NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

REPORT No. 789

ON THE FLOW OF A COMPRESSIBLE FLUID BY THE HODOGRAPH METHOD I-UNIFICATION AND EXTENSION OF PRESENT-DAY RESULTS

By I. E. GARRICK and CARL KAPLAN

1944

AERONAUTIC SYMBOLS

1. FUNDAMENTAL AND DERIVED UNITS

	Symbol	Metric		English	
		Unit	Abbreviation	Unit	Abbreviation
Length	l	meter	m	foot (or mile)	ft (or mi)
Time	t	second	s	second (or hour)	sec (or hr)
Force	F	weight of 1 kilogram	kg	weight of 1 pound	lb
Power	P	horsepower (metric)		horsepower	hp
Speed	V	kilometers per hour	kph	miles per hour	mph
		meters per second	mps	feet per second	fps

2. GENERAL SYMBOLS

W Weight $=mg$

g Standard acceleration of gravity $=9.80665$ m/s^2 or 32.1740 ft/sec^2

m Mass $=\frac{W}{g}$

I Moment of inertia $=mk^2$. (Indicate axis of radius of gyration k by proper subscript.)

μ Coefficient of viscosity

ν Kinematic viscosity

ρ Density (mass per unit volume)

Standard density of dry air, 0.12497 kg-m^{-4}-s^2 at 15° C and 760 mm; or 0.002378 lb-ft^{-4} sec^2

Specific weight of "standard" air, 1.2255 kg/m^3 or 0.07651 lb/cu ft

3. AERODYNAMIC SYMBOLS

S Area

S_w Area of wing

G Gap

b Span

c Chord

A Aspect ratio, $\frac{b^2}{S}$

V True air speed

q Dynamic pressure, $\frac{1}{2}\rho V^2$

L Lift, absolute coefficient $C_L=\frac{L}{qS}$

D Drag, absolute coefficient $C_D=\frac{D}{qS}$

D_0 Profile drag, absolute coefficient $C_{D0}=\frac{D_0}{qS}$

D_i Induced drag, absolute coefficient $C_{Di}=\frac{D_i}{qS}$

D_p Parasite drag, absolute coefficient $C_{Dp}=\frac{D_p}{qS}$

C Cross-wind force, absolute coefficient $C_C=\frac{C}{qS}$

i_w Angle of setting of wings (relative to thrust line)

i_t Angle of stabilizer setting (relative to thrust line)

Q Resultant moment

Ω Resultant angular velocity

R Reynolds number, $\rho\frac{Vl}{\mu}$ where l is a linear dimension (e.g., for an airfoil of 1.0 ft chord, 100 mph, standard pressure at 15° C, the corresponding Reynolds number is 935,400; or for an airfoil of 1.0 m chord, 100 mps, the corresponding Reynolds number is 6,865,000)

α Angle of attack

ϵ Angle of downwash

α_0 Angle of attack, infinite aspect ratio

α_i Angle of attack, induced

α_a Angle of attack, absolute (measured from zero-lift position)

γ Flight-path angle

E R R A T A

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Page 6,

Column 1, lines 5 and 8: Change $\tau = 2$ to $\gamma = 2$ and $\tau = \frac{3}{2}$ to $\gamma = \frac{3}{2}$.

Column 1, lines 20 and 21: Insert zero for the lower limit on the integral sign in the equation beginning "$g(q/a_0) = \ldots$"

Page 7, column 2, equation (37): The exponent on e in the denominator on the right-hand side of the equation should read

$$\frac{1}{2}\left[f(\tau_1)+g(\tau_1)\right]$$

Page 11, figure 3: The first part of the main legend should read "Pressure coefficients C_{p,M_1} and $C_{p,0}$ against local Mach number . . . ."

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National Advisory Committee for Aeronautics

Headquarters, 1500 New Hampshire Avenue NW., Washington 25, D. C.

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SUMMARY

Elementary basic solutions of the equations of motion of a compressible fluid in the hodograph variables are developed and used to provide a basis for comparison, in the form of velocity correction formulas, of corresponding compressible and incompressible flows. The known approximate results of Chaplygin, von Kármán and Tsien, Temple and Yarwood, and Prandtl and Glauert are unified by means of the analysis of the present paper. Two new types of approximations, obtained from the basic solutions, are introduced; they possess certain desirable features of the other approximations and appear preferable as a basis for extrapolation into the range of high stream Mach numbers and large disturbances to the main stream. Tables and figures giving velocity and pressure-coefficient correction factors are included in order to facilitate the practical application of the results.

INTRODUCTION

The present paper is concerned with a theoretical study of the hydrodynamical equations of a perfect compressible fluid in two dimensions, in which the so-called hodograph variables are used as the independent variables. It is hoped to achieve herein a unification of the present-day results obtained in this field and also to provide a working basis for further developments. The earliest contributors to the hodograph method for treating compressible fluids were Molenbroek (reference 1) and Chaplygin (reference 2). The remarkable work of Chaplygin on gas jets appeared in Russian in 1904 but remained relatively unnoticed. In recent years contributions to the hodograph method have been made chiefly by Demtchenko (reference 3), von Kármán (reference 4), Tsien (reference 5), Ringleb (reference 6), and Temple and Yarwood (reference 7).

The chief reason, and perhaps the only reason, for preferring the hodograph variables to the physical plane coordinates is that the equations of motion in the hodograph variables are linear. This simplification is achieved, however, at the cost of more difficult boundary conditions and at a loss of physical insight. The great simplification in the mathematics due to linearity nevertheless makes it desirable to pursue this line of attack as long as it appears profitable to do so.

The mathematics for handling the flow equations received a substantial impetus by the work of Bers and Gelbart (reference 8), who developed a new function theory analogous to ordinary analytic function theory. The present paper utilizes the methods of this new function theory to develop certain functions essential to the compressible-flow problem. It is of historical interest that ideas similar to those of Bers and Gelbart were explored by the renowned mathematician Hilbert (reference 9) in the early part of this century but do not appear to have been further developed at the time.

The material to be treated is conveniently separated into two parts. In part I, the present paper, basic particular solutions of the hodograph flow equations are developed and employed in unifying and extending the results obtained by Chaplygin, von Kármán, and Temple and Yarwood. The results obtained in part I are of immediate practical application and are given in the form of tables and graphs of velocity and pressure-coefficient correction factors. In part II, general particular solutions of the hodograph flow equations are developed and discussed. The material in part II, it is hoped, will lead to a method for handling the actual boundary problem of the flow of a compressible fluid past a prescribed body.

ANALYSIS

FLOW EQUATIONS OF AN INCOMPRESSIBLE FLUID

It is well known that the relations between the velocity potential ϕ and the stream function ψ for the steady irrotational two-dimensional motion of a perfect incompressible fluid are

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x}&=\frac{\partial\psi}{\partial y}\\ \frac{\partial\phi}{\partial y}&=-\frac{\partial\psi}{\partial x}\end{aligned}\right\}\tag{1}$$

These equations are the Cauchy-Riemann equations and therefore $\phi+i\psi$ is an analytic function $f(z)$ of the complex variable $z=x+iy$.

The complex velocity or reflected velocity vector $u-iv$ is obtained from the complex potential $f(z)$ by differentiation. Thus,

$$\begin{aligned} u-iv &= \frac{df(z)}{dz} \\ &= qe^{-i\theta} \\ &= e^{-i(\theta+i\log q)} \end{aligned} \tag{2}$$

where q is the magnitude of the velocity vector and θ is the angle the vector makes with the positive direction of the x-axis.

The variables θ and q are sometimes referred to as "the hodograph variables." The flow equations in the variables θ and q can be readily derived by introducing $\theta+i\log q$ as the independent complex variable in place of $x+iy$. Then, in analogy with equation (1),

$$\left.\begin{aligned} \frac{\partial\phi}{\partial\theta} &= \frac{\partial\psi}{\partial \log q} \\ \frac{\partial\phi}{\partial \log q} &= -\frac{\partial\psi}{\partial\theta} \end{aligned}\right\} \tag{3}$$

or

$$\left.\begin{aligned} \frac{\partial\phi}{\partial\theta} &= q\frac{\partial\psi}{\partial q} \\ \frac{\partial\phi}{\partial q} &= -\frac{1}{q}\frac{\partial\psi}{\partial\theta} \end{aligned}\right\} \tag{4}$$

These equations are known as the hodograph equations for the flow of an incompressible fluid.

FLOW EQUATIONS OF A COMPRESSIBLE FLUID

The equations corresponding to equation (1) are, for a compressible fluid,

$$\left.\begin{aligned} \frac{\partial\phi}{\partial x} &= \frac{\rho_0}{\rho}\frac{\partial\psi}{\partial y} \\ \frac{\partial\phi}{\partial y} &= -\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial x} \end{aligned}\right\} \tag{5}$$

where ρ is the density of the fluid at any point (x,y) and ρ_0 is a constant density, which for convenience is referred to a stagnation point.

A short way to derive the hodograph equations for a compressible fluid, attributed to Molenbroek, is as follows: According to equations (5), with $u=\frac{\partial\phi}{\partial x}$ and $v=\frac{\partial\phi}{\partial y}$

$$\begin{aligned} d\phi+i\frac{\rho_0}{\rho}d\psi &= (u\,dx+v\,dy)+i(-v\,dx+u\,dy) \\ &= (u-iv)(dx+i\,dy) \\ &= qe^{-i\theta}\,dz \end{aligned}$$

or

$$dz=\frac{1}{q}e^{i\theta}\left(d\phi+i\frac{\rho_0}{\rho}d\psi\right) \tag{6}$$

It follows from equation (6), by considering θ and q as independent variables, that

$$\frac{\partial z}{\partial\theta}=\frac{1}{q}e^{i\theta}\left(\frac{\partial\phi}{\partial\theta}+i\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial\theta}\right)$$

and

$$\frac{\partial z}{\partial q}=\frac{1}{q}e^{i\theta}\left(\frac{\partial\phi}{\partial q}+i\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial q}\right)$$

Then, by assuming that ρ is a function of only q (equivalent to assuming that the pressure is a function of only the density),

$$\frac{\partial^2 z}{\partial q\partial\theta}=e^{i\theta}\left[-\frac{1}{q^2}\frac{\partial\phi}{\partial\theta}+i\frac{d(\rho_0/\rho q)}{dq}\frac{\partial\psi}{\partial\theta}\right]+\frac{1}{q}e^{i\theta}\left(\frac{\partial^2\phi}{\partial q\partial\theta}+i\frac{\rho_0}{\rho}\frac{\partial^2\psi}{\partial q\partial\theta}\right)$$

and

$$\frac{\partial^2 z}{\partial\theta\partial q}=\frac{i}{q}e^{i\theta}\left(\frac{\partial\phi}{\partial q}+i\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial q}\right)+\frac{1}{q}e^{i\theta}\left(\frac{\partial^2\phi}{\partial\theta\partial q}+i\frac{\rho_0}{\rho}\frac{\partial^2\psi}{\partial\theta\partial q}\right)$$

Since, by continuity, these two expressions are identical, it follows that

$$\frac{i}{q}e^{i\theta}\left(\frac{\partial\phi}{\partial q}+i\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial q}\right)=e^{i\theta}\left[-\frac{1}{q^2}\frac{\partial\phi}{\partial\theta}+i\frac{d(\rho_0/\rho q)}{dq}\frac{\partial\psi}{\partial\theta}\right]$$

Hence, by equating real and imaginary parts,

$$\left.\begin{aligned} \frac{\partial\phi}{\partial\theta} &= \frac{\rho_0 q}{\rho}\frac{\partial\psi}{\partial q} \\ \frac{\partial\phi}{\partial q} &= q\frac{d(\rho_0/\rho q)}{dq}\frac{\partial\psi}{\partial\theta} \end{aligned}\right\} \tag{7}$$

These are the hodograph equations, first obtained by Molenbroek, for the flow of a compressible fluid and are independent of the form of the pressure-density relation. It is observed that, when $\rho=\rho_0=$Constant, equations (7) reduce to equations (4). Equations (7), in contrast with equations (5), are linear in the dependent variables.

BERNOULLI'S EQUATION AND EQUATION OF STATE

In the present section there is listed a collection of formulas and definitions necessary in the analysis.

Bernoulli's equation for a compressible fluid is

$$\int_{p_0}^{p}\frac{dp}{\rho}+\frac{1}{2}q^2=0 \tag{8}$$

where

- p static pressure in fluid
- p_0 static pressure at stagnation point $(q=0)$
- ρ density of fluid
- q magnitude of velocity of fluid

The adiabatic relation between the pressure and the density is

$$\frac{p}{p_0}=\left(\frac{\rho}{\rho_0}\right)^\gamma \tag{9}$$

where

- γ adiabatic index (approx. 1.4 for air)
- ρ_0 density of fluid at stagnation point $(q=0)$

The local velocity of sound a is obtained from

$$a^2=\frac{dp}{d\rho}$$

For the adiabatic case,

$$a^2=\gamma\frac{p}{\rho} \tag{10}$$

From Bernoulli's equation (8) and from equations (9) and (10), the following relations may be obtained:

$$\left.\begin{aligned} a^2&=a_0^2-\frac{1}{2}(\gamma-1)q^2 \\ \rho&=\rho_0\left[1-\frac{1}{2}(\gamma-1)\frac{q^2}{a_0^2}\right]^{\frac{1}{\gamma-1}} \\ p&=p_0\left[1-\frac{1}{2}(\gamma-1)\frac{q^2}{a_0^2}\right]^{\frac{\gamma}{\gamma-1}} \end{aligned}\right\} \tag{11}$$

where a_0 is the velocity of sound at stagnation point ($q=0$).

From equations (11) for $\gamma>1$, a maximum velocity $q=q_m$ is obtained for the limiting conditions $p=\rho=a=0$. Thus,

$$\begin{aligned} q_m^2&=\frac{2}{\gamma-1}a_0^2 \\ &=2\beta a_0^2 \end{aligned} \tag{12}$$

where

$$\beta=\frac{1}{\gamma-1}$$

The fundamental nondimensional speed variable, in general, is q/a_0 but it is found useful in the analysis to employ a nondimensional speed variable τ defined as

$$\tau=\frac{q^2}{q_m^2} \tag{13}$$

For $\gamma>1$, the range of the variable τ is $0\leqq\tau\leqq 1$. The value $\tau=0$ has a dual meaning; $\tau=0$ in the case of a compressible fluid corresponds to a stagnation point ($q=0$), or $\tau=0$ may mean the limiting case of an incompressible fluid ($a_0=\infty$).

With the definitions of τ and β, equations (11) become

$$\left.\begin{aligned} a&=a_0(1-\tau)^{1/2} \\ \rho&=\rho_0(1-\tau)^{\beta} \\ p&=p_0(1-\tau)^{\beta+1} \end{aligned}\right\} \tag{14}$$

The local Mach number $M=\frac{q}{a}$ may be expressed in terms of the speed variable τ in the following way:

$$\begin{aligned} M^2&=\frac{q^2}{q_m^2}\frac{q_m^2}{a_0^2}\frac{a_0^2}{a^2} \\ &=\frac{2\beta\tau}{1-\tau} \end{aligned} \tag{15}$$

or, by solving for τ in terms of M,

$$\tau=\frac{M^2}{2\beta+M^2} \tag{16}$$

The value of τ for which the local velocity of the fluid equals the local velocity of sound ($M=1$) is given by

$$\tau_s=\frac{1}{2\beta+1} \tag{17}$$

In the case of uniform flow past a fixed boundary, the pressure coefficient is defined as

$$C_{p,M_1}=\frac{p-p_1}{\frac{1}{2}\rho_1 q_1^2}$$

where the subscript 1 refers to the undisturbed stream. The pressure coefficient for the incompressible case ($M=0$) is

$$C_{p,0}=1-\left(\frac{q}{q_1}\right)_i^2 \tag{18a}$$

The pressure coefficient for the compressible case is

$$C_{p,M_1}=\frac{2}{\gamma M_1^2}\left(-1+\left\{1+\frac{1}{2}(\gamma-1)M_1^2\left[1-\left(\frac{q}{q_1}\right)_c^2\right]\right\}^{\frac{\gamma}{\gamma-1}}\right) \tag{18b}$$

For $q=q_s$ (*sonic*),

$$(C_{p,M_1})_s=\frac{2}{\gamma M_1^2}\left\{-1+\left[\frac{2+(\gamma-1)M_1^2}{\gamma+1}\right]^{\frac{\gamma}{\gamma-1}}\right\} \tag{18c}$$

For $q=q_m$ (*vacuum*),

$$(C_{p,M_1})_m=-\frac{2}{\gamma M_1^2} \tag{18d}$$

BASIC SOLUTIONS OF HODOGRAPH EQUATIONS

Consider the incompressible case represented by equations (3) or (4). It is clear that $\phi=\theta$ and $\psi=\log q$ satisfy these equations. In fact, any convergent power series in $w=\theta+i\log q$ represents an analytic function of which the real and imaginary parts satisfy equations (3) or (4). The class of analytic functions in w (and the concept of analytic continuation) then yields all the particular solutions of these equations.

The particular solution $w=\theta+i\log q$ can be obtained by means of an integration that is instructive in the generalization to the compressible case. It is well known that

$$F(w)=\int f(w)\,dw$$

can be represented as the sum of two line integrals

$$F(w)=\int(P\,d\theta-Q\,d\log q)+i\int(Q\,d\theta+P\,d\log q)$$

where

$$f(w)=P+iQ$$

Thus, given a pair of functions P and Q that satisfy equations (3) or (4), this process yields another pair of solutions, namely, the real and the imaginary parts of $F(w)$. For example, if $P=1$ and $Q=0$,

$$F(w)=w=\theta+i\log q \tag{19}$$

Again, if $P=0$ and $Q=1$,

$$F(w)=iw=-\log q+i\theta \tag{20}$$

The physical interpretation of equations (19) and (20), considered as flow patterns, is of some interest in connection with later developments. It is clear that equations (19) and (20) represent a vortex and a source located at the origin, respectively.

The generalization to the compressible case of the foregoing elementary results was accomplished by Bers and Gelbart (reference 8) by means of simple yet fertile ideas. Bers and Gelbart treat equations of the form

$$\left.\begin{aligned}\frac{\partial\phi}{\partial\theta}&=\lambda_1(q)\,\frac{\partial\psi}{\partial q}\\ \frac{\partial\phi}{\partial q}&=-\lambda_2(q)\,\frac{\partial\psi}{\partial\theta}\end{aligned}\right\}\tag{21}$$

and show as is readily verified that, if P and Q are a pair of solutions, the real and imaginary parts of the following sum of line integrals

$$\int[P\,d\theta-\lambda_2(q)Q\,dq]+i\int\left[Q\,d\theta+\frac{1}{\lambda_1(q)}\,P\,dq\right]\tag{22}$$

are also solutions of equations (21).

In particular, corresponding to the pair of solutions $P=1$ and $Q=0$, there is obtained

$$W=\theta+i\int\frac{1}{\lambda_1(q)}\,dq\tag{23}$$

and, for $P=0$ and $Q=1$,

$$i\tilde{W}=i[\theta+i\int\lambda_2(q)\,dq]\tag{24}$$

By repeated application of the process of integration, indicated by expression (22), a general set of particular solutions of equations (21) may be obtained. These particular solutions are discussed in part II; in the present paper, only the solutions given by equations (23) and (24) are needed.

The general hodograph equations (7) are of the form of equations (21) with

$$\lambda_1(q)=\frac{\rho_0 q}{\rho}$$

and

$$\lambda_2(q)=-q\,\frac{d(\rho_0/\rho q)}{dq}$$

For the rest of this paper, the adiabatic pressure-density relation (9) is used. By means of equations (9) and (14) and the relation

$$\frac{d\rho}{dq}=-\frac{\rho}{q}\,M^2$$

obtained from the differential form of Bernoulli's equation (8), it follows that

$$\left.\begin{aligned}\lambda_1(q)&=\frac{q}{(1-\tau)^{\beta}}\\ \lambda_2(q)&=\frac{1-(2\beta+1)\tau}{q(1-\tau)^{\beta+1}}\end{aligned}\right\}\tag{25}$$

The evaluation of the integrals in equations (23) and (24) is made unique by requiring that the results reduce to the incompressible case when the speed of sound is infinite (that is, when $\tau=0$). Then,

$$\begin{aligned}L&=\int(1-\tau)^{\beta}\,\frac{dq}{q}\\ &=\log q+f(\tau)\end{aligned}\tag{26}$$

where

$$f(\tau)=\frac{1}{2}\int_0^{\tau}[(1-\tau)^{\beta}-1]\frac{d\tau}{\tau}$$

and

$$\begin{aligned}\tilde{L}&=\int\frac{1-(2\beta+1)\tau}{(1-\tau)^{\beta+1}}\,\frac{dq}{q}\\ &=\log q+g(\tau)\end{aligned}\tag{27}$$

where

$$g(\tau)=\frac{1}{2}\int_0^{\tau}\left[\frac{1-(2\beta+1)\tau}{(1-\tau)^{\beta+1}}-1\right]\frac{d\tau}{\tau}$$

and it is observed that the functions $f(\tau)$ and $g(\tau)$ vanish for $\tau=0$.

Equations (23) and (24) can be written in the form

$$W=\theta+iL$$

and

$$i\tilde{W}=i(\theta+i\tilde{L})$$

It is important to note that, in the incompressible case, W and $i\tilde{W}$ reduce to w and iw, since L and $\tilde{L}$ reduce to $\log q$. Thus, there are in the compressible case two basic functions L and $\tilde{L}$ corresponding to the one function $\log q$ in the incompressible case. It is of interest to mention that the functions W and $i\tilde{W}$, considered as flow patterns in a compressible fluid, can again be interpreted as a vortex and a source.

EVALUATION OF FUNCTIONS $f(\tau)$ AND $g(\tau)$ FOR VARIOUS VALUES OF β

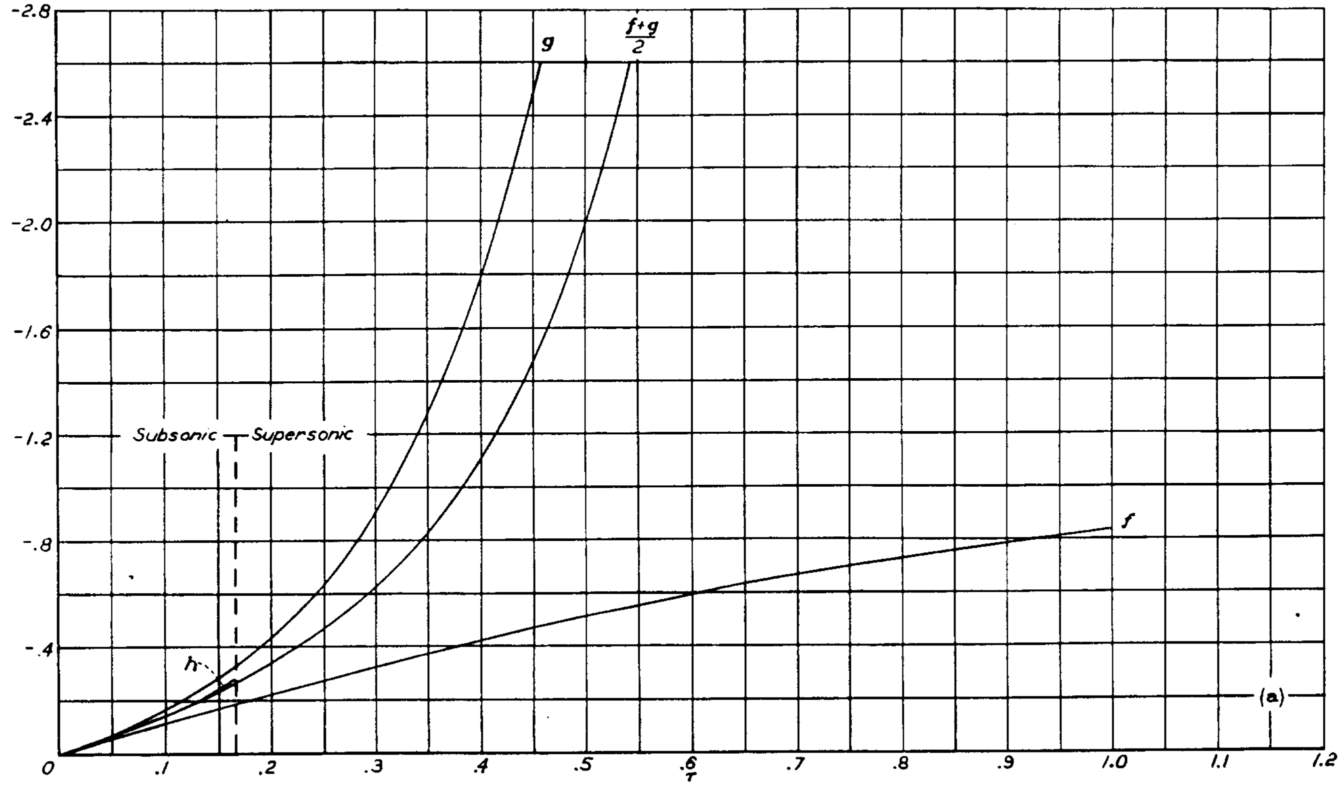
In general, the integrals in equations (26) and (27) representing the functions $f(\tau)$ and $g(\tau)$ are expressible by infinite series. For the important case of air, however, with the adiabatic index γ put equal to 1.4 instead of the usual value 1.408, these functions can be obtained in closed forms. Thus, with $\beta=2.5$,

$$\begin{aligned}f(\tau)&=\frac{1}{2}\int_0^{\tau}[(1-\tau)^{5/2}-1]\,\frac{d\tau}{\tau}\\ &=\frac{1}{5}\,(1-\tau)^{5/2}+\frac{1}{3}\,(1-\tau)^{3/2}\\ &\quad+(1-\tau)^{1/2}-\frac{23}{15}-\log\frac{1+(1-\tau)^{1/2}}{2}\end{aligned}\tag{28}$$

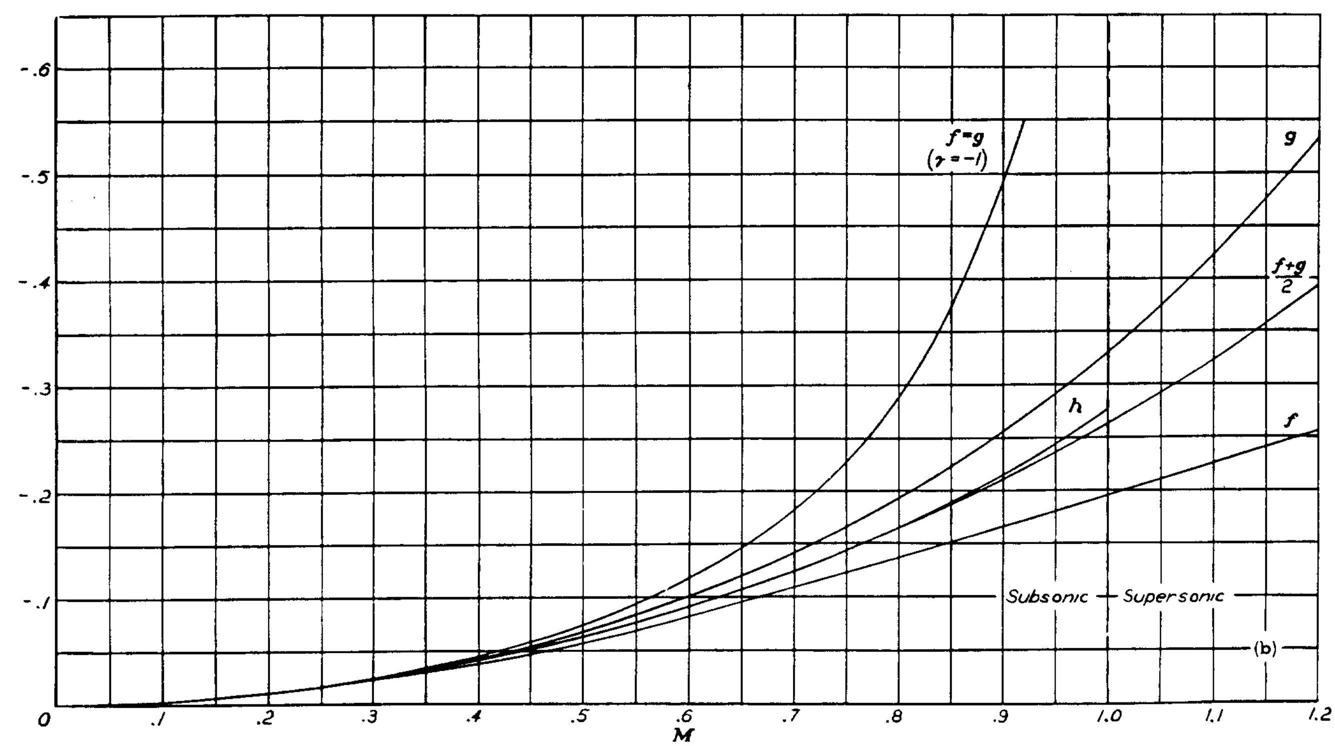
and

$$\begin{aligned}g(\tau)&=\frac{1}{2}\int_0^{\tau}\left[\frac{1-6\tau}{(1-\tau)^{7/2}}-1\right]\frac{d\tau}{\tau}\\ &=-\frac{1}{(1-\tau)^{5/2}}+\frac{1}{3}\,\frac{1}{(1-\tau)^{3/2}}\\ &\quad+\frac{1}{(1-\tau)^{1/2}}-\frac{1}{3}-\log\frac{1+(1-\tau)^{1/2}}{2}\end{aligned}\tag{29}$$

Table 1 contains values of $f(\tau)$ and $g(\tau)$, and figure 1(a) shows these functions plotted against τ. Observe that $f(\tau)$ and $g(\tau)$ are well-behaved functions in the range $0\leqq\tau<1$. In figure 1(b), these functions are plotted against the local Mach number M in the practical speed range.



(a) τ-curves.



(b) M-curves.

FIGURE 1.—The functions f, g, $\frac{f+g}{2}$, and h against τ and M for $\gamma=1.4$; the function $f=g$ against M for $\gamma=-1.0$.

Other interesting cases for which the functions $f(\tau)$ and $g(\tau)$ can be expressed in closed forms are $\gamma=\infty$, $\gamma=2$, $\gamma=\frac{3}{2}$, and $\gamma=-1$. For $\gamma=\infty$ ($\beta=0$, $a=\infty$, incompressible case),

$$f(\tau)=g(\tau)=0$$

For $\tau=2$ ($\beta=1$),

$$f(\tau)=-\frac{1}{2}\tau$$

$$g(\tau)=1-\frac{1}{1-\tau}-\frac{1}{2}\log(1-\tau)$$

For $\tau=\frac{3}{2}$ ($\beta=2$)

$$f(\tau)=-\tau+\frac{1}{4}\tau^2$$

$$g(\tau)=\frac{1}{2}-\frac{1}{2}\frac{1+\tau}{(1-\tau)^2}-\frac{1}{2}\log(1-\tau)$$

For $\gamma=-1\left(\beta=-\frac{1}{2}\right)$,

$$f(\tau)=g(\tau)=-\log\frac{1+(1-\tau)^{1/2}}{2}$$

For the isothermal case $\gamma=1(\beta=\infty)$, the velocity of sound $a=a_0=$Constant and the functions f and g are obtained as infinite series in the ratio q/a_0. Thus, in the limit $\beta\rightarrow\infty$,

$$\begin{aligned} f(q/a_0)&=\lim_{\beta\rightarrow\infty}\frac{1}{2}\int_0^{q/a_0}\left[\left(1-\frac{q^2}{2\beta a_0^2}\right)^\beta-1\right]\frac{d(q/a_0)}{q/a_0}\\ &=\frac{1}{2}\int_0^{q/a_0}\left(e^{-\frac{1}{2}\frac{q^2}{a_0^2}}-1\right)\frac{d(q/a_0)}{q/a_0}\\ &=\sum_{n=1}^{\infty}(-1)^n\frac{(q^2/a_0^2)^n}{2^{n+1}n\,n!}\end{aligned}$$

and

$$\begin{aligned} g(q/a_0)&=\lim_{\beta\rightarrow\infty}\frac{1}{2}\int^{q/a_0}\left[\frac{1-\left(1+\frac{1}{2\beta}\right)\frac{q^2}{a_0^2}}{\left(1-\frac{q^2}{2\beta a_0^2}\right)^{\beta+1}}-1\right]\frac{d(q/a_0)}{q/a_0}\\ &=\frac{1}{2}\int^{q/a_0}\left[\left(1-\frac{q^2}{a_0^2}\right)e^{\frac{1}{2}\frac{q^2}{a_0^2}}-1\right]\frac{d(q/a_0)}{q/a_0}\\ &=1-e^{\frac{1}{2}\frac{q^2}{a_0^2}}+\sum_{n=1}^{\infty}\frac{(q^2/a^2{}_0)^n}{2^{n+1}n\,n!}\end{aligned}$$

For arbitrary values of γ (or β) the expressions for $f(\tau)$ and $g(\tau)$, obtained with the aid of the binomial expansion, are

$$\begin{aligned} f(\tau)&=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\binom{\beta}{n}\frac{\tau^n}{n}\\ &=-\frac{1}{2}\beta\tau+\frac{1}{8}\beta(\beta-1)\tau^2-\ \ldots\\ &=-\frac{1}{4}\frac{q^2}{a_0^2}+\frac{1}{32}(2-\gamma)\left(\frac{q^2}{a_0^2}\right)^2-\ \ldots\end{aligned}$$

and

$$\begin{aligned} g(\tau)&=-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\binom{-\beta}{n}\frac{2n-1}{n}\tau^n\\ &=-\frac{1}{2}\beta\tau-\frac{3}{8}\beta(\beta+1)\tau^2-\ \ldots\\ &=-\frac{1}{4}\frac{q^2}{a_0^2}-\frac{3}{32}\gamma\left(\frac{q^2}{a_0^2}\right)^2-\ \ldots\end{aligned}$$

The significant feature of this general result is that, if powers of q/a_0 higher than the third are neglected,

$$f(\tau)=g(\tau)\approx-\frac{1}{4}\frac{q^2}{a_0^2} \tag{30}$$

and does not involve explicitly the adiabatic index γ. This circumstance underlies the present-day approximate methods for obtaining velocity and pressure-coefficient correction factors; in the following sections, this point is brought out more clearly.

APPLICATION OF BASIC FUNCTIONS L AND $\tilde{L}$

In this section, the basic functions L and $\tilde{L}$ are employed to set up relations between velocities in "corresponding" compressible and incompressible flows. These relations are of the nature of "stretching factors" or velocity correction formulas and contain the results of Chaplygin, von Kármán, Temple and Yarwood, and Glauert and Prandtl. It is important to recognize at the outset that no single velocity correction formula can represent in an exact way the correspondence of flow patterns past a prescribed body in a compressible and an incompressible fluid. A single velocity correction formula is actually feasible in only two cases: (1) The stream Mach number is small (even though the disturbance to the main stream due to the presence of the body may be large) so that the compressible-flow pattern differs only slightly from the incompressible-flow pattern or (2) the disturbance to the main stream is vanishingly small (even though the stream Mach number may be high) so that the effect of the shape of the solid boundary is small. The various velocity correction formulas discussed in the present paper differ essentially only in the degree to which the requirements of these two cases are satisfied. Despite their limitations, single velocity correction formulas are extrapolated, in view of the lack of more rigorous solutions, into the range of large disturbances to the main stream and high Mach numbers. This extrapolation can be justified by further theoretical investigations and by comparison with experimental results.

Consider again the corresponding pairs of functions

$$\left.\begin{aligned} w&=\theta+i\log q\\ W&=\theta+iL\end{aligned}\right\} \tag{31}$$

and

$$\left.\begin{aligned} iw&=i(\theta+i\log q)\\ i\tilde{W}&=i(\theta+i\tilde{L})\end{aligned}\right\} \tag{32}$$

It has previously been noted that the pairs of functions in equations (31) and (32) denote respectively a vortex and a source in an incompressible and a compressible fluid. Each pair of functions can be employed to define a correspondence of flow patterns in which corresponding points are identified by the same values (ϕ, ψ). Thus, in the case of the vortex (equations (31)),

$$\phi_i = \phi_c = \theta$$

$$\psi_i = \psi_c = \log q_i = L$$

where the subscripts i and c refer to the incompressible and to the compressible case, respectively. It follows that

$$\begin{aligned} q_i &= e^L \\ &= q_c e^{f(\tau)} \end{aligned} \tag{33}$$

Similarly, in the case of the source (equations (32)),

$$\phi_i = \phi_c = -\log q_i = -\tilde{L}$$

$$\psi_i = \psi_c = \theta$$

and

$$\begin{aligned} q_i &= e^{\tilde{L}} \\ &= q_c e^{g(\tau)} \end{aligned} \tag{34}$$

At the end of the preceding section it was pointed out that, to a first approximation, the functions $f(\tau)$ and $g(\tau)$ are equal. This fact implies that, to a first approximation, a single velocity correction formula is feasible. The assumption is now made that either equation (33) or equation (34) can be adopted to provide a correspondence of flow patterns in the case of uniform flow past a body in an incompressible and a compressible fluid. With the undisturbed streams as convenient references, the following nondimensional forms of equations (33) and (34) can be written:

$$\left(\frac{q}{q_1}\right)_i = \left(\frac{q}{q_1}\right)_c \frac{e^{f(\tau)}}{e^{f(\tau_1)}} \tag{35}$$

and

$$\left(\frac{q}{q_1}\right)_i = \left(\frac{q}{q_1}\right)_c \frac{e^{g(\tau)}}{e^{g(\tau_1)}} \tag{36}$$

where the subscript 1 refers to the undisturbed stream. The use of the undisturbed stream as reference in the nondimensional form of the velocity correction formula was introduced by Tsien in reference 5, where also the details of the von Kármán approximation are developed. It is shown in the following section that either of equations (35) or (36) contains the result of Chaplygin, von Kármán, and Temple and Yarwood. As has been previously pointed out, the concept of a single velocity correction formula is feasible in only two cases, namely, small stream Mach numbers and vanishingly small disturbances to the main stream. It is desirable then to seek a single velocity correction formula that combines the features of these two cases. From this point of view, equation (35) or equation (36) is not the best choice. A better choice of a single velocity correction formula appears to be the following combination of equations (35) and (36), based on the arithmetic mean of $f(\tau)$ and $g(\tau)$:

$$\left(\frac{q}{q_1}\right)_i = \left(\frac{q}{q_1}\right)_c \frac{e^{\frac{1}{2}[f(\tau)+g(\tau)]}}{e^{\frac{1}{2}[f(\tau_1)+g(\tau_1)]}} \tag{37}$$

In a later section, still another combination referred to as "the geometric-mean type of approximation" is introduced; in the section dealing with the Glauert-Prandtl approximation, certain features of the foregoing arithmetic-mean type of approximation and of the geometric-mean type are discussed.

At this point it is desirable to discuss the practical application of equation (37). According to equation (16),

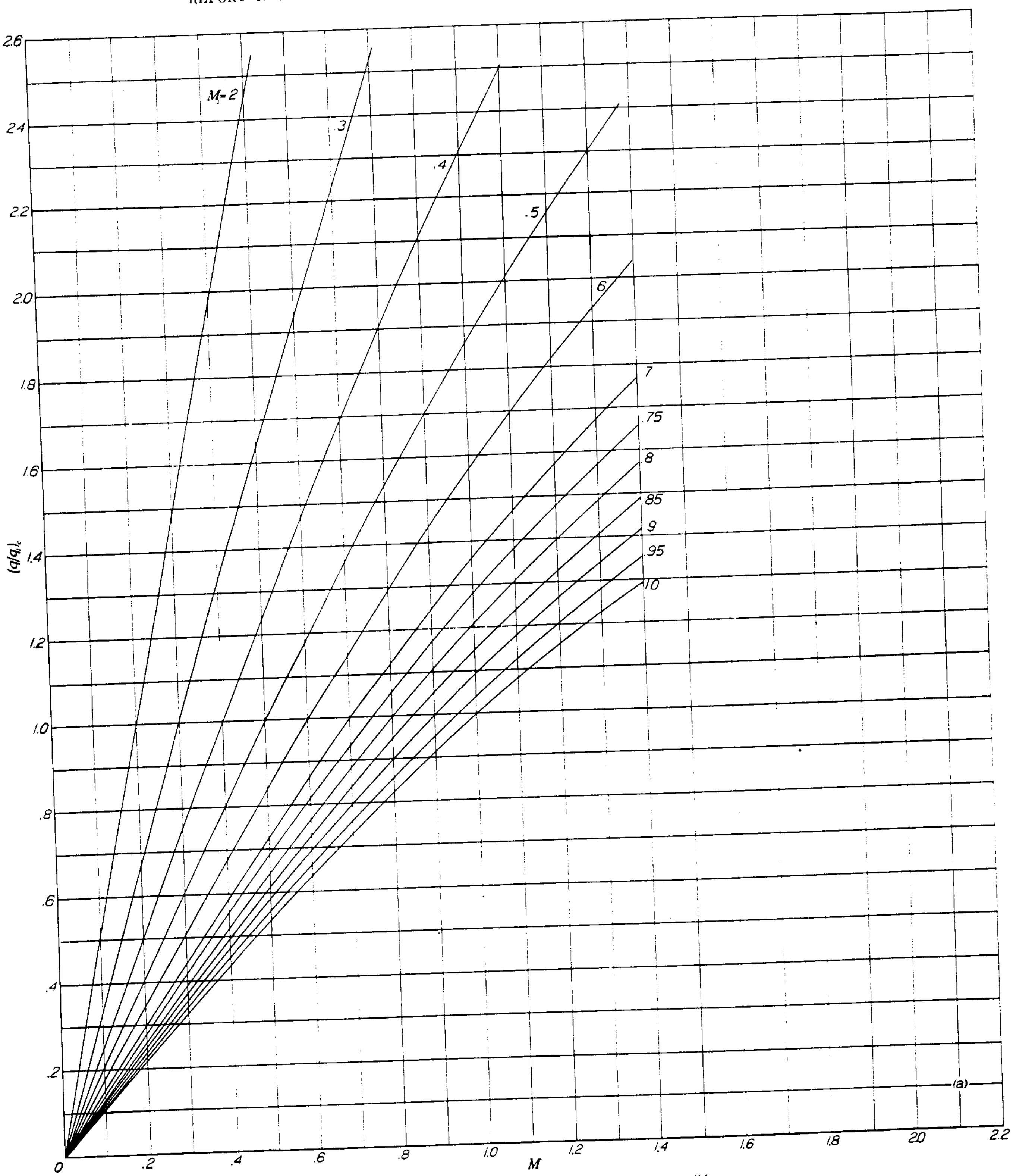
$$\tau = \frac{M^2}{2\beta + M^2}$$

$$\tau_1 = \frac{M_1^2}{2\beta + M_1^2}$$

and

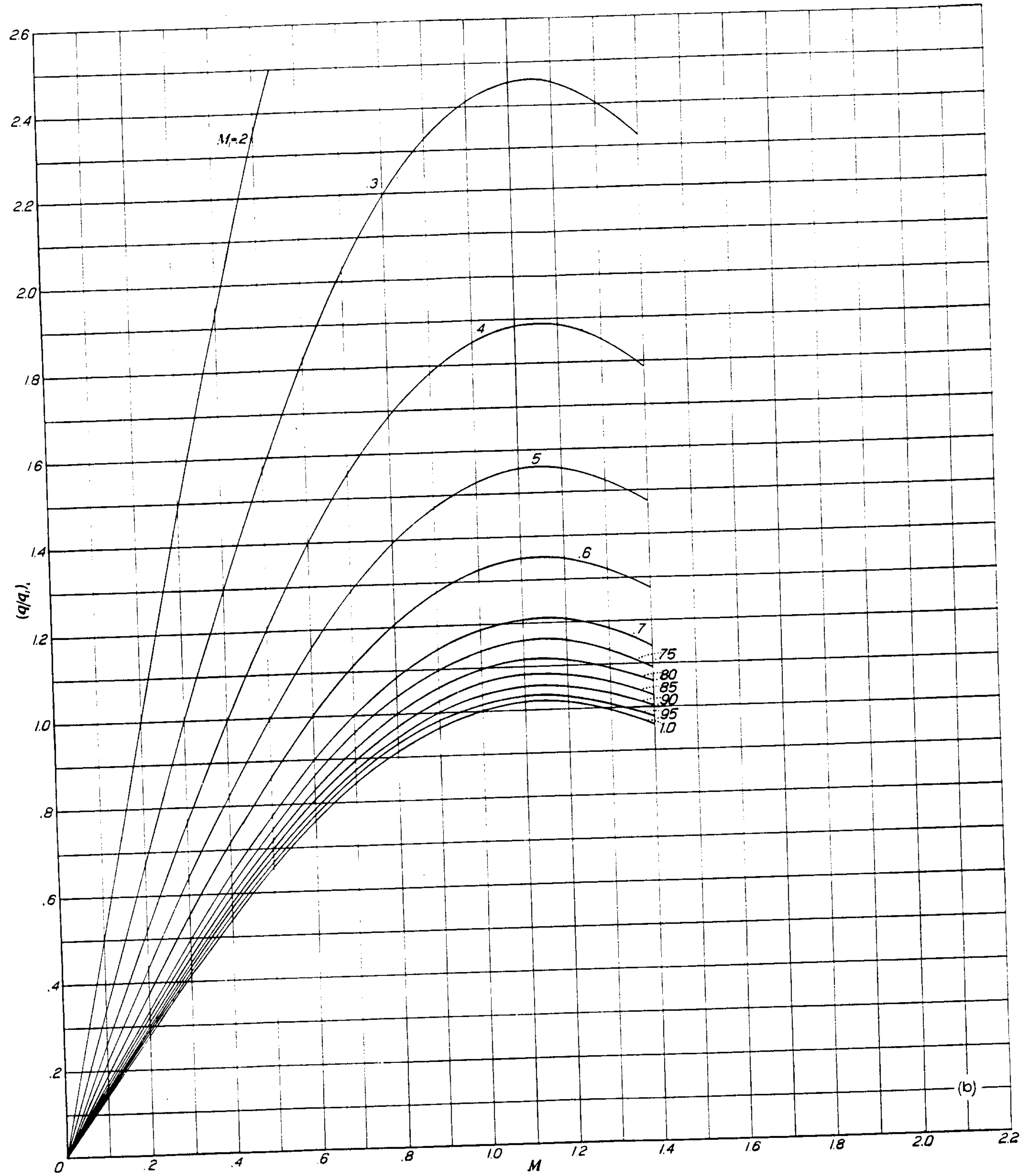
$$\begin{aligned} \left(\frac{q}{q_1}\right)_c &= \left(\frac{\tau}{\tau_1}\right)^{1/2} \\ &= \frac{M}{M_1}\left(\frac{2\beta + M_1^2}{2\beta + M^2}\right)^{1/2} \end{aligned} \tag{38}$$

Equation (37) then yields, for a given set of values of the stream Mach number M_1 and the local Mach number M, a value for the ratio $(q/q_1)_i$ of the local velocity q and the stream velocity q_1 in an incompressible fluid. Table 2 shows corresponding values of $(q/q_1)_c$ and $(q/q_1)_i$ for various values of the stream Mach number M_1 with $\gamma = 1.4$ $(\beta = 2.5)$. This tabulation is performed, for the purpose of comparison, for the three cases represented by equations (35), (36), and (37). Values of $(q/q_1)_i$, $(q/q_1)_c$, and $\frac{(q/q_1)_c}{(q/q_1)_i}$, obtained from equations (37) and (38), are plotted against the local Mach number M in figure 2 for various values of the stream Mach number M_1. Table 2 also shows values of the pressure coefficients $C_{p,0}$ and C_{p,M_1} calculated by equations (18a) and (18b) for these corresponding values of $(q/q_1)_i$ and $(q/q_1)_c$. Figure 3 shows the curves of pressure coefficients corresponding to the curves of velocities of figure 2. Useful cross plots of the curves in figure 3 are shown in figure 4, in which C_{p,M_1} is plotted against M_1 for various values of $C_{p,0}$. In addition, curves are shown in figure 4 for $(C_{p,M_1})_s$ and $(C_{p,M_1})_m$ calculated by equations (18c) and (18d), respectively. The curve for $(C_{p,M_1})_s$ corresponds to the sonic value $M=1$ or $\tau = \tau_s = \frac{1}{6}$ and in effect divides the region of flow into a subsonic and a supersonic part. The curve of $(C_{p,M_1})_m$ corresponds to the maximum value $M = \infty$ or $\tau = 1$ and represents the outer limit of the supersonic region (or a perfect vacuum). In order to exhibit the main differences between the various correction formulas (35), (36), and (37), the ratios of the sonic values $(C_{p,M_1})_s$ and the corresponding incompressible values $C_{p,0}$ are plotted against the stream Mach number M_1 in figure 5.



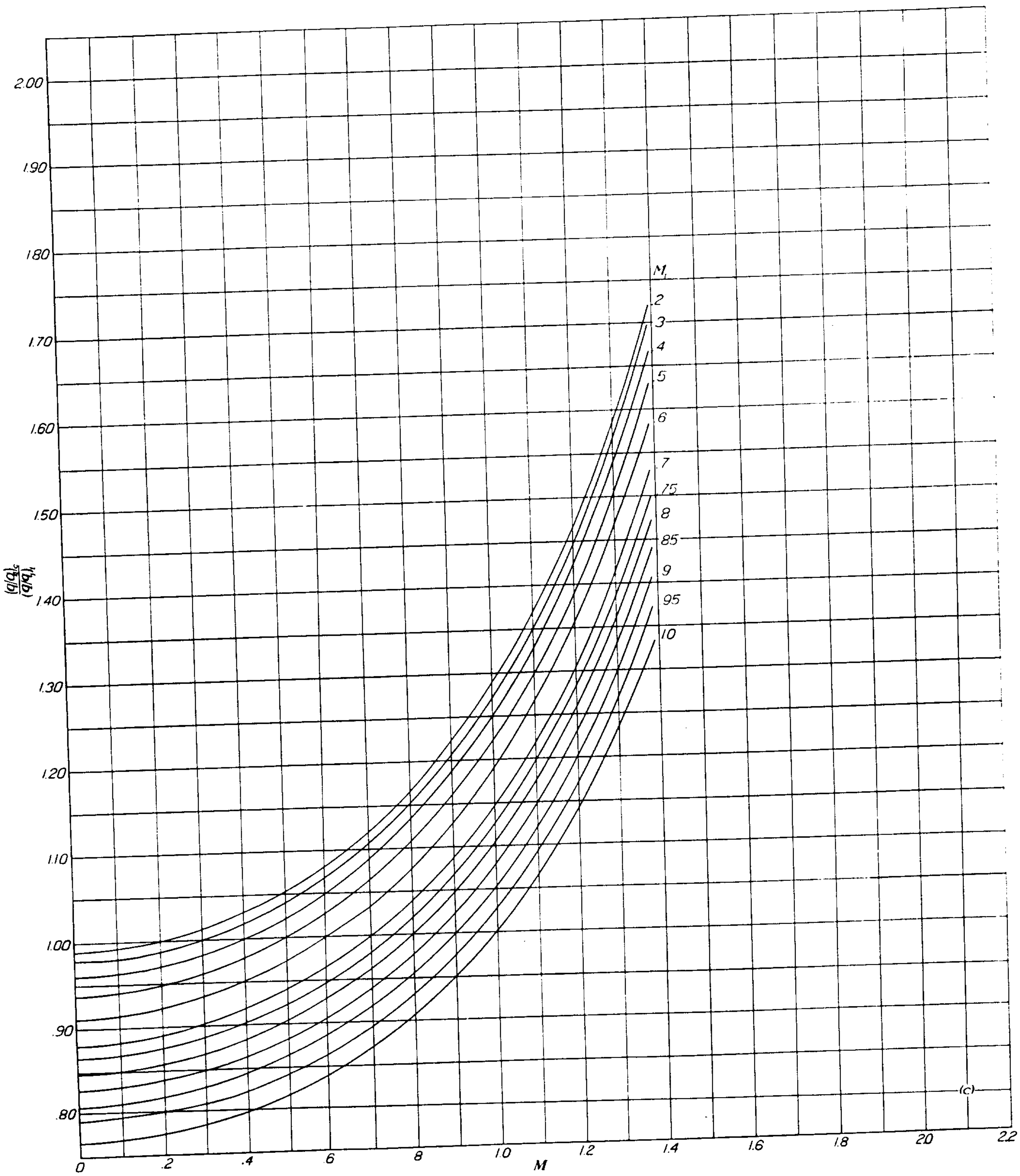
(a) $(q/q_1)_c$, ratio of local velocity to stream velocity, compressible.

FIGURE 2.—Velocity ratios $(q/q_1)_c$, $(q/q_1)_i$, and $(q/q_1)_c/(q/q_1)_i$ against local Mach number for various values of stream Mach number M_1. Corresponding values of $(q/q_1)_c$ and $(q/q_1)_i$ in (a) and (b) are given by the same pair of values M, M_1.



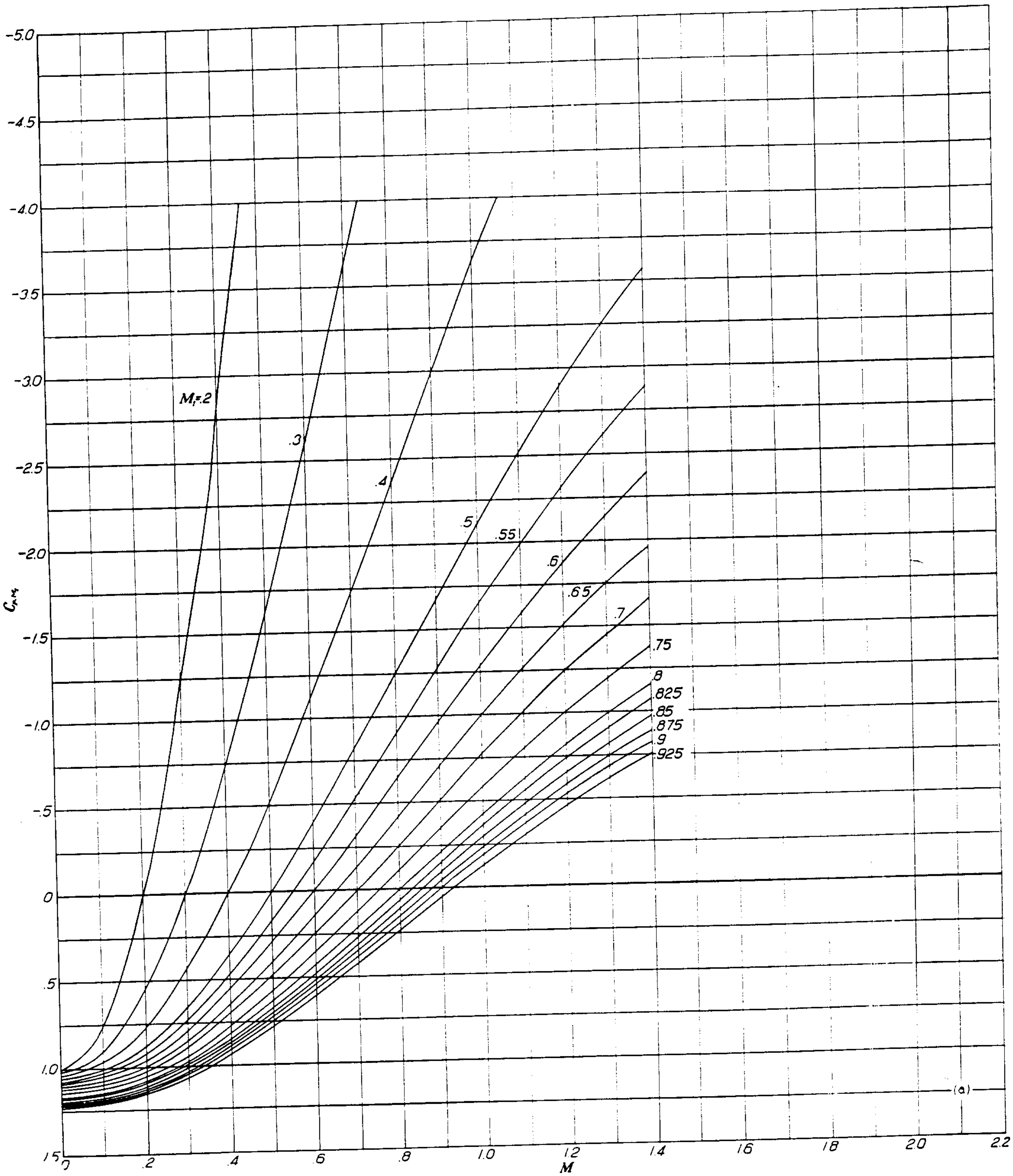
(b) $(q/q_1)_i$, ratio of local velocity to stream velocity, incompressible.

FIGURE 2.—Continued.



(c) Curves of $(q/q_1)_s/(q/q_1)_i$.

FIGURE 2.—Concluded.



(a) C_{p,M_1}-curves.

FIGURE 3.—Pressure coefficients C_{p,M_1} and $C_{p,0}$, against local Mach number M for various values of stream Mach number M_1. Corresponding values of C_{p,M_1} and $C_{p,0}$ in (a) and (b) are given by the same pair of values M, M_1.

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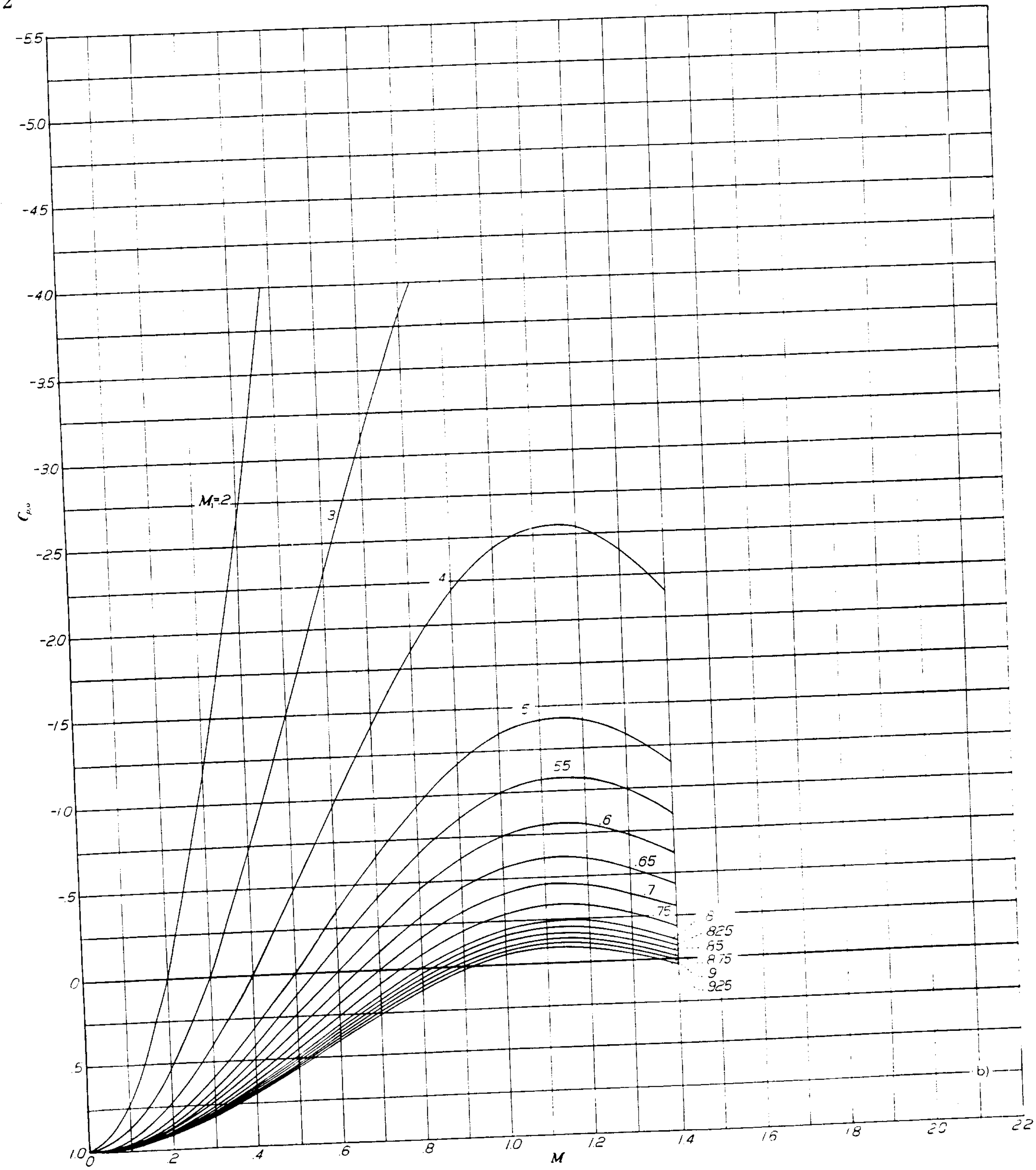

(b) $C_{p,0}$-curves.
FIGURE 3.—Concluded.

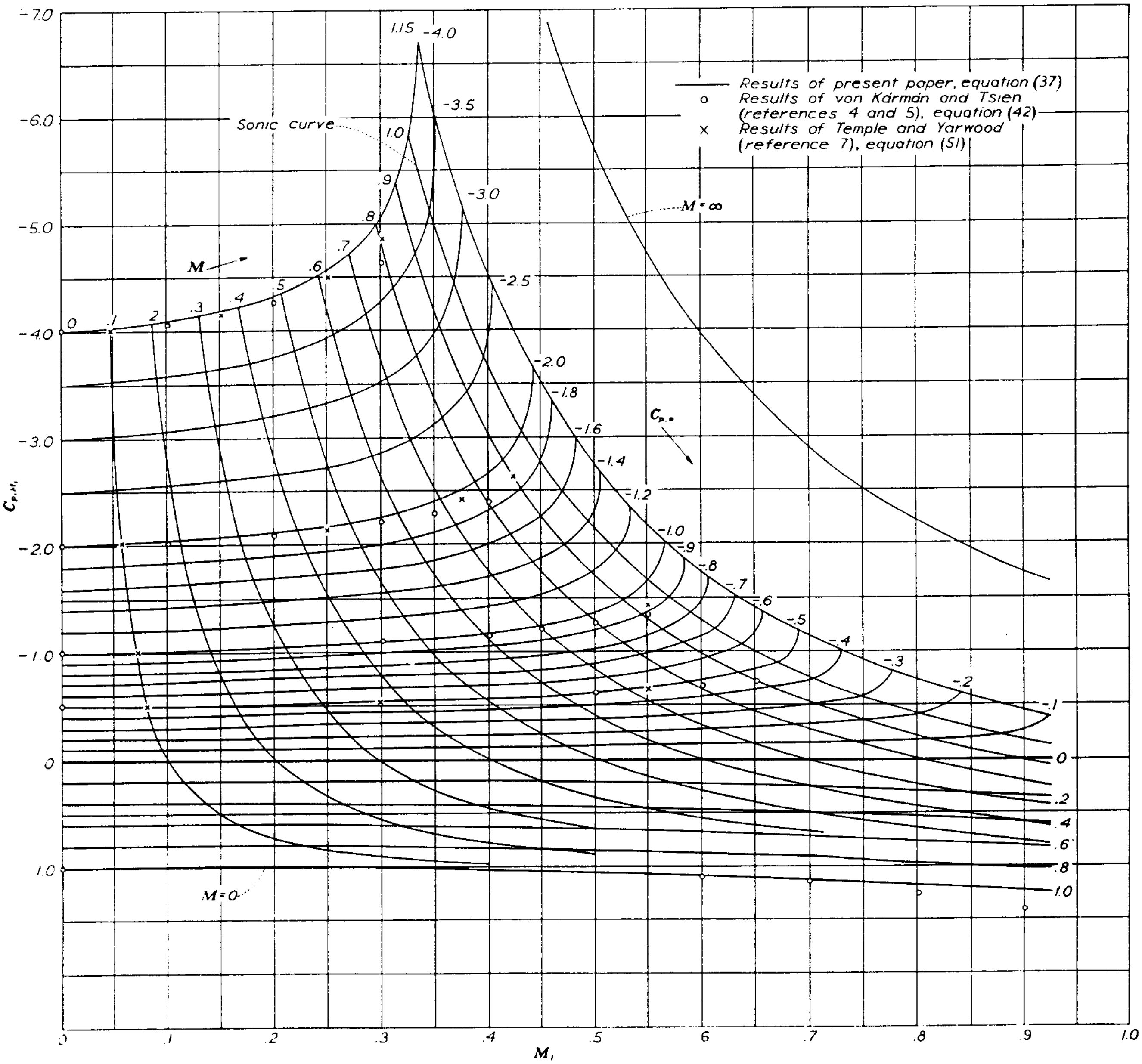

FIGURE 4.—Pressure coefficient C_{p,M_1} against stream Mach number M_1 with corresponding values of $C_{p,0}$; and lines of constant local Mach number M.

Observe in figure 2 that the $(q/q_1)_i$-curves have maximum points. This fact means that the value of $(q/q_1)_c$ associated with a value of $(q/q_1)_i$ is not unique. Analytically, the criterion for the maximum point is equivalent to

$$\frac{d(q/q_1)_i^2}{d\tau}=0 \tag{39}$$

or, from velocity correction formula (37),

$$(1-\tau)^{2\beta+1}-(2\beta+1)\tau+1=0$$

For $\beta=2.5$ this equation has only one positive root, $\tau\approx\frac{5}{24}$ or $M\approx 1.15$. It is interesting to note that velocity correction formula (36) yields as the criterion for the maximum point

$$1-(2\beta+1)\tau=0$$

The root of this equation is $\tau=\tau_s=\frac{1}{2\beta+1}$ and, for $\beta=2.5$, is $\tau=\frac{1}{6}$ or $M=1$. Velocity correction formula (35) yields no maximum value of τ or M.

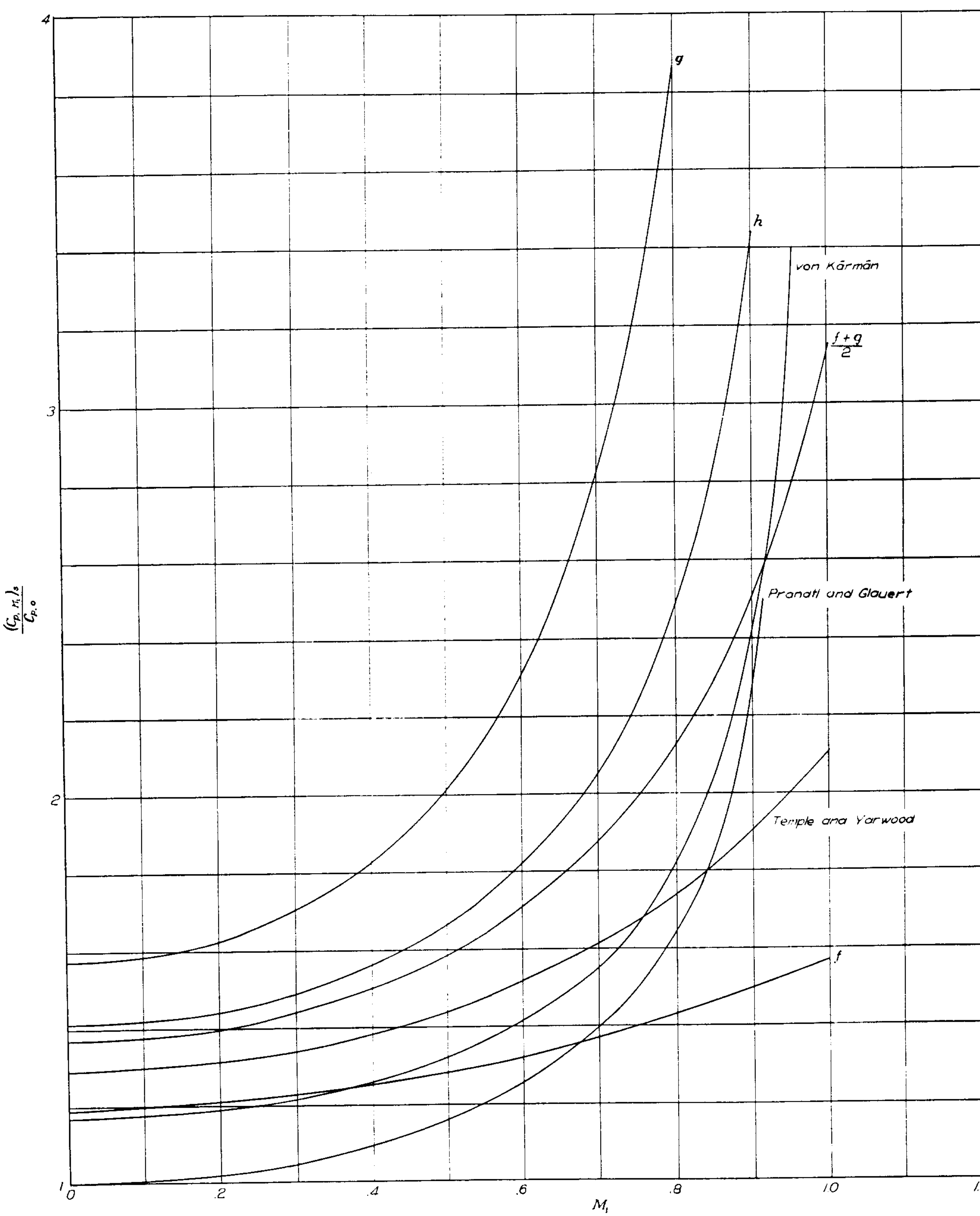

FIGURE 5.—The ratio of $(C_{p,M_1})_s$ to $C_{p,0}$ against M_1 for the various approximations.

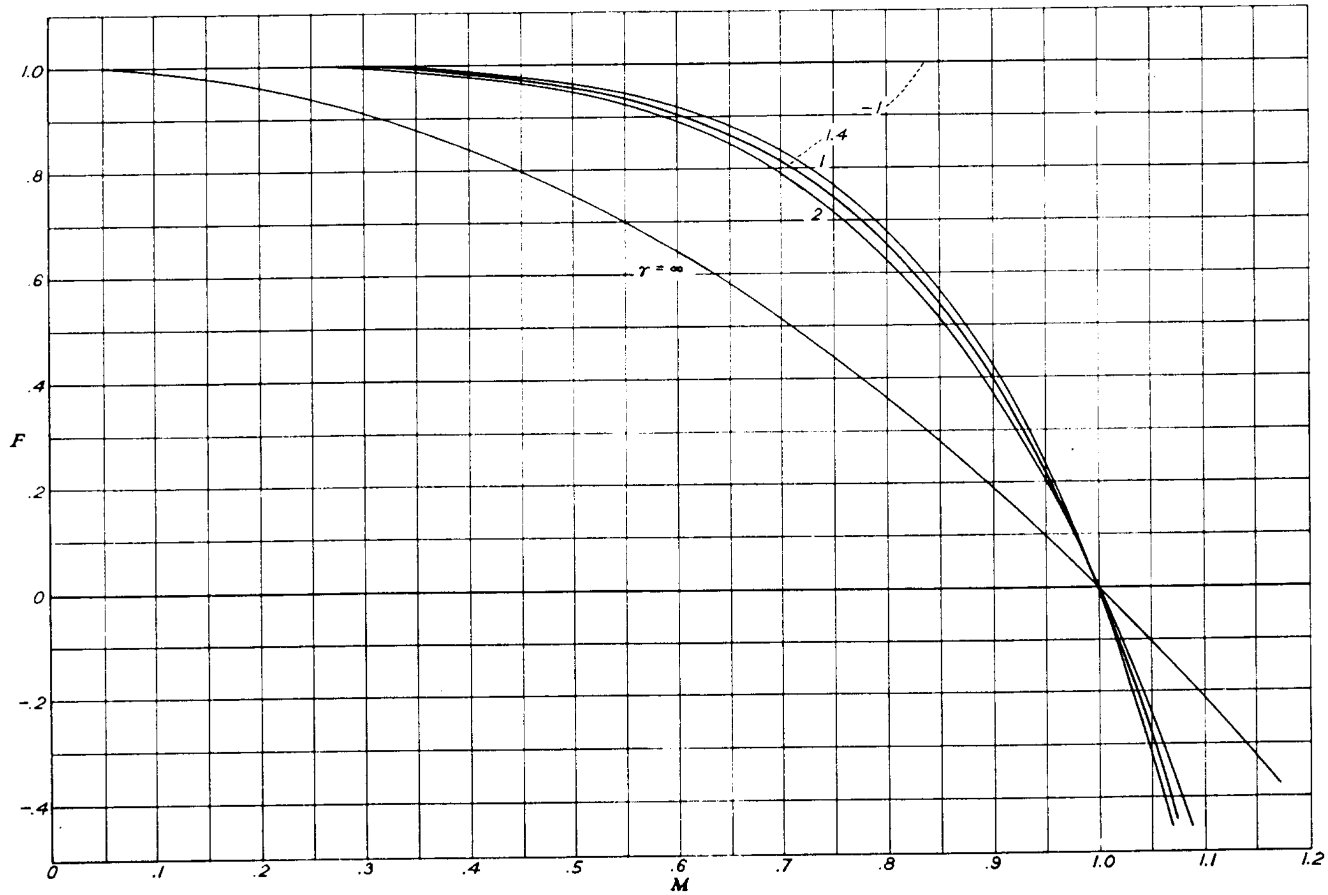

FIGURE 6.—The function F against Mach number M for several values of the adiabatic index γ.

Meaning can be given to the value $\tau=\frac{1}{6}$ $(M=1)$ in the case of equation (34) with reference to the original interpretation of the flow pattern as that of a source. It can be shown that the acceleration $\left(q\frac{dq}{ds}\right)_c$ along a streamline is infinite at all points for which the local Mach number is unity $\left(\tau=\frac{1}{6}\right)$ and that a flow discontinuity exists there. In the case of the vortex flow pattern (equation (33)), no flow discontinuity occurs for $M<\infty$. The velocity correction formula (37) suggests a "limiting" value $M\approx 1.15$ for a spiral flow, since equation (39) is analogous to a condition of infinite acceleration. Thus, the existence of a mixed subsonic and supersonic region of flow without discontinuities is indicated. Since the occurrence of this limiting value of M is a consequence of the simple form assumed for the velocity correction formula, no undue significance should be attached to any particular value at the present time.

THE CHAPLYGIN APPROXIMATION

From the point of view of the present paper, Chaplygin's approximation for subsonic speeds assumes a simple and lucid form. Chaplygin introduces in place of q a new independent speed variable η equivalent to the quantity given on the right-hand side of equation (33), namely,

$$\eta=qe^{f(\tau)}$$

The hodograph flow equations (7) then assume the form

$$\left.\begin{aligned}\frac{\partial\phi}{\partial\theta}&=\eta\,\frac{\partial\psi}{\partial\eta}\\ \eta\,\frac{\partial\phi}{\partial\eta}&=-F(\tau)\,\frac{\partial\psi}{\partial\theta}\end{aligned}\right\}\tag{40}$$

where

$$F(\tau)=\frac{1-(2\beta+1)\tau}{(1-\tau)^{2\beta+1}}$$

$$=1-\beta(2\beta+1)\tau^2-\frac{2}{3}\,\beta(2\beta+1)(2\beta+2)\tau^3-\ .\ .\ .$$

Values of the function $F(\tau)$, for several values of γ (or β), are given in table 3 and are plotted in figure 6 against the local Mach number M. Chaplygin noted that, in the case of air ($\beta=2.5$), $F(\tau)$ differs but little from unity over about one-half the subsonic range $0\leqq\tau\leqq\frac{1}{6}$. His approximation in the range of low subsonic speeds consists in neglecting powers of τ higher than the first or in replacing $F(\tau)$ by unity.

Equations (40) can then be written in the Cauchy-Riemann form

$$\frac{\partial\phi}{\partial\theta}=\frac{\partial\psi}{\partial\log\eta}$$

$$\frac{\partial\phi}{\partial\log\eta}=-\frac{\partial\psi}{\partial\theta}$$

and $\phi+i\psi$ therefore is an analytic function of the complex variable $\theta+i\log\eta$. Chaplygin's approximation thus leads to the velocity correction formula

$$\left(\frac{q}{q_1}\right)_i=\left(\frac{q}{q_1}\right)_c^{\frac{1-\frac{5}{4}\tau}{1-\frac{5}{4}\tau_1}} \tag{41}$$

where powers of τ higher than the first are neglected throughout. The use of equation (34) instead of equation (33) also leads to this result to the same order of approximation.

THE VON KÁRMÁN APPROXIMATION

Von Kármán's approximation corresponds to the case $\gamma=-1$ $\left(\text{or } \beta=-\frac{1}{2}\right)$. It follows at once from the integral expressions for $f(\tau)$ and $g(\tau)$ given by equations (26) and (27), respectively, that for this case

$$f(\tau)=g(\tau)=-\log\frac{1+(1-\tau)^{1/2}}{2}$$

or, with the use of equation (16),

$$f(\tau)=g(\tau)=-\log\frac{1}{2}\left[1+\frac{1}{(1-M^2)^{1/2}}\right]$$

This function, plotted against M, is included in figure 1(b). Corresponding to equations (35) and (36), there is a single equation

$$\left(\frac{q}{q_1}\right)_i=\left(\frac{q}{q_1}\right)_c\frac{1+(1-\tau_1)^{1/2}}{1+(1-\tau)^{1/2}}$$

Replacing τ by $\tau_1\left(\frac{q}{q_1}\right)_c^2$ and τ_1 by $\frac{M_1^2}{M_1^2-1}$ according to equation (16) yields

$$\left(\frac{q}{q_1}\right)_i=\left(\frac{q}{q_1}\right)_c\frac{1+(1-M_1^2)^{1/2}}{(1-M_1^2)^{1/2}+\left[1-M_1^2+M_1^2\left(\frac{q}{q_1}\right)_c^2\right]^{1/2}} \tag{42}$$

Then, by solving for $(q/q_1)_c$ in terms of $(q/q_1)_i$ and the stream Mach number M_1,

$$\left(\frac{q}{q_1}\right)_c=\left(\frac{q}{q_1}\right)_i\frac{1-\mu}{1-\mu\left(\frac{q}{q_1}\right)_i^2} \tag{43}$$

where

$$\mu=\frac{M_1^2}{[1+(1-M_1^2)^{1/2}]^2}$$

The pressure coefficient C_{p,M_1}, expressed in terms of the incompressible pressure coefficient $C_{p,0}$, is easily obtained from the general formula (18b) by putting $\gamma=-1$ and making use of equations (43) and (18a). Thus,

$$C_{p,M_1}=C_{p,0}\frac{1}{(1-M_1^2)^{1/2}+\frac{M_1^2}{1+(1-M_1^2)^{1/2}}\frac{C_{p,0}}{2}} \tag{44}$$

Observe that for this case the function $F(\tau)$ introduced by Chaplygin and given in equation (40) is exactly equal to unity. From the point of view of the present paper then, von Kármán's approximation appears to be equivalent to that of Chaplygin, who approximates $F(\tau)$ by unity. It follows that the range of validity of von Kármán's approximation and that of Chaplygin, in a strict sense, coincide. Furthermore, it is pointed out that the von Kármán approximation does not permit a supersonic region. Von Kármán's choice of $\gamma=-1$ has the advantage, however, of yielding simple explicit expressions for $(q/q_1)_c$ in terms of $(q/q_1)_i$ and for C_{p,M_1} in terms of $C_{p,0}$. Several values of C_{p,M_1} calculated by equation (44) are included in figure 4. For the purpose of comparison with the other approximations, there is plotted in figure 5 the ratio of $(C_{p,M_1})_s$ to $C_{p,0}$ against the stream Mach number M_1 in the case of von Kármán's approximation. The values of $C_{p,0}$ are obtained with the use of velocity correction formula (42) for the local Mach number $M=1$, but the values of $(C_{p,M_1})_s$ are calculated with $\gamma=1.4$.

THE TEMPLE-YARWOOD APPROXIMATION

The functions ϕ and ψ related by the first-order simultaneous equations (21) separately satisfy the second-order equations

$$\left.\begin{aligned}\frac{\partial^2\phi}{\partial\theta^2}+\lambda_1(q)\frac{\partial}{\partial q}\left[\frac{1}{\lambda_2(q)}\frac{\partial\phi}{\partial q}\right]=0\\ \frac{\partial^2\psi}{\partial\theta^2}+\frac{1}{\lambda_2(q)}\frac{\partial}{\partial q}\left[\lambda_1(q)\frac{\partial\psi}{\partial q}\right]=0\end{aligned}\right\} \tag{45}$$

In terms of the nondimensional speed variable τ and with the values of $\lambda_1(q)$ and $\lambda_2(q)$ for the adiabatic case given by equations (25), these equations take the form

$$\left.\begin{aligned}\frac{1}{4}\frac{(1-\tau)^\beta}{\tau}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial}{\partial\tau}\left[\frac{\tau(1-\tau)^{\beta+1}}{1-(2\beta+1)\tau}\frac{\partial\phi}{\partial\tau}\right]=0\\ \frac{1}{4}\frac{1-(2\beta+1)\tau}{\tau(1-\tau)^{\beta+1}}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial}{\partial\tau}\left[\frac{\tau}{(1-\tau)^\beta}\frac{\partial\psi}{\partial\tau}\right]=0\end{aligned}\right\} \tag{46}$$

Formal solutions of these equations were given by Chaplygin in the form of two infinite series

$$\left.\begin{aligned}\psi=B\theta+\sum_{m=1}^{\infty}B_m\psi_m(\tau)\sin(m\theta+\epsilon_m)\\ \phi=-B\phi_0(\tau)-\sum_{m=1}^{\infty}B_m\phi_m(\tau)\cos(m\theta+\epsilon_m)\end{aligned}\right\} \tag{47}$$

where the functions $\psi_m(\tau)$ and $\phi_m(\tau)$ are obtained from hypergeometric series and B, B_m, and ϵ_m are arbitrary constants.

A disadvantage of the formal solution, as remarked by Temple and Yarwood, is that it is unsuitable for numerical computation because the hypergeometric functions involved are complicated and are not tabulated. Temple and Yarwood therefore looked for approximations that are of practical value in calculations of compressible flows. By means of a skillful analysis, they found such approximations and showed that the simplest forms for ψ_m and ϕ_m are of the type

$$\left.\begin{aligned}\psi_m(\tau)&\approx[\eta(\tau)]^m\\ \phi_m(\tau)&\approx[\xi(\tau)]^m\\ \phi_0(\tau)&\approx\log\xi(\tau)\end{aligned}\right\}\tag{48}$$

where $\eta(\tau)$ and $\xi(\tau)$, independent of the index m, are

$$\eta=\xi=\left(1-\frac{5}{4}\tau\right)q\tag{49}$$

Significantly, from the point of view of the analysis of the present paper, the functions η and ξ, approximated by $\left(1-\frac{5}{4}\tau\right)q$ are none other than the functions defined on the right-hand sides of equations (33) and (34). The approximation of Temple and Yarwood then leads to the same velocity correction relation as was obtained by means of Chaplygin's approximation (equation (41)).

The velocity and pressure-coefficient correction formulas obtained by Temple and Yarwood are more involved than the explicit expressions (43) and (44) obtained by von Kármán. Replacing τ in equation (41) by $\tau_1\left(\frac{q}{q_1}\right)_c^2$ thus yields

$$\left(\frac{q}{q_1}\right)_i=\left(\frac{q}{q_1}\right)_c\frac{1-\frac{5}{4}\tau_1\left(\frac{q}{q_1}\right)_c^2}{1-\frac{5}{4}\tau_1}\tag{50}$$

where

$$\tau_1=\frac{M_1^2}{5+M_1^2}$$

The solution of this cubic equation for $(q/q_1)_c$ is

$$\left(\frac{q}{q_1}\right)_c=\left(\frac{q}{q_1}\right)_i 3\left(1-\frac{5}{4}\tau_1\right)\frac{\cos\frac{1}{3}(\pi+\sigma)}{\cos\sigma}\tag{51}$$

where

$$\cos\sigma=\frac{3\sqrt{3}}{2}\left(1-\frac{5}{4}\tau_1\right)\left(\frac{5}{4}\tau_1\right)^{1/2}\left(\frac{q}{q_1}\right)_i$$

and $0<\sigma\leqq\frac{\pi}{2}$. The pressure coefficient C_{p,M_1} is then calculated by equation (18b). Some values of the pressure coefficient C_{p,M_1} calculated with the aid of equation (51) are shown in figure 4; a curve of $\frac{(C_{p,M_1})_s}{C_{p,0}}$ plotted against M_1 is included in figure 5. It is remarked that, with the use of equation (39), the velocity correction formula (50) yields a limiting value $M\approx1.35$.

APPROXIMATION BASED ON GEOMETRIC MEAN OF dL AND $d\tilde{L}$

Without going into its deep significance in the present paper, it is of interest to introduce another function related to L and $\tilde{L}$ and to the general particular solutions. This function, which like L and $\tilde{L}$ reduces to log q for $\tau=0$, is defined by

$$H(\tau)=\int(dL\,d\tilde{L})^{1/2}\tag{52}$$

It is remarked that $H(\tau)$ is closely related to a function $K(\tau)$ employed by Temple and Yarwood (reference 7) in the determination of their approximation. In the next section, it will be seen that the function $H(\tau)$ plays an important role in connection with the Prandtl-Glauert approximation.

From equations (26) and (27),

$$dL=\frac{1}{\lambda_1}dq=(1-\tau)^\beta\frac{dq}{q}$$

and

$$d\tilde{L}=\lambda_2 dq=\frac{1-(2\beta+1)\tau}{(1-\tau)^{\beta+1}}\frac{dq}{q}$$

Then,

$$(dL\,d\tilde{L})^{1/2}=\left(\frac{\lambda_2}{\lambda_1}\right)^{1/2}dq=\left[\frac{1-(2\beta+1)\tau}{1-\tau}\right]^{1/2}\frac{dq}{q}\tag{53}$$

and, from equation (52),

$$H(\tau)=\log q+h(\tau)\tag{54}$$

where

$$h(\tau)=\frac{1}{2}\int_0^\tau\left\{\left[\frac{1-(2\beta+1)\tau}{1-\tau}\right]^{1/2}-1\right\}\frac{d\tau}{\tau}$$

The function $h(\tau)$ can be obtained in a closed form for any value of γ (or β) and is

$$h(\tau)=-\log\frac{\left[(1-\tau)^{1/2}+\left(1-\frac{\tau}{\tau_s}\right)^{1/2}\right]\left[(1-\tau)^{1/2}-(\tau_s-\tau)^{1/2}\right]^{\frac{1}{\sqrt{\tau_s}}}}{2(1-\sqrt{\tau_s})^{\frac{1}{\sqrt{\tau_s}}}}\tag{55a}$$

where $\tau_s=\frac{1}{2\beta+1}$ and where this expression is valid in the subsonic range $0\leqq\tau\leqq\tau_s$. With τ replaced by $\frac{M^2}{2\beta+M^2}$ and $0\leqq M\leqq1$, the expression for $h(\tau)$ becomes

$$h(\tau)=-\log\frac{1+(1-M^2)^{1/2}}{2}-\frac{1-\sqrt{\tau_s}}{2\sqrt{\tau_s}}\log\frac{1-\sqrt{\tau_s}(1-M^2)^{1/2}}{1-\sqrt{\tau_s}}$$
$$+\frac{1+\sqrt{\tau_s}}{2\sqrt{\tau_s}}\log\frac{1+\sqrt{\tau_s}(1-M^2)^{1/2}}{1+\sqrt{\tau_s}}\tag{55b}$$

It is observed that, for the supersonic region $\tau_s\leqq\tau\leqq1$ or $M>1$, $H(\tau)$ as defined by equation (52) becomes a complex function; but, for present purposes, only the real function of the subsonic range is utilized.

The function $H(\tau)$ may be utilized to obtain a velocity correction formula in the same manner as the functions $L(\tau)$ and $\tilde{L}(\tau)$. Thus, analogous to equation (35), (36), or (37).

$$\left(\frac{q}{q_1}\right)_i=\left(\frac{q}{q_1}\right)_c\frac{e^{h(\tau)}}{e^{h(\tau_1)}}\tag{56}$$

It is instructive to compare equation (56) with the approximation given by equation (37). Equation (37) may be written as

$$\left(\frac{q}{q_1}\right)_i=\frac{e^{\int \frac{1}{2}(dL+d\bar{L})}}{\left[e^{\int \frac{1}{2}(dL+d\bar{L})}\right]_{\tau=\tau_1}}$$

and equation (56) may be written as

$$\left(\frac{q}{q_1}\right)_i=\frac{e^{\int (dL\,d\bar{L})^{1/2}}}{\left[e^{\int (dL\,d\bar{L})^{1/2}}\right]_{\tau=\tau_1}}$$

Thus, the power of the exponential is in one case the integral of the arithmetic mean $\frac{dL+d\bar{L}}{2}$ and in the other case the integral of the geometric mean $(dL\,d\bar{L})^{1/2}$. Table I shows values of the functions $\frac{f(\tau)+g(\tau)}{2}$ and $h(\tau)$ in the case of air $\left(\gamma=1.4,\ \beta=2.5,\text{ and }\tau_s=\frac{1}{6}\right)$ and figures 1(a) and 1(b) show these functions plotted against τ and M, respectively. Observe that these functions, and consequently the velocity correction formulas (37) and (56), differ only slightly in the subsonic range $0<M<1$. Figure 5 exhibits graphically a comparison of the velocity correction formulas (37) and (56) for $M=1$. The limiting value of M (defined by equation (39)) is $M=1$ in the case of equation (56) as compared with $M\approx 1.15$ in the case of equation (37).

COMPARISON OF RESULTS OF PRESENT PAPER WITH PRANDTL-GLAUERT APPROXIMATION

The well-known Prandtl-Glauert approximation is based on the assumption of vanishingly small disturbances to the main stream. The Prandtl-Glauert velocity correction formula may be expressed as

$$\frac{\left(\frac{q-q_1}{q_1}\right)_c}{\left(\frac{q-q_1}{q_1}\right)_i}=\frac{1}{(1-M_1^2)^{1/2}} \tag{57}$$

where $q-q_1$ is vanishingly small. The left-hand side of this equation is actually the differential coefficient $\frac{d(q/q_1)_c}{d(q/q_1)_i}$ evaluated at the main stream velocity $q=q_1$ (or $\tau=\tau_1$). An exact form of the Prandtl-Glauert approximation then is

$$\left[\frac{d(q/q_1)_c}{d(q/q_1)_i}\right]_{\tau=\tau_1}=\frac{1}{(1-M_1^2)^{1/2}} \tag{58}$$

The differential coefficient in equation (58) is now evaluated for the various approximations treated in the present paper.

For the arithmetic-mean approximation of the present paper given by equation (37) (γ or β arbitrary),

$$\left[\frac{d(q/q_1)_c}{d(q/q_1)_i}\right]_{\tau=\tau_1}=\frac{2}{(1-\tau_1)^{\beta}+\frac{1-(2\beta+1)\tau_1}{(1-\tau_1)^{\beta+1}}}$$

$$=\frac{2\left(1+\frac{M_1^2}{2\beta}\right)^{\beta}}{1+(1-M_1^2)\left(1+\frac{M_1^2}{2\beta}\right)^{2\beta}}$$

$$=1+\frac{1}{2}M_1^2+\frac{3}{8}M_1^4+\frac{5}{16}M_1^6$$

$$+\frac{5}{128}\ \frac{11\beta^2+4\beta+1}{\beta^2}M_1^8+\ .\ .\ . \tag{59}$$

For the Chaplygin or the Temple-Yarwood approximation given by equation (41) ($\gamma=1.4$ or $\beta=2.5$),

$$\left[\frac{d(q/q_1)_c}{d(q/q_1)_i}\right]_{\tau=\tau_1}=\frac{1-\frac{5}{4}\tau_1}{1-\frac{15}{4}\tau_1}$$

$$=\frac{1-\frac{1}{20}M_1^2}{1-\frac{11}{20}M_1^2}$$

$$=1+\frac{1}{2}M_1^2+\frac{11}{40}M_1^4+\ .\ .\ . \tag{60}$$

For the von Kármán approximation given by equation (42) $\left(\gamma=-1\text{ or }\beta=-\frac{1}{2}\right)$,

$$\left[\frac{d(q/q_1)_c}{d(q/q_1)_i}\right]_{\tau=\tau_1}=(1-\tau)^{1/2}$$

$$=\frac{1}{(1-M_1^2)^{1/2}} \tag{61}$$

For the geometric-mean approximation of the present paper given by equation (56) (γ or β arbitrary),

$$\left[\frac{d(q/q_1)_c}{d(q/q_1)_i}\right]_{\tau=\tau_1}=\left[\frac{1-\tau_1}{1-(2\beta+1)\tau_1}\right]^{1/2}$$

$$=\frac{1}{(1-M_1^2)^{1/2}} \tag{62}$$

Equation (62) is independent of the value of the adiabatic index γ and includes the von Kármán approximation. Observe that the geometric-mean approximation yields the Prandtl-Glauert result exactly, whereas the arithmetic-mean approximation yields the Prandtl-Glauert result insofar as terms inclusive of M_1^6 are concerned. The Chaplygin or the Temple-Yarwood approximation contains the Prandtl-Glauert result only insofar as the M_1^2-term is concerned.

RÉSUMÉ AND CONCLUDING REMARKS

1. Basic elementary solutions of the hodograph equations have been employed to provide a basis for comparison, in the form of velocity correction formulas, of corresponding compressible and incompressible flows.

2. The velocity correction formulas obtained by Chaplygin, by von Kármán, and by Temple and Yarwood have been unified by means of these basic solutions and shown to be essentially equivalent.

3. In the present paper two types of approximations have been introduced by means of the basic elementary solutions, namely, the "arithmetic-mean" type and the "geometric-mean" type. These approximations include those obtained by Chaplygin, by von Kármán, and by Temple and Yarwood.

4. The approximations discussed in the present paper have been compared with the well-known results of Prandtl and Glauert. For this purpose, it has been emphasized that the Prandtl-Glauert result is valid for vanishingly small disturbances and, in a strict sense, is the slope term in a Taylor expansion in a quantity which measures the disturbance. It was found that the arithmetic-mean type yields the Prandtl-Glauert result to a higher order of approximation than the Chaplygin or the Temple-Yarwood type and that the geometric-mean type contains the Prandtl-Glauert result exactly. The two types of approximations introduced in the present paper then appear to be preferable to the others as a basis for extrapolation into the range of high stream Mach numbers and large disturbances to the main stream.

5. The results of the present paper have been obtained without consideration of any particular boundary. The actual boundary problem of determining the flow past a prescribed body is of a high order of difficulty and involves in general all the particular solutions of the hodograph equations.

6. The particular solutions discussed in the present paper are well-behaved functions in both the subsonic and the supersonic regions. The hodograph equations give no reason, in general, to suppose that a discontinuity necessarily occurs in the solution when local sound speed is attained. Rather, it appears that the first breakdown of the solution is associated with the vanishing of the Jacobian of the transformation from the physical to the hodograph variables. Indeed, von Kármán has made an equivalent suggestion in that the appearance of infinite accelerations in the flow solution is a condition for flow discontinuities. Interesting speculations on this matter are suggested by the results of the present paper since the "limiting" curves discussed in the present paper are defined by a condition that is equivalent to the condition for infinite acceleration. The arithmetic-mean type of approximation thus yields a limiting value of the local Mach number $M\approx1.15$, and the geometric-mean type of approximation yields a limiting value of the local Mach number $M=1$. The value $M=1$ appears to be exact for vanishingly small disturbances; that is, local Mach number M=stream Mach number $M_1=1$ (Prandtl-Glauert approximation). However, for finite disturbances to the main flow due to the presence of a body in the fluid, infinite accelerations may occur, for stream Mach numbers less than unity, in regions where the local Mach number is greater than unity. In this regard, the arithmetic-mean type of approximation, considered as an extension of the Prandtl-Glauert relation to finite disturbances, indicates the possibility of a mixed subsonic and supersonic flow without discontinuities. It is important, however, to recognize that in general the limiting value of the local Mach number M is a function of shape parameters and is a result of the blending of many particular solutions of the hodograph flow equations according to the boundary conditions.

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TABLE 1.—VALUES OF f, g, $\frac{f+g}{2}$, h, AND THEIR EXPONENTIALS FOR $\gamma=1.4$

M	τ	f	g	$\frac{f+g}{2}$	h	e^f	e^g	$e^{\frac{f+g}{2}}$	e^h
0	0	0	0	0	0	1.00	1.00	1.00	1.00
.1	.00200	−.00250	−.00251	−.00251	−.00250	.99750	.99749	.99750	.99750
.2	.00794	−.00989	−.01013	−.01001	−.01001	.99016	.98992	.99004	.99004
.3	.01768	−.02196	−.02316	−.02256	−.02256	.97828	.97711	.97769	.97769
.4	.03101	−.03831	−.04208	−.04020	−.04020	.96241	.95879	.96060	.96060
.5	.04762	−.05847	−.06760	−.06304	−.06306	.94321	.93463	.93890	.93889
.6	.06716	−.08186	−.10059	−.09123	−.09133	.92140	.90430	.91281	.91271
.65	.07792	−.09457	−.12023	−.10740	−.10768	.90977	.88671	.89817	.89792
.70	.08925	−.10787	−.14216	−.12502	−.12541	.89775	.86748	.88248	.88213
.75	.10112	−.12167	−.16657	−.14412	−.14484	.88544	.84656	.86578	.86516
.80	.11348	−.13588	−.19363	−.16476	−.16605	.87295	.82397	.84810	.84700
.825	.11982	−.14313	−.20822	−.17568	−.17740	.86664	.81202	.83889	.83744
.850	.12626	−.15045	−.22354	−.18700	−.18927	.86032	.79968	.82944	.82756
.875	.13279	−.15785	−.23964	−.19875	−.20173	.85398	.78691	.81975	.81732
.900	.13942	−.16530	−.25652	−.21091	−.21487	.84764	.77374	.80985	.80664
.925	.14612	−.17281	−.27423	−.22352	−.22876	.84130	.76016	.79969	.79552
.950	.15290	−.18036	−.29280	−.23658	−.24353	.83497	.74617	.78933	.78386
.96	.15563	−.18339	−.30047	−.24193	−.24976	.83245	.74047	.78511	.77899
.98	.16113	−.18946	−.31624	−.25285	−.26292	.82741	.72888	.77658	.76880
1.00	.16667	−.19556	−.33261	−.26409	−.27757	.82237	.71705	.76791	.75762
1.02	.17224	−.20166	−.34958	−.27562	....	.81737	.70498	.75910	....
1.04	.17785	−.20778	−.36718	−.28748	....	.81239	.69268	.75015	....
1.06	.18349	−.21391	−.38542	−.29967	....	.80742	.68017	.74106	....
1.08	.18915	−.22003	−.40432	−.31218	....	.80250	.66743	.73185	....
1.10	.19485	−.22616	−.42390	−.32503	....	.79759	.65449	.72251	....
1.12	.20056	−.23228	−.44417	−.33823	....	.79273	.64136	.71303	....
1.15	.20917	−.24144	−.47594	−.35869	....	.78550	.62130	.69859	....
1.18	.21782	−.25059	−.50938	−.37999	....	.77834	.60087	.68387	....
1.20	.22360	−.25665	−.53263	−.39464	....	.77364	.58706	.67392	....
1.30	.25262	−.28673	−.66139	−.47406	....	.75072	.51613	.62247	....
1.40	.28161	−.31604	−.81292	−.56448	....	.72903	.44356	.56865	....
1.50	.31034	−.34438	−.99030	−.66734	....	.70866	.37147	.51307	....
2.00	.44444	−.46775	−2.39742	−1.43259	....	.62641	.09095	.23869	....
2.50	.55556	−.55925	−5.12014	−2.83970	....	.57164	.00598	.05844	....
3.00	.64286	−.62470	−9.99177	−5.30824	....	.53542	.00004	.00495	....
4.00	.76190	−.70538	−31.27238	−15.98888	....	.49392	.00000	.00000	....
5.00	.83333	−.74934	−80.81740	−40.78337	....	.47268	.00000	.00000	....
∞	1.00	−.84019	−∞	−∞	....	.43162	0	0	....

TABLE 2.—VALUES OF $(q/q_1)_e$, $(q/q_1)_i$, $\frac{(q/q_1)_e}{(q/q_1)_i}$, $C_{p,0}$, AND C_{p,M_1}

FOR $\gamma=1.4$ AND FOR VARIOUS VALUES OF M_1

M		0.2	0.3	0.4	0.5	0.55	0.6	0.7	0.8	0.9	1.0	1.1	1.2
τ		0.00794	0.01768	0.03101	0.04762	0.05705	0.06716	0.08925	0.11348	0.13942	0.16667	0.19485	0.22360
$M_1=0.2$													
$(q/q_1)_e$		1.00	1.49262	1.97660	2.44948	2.68106	2.90907	3.35349	3.78124	4.19121	4.58255	4.95484	5.30790
$(q/q_1)_i$	Eq. (35)	1.00	1.47471	1.92120	2.33332	2.52518	2.70706	3.04051	3.33365	3.58793	3.80599	3.99122	4.14722
	Eq. (36)	1.00	1.47331	1.91444	2.31268	2.49242	2.65746	2.93870	3.14735	3.27593	3.31937	3.27590	3.14780
	Eq. (37)	1.00	1.47401	1.91782	2.32296	2.50875	2.68213	2.98917	3.23912	3.42841	3.55441	3.61594	3.61309
$\frac{(q/q_1)_e}{(q/q_1)_i}$	Eq. (37)	1.00	1.01263	1.03065	1.05446	1.06868	1.08461	1.12188	1.16737	1.22249	1.28926	1.37028	1.46907
$C_{p,0}$ (Eq. (18a))		0	−1.17271	−2.67803	−4.39614	−5.29383	−6.19382	−7.93514	−9.49190	−10.75400	−11.63383	−12.07502	−12.05442
C_{p,M_1} (Eq. (18b))		0	−1.21250	−2.82393	−4.75500	−5.81393	−6.92214	−9.23893	−11.62214	−14.00071	−16.31357	−18.51393	−20.57000
$M_1=0.3$													
$(q/q_1)_e$		0.66997	1.00	1.32425	1.64107	1.79621	1.94898	2.24672	2.53331	2.80797	3.07016	3.31958	3.55612
$(q/q_1)_i$	Eq. (35)	0.67810	1.00	1.30277	1.58224	1.71233	1.83567	2.06177	2.26055	2.43299	2.58087	2.70646	2.81225
	Eq. (36)	0.67875	1.00	1.29942	1.56971	1.69172	1.80374	1.99463	2.13626	2.22355	2.25304	2.22352	2.13655
	Eq. (37)	0.67843	1.00	1.30110	1.57596	1.70200	1.81965	2.02793	2.19752	2.32593	2.41140	2.45317	2.45123
$\frac{(q/q_1)_e}{(q/q_1)_i}$	Eq. (37)	0.98753	1.00	1.01779	1.04131	1.05535	1.07107	1.10789	1.15280	1.20725	1.27319	1.35318	1.45075
$C_{p,0}$ (Eq. (18a))		0.53973	0	−0.69286	−1.48365	−1.89680	−2.31113	−3.11250	−3.82909	−4.40995	−4.81485	−5.01804	−5.00853
C_{p,M_1} (Eq. (18b))		0.55794	0	−0.74111	−1.62984	−2.11683	−2.62651	−3.69238	−4.78921	−5.88302	−6.94746	−7.95968	−8.90571
$M_1=0.4$													
$(q/q_1)_e$		0.50592	0.75514	1.00	1.23924	1.35640	1.47174	1.69659	1.91300	2.12040	2.31840	2.50674	2.68536
$(q/q_1)_i$	Eq. (35)	0.52051	0.76759	1.00	1.21451	1.31438	1.40903	1.58260	1.73519	1.86754	1.98105	2.07744	2.15865
	Eq. (36)	0.52235	0.76957	1.00	1.20801	1.30190	1.38811	1.53503	1.64401	1.71116	1.73387	1.71115	1.64422
	Eq. (37)	0.52143	0.76857	1.00	1.21124	1.30813	1.39853	1.55862	1.68897	1.78765	1.85336	1.88542	1.88394
$\frac{(q/q_1)_e}{(q/q_1)_i}$	Eq. (37)	0.97025	0.98253	1.00	1.02312	1.03690	1.05235	1.08852	1.13264	1.18614	1.25092	1.32954	1.42540
$C_{p,0}$ (Eq. (18a))		0.72811	0.40930	0	−0.46710	−0.71120	−0.95589	−1.42930	−1.85262	−2.19569	−2.43494	−2.55481	−2.54923
C_{p,M_1} (Eq. (18b))		0.76643	0.43714	0	−0.52420	−0.81188	−1.11259	−1.74152	−2.38866	−3.03429	−3.66205	−4.25938	−4.81750
$M_1=0.5$													
$(q/q_1)_e$		0.40825	0.60936	0.80695	1.00	1.09454	1.18763	1.36906	1.54370	1.71106	1.87084	2.02281	2.16694
$(q/q_1)_i$	Eq. (35)	0.42857	0.63202	0.82338	1.00	1.08222	1.16017	1.30307	1.42871	1.53769	1.63115	1.71051	1.77737
	Eq. (36)	0.43240	0.63706	0.82781	1.00	1.07773	1.14909	1.27069	1.36092	1.41652	1.43531	1.41651	1.36110
	Eq. (37)	0.43049	0.63454	0.82560	1.00	1.07998	1.15462	1.28679	1.39440	1.47588	1.53012	1.55662	1.55539
$\frac{(q/q_1)_e}{(q/q_1)_i}$	Eq. (37)	0.94834	0.96032	0.97741	1.00	1.01348	1.02859	1.06393	1.10707	1.15935	1.22267	1.29941	1.39318
$C_{p,0}$ (Eq. (18a))		0.81468	0.59736	0.31838	0	−0.16636	−0.33315	−0.65583	−0.94435	−1.17822	−1.34127	−1.42307	−1.41924
C_{p,M_1} (Eq. (18b))		0.87771	0.65366	0.35646	0	−0.19554	−0.40000	−0.82766	−1.26754	−1.70657	−2.13343	−2.53960	−2.91903
$M_1=0.55$													
$(q/q_1)_e$		0.37299	0.55672	0.73725	0.91363	1.00	1.08504	1.25081	1.41036	1.56326	1.70923	1.84810	1.97978
$(q/q_1)_i$	Eq. (35)	0.39601	0.58399	0.76083	0.92404	1.00	1.07202	1.20408	1.32017	1.42086	1.50722	1.58057	1.64235
	Eq. (36)	0.40122	0.59110	0.76811	0.92788	1.00	1.06621	1.17906	1.26278	1.31434	1.33178	1.31435	1.26294
	Eq. (37)	0.39861	0.58753	0.76446	0.92595	1.00	1.06911	1.19150	1.29114	1.36657	1.41680	1.44133	1.44019
$\frac{(q/q_1)_e}{(q/q_1)_i}$	Eq. (37)	0.93573	0.94756	0.96441	0.98669	1.00	1.01490	1.04978	1.09234	1.14393	1.20640	1.28222	1.37467
$C_{p,0}$ (Eq. (18a))		0.84111	0.65481	0.41560	0.14262	0	−0.14300	−0.41967	−0.66704	−0.86751	−1.00732	−1.07743	−1.07415
C_{p,M_1} (Eq. (18b))		0.91835	0.72685	0.47249	0.16737	0	−0.17497	−0.54078	−0.91731	−1.29299	−1.65828	−2.00590	−2.33058

TABLE 2.—Continued

M		0.4	0.6	0.65	0.7	0.75	0.8	0.85	0.9	1.0	1.1	1.2	1.3
τ		0.03101	0.06716	0.07792	0.08925	0.10112	0.11348	0.12626	0.13942	0.16667	0.19485	0.22350	0.25252

$M_1=0.6$

$(q/q_1)_e$		0.67947	1.00	1.07707	1.15277	1.22704	1.29981	1.37107	1.44093	1.57527	1.70324	1.82460	1.93339
$(q/q_1)_i$	Eq. (35)	0.70971	1.00	1.06348	1.12318	1.17914	1.23146	1.28018	1.32559	1.40596	1.47438	1.53201	1.58014
	Eq. (36)	0.72041	1.00	1.05612	1.10583	1.14869	1.18435	1.21245	1.23289	1.24908	1.23272	1.18451	1.10691
	Eq. (37)	0.71504	1.00	1.05979	1.11447	1.16382	1.20767	1.24585	1.27841	1.32521	1.34815	1.34708	1.32253
$\frac{(q/q_1)_e}{(q/q_1)_i}$	Eq. (37)	0.95025	1.00	1.01631	1.03437	1.05432	1.07630	1.10051	1.12713	1.18869	1.26339	1.35449	1.46642
$C_{p,0}$ (Eq. (18a))		0.48872	0	−0.12315	−0.24204	−0.35448	−0.45847	−0.55214	−0.63433	−0.75618	−0.81751	−0.81462	−0.74909
C_{p,M_1} (Eq. (18b))		0.56492	0	−0.15786	−0.31929	−0.48310	−0.64774	−0.81238	−0.97599	−1.29437	−1.59762	−1.88099	−2.14151

$M_1=0.65$

$(q/q_1)_e$		0.63084	0.92843	1.00	1.07029	1.13924	1.20381	1.27296	1.33765	1.46255	1.58138	1.69405	1.80060
$(q/q_1)_i$	Eq. (35)	0.66734	0.94030	1.00	1.05615	1.10378	1.15797	1.20377	1.24630	1.32204	1.38638	1.44057	1.48582
	Eq. (36)	0.68212	0.94685	1.00	1.04708	1.08766	1.12142	1.14802	1.16723	1.18271	1.16723	1.12158	1.04808
	Eq. (37)	0.67469	0.94356	1.00	1.05159	1.09816	1.13953	1.17555	1.20612	1.25044	1.27209	1.27110	1.24789
$\frac{(q/q_1)_e}{(q/q_1)_i}$	Eq. (37)	0.93501	0.98396	1.00	1.01778	1.03741	1.05904	1.08286	1.10905	1.16963	1.24314	1.33274	1.44292
$C_{p,0}$ (Eq. (18a))		0.54479	0.10969	0	−0.10584	−0.20596	−0.29853	−0.38192	−0.45473	−0.56360	−0.61821	−0.61570	−0.55723
C_{p,M_1} (Eq. (18b))		0.64074	0.14002	0	−0.14333	−0.28862	−0.43476	−0.58086	−0.72571	−1.00852	−1.27707	−1.52913	−1.76024

$M_1=0.7$

$(q/q_1)_e$		0.58942	0.86747	0.93433	1.00	1.06442	1.12756	1.18936	1.24980	1.36650	1.47752	1.58280	1.68235
$(q/q_1)_i$	Eq. (35)	0.63186	0.89032	0.94683	1.00	1.04983	1.09641	1.13978	1.18003	1.25175	1.31268	1.36398	1.40681
	Eq. (36)	0.65146	0.90429	0.95504	1.00	1.03875	1.07099	1.09640	1.11475	1.12953	1.11475	1.07114	1.00096
	Eq. (37)	0.64159	0.89728	0.95094	1.00	1.04429	1.08362	1.11788	1.14695	1.18909	1.20969	1.20874	1.18666
$\frac{(q/q_1)_e}{(q/q_1)_i}$	Eq. (37)	0.91869	0.96678	0.98253	1.00	1.01928	1.04055	1.06394	1.08967	1.14920	1.22140	1.30946	1.41772
$C_{p,0}$ (Eq. (18a))		0.58836	0.19489	0.09571	0	−0.09054	−0.17423	−0.24966	−0.31549	−0.41394	−0.46335	−0.46105	−0.40816
C_{p,M_1} (Eq. (18b))		0.70580	0.25516	0.12904	0	−0.13082	−0.26254	−0.39397	−0.52440	−0.77907	−0.02140	−1.24781	−1.45589

$M_1=0.75$

$(q/q_1)_e$		0.55374	0.81497	0.87778	0.93947	1.00	1.05931	1.11737	1.17416	1.28380	1.38810	1.48701	1.58054
$(q/q_1)_i$	Eq. (35)	0.60187	0.84807	0.90191	0.95253	1.00	1.04436	1.08568	1.12404	1.19235	1.25038	1.29923	1.34006
	Eq. (36)	0.62715	0.87056	0.91942	0.96268	1.00	1.03105	1.05550	1.07317	1.08740	1.07317	1.03120	0.96362
	Eq. (37)	0.61439	0.85924	0.91062	0.95759	1.00	1.03768	1.07047	1.09831	1.13868	1.15840	1.15749	1.13636
$\frac{(q/q_1)_e}{(q/q_1)_i}$	Eq. (37)	0.90128	0.94848	0.96394	0.98108	1.00	1.02084	1.04381	1.06906	1.12745	1.19829	1.28468	1.39088
$C_{p,0}$ (Eq. (18a))		0.62252	0.26171	0.17077	0.08302	0	−0.07678	−0.14591	−0.20628	−0.29659	−0.34189	−0.33978	−0.29131
C_{p,M_1} (Eq. (18b))		0.76361	0.35197	0.23700	0.11937	0	−0.12005	−0.23997	−0.35893	−0.59124	−0.81227	−1.01872	−1.20853

$M_1=0.8$

$(q/q_1)_e$		0.52274	0.76934	0.82864	0.88687	0.94402	1.00	1.05482	1.10842	1.21193	1.31038	1.40375	1.49204
$(q/q_1)_i$	Eq. (35)	0.57631	0.81205	0.86359	0.91207	0.95753	1.00	1.03956	1.07629	1.14171	1.19725	1.24403	1.28312
	Eq. (36)	0.60827	0.84435	0.89172	0.93371	0.96990	1.00	1.02372	1.04085	1.05467	1.04035	1.00014	0.93460
	Eq. (37)	0.59209	0.82805	0.87756	0.92283	0.96370	1.00	1.03162	1.05843	1.09734	1.11633	1.11545	1.09510
$\frac{(q/q_1)_e}{(q/q_1)_i}$	Eq. (37)	0.88287	0.92910	0.94425	0.96103	0.97958	1.00	1.02249	1.04723	1.10443	1.17383	1.25846	1.36247
$C_{p,0}$ (Eq. (18a))		0.64943	0.31433	0.22989	0.14838	0.07128	0	−0.05424	−0.12027	−0.20416	−0.24619	−0.24423	−0.19924
C_{p,M_1} (Eq. (18b))		0.81520	0.43547	0.32940	0.22083	0.11074	0	−0.11065	−0.22036	−0.43464	−0.63857	−0.82904	−1.00411

TABLE 2.—Concluded

M		0.4	0.6	0.825	0.85	0.875	0.9	0.925	0.96	1.0	1.1	1.2	1.3
τ		0.03101	0.06716	0.11982	0.12626	0.13279	0.13942	0.14612	0.15563	0.16667	0.19485	0.22360	0.25262
$M_1=0.825$													
$(q/q_1)_e$		0.50872	0.74871	1.00	1.02653	1.05276	1.07870	1.10434	1.13972	1.17942	1.27524	1.36610	1.45203
$(q/q_1)_i$	Eq. (35)	0.56494	0.79602	1.00	1.01905	1.03738	1.05506	1.07205	1.09475	1.11917	1.17363	1.21950	1.25781
	Eq. (36)	0.60068	0.83380	1.00	1.01093	1.02021	1.02785	1.03380	1.03929	1.04147	1.02784	0.98764	0.92292
	Eq. (37)	0.58253	0.81469	1.00	1.01497	1.02874	1.04136	1.05273	1.06665	1.07962	1.09833	1.09746	1.07744
$\frac{(q/q_1)_e}{(q/q_1)_i}$	Eq. (37)	0.87329	0.91901	1.00	1.01139	1.02335	1.03586	1.04902	1.06850	1.09244	1.16107	1.24478	1.34767
$C_{p,0}$ (Eq. (18a))		0.66066	0.33628	0	−0.03016	−0.05831	−0.08443	−0.10824	−0.13774	−0.16558	−0.20633	−0.20442	−0.16088
C_{p,M_1} (Eq. (18b))		0.83950	0.47330	0	−0.05329	−0.10631	−0.15910	−0.21149	−0.28409	−0.36571	−0.56234	−0.74599	−0.91487
$M_1=0.85$													
$(q/q_1)_e$		0.49558	0.72936	0.97416	1.00	1.02555	1.05082	1.07580	1.11026	1.14894	1.24228	1.33080	1.41451
$(q/q_1)_i$	Eq. (35)	0.55438	0.78114	0.98132	1.00	1.01799	1.03532	1.05202	1.07430	1.09826	1.15171	1.19672	1.23432
	Eq. (36)	0.59418	0.82478	0.98920	1.00	1.00917	1.01673	1.02264	1.02805	1.03023	1.01673	0.97697	0.91295
	Eq. (37)	0.57394	0.80267	0.98526	1.00	1.01357	1.02599	1.03721	1.05091	1.06371	1.08212	1.08128	1.06155
$\frac{(q/q_1)_e}{(q/q_1)_i}$	Eq. (37)	0.86347	0.90867	0.98873	1.00	1.01182	1.02420	1.03721	1.05647	1.08013	1.14801	1.23076	1.33249
$C_{p,0}$ (Eq. (18a))		0.67059	0.35572	0.02926	0	−0.02732	−0.05206	−0.07580	−0.10441	−0.13148	−0.17098	−0.16917	−0.12689
C_{p,M_1} (Eq. (18b))		0.86288	0.50895	0.05147	0	−0.05129	−0.10226	−0.15294	−0.22305	−0.30199	−0.49202	−0.66952	−0.83274
$M_1=0.875$													
$(q/q_1)_e$		0.48322	0.71118	0.94988	0.97508	1.00	1.02463	1.04898	1.08259	1.12031	1.21133	1.29763	1.37926
$(q/q_1)_i$	Eq. (35)	0.54458	0.76734	0.96396	0.98232	1.00	1.01703	1.03340	1.05530	1.07885	1.13134	1.17555	1.21248
	Eq. (36)	0.58877	0.81727	0.98020	0.99091	1.00	1.00748	1.01333	1.01870	1.02085	1.00749	0.96807	0.90464
	Eq. (37)	0.56626	0.79192	0.97207	0.98661	1.00	1.01225	1.02331	1.03684	1.04947	1.06764	1.06678	1.04733
$\frac{(q/q_1)_e}{(q/q_1)_i}$	Eq. (37)	0.85335	0.89805	0.97717	0.98831	1.00	1.01223	1.02509	1.04412	1.06750	1.13459	1.21640	1.31693
$C_{p,0}$ (Eq. (18a))		0.67935	0.37286	0.05508	0.02660	0	−0.02465	−0.04716	−0.07504	−0.10139	−0.13986	−0.13802	−0.09690
C_{p,M_1} (Eq. (18b))		0.88564	0.54277	0.09960	0.04971	0	−0.04943	−0.09846	−0.16642	−0.24286	−0.42701	−0.59898	−0.75712
$M_1=0.9$													
$(q/q_1)_e$		0.47160	0.69409	0.92705	0.95164	0.97595	1.00	1.02376	1.05657	1.09338	1.18220	1.26644	1.34609
$(q/q_1)_i$	Eq. (35)	0.53546	0.75449	0.94783	0.96587	0.98325	1.00	1.01610	1.03764	1.06079	1.11240	1.15588	1.19218
	Eq. (36)	0.58439	0.81121	0.97291	0.98355	0.99257	1.00	1.00579	1.01113	1.01327	1.00000	0.96089	0.89792
	Eq. (37)	0.55939	0.78233	0.96029	0.97466	0.98788	1.00	1.01091	1.02429	1.03676	1.05470	1.05387	1.03463
$\frac{(q/q_1)_e}{(q/q_1)_i}$	Eq. (37)	0.84306	0.88721	0.96539	0.97638	0.98792	1.00	1.01271	1.03151	1.05461	1.12089	1.20170	1.30104
$C_{p,0}$ (Eq. (18a))		0.68708	0.38796	0.07784	0.05004	0.02409	0	−0.02194	−0.04917	−0.07487	−0.11239	−0.11064	−0.07046
C_{p,M_1} (Eq. (18b))		0.90787	0.57490	0.14460	0.09617	0.04799	0	−0.04762	−0.11365	−0.18788	−0.36660	−0.53362	−0.68711
$M_1=0.925$													
$(q/q_1)_e$		0.46066	0.67798	0.90552	0.92955	0.95331	0.97678	1.00	1.03205	1.06800	1.15476	1.23704	1.31486
$(q/q_1)_i$	Eq. (35)	0.52697	0.74253	0.93279	0.95057	0.96768	0.98415	1.00	1.02119	1.04397	1.09476	1.13756	1.17329
	Eq. (36)	0.58103	0.80654	0.96729	0.97787	0.98686	0.99423	1.00	1.00532	1.00743	0.99424	0.95534	0.89276
	Eq. (37)	0.55335	0.77387	0.94991	0.96413	0.97721	0.98919	1.00	1.01323	1.02556	1.04332	1.04249	1.02347
$\frac{(q/q_1)_e}{(q/q_1)_i}$	Eq. (37)	0.83249	0.87609	0.95327	0.96413	0.97554	0.98745	1.00	1.01857	1.04138	1.10681	1.18662	1.28471
$C_{p,0}$ (Eq. (18a))		0.69380	0.40113	0.09767	0.07045	0.04506	0.02150	0	−0.02664	−0.05177	−0.08852	−0.08679	−0.04749
C_{p,M_1} (Eq. (18b))		0.92963	0.60574	0.18708	0.13991	0.09301	0.04633	0	−0.06425	−0.13642	−0.31038	−0.47285	−0.62222

TABLE 3.—VALUES OF $F(\tau)$ FOR SEVERAL VALUES OF γ

$$\left[F(\gamma)=\frac{1-(2\beta+1)\tau}{(1-\tau)^{2\beta+1}}=\frac{1-M^2}{\left(1+\frac{M^2}{2\beta}\right)^{-2\beta}}\right]$$

M	F			
	$\gamma=1.4$ ($\beta=2.5$) Adiabatic	$\gamma=1$ ($\beta\to\infty$) Isothermal [1]	$\gamma=2$ ($\beta=1$) Hydraulic analogy	$\gamma=\infty$ ($\beta\to 0$) Limiting incompressible [2]
0	1.00	1.00	1.00	1.00
.2	.99901	.99918	.99879	.9600
.4	.98328	.98575	.97977	.8400
.6	.90606	.91733	.89113	.6400
.65	.86634	.88113	.84726	.5775
.70	.81394	.83248	.79050	.5100
.75	.74558	.76783	.71822	.4375
.80	.65738	.68273	.62728	.3600
.85	.54489	.57153	.51421	.2775
.90	.40258	.42710	.37504	.1900
.95	.22355	.24041	.20534	.0975
1.00	0	0	0	0
1.05	−.27752	−.30870	−.24667	−.1025
1.10	−.62069	−.70423	−.54102	−.2100
1.20	−1.55960	−1.85711	−1.30158	−.4400
1.30	−2.95915	−3.73944	−2.34862	−.6900
1.50	−8.01227	−11.8597	−5.64466	−1.2500
2.00	−56.6884	−163.79	−26.9981	−3.0000

[1] $\gamma=1$, $F=(1-M^2)e^{M^2}$
[2] $\gamma=\infty$, $F=1-M^2$

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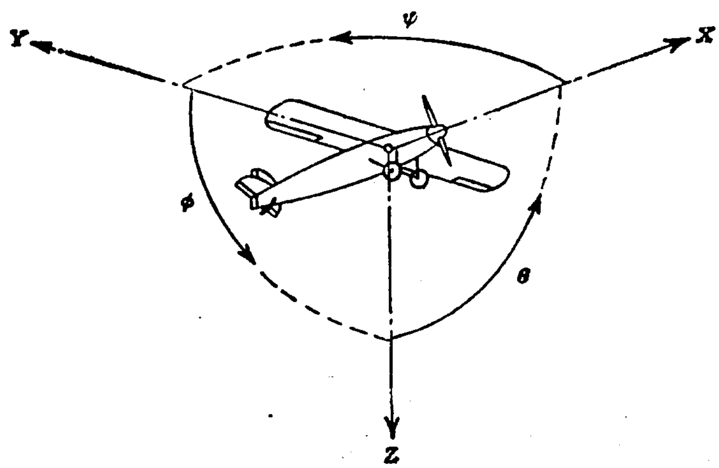

Positive directions of axes and angles (forces and moments) are shown by arrows

Axis		Force (parallel to axis) symbol	Moment about axis			Angle		Velocities	
Designation	Symbol		Designation	Symbol	Positive direction	Designation	Symbol	Linear (component along axis)	Angular
Longitudinal	X	X	Rolling	L	$Y \longrightarrow Z$	Roll	φ	u	p
Lateral	Y	Y	Pitching	M	$Z \longrightarrow X$	Pitch	θ	v	q
Normal	Z	Z	Yawing	N	$X \longrightarrow Y$	Yaw	ψ	w	r

Absolute coefficients of moment

$C_l=\frac{L}{qbS}$ (rolling) $\quad C_m=\frac{M}{qcS}$ (pitching) $\quad C_n=\frac{N}{qbS}$ (yawing)

Angle of set of control surface (relative to neutral position), δ. (Indicate surface by proper subscript.)

4. PROPELLER SYMBOLS

D Diameter
p Geometric pitch
p/D Pitch ratio
V' Inflow velocity
V_s Slipstream velocity
T Thrust, absolute coefficient $C_T=\frac{T}{\rho n^2 D^4}$
Q Torque, absolute coefficient $C_Q=\frac{Q}{\rho n^2 D^5}$
P Power, absolute coefficient $C_P=\frac{P}{\rho n^3 D^5}$
C_s Speed-power coefficient $=\sqrt[5]{\frac{\rho V^5}{P n^2}}$
η Efficiency
n Revolutions per second, rps
Φ Effective helix angle $=\tan^{-1}\left(\frac{V}{2\pi r n}\right)$

5. NUMERICAL RELATIONS

1 hp=76.04 kg-m/s=550 ft-lb/sec
1 metric horsepower=0.9863 hp
1 mph=0.4470 mps
1 mps=2.2369 mph
1 lb=0.4536 kg
1 kg=2.2046 lb
1 mi=1,609.35 m=5,280 ft
1 m=3.2808 ft