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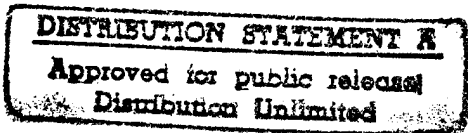
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**ANALYSIS AND FINITE ELEMENT METHOD FOR A
FLUID-SOLID INTERACTION PROBLEM IN ONE DIMENSION**



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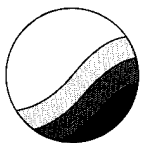
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ANALYSIS AND FINITE ELEMENT METHODS FOR A FLUID-SOLID INTERACTION PROBLEM IN ONE DIMENSION

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ABSTRACT. In this paper we study a time-harmonic fluid-solid interaction model problem in one dimension. This is a Helmholtz-type system equipped with boundary and transmission conditions. We show the existence of a unique solution to this problem and study its stability and regularity properties. We analyze the convergence of finite element methods with respect to appropriate energy norms. Computational results are also presented.

1. INTRODUCTION

We shall analyze a model problem for a fluid solid interaction process in one dimension. Here, we assume that we are given a layered fluid-solid-fluid medium with configuration Ω , $\Omega = [0, L]$, $L > 0$. Ω is divided in three subintervals $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ where $\Omega_1 = [0, x_1]$ and $\Omega_3 = [x_2, L]$ will be the "fluid" parts and $\Omega_2 = [x_1, x_2]$ will be the "solid" part. We assume that the time-harmonic acoustic wave equations are satisfied in the fluid and the time-harmonic elastic wave equations in the solid:

$$(1.1) \quad \begin{aligned} p_{xx} + k^2 p &= -g_1, & \text{in } \Omega_1 \\ (au_x)_x + k^2 gu &= -f, & \text{in } \Omega_2 \\ \tilde{p}_{xx} + k^2 \tilde{p} &= -g_2, & \text{in } \Omega_3. \end{aligned}$$

Here k , $k \in \mathbb{R}$, $k \geq k_0 > 0$ denotes the wave number and $p = p(x)$, $\tilde{p} = \tilde{p}(x)$ denote the pressure in Ω_1 and Ω_3 respectively and $u = u(x)$ the displacement in Ω_2 . We assume that a and g are positive constants that characterize the materials, and g_1, g_2, f are given functions. We assume radiation conditions at the boundary of Ω

$$(1.2a) \quad \begin{aligned} p_x(0) + ikp(0) &= 0, \\ \tilde{p}_x(L) - ik\tilde{p}(L) &= 0. \end{aligned}$$

Also the continuity requirement for the pressure and for the displacement yields the transmission conditions at x_1 :

$$(1.2b) \quad \begin{aligned} p_x(x_1) - k^2 u(x_1) &= 0, \\ p(x_1) + au_x(x_1) &= 0, \end{aligned}$$

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and at x_2 :

$$(1.2c) \quad \begin{aligned} \tilde{p}_x(x_2) - k^2 u(x_2) &= 0, \\ \tilde{p}(x_2) + au_x(x_2) &= 0. \end{aligned}$$

(1.1-2) is a system of three equations in one dimension equipped with boundary and transmission conditions. Each one of these is a Helmholtz-type equation where the wave number k appears as a common parameter of the system. We shall propose a weak formulation, show the existence of a unique solution for this problem and study its stability and regularity properties. Further we consider finite element methods for the approximation of the solution and analyze the convergence behavior of these methods with respect to appropriate energy norms. Throughout this work a particular emphasis is given in the dependence of the constants on the frequency parameter k .

We shall state a weak formulation of this problem below. First, we introduce some notation: Let $H^m(S)$, m positive integer, be the usual Sobolev space of complex valued functions which have derivatives in the sense of distributions up to the order m . Let $\|\cdot\|$ and (\cdot, \cdot) be the $L^2(S)$ -norm and inner product, respectively. We shall denote by \mathcal{H} the space $H^1(\Omega_1) \times H^1(\Omega_2) \times H^1(\Omega_3)$. If q, v and \tilde{q} are three smooth test functions defined on Ω_1, Ω_2 and Ω_3 , we see that (1.1) and (1.2) give

$$(1.3) \quad \int_{\Omega_1} p_x \bar{q}_x dx - k^2 \int_{\Omega_1} p \bar{q} dx - k^2 u(x_1) \bar{q}(x_1) - ikp(0) \bar{q}(0) = \int_{\Omega_1} g_1 \bar{q} dx,$$

$$(1.4) \quad \int_{\Omega_2} au_x \bar{v}_x dx - k^2 \int_{\Omega_2} gu \bar{v} dx - p(x_1) \bar{v}(x_1) + \tilde{p}(x_2) \bar{v}(x_2) = \int_{\Omega_2} f \bar{v} dx,$$

and

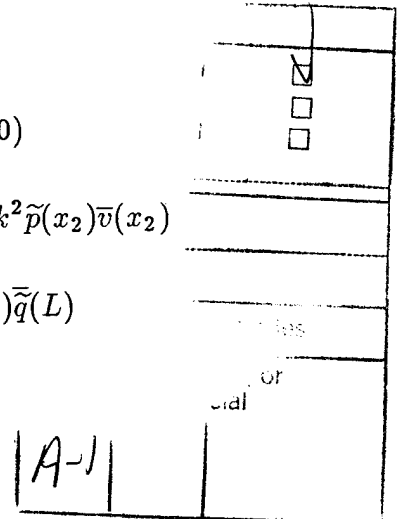
$$(1.5) \quad \int_{\Omega_3} \tilde{p}_x \bar{\tilde{q}}_x dx - k^2 \int_{\Omega_3} \tilde{p} \bar{\tilde{q}} dx + k^2 u(x_2) \bar{\tilde{q}}(x_2) - ik\tilde{p}(L) \bar{\tilde{q}}(L) = \int_{\Omega_3} g_2 \bar{\tilde{q}} dx.$$

Multiplying (1.4) by k^2 and summing (1.3), (1.4) and (1.5) we arrive to the following weak formulation of problem (1.1-2): Find $\mathcal{U} = (p, u, \tilde{p}) \in \mathcal{H}$ such that

$$(1.6) \quad B(\mathcal{U}, \mathcal{V}) = (\mathcal{F}, \mathcal{V})_{0, \mathcal{H}}, \quad \forall \mathcal{V} \in \mathcal{H}$$

where for $\mathcal{V} = (q, v, \tilde{q}) \in \mathcal{H}$,

$$(1.6a) \quad \begin{aligned} B(\mathcal{U}, \mathcal{V}) &= \int_{\Omega_1} p_x \bar{q}_x dx - k^2 \int_{\Omega_1} p \bar{q} dx - k^2 u(x_1) \bar{q}(x_1) - ikp(0) \bar{q}(0) \\ &+ k^2 \int_{\Omega_2} au_x \bar{v}_x dx - k^4 \int_{\Omega_2} gu \bar{v} dx - k^2 p(x_1) \bar{v}(x_1) + k^2 \tilde{p}(x_2) \bar{v}(x_2) \\ &+ \int_{\Omega_3} \tilde{p}_x \bar{\tilde{q}}_x dx - k^2 \int_{\Omega_3} \tilde{p} \bar{\tilde{q}} dx + k^2 u(x_2) \bar{\tilde{q}}(x_2) - ik\tilde{p}(L) \bar{\tilde{q}}(L) \end{aligned}$$



and

$$(1.6b) \quad (\mathcal{F}, \mathcal{V})_{0, \mathcal{H}} = (g_1, q) + k^2(f, v) + (g_2, \tilde{q}).$$

In the sequel we shall see that (1.6) is indeed a suitable variational formulation for the problem at hand. We show that problem (1.6) admits a unique solution, we study its stability properties and we analyze the convergence of numerical approximations to it. In order to describe the results of this work we shall need some more notation. For $k, k > 0$, fixed and $\mathcal{U} = (p, u, \tilde{p}), \mathcal{V} = (q, v, \tilde{q}) \in \mathcal{H}$, let

$$(1.7) \quad (\mathcal{U}, \mathcal{V})_{0, \mathcal{H}} = (p, q) + k^2(u, v) + (\tilde{p}, \tilde{q}).$$

Of course $(\cdot, \cdot)_{0, \mathcal{H}}$ defines an inner product in $\mathbf{L}^2 := L^2(\Omega_1) \times L^2(\Omega_2) \times L^2(\Omega_3)$ equivalent to the standard inner product of this space. If \mathcal{V} is an element of \mathcal{H} , by \mathcal{V}_x we denote (q_x, v_x, \tilde{q}_x) where q, v and \tilde{q} are the components of \mathcal{V} . Further by $(\cdot, \cdot)_{1, \mathcal{H}}$ we shall denote the inner product in \mathcal{H} defined by

$$(1.8) \quad (\mathcal{U}, \mathcal{V})_{1, \mathcal{H}} = (\mathcal{U}_x, \mathcal{V}_x)_{0, \mathcal{H}} + k^2(\mathcal{U}, \mathcal{V})_{0, \mathcal{H}}.$$

Finally $\|\cdot\|_{0, \mathcal{H}}$ and $\|\cdot\|_{1, \mathcal{H}}$ shall be the norms induced by the inner products $(\cdot, \cdot)_{0, \mathcal{H}}$ and $(\cdot, \cdot)_{1, \mathcal{H}}$. These norms are better suited to the nature of our problem and furthermore help us to reduce the dependence on k of the constants in the stability estimates.

In Section 2 we study the continuous problem (1.6). Using variational techniques, we show that if $\mathcal{F} \in \mathbf{L}^2$, and \mathcal{U} is a solution of (1.6), then

$$(1.9) \quad \|\mathcal{U}\|_{1, \mathcal{H}} \leq C_1 \|\mathcal{F}\|_{0, \mathcal{H}},$$

where C_1 is a constant *independent of* k . The estimate (1.9) establishes the uniqueness of the solution of (1.6) and, combined with the fact that $B(\cdot, \cdot)$ satisfies a Gårding-type inequality, yields the existence of \mathcal{U} as well, by a standard alternative argument of the theory of differential equations. Using an estimate of the type (1.9) for a properly chosen auxiliary problem we prove the Babuška - Brezzi (BB) condition for the bilinear form $B(\cdot, \cdot)$:

$$(1.10) \quad \sup_{\mathcal{V} \in \mathcal{H}} \frac{\operatorname{Re} B(\mathcal{U}, \mathcal{V})}{\|\mathcal{V}\|_{1, \mathcal{H}}} \geq \gamma \frac{1}{k} \|\mathcal{U}\|_{1, \mathcal{H}}, \quad \forall \mathcal{U} \in \mathcal{H},$$

where γ is a positive constant independent of k . Furthermore, using (1.9) and (1.10) we derive other similar "regularity" estimates that are very useful in the convergence analysis of the numerical methods for our problem.

In Section 3 we consider Galerkin-finite element methods for approximating the solution of (1.1-2). If \mathcal{S}_h is a suitable finite element subspace of \mathcal{H} consisting of piecewise polynomial functions, we define the Galerkin approximation $\mathcal{U}_h \in \mathcal{S}_h$ to \mathcal{U} as the solution of

$$(1.11) \quad B(\mathcal{U}_h, \Phi) = (\mathcal{F}, \Phi)_{0, \mathcal{H}}, \quad \forall \Phi \in \mathcal{S}_h.$$

We establish existence of a unique solution of (1.11) provided that the quantity $\frac{h}{s_0} k^2$ is small enough, where h is the maximum of the mesh sizes and s_0 is the polynomial degree of the functions of \mathcal{S}_h , cf. Section 3. Further, under the same hypothesis on h , we show that the discrete analog of the BB condition (1.10) is satisfied on \mathcal{S}_h and a quasioptimal estimate of the form

$$(1.12) \quad \|\mathcal{U} - \mathcal{U}_h\|_{1,\mathcal{H}} \leq C_* \inf_{\Phi \in \mathcal{S}_h} \|\mathcal{U} - \Phi\|_{1,\mathcal{H}},$$

is valid, where C_* is a positive constant independent of h and k . Thus, since the approximation properties of \mathcal{S}_h will be known, (1.12) will imply convergence of optimal order of the finite element approximations in the $\|\cdot\|_{1,\mathcal{H}}$ norm. This result is independent of the choice of the particular method and of the finite element spaces (h version, p version or $h-p$ version). Note that stability estimates similar to (1.9), derived in section 2 are needed in the analysis of the discrete problem and, in particular, in the proof of (1.12).

In Section 4 we present numerical results on the behavior of the finite element approximations to (1.1-2) (coupled problem). As it is expected from the theoretical analysis, increasing the polynomial degree s_0 , one decreases the influence of the ‘‘pollution effect’’ on the error and, of course, increases the rate of convergence. These results are compared to analogous computational results obtained previously for the single reduced wave equation (uncoupled problem), [IB 1, 2], [BS]. We also investigate whether the restrictive condition ‘‘ $\frac{h}{s_0} k^2$ is small’’, mentioned above, is actually needed or not. The experiments of Section 4 provide numerical evidence of the fact (for $s_0 = 1$) that the restriction $hk^2 \leq \alpha$, α constant, is necessary for the quasioptimal estimate (1.12) to hold. On the other hand, the results of [IB1,2] for the single reduced wave equation indicate that this restriction is rather pessimistic for establishing existence-uniqueness and stability (BB condition) of the finite element approximations. Indeed, for the uncoupled problem and in the case where \mathcal{S}_h consists of piecewise polynomial functions in a uniform grid, it is shown in [IB1,2] that the discrete BB constant is proportional to $\frac{1}{k}$ provided that kh remains bounded.

Work on Galerkin approximations for a model problem of fluid-solid interaction has been also done by Demkowicz, [D]. A general variational setting was considered, for which the asymptotic convergence of the Galerkin approximations was established. Also, in a numerical evaluation of a one dimensional coupled problem it was observed that the discrete (and continuous) inf-sup constant decreases algebraically with the wave number k . Particular attention is given in [D] to the sensitivity of the stability constant with respect to wave damping at the interface of the two media. The approach we take here is different and focusses mainly on the stability of the continuous problem (1.1-2) and on the stability and convergence of the discrete problem (1.11) with respect to the wave number k . One may say that, for the coupled problem and without special assumptions on the finite element method, our results are an extension of similar results known for the one dimensional single reduced wave equation, cf. eg. [BGT], [AKS], [DS], [IB 1,2]. Note however that we do not use explicit representations of \mathcal{U} and \mathcal{U}_h in terms of Green’s functions to derive our stability results. Other works for the Helmholtz equation with

radiation boundary conditions of the type (1.2a) —in higher dimensions too— include [AK], [G], [HH1], [BS] [IB3], [BIPS]. Among these works, [AKS], [HH], [BS] and [BIPS] are devoted to the analysis of non-standard finite element methods which have been designed to minimize the “pollution effect” observed for the standard Galerkin method for moderate to high wave numbers. A detailed analysis of the behavior of the Galerkin finite element approximations in the case of high wave numbers is presented in [IB1-3] and [BS]. For a comparison of boundary element and finite element methods for time-harmonic acoustic wave equations cf. [HH2].

2. ANALYSIS OF THE CONTINUOUS PROBLEM

We first observe that if \mathcal{U} is a solution of (1.6) then its components satisfy (1.1-2):

Lemma 2.1. *Let $\mathcal{F} \in \mathbf{L}^2$. Assume that (1.6) has a solution $\mathcal{U} = (p, u, \tilde{p}) \in \mathcal{H}$. Then $\mathcal{U} \in \mathcal{H}^2 := H^2(\Omega_1) \times H^2(\Omega_2) \times H^2(\Omega_3)$, and p, u and \tilde{p} satisfy (1.1-2), where the derivatives in these equations are derivatives in the sense of distributions.*

The main part of this section is devoted to the proof of the following stability result:

Proposition 2.1. *If \mathcal{U} is a solution of problem (1.6) for $\mathcal{F} \in \mathbf{L}^2$ then*

$$(2.1a) \quad \|\mathcal{U}\|_{1,\mathcal{H}} \leq C_1 \|\mathcal{F}\|_{0,\mathcal{H}},$$

where C_1 is a positive constant independent of k . Also, for $\mathcal{F} \in \mathcal{H}$

$$(2.1b) \quad \|\mathcal{U}\|_{1,\mathcal{H}} \leq C_1 \frac{1}{k} \|\mathcal{F}\|_{1,\mathcal{H}},$$

We shall state two consequences of this result as Theorems 2.1 and 2.2 below. The proof of Proposition 2.1 is given after the proof of Theorem 2.2.

Theorem 2.1. *Let $\mathcal{F} \in \mathbf{L}^2$. Then there exists a unique solution of (1.6). Furthermore $\mathcal{U} \in \mathcal{H}^2$, and*

$$(2.2) \quad \|\mathcal{U}_{xx}\|_{0,\mathcal{H}} \leq C_2(1+k) \|\mathcal{F}\|_{0,\mathcal{H}}.$$

Proof. The uniqueness follows from Proposition 2.1. Now, using the trace inequality:

$$(2.2) \quad |v(\alpha)|^2 \leq \varepsilon \|v_x\|_{L^2(I)}^2 + \frac{1}{\varepsilon} C \|v\|_{L^2(I)}^2, \quad v \in H^1(I), \varepsilon > 0$$

where α is one of the endpoints of the interval I , and the arithmetic-geometric mean inequality, we easily verify that $B(\cdot, \cdot)$ satisfies

$$(2.3) \quad \operatorname{Re} B(\mathcal{U}, \mathcal{U}) \geq C_0 \|\mathcal{U}\|_{1,\mathcal{H}}^2 - \mu_0(1+k^2) \|\mathcal{U}\|_{0,\mathcal{H}}^2 \quad \forall \mathcal{U} \in \mathcal{H},$$

with positive constants C_0, μ_0 independent of k . Therefore $B(\cdot, \cdot)$ satisfies a Gårding-type inequality. It is well known that (2.3) implies the alternative statement: Either the problem

$$B(\mathcal{U}, \mathcal{V}) = 0 \quad \forall \mathcal{V} \in \mathcal{H},$$

has a nontrivial solution or (1.6) has a solution. Therefore since we have established uniqueness, (2.3) implies existence of the solution for the problem (1.6). The estimate (2.2) follows from Lemma 2.1, Proposition 2.1 and (1.1). \square

Let $\|\cdot\|'_{1, \mathcal{H}}$ be the dual norm of $\|\cdot\|_{1, \mathcal{H}}$ defined by $\|\mathcal{V}\|'_{1, \mathcal{H}} = \sup_{\Phi \in \mathcal{H}} \frac{(\mathcal{V}, \Phi)_{0, \mathcal{H}}}{\|\Phi\|_{1, \mathcal{H}}}$. We have the following result:

Theorem 2.2. *The bilinear form $B(\cdot, \cdot)$ defined on $\mathcal{H} \times \mathcal{H}$ satisfies*

$$(2.4) \quad |B(\mathcal{U}, \mathcal{V})| \leq \beta \|\mathcal{U}\|_{1, \mathcal{H}} \|\mathcal{V}\|_{1, \mathcal{H}} \quad \forall \mathcal{U}, \mathcal{V} \in \mathcal{H},$$

and

$$(2.5) \quad \sup_{\mathcal{V} \in \mathcal{H}} \frac{\operatorname{Re} B(\mathcal{U}, \mathcal{V})}{\|\mathcal{V}\|_{1, \mathcal{H}}} \geq \gamma \frac{1}{k} \|\mathcal{U}\|_{1, \mathcal{H}}, \quad \forall \mathcal{U} \in \mathcal{H},$$

where β and γ are two positive constants independent of k . Furthermore, if \mathcal{U} is the solution of (1.6), we have the estimate

$$(2.6) \quad \|\mathcal{U}\|_{1, \mathcal{H}} \leq k \frac{1}{\gamma} \|\mathcal{F}\|'_{1, \mathcal{H}}.$$

Proof. The inequality (2.4) follows (similarly as (2.3)) from (2.2), the arithmetic-geometric mean and Cauchy-Schwarz inequalities. For the proof of (2.5), let \mathcal{U} be a given element of \mathcal{H} . If $\mathcal{V} = \frac{1}{k^2} \mathcal{U} + Z$, where $Z \in \mathcal{H}$ shall be appropriately chosen bellow, we have

$$\begin{aligned} B(\mathcal{U}, \mathcal{V}) &= \frac{1}{k^2} B(\mathcal{U}, \mathcal{U}) + B(\mathcal{U}, Z) \\ &= \frac{1}{k^2} (B(\mathcal{U}, \mathcal{U}) + \mu_0(1 + k^2) \|\mathcal{U}\|_{0, \mathcal{H}}^2) + \left(B(\mathcal{U}, Z) - \mu_0 \frac{1 + k^2}{k^2} \|\mathcal{U}\|_{0, \mathcal{H}}^2 \right) \\ &=: I + II. \end{aligned}$$

We choose Z such that the part II vanishes. I.e., Z shall be the solution of

$$(2.7) \quad B(\Phi, Z) = \mu_0 \frac{1 + k^2}{k^2} (\Phi, \mathcal{U})_{0, \mathcal{H}} \quad \forall \Phi \in \mathcal{H}.$$

(2.7) is a dual problem to (1.6) and one can verify that the results of Proposition 2.1 and Theorem 2.1 hold for this problem as well. Therefore Z is uniquely determined as solution of (2.7), and furthermore satisfies the estimate (since $\mathcal{U} \in \mathcal{H}$)

$$(2.7) \quad \|Z\|_{1, \mathcal{H}} \leq C_1 \frac{1}{k} \mu_0 \frac{1 + k^2}{k^2} \|\mathcal{U}\|_{1, \mathcal{H}}.$$

With this choice of Z we have

$$(2.8) \quad \operatorname{Re} B(\mathcal{U}, \mathcal{V}) \geq \frac{1}{k^2} C_0 \|\mathcal{U}\|_{1, \mathcal{H}}^2,$$

where $\mathcal{V} \in \mathcal{H}$ satisfies

$$\begin{aligned} \|\mathcal{V}\|_{1, \mathcal{H}} &\leq \frac{1}{k^2} \|\mathcal{U}\|_{1, \mathcal{H}} + \|Z\|_{1, \mathcal{H}} \\ &\leq \frac{1}{k^2} \left(C_1 \mu_0 \frac{1+k^2}{k} + 1 \right) \|\mathcal{U}\|_{1, \mathcal{H}} \\ &\leq \frac{1}{k^2} C_2 k \|\mathcal{U}\|_{1, \mathcal{H}}. \end{aligned}$$

Therefore (2.8) implies

$$\begin{aligned} \operatorname{Re} B(\mathcal{U}, \mathcal{V}) &\geq \frac{1}{k^2} C_0 \|\mathcal{U}\|_{1, \mathcal{H}}^2 \\ &\geq C_0 \|\mathcal{U}\|_{1, \mathcal{H}} \frac{1}{C_2 k} \|\mathcal{V}\|_{1, \mathcal{H}} = \gamma \frac{1}{k} \|\mathcal{U}\|_{1, \mathcal{H}} \|\mathcal{V}\|_{1, \mathcal{H}}, \end{aligned}$$

and (2.5) follows. The estimate (2.6) is a consequence of [BA, Theorem 5.2.1] and of (2.4), (2.5). \square

We now prove Proposition 2.1:

Proof of Proposition 2.1. We write $B(\mathcal{U}, \mathcal{V}) = B_1(\mathcal{U}, \mathcal{V}) + B_2(\mathcal{U}, \mathcal{V}) + B_3(\mathcal{U}, \mathcal{V})$, where

$$(2.9) \quad \begin{aligned} B_1(\mathcal{U}, \mathcal{V}) &= \int_{\Omega_1} p_x \bar{q}_x dx - k^2 \int_{\Omega_1} p \bar{q} dx - k^2 u(x_1) \bar{q}(x_1) - ikp(0) \bar{q}(0), \\ B_2(\mathcal{U}, \mathcal{V}) &= k^2 \int_{\Omega_2} au_x \bar{v}_x dx - k^4 \int_{\Omega_2} gu \bar{v} dx - k^2 p(x_1) \bar{v}(x_1) + k^2 \tilde{p}(x_2) \bar{v}(x_2), \\ B_3(\mathcal{U}, \mathcal{V}) &= \int_{\Omega_3} \tilde{p}_x \bar{\tilde{q}}_x dx - k^2 \int_{\Omega_3} \tilde{p} \bar{\tilde{q}} dx + k^2 u(x_2) \bar{\tilde{q}}(x_2) - ik\tilde{p}(L) \bar{\tilde{q}}(L). \end{aligned}$$

In the sequel we assume that \mathcal{U} is a solution of (1.6) and $\mathcal{F} \in \mathbf{L}^2$. By Lemma 2.1 we have $\mathcal{U} \in \mathcal{H}^2$, and the components of \mathcal{U} satisfy (1.1-2). Therefore the function \mathcal{V} with components

$$(2.10) \quad \begin{aligned} q &= (x - x_1)p_x, \\ v &= (x - x_2)u_x, \\ \tilde{q} &= (x - x_2)\tilde{p}_x, \end{aligned}$$

is an element of \mathcal{H} . We shall evaluate $\operatorname{Re} B(\mathcal{U}, \mathcal{V})$. We observe first that ($q(x_1) = 0$):

$$\begin{aligned} B_1(\mathcal{U}, \mathcal{V}) &= \int_{\Omega_1} p_x \bar{q}_x dx - k^2 \int_{\Omega_1} p \bar{q} dx - k^2 u(x_1) \bar{q}(x_1) - ikp(0) \bar{q}(0), \\ &= \int_{\Omega_1} p_x (x - x_1) \bar{p}_{xx} dx + \int_{\Omega_1} |p_x|^2 - k^2 \int_{\Omega_1} p(x - x_1) \bar{p}_x dx + ikp(0)x_1 \bar{p}_x(0); \end{aligned}$$

hence, using (1.2),

$$\begin{aligned}
\operatorname{Re} B_1(\mathcal{U}, \mathcal{V}) &= \int_{\Omega_1} |p_x|^2 dx + \operatorname{Re} \frac{1}{2} \int_{\Omega_1} (x - x_1)(|p_x|^2)_x dx \\
&\quad - \operatorname{Re} \frac{1}{2} k^2 \int_{\Omega_1} (x - x_1)(|p|^2)_x dx - x_1 |p_x(0)|^2 \\
&= \int_{\Omega_1} |p_x|^2 dx + \operatorname{Re} \frac{1}{2} \left\{ - \int_{\Omega_1} |p_x|^2 dx + x_1 |p_x(0)|^2 \right\} \\
&\quad - \operatorname{Re} \frac{1}{2} k^2 \left\{ - \int_{\Omega_1} |p|^2 dx + x_1 |p(0)|^2 \right\} - x_1 |p_x(0)|^2 \\
&= \frac{1}{2} \int_{\Omega_1} |p_x|^2 dx + \frac{1}{2} k^2 \int_{\Omega_1} |p|^2 dx - \frac{1}{2} x_1 |p_x(0)|^2 - \frac{1}{2} k^2 x_1 |p(0)|^2.
\end{aligned}$$

Since $|p_x(0)| = k|p(0)|$ we conclude

$$(2.11) \quad \operatorname{Re} B_1(\mathcal{U}, \mathcal{V}) = \frac{1}{2} \int_{\Omega_1} |p_x|^2 dx + \frac{1}{2} k^2 \int_{\Omega_1} |p|^2 dx - x_1 k^2 |p(0)|^2.$$

Similarly,

$$\begin{aligned}
B_2(\mathcal{U}, \mathcal{V}) &= k^2 \int_{\Omega_2} a u_x \bar{v}_x dx - k^4 \int_{\Omega_2} g u \bar{v} dx - k^2 p(x_1) \bar{v}(x_1) + k^2 \tilde{p}(x_2) \bar{v}(x_2), \\
&= k^2 \int_{\Omega_2} a u_x (x - x_2) \bar{u}_{xx} dx + k^2 \int_{\Omega_2} a |u_x|^2 dx \\
&\quad - k^4 \int_{\Omega_2} g u (x - x_2) \bar{u}_x dx - k^2 p(x_1) (x_1 - x_2) \bar{u}_x(x_1);
\end{aligned}$$

hence, using (1.2),

$$\begin{aligned}
\operatorname{Re} B_2(\mathcal{U}, \mathcal{V}) &= k^2 \int_{\Omega_2} a |u_x|^2 dx + \operatorname{Re} \frac{1}{2} k^2 \int_{\Omega_2} a (x - x_2) (|u_x|^2)_x dx \\
&\quad - \operatorname{Re} \frac{1}{2} k^4 \int_{\Omega_2} g (x - x_2) (|u|^2)_x dx - \operatorname{Re} k^2 p(x_1) (x_1 - x_2) \bar{u}_x(x_1) \\
&= \frac{1}{2} k^2 \int_{\Omega_2} a |u_x|^2 dx + \frac{1}{2} k^2 a (x_2 - x_1) |u_x(x_1)|^2 \\
&\quad + \frac{1}{2} k^4 \int_{\Omega_2} g |u|^2 dx - \frac{1}{2} k^4 g (x_2 - x_1) |u(x_1)|^2 + k^2 a (x_1 - x_2) |u_x(x_1)|^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
(2.12) \quad \operatorname{Re} B_2(\mathcal{U}, \mathcal{V}) &= \frac{1}{2} k^2 \int_{\Omega_2} a |u_x|^2 dx + \frac{1}{2} k^4 \int_{\Omega_2} g |u|^2 dx \\
&\quad - \frac{1}{2} k^4 g (x_2 - x_1) |u(x_1)|^2 - \frac{1}{2} k^2 a (x_2 - x_1) |u_x(x_1)|^2.
\end{aligned}$$

It remains to evaluate $B_3(\mathcal{U}, \mathcal{V})$:

$$B_3(\mathcal{U}, \mathcal{V}) = \int_{\Omega_1} \tilde{p}_x(x-x_2)\tilde{p}_{xx}dx + \int_{\Omega_3} |\tilde{p}_x|^2 - k^2 \int_{\Omega_3} \tilde{p}(x-x_2)\tilde{p}_x dx - ik\tilde{p}(L)(L-x_2)\tilde{p}_x(L),$$

and, using (1.2),

$$\begin{aligned} \operatorname{Re} B_3(\mathcal{U}, \mathcal{V}) &= \int_{\Omega_3} |\tilde{p}_x|^2 dx + \operatorname{Re} \frac{1}{2} \left\{ - \int_{\Omega_3} |\tilde{p}_x|^2 dx + (L-x_2)|\tilde{p}_x(L)|^2 \right\} \\ &\quad - \operatorname{Re} \frac{1}{2} k^2 \left\{ - \int_{\Omega_3} |\tilde{p}|^2 dx + (L-x_2)|\tilde{p}(L)|^2 \right\} - (L-x_2)|\tilde{p}_x(L)|^2 \\ &= \frac{1}{2} \int_{\Omega_3} |\tilde{p}_x|^2 dx + \frac{1}{2} k^2 \int_{\Omega_3} |\tilde{p}|^2 dx \\ &\quad - \frac{1}{2} (L-x_2)|\tilde{p}_x(L)|^2 - \frac{1}{2} k^2 (L-x_2)|\tilde{p}(L)|^2, \end{aligned}$$

or, since $|\tilde{p}_x(L)| = k|\tilde{p}(L)|$,

$$(2.13) \quad \operatorname{Re} B_1(\mathcal{U}, \mathcal{V}) = \frac{1}{2} \int_{\Omega_3} |\tilde{p}_x|^2 dx + \frac{1}{2} k^2 \int_{\Omega_3} |\tilde{p}|^2 dx - (L-x_2)k^2 |\tilde{p}(L)|^2.$$

From equations (2.11) and (2.13) it is clear that we need to estimate $|p(0)|$, $|\tilde{p}(L)|$. To this end, we observe that

$$(2.14) \quad \operatorname{Im} B(\mathcal{U}, \mathcal{U}) = k|p(0)|^2 + k|\tilde{p}(L)|^2,$$

and therefore

$$(2.15) \quad k^2|p(0)|^2 + k^2|\tilde{p}(L)|^2 \leq k|(\mathcal{F}, \mathcal{U})_{0, \mathcal{H}}|.$$

Now, let $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3$ where

$$\mathcal{V}_1 = (q, 0, 0), \quad \mathcal{V}_2 = (0, v, 0), \quad \mathcal{V}_3 = (0, 0, \tilde{q})$$

(q, v and \tilde{q} are given by (2.10)). The relations (2.11) and (2.13) are still valid if we replace \mathcal{V} by \mathcal{V}_1 and \mathcal{V}_3 , respectively. Hence, in view of (2.15), we obtain

$$(2.16) \quad \begin{aligned} \frac{1}{2} \|p_x\|^2 + \frac{1}{2} k^2 \|p\|^2 &\leq |(\mathcal{F}, \mathcal{V}_1)_{0, \mathcal{H}}| + x_1 k |(\mathcal{F}, \mathcal{U})_{0, \mathcal{H}}|, \\ \frac{1}{2} \|\tilde{p}_x\|^2 + \frac{1}{2} k^2 \|\tilde{p}\|^2 &\leq |(\mathcal{F}, \mathcal{V}_3)_{0, \mathcal{H}}| + (L-x_2)k |(\mathcal{F}, \mathcal{U})_{0, \mathcal{H}}|. \end{aligned}$$

We turn now to (2.12). In order to estimate the terms at x_1 , we rewrite them using the transmission condition (1.2b) as

$$(2.17) \quad \begin{aligned} \frac{1}{2} k^4 g(x_2 - x_1) |u(x_1)|^2 + \frac{1}{2} k^2 a(x_2 - x_1) |u_x(x_1)|^2 \\ = \frac{1}{2} g(x_2 - x_1) |p_x(x_1)|^2 + \frac{1}{2a} k^2 (x_2 - x_1) |p(x_1)|^2. \end{aligned}$$

To bound the right-hand side of this equation, we proceed as follows: Let $q' = xp_x \in H^1(\Omega_1)$ and $\mathcal{V}'_1 = (q', 0, 0)$. Using the relation $k^2 u(x_1) = p_x(x_1)$ and following the steps of the proof of (2.11) we conclude

$$\begin{aligned} \operatorname{Re} B_1(\mathcal{U}, \mathcal{V}'_1) &= \int_{\Omega_1} |p_x|^2 dx + \operatorname{Re} \frac{1}{2} \left\{ - \int_{\Omega_1} |p_x|^2 dx + x_1 |p_x(x_1)|^2 \right\} \\ &\quad - \operatorname{Re} \frac{1}{2} k^2 \left\{ - \int_{\Omega_1} |p|^2 dx + |p(x_1)|^2 \right\} - \operatorname{Re} k^2 x_1 u(x_1) \bar{p}_x(x_1) \\ &= \frac{1}{2} \int_{\Omega_1} |p_x|^2 dx + \frac{1}{2} k^2 \int_{\Omega_1} |p|^2 dx - \frac{1}{2} x_1 |p_x(x_1)|^2 - \frac{1}{2} k^2 x_1 |p(x_1)|^2. \end{aligned}$$

Hence, using (2.15), we obtain

$$(2.18) \quad \begin{aligned} |p_x(x_1)|^2 + k^2 |p(x_1)|^2 &\leq \frac{1}{x_1} \|p_x\|^2 + \frac{1}{x_1} k^2 \|p\|^2 + \frac{1}{2x_1} |(\mathcal{F}, \mathcal{V}'_1)_{0,\mathcal{H}}| \\ &\leq \frac{1}{2x_1} (4|(\mathcal{F}, \mathcal{V}_1)_{0,\mathcal{H}}| + |(\mathcal{F}, \mathcal{V}'_1)_{0,\mathcal{H}}|) + 2k |(\mathcal{F}, \mathcal{U})_{0,\mathcal{H}}|. \end{aligned}$$

Summarizing, (2.12), (2.17) and (2.18) give the estimate

$$\begin{aligned} \frac{1}{2} k^2 \|u_x\|^2 + \frac{1}{2} k^4 \|u\|^2 &\leq |(\mathcal{F}, \mathcal{V}_2)_{0,\mathcal{H}}| + C_* |(\mathcal{F}, \mathcal{V}_1)_{0,\mathcal{H}}| \\ &\quad + C_* |(\mathcal{F}, \mathcal{V}'_1)_{0,\mathcal{H}}| + C_* k |(\mathcal{F}, \mathcal{U})_{0,\mathcal{H}}|, \end{aligned}$$

where C_* is a constant (not necessarily the same in two different places) independent of k . Recalling the definitions of $\mathcal{V}_1, \mathcal{V}'_1$ we have

$$(2.19) \quad k^2 \|u_x\|^2 + k^4 \|u\|^2 \leq C_*(k^2 \|f\| \|u_x\| + \|g_1\| \|p_x\|) + C_* k |(\mathcal{F}, \mathcal{U})_{0,\mathcal{H}}|,$$

where

$$(2.20) \quad \begin{aligned} k |(\mathcal{F}, \mathcal{U})_{0,\mathcal{H}}| &\leq k \|g_1\| \|p\| + k^3 \|f\| \|u\| + k^2 \|g_2\| \|\tilde{p}\| \\ &\leq C_\varepsilon (\|g_1\|^2 + k^2 \|f\| + \|g_2\|) + \varepsilon (k^2 \|p\| + k^4 \|u\| + k^2 \|\tilde{p}\|) \end{aligned}$$

Handling analogously the terms of (2.16) and summing we conclude:

$$\begin{aligned} \|p_x\|^2 + k^2 \|p\|^2 + k^2 \|u_x\|^2 + k^4 \|u\|^2 + \|\tilde{p}_x\|^2 + k^2 \|\tilde{p}\|^2 \\ \leq C_*(\|g_1\|^2 + k^2 \|f\| + \|g_2\|), \end{aligned}$$

or

$$\|\mathcal{U}\|_{1,\mathcal{H}}^2 \leq C_* \|\mathcal{F}\|_{0,\mathcal{H}}^2,$$

which is the relation (2.1a). If $\mathcal{F} \in \mathcal{H}$ we have $\|\mathcal{F}\|_{0,\mathcal{H}} \leq \frac{1}{k} \|\mathcal{F}\|_{1,\mathcal{H}}$ and (2.1b) follows. \square

Remark. It is clear from the proof of Proposition 2.1 (cf. (2.12), (2.17-9)) that the constant C_1 of (2.1a,b) tends to infinity as $a \rightarrow 0$, in particular $C_1 = O(\frac{1}{a})$.

2. FINITE ELEMENT APPROXIMATIONS

In this section we shall consider numerical approximations to the solution of problem (1.6). If S_h^i , $i = 1, 2, 3$, are three finite dimensional subspaces of $H^1(\Omega_i)$, $i = 1, 2, 3$, and \mathcal{S}_h is the space $\mathcal{S}_h = S_h^1 \times S_h^2 \times S_h^3$ we have $\mathcal{S}_h \subset \mathcal{H}$. We shall seek approximation $\mathcal{U}_h \in \mathcal{S}_h$ to \mathcal{U} defined as the solution of the problem: Find $\mathcal{U}_h \in \mathcal{S}_h$ such that

$$(3.1) \quad B(\mathcal{U}_h, \Phi) = (\mathcal{F}, \Phi)_{0, \mathcal{H}} \quad \forall \Phi \in \mathcal{S}_h,$$

where $B(\cdot, \cdot)$ is defined in (1.6b).

The finite element spaces. Our basic assumption on the finite element spaces S_h^i , $i = 1, 2, 3$, is that they consist of continuous piecewise polynomials of degree s , $s \geq s_0 \geq 1$. These functions are constructed using a fixed but arbitrary partition of Ω_i into intervals $\{I_j^i\}_j$. Note that the degree of the polynomials may vary in the intervals $\{I_j^i\}_j$. Let $h^i = \max_j |I_j^i|$ and $h = \max\{h^1, h^2, h^3\}$. A consequence of standard approximation properties for these spaces is the relation: For any $\mathcal{V} \in \mathcal{H}^2$ there is an element $\Phi \in \mathcal{S}_h$ such that

$$(3.2) \quad \|\mathcal{V}^{(m)} - \Phi^{(m)}\|_{0, \mathcal{H}} \leq C \frac{h^{2-m}}{s_0} \|\mathcal{V}_{xx}\|_{0, \mathcal{H}}, \quad m = 0, 1,$$

where $\mathcal{V}^{(0)} = \mathcal{V}$ and $\mathcal{V}^{(1)} = \mathcal{V}_x$. Of course (3.2) gives a crude bound for $\inf_{\Phi \in \mathcal{S}_h} \|\mathcal{V} - \Phi\|_{1, \mathcal{H}}$, but we use this estimate as a tool in the proofs below. In particular, we first show that the bilinear form $B(\cdot, \cdot)$ satisfies a discrete analog of the BB condition (1.10) (Theorem 3.1), and also that two quasioptimal estimates of the type (1.12) are satisfied ((3.9) and Theorem 3.2). An essential tool in the proofs of these results is the stability estimate of Proposition 2.1.

Theorem 3.1. *Assume that h is small enough such that $k^2 \frac{h}{s_0} \leq c$, where c is an appropriately chosen constant. Then there exists a constant $\tilde{\gamma} > 0$, independent of h and k , such that*

$$(3.3) \quad \sup_{\mathcal{V}_h \in \mathcal{S}_h} \frac{\operatorname{Re} B(\mathcal{U}_h, \mathcal{V}_h)}{\|\mathcal{V}_h\|_{1, \mathcal{H}}} \geq \tilde{\gamma} \frac{1}{k} \|\mathcal{U}_h\|_{1, \mathcal{H}}, \quad \forall \mathcal{U}_h \in \mathcal{S}_h.$$

Proof. We first consider the auxiliary bilinear form

$$(3.4) \quad A(\mathcal{U}, \mathcal{V}) = B(\mathcal{U}, \mathcal{V}) + \mu_0(1 + k^2)(\mathcal{U}, \mathcal{V})_{0, \mathcal{H}}, \quad \mathcal{U}, \mathcal{V} \in \mathcal{H},$$

where μ_0 is the constant of (2.3). It is easy to see that there is a constant $\beta_A > 0$, independent of k , such that

$$|A(\mathcal{U}, \mathcal{V})| \leq \beta_A \|\mathcal{U}\|_{1, \mathcal{H}} \|\mathcal{V}\|_{1, \mathcal{H}}.$$

Also (2.3) implies that $A(\cdot, \cdot)$ is coersive with respect to $\|\cdot\|_{1,\mathcal{H}}$. Therefore the projection $P_A : \mathcal{H} \rightarrow \mathcal{S}_h$ is well defined by

$$(3.5) \quad A(\Phi, P_A \mathcal{V}) = A(\Phi, \mathcal{V}) \quad \forall \Phi \in \mathcal{S}_h.$$

For the proof of (3.3), let \mathcal{U}_h be a given element of \mathcal{S}_h . Let $\mathcal{V}_h = \frac{1}{k^2} \mathcal{U}_h + P_A Z$, where $Z \in \mathcal{H}$ shall be defined below. For this choice of \mathcal{V}_h we have

$$(3.6) \quad \begin{aligned} B(\mathcal{U}_h, \mathcal{V}_h) &= \frac{1}{k^2} B(\mathcal{U}_h, \mathcal{U}_h) + B(\mathcal{U}_h, P_A Z) \\ &= \frac{1}{k^2} (B(\mathcal{U}_h, \mathcal{U}_h) + \mu_0(1+k^2) \|\mathcal{U}_h\|_{0,\mathcal{H}}^2) + \left(B(\mathcal{U}_h, P_A Z) - \mu_0 \frac{1+k^2}{k^2} \|\mathcal{U}_h\|_{0,\mathcal{H}}^2 \right) \\ &= \frac{1}{k^2} (B(\mathcal{U}_h, \mathcal{U}_h) + \mu_0(1+k^2) \|\mathcal{U}_h\|_{0,\mathcal{H}}^2) + \left(B(\mathcal{U}_h, Z) - \mu_0 \frac{1+k^2}{k^2} \|\mathcal{U}_h\|_{0,\mathcal{H}}^2 \right) \\ &\quad + \mu_0(1+k^2) (\mathcal{U}_h, Z - P_A Z)_{0,\mathcal{H}}. \end{aligned}$$

Choosing now $Z \in \mathcal{H}$ to be the solution of

$$(3.6') \quad B(\Phi, Z) = \mu_0 \frac{1+k^2}{k^2} (\Phi, \mathcal{U}_h)_{0,\mathcal{H}} \quad \forall \Phi \in \mathcal{H},$$

the second term of the last equality of (3.6) vanishes. (As we have seen in the proof of Theorem 2.1 Z is uniquely determined as solution of (3.6'), and furthermore satisfies the estimate $\|Z\|_{1,\mathcal{H}} \leq C_1 \frac{1}{k} \mu_0 \frac{1+k^2}{k^2} \|\mathcal{U}_h\|_{1,\mathcal{H}}$.) Therefore it remains to estimate the term

$$(\mathcal{U}_h, Z - P_A Z)_{0,\mathcal{H}}.$$

We shall use the following lemma, the proof of which is given below.

Lemma 3.1. *If $Z \in \mathcal{H}$ and $P_A Z$ is defined by (3.5), then we have the estimate*

$$\|Z - P_A Z\|_{0,\mathcal{H}} \leq C \frac{h}{s_0} \|Z\|_{1,\mathcal{H}},$$

where the constant $C > 0$ is independent of h and k .

Now, since $\|Z\|_{1,\mathcal{H}} \leq C \frac{1}{k} \|\mathcal{U}_h\|_{1,\mathcal{H}}$ and $\|\mathcal{U}_h\|_{0,\mathcal{H}} \leq \frac{1}{k} \|\mathcal{U}_h\|_{1,\mathcal{H}}$, we obtain

$$\begin{aligned} \mu_0(1+k^2) |(\mathcal{U}_h, Z - P_A Z)_{0,\mathcal{H}}| &\leq C(1+k^2) \|\mathcal{U}_h\|_{0,\mathcal{H}} \frac{h}{s_0} \|Z\|_{1,\mathcal{H}} \\ &\leq C \frac{(1+k^2)}{k^2} \frac{h}{s_0} \|\mathcal{U}_h\|_{1,\mathcal{H}}^2 = C \frac{h}{s_0} \|\mathcal{U}_h\|_{1,\mathcal{H}}^2. \end{aligned}$$

Therefore, if $k^2 \frac{h}{s_0}$ is small enough, we obtain

$$(3.7) \quad \begin{aligned} \operatorname{Re} B(\mathcal{U}_h, \mathcal{V}_h) &\geq \frac{1}{k^2} C_0 \|\mathcal{U}\|_{1,\mathcal{H}}^2 - C \frac{h}{s_0} \|\mathcal{U}\|_{1,\mathcal{H}}^2 \\ &\geq \frac{1}{k^2} C \|\mathcal{U}\|_{1,\mathcal{H}}^2, \end{aligned}$$

where $\mathcal{V}_h \in \mathcal{S}_h$ satisfies

$$\begin{aligned}\|\mathcal{V}_h\|_{1,\mathcal{H}} &\leq \frac{1}{k^2}\|\mathcal{U}_h\|_{1,\mathcal{H}} + \|P_A Z\|_{1,\mathcal{H}} \leq \frac{1}{k^2}\|\mathcal{U}_h\|_{1,\mathcal{H}} + C\|Z\|_{1,\mathcal{H}} \\ &\leq \frac{1}{k^2}(Ck + 1)\|\mathcal{U}_h\|_{1,\mathcal{H}} \\ &\leq \frac{1}{k^2}C_3 k\|\mathcal{U}_h\|_{1,\mathcal{H}}.\end{aligned}$$

Therefore, (3.7) implies

$$\begin{aligned}\operatorname{Re}B(\mathcal{U}_h, \mathcal{V}_h) &\geq \frac{1}{k^2}C\|\mathcal{U}_h\|_{1,\mathcal{H}}^2 \\ &\geq C\|\mathcal{U}_h\|_{1,\mathcal{H}}\frac{1}{C_3 k}\|\mathcal{V}_h\|_{1,\mathcal{H}} = \tilde{\gamma}\frac{1}{k}\|\mathcal{U}_h\|_{1,\mathcal{H}}\|\mathcal{V}_h\|_{1,\mathcal{H}},\end{aligned}$$

which yields (3.3). \square

Proof of Lemma 3.1. We shall use a classical duality argument. Let $\rho := Z - P_A Z$ and $\Psi \in \mathcal{H}$ be the solution of

$$(3.8) \quad A(\Psi, \Phi) = k^2(\rho, \Phi)_{0,\mathcal{H}}, \quad \forall \Phi \in \mathcal{H}.$$

This problem has a unique solution with the regularity $\Psi \in \mathcal{H}^2$. Also, using the coersivity of $A(\cdot, \cdot)$ and equation (1.1), we easily conclude that

$$\|\Psi_{xx}\|_{0,\mathcal{H}} \leq k^2\|\rho\|_{0,\mathcal{H}}.$$

Therefore, using (3.2) for a properly chosen $\Psi_h \in \mathcal{S}_h$,

$$\begin{aligned}k^2\|\rho\|_{0,\mathcal{H}}^2 &= A(\Psi, \rho) = A(\Psi - \Psi_h, \rho) \\ &\leq \beta_A\|\Psi - \Psi_h\|_{1,\mathcal{H}}\|\rho\|_{1,\mathcal{H}} \leq C\frac{h}{s_0}\|\Psi_{xx}\|_{0,\mathcal{H}}\|\rho\|_{1,\mathcal{H}} \\ &\leq Ck^2\frac{h}{s_0}\|\rho\|_{0,\mathcal{H}}\|\rho\|_{1,\mathcal{H}};\end{aligned}$$

hence,

$$\|\rho\|_{0,\mathcal{H}} \leq C\frac{h}{s_0}\|\rho\|_{1,\mathcal{H}} \leq C\frac{h}{s_0}\|Z\|_{1,\mathcal{H}},$$

since from the definition of P_A we have $\|P_A Z\|_{1,\mathcal{H}} \leq C\|Z\|_{1,\mathcal{H}}$. \square

An application of Theorems 2.1 and 3.1 and of [BA, Theorem 6.2.1] gives the following: If the hypotheses of Theorem 3.1 hold, then \mathcal{U}_h is uniquely determined as solution of problem (3.1) and

$$(3.9) \quad \|\mathcal{U} - \mathcal{U}_h\|_{1,\mathcal{H}} \leq Ck \inf_{\Phi \in \mathcal{S}_h} \|\mathcal{U} - \Phi\|_{1,\mathcal{H}}, \quad C > 0 \text{ independent of } k,$$

where \mathcal{U} is the solution of (1.6).

The estimate (3.9) is a quasioptimal result of the finite element approximation with respect to $\|\cdot\|_{1,\mathcal{H}}$. It is possible however to improve this result in terms of the dependence of the bound on k . Indeed, in the proof of the following result we shall replace the constant Ck in (3.9) by a constant independent of k . A comparison of (3.9) and (3.10) below indicates that (3.9) might be valid under a weaker restriction than $k^2h \leq c$, c small. This assumption is consistent with available results for the single reduced wave equation, cf. [IB1,2], where the analog of (3.3) and therefore of (3.9) is proved (in a uniform mesh) under the weaker condition that kh remains bounded. The proof of the Theorem 3.2 is based on a argument due to Schatz, [S]. The use of the stability estimates of Section 2 is essential here.

Theorem 3.2. *Assume that the hypotheses of Theorem 3.1 hold. Then*

$$(3.10) \quad \|\mathcal{U} - \mathcal{U}_h\|_{1,\mathcal{H}} \leq C_* \inf_{\Phi \in \mathcal{S}_h} \|\mathcal{U} - \Phi\|_{1,\mathcal{H}}, \quad C_* > 0 \text{ independent of } k,$$

where \mathcal{U} and \mathcal{U}_h are the solutions of (1.6) and (3.1), respectively.

Proof. Let $\Psi \in \mathcal{H}$ be the solution of the problem ($e = \mathcal{U} - \mathcal{U}_h$)

$$(3.11) \quad B(\Phi, \Psi) = k^2(e, \Phi)_{0,\mathcal{H}}, \quad \forall \Phi \in \mathcal{H}.$$

As we have seen in Section 2, this problem has a unique solution. Now, for any $\Psi_h \in \mathcal{S}_h$,

$$\begin{aligned} k^2 \|e\|_{0,\mathcal{H}}^2 &= B(e, \Psi) = B(e, \Psi - \Psi_h) \\ &\leq \beta \|e\|_{1,\mathcal{H}} \|\Psi - \Psi_h\|_{1,\mathcal{H}}. \end{aligned}$$

The results of Theorem 2.1 hold for Ψ as solution of an adjoint problem to (1.6). Therefore $\Psi \in \mathcal{H}^2$ and

$$\|\Psi_{xx}\|_{0,\mathcal{H}} \leq C(1+k)k^2 \|e\|_{0,\mathcal{H}}.$$

Hence, by (3.2),

$$\begin{aligned} k^2 \|e\|_{0,\mathcal{H}}^2 &\leq Ck^2 \frac{h}{s_0} \|\Psi_{xx}\|_{0,\mathcal{H}} \|e\|_{1,\mathcal{H}} \\ &\leq Ck^2 \frac{h}{s_0} (1+k) \|e\|_{0,\mathcal{H}} \|e\|_{1,\mathcal{H}}, \end{aligned}$$

or

$$(3.12) \quad \|e\|_{0,\mathcal{H}} \leq C(1+k) \frac{h}{s_0} \|e\|_{1,\mathcal{H}}.$$

Since the bilinear form $B(\cdot, \cdot)$ satisfies (2.3), we have, for any $\Phi \in \mathcal{S}_h$,

$$\begin{aligned} C_0 \|e\|_{1,\mathcal{H}}^2 &= B(e, e) + \mu_0(1+k^2) \|e\|_{0,\mathcal{H}}^2 \\ &= B(e, \mathcal{U} - \Phi) + \mu_0(1+k^2) \|e\|_{0,\mathcal{H}}^2 \\ &\leq \beta \|e\|_{1,\mathcal{H}} \|\mathcal{U} - \Phi\|_{1,\mathcal{H}} + \mu_0(1+k^2)(1+k)^2 \frac{h^2}{s_0^2} \|e\|_{1,\mathcal{H}}^2, \end{aligned}$$

and therefore, since $k^2 \frac{h}{s_0}$ is small enough,

$$\|e\|_{1,\mathcal{K}}^2 \leq C \|e\|_{1,\mathcal{K}} \|\mathcal{U} - \Phi\|_{1,\mathcal{K}},$$

or

$$\|\mathcal{U} - \mathcal{U}_h\|_{1,\mathcal{K}} \leq C \|\mathcal{U} - \Phi\|_{1,\mathcal{K}}, \quad \forall \Phi \in \mathcal{S}_h,$$

which is the desired result. \square

4. NUMERICAL EVALUATION

To demonstrate the principal numerical effects of Galerkin FE-solutions to the Helmholtz equation, we first consider a related uncoupled problem, [IB1],

$$(4.1) \quad \begin{aligned} u'' + k^2 u &= 1 \quad \text{on } \Omega = (0, 1) \\ u(0) &= 0 \\ u'(1) - iku(1) &= 0. \end{aligned}$$

For the finite element solution, Ω is covered by uniform mesh X_h of n elements with stepsize $h = n^{-1}$. The subspace of piecewise polynomial functions of uniform degree p with nodes in X_h is called S_h^p . The error is measured in H^1 -seminorm $|\cdot|_1$ which, for this problem, is equivalent to the full H^1 -norm. The optimal convergence rate is then displayed by the H^1 -projection (in the seminorm) of the exact solution onto S_h^p . It is well known that this best approximation interpolates the exact solution on X_h . In Fig. 1, the errors of best approximation and finite element solution are shown for $k = 2, k = 10$ and $k = 100$.

We observe that for all k the Galerkin FE-solution reaches optimal convergence if h becomes small. However, for medium and large k , the range of optimal convergence is preceded by a range in which the FE-solution is numerically polluted, cf. [BS], [IB3]. In this range, the finite element error differs significantly from the optimal approximation. In particular, it was shown by numerical experiment that a constraint of the form $h^2 k = \text{const}$ is indeed necessary for quasioptimal behaviour of the FE-solution if $s_o = 1$, cf. [IB1] – see Fig. 2.

For sufficiently regular solution, oscillating with frequency k , the relative error in H^1 -norm can be estimated, generally, as [IB2]

$$\bar{\epsilon}_1 \leq C_1(hk)^{s_o} + C_2 k(kh)^{2s_o}.$$

It was shown in numerical experiments that this estimate is sharp in the preasymptotic range of convergence. Hence, raising the order of polynomial approximation s_o with $hk/s_o = \text{const}$, the gain in performance of the Galerkin FEM for numerical solution of the Helmholtz equation is twofold: increase of the order of asymptotic convergence and decrease of the pollution in the preasymptotic range. This is illustrated by the computational results shown in Fig. 3. The vertical line at the relative error $\bar{\epsilon}_1 = 0.1$ indicates the decrease in the number of elements, needed for a given accuracy as the degree of approximation is increased. For a more detailed discussion of the numerical effort involved, see [IB2]. The “bumps” in the plots are an effect of local approximation if the meshsize coincides with the wavelength, [IB2].

Turn now to computational results for the model problem (1.1-2) of fluid-structure interaction. The numerical evaluation has been carried out with the data:

- The domains are $\Omega_1 = (0, 3)$; $\Omega_2 = (3, 6)$; $\Omega_3 = (6, 9)$.

- In the first region, a source is present in the form of a step-function

$$g_1 = \begin{cases} 0 & \text{in } (0,1) \\ 1 & \text{in } [1,2] \\ 0 & \text{in } (2,3) \end{cases}$$

- No load is given in the solid and in the 'right' fluid region:

$$f = g_2 = 0.$$

All domains are covered by uniform (in h and s) mesh; the computations are carried out with 10, 20, 40, ..., 320 elements per region. The degree d of polynomial approximation was varied from 1 to 7. The errors are measured

1. pointwise as absolute values in $x_1 = 1.5$, $x_2 = 4.5$ and $x_3 = 7.5$.
2. In the H_1 -norm on the first region.

In the second case, the FE-error was compared to the error of the H^1 -projection. A review of the plots for all computations shows that the coupled problem displays the same principal effects as the uncoupled problem:

1. The error is smaller than 100 % only if the meshsize exceeds a critical number which grows over-proportionally with the wavenumber k - see Fig. 4.
2. On coarse mesh and for medium or large k , the pollution term dominates the FE-error. For small h , the FE-solution is quasioptimal with an optimality constant not depending on k (Theorem 3.2). This behaviour is shown in Fig. 5 where the relative errors of the H^1 -projection and the FE-solution are compared for different k and different degrees of approximation. The quasioptimal behaviour of the solution for sufficiently small $\frac{h}{s_0}$ can be seen in Fig. 6.
3. Increasing the polynomial degree s_0 , one decreases the influence of the pollution term on the error and increases the rate of convergence - see Fig. 7.

The graphics shown here for demonstration are representative for the whole series of computational results. In particular, no significant dependence of the error on the region Ω_i was observed.

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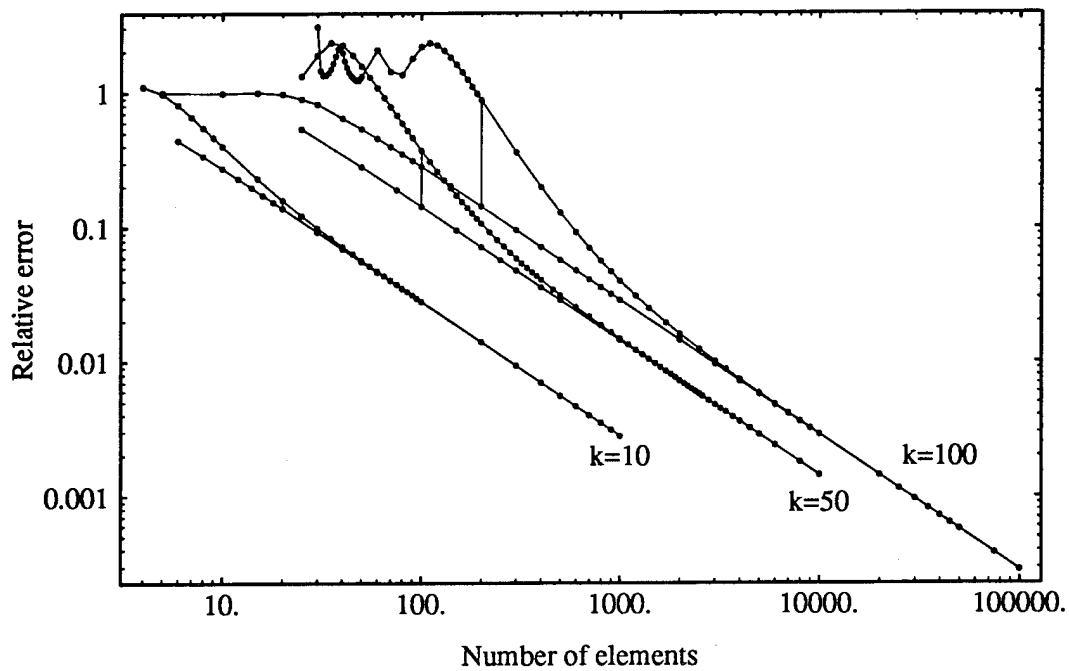


Figure 1: Errors of H^1 -projection versus error of finite element solution for Dirichlet problem; error in H^1 -seminorm; wavenumbers $k = 10$, $k = 50$ and $k = 100$.

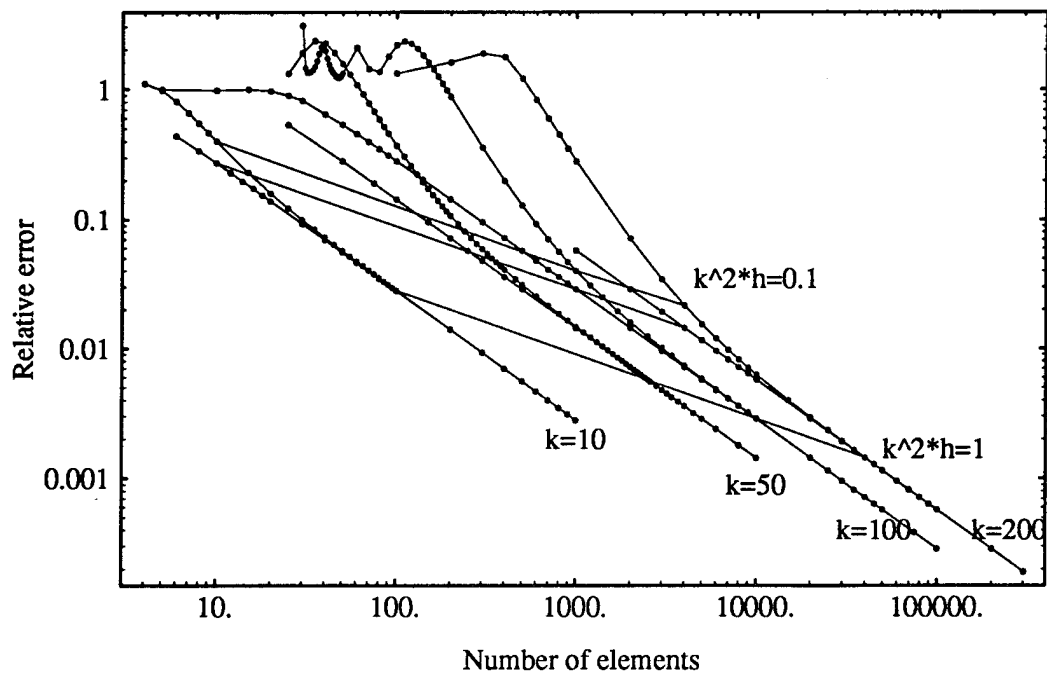


Figure 2: Constraints of the form $k^2 h = \alpha$ for $\alpha = 1$ and $\alpha = 0.1$ in error plots for $k = 10, 50, 100$ and $k = 200$.

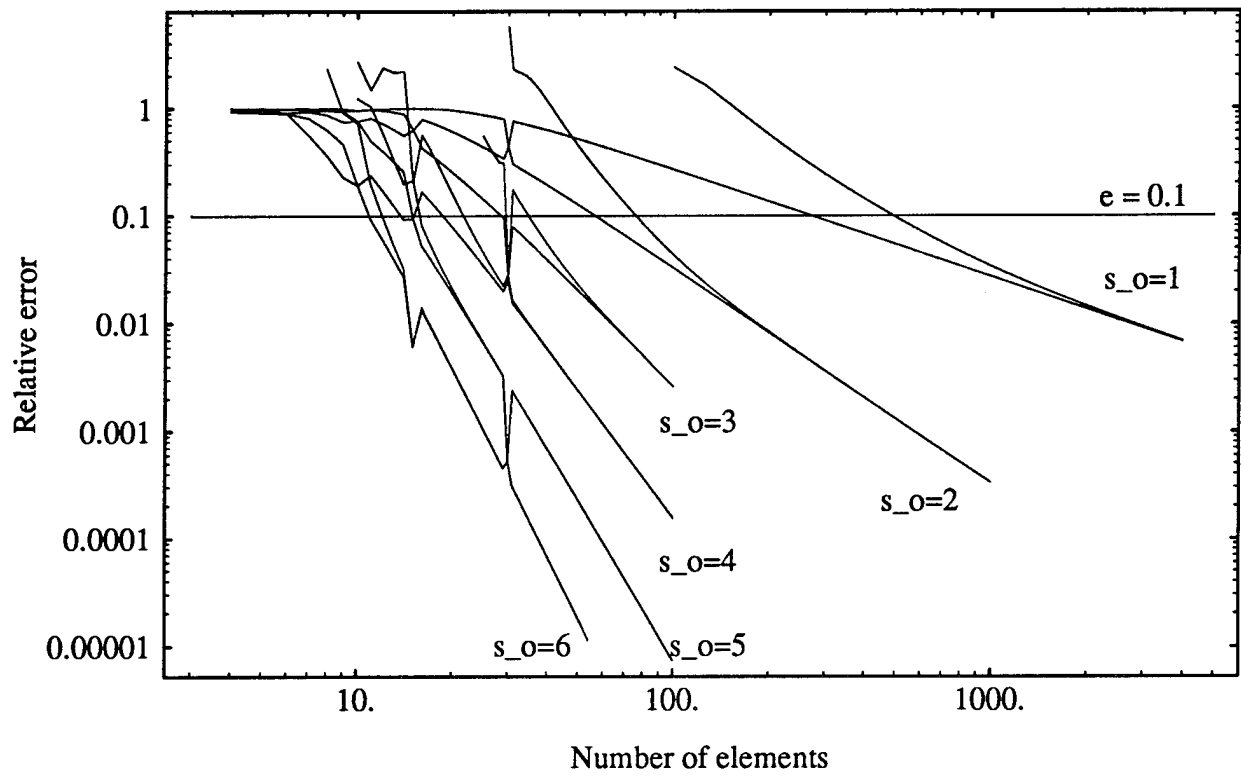


Figure 3: Relative error of the FE-solution versus error of H^1 -projection for $k = 50$ and $s_o = 1, 2, \dots, 6$.

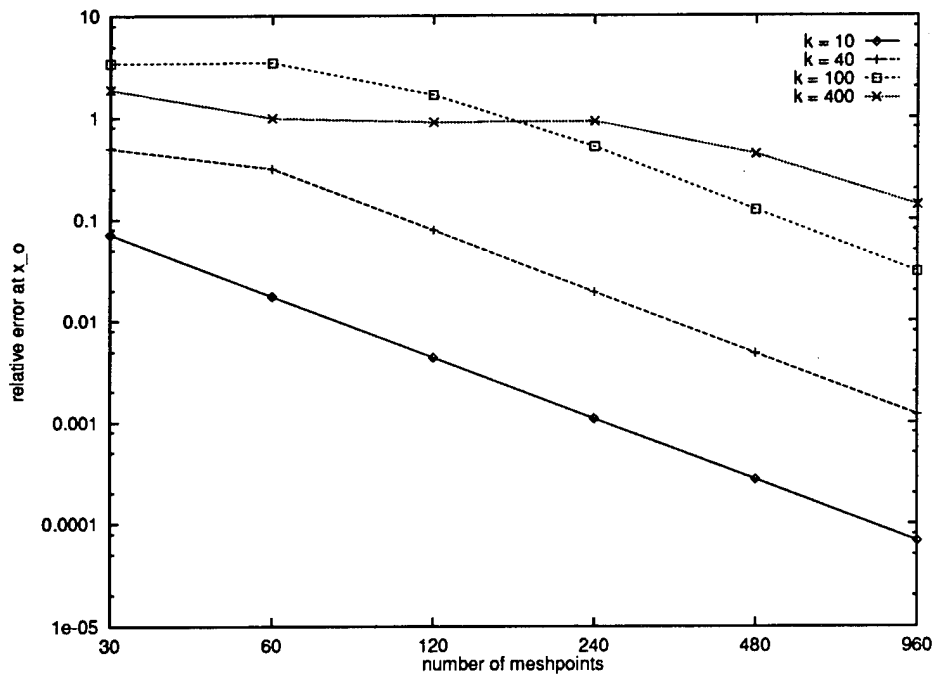


Figure 4: Relative error of the finite element solution $\tilde{e} = |u_{fe}(x_0) - u_{ex}(x_0)|/|u_{ex}(x_0)|$, for the coupled problem at $x_0 = 1.5$, for $k = 10, 40, 100, 400$. (a) $s_0 = 1$.

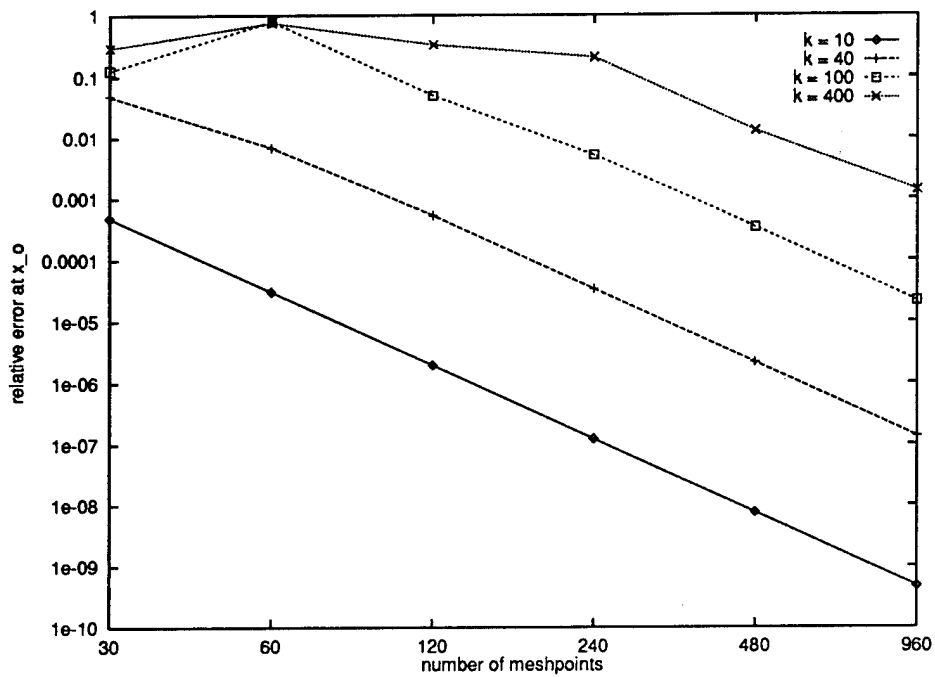


Figure 4: (b) $s_0 = 3$.

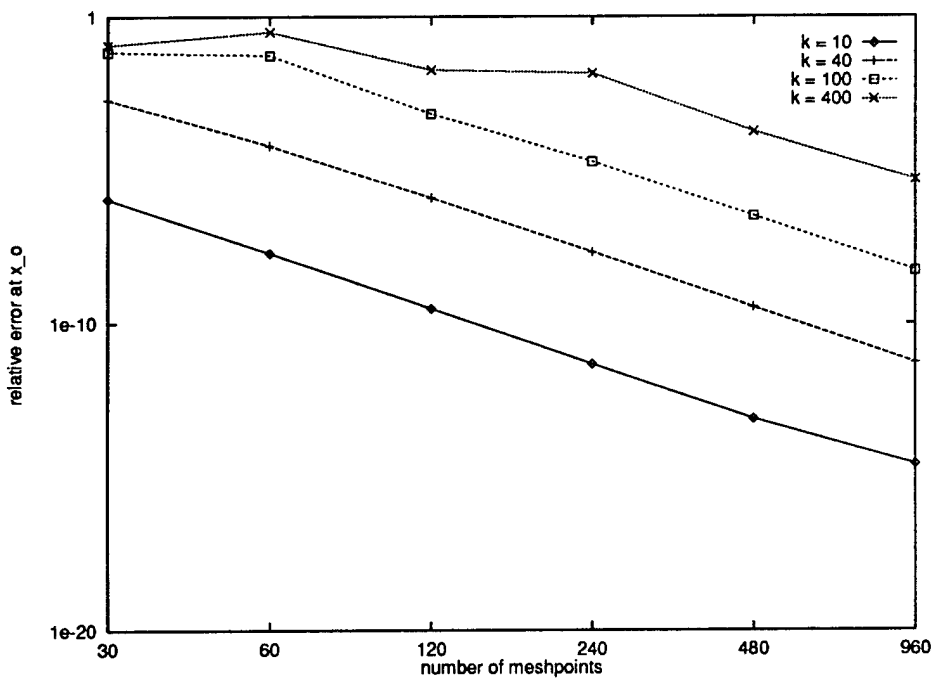


Figure 4: (c) $s_0 = 5$.

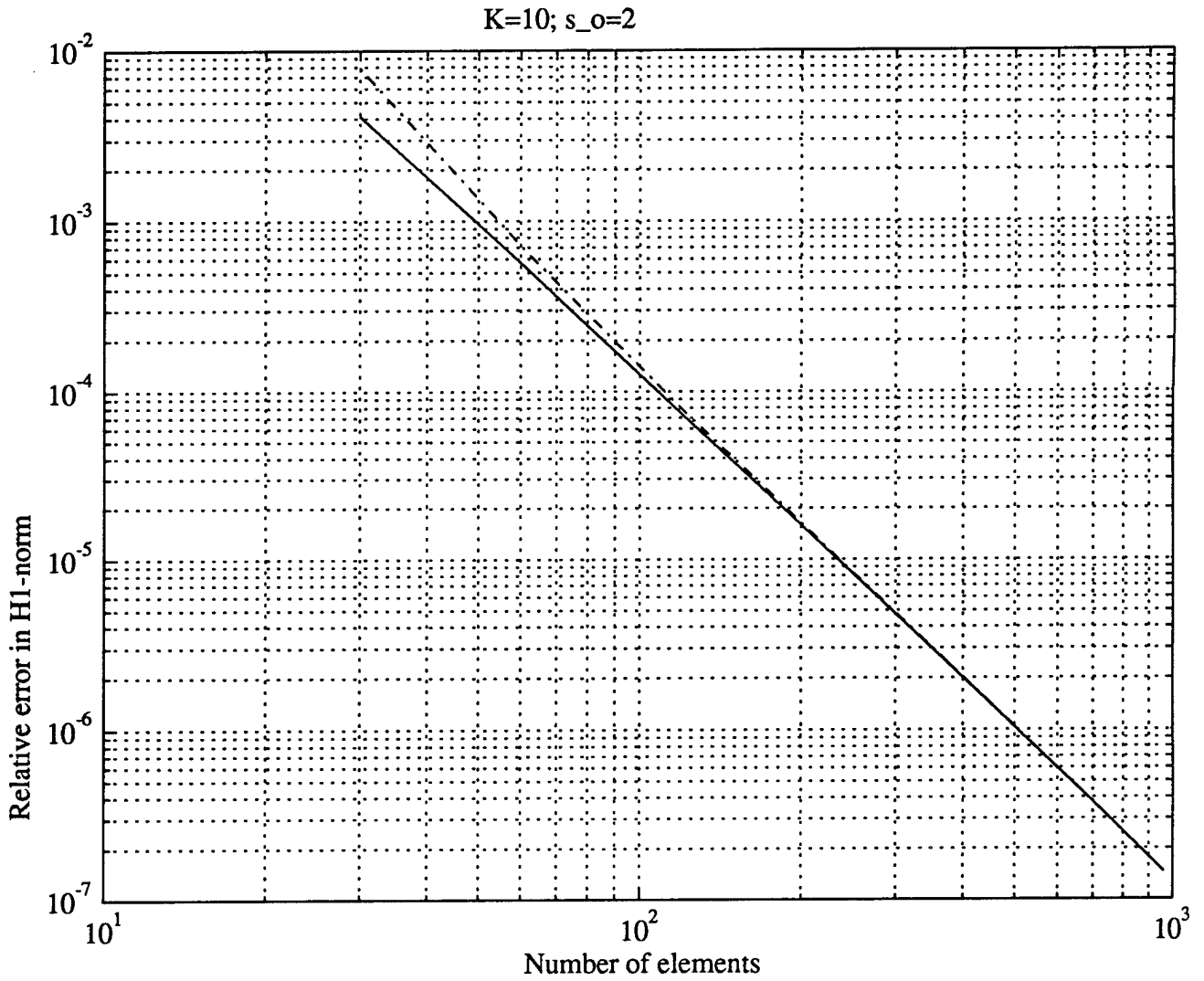


Figure 5: Pollution effect: Relative error of H^1 -projection and finite element solution for the coupled problem; error in H^1 -norm for $k = 10, 30, 100$ and $k = 500$. (a) $k = 10, s_o = 2$

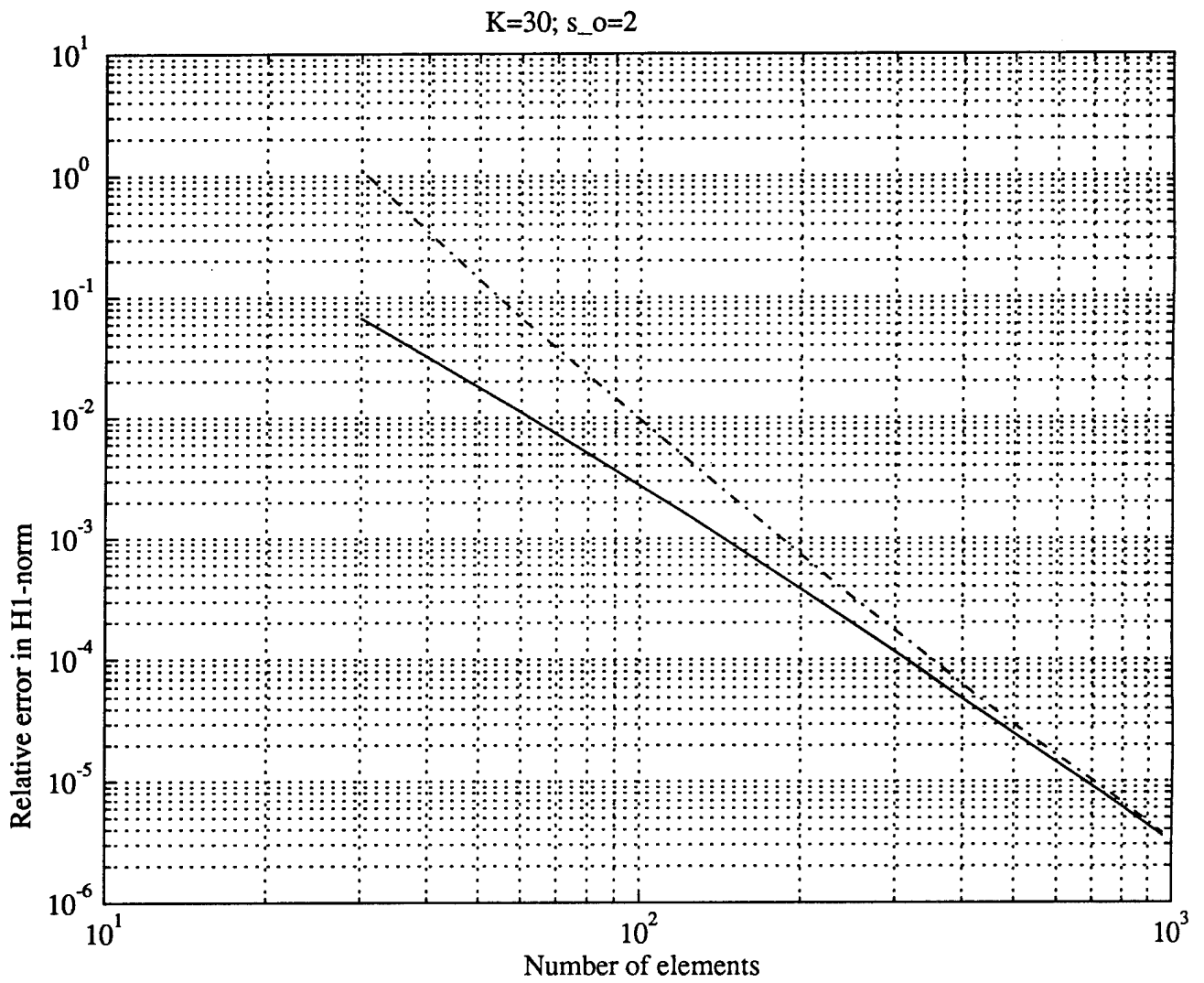


Figure 5: (b) $k = 30, s_o = 2$

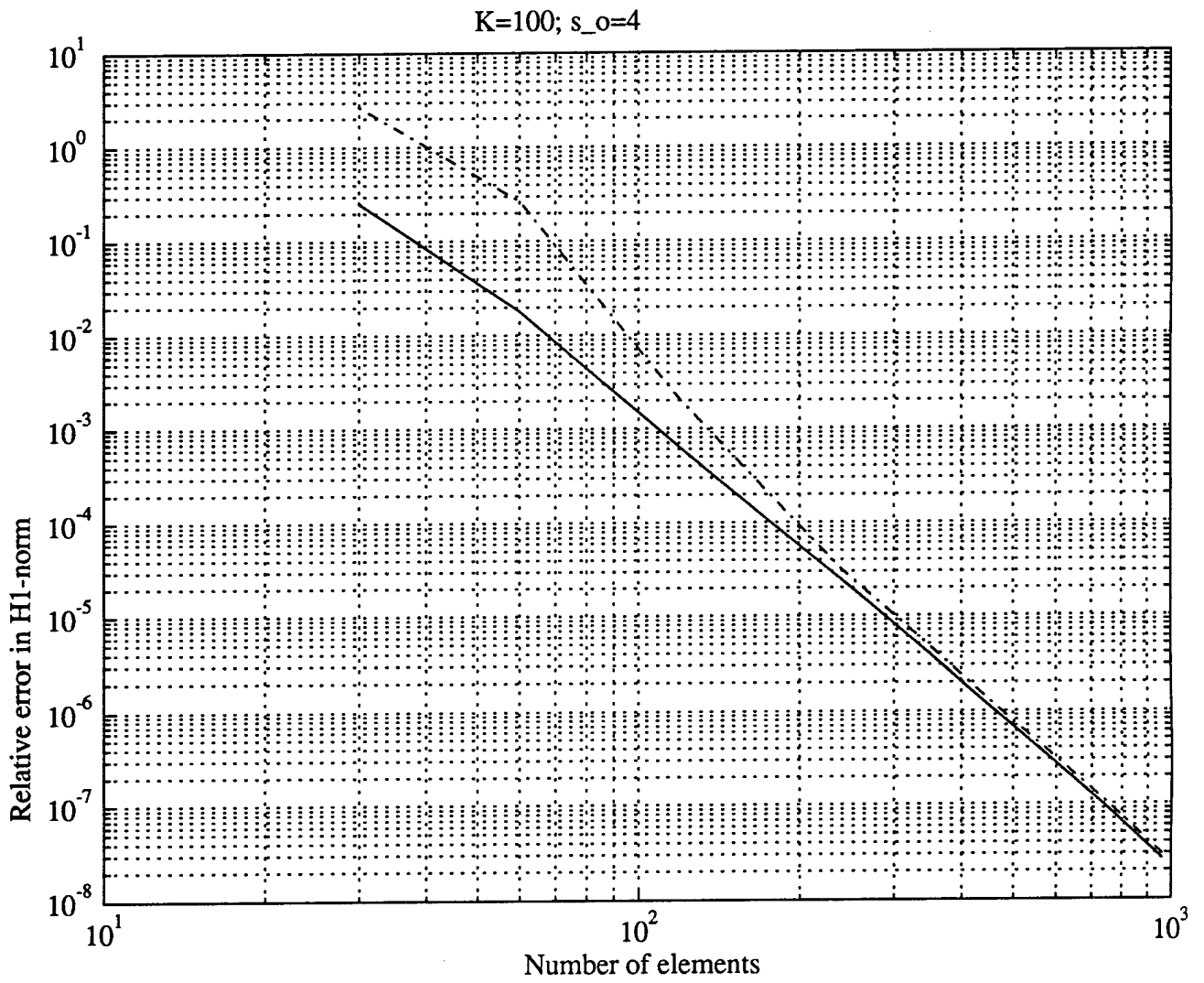


Figure 5: (c) $k = 100, s_o = 4$

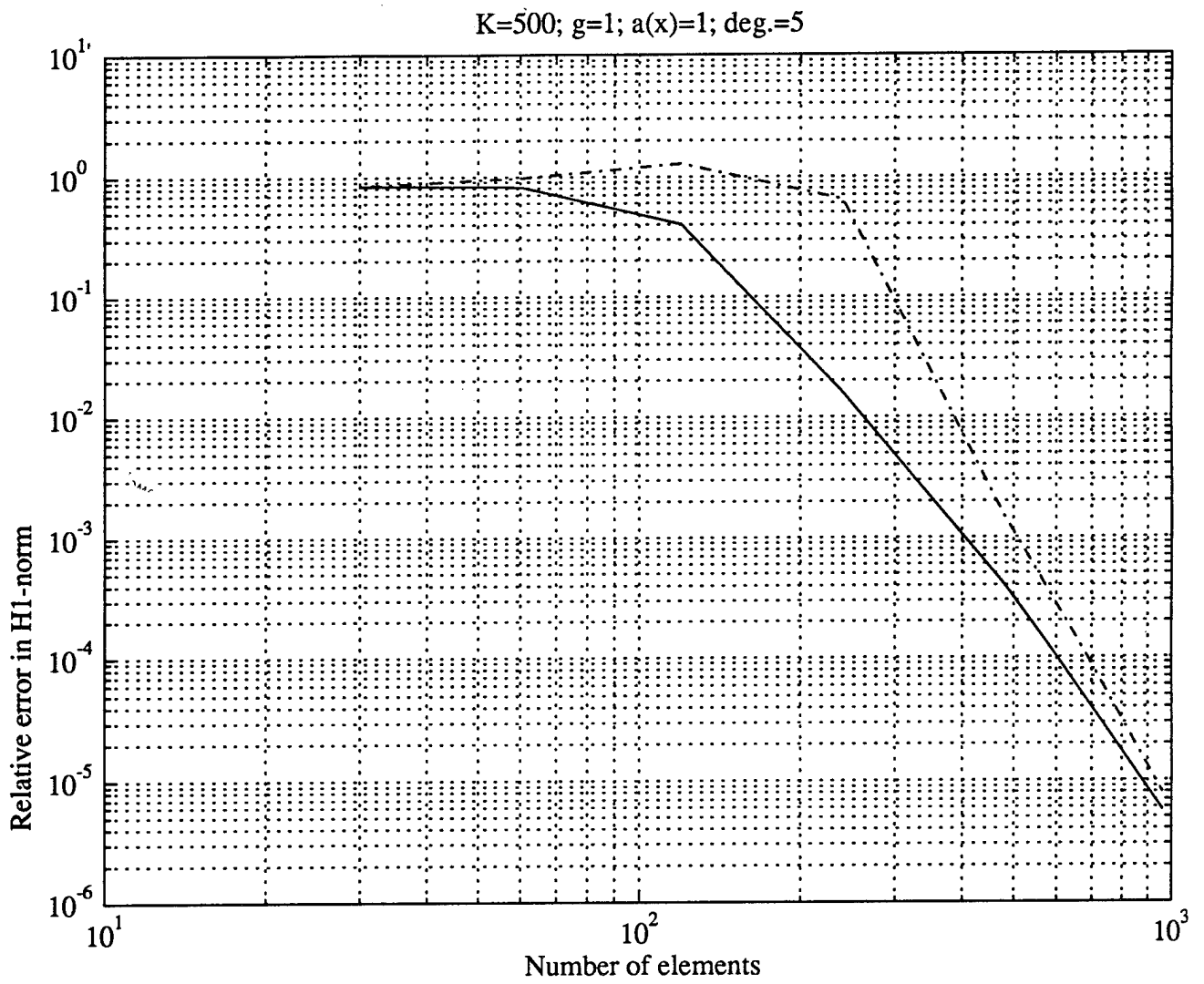


Figure 5: (d) $k = 500, s_o = 5$

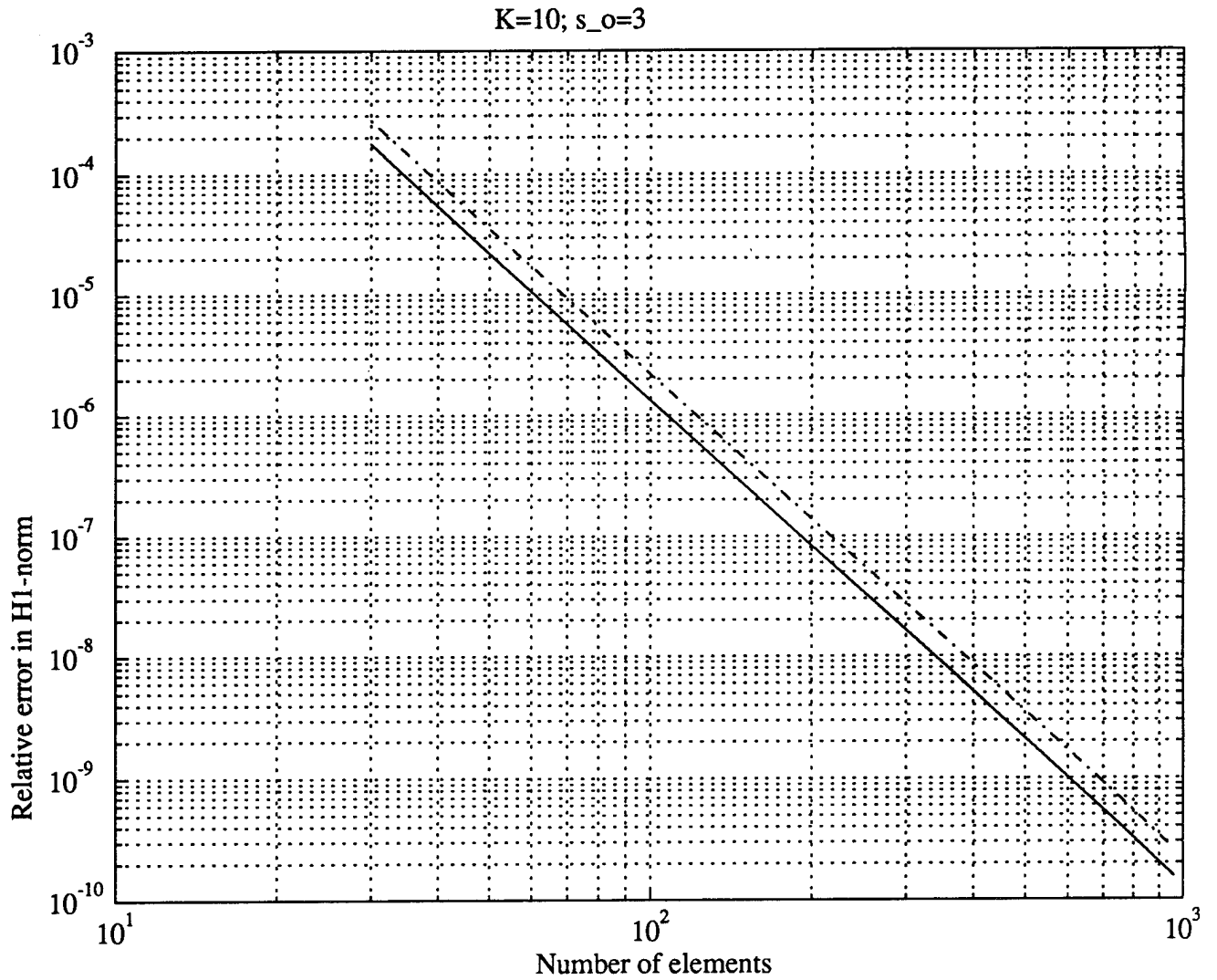


Figure 6: Asymptotic convergence: Relative error of the finite element solution for the coupled problem for $k = 10, s_o = 3$.

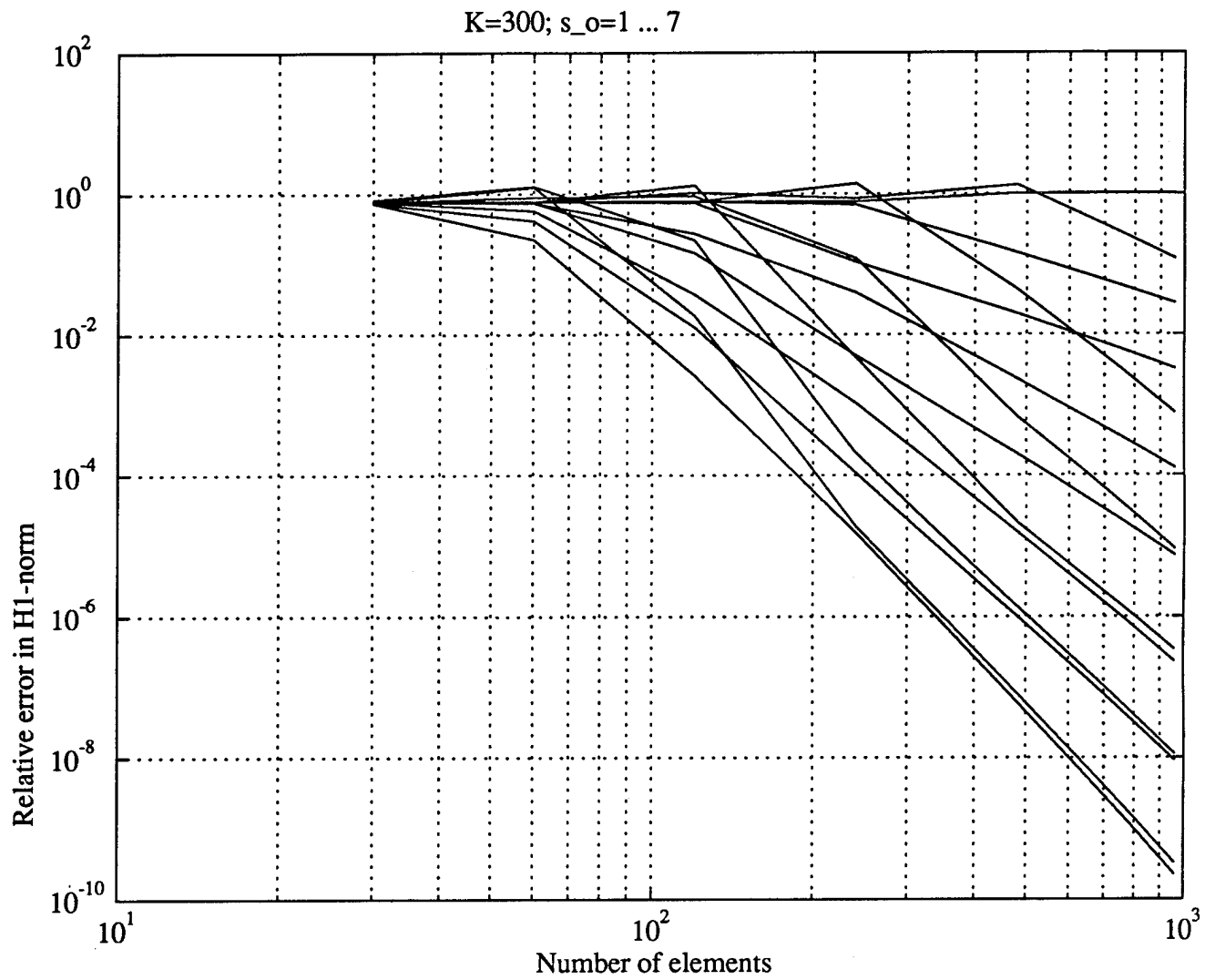


Figure 7: p -convergence: Relative error in H^1 -norm: error of the finite element solution vs. error of best approximation for $k = 300$ and $s_o = 1 \dots 7$

List of Figures

1	Errors of H^1 -projection versus error of finite element solution for Dirichlet problem; error in H^1 -seminorm; wavenumbers $k = 10$, $k = 50$ and $k = 100$.	19
2	Constraints of the form $k^2 h = \alpha$ for $\alpha = 1$ and $\alpha = 0.1$ in error plots for $k = 10, 50, 100$ and $k = 200$.	20
3	Relative error of the FE-solution versus error of H^1 -projection for $k = 50$ and $s_o = 1, 2, \dots, 6$.	21
4	Relative error of the finite element solution $\tilde{e} = u_{fe}(x_o) - u_{ex}(x_o) / u_{ex}(x_o) $, for the coupled problem at $x_o = 1.5$, for $k = 10, 40, 100, 400$. (a) $s_o = 1$.	22
4	(b) $s_o = 3$.	23
4	(c) $s_o = 5$.	23
5	Pollution effect: Relative error of H^1 -projection and finite element solution for the coupled problem; error in H^1 -norm for $k = 10, 30, 100$ and $k = 500$.	
	(a) $k = 10, s_o = 2$.	24
5	(b) $k = 30, s_o = 2$.	25
5	(c) $k = 100, s_o = 4$.	26
5	(d) $k = 500, s_o = 5$.	27
6	Asymptotic convergence: Relative error of the finite element solution for the coupled problem for $k = 10, s_o = 3$.	28
7	p -convergence: Relative error in H^1 -norm: error of the finite element solution vs. error of best approximation for $k = 300$ and $s_o = 1 \dots 7$.	29

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To conduct research in the mathematical theory and computational implementation of numerical analysis and related topics, with emphasis on the numerical treatment of linear and nonlinear differential equations and problems in linear and nonlinear algebra.

To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.

To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.

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