BD 14,039

MTP-71

PHYSICAL OPTICS INVERSE DIFFRACTION

R. M. LEWIS

THE MITRE CORPORATION

November 1967

AD 14,039

MTP-71

PHYSICAL OPTICS INVERSE DIFFRACTION

by

Robert M. Lewis

Accesion For	
NTIS CRA&I	☑
DTIC TAB	☐
Unannounced	☐
Justification	
By	
Distribution /	
Availability Codes	
Dist	Avail and / or Special
A-1	

November 12, 1967

THE MITRE CORPORATION
BOX 208
BEDFORD, MASSACHUSETTS

AnnoAo 5-96

5-22-68-dw
JB

This document has been released for public dissemination.

DS-1

Contract No. AF 19(628)-5165
Project 8051

Rec'd 5-6-68
The MITRE Corp.

ABSTRACT

A general method for solving the inverse diffraction problem is presented. It is based on an identity of Bojarski which states that $\gamma(\underline{x})$ and $\Gamma(\underline{p})$ are a Fourier transform pair. Here $\gamma(\underline{x})$ is the characteristic function of the target ($\gamma = 1$ inside the target, $\gamma = 0$ outside), $\underline{p} = \frac{2\omega}{c}\underline{J}$, ω is the frequency, $\underline{J}$ is a unit vector specifying the aspect, and $\Gamma(\underline{p})$ can be obtained by measurement of the backscattered electromagnetic far-field at frequency $\omega = \frac{c}{2}|\underline{p}|$ and aspect $\underline{J} = |\underline{p}|^{-1}\underline{p}$. If data is obtained in any subset D of $\underline{p}$-space, the method yields partial or complete information about the target geometry. It is used to rederive earlier results very simply, and to obtain a significant new solution, in which the target geometry is completely determined using frequencies only in a practical frequency band and aspects in a narrow cone.

TABLE OF CONTENTS

SECTION I

INTRODUCTION

The inverse diffraction problem requires the determination of the size and shape of a scattering target from given data. For example, the far-field backscattered at certain aspects and frequencies may be given. Any approach to the problem must be based on either the exact or some approximate theory of direct scattering.

As in other approaches to the inverse problem, the theory developed here will be based on the physical optics or Kirchhoff approximation [1] for direct scattering. Our starting point will be a remarkable identity recently obtained by N. Bojarski.[2] A derivation of the identity is given in Section II. There we introduce a three-dimensional space of vectors $\underline{p}$ whose direction coincides with the aspect direction and whose magnitude is $\underline{p} = |\underline{p}| = \frac{2\omega}{c}$ where ω is the frequency. We also introduce the characteristic function $\gamma(\underline{x})$ of the target and a function $\rho(\underline{p})$ which can be obtained by measuring the far-field backscattered in the direction of $\underline{p}$ at the frequency $\omega = \frac{c}{2}|\underline{p}|$. The function $\gamma(\underline{x})$ is 1 inside the target and 0 outside of it. Then Bojarski's identity states that the functions $\gamma(\underline{x})$ and $\Gamma(\underline{p}) = 2\sqrt{\pi}\,(p^2)^{-1}\left[\rho(\underline{p}) + \rho^*(-\underline{p})\right]$ are a Fourier transform pair. Thus if $\rho(\underline{p})$ could be measured in all of $\underline{p}$-space we could immediately obtain $\gamma(\underline{x})$ and hence the target.

In practice $\rho(\underline{p})$ and hence $\Gamma(\underline{p})$ can be measured only in a restricted range of aspects and frequencies, i.e., in a subset D of $\underline{p}$-space. In the general theory presented in Section III we show how these restricted scattering data can be used to obtain information about the geometry of the target. In some cases, even when D is a relatively small subset of $\underline{p}$-space, the target can be completely determined. In other cases, partial but often useful information, such as the width function, cross-sectional area function, volume, etc., can be obtained.

This suggests a possible reformulation of the inverse diffraction problem itself. In the usual formulations some data domain D is prescribed and one asks for proofs of the existence and uniqueness of a target that produces the specified data, as well as for constructive algorithms which yield the target geometry. In the reformulated version, for each prescribed data domain D, one asks for the maximum determination of geometric parameters that can be obtained and for algorithms for constructing these parameters. For relatively "small" domains D these parameters may not determine the target uniquely. In such cases we obtain only the class of targets (e.g., all targets with a given volume) with the specified parameters. For unique determination of the target, or for a given type of partial specification, one may also ask for the "minimum" or most "accessible" data domain D needed.

In Section IV we apply the general theory to four elementary examples. In these examples D is first all of $\underline{p}$-space, then a plane, a line, and finally a point in $\underline{p}$-space. To readers familiar with physical optics methods for the inverse diffraction problem the nature of the results will be familiar, but the ease and simplicity with which they are obtained demonstrate the power of the present method. In Section V we obtain a complete target specification using data in and infinitesimally near an infinite plane. Since this plane passes through the origin, it contains all frequencies, and all aspects in some "equatorial plane" of the target. Here, too, the results have been obtained earlier by a different method [3] but are now derived with great simplicity and elegance.

Existing radar techniques are restricted to a frequency band of moderate width. This band corresponds in $\underline{p}$-space to the annular region between two concentric spheres of minimum and maximum frequency. Even if this annular region could be greatly enlarged by future technological developments, there would remain a "forbidden" low-frequency sphere about the origin in which the physical optics approximation, which forms the basis of our theory, fails. It is clear that to obtain a practical method, the domain D must lie within the annular region. Those examples in Sections IV and V that yield useful target information all violate this requirement. Therefore in Section VI we present an application of the general theory which yields a complete determination of the target, in which the domain D lies entirely within the annular region. It consists of a disc-shaped portion of a plane which is tangent to the inner sphere and truncated by the outer sphere. Thus the frequencies lie within the usable band and the aspects lie within a cone of semi-angle θ. For a sufficiently large upper frequency limit, θ can be made arbitrarily small. Thus the results of Section VI might form the theoretical basis for a workable radar recognition system.

We note that since $\Gamma(\underline{p}) = 2\sqrt{\pi}(p^2)^{-1}\left[\rho(\underline{p}) + \rho^*(-\underline{p})\right]$, measurement of $\Gamma(\underline{p})$ in a domain D requires measurement of ρ in D and in D', which is the reflection of D through the origin. Thus if D corresponds to aspects near the "front" of the target, ρ must unfortunately be measured also near the "back." Since this would, in many applications, be a severe limitation, we have attempted to eliminate the domain D'. In Section VII we present an extension of our general theory which, under suitable conditions, yields the "front half" of the target using measurements of ρ in D. This extension can be utilized in connection with the method of Section VI. In Section VIII we derive a simplified version of our general theory suitable for targets which are bodies of revolution. By introducing cylindrical coordinates we can, in effect, reduce the dimensions of both $\underline{p}$-space and $\underline{x}$-space from three to two. Some elementary examples of this simplified theory are presented.

Although we have used the general theory to obtain old results easily, and have obtained one significantly new result in Section VI, we have not begun to exhaust the potential usefulness of the method. There is good reason to hope that other useful applications of the theory and/or generalizations of the method, either by the present author or by others, may be given in sequels to this paper.

SECTION II

BOJARSKI'S IDENTITY

The identity which forms the basis of our general theory is derived* from a representation for the electromagnetic far-field backscattered from a perfectly conducting target. We consider an incident-plane, time-harmonic, electromagnetic wave with electric vector.

$$\underline{E}_i\,(t\,,\,\underline{x}) \;=\; \underline{E}_o\; e^{ik(\underline{I}\,\cdot\,x\;+\;R)\,-i\omega t} \quad . \tag{1}$$

Here ω is the frequency and $k = \omega/c$ is the wave number. The constant R is the range, $\underline{I}$ is a unit vector in the direction of incidence, and $\underline{E}_o$ is a constant vector orthogonal to $\underline{I}$. According to the Kirchhoff, or physical optics approximation [see (3.41) of Reference 1] , the backscattered far-field at the point $\underline{x} = -R\underline{I}$ is given by

$$\underline{E}_s \;=\; \frac{e^{2ikR-i\omega t}}{2\sqrt{\pi}R}\;\rho\,\underline{E}_o \;, \tag{2}$$

where

$$\rho \;=\; \frac{-ik}{\sqrt{\pi}}\int_L \underline{I}\cdot\underline{n}\; e^{2ik\underline{I}\,\cdot\,\underline{x}}\, dS\,(\underline{x}) \quad . \tag{3}$$

The surface integral in Equation (3) is over the illuminated portion L of the target, and $\underline{n}$ is the outward unit vector normal to the surface. The power cross section σ is related to ρ by the equation,

$$\sigma \;=\; 4\pi\; R^2\; |\underline{E}_s|^2/|\underline{E}_i|^2 \;=\; |\rho|^2 \;. \tag{4}$$

We introduce the vector

$$\underline{p} = -2k\underline{I} \qquad \left(k = \frac{\omega}{c} \geq 0\right) \tag{5}$$

which is defined for non-negative frequencies ω. The magnitude of $\underline{p}$ is $p = 2k = 2\,\omega/c$, and its direction is that of the unit vector $\underline{J} = -\underline{I}$ directed from an origin in the target to the transmitter-receiver. From (3) we see that

$$\rho = \rho(\underline{p}) = \frac{i}{2\sqrt{\pi}} \int_{p \cdot n > 0} p \cdot n \, e^{-i\underline{p}\cdot\underline{x}} dS(\underline{x}) \ . \tag{6}$$

Here we have replaced the restriction to the illuminated region L by the condition $\underline{p} \cdot \underline{n} > 0$. The replacement is strictly valid for all $\underline{p}$ only if the target is convex. In general we introduce an additional approximation in replacing (3) by (6).

We now note that

$$\rho(\underline{p}) + \rho^*(-\underline{p}) = \frac{i}{2\sqrt{\pi}} \int_S \underline{p} \cdot \underline{n} \, e^{-i\underline{p}\cdot\underline{x}} dS \ . \tag{7}$$

Here the star denotes the complex conjugate, and the integral is over the entire surface S of the target B. It follows from the divergence theorem that

$$\rho(\underline{p}) + \rho^*(-\underline{p}) = \frac{i}{2\sqrt{\pi}} \int_B \nabla \cdot \left(\underline{p} e^{-i\underline{p}\cdot\underline{x}}\right) d\underline{x} = \frac{p^2}{2\sqrt{\pi}} \int_B e^{-i\underline{p}\cdot\underline{x}} d\underline{x} \ . \tag{8}$$

We introduce the characteristic function of B defined by

$$\gamma(\underline{x}) = \begin{Bmatrix} 1 \ , & \underline{x} \text{ in B} \\ \\ 0 \ , & \underline{x} \text{ not in B} \end{Bmatrix} . \tag{9}$$

Then

$$\Gamma(\underline{p}) = 2\sqrt{\pi} \frac{\rho(\underline{p}) + \rho^{*}(-\underline{p})}{p^{2}} \int_{B} e^{-i\underline{p} \cdot \underline{x}} d\underline{x} = \int \gamma(\underline{x}) e^{-i\underline{p} \cdot \underline{x}} d\underline{x} . \tag{10}$$

This identity was first derived by Bojarski. [2] It shows that $\Gamma(\underline{p})$, which is defined in terms of $\rho(\underline{p})$, is the Fourier transform of the characteristic function $\gamma(\underline{x})$. It follows that

$$\gamma(\underline{x}) = (2\pi)^{-3} \int \Gamma(\underline{p}) e^{i\underline{p} \cdot \underline{x}} d\underline{x} . \tag{11}$$

If the back-scattered field could be measured at all frequencies and all aspects $\underline{I}$, then $\rho(\underline{p})$ would be known for all $\underline{p}$, and (11) would yield the full solution of the inverse diffraction problem, i.e., the size and shape of B. In general $\rho(\underline{p})$ is measurable only for restricted frequencies and aspects, i.e., in a restricted portion of $\underline{p}$-space. The general theory presented in the next section, and illustrated in succeeding sections, shows how the restricted information about $\rho(\underline{p})$ can be used to yield complete or partial information about the target.

SECTION III

THE GENERAL THEORY

Suppose $\Gamma(\underline{p})$ can be obtained [by measurement of $\rho(\underline{p})$] in some portion D of $\underline{p}$-space. A convenient way of stating this is that we can measure the function $K(\underline{p})\,\Gamma(\underline{p})$ where $K(\underline{p})$ is the characteristic function of D (i.e., K = 1 in D, K = 0 outside D). Alternatively, $K(\underline{p})$ could be any function which is non-zero in D and zero outside D. We introduce the transform pair

$$\mathcal{K}(\underline{x}) = (2\pi)^{-3} \int K(\underline{p})\, e^{i\underline{p}\cdot\underline{x}}\, d\underline{p} \quad , \quad K(p) = \int \mathcal{K}(\underline{x}) e^{-i\underline{p}\cdot\underline{x}}\, d\underline{x} \ . \tag{12}$$

Since $K(\underline{p})\ \Gamma(\underline{p})$ can be measured, we can construct the function

$$f(\underline{x}) = (2\pi)^{-3} \int K(\underline{p})\ \Gamma\ (\underline{p})\ e^{i\underline{p}\cdot\underline{x}}\, d\underline{p}$$

$$= 2^{-2} \pi^{-5/2} \int \frac{K(\underline{p})}{p^2} \left[\rho(\underline{p}) + \rho^*(-\underline{p}) \right] e^{i\underline{p}\cdot\underline{x}}\, d\underline{p} \ . \tag{13}$$

On the other hand, it follows from the convolution theorem for Fourier transforms that

$$f(x) = \int \gamma(\underline{x}')\, \mathcal{K}(\underline{x} - \underline{x}')\, d\underline{x}' = \int \mathcal{K}(\underline{x}')\, \gamma(\underline{x} - \underline{x}')\, d\underline{x}' \ . \tag{14}$$

Since f can be constructed from measurement and $\mathcal{K}$ is known from (12), (14) is an equation for $\gamma(\underline{x})$. We shall see that in some cases it can be solved to determine $\gamma(\underline{x})$, i.e., the target B. In other cases it yields partial information about B. It is not surprising that (14) can sometimes be fully solved even when K vanishes in much of $\underline{p}$-space, because we know in advance that γ is a characteristic function. In fact, since γ is completely determined by a two-dimensional manifold, i.e., the surface of B, we might expect that (14) can be solved even when $\Gamma(\underline{p})$ can be measured only on a surface, i.e., a two-dimensional manifold in $\underline{p}$-space. This expectation will be confirmed in Sections V and VI.

It is interesting to note that if

$$K(-\underline{p}) = K^{*}(\underline{p}) \tag{15}$$

then $\mathcal{K}(\underline{x})$ is real and

$$f(\underline{x}) = 2^{-2}\pi^{-5/2}\left\{\int \frac{K(\underline{p})}{p^2}\,\rho(\underline{p})e^{i\underline{p}\cdot\underline{x}}\,d\underline{p} + \int \frac{K^{*}(\underline{p})}{p^2}\,\rho^{*}(\underline{p})e^{-i\underline{p}\cdot\underline{x}}\,d\underline{p}\right\}$$

$$= 2^{-1}\pi^{-5/2}\,\mathcal{R}e\int \frac{K(\underline{p})\rho(\underline{p})}{p^2}\,e^{i\underline{p}\cdot\underline{x}}\,d\underline{p}\ . \tag{16}$$

Then the determination of $\mathcal{K}$ and f is somewhat simplified.

In general we proceed by choosing a domain D in $\underline{p}$-space and a convenient function $K(\underline{p})$ which is non-zero only in D. Then we determine $\mathcal{K}$ from (12) and f (in terms of ρ) from (13). Finally, we use (14) and the fact that γ is a characteristic function to obtain a partial or complete determination of B. The procedure will be illustrated in Sections IV, V, and VI. It should be emphasized that those cases that yield only a partial determination of B should not be considered a failure to solve the inverse diffraction problem. Rather, it might be considered remarkable that partial, and often useful, information can be obtained from inadequate data.

SECTION IV

ELEMENTARY EXAMPLES

In this section we will apply the method of Section III to several simple cases. The results are not of practical importance, primarily because the domain D includes low-frequency regions where physical optics fails and both high and low-frequency regions where ρ cannot be measured by existing techniques. Nevertheless, the examples are useful in clarifying the application of the theory. In each case we give first the domain D and $K(\underline{p})$. Then we determine $\mathcal{K}(\underline{x})$ from (12) and $f(\underline{x})$ in terms of ρ from (13) or (16) when (15) is satisfied. Finally we evaluate the right side of (14) and discuss the resulting information about the target B.

Example 1:

$$\text{D: all of } \underline{p}\text{-space; } K(\underline{p}) \equiv 1; \ \mathcal{K}(\underline{x}) = \delta(\underline{x}) \ , \tag{17}$$

$$f(\underline{x}) = 2^{-1}\pi^{-5/2}\, \mathcal{R}e \int \frac{\rho(\underline{p})}{p^2}\, e^{i\underline{p}\cdot\underline{x}}\, d\underline{p} \ , \tag{18}$$

$$f(\underline{x}) = \int \gamma(\underline{x}')\, \delta(\underline{x} - \underline{x}')\, d\underline{x}' = \gamma(\underline{x}) \ . \tag{19}$$

Here $\gamma(\underline{x})$ is completely determined by eliminating f in (19) and (18).

Example 2:

$$\text{D: the plane } p_1 = 0\,; \ K(\underline{p}) = \delta(p_1) \tag{20}$$

$$\mathcal{K}(\underline{x}) = (2\pi)^{-3} \int e^{i(p_2x_2 + p_3x_3)}\, dp_2dp_3 = (2\pi)^{-1}\, \delta(x_2)\, \delta(x_3) \ . \tag{21}$$

$$f(\underline{x}) = 2^{-1}\pi^{-5/2}\,\Re e \int \frac{\rho\left(0,\, p_2\,,\, p_3\right)}{p_2^{\,2} + p_3^{\,2}}\, e^{i\left(p_2 x_2 \,+\, p_3 x_3\right)}\, dp_2\, dp_3\;, \qquad (22)$$

$$f(\underline{x}) = (2\pi)^{-1} \int \gamma\,(\underline{x}')\,\delta\left(x_2 - x_2'\right)\delta\left(x_3 - x_3'\right) d\underline{x}'$$

$$= (2\pi)^{-1} \int \gamma\left(x_1'\,,\, x_2\,,\, x_3\right) dx_1' \;=\; (2\pi)^{-1}\, w\left(x_2\,,\, x_3\right). \qquad (23)$$

Here $w\left(x_2\,,\, x_3\right)$ denotes the "width" of the target at the point $\left(x_2\,,\, x_3\right)$, i.e., the length of that part of the line parallel to the x_1-axis and passing through the point $\left(0,\, x_2\,,\, x_3\right)$ which is contained in the target. Thus a partial description of B is given by (23) and (22). The description is incomplete because many different targets can have the same width-function w.

Example 3:

$$\text{D: the line } p_2 = p_3 = 0; \quad K(\underline{p}) = \delta\left(p_2\right)\delta\left(p_3\right); \qquad (24)$$

$$\mathcal{K}(\underline{x}) = (2\pi)^{-3} \int e^{ip_1 x_1}\, dp_1 = (2\pi)^{-2}\, \delta\left(x_1\right)\;, \qquad (25)$$

$$f(\underline{x}) = 2^{-1}\pi^{-5/2}\,\Re e \int \frac{\rho\left(p_1\,,\, 0\,,\, 0\right)}{p_1^{\,2}}\, e^{ip_1 x_1}\, dp_1\;, \qquad (26)$$

$$f(\underline{x}) = (2\pi)^{-2} \int \gamma\,(\underline{x}')\,\delta\left(x_1 - x_1'\right) d\underline{x}'$$

$$= (2\pi)^{-2} \int \gamma\left(x_1\,,\, x_2'\,,\, x_3'\right) dx_2'\, dx_3'$$

$$= a\left(x_1\right)\,. \qquad (27)$$

Here $a(x_1)$ denotes the cross-sectional area of the target, i.e., the area of that part of the plane $x_1 =$ constant which is contained in the target. Thus (26) and (27) yield a partial description of B.

Example 4:

$$D: \text{ the point } \underline{p} = (p_1, p_2, p_3) = (p_0, 0, 0);$$

$$K(\underline{p}) = \delta(p_2)\,\delta(p_3)\,\delta(p_1 - p_0); \tag{28}$$

$$\mathcal{K}(\underline{x}) = (2\pi)^{-3} e^{ip_0 x_1}, \tag{29}$$

$$f(\underline{x}) = 2^{-2}\pi^{-5/2} \frac{\rho(p_0, 0, 0) + \rho^*(-p_0, 0, 0)}{p_0^2} e^{ip_0 x_1}, \tag{30}$$

$$f(\underline{x}) = (2\pi)^{-3} \int \gamma(\underline{x}')\, e^{ip_0(x_1 - x_1')} d\underline{x}'$$

$$= (2\pi)^{-3} e^{ip_0 x_1} \int a(x_1') e^{ip_0 x_1'} dx_1'. \tag{31}$$

From (30) and (31) we obtain

$$\int a(x_1)\, e^{-ip_0 x_1} dx_1 = 2\pi^{1/2} \frac{\rho(p_0, 0, 0) + \rho^*(-p_0, 0, 0)}{p_0^2}. \tag{32}$$

In this case the description obtained is only an integral of the area function $a(x_1)$ and thus yields very little information. For $p_0 = 0$ the left side of (32) becomes the volume of the target, but the right side is indeterminate. However, the result is of little interest because of the failure of the physical optics approximation at low frequencies.

SECTION V

THE FULL-FREQUENCY PLANE

As in Example 2 of the preceding section, we take D to be the plane $p_1 = 0$, but now we choose

$$K(\underline{p}) = i\delta'(p_1) \quad . \tag{33}$$

Then

$$\mathcal{K}(\underline{x}) = i(2\pi)^{-3}\int \delta'(p_1)\, e^{i\underline{p}\cdot\underline{x}} d\underline{p} = i(2\pi)^{-1}\,\delta(x_2)\,\delta(x_3)\int \delta'(p_1)\, e^{ip_1x_1} dp_1$$

$$= (2\pi)^{-1}\, x_1\, \delta(x_2)\, \delta(x_3) \quad . \tag{34}$$

Since (15) is satisfied we may use (16) to determine f . Thus

$$f(\underline{x}) = 2^{-1}\pi^{-5/2}\,\Re e\int i\delta'(p_1)\,\frac{\rho(\underline{p})}{p^2}\, e^{i\underline{p}\cdot\underline{x}}\, d\underline{p}$$

$$= 2^{-1}\pi^{-5/2}\,\Re e\int -i\,\delta'(p_1)\, e^{i\underline{p}\cdot\underline{x}}\left[ix_1\,\frac{\rho}{p^2} + \lambda(\underline{p})\right] d\underline{p} \tag{35}$$

where

$$\lambda(\underline{p}) = \frac{\partial}{\partial p_1}\,\frac{\rho(\underline{p})}{p^2} \quad . \tag{36}$$

It follows that

$$f(\underline{x}) = x_1\alpha(x_2\, ,\, x_3) + \beta(x_2\, ,\, x_3) \tag{37}$$

where

$$\alpha(x_2, x_3) = 2^{-1}\pi^{-5/2}\,\mathrm{Re}\int \frac{\rho(0, p_2, p_3)}{p_2^2 + p_3^2}\, e^{i(p_2x_2 + p_3x_3)}\, dp_2 dp_3 \tag{38}$$

and

$$\beta(x_2, x_3) = 2^{-1}\pi^{-5/2}\,\mathrm{Im}\int \lambda(0, p_2, p_3)\, e^{i(p_2x_2 + p_3x_3)}\, dp_2 dp_3 \ . \tag{39}$$

By inserting (34) in (14) we obtain

$$f(\underline{x}) = (2\pi)^{-1}\int \gamma(\underline{x}')\,(x_1' - x_1)\,\delta(x_2' - x_2)\,\delta(x_3' - x_3)\, d\underline{x}'$$

$$= (2\pi)^{-1}\int \gamma(x_1', x_2, x_3)\,(x_1 - x_1')\, dx_1' \ . \tag{40}$$

It follows that (37) is satisfied with

$$\alpha(x_2, x_3) = (2\pi)^{-1}\left[y_2(x_2, x_3) - y_1(x_2, x_3)\right] \tag{41}$$

and

$$\beta(x_2, x_3) = -(2\pi)^{-1}\int_{y_1}^{y_2} x_1' dx_1' = -(4\pi)^{-1}\left[y_2^2(x_2, x_3) - y_1^2(x_2, x_3)\right]. \tag{42}$$

Here $x_1 = y_1(x_2, x_3)$ and $x_1 = y_2(x_2, x_3)$ are the functions that define the two surfaces enclosing B as illustrated in Figure 1.

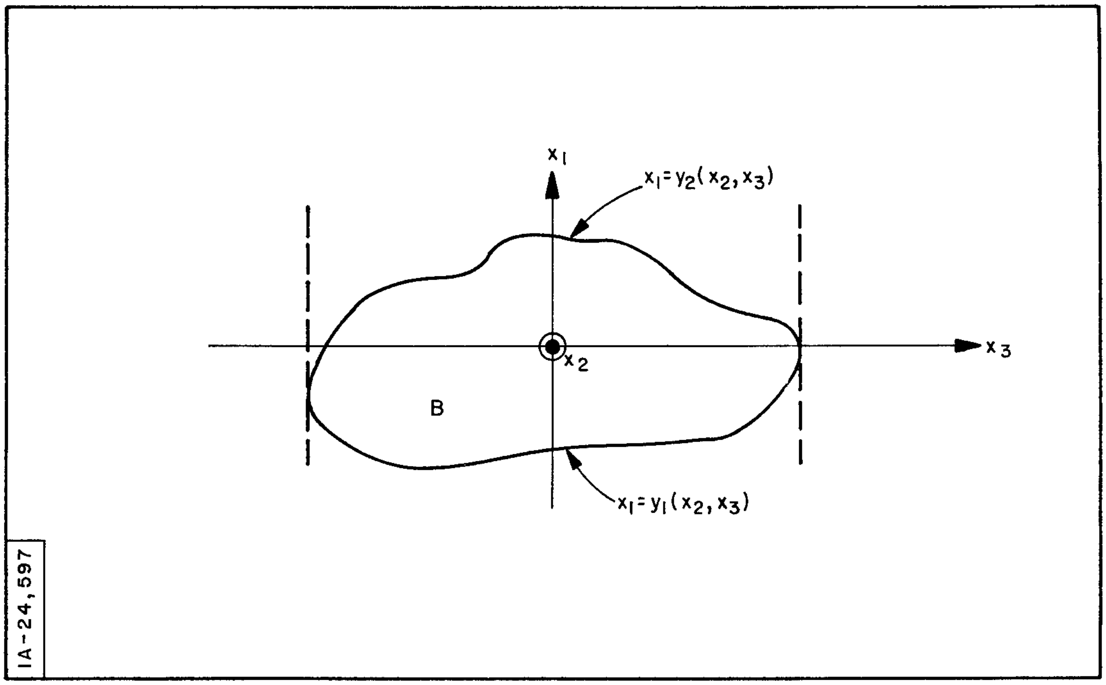

Figure 1. Functional Description of Body Geometry

We have tacitly assumed that each line parallel to the x_1-axis cuts the surface of the target at most twice. Since $y_2^2 - y_1^2 = (y_2 - y_1)(y_2 + y_1)$, it follows that $y_2 + y_1 = -2\beta/\alpha$ and $y_2 - y_1 = 2\pi\alpha$. These equations are easily solved to yield

$$y_1 = -\frac{\pi\alpha^2 + \beta}{\alpha}, \quad y_2 = \frac{\pi\alpha^2 - \beta}{\alpha} . \tag{43}$$

Since α and β are given by (41), (42) and (36) in terms of $\rho(0, p_2, p_3)$ and $\frac{\partial}{\partial p_1}\rho(0, p_2, p_3)$ we have obtained a complete determination of the target geometry. We note that the width function is given by

$$w(x_1, x_3) = y_2 - y_1 = 2\pi\alpha$$

$$= \pi^{-3/2}\, \Re e \int \frac{\rho(p_2, p_3)}{p_2^{\,2} + p_3^{\,2}}\, e^{i(p_2x_2 + p_3x_3)} dp_2dp_3 \ . \qquad (44)$$

This result agrees exactly with the result (22), (23) obtained by a different method.

Our complete solution (43) requires a knowledge of ρ and $\frac{\partial\rho}{\partial p_1}$ in the "equatorial plane" $p_1 = 0$. Thus all frequencies and all aspects in this plane are required. To evaluate $\frac{\partial\rho}{\partial p_1}(0, p_2, p_3)$ we also need aspects infinitesimally near the equatorial plane. This result is very closely related to that obtained in Reference 3 by an entirely different method. Since all frequencies are required we refer to the plane $p_1 = 0$ as a "full-frequency plane".

SECTION VI

THE BAND-LIMITED PLANE

Let us recall that the magnitude p of the vector $\underline{p}$ is $p = \frac{2\omega}{c}$, where ω is the frequency and c is the speed of light. Thus points near the origin in $\underline{p}$-space correspond to low frequencies, and distant points correspond to high frequencies. The frequency is constant on spheres centered at the origin. In most practical radar applications the usable frequency band is limited by minimum and maximum values ω_1 and ω_2. If we set

$$m = \frac{2\omega_1}{c} \ , \quad M = \frac{2\omega_2}{c} \ , \tag{45}$$

then the usable portion of $\underline{p}$-space corresponds to the annular region $m \le p \le M$ between the two spheres illustrated in Figure 2. It is important to note that the frequency limitation is due not only to technological factors but to the failure of the physical optics approximation, on which our theory is based, at the low frequencies. Therefore, in applying the general theory of Section III the region D should certainly not intersect a "forbidden sphere" about the origin in which the physical optics approximation fails. With the exception of example 4 of Section IV, all of the cases examined so far have violated this criterion, and the exceptional case yielded very little information about the target. The "full-frequency plane" utilized in Section V passes through the origin and thus intersects the forbidden sphere.

In this section we will take D to be the disc consisting of that part of the plane $p_1 = m$ which lies within the outer sphere $p = M$ of Figure 2. The disc is illustrated by the heavy line in Figure 2. Its radius is

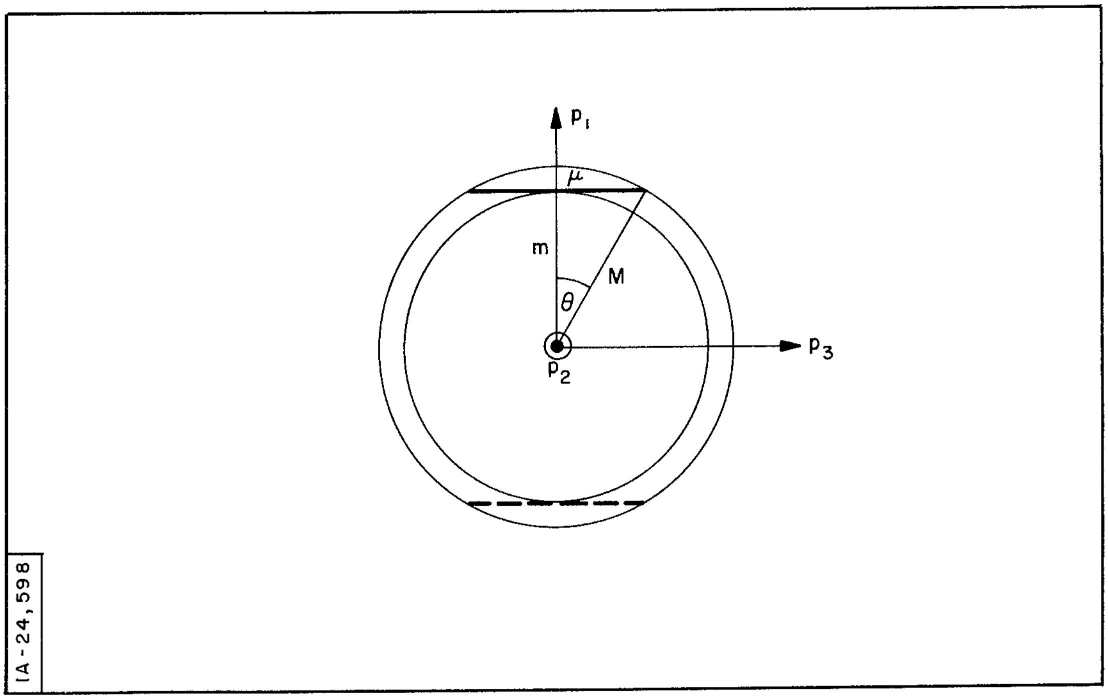

Figure 2. Usable Portions of p-Space

$$\mu = \sqrt{M^2 - m^2} \quad . \tag{46}$$

Thus the region D not only avoids the forbidden low-frequency sphere but lies entirely within the usable frequency band. The function $K(\underline{p})$, which must vanish outside D , is taken to be

$$K(\underline{p}) = \left\{ \begin{array}{ll} \delta(p_1 - m) & , \; p^2 < M^2 \\ 0 & , \; p^2 > M^2 \end{array} \right\} . \tag{47}$$

Then, from (12),

$$\mathbb{K}(\underline{x}) = (2\pi)^{-3} \int_{p^2 < M^2} \delta\left(p_1 - m\right) e^{i\underline{p} \cdot \underline{x}} d\underline{p} = (2\pi)^{-1} e^{imx_1} \delta_\mu\left(x_2 , x_3\right), \tag{48}$$

where

$$\delta_\mu\left(x_2 , x_3\right) = (2\pi)^{-2} \int_{p_2^2 + p_3^2 < \mu^2} e^{i\left(p_2 x_2 + p_3 x_3\right)} dp_2 dp_3 . \tag{49}$$

We note that for $M \to \infty$, μ approaches infinity and δ_μ approaches the two-dimensional delta function,

$$\delta\left(x_2 , x_3\right) = \delta\left(x_2\right) \delta\left(x_3\right) . \tag{50}$$

Thus δ_μ is an approximation to (50). In fact, if we set

$$\left(p_2 , p_3\right) = b(\cos\beta , \sin\beta) , \left(x_2 , x_3\right) = r(\cos\alpha , \sin\alpha) , \beta' = \beta - \alpha , \tag{51}$$

we see that

$$\delta_\mu\left(x_2 , x_3\right) = (2\pi)^{-2} \int_0^{2\pi} d\beta' \int_0^{\mu} b db e^{ibr \cos\beta'}$$

$$= (2\pi)^{-1} \int_0^{\mu} b J_0 (rb) db = \left(2\pi r^2\right)^{-1} \int_0^{r\mu} z J_0(z) dz$$

$$= \frac{\mu}{2\pi r} J_1 (\mu r); \quad r = \sqrt{x_2^2 + x_3^2} . \tag{52}$$

Here J_0 and J_1 are the Bessel functions of order zero and one, and we have used the identity

$$\frac{d}{dz}\left[z\, J_1(z)\right] = z\, J_0(z) \ . \tag{53}$$

From (52) it is easy to show that $\delta_\mu(0\,,\,0) = \frac{\mu^2}{4\pi}$, and the width of the large central lobe is of order $\frac{1}{\mu}$. Thus δ_μ will be a good approximation to the delta-function (50) provided

$$\frac{1}{\mu} \ll L \ , \tag{54}$$

where L is a typical target dimension.

Assuming that (54) is satisfied, we may approximate (48) by

$$\mathcal{K}(\underline{x}) = (2\pi)^{-1}\, e^{imx_1}\, \delta\left(x_2\right)\, \delta\left(x_3\right). \tag{55}$$

Proceeding with the method of Section III we find from (13) that

$$f(\underline{x}) = 2^{-2}\pi^{-5/2} \int_{p^2 < M^2} \delta\left(p_1 - m\right) \frac{\rho(\underline{p}) + \rho^*(-\underline{p})}{p^2}\, e^{i\underline{p}\cdot\underline{x}}\, d\underline{p}$$

$$= (\pi m)^{-1}\, e^{imx_1}\, F\left(m\,,\, x_2\,,\, x_3\right) \ , \tag{56}$$

where

$$F\left(m\,,\, x_2\,,\, x_3\right) = 2^{-2}\pi^{-3/2} m \int\limits_{p_2^2 + p_3^2 < M^2} \frac{\rho\left(m\,,\, p_2\,,\, p_3\right) + \rho^*\left(-m\,,\, p_2\,,\, p_3\right)}{m^2 + p_2^2 + p_3^2}$$

$$\times\ e^{i\left(p_2 x_2 \,+\, p_3 x_3\right)}\, dp_2 dp_3\ . \tag{57}$$

By inserting (55) in (14) we obtain

$$f(\underline{x}) = (2\pi)^{-1} \int \gamma\,(\underline{x}')e^{im\left(x_1 \,-\, x_1'\right)}\, \delta\left(x_2 - x_2'\right)\, \delta\left(x_3 - x_3'\right)\, d\underline{x}'$$

$$= (2\pi)^{-1}\, e^{imx_1} \int \gamma\left(x_1'\,,\, x_2\,,\, x_3\right)\, e^{-imx_1'}\, dx_1'\ . \tag{58}$$

We again introduce the functions $y_1\left(x_2\,,\, x_3\right)$ and $y_2\left(x_2\,,\, x_3\right)$ illustrated in Figure 1. It is convenient to replace them by functions $a\left(x_2\,,\, x_3\right)$ and $b\left(x_2\,,\, x_3\right)$ defined by

$$y_1 = a - b\,,\ y_2 = a + b\,,\ a = \frac{1}{2}\left(y_1 + y_2\right)\,,\ b = \frac{1}{2}\left(y_2 - y_1\right) = \frac{1}{2}\,w\ . \tag{59}$$

Here w is again the "width function". Now, (58) yields

$$F\left(m\,,\, x_2\,,\, x_3\right) = \pi m e^{-imx_1}\, f(\underline{x}) = \frac{1}{2}\int_{a-b}^{a+b} e^{-imx_1'}\, dx_1'$$

$$= e^{-ima\left(x_2\,,\, x_3\right)}\, \sin\left[mb\left(x_2\,,\, x_3\right)\right]. \tag{60}$$

We see from (57) that $F\left(m, x_2, x_3\right)$ can be obtained from measurement of ρ in the disc D and its image D' indicated by the dashed line in Figure 2*. Then the functions a and b, which determine the target completely, can be obtained from (60) in a variety of ways. One way is as follows. We note that

$$|F| = \pm \sin(mb) \tag{61}$$

hence

$$\frac{d}{dm}|F| = \pm b \cos(mb) . \tag{62}$$

We see that (62) consists of a rapidly fluctuating "carrier" modulated by the slowly varying function b. Thus the (positive) function b is given by

$$b\left(x_2, x_3\right) = \text{envelope of } \frac{d}{dm}\left|F\left(m, x_2, x_3\right)\right| . \tag{63}$$

Furthermore, from (60)

$$\left|\frac{dF}{dm}\right|^2 = \left|e^{-ima}\left[b \cos mb - ia \sin mb\right]\right|^2$$

$$= b^2 \cos^2(mb) + a^2 \sin^2(mb) . \tag{64}$$

Hence

$$a^2 \sin^2(mb) = \left|\frac{dF}{dm}\right|^2 - \left(\frac{d|F|}{dm}\right)^2 , \tag{65}$$

and

$$a^2\left(x_2, x_3\right) = \text{envelope of } \left\{\left|\frac{dF}{dm}\right|^2 - \left(\frac{d|F|}{dm}\right)^2\right\}. \tag{66}$$

*In the next section we will attempt to eliminate the need to measure ρ in D'.

This determines $a(x_2, x_3)$ except for an ambiguity of sign which can be resolved by noting that

$$a = -\frac{1}{m} \arg\left[\frac{F}{\sin(mb)}\right] . \tag{67}$$

It is interesting to note that the condition (54) for the validity of our approximation of δ_μ by δ can be stated in the form

$$1 \ll L\mu = L\sqrt{M^2 - m^2} = LM\sqrt{1 - \frac{m^2}{M^2}} = LM\sqrt{1 - \cos^2\theta}$$

$$= ML \sin\theta = \frac{2\omega_2}{c} L \sin\theta \ . \tag{68}$$

Here ω_2 is the upper frequency limit and θ is the semi-angle of the cone of required aspect directions illustrated in Figure 2. We note that θ can be made arbitrarily small provided ω_2 is sufficiently large. There is, of course, no need to take $m = 2\omega_1/c$ to correspond to the low-frequency limit. We need only require that ω_1 be greater than or equal to this lower limit. Thus the disc D will still lie in the usable frequency band.

SECTION VII

HALF-TARGET RECONSTRUCTION

If we think of the p_1-axis of Figure 2 as pointing in the direction of a transmitter as seen from the target, there are many practical situations in which one would like to eliminate the need of measuring ρ in the region D' corresponding to aspects in the "back" of the target. One might hope to obtain a representation of the "front" half of the target without these measurements. The purpose of this section is to show that this "half-target reconstruction" can, in principle, be accomplished by a modification of our method.

For a given target B and a given unit vector $\underline{J}_o$ we introduce the symmetric target B_o obtained by reflecting the illuminated part L_o (at aspect $\underline{J}_o$) of the surface of B through the origin. A typical case is illustrated in Figure 3. The resulting B_o depends of course on the choice of the origin o in B.

It is clear from (3) that if ρ_o corresponds to B_o and ρ to B, then

$$\rho(\underline{p}) \approx \rho_o(\underline{p}) \quad \text{for} \quad \underline{J} \approx \underline{J}_o \ . \tag{69}$$

Here $\underline{J} = \frac{1}{p}\underline{p} = -\underline{I}$ [see Equation (5)] . Furthermore, because of the symmetry of B_o,

$$\rho_o(-\underline{p}) = \rho_o(\underline{p}) \ . \tag{70}$$

Using subscripts "o" to refer to B_o, we see from (13) and (70) that

$$f_o(\underline{x}) = 2^{-2}\pi^{-5/2} \int \frac{K(p)}{p^2} \left[\rho_o(\underline{p}) + \rho_o^{*}(\underline{p})\right] e^{i\underline{p}\cdot\underline{x}} d\underline{p} \ . \tag{71}$$

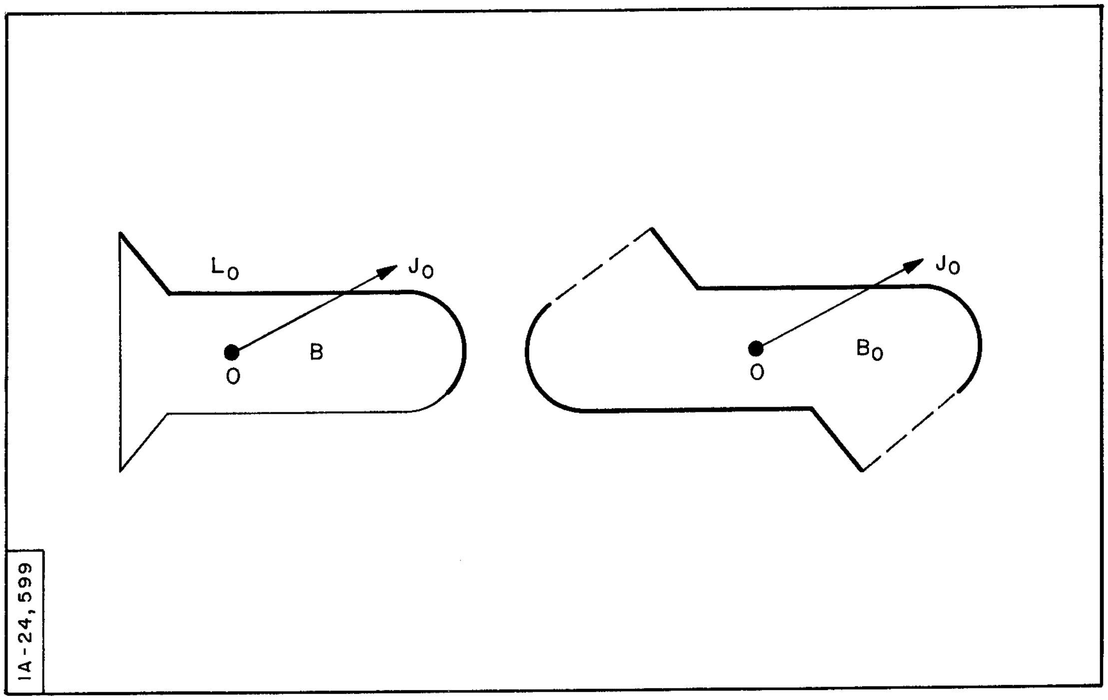

Figure 3. Representation of the Symmetric Target

We now assume that $K(\underline{p})$ vanishes except for $\underline{J} = \frac{1}{p}\,\underline{p}$ near $\underline{J}_o$. Then from (69)

$$f_o(\underline{x}) \approx 2^{-2}\pi^{-5/2} \int \frac{K(\underline{p})}{p^2} \left[\rho(\underline{p}) + \rho^*(\underline{p})\right] e^{i\underline{p}\cdot\underline{x}}\, d\underline{p} \ . \qquad (72)$$

But from (14)

$$f_o(\underline{x}) = \int \gamma_o(\underline{x}')\, \mathcal{K}(\underline{x} - \underline{x}')\, d\underline{x}' = \gamma_o * \mathcal{K} \ , \qquad (73)$$

and we see that, by eliminating $f_o(\underline{x})$ in (72) and (73), we obtain an equation not for γ but for the characteristic function γ_o of B_o . If we can use this equation to determine B_o , then we have determined the illuminated part

L_o of B . The input data in (72) requires the measurement of

$$\rho(\underline{p}) + \rho^*(\underline{p}) = 2\,\mathcal{R}e\,\rho(\underline{p}) \tag{74}$$

in the region D where $K(\underline{p})$ does not vanish. By assumption this region involves aspects $\underline{J}$ only near $\underline{J}_o$.

If, for example, we apply the method of this section to that of Section VI we take $\underline{J}_o = (1, 0, 0)$ to be the unit vector in the direction of the p_1-axis. Then we merely replace $\rho^*(-\underline{p})$ by $\rho^*(\underline{p})$ in (56) and (57). Thus we obtain

$$F_o\left(m, x_2, x_3\right) = 2^{-1}\pi^{-3/2} m \int_{p_2^2 + p_3^2 < \mu^2} \frac{\mathcal{R}e\left(\rho\ \ m, p_2, p_3\right)}{m^2 + p_2^2 + p_3^2}$$

$$\times\, e^{i\left(p_2 x_2 + p_3 x_3\right)}\, dp_2 dp_3 , \tag{75}$$

and by using F_o instead of F in the succeeding equations of Section VI we obtain not the target B, but its symmetric replacement B_o . We note that the input data for (75) require the measurement of ρ on the disc D but not on its image D' .

SECTION VIII

THE GENERAL THEORY FOR BODIES OF REVOLUTION

In this section we will show that the general theory developed in Section III can be considerably simplified if the target B is axially symmetric. In effect we will reduce the complexity of our formulas from three to two dimensions.

We begin by introducing cylindrical coordinates in both $\underline{x}$-space and $\underline{p}$-space defined by

$$\underline{x} = \left(x_1 , r \cos \alpha , r \sin \alpha\right) , \underline{p} = \left(p_1 , b \cos \beta , b \sin \beta\right) . \quad (76)$$

Thus

$$d\underline{x} = dx_1 \, r dr d\alpha , \; d\underline{p} = dp_1 \, b db d\beta , \quad (77)$$

and

$$\underline{p} \cdot \underline{x} = p_1 x_1 + br \cos (\beta - \alpha) . \quad (78)$$

We take the x_1-axis to be the symmetry axis of the target. Then $\rho(\underline{p})$ is axially symmetric (independent of β) , i.e.,

$$\rho = \rho\left(p_1 , b\right) , \quad (79)$$

and the characteristic function is also axially symmetric (independent of α), i.e.,

$$\gamma = \gamma\left(x_1 , r\right) . \quad (80)$$

If we use the integral representation of the Bessel function

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz\cos\theta} \, d\theta \,, \tag{81}$$

and take $K = K(p_1 \,,\, b)$, we find that Equation (12) becomes

$$\mathcal{K}(x_1 \,,\, r) = (2\pi)^{-3} \int dp_1 \int_0^{2\pi} d\beta \int_0^{\infty} bdb \, K(p_1 \,,\, b) \, e^{i\left[p_1 x_1 + br\cos(\beta - \alpha)\right]}$$

$$= (2\pi)^{-2} \int dp_1 \int_0^{\infty} bdb \, K(p_1 \,,\, b) \, e^{ip_1 x_1} \, J_0(br) \;. \tag{82}$$

Since

$$\begin{aligned} (x_2 - x_2')^2 + (x_3 - x_3')^2 &= (r\cos\alpha - r'\cos\alpha')^2 \\ &\quad + (r\sin\alpha - r'\sin\alpha')^2 \\ &= r^2 + (r')^2 - 2rr'\cos(\alpha' - \alpha) \,, \end{aligned} \tag{83}$$

Equation (14) becomes

$$\begin{aligned} f(x_1 \,,\, r) &= \int dx_1' r' dr' d\alpha' \gamma \, (x_1' \,,\, r') \\ &\quad \times \mathcal{K}\left[x_1 - x_1' \,,\, \sqrt{(x_2 - x_2')^2 + (x_3 - x_3')^2}\,\right] \\ &= \int dx_1' \int_0^{\infty} r' dr' \gamma \, (x_1' \,,\, r') \int_0^{2\pi} d\alpha' \\ &\quad \times \mathcal{K}\left[x_1 - x_1' \,,\, \sqrt{r^2 + (r')^2 - 2rr'\cos\alpha'}\,\right] . \end{aligned} \tag{84}$$

We will make use of the "addition theorem" for Bessel functions,

$$J_0\left(\sqrt{R^2 + P^2 - 2RP\cos\theta}\right) = J_0(R)J_0(P) + 2\sum_{m=1}^{\infty} J_m(R)J_m(P)\cos m\theta , \tag{85}$$

from which it follows that

$$\int_0^{2\pi} J_0\left(\sqrt{R^2 + P^2 - 2RP\cos\theta}\right) d\theta = 2\pi\, J_0(R)J_0(P) . \tag{86}$$

Then from (84) we see that

$$f\left(x_1 , r\right) = \int dx_1' \int_0^{\infty} r'dr' \gamma\left(x_1' , r'\right) Q\left(x_1 - x_1' , r, r'\right) \tag{87}$$

where

$$Q\left(x_1 , r , r'\right) = \int_0^{2\pi} d\alpha' \; \mathcal{K}\left[x_1 , \sqrt{r^2 + (r')^2 - 2rr'\cos\alpha'}\right]$$

$$= (2\pi)^{-1} \int dp_1 \int_0^{\infty} bdb\, K\left(p_1 , b\right) e^{ip_1x_1} J_0(br)\, J_0(br') . \tag{88}$$

On the other hand, Equation (13) becomes

$$f\left(x_1\,,\,r\right) = 2^{-2}\pi^{-5/2}\int dp_1 \int_0^\infty bdb$$

$$\times\int_0^{2\pi} d\beta\,\frac{K\left(p_1\,,\,b\right)}{p_1^{\,2}+b^2}\left[\rho\left(p_1\,,\,b\right) + \rho^*\left(-p_1\,,\,b\right)\right]e^{i\left(p_1x_1 + br\cos\beta\right)}$$

$$= 2^{-1}\pi^{-3/2}\int dp_1 \int_0^\infty bdb\,\frac{K\left(p_1\,,\,b\right)}{p_1^{\,2}+b^2}\left[\rho\left(p_1\,,\,b\right) + \rho^*\left(-p_1\,,\,b\right)\right]$$

$$\times\, e^{ip_1x_1}\,J_0\,(br)\ . \tag{89}$$

The application of these formulas is much the same as in Section III. We choose a domain D in $\left(p_1\,,\,b\right)$-space and a convenient function $K\left(p_1\,,\,b\right)$ which is non-zero only in D. Then we determine Q from Equation (88) and f in terms of measurable values of ρ from Equation (89). Finally, we use (87) and the fact that $\gamma\,\left(x_1\,,\,r\right)$ is a characteristic function to obtain a partial or complete determination of B. The method may be illustrated by the following elementary examples:

Example 1:

$$\text{D: all of } \left(p_1\,,\,b\right)\text{-space; } K\left(p_1\,,\,b\right) \equiv 1\ . \tag{90}$$

Since*

$$\int_0^\infty bdb\,J_0(br)\,J_0(br') = \frac{1}{r'}\,\delta\,(r - r') \tag{91}$$

* This identity is equivalent to the inversion theorem for Hankel transforms.

it follows that

$$Q\left(x_1 , r , r'\right) = \delta\left(x_1\right) \frac{\delta\left(r - r'\right)}{r'} \quad . \tag{92}$$

Then (87) yields

$$f\left(x_1 , r\right) = \gamma\left(x_1 , r\right) \; ; \tag{93}$$

i.e., the characteristic function is given by (93) where f is obtained by setting $K \equiv 1$ in (89) .

<u>Example 2:</u>

$$\text{D: the axial line } b = 0; \; K\left(p_1 , b\right) = \frac{1}{b}\, \delta(b) \quad . \tag{94}$$

From (88),

$$Q\left(x_1 , r , r'\right) = \delta\left(x_1\right) J_0^{\,2}(0) = \delta\left(x_1\right) \quad , \tag{95}$$

and (89) yields

$$f\left(x_1 , r\right) = 2^{-1}\pi^{-3/2} \int dp_1 \, \frac{\rho\left(p_1 , 0\right) + \rho^{*}\left(-p_1 , 0\right)}{p_1^{\,2}} \, e^{ip_1 x_1} \quad . \tag{96}$$

By inserting (95) in (87) we obtain

$$f\left(x_1 , r\right) = \int_0^{\infty} \gamma\left(x_1 , r'\right) r'dr' = \int_0^{r_o\left(x_1\right)} r'dr' = \frac{1}{2} r_o^{\,2}\left(x_1\right) \quad . \tag{97}$$

Here $r = r_o\left(x_1\right)$ is the (non-negative) function which describes the "profile curve" of the target B. Thus the target is completely determined by eliminating f in (96) and (97).

Both of the examples discussed here suffer from the defect that ρ must be measured at forbidden low frequencies.

REFERENCES

1. R. M. Lewis, Notes on the Kirchhoff Method in Scattering Problems, 1963 (unpublished).

2. N. N. Bojarski, Three-Dimensional Electromagnetic Short Pulse Inverse Scattering, Syracuse University Research Corporation, Syracuse, February 1967.

3. R. M. Lewis, Short Pulse Inversion Diffraction Theory, MTR-322, 18 November 1966, and Supplement 1, January 30, 1967, The MITRE Corporation, Bedford, Massachusetts.