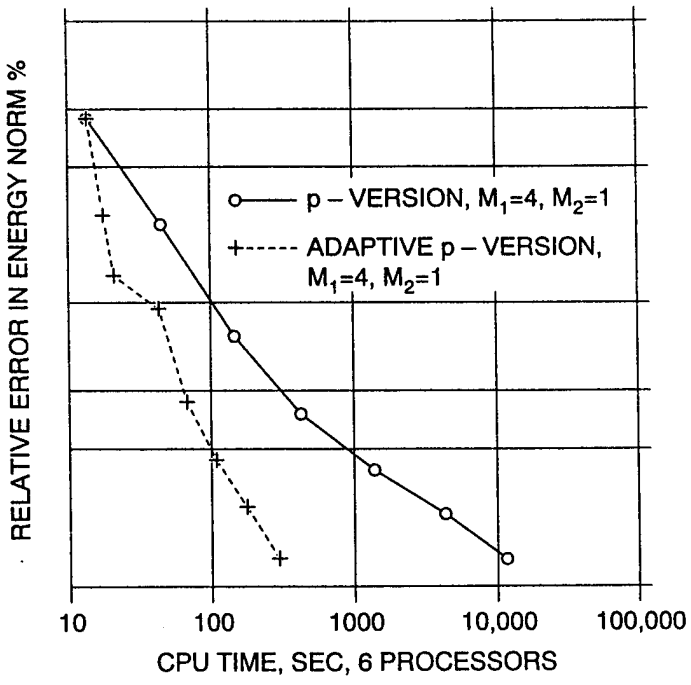

Figure 4.5: The CPU time for the adaptive p version and uniform p version.

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