HARVARD UNIVERSITY
GRADUATE SCHOOL OF ENGINEERING

OPERATOR METHODS IN THE THEORY OF COMPRESSIBLE FLUIDS

WORK PERFORMED UNDER CONTRACT NORD 8555 TASK F.

TECHNICAL REPORT NO. 10

Two-Dimensional Transonic Flow Patterns

BY STEFAN BERGMAN

CAMBRIDGE, MASS.

1948

TWO-DIMENSIONAL TRANSONIC FLOW PATTERNS.*[1]

By STEFAN BERGMAN.

1. Introduction. The successful development of the mathematical theory of two-dimensional steady flows of an ideal incompressible fluid is largely due to the fact that the complex potentials of such flows are analytic functions of a complex variable. Function-theoretical methods may therefore be used in order to obtain and to investigate flow patterns possessing at given points the assigned singular character and satisfying given boundary conditions.[2]

In investigating the flow of an incompressible fluid by theoretical methods, two alternative treatments have been used. The behavior of the solutions has been studied either in the so-called physical, i. e., the plane in which the motion actually occurs, or in the hodograph plane, i. e., the plane whose cartesian coordinates are the velocity components. In the incompressible case, both the potential and the streamfunction are harmonic functions, irrespective of whether the motion is considered in the physical—or the hodograph plane.

In the compressible case, both the potential and the streamfunction considered in the physical plane satisfy complicated non-linear equations. By considering the motion in the hodograph plane, and making a few appropriate transformations, it is possible to linearize these equations. (See [11] and **2** of the present paper.[3])

The treatment of the problem in the physical plane has, of course, the fundamental advantage of making the boundary conditions enter in an obvious way. The gain in simplicity due to the linearization is, however, so considerable as to make the hodograph method vastly superior for various purposes.

In the present paper the theory of compressible fluid flow is developed

* Received August 2, 1947.

[1] Research paper done under Navy Contract NOrd 8555-Task F, at Harvard University. The ideas expressed in this paper represent the personal views of the author, and are not necessarily those of the Bureau of Ordnance.

[2] Poles, logarithmic singularities, etc., of the complex potential represent doublets, sinks, sources, vortices, etc., of the flow.

[3] The numbers in brackets refer to the bibliography at the end of the paper. Acquaintance with the contents of these publications is not assumed in the present paper.

by studying the linearized equation and applying to it the operator method developed in [**1-10**]. Generalizing the procedure: "taking the real part," this method leads to the determination of certain linear operators which transform functions of one variable into stream (or potential) functions of compressible fluid flows preserving many fundamental properties of functions to which the operator is applied. In the present paper, two operators of this kind are considered: the so-called integral operator of the first kind (**3**) and that of the second kind (**4-6**). For some fluid dynamical applications, and in particular for the study of the behavior of the flow near the sonic line, the second operator seems to be more appropriate. In this case, however, the relations between the properties of functions to which the operator is applied and those of generated functions is more hidden. This is why it is useful at first to investigate the special case which is obtained by assuming a simplified equation of state. This simplifying assumption results in the equation for the stream function taking the form

$$-C\mathrm{H}(\partial^2\psi/\partial\theta) + (\partial^2\psi/\partial \mathrm{H}^2) = 0, \qquad C > 0 \tag{1.1}$$

where H is a function of the Mach number, M, which is negative for $M < 1$ and positive for $M > 1$. (See (2.1) and (2.3)).

In an important investigation, Tricomi [**16**] studied the boundary value problem of equation (1.1) and showed that if we consider a finite domain D, bounded in the supersonic region by two characteristics, say BA and CA, and by a curve BmC in the subsonic region, and if the boundary values are prescribed in BmC and on one of the characteristics, say BA, then the boundary value problem has a unique solution. Frankl [**13**] considered questions allied with Tricomi's investigations in the case of the exact compressibility equation.

The questions arising in our approach are, however, of a somewhat different nature than those considered by the above-mentioned authors.

In the first place, we seek to find conditions for a function, say f, of one variable in order that the generated function $\mathrm{P}(f)$ will be defined in a prescribed domain B, which in general lies partially in the subsonic and partially in the supersonic region.

Secondly, and this is the most essential difference, we are considering solutions of the compressibility equation which possess singularities (e.g., branchpoints) in the hodograph plane. In the applications of the theory, the consideration of this type of solution cannot be dispensed with, since, notwithstanding the singularities in the hodograph plane, the behavior of the solutions in the physical plane can be perfectly regular. Furthermore, certain singularities in the physical plane have a hydrodynamical meaning and must be considered in investigating flow patterns.

The methods employed in the case of equation (1.1) are to some extent capable of generalization to the case in which the coefficient $(-C\mathrm{H})$ is replaced by an arbitrary function $l(\mathrm{H})$. (See **5**). This includes, in particular, the exact, i. e., non-simplified compressibility equation.

In **3**, we introduce the so-called integral operator of the first kind, P_1. This operator yields a streamfunction of a subsonic, compressible fluid flow in terms of an arbitrary function of one complex variable. The representation holds for [4] $\mathrm{E}[M < 1, -\infty < \theta < \infty]$. Here M is the Mach number and θ the angle which the velocity vector forms with the positive x-axis.

An analogous representation for the streamfunctions of supersonic flows in terms of two differentiable functions of one real variable holds for $\mathrm{E}[M > 1, -\infty < \theta < \infty]$. Using the integral operator of the second kind, we obtain (**4, 5**) four analogous representations in terms of arbitrary functions of one variable. These four representations are valid in four adjacent domains of the M, θ-plane, namely

$$D_1 = \mathrm{E}[M < 1, \theta > 3^{\frac{1}{2}} \,|\, \lambda(M) |] + \mathrm{E}[M > 1, \theta > \Lambda(M)],$$
$$D_2 = \mathrm{E}[M < 1, |\, \theta \,| < 3^{\frac{1}{2}} \,|\, \lambda(M) |],$$

$$D_3 = \mathrm{E}[M < 1, \theta < -3^{\frac{1}{2}} \,|\, \lambda(M) |] + \mathrm{E}[M > 1, \theta < -3\Lambda(M)],$$
$$D_4 = \mathrm{E}[M > 1, -3\Lambda(M) < \theta < \Lambda(M)],$$

respectively. Here $\theta = \pm 3^{\frac{1}{2}}\lambda(M)$ and $\theta = (-1 \pm 2)\Lambda(M)$ are certain curves which pass through the point $M = 1$, $\theta = 0$ and which lie in the subsonic and supersonic region respectively (see fig. 1).

In the simplified case these four representations can be combined into one, yielding a representation which holds in the whole M, θ-plane. This result is based on certain theorems of the Fuchs theory of ordinary differential equations with singular coefficients. The question of combining the analogues of the above four solutions in the exact case, and generally, the study of the solutions leads to the investigation of *partial* differential equations with singular coefficients, which, when solutions are continued to complex values of the arguments, can be attacked by methods representing a generalization

[4] The functions $\psi = \mathrm{P}_1(f)$ may be multi-valued functions which may possess singularities. The statement that $\mathrm{P}_1(f)$ is defined in $\mathrm{E}[M < 1]$ means that the projections of the domain in which $\mathrm{P}_1(f)$ is defined on the schlicht M, θ-plane, lies in $\mathrm{E}[M < 1]$.

$\mathrm{E}[\ \ ]$ denotes the set of points whose coordinates satisfy conditions indicated in the brackets.

of the Fuchs theory for ordinary differential equations. These questions, and in particular the problem of combining the four above representations into one, will be treated in a subsequent paper.

In **6**, we determine the "associate" function for $P_2(f)$ in terms of the values of the stream function $\psi = \mathrm{Im}[P_2(f)]$, and its derivative with respect to M on the line $M = 1$ (sonic line).

The author would like to take this opportunity to thank Bernard Epstein and A. Zeichner for helpful advice and aid in connection with the present

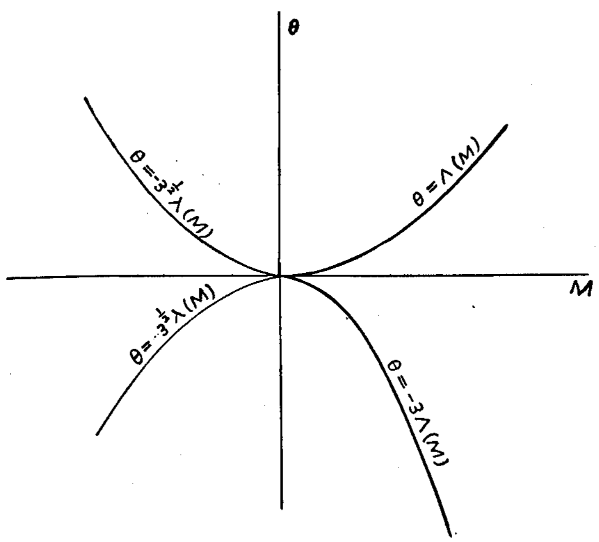

Fig. 1.

paper. He would also like to thank Z. Nehari and M. Schiffer for a number of helpful suggestions.

2. Equation for the streamfunction of a compressible fluid flow. Exact and simplified equations. Assuming that the thermodynamical equation of state of the fluid has the form $p = \sigma\rho^k$, where σ and k are constants, and ρ and p the density and pressure respectively, and introducing as new variables

$$\text{(2.1)} \qquad \mathrm{H} = \int_{q_1}^{q} \rho(dq/q) = \int_{q_1}^{q} q^{-1}[1 - \tfrac{1}{2}(k-1)q^2]^{1/k-1}dq,$$

and θ, where q is the speed and θ the angle which the velocity vector makes with some fixed direction (say the positive x-axis,), we obtain the following linear equation for the streamfunction:

$$(2.2)\quad S(\psi) \equiv l(\mathrm{H})(\partial^2\psi/\partial\theta^2) + (\partial^2\psi/\partial \mathrm{H}^2) = 0, \qquad l(\mathrm{H}) = (1 - M^2)/\rho^2,$$

where

$$(2.3)\qquad M = q/[1 - \tfrac{1}{2}(k-1)q^2]^{\frac{1}{2}}$$

is the Mach number. The denominator in (2.3), being the local velocity of sound, M will be smaller or larger than 1 according as the flow is subsonic or supersonic respectively. The differential equation (2.2) will therefore be of ellipitic or hyperbolic type, corresponding to the subsonic or supersonic character of the flow.

A formal computation [5] shows that the Taylor development of $l(\mathrm{H})$ in the neighborhood of $\mathrm{H} = 0$ is

$$(2.4)\quad l(\mathrm{H}) = [2/(k-1)]^{(2-k)/(k-1)}[(-2\mathrm{H}) - ((2k+5)/(2k+2))((k+1)/2)^{2k/(k-1)}(-2\mathrm{H})^2$$
$$+ \frac{k^4 + (43/6)k^3 + 16k^2 + (31/2)k + 31/6}{(k+1)^4}((k+1)/2)^{k/(k-1)}(-2\mathrm{H})^3 + \cdots].$$

In considering a flow (or a portion of a flow) which is purely subsonic, it has some advantages to replace H by the variable λ defined by

$$(2.5)\qquad -\lambda = \int_{\tau=0}^{-\mathrm{H}} [l(-\tau)]^{\frac{1}{2}} d\tau.$$

λ can be expressed in a closed form as a function of M; a formal computation yields

$$(2.6)\quad \lambda = \tfrac{1}{2}\log\left[\frac{1-(1-M^2)^{\frac{1}{2}}}{1+(1-M^2)^{\frac{1}{2}}}\left(\frac{1+h(1-M^2)^{\frac{1}{2}}}{1-h(1-M^2)^{\frac{1}{2}}}\right)^{1/h}\right],$$
$$h = [(k-1)/(k+1)]^{\frac{1}{2}}.$$

Now

$$-\psi_\lambda \lambda_{\mathrm{H}} = -\psi_\lambda [il(\mathrm{H})]^{\frac{1}{2}},$$

$$(2.7)\quad \psi_{\mathrm{HH}} = \psi_{\lambda\lambda} l + \psi_\lambda (l^{\frac{1}{2}})_{\mathrm{H}} = \psi_{\lambda\lambda} l - \tfrac{1}{2}\psi_\lambda l^{-\frac{1}{2}} l_{\mathrm{H}} = l[\psi_{\lambda\lambda} + \tfrac{1}{2} l^{-3/2} l_{\mathrm{H}}]$$

so that (2.2) becomes

$$(2.8a)\qquad \psi_{\lambda\lambda} + \psi_{\theta\theta} + 4N\psi_\lambda = 0,$$

[5] A detailed account of formal derivations of some expressions used in the present paper can be found in the Appendix to Technical Report 10, of the series "Operator methods in the theory of compressible fluids," Harvard University, 1948.

or, in complex notation,

$$4\psi_{Z\bar{Z}} + 4N(\psi_Z + \psi_{\bar{Z}}) = 0, \tag{2.8b}$$

where

$$N = \tfrac{1}{8}l^{-3/2}l_{\mathrm{H}} = -[(k+1)/8]\frac{M^4}{(1-M^2)^{3/2}}, \quad Z = \lambda + i\theta, \bar{Z} = \lambda - i\theta. \tag{2.9}$$

See [**6**, (46)]. It should be noted that the interval $-\infty < \lambda < 0$ corresponds to the interval $-\infty < \mathrm{H} < 0$.

In the supersonic case (i.e., for $M > 1$) the right-hand side of (2.6) becomes purely imaginary. If we introduce the new variable Λ defined by

$$\Lambda = i\lambda, \tag{2.10}$$

it is easily confirmed that

$$\Lambda = h^{-1}\arctan[h(M^2-1)^{\frac{1}{2}}] - \arctan[(M^2-1)^{\frac{1}{2}}]. \tag{2.11}$$

In this case, (2.8a) will take the form

$$\psi_{\Lambda\Lambda} - \psi_{\theta\theta} + 4N_1\psi_\Lambda = 0, \quad N_1 = \frac{k+1}{8}\frac{M^4}{(M^2-1)^{3/2}}. \tag{2.12}$$

REMARK. Equations (2.8) and (2.12) can be simplified. If ψ is replaced by

$$\psi^* = \psi/R \tag{2.13}$$

where $(\partial R/\partial\bar{Z}) = N$, then ψ^* satisfies the equations

$$\psi^*_{\lambda\lambda} + \psi^*_{\theta\theta} + 4F\psi^* = 0 \text{ and } \psi^*_{\Lambda\Lambda} - \psi^*_{\theta\theta} - 4F_1\psi^* = 0, \tag{2.14}$$

where

$$F = F_1 = \frac{(k+1)M^4}{64}\left[\frac{-(3k-1)M^4 - 4(3-2k)M^2 + 16}{(1-M^2)^3}\right]. \tag{2.15}$$

For subsequent use, we write down the expansions of N and F in the neighborhood of $\lambda = 0$

$$N = (1/12\lambda)[1 - \tfrac{1}{4}(k+1)^{\frac{1}{3}}[(2+\tfrac{3}{5}2^{\frac{1}{3}})k + 5\cdot 2^{\frac{1}{3}} - 2] \times [2^{-1/6}3^{2/3}(k+1)^{-5/6}(-\lambda)^{2/3}] + \cdots], \tag{2.16}$$

$$F = (5/144)(-\lambda)^{-2} + A_{-2}(-\lambda)^{-2/3} + A_0 + A_2(-\lambda)^{2/3} + \cdots = 5/36(-2\lambda)^2 + \cdots. \tag{2.17}$$

In the vicinity of $\mathrm{H} = 0$, i.e., the sonic line, $l(\mathrm{H})$ may be replaced by the first term in its expansion (2.4). Using this value of $l(\mathrm{H})$ in (2.2), we obtain the so-called "simplified" compressibility equation (1.1). In considering transonic flows, the solutions of the simplified equation will therefore give a fair approximation—in a certain neighborhood of the sonic line—of the exact streamfunction.

The expression for N, F, and H will, in this case, reduce to

$$\begin{aligned} N &= N\dagger = -(1/6)(1/(-2\lambda)), \\ F &= F\dagger = (5/36)(1/(-2\lambda)^2), \\ \mathrm{H} &= \mathrm{H}\dagger = (3^{2/3}/2)(2/(k-1))^{(k-2)/(3k-3)}(-\lambda)^{2/3}, \end{aligned} \tag{2.18}$$

respectively.

REMARK. We are using the same variable λ in both the exact and the simplified case, as this facilitates the comparison of the respective flow patterns.

3. Application of integral operators to the compressibility equation. Integral operator of the first kind. The use of integral operators in the theory of the compressibility equation is based on the following theorem:

THEOREM 3.1. *Let $E(Z,\bar{Z},t)$ be a function of two real and one complex variables, λ, θ, t, which is defined for t along a curve connecting $t=-1$ and $t=1$, and for $(\lambda,\theta)\ \varepsilon\ G$. G denotes here a sufficiently small neighborhood of the origin.*

Let E satisfy the following conditions:

1. *E possesses continuous partial derivatives with respect to all three of its arguments, up to the second order.*

2. *The expression*

$$[(1-t^2)E(Z,\bar{Z},t)/Zt]\partial/\partial\bar{Z} \tag{3.1}$$

is continuous for $Z=0$, and approaches zero, uniformly with respect to $(\lambda,\theta)\ \varepsilon\ G$ as $t\to-1$ or $t\to 1$.

3. *E satisfies the partial differential equation*

$$\boldsymbol{G}(E)\equiv(1-t^2)(E_{\bar{Z}t}+NE_t)-(1/t)E_{\bar{Z}}+2tZ\boldsymbol{L}(E)=0 \tag{3.2}$$

where

$$\boldsymbol{L}(E)\equiv E_{Z\bar{Z}}+N(E_Z+E_{\bar{Z}})\equiv(1/4)E_{\lambda\lambda}+(1/4)E_{\theta\theta}+NE_\lambda. \tag{3.3}$$

If $f(\zeta/2)$ is an analytic function of ζ defined in a simply-connected domain P, which includes the origin, then the expression $u(Z,\bar{Z})$, given by

(3.4a) $$u(Z, \bar{Z}) = \mathrm{P}(f),$$

(3.4b) $$\mathrm{P}(f) \equiv \int_{-1}^{1} E(Z, \bar{Z}, t) f(\tfrac{1}{2}Z(1-t^2))\, dt/(1-t^2)^{\frac{1}{2}}$$

is defined in a simply-connected domain which lies in $G \cap P$, *and satisfies the equation* $\boldsymbol{L}(u) = 0$.

The proof of this theorem is given in [**1**, § 1 and **6**, pp. 34-39]. The function f will be called the associate of $\mathrm{P}(f)$ with regard to the operator **P**.

Since (3.2) has an infinity of solutions, there exist infinitely many integral operators; a closer investigation of their properties will show which type of operator is best suited for the purpose on hand. The following property of the integral operator plays an important role in some of the applications.

As is well-known, a harmonic function $G_1(\lambda', \theta') = G(Z', \bar{Z}')$ may be written in the form

$$G(Z', \bar{Z}') = (1/2i)[g(Z') - g(\bar{Z}')].$$

Here [6]

$$Z' = \lambda' + i\theta', \quad \bar{Z}' = \lambda' - i\theta', \quad \lambda' = \lambda - \lambda_0, \quad \theta' = \theta - \theta_0,\ \lambda_0 < 0.$$

If now the harmonic function $G_1(\lambda', \theta')$ is continued to the complex values of the arguments λ' and θ', i. e., if we assume that Z' and $\bar{Z}'$ are not necessarily conjugate to each other, and if, in particular, we consider G and g in the so-called characteristic planes $Z' = 0$ or $\bar{Z}' = 0$, then we see that the analytic function of a complex variable and the continuation of the real harmonic function differ only by constants; [7] indeed,

$$G(Z', 0) = (1/2i)[g(Z') - g(0)].$$

An integral operator generates complex solutions of $\boldsymbol{L}$, and we may demand that these complex solutions possess an analogous property. We shall show that there exists an integral operator $\boldsymbol{p}$ (which is connected by relations (3.11), (3.12) with **P**) such that the complex solution of (3.3)

[6] In the discussions, it will be useful to consider a shift of origin to the point λ_0, θ_0 (point of reference of the operator).

In this section Z' and $\bar{Z}'$ will be treated as independent variables.

[7] Analytic functions of a complex variable represent a very special subclass of complex harmonic functions (i. e., the totality of functions $G + iH$, where G and H are two *arbitrary* real harmonic functions). Operator (3.4a), (3.4b) generates also a subclass of *complex* solutions of the equation $\boldsymbol{L}$.

$$u(Z', \bar{Z}') = \boldsymbol{p}[g(Z')]$$

and the real solution

$$\psi(Z', \bar{Z}') = \mathrm{Im}[u(Z', \bar{Z}')]$$

are connected by the following relations

$$(3.5) \quad \psi(Z', 0) = (1/2i)[u(Z', 0) - R(Z', 0) \text{ const.}], \quad u(Z',0) = g(Z')$$

where $R(Z', \bar{Z}')$ is a given function defined in (3.7). This operator $\mathbf{P}$ will be called "integral operator of the first kind" and will be denoted by $\mathbf{P}_1$.

Defining the generating function $E_1(Z', \bar{Z}', t)$ of the operator of the first kind by the requirement that

$$(3.6\text{a}) \qquad E_1(Z', 0, t) = 1$$

and

$$(3.6\text{b}) \qquad E_1(0, \bar{Z}', t) = \exp\left[-\int_0^{\bar{Z}'} N d\bar{Z}'\right],$$

we shall show that these relations imply the property (3.5).

Writing the generating function of the first kind E_1 in the form

$$(3.7) \quad E_1 = R(Z', \bar{Z}') E^*_1(Z', \bar{Z}', t);$$

$$R(Z', \bar{Z}') = \exp\left[-\int_0^{\bar{Z}'} N(Z' + \bar{Z}'_1) d\bar{Z}'_1\right]$$

and assuming that E^*_1 has the development

$$(3.8) \qquad E^*_1 = 1 + \sum_{n=1}^{\infty} Z'^n t^{2n} P^{(n)}(Z', \bar{Z}'),$$

it is found that (3.7) satisfies the relation (3.6b). Substituting this into equation (3.2) we find that the $P^{(n)}$ satisfy the following recurrence relations:

$$(3.9) \quad P_{\bar{Z}'}^{(1)} + 2F = 0, \quad (2n+1) P_{\bar{Z}'}^{(n+1)} + 2P_{Z'\bar{Z}'}^{(n)} + 2F P^{(n)} = 0, \qquad n = 1, 2, 3, \cdots.$$

Finally, (3.6a) is satisfied by imposing upon the $P^{(n)}$ the initial conditions

$$(3.10) \qquad P^{(n)}(Z', 0) = 0, \qquad n = 1, 2, 3, \cdots.$$

By the above requirement, the $P^{(n)}$ and hence the generating functions E_1 (of the first kind) are uniquely determined. Applying the considerations of [**1**, pp. 1173-76] it can be shown that the series (3.8) converges absolutely and uniformly in a sufficiently small neighborhood of the origin $Z' = 0$,

$\bar{Z}' = 0$. The existence of an integral operator of the first kind thus is assured. Assuming that the associate function f is regular in a sufficiently large domain, we prove that by applying to it the integral operator of the first kind, we obtain a solution $u(Z, \bar{Z})$ (see (3. 4a), (3. 4b)) of (2. 8b) defined in a sufficiently small neighborhood of the origin. We shall show in the following that if f is regular for $M < 1$, this solution can be continued throughout the whole subsonic region.

REMARK. Integral operators of the first kind can also be written in a somewhat different form which is useful for various purposes. Namely (as can be shown by a straightforward computation, see [**4**, pp. 618-619]) we have [8]

(3. 11) $\quad \mathrm{P}(f) = \boldsymbol{p}(g)$

$$= R(Z', \bar{Z}')[g(Z') + \sum_{n=1}^{\infty} 2^{-2n} \frac{\Gamma(2n+1)}{\Gamma(n+1)} P^{(n)}(Z', \bar{Z}') g^{[n]}(Z')]$$

$$g^{[n]}(Z') = \int_0^{Z'} \int_0^{Z_1} \cdots \int_0^{Z_{n-1}} g(Z_n)\, dZ_n \cdots dZ_1$$

$$= \frac{(-1)^{n-1}}{(n-1)\,!} \int_{\zeta=0}^{\zeta=Z'} (Z' - \zeta)^{n-1} g(\zeta)\, d\zeta$$

where

$$g(Z') = \int_{t=-1}^{1} f[\tfrac{1}{2}Z'(1-t^2)]\, dt/(1-t^2)^{\frac{1}{2}}. \tag{3. 12}$$

(3.11), (3. 7) and (3. 10) imply (3. 5).

We proceed now to the proof that every *real* solution of equation (2. 8b) can be represented in a sufficiently small neighborhood of the origin $Z' = 0$, $\bar{Z}' = 0$ as the imaginary part of the right-hand side of (3. 11) with suitably chosen associate function f (or g).

Let $\psi(Z', \bar{Z}')$ ($\bar{Z}'$ being conjugate to Z') be a real solution of equation (2. 8b) which is regular in a sufficiently small neighborhood of the origin. Since this equation is of elliptic type and its coefficient N is an analytic function of two variables, $\psi(Z', \bar{Z}')$ can be written in the form of a power series

$$\psi(Z', \bar{Z}') = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} D_{mn} Z'^m \bar{Z}'^n, \qquad D_{mn} = \bar{D}_{nm},$$

which converges in a sufficiently small neighborhood of the origin.

---

[8] f and g are associates of the same solution of (2. 8b), the first with respect to the operator P, the second with respect to $\boldsymbol{p}$. In order to avoid any confusion we shall speak about "$\boldsymbol{p}$-associate" and "P-associate."

In the plane $\bar{Z}' = 0$, we have:

$$\psi(Z', 0) = \sum_{m=0}^{\infty} D_{m0} Z'^m = G_1(Z') \tag{3.13}$$

and in the plane $Z' = 0$,

$$\psi(0, \bar{Z}') = \sum_{m=0}^{\infty} D_{0m} \bar{Z}'^m = G_2(\bar{Z}')\,; \tag{3.14}$$

G_1 and G_2 are two analytic functions of one complex variable Z' and $\bar{Z}'$, respectively, which are regular in a sufficiently small neighborhood of the origin. (We note that $G_1(0) = G_2(0)$ and $G_2(\bar{Z}') = \overline{G_1(Z')}$ since for $\bar{Z}'$ conjugate to Z', ψ is real.) On the other hand, by classical results (the initial value problem in the theory of partial differential equations), it is known that if functions $G_1(Z')$, $G_2(\bar{Z}')$, $G_1(0) = G_2(0)$, are given, there exists *one and only one* solution $\psi(Z', \bar{Z}')$ of equation (2.8b) such that (3.13) and (3.14) hold. The integral operator (3.4) enables us to write down the solution. Indeed, let us determine two functions say $g_1(Z')$ and $g_2(\bar{Z}')$, $g_2(\bar{Z}') = \overline{g_1(Z')}$, such that

$$g_1(Z') + g_2(0)\overline{R(0, \bar{Z}')} = G_1(Z') \tag{3.15}$$

and therefore:

$$g_2(\bar{Z}') + g_1(0) R(0, \bar{Z}') = G_2(\bar{Z}').$$

Now

$$\begin{aligned} (3.16)\quad & R(Z', \bar{Z}')[g_1(Z') \\ & + \sum_{n=1}^{\infty} \frac{\Gamma(2n+1)}{2^{2n}\Gamma(n+1)} P^{(n)}(Z', \bar{Z}') \int_0^{Z'} \cdots \int_0^{Z_{n-1}} g_1(Z_n)\, dZ_n \cdots dZ_1] \\ & + \bar{R}(\bar{Z}', Z')[g_2(\bar{Z}') \\ & + \sum_{n=1}^{\infty} \frac{\Gamma(2n+1)}{2^{2n}\Gamma(n+1)} \bar{P}^{(n)}(\bar{Z}', Z') \int_0^{\bar{Z}'} \cdots \int_0^{\bar{Z}_n} g_2(\bar{Z}_n)\, d\bar{Z}_n \cdots d\bar{Z}_1] \end{aligned}$$

will represent a solution of (2.8b) which satisfies the conditions (3.13) and (3.14); our assertion that every (real) solution can be represented as the imaginary part of (3.13) is therefore proved.

As already mentioned, (3.18) is a priori only defined in a sufficiently small neighborhood of the origin. We shall prove, however, that provided f is regular for $\lambda' < \lambda_0$, the solution $\psi(Z', \bar{Z}')$ obtained in this way can be continued into the whole region $\mathrm{E}[\mathrm{Re}(Z' + \bar{Z}') < 2\lambda_0, |Z'| < \infty, |\bar{Z}'| < \infty]$. As shown in [**6**, pp. 56 ff.; **8**, p. 48 ff.], the quantity $F = F(\frac{1}{2}(Z' + \bar{Z}'))$

introduced in (2.15) is a function of two complex variables Z', $\bar{Z}'$ which is defined in the above region. Therefore the expression [9]

$$(3.17)\quad R(Z',\bar{Z}')[g_1(Z') - \int_0^{Z'}\int_0^{\bar{Z}'} Fg_1 dZ_1 d\bar{Z}_1 + \int_0^{Z'}\int_0^{\bar{Z}'} F[\int_0^{Z_1}\int_0^{\bar{Z}_1} Fg_1 dZ_2 d\bar{Z}_2] dZ_1 d\bar{Z}_1 + \cdots] + \bar{R}(\bar{Z}',Z')[g_2(\bar{Z}') - \int_0^{Z'}\int_0^{\bar{Z}'} Fg_2 dZ'_1 d\bar{Z}'_1 + \int_0^{Z'}\int_0^{\bar{Z}'}[F\int_0^{Z_1}\int_0^{\bar{Z}_1} Fg_2 dZ_2 d\bar{Z}_2] dZ_1 d\bar{Z}_1 + \cdots]$$

satisfies the differential equation (2.8b) and the initial conditions (3.13), (3.14). It is evident that the series (3.17) converges in any simply-connected domain which includes the origin $Z' = 0$, $\bar{Z}' = 0$ and which is common to the regularity domains of F, g_1, g_2. Since by the above requirements a solution of (2.8b) is uniquely determined, the expressions (3.16) and (3.17) must coincide, so that they are two different representations of the same function.

We proceed now to the discussion of the relations between the domains of regularity, a and k, of the $\mathbf{p}_1$-associate function g and the generated solution $\mathbf{p}_1(g)$ in the real plane, i. e., for $\bar{Z}'$ conjugate to Z'.

THEOREM 3.2. *Let B be a bounded region of the (real) λ', θ'-plane, situated in* $\mathrm{E}[\lambda' < \lambda_0]$. *If g is regular in B, then $\mathbf{p}_1(g)$ is also regular in B; conversely, the regularity of $\mathbf{p}_1(g)$ in B implies that g is regular there.*

Proof. In order to prove our statement we investigate the relations which exist between the regularity domain of a solution $\psi(Z',\bar{Z}')$ of (2.8b) in the real λ', θ'-plane (i. e., for $\bar{Z}'$ conjugate to Z') and in the space of two complex variables, λ, θ (i. e., when $\bar{Z}'$ and Z' are two independent complex variables). Let B be a domain in the λ, θ-plane. We denote as the hull [10]

[9] The integral operator of the first kind may be regarded as a generalization of the Riemann formula in the theory of linear hyperbolic equations $u_{XY} + a(X,Y)u_X + b(X,Y)u_Y + c(X,Y) = 0$ to the elliptic case, where the real variables X, Y are replaced by two independent complex variables, Z' and Z' respectively. If we consider the solution in the real plane, i. e., for Z', $\bar{Z}'$ conjugate, we thus obtain solutions of elliptic equations. See [**5**, pp. 317-318].

[10] The superscript indicates the dimension of the manifold under consideration. In the case where the manifolds are one- or two-dimensional and are situated in the (real) λ, θ-plane, these superscripts are omitted.

$H^4(B)$ of B the intersection $B_1{}^4 \cap B_2{}^4$ of two cylinders of (four-dimensional) space, $B_1{}^4 = \mathrm{E}[Z' \varepsilon B,\ \bar{Z}' \text{ arbitrary}]$, $B_2{}^4 = \mathrm{E}[Z' \text{ arbitrary},\ \bar{Z}' \varepsilon B]$.

It is well-known that a solution $\psi^*(Z', \bar{Z}')$ of the first equation of (2.14) can be represented in the domain B of the real λ, θ-plane in the form

$$(3.18) \quad \psi^*(Z', \bar{Z}') = \int_b \left[\frac{\partial \psi^*(\zeta, \bar{\zeta})}{\partial n_\zeta} \phi^*(Z', \bar{Z}'; \zeta, \bar{\zeta}) - \psi^*(\zeta, \bar{\zeta}) \frac{\partial \phi^*(Z', \bar{Z}'; \zeta, \bar{\zeta})}{\partial n_\zeta} \right] ds_\zeta,$$

where b denotes the boundary of B, n_ζ the interior normal to b, ds_ζ the line element of b. Here ϕ^* is a fundamental solution of (2.14), and we have

$$(3.19) \quad \phi^* = \tfrac{1}{2}\chi(Z', \bar{Z}'; \zeta, \bar{\zeta})\, [lg(Z' - \zeta) + lg(\bar{Z}' - \bar{\zeta})] + v(Z', \bar{Z}'; \zeta, \bar{\zeta})$$

where [11]

$$(3.20) \quad \chi(Z', \bar{Z}'; \zeta, \bar{\zeta}) = 1 - \int_\zeta^{Z'} \int_{\bar{\zeta}}^{\bar{Z}'} F dZ_1 d\bar{Z}_1 + \int_\zeta^{Z'} \int_{\bar{\zeta}}^{\bar{Z}'} F \left[\int_\zeta^{Z_1} \int_{\bar{\zeta}}^{\bar{Z}_1} F dZ_2 d\bar{Z}_2 \right] dZ_1 d\bar{Z}_1 + \cdots$$

$$(3.21) \quad v(Z', \bar{Z}'; \zeta, \bar{\zeta}) = \int_\zeta^{Z'} \int_{\bar{\zeta}}^{\bar{Z}'} G dZ_1 d\bar{Z}_1 - \int_\zeta^{Z'} \int_{\bar{\zeta}}^{\bar{Z}'} F \left[\int_\zeta^{Z_1} \int_{\bar{\zeta}}^{\bar{Z}_1} G dZ_2 d\bar{Z}_2 \right] dZ_1 d\bar{Z}_1 + \cdots,$$

$$G = -(1/(\bar{Z}' - \bar{\zeta}))(\partial\chi/\partial Z') - (1/(Z' - \zeta))(\partial\chi/\partial \bar{Z}')$$

Since F is defined for all values: $\mathrm{Re}((Z' + \bar{Z}')/2) < \lambda_0$, $|Z| < \infty$, $|\bar{Z}| < \infty$, it follows from (3.17) that if g_1 is regular in B, $B = \mathrm{E}[\lambda' < \lambda_0]$, then $\boldsymbol{p}_1(g_1)$ is regular in $H^4(B)$, and therefore also in the domain B, which is the intersection of $H^4(B)$ with the real λ, θ-plane. This is the first assertion of Theorem 3.2.

From (3.18), (3.19), (3.20), (3.21) it follows that every solution which is regular in the domain B of the real λ, θ-plane can be extended to the complex values in $H^4(B)$. Since B represents the intersection of $H^4(B)$ with the plane $Z' = 0$ as well as with $\bar{Z}' = 0$, $\psi(Z', 0)$ and $\psi(0, \bar{Z}')$ are regular in B. Since $R(Z', \bar{Z}')$ is regular in $\mathrm{E}[\mathrm{Re}((Z' + \bar{Z}')/2) < \lambda_0,\ |Z'| < \infty,\ |\bar{Z}'| < \infty]$, it follows from (3.17) that $G_1(Z')$, (as well as $G_2(\bar{Z}') = G_1(Z')$ see (3.15)) is regular in B. See [**1**, § 1 and **2**, § 2]. This completes the proof of Theorem 3.2.

[11] We note that $\chi(Z', \bar{Z}'; 0, 0) = \boldsymbol{p}_1(1)$.

As has been shown in [**5**, pp. 318-319], the function generated by integral operators $\mathrm{P}(f)$ defined in (3.11), will have branchpoints of finite order,[12] at the same points as the function

$$\mathrm{P}_0(f) = \int_{t=-1}^{1} f[\tfrac{1}{2}Z(1-t^2)]dt/(1-t^2)^{\frac{1}{2}}. \tag{3.22}$$

As a consequence, the following holds.

THEOREM 3.3. *Suppose that the function*

$$g(Z) = \mathrm{P}_0(f) \tag{3.23}$$

is defined and regular in a region R situated on a Riemann surface, which possesses in its interior a finite number of branchpoints, each of finite order. Let further the projection of R on the schlicht λ, θ-plane lie in $\mathrm{E}[-\infty < \lambda' < \lambda_0]$.

Then the function

$$\mathrm{P}_1(f) = \int_{t=-1}^{1} E_1(Z', \bar{Z}', t) f[\tfrac{1}{2}Z(1-t^2)]dt/(1-t^2)^{\frac{1}{2}} \tag{3.24}$$

(see (3.2), (3.7), (3.8), (3.9), (3.10)), *is a solution of* (2.8b) *defined in R, possessing branchpoints at the same points and of the same orders as g.*

$\psi = \mathrm{Im}\,[\mathrm{P}_1(f)]$ *satisfies equation* (2.8a) *and can be interpreted as a streamfunction* (*in the λ, θ-plane*) *of a* (*possible*) *flow pattern of a compressible fluid.*

Making an obvious modification, we can in a similar way extend the definition of the operator P_1 so that it can be applied to functions of one variable $\Lambda + \theta$ and $\Lambda - \theta$ respectively, thus generating solutions of equation (2.12). The expression

$$\mathrm{P}_1(f_1(\Lambda + \theta)) + \mathrm{P}_1(f_2(\Lambda - \theta)) \tag{3.25}$$

where f_1 and f_2 are two linearly independent functions, will represent a (possible) streamfunction of a supersonic flow pattern.

4. The integral operator of the second kind in the case of the simplified compressibility equation. The integral operator of the first kind, convenient though it is for many purposes, has the disadvantage that it does

[12] At poles and logarithmic singularities, operators $\mathrm{P}(f)$ do not, in general, preserve certain properties of $\mathrm{P}_0(f)$ which are essential in aerodynamical applications; in these cases, we have to use other means (see [**3**, **9**]), in order to produce the necessary singularities of ψ.

not represent solutions of the compressibility equation in the neighborhood of the sonic line. Furthermore, it has the disadvantage from the practical point of view that the $P^{(n)}(Z', \bar{Z}')$ are functions of two variables which makes tabulation of the values of $P^{(n)}$ very time-consuming.

These two disadvantages can be removed by the use of another operator—to be termed " operator of the second kind "—for which the $P^{(n)}$ are functions of one variable only and which yields a representation of the streamfunction in the neighborhood of the sonic line. This operator has a number of other distinctive features which will best be elucidated by the detailed discussion of the so-called " simplified " compressibility equation, i. e., where $N = N\dagger = (12\lambda)^{-1}$ in equation (2. 8a) or alternatively, where $F = F\dagger = 5/144\lambda^2$ in equation (2. 14).

According to Theorem 3. 1, any function E of the form

$$E = HE^*, \tag{4.1}$$

where E^* is a solution of the equation

$$\boldsymbol{G}_2(E^*) = (1 - t^2)E_{\bar{Z}t} - (1/t)E^*_{\bar{Z}} + 2ZtE^*_{Z\bar{Z}} + 2ZtFE^* = 0, \tag{4.2}$$

and H is defined by

$$\begin{aligned} H(2\lambda) &= \exp\left[-\int_{-\infty}^{2\lambda} N(\tau)\,d\tau\right] \\ &= (1 - M^2)^{-1/4}[1 + \tfrac{1}{2}(k-1)M^2]^{-1/2(k-1)} \\ &= S_0(-2\lambda)^{-1/6}p((-2\lambda)^{2/3}) \end{aligned} \tag{4.3}$$

with

$$p((-2\lambda)^{2/3}) = 1 + S_1(-2\lambda)^{2/3} + S_2(-2\lambda)^{4/3} + \cdots,$$

$$S_0 = 2^{(2k+1)/(6k-6)}3^{-1/6}(k+1)^{(2-k)/(6k+6)}$$

$$\begin{aligned} S_1 &= (1/10)(3/4)^{2/3}(k+1)^{-1/3}(2k+5), \\ S_2 &= -(1/1400)(3/4)^{4/3}(k+1)^{-2/3}(64k^2 + 70k + 75), \cdots \end{aligned} \tag{4.4}$$

may be used as a generating function of our operator. In the case of the simplified equation, we have $N\dagger = 1/(12\lambda)$, $p((-2\lambda)^{2/3}) = 1$, $H\dagger(2\lambda) = S_0(-2\lambda)^{-1/6}$, $F\dagger = 5/144\lambda^2$.

We now introduce a new variable

$$u = t^2 Z/(Z + \bar{Z}), \tag{4.5}$$

and we shall show that in the simplified case there exist solutions of (4. 2) which are functions of u alone.

LEMMA 4.1.

$$(4.6)\quad E^*\dagger(\lambda,\theta,t)=A_1\mathrm{F}(1/6,\ 5/6,\ 1/2,\ -t^2(\lambda+i\theta)/-2\lambda)$$
$$+B_1(-t^2(\lambda+i\theta)/-2\lambda)^{\frac{1}{2}}\mathrm{F}(2/3,\ 4/3,\ 3/2,\ -t^2(\lambda+i\theta)/-2\lambda)$$

(where $\mathrm{F}(\alpha,\ \beta,\ \gamma,\ X)$ denotes the hypergeometric function and A_1 and B_1 are arbitrary constants) is the most general solution of (4.2) which is a function of u alone. Let us note that there exist other solutions of (4.2) which are functions of *one* variable, see (4.20).)

Proof. We shall show that (4.2) can be reduced to an ordinary differential equation whose solution is (4.6). A formal computation yields

$$\partial u/\partial t=2u/t,\qquad \partial u/\partial Z=(t^2u-u^2)/t^2Z,\qquad \partial u/\partial\bar{Z}=-u^2/t^2Z,$$
$$E^*\dagger_Z=-u^2t^{-2}Z^{-1}E^*\dagger_u,\quad E^*\dagger_{Zt}=-2u^2t^{-3}Z^{-1}[uE^*\dagger_{uu}+E^*\dagger_u]$$
$$E^*\dagger_{Z\bar{Z}}=-u^2t^{-4}Z^{-2}[(ut^2-u^2)E^*\dagger_{uu}+(t^2-2u)E^*\dagger_u],$$
$$F\dagger=(5/36)u^2t^{-4}Z^{-2}.$$

Substituting the above expressions into (4.2) we obtain

$$\boldsymbol{G}_2(E^*\dagger)=-2u^2t^{-3}Z[u(1-u)E^*\dagger_{uu}+(\tfrac{1}{2}-2u)E^*\dagger_u-(5/36)E^*\dagger]=0.$$

The equation

$$(4.7)\qquad u(1-u)E^*\dagger_{uu}+(\tfrac{1}{2}-2u)E^*\dagger_u-(5/36)E^*\dagger=0$$

is a hypergeometric equation whose general solution can be represented in the form

$$(4.8)\quad E^*\dagger=A_1\mathrm{F}(1/6,5/6,1/2,u)+B_1u^{\frac{1}{2}}\mathrm{F}(2/3,4/3,3/2,u)\qquad |u|<1$$

$$(4.8')\qquad =A_2u^{-1/6}\mathrm{F}(1/6,2/3,1/3,1/u)$$
$$+B_2u^{-5/6}\mathrm{F}(5/6,4/3,5/3,1/u),\qquad |u|>1.$$

Replacing u by the right-hand side of (4.5), we arrive at (4.6).

Thus, combining (4.1) and (4.6), we obtain for the generating function in the simplified case [13]

[13] We note that in many instances we may omit the second term on the right-hand side of (4.9) since

$$\int_{t=-1}^{1}t\mathrm{F}(2/3;\ 4/3,3/2,-t^2(\lambda+i\theta)/-2\lambda)f(\tfrac{1}{2}Z(1-t^2))\,dt/(1-t^2)^{\frac{1}{2}}=0$$

if f is regular at $Z=0$.

$$(4.9)\quad E^{\dagger}(\lambda,\theta,t)=A_1S_0(-2\lambda)^{-1/6}\mathrm{F}(1/6,5/6,1/2,\,-t^2(\lambda+i\theta)/-2\lambda)$$
$$+B_1S_0(-2\lambda)^{-2/3}[-t^2(\lambda+i\theta)]^{\frac{1}{2}}\mathrm{F}(2/3,4/3,3/2,\,-t^2(\lambda+i\theta)/-2\lambda),$$
$$|-t^2(\lambda+i\theta)/-2\lambda|<1$$

$$(4.9')\quad =A_2S_0[(-t^2)(\lambda+i\theta)]^{-1/6}\mathrm{F}(1/6,2/3,1/3,\,-2\lambda/-t^2(\lambda+i\theta)]$$
$$+B_2S_0(-2\lambda)^{-2/3}[(-t^2)(\lambda+i\theta)]^{-5/6}\mathrm{F}(5/6,4/3,5/3,\,-2\lambda/-t^2(\lambda+i\theta)),$$
$$|-2\lambda/-t^2(\lambda+i\theta)|<1.$$

In Theorem 3.1 we proved the existence of a generating function by means of which we can obtain solutions of the compressibility equation in a sufficiently small neighborhood of the origin. We shall now show that in the case of the simplified equation and the operator of the second type this result in the small can be replaced by a result in the large. This result enables us, from the behavior of the associate f, to make conclusions concerning the behavior of the generated solution of the compressibility equation in its entire domain of definition. An exact formulation of this statement, at first for the subsonic region, is given in the following theorem:

THEOREM 4.1. *Suppose that the function*

$$(4.10)\qquad g(Z)=\int_{t=-1}^{1} f[\tfrac{1}{2}Z(1-t^2)]\,dt/(1-t^2)^{\frac{1}{2}}$$

is regular in a region B (situated on a Riemann surface) which possesses in its interior a finite number of branchpoints,[14] *each of finite order. Let further the projection of B on the schlicht λ,θ-plane lie in* $\mathrm{E}[-\infty<\lambda<0]$.

The function

$$(4.11)\quad \psi=\mathrm{Im}[\mathrm{P}^{\dagger}_2(f)],$$
$$\mathrm{P}^{\dagger}_2(f)=\int_C E^{\dagger}(\lambda,\theta,t)f[\tfrac{1}{2}Z(1-t^2)]\,dt/1-t^2)^{\frac{1}{2}},\qquad Z=\lambda+i\theta$$

(where C is a suitably chosen curve in the complex t-plane connecting the points $t=-1$ and $t=1$) is a solution of (2.8) *with $N=N^{\dagger}=1/12\lambda$; this solution is defined in B and possesses branchpoints at the same points and of the same order as* (4.10). *$\psi=\mathrm{Im}[\mathrm{P}^{\dagger}_2(f)]$ can be interpreted as a streamfunction (in the λ,θ-plane) of a (possible) flow pattern of a compressible fluid (for the simplified compressibility equation).*

[14] We assume here that the only singularities of g in B are branchpoints. In applying the integral operator method in the case where g has poles or logarithmic singularities, certain modifications, indicated in [9, p. 469, footnote 14], are needed.

Proof. In order to prove our statement, we have to show that by (4.6) and (4.10) and slight modifications of these formulas, $E\dagger_2(\lambda, \theta, t)$ is defined for all values $\lambda < 0$, $-\infty < \theta < \infty$ and for values t belonging to a suitable simple, sufficiently smooth curve in the complex t-plane which connects $t = -1$ and $t = 1$. Obviously, this curve has to avoid the points $t = 0$, $t = \pm (2\lambda/(\lambda + i\theta))^{\frac{1}{2}}$ as these would give rise to singularities of the hypergeometric function. On the other hand, any such curve will be suitable for our purposes. However, with a view to the subsequent generalization of our procedure to the "exact" case, we shall use two special paths of integration, C_1 and C_2, the former apart from its terminals $t = \pm 1$, inside $\mathrm{E}[\,|\,t\,| < 1]$, and the latter outside $\mathrm{E}[\,|\,t\,| \leq 1]$. C_1 will be used for values λ, θ satisfying $|(\lambda + i\theta)/2\lambda| < 1$ and C_2 for the case $|(\lambda + i\theta)/2\lambda| > 1$.

It should be noted that the two terminals of the integration path, viz., $t = \pm 1$, will never be singularities of $E\dagger_2(\lambda, \theta, t)$ since, for $t = \pm 1$, $(\lambda + i\theta)/2\lambda \neq 1$ for real λ and θ.

The expressions thus obtained will not necessarily be analytical continuations of each other (*qua* functions of λ, θ). Since, however, the hypergeometric equation has only two linearly independent solutions, the constants, A_1, B_1 and A_2, B_2 can always be so adjusted as to make these two solutions analytical continuations of each other.

REMARK. It would, of course, also be possible to characterize the path of integration in a manner which is topologically invariant with regard to the way the singular points of $E\dagger_2(\lambda, \theta, t)$ are by-passed; using this definition we would, for any value (λ, θ), obtain one and the same function $\psi(\lambda, \theta)$. However, although this procedure has some theoretical advantages, its actual carrying out may give rise to certain difficulties; in practical applications it is much easier to assure analytical continuation by the determination of the constants A_2, B_2 if A_1, B_1 are given, or vice versa.

The integral representation (4.11) can be immediately generalized to the supersonic case where it will produce, in an analogous manner, solutions of (2.12) with $N_1 = N\dagger_1$ (see (2.18)). Indeed, replacing λ by the variable $\omega = \lambda + i\Lambda$ and considering the solution $\psi(\omega, \theta)$ of (2.8a) in the plane $\lambda = 0$, it is seen that $\psi(i\Lambda, \theta)$ satisfies equation (2.12) with $N = 1/12\Lambda$. Repeating the procedure which led to the generating function (4.9) in the subsonic case, we now obtain the generating function

$$(4.12)\qquad E\dagger = (a_1 S_0/(2\Lambda)^{1/6})\mathrm{F}(1/6, 5/6, 1/2, t^2(\Lambda + \theta)/2\Lambda)$$
$$+ (b_1 S_0(\Lambda + \theta)^{\frac{1}{2}} t/(2\Lambda)^{2/3})\mathrm{F}(2/3, 4/3, 3/2, t^2(\Lambda + \theta)/2\Lambda),$$
$$|\,t^2(\Lambda + \theta)/2\Lambda\,| < 1.$$

If $\mathrm{F}(\alpha_1, \beta_1, \gamma_1, X)$ denotes only the hypergeometric series and not the hypergeometric function, (4.12) has to be replaced, for $|t^2(\Lambda+\theta)/2\Lambda| > 1$, by

$$(4.13)\qquad E\dagger = (a_2 S_0/(t^2(\Lambda+\theta))^{1/6})\mathrm{F}(1/6, 2/3, 1/3, 2\Lambda/t^2(\Lambda+\theta)) + (b_2 S_0(2\Lambda)^{2/3}/(t^2(\Lambda+\theta))^{5/6})\mathrm{F}(5/6, 4/3, 5/3, 2\Lambda/t^2(\Lambda+\theta)),$$

where the constants a_2, b_2 are easily expressible in terms of a_1, b_1.

If $\theta = \Lambda$ and $t^2 = 1$, there arise certain difficulties, since the hypergeometric functions in (4.12) will then become singular. By the transformation formulas of the hypergeometric function, (4.12) may be written in the neighborhood of $((\Lambda+\theta)/2\Lambda)t^2 = 1$, in the form

$$(4.14)\qquad E\dagger = (a_3 S_0/(2\Lambda)^{1/6})\mathrm{F}(1/6, 5/6, 3/2,\ 1 - t^2(\Lambda+\theta)/2\Lambda)$$
$$b_3 S_6(2\Lambda/(2\Lambda - t^2(\Lambda+\theta)))^{\frac{1}{2}}\mathrm{F}(1/3, -1/3, 1/2, 1 - t^2(\Lambda+\theta)/2\Lambda).$$

In order to avoid the complications which arise from the fact that the second term of (4.14) has a singularity for $\Lambda = \theta$, $t^2 = 1$, we shall therefore take $b_3 = 0$. The function $E\dagger$ will accordingly be of the form

$$(4.15)\qquad E\dagger = (a_3 S_0/(2\Lambda)^{1/6})\mathrm{F}(1/6, 5/6, 3/2,\ 1 - t^2(\Lambda+\theta)/2\Lambda).$$

We note further that all these considerations can be repeated with Λ replaced by $-\Lambda$. Our operator will therefore yield two independent types of solutions, depending on whether the argument of the associate function is taken as $\Lambda+\theta$ or $\Lambda-\theta$.

The exact conditions under which our operator can generate solutions of the compressibility equation in the supersonic case are given in the following theorem:

THEOREM 4.2. *Suppose* $f_s(\zeta)$, $s = 1, 2$, *are real functions of the real variable* ζ *and everywhere differentiable with the possible exception of* $\zeta = 0$; *suppose further that in a fixed neighborhood of* $\zeta = 0$, f_s *can be approximated to any prescribed degree of accuracy by the expressions of the form* $\sum_{n=1}^{N} A_n{}^{(N)}\zeta^{K_n}$, $K_n \geq 1$. *Then*

$$(4.16)\qquad \psi(\Lambda, \theta) = \boldsymbol{R}\dagger_1(f_1) + \boldsymbol{R}\dagger_2(f_2),$$

$$\boldsymbol{R}\dagger_s(f_s) = \int_{t=-1}^{1} E\dagger_2(\Lambda, -(-1)^s\theta, t) f_s[\tfrac{1}{2}(\Lambda - (-1)^s\theta)(1-t^2)]\,dt/(1-t^2)^{\frac{1}{2}}, \qquad s = 1, 2$$

represents a solution of the compressibility equation, which is defined for any $\Lambda > 0$ *and can be interpreted as a streamfunction of a (possible) supersonic flow pattern.*

Proof. By (4.12) we have

$$(4.17)\quad \boldsymbol{R}\dagger_1(f_1) = a_1/(2\Lambda)^{1/6}$$
$$\times \int_{-1}^{1} \mathrm{F}(1/6, 5/6, 1/2, t^2((\Lambda+\theta)/2\Lambda) f((\Lambda+\theta)(1-t^2)/2)(1-t^2)^{-\frac{1}{2}}dt$$
$$+ c_1(\Lambda+\theta)^{\frac{1}{2}}(2\Lambda)^{-2/3}$$
$$\times \int_{-1}^{1} t\mathrm{F}(2/3, 4/3, 3/2, t^2(\Lambda+\theta)/2\Lambda) f((\Lambda+\theta)(1-t^2)/2)(1-t^2)^{-\frac{1}{2}}dt.$$

In view of $\Lambda > 0$, the only values of (Λ, θ) for which this expression may not be differentiable are those for which $\Lambda + \theta = 0$ or for which $u = t^2(\Lambda+\theta)/2\Lambda$ coincides with either of the values $0, 1, \infty$ for $-1 < t < 1$. The case $u = \infty$ is ruled out because of $\Lambda > 0$; the case $u = 1$, although corresponding to a singularity of the hypergeometric equation, does not give rise to a singularity of $\boldsymbol{R}$, by virtue of our particular choice (4.15) of $E\dagger$. The case $u = 0$ need only to be considered for $\Lambda + \theta = 0$, as $t = 0$ obviously does not cause any difficulties.

Under our assumptions, it is sufficient to consider the special case

$$(4.18)\qquad f_1(\zeta) = f_{1,\kappa}(\zeta) = 2^{\kappa}\zeta^{\kappa}, \qquad \kappa \geq 1,$$

in which (4.17) reduces to

$$(4.19)\quad \boldsymbol{R}\dagger_{2,1}(f_{1,\kappa}) = a_1/(2\Lambda)^{1/6}$$
$$\times \int_{-1}^{1} \mathrm{F}(1/6, 5/6, 1/2, \ t^2(\Lambda+\theta)/2\Lambda)(\Lambda+\theta)^{\kappa}(1-t^2)^{\kappa-\frac{1}{2}}dt$$
$$+ c_1(2\Lambda)^{-2/3} \int_{-1}^{1} t\mathrm{F}(2/3, 4/3, 3/2, \ t^2(\Lambda+\theta)/2\Lambda)(\Lambda+\theta)^{\kappa+\frac{1}{2}}(1-t^2)^{\kappa-\frac{1}{2}}dt.$$

For $\Lambda + \theta \to 0$, this expression is obviously continuous. The same is also true of the derivative $\partial\boldsymbol{R}\dagger_{2,1}/\partial\mathrm{H}\dagger$. Indeed, $\partial\boldsymbol{R}\dagger_{2,1}/\partial\mathrm{H}\dagger$ (see (2.18)) behaves in the neighborhood of $\Lambda + \theta = 0$ like $(\Lambda+\theta)^{\kappa-1}$; in view of $\kappa \geq 1$, it therefore remains continuous there. This completes the proof of Theorem 4.1.

Before we proceed to investigate the behavior of our solutions on the sonic line, we note that (4.6) is not the only solution which depends only on one variable. If we write

$$(4.20)\qquad \mu = t^2(\lambda - \lambda_0 + i\theta)/2\lambda$$

and try to solve (4.2) by a function of one variable μ, a formal computation shows that $E^*\dagger$ as a function of μ must satisfy the hypergeometric equation (4.7) with u replaced by μ.

Accordingly, the generating function (4.6) for the operator of the second kind may be replaced by

$$(4.21)\quad E\dagger(\lambda, \theta, t) = A_1(-2\lambda)^{1/6}F(1/6, 5/6, 1/2, t^2(\lambda - \lambda_0 + i\theta)/2\lambda).$$

Besides its greater generality, the generating function (4.21 with $\lambda_0 \neq 0$ has a number of additional features, which make it superior to (4.6) in many cases. In (4.6) the point $\lambda = 0$, $\theta = 0$ is a singularity since, by approaching this point in a suitable manner, the argument of the hypergeometric function can be given an arbitrary value. In (4.21) such a singularity does not exist, since λ and $\lambda - \lambda_0 + i\theta$ cannot vanish simultaneously if $\lambda_0 \neq 0$.

Another singularity, which can be removed by using (4.21) with $\lambda_0 \neq 0$ instead of (4.6), occurs in the supersonic case. For $t^2 = 1$ and $\Lambda = \theta$, the second term of the right-hand side of (4.14) becomes singular. In order to allow for this case, we had to assume that $b_3 = 0$, thus somewhat restricting the generality of the solutions we could obtain. Setting $\lambda = i\Lambda$, it is seen that in the supersonic case the argument of the hypergeometric function in (4.21) becomes

$$(4.22)\quad \mu = t^2(\Lambda + i\lambda_0 + \theta)/2\Lambda.$$

For $t = \pm 1$, we have $\mu \neq 1$ for real values of Λ, θ; the singularity in question is therefore removed and the constant b_3 in (4.14) may now take any arbitrary value. Moreover, (4.22) shows that $\mu \neq 0$ for $\Lambda + \theta = 0$; as a consequence, the line $\Lambda + \theta = 0$ loses its singular character and the discussion of the equivalent of (4.17) is considerably simplified.

Thus, by the use of integral operators of the second kind with the generating functions (4.9) and (4.12) as well as (4.21) we obtain solutions $\psi = \operatorname{Im}[\boldsymbol{p}_2(g)]$ which are defined in the subsonic and supersonic regions respectively. If g is defined in a (not necessarily schlicht) domain G_1 in the subsonic region and has, as its only singularities in G_1, branchpoints of finite order (but not poles or logarithmic singularities), then the generated function is again defined in G_1. (In particular, it has branchpoints at the same points and of the same order as g).

If g is a twice differentiable function of one real variable in a schlicht domain G_2 in the supersonic region, the generated function will be a solution of (2.18).

Thus by this procedure, we can generate solutions of (2.2) which are defined in adjacent domains, one in the subsonic, the other in the supersonic region. Our object is to show that these solutions can be continued across the sonic line.

This fact is immediately seen if we introduce a new variable,[15]

$$(4.23)\qquad \begin{aligned} s &= (-\lambda)^{2/3} \quad \text{for} \quad \lambda < 0, \\ s &= -\Lambda^{2/3} \quad \text{for} \quad \Lambda > 0. \end{aligned}$$

The variable $s = s(M\dagger)$ considered as a function of Mach number $M\dagger$, possesses the property that $s(1) = 0$, and that

$$(4.24)\qquad ds(M\dagger)/dM\dagger = -2^{5/3}(3k+3)^{-2/3}M\dagger + O(1 - M\dagger^2)$$

is non-vanishing and bounded in a sufficiently small neighborhood of $M\dagger = 1$.

If $\lambda_0 \neq 0$, and if g is regular for $(-\theta_0 \leq \theta \leq \theta_0,\ s = \lambda = \Lambda = 0)$ then the generated function is an analytic function of θ and s (and therefore of θ and $M\dagger$) for $(-\theta_0 \leq \theta \leq \theta_0,\ -s^{(0)} \leq s \leq s^{(0)})$, $s^{(0)}$ sufficiently small.

If $\lambda_0 = 0$, we have to assume that $\lim_{Z=0} Z^{-5/6}f(Z)$ exists (or alternately that $\lim_{Z=0} Z^{1/6}g'(Z)$ exists) in order to assume that the generated function $\psi(\theta, s)$ is regular also at the point $\theta = 0$, $s = 0$.

5. Integral operator of the second kind in the case of the "exact" compressibility equation. As we mentioned before, any solution E^* of the equation

$$(5.1)\qquad E^*_{\bar{Z}t} - t^2E^*_{\bar{Z}t} - (1/t)E^*_{\bar{Z}} + 2ZtE^*_{\bar{Z}Z} + 2ZtFE^* = 0,$$

multiplied by $\exp\left[-\int_{-\infty}^{Z+\bar{Z}} N d\tau\right]$, (see (4.3)), is a generating function of an integral operator $\mathrm{P}(f)$ (see (3.4b) which produces solutions of equation (2.8b). The series $1 + \sum_{n=1}^{\infty} (t^2Z)^n Q^{(n)}(2\lambda)$, where the $Q^{(n)}$'s are solutions of the system:

$$(5.2)\qquad (2n+1)Q_\lambda^{(n+1)} + Q_{\lambda\lambda}^{(n)} + 4FQ^{(n)} = 0,\ Q^{(1)} = -4\int_{-\infty}^{2\lambda} F d\lambda$$

(see (87) of [**6**]), is a solution of (5.1). The above series converges for $|Z| < 2|\lambda|$ and reduces to the first summand of (4.6) for $F = F\dagger$.

It is therefore desirable to obtain solutions of (5.1) which are defined in $|Z| > 2|\lambda|$. In this section we shall determine two power series $E^{*(\kappa)}$ $(\kappa = 1, 2)$, both converging in $|Z| > 2|\lambda|$, such that in the simplified case, $A_2RE^{*(1)} + B_2RE^{*(2)}$ reduces to (4.9′).

[15] We note that in the simplified case we have

$$s = -2\cdot 3^{-2/3}(2/(k-1))^{(2-k)/(3k-3)}\mathrm{H}\dagger.$$

THEOREM 5.1. *Let*

$$q^{(n,\kappa)} \equiv \sum_{\nu=0}^{\infty} C_\nu^{(n,\kappa)} (-\lambda)^{n-\frac{1}{2}+(2/3)(\kappa+\nu)} \qquad \kappa = 1, 2, \tag{5.3}$$

be a set of functions which are connected by the relations

$$q_{\lambda\lambda}^{(0,\kappa)} + 4Fq^{(0,\kappa)} = 0 \tag{5.4a}$$

$$2(n + \tfrac{2}{3}\kappa) q_\lambda^{(n,\kappa)} + q_{\lambda\lambda}^{(n+1,\kappa)} + 4Fq^{(n+1,\kappa)} = 0, \qquad n = 1, 2, \cdots, \; \kappa = 1, 2. \tag{5.4b}$$

Then each of the functions

$$E^{*(\kappa)} = \sum_{n=0}^{\infty} q^{(n,\kappa)} / (-t^2 Z)^{n-\frac{1}{2}+(2/3)\kappa} \tag{5.5}$$

is a solution of (5.1). *Each of the series converges in* $E[2\,|\,\lambda\,| < |\,Z\,|\,]$.

Proof. Substituting the series (5.5) into (5.1) and equating the coefficients of $t^{-(6n+4)/3}(-Z)^{-(6n+1)/6}$, $n = -1, 0, 1, \cdots$, to 0, we obtain the following set of equations:

$$q_{\bar{Z}Z}^{(0,\kappa)} + Fq^{(0,\kappa)} = 0, \; (n + \tfrac{2}{3}\kappa) q_{\bar{Z}}^{(n,\kappa)} + q_{\bar{Z}Z}^{(n+1,\kappa)} + Fq^{(n+1,\kappa)} = 0, \qquad n = 0, 1, \cdots, \tag{5.6}$$

which, if we assume that the $q^{(n,\kappa)}$ are functions of λ alone, result in the equations (5.4a) and (5.4b).

According to (2.17), F can be written in the form

$$F = s^{-3} S(s), \qquad s = (-\lambda)^{2/3}, \tag{5.7}$$

where $S(s) = \alpha_0 + \alpha_1 s + \alpha_2 s^2 + \cdots$, $\alpha_0 = 5/144$, $\alpha_1 = 0, \cdots$, see (2.17), considered as a function of s, is regular in a circle of radius s_0 say, with center at the origin.

Introducing the variable s, we obtain

$$q_\lambda^{(n,\kappa)} = -(ds/d(-\lambda))(dq^{(n,\kappa)}/ds) = -\tfrac{2}{3}(-\lambda)^{-1/3}(dq^{(n,\kappa)}/ds) = -\tfrac{2}{3}s^{-\frac{1}{2}}(dq^{(n,\kappa)}/ds)$$

$$q_{\lambda\lambda}^{(n+1,\kappa)} = (4/9)[-\tfrac{1}{2}s^{-2}q_s^{(n+1,\kappa)} + s^{-1}q_{ss}^{(n+1,\kappa)}].$$

Hence, the system (5.4a), (5.4b) assumes the form

$$s^2 q_{ss}^{(0,\kappa)} - \tfrac{1}{2}s q_s^{(0,\kappa)} + 9S(s) q^{(0,\kappa)} = 0 \tag{5.9a}$$

$$-3(n + \tfrac{2}{3}\kappa) s^{5/2} q_s^{(n,\kappa)} + s^2 q_{ss}^{(n+1,\kappa)} - \tfrac{1}{2}s q_s^{(n+1,\kappa)} + 9S(s) q^{(n+1,\kappa)} = 0, \qquad n = 0, 1, 2, 3, \cdots. \tag{5.9b}$$

LEMMA 5.1. There exists a system of solutions $q^{(n,\kappa)}$, $n = 0, 1, 2, \cdots$, of (5.9a), (5.9b) of the form

$$(5.10) \quad q^{(n,\kappa)} = s^{(3/2)(n-\frac{1}{2}+(2/3)\kappa)} T_\kappa^{(n)}(s), \ T_\kappa^{(n)}(s) = \sum_{\nu=0}^{\infty} C_\nu^{(n,\kappa)} s^\nu, \ C_0^{(n,\kappa)} \neq 0$$

where each $T_\kappa^{(n)}(s)$ is a power series which converges in the circle $|s| < s_0$.

Proof. We shall first prove the above lemma for $q^{(0\kappa)}$ and then for arbitrary n by induction on n, i.e., we shall show that if the lemma holds for $q^{(n,\kappa)}$, then it must hold for $q^{(n+1,\kappa)}$. Let us first consider the homogeneous equation

$$(5.11) \qquad s^2 w_{ss} - \tfrac{1}{2} s w_s + 9S(s) w = 0,$$

The indicial equation (see [15], p. 225) is

$$(5.12) \qquad \rho(\rho - 1) - \tfrac{1}{2}\rho + 5/16 = 0, \text{ or } \rho_1 = 5/4, \ \rho_2 = 1/4.$$

By substituting $w = s^{1/4} y$, we obtain the equation

$$(5.13) \qquad y'' + 9s^{-2}[S(s) - 5/144] y = 0.$$

Since $s^{-2}[S(s) - 5/144]$ is a regular function of s for $|s| < s_0$, (see 5.7)) we may choose the two particular solutions of (5.13) as

$$y_1 = C_0^{(0,1)} + C_2^{(0,1)} s^2 + C_3^{(0,1)} s^3 + \cdots$$
$$y_2 = C_0^{(0,2)} s + C_1^{(0,2)} s^2 + C_2^{(0,2)} s^3 + \cdots$$

which yield

$$q^{(0,1)} = C_0^{(0,1)} s^{1/4} + C_2^{(0,1)} s^{9/4} + C_3^{(0,1)} s^{13/4} + \cdots \equiv s^{1/4} W^{(1)}(s)$$
$$q^{(0,2)} = C_0^{(0,2)} s^{5/4} + C_1^{(0,2)} s^{9/4} + C_2^{(0,1)} s^{13/4} + \cdots \equiv s^{5/4} W^{(2)}(s).$$

In the case of the simplified equation we have [see (4.9′), (4.1), (4.3)]

$$q\dagger^{(0,1)} = C\dagger_0^{(0,1)} (-\lambda)^{1/6}, \ C\dagger_0^{(0,1)} = 2^{1/6},$$
$$q\dagger^{(0,2)} = C\dagger_0^{(0,2)} (-\lambda)^{5/6}, \ C\dagger^{(0,2)} = 2^{5/6}.$$

In order to obtain analogous series for the "exact" case we choose

$$(5.14) \qquad C_0^{(0,1)} = C\dagger_0^{(0,1)} = 2^{1/6}, \qquad C_0^{(0,2)} = C\dagger_0^{(0,2)} = 2^{5/6}.$$

Let us now instead of (5.9a) consider the solution of the non-homogeneous equation (5.9b) and let us assume that for $n \geq 0$, we have already proved that $q^{(n,\kappa)}$ has the form (5.10) and $T_\kappa^{(n)}(s)$ converges for $|s| < s_0$.

In order to prove that there exists a solution of the equation (5.9b) of the form $q^{(n+1,\kappa)} = s^{(3/2)(n+\frac{1}{2}+(2/3)\kappa)} T_\kappa^{(n+1)}(s)$, we proceed as follows:

Let

$$q^{(n+1,\kappa)}(s) = w(s)u(s), \tag{5.15}$$

where $w(s) = s^{\kappa-3/4}W^{(\kappa)}(s)$ is a solution (see above) of the homogeneous equation (5.11). Then u will satisfy the equation

$$wu_{ss} + (2w_s - w/2s)u_s = 3(n+\tfrac{2}{3}\kappa)s^{\frac{1}{2}}q_s^{(n,\kappa)}. \tag{5.16}$$

The particular solution of this equation is easily verified to be

$$u = 3(n+\tfrac{2}{3}\kappa)\int_0^s w^{-2}s^{\frac{1}{2}}\left(\int_0^s wq_s^{(n,\kappa)}ds\right)ds \tag{5.17}$$

Expanding $q^{(n+1,\kappa)}$, we obtain

$$\begin{aligned} q^{(n+1,\kappa)}(s) &= s^{(3/2)(n+\frac{1}{2}+(2/3)\kappa)}T_\kappa^{(n+1)}(s), \\ T_\kappa^{(n+1)}(0) & \\ &= [(6n-3+4\kappa)(3n+2\kappa)/(3n+3)(3n+4\kappa-3)]T_\kappa^{(n)}(0) \neq 0 \end{aligned} \tag{5.19}$$

the series for $T_\kappa^{(n+1)}(s)$ converging for $|s| < s_0$.

This completes the proof of Lemma 5.1. In order to prove Theorem 5.1, it remains to be shown that it is legitimate to interchange the order of summation in the double series

$$\sum_{n=0}^{\infty}(-t^2Z)^{-(n-\frac{1}{2}+(2/3)\kappa)}\sum_{\nu=0}^{\infty}C_\nu^{(n,\kappa)}(-\lambda)^{n-\frac{1}{2}+(2/3)(\kappa+\nu)} \tag{5.20}$$

for $2|\lambda| < |Z|$. For this purpose, we shall prove [16]

LEMMA 5.2.

$$|C_\mu^{(n+1,1)}| \leqq 2^{n+1}\mathrm{M}/s_1^{\mu} \tag{5.21}$$

for $n + \frac{2}{3}\mu \geqq p_0$, where $C_\mu^{(n+1,1)}$ are the coefficients of the series (5.2). Here, p_0 and M are sufficiently large constants, and $s_1 = s_0(1+\epsilon)^{-1}$.

Proof. We shall give a proof by induction. Consider at first the $C_0^{(n,1)}$, $n = 0,1,2,\cdots$. If we substitute the power series (5.3) into (5.4b), it becomes evident that the $C_0^{(n,1)}$ depend only upon α_0, and are independent of the remaining coefficients α_n, $n > 0$, of the series expansion of $S(s)$, (see (5.7)). On the other hand, if we substitute $\alpha_0 = 5/144$, $\alpha_n = 0$ for $n > 0$, we obtain the simplified case considered in (4.9′), where we had a representation for $E\dagger$ as a power series in $(2\lambda/-t^2Z)$. Using the fact that

[16] We shall consider here only the case when $\kappa = 1$. Exactly the same proof holds for $\kappa = 2$.

$E^*\dagger = E\dagger/H\dagger = S_0^{-1}(-2\lambda)^{1/6}E\dagger$ (see (4.1) ff.) we obtain two power series for $E^*\dagger$; the first of which corresponds to the case $\kappa = 1$ and the second to the case $\kappa = 2$. Thus, setting $A_2 = 1$ in (4.9′), we have

$$(5.22) \qquad E^*\dagger = (-2\lambda/-t^2Z)^{1/6}\mathrm{F}(1/6, 2/3, 1/3, -2\lambda/-t^2Z)$$
$$= (-2\lambda/-t^2Z)^{1/6}$$
$$+\sum_{n=1}^{\infty}\frac{\frac{1}{6}(\frac{1}{6}+1)\cdots(\frac{1}{6}+n)\cdot\frac{2}{3}(\frac{2}{3}+1)(\frac{2}{3}+n)}{\frac{1}{3}(\frac{1}{3}+1)\cdots(\frac{1}{3}+n)\cdot n!}(-2\lambda/-t^2Z)^{n+1/6}$$

whence

$$(5.23) \qquad C_0^{(0,1)} = C\dagger_0^{(0,1)} = 2^{1/6},\ C_0^{(n,1)} = C\dagger_0^{(n,1)}$$
$$= \frac{\frac{1}{6}(\frac{1}{6}+1)\cdots(\frac{1}{6}+n)(\frac{2}{3})(\frac{2}{3}+1)\cdots(\frac{2}{3}+n)2^{n+1/6}}{\frac{1}{3}(\frac{1}{3}+1)\cdots(\frac{1}{3}+n)\cdot n!}, \qquad n = 1, 2, 3, \cdots,$$

from which the inequality (5.21) for $\mu = 0$ follows.

Remark. Since in this case $\nu = \mu = 0$, the number s_0 in (5.21) where ν is replaced by μ) may be given any positive value. For $n = -1$, and an arbitrary μ, the inequality (5.21) follows from the fact that $q^{(0,1)} = \sum_{\mu=0}^{\infty} C_\mu^{(0,1)}s^{\mu+1/4}$ is a solution of (5.9a) and the fact that $S(s) = \sum_{\mu=0}^{\infty}\alpha_\mu s^\mu$ is regular in the circles of the radius s_0 (see also (5.13)); accordingly,

$$(5.24) \qquad |\alpha_\mu| \leq \Gamma/s_0^{\mu}$$

holds, if Γ is a sufficiently large constant.

We now proceed to the proof (by induction) of (5.21) for $n > -1$ and $\mu > 0$. Let us assume that this inequality holds for some $n+1$, and $\mu \leq \nu - 1$ as well as for $N \leq n$ and $\mu \leq \nu + 1$. We shall prove that (5.21) then holds for $N = n+1$, $\mu = \nu$.

If we substitute the series (5.3) (with $\kappa = 1$) into (5.4b), we obtain the relation

$$(5.25) \qquad \begin{aligned} &-2(n+\tfrac{2}{3})(n+\tfrac{1}{6}+\tfrac{2}{3}\nu)C_\nu^{(n,1)} + (n+\tfrac{7}{6}+\tfrac{2}{3}\nu)(n+\tfrac{1}{6}+\tfrac{2}{3}\nu)C_\nu^{(n+1,1)} \\ &\qquad + 4\alpha_0 C_\nu^{(n+1,1)} + \sum_{\mu=0}^{\nu-1}4C_\mu^{(n+1,1)}\alpha_{\nu-\mu} = 0. \end{aligned}$$

Hence,

$$(5.26) \qquad |C_\nu^{(n+1,1)}[1 + (4\alpha_0/(n+\tfrac{7}{6}+\tfrac{2}{3}\nu)(n+\tfrac{1}{6}+\tfrac{2}{3}\nu))]|$$
$$\leq 2^{n+1}\mathrm{M}/s_0 s_1^{\nu-1}\{[(1+2/3n)/(1+7/6n+(2/3)(\nu/n))]$$
$$+ 4\nu\Gamma/(n+\tfrac{7}{6}+\tfrac{2}{3}\nu)(n+\tfrac{1}{6}+\tfrac{2}{3}\nu)\}.$$

If $n + \frac{2}{3}\nu > p_0$, then

$$(5.27) \qquad | C_\nu^{(n+1,1)} [1 + 4\alpha_0/p_0^2] | \leqq 2^{n+1}\mathrm{M}/s_0 s_1^{\nu-1} [1 + 6\Gamma/p_0],$$

or

$$| C_\nu^{(n+1,1)} | \leqq 2^{n+1}\mathrm{M}(1+\epsilon)/s_0 s_1^{\nu-1} = 2^{n+1}\mathrm{M}/s_1^\nu.$$

This completes the proof of Lemma 5. 2.

In order to show that it is legitimate to interchange the order of summation in (5. 20), we shall show that for

$$| \lambda |^{2/3} \leqq s_0 - \epsilon, \ | -2\lambda/-t^2 Z | \leqq 1 - \epsilon, \text{ where } \epsilon > 0,$$

the series converges absolutely and uniformly. Indeed, by Lemma 5. 2 we obtain

$$(5.28) \qquad | \sum_{n=0}^{\infty} (t^2 Z)^{-(1/6)-n} \sum_{\nu=0}^{\infty} {}_{\nu}^{(n,1)} (-\lambda)^{n+(1/6)+(2/3)\nu} |$$
$$\leqq \mathrm{M} \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} 2^n (-\lambda)^{n+(1/6)+(2/3)\nu}/s_1^\nu \, | t^2 Z |^{(1/6)+n}$$
$$= \mathrm{M} \, | -2\lambda/-t^2 Z |^{1/6} [1 - | -2\lambda/-t^2 Z |]^{-1} [1 - | \lambda^{2/3}/s_1 |]^{-1}$$

which for $| \lambda |^{2/3} < s_0 - \epsilon$, $| -2\lambda/-t^2 Z | < 1 - \epsilon$, $0 < \epsilon < 1$, becomes smaller than $\mathrm{M}(1-\epsilon)^{1/6}/\epsilon^2$, which shows that the series converges absolutely and uniformly.

In order to obtain a continuation of a given streamfunction to the supersonic region, i. e., for imaginary values of λ, it is convenient to replace λ by the variable $s = (-\lambda)^{2/3}$ used before in a different context (see (4. 23)). Then the generating function in the subsonic case may be written as

$$(5.29) \qquad E^{(\kappa)}(\lambda, \theta, t) = H(2\lambda) E^{*(\kappa)}(\lambda, \theta, t)$$
$$= S_0 2^{-1/6} p(s) s^{\kappa-1} \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} C_\nu^{(n,\kappa)} s^{(3/2)n} s^\nu s^\nu / (-t^2 \mathrm{Z})^{n-\frac{1}{2}+(2/3)\kappa},$$
$$| s | < s_0, \ 2 | s |^{3/2} < | Z |, \ \kappa = 1, 2,$$

(see (4. 3) and (5. 5)) where now

$$(5.29\text{a}) \qquad Z = -s^{3/2} + i\theta.$$

By replacing λ by $i\Lambda$, we see that $(-\lambda)^{2/3}$ is changed to $-\Lambda^{2/3}$. Thus we obtain as the generating function in the supersonic case

$$(5.30) \qquad E^{(\kappa)}(\Lambda, \theta, t)$$
$$= 2^{-1/6} S_0 p(s) s^{\kappa-1} \sum \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} i^n C_\nu^{(n,\kappa)} (-s)^{(3/2)n} s^\nu / (-t^2 Z)^{n-\frac{1}{2}+(2/3)\kappa}$$
$$| s | < s_0, \ 2 | s |^{3/2} < | Z |,$$

where

(5.30a) $$Z = i(-s)^{3/2} + i\theta.$$

Let us now restrict ourselves to a neighborhood of the point $\lambda = 0$, $\theta = \theta_0$ ($\theta_0 \neq 0$) lying entirely in the domain $D = \mathrm{E}[\,|s| < s_0,\ |s| < 3^{-1/3}|\theta|^{2/3}]$.

For $(s,\theta)\ \varepsilon\ \mathrm{E}[s^2 + (\theta - \theta_0)^2 < \theta_0^2/25]$, we have

(5.31a) $$Z^\gamma = (i\theta_0)^\gamma[1 - (s^{3/2} - i(\theta - \theta_0))/i\theta_0]^\gamma = \sum_{m,n} a_{mn}(\theta - \theta_0)^m s^{(3/2)n} \qquad \text{for } M < 1,$$

(5.31b) $$Z^\gamma = (i\theta_0)^\gamma[1 + ((-s)^{3/2} + (\theta - \theta_0))/\theta]^\gamma = \sum_{m,n} b_{mn}(\theta - \theta_0)^m s^{(3/2)n} \qquad \text{for } M > 1$$

because then both $|(s^{3/2} - i(\theta - \theta_0))/i\theta_0|$ and $|((-s)^{3/2} + i(\theta - \theta_0))/\theta_0|$ are less than

$$(|s|^{3/2} + |\theta - \theta_0|)/|\theta_0| < (3^{-1/2}|\theta| + |\theta - \theta_0|)/|\theta_0| < (2 \cdot 3^{1/2} + 1)/5 < 1.$$

Here a_{mn} and b_{mn} are suitably chosen constants. Thus both the subsonic generating function and the associate function, and therefore $\mathrm{P}[f(-s^{3/2} + i\theta)]$, may be expanded in integral powers of $(\theta - \theta_0)$ and $s^{1/2}$. Similarly, in the supersonic case, $\mathrm{P}[f(i(-s)^{3/2} + i\theta)]$ may be expanded in integral powers of $(\theta - \theta_0)$ and $(-s)^{1/2}$. Both functions are determined and equal to each other for $s = 0$.

We shall show that if f is regular in D, the coefficients of $s^{\nu/2}$, ν odd, vanish, and therefore $\mathrm{P}[f(-s^{3/2} + i\theta)]$ is an analytic function of s and $(\theta - \theta_0)$ at $(0, \theta_0)$.

THEOREM 5.2. *If f is regular in the domain $D = \mathrm{E}[\,|s| < s_0$, $|s| < 3^{-1/3}|\theta|^{2/3}]$, then the solution $\mathrm{P}[f(-s^{3/2} + i\theta)]$ is a regular function of s and $(\theta - \theta_0)$ in D.*

Proof. We have shown that $\mathrm{P}(f)$ may be expresed as a power series in $s^{1/2}$ of the form

(5.32) $$\mathrm{P}(f) = \sum_{n=0}^{\infty} h_n(\theta - \theta_0)s^{n/2}.$$

Using the variable s, and noting the fact that $N = (1/12)s^{-3/2}(1 + \sum_{n=0}^{\infty} a_{n+1}s^{n+1})$ (see (2.16)) equation (2.8a) becomes

(2.8′) $$4\psi_{ss} + 9s\psi_{\theta\theta} + 2\psi_s \sum_{n=0}^{\infty} a_{n+1}s^n = 0.$$

By substituting (5.32) in (2.8′) we obtain the recursion formulae:

(5.33a) $$h_1 = h_3 = 0$$

(5.33b) h_{n+4}

$$= -[1/(n+4)(n+2)][9h''_{n-2} + 2\sum_{j=0}^{(n-1)/2}((3/2)+j)a_{(n/2)+\frac{1}{2}-j}h_{2j+3}] \quad \text{for } n \text{ odd},$$

(5.33c) h_{n+4}

$$= -[1/(n+4)(n+2)][9h''_{n-2} + 2\sum_{j=0}^{n/2}(1+j)a_{(n/2)+1-j}h_{2j+2}] \quad \text{for } n \text{ even}.$$

Since h_n, where n is odd, depends only on the previous h_m for odd m, we see from (5.33a) that $h_n = 0$ for all odd n.

Thus (5.32) becomes

$$\text{(5.32}'\text{)} \qquad \mathrm{P}[f(-s^{3/2}+i\theta)] = \sum_{n=0}^{\infty} f_n(\theta-\theta_0)s^n.$$

Since s is unchanged by the result of putting $i\Lambda$ for λ, we obtain

$$\text{(5.34)} \qquad \mathrm{P}[f(i(-s)^{3/2}+i\theta)] = \sum_{n=0}^{\infty} f_n(\theta-\theta_0)s^n.$$

The expressions $\mathrm{P}[f(-s^{3/2}+i\theta)]$ and $\mathrm{P}[f(i(-s)^{3/2}+i\theta)]$, *qua* functions of s and $(\theta-\theta_0)$, are analytic continuations of each other across the sonic line.

Thus, assuming that the associate function is regular in a sufficiently large domain, and applying the integral operator of the second kind, we obtain solutions of a compressibility equation defined in four adjacent domains,

$$D_1 = \mathrm{E}[M<1, \theta > 3^{1/2}\,|\lambda(M)|] + \mathrm{E}[M>1, \theta > \Lambda(M)],$$
$$D_2 = \mathrm{E}[M<1, |\theta| < 3^{1/2}\,|\lambda(M)|],$$
$$D_3 = \mathrm{E}[M<1, \theta < -3^{1/2}\,|\lambda(M)|] + \mathrm{E}[M>1, \theta < -3\Lambda(M)],$$
$$D_4 = \mathrm{E}[M>1, -3\Lambda(M) < \theta < \Lambda(M)].$$

(See fig. 1, p. 859.) The solutions defined in D_1 and D_3 were derived in the present paper, while those defined in D_2 and D_4 were derived in [**6**, § 11].

In the simplified case, using the theory of hypergeometric equations, it was possible to combine these representations into one, which yields solutions of (2.2) defined in the whole M-θ-plane. In the exact compressibility equation the problem remains of combining these four representations into one. This problem can be attacked by using the integral operator of the first kind in addition to that of the second kind, and, in analogy to the simplified case, developing a theory of differential equations with singular coefficients, which would furnish us with information corresponding to that used in the simplified case. As will be shown elsewhere, the methods of the Fuchs theory for ordinary differential equations can be generalized to the case of partial differential equations of type (2. 14) with F being an analytic function of two complex variables and possessing certain singularities. In particular, if the singularity surfaces are linear, the following results which will be proved in a subsequent paper, will be valid:

THEOREM 5. 3. *Let the coefficient F of ψ in equation*

$$(5.35)\qquad (\partial^2\psi/\partial Z_1{}^2) + (\partial^2\psi/\partial Z_2{}^2) + F\psi = 0, \quad Z_1 = \lambda + i\Lambda, \quad Z_2 = \theta + i\theta,$$

have the form

$$(5.36)\qquad F = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{m-1,n-1}W^{m-1}Z^{n-1},\ A_{-1,-1} \neq 0,\qquad W = Z_1 + iZ_2,\quad Z = Z_1 - iZ_2$$

where the coefficients $A_{m,n}$ satisfy the inequalities

$$(5.37)\qquad |A_{m-1,n-1}| < A/\rho_1{}^m\rho_2{}^n,\quad A \geq 1,\quad \rho_1 > 0,\quad \rho_2 > 0,$$

being suitably chosen constants.

If $r \neq 0$ is a complex constant which satisfies the inequalities

$$(5.38a)\qquad r \neq -(m/2)[1 \pm (1 + A_{-1,-1}/mn)^{\frac{1}{2}}],\qquad 0 \leq \arg(1 + A_{-1,-1}/mn)^{\frac{1}{2}} < \pi/2,$$

$$(5.38b)\qquad r \neq A_{-1,-1}/4n,$$

$$(5.38c)\qquad r \neq -m$$

for $m = 0, 1, \cdots; n = 0, 1, \cdots; (m, n) \neq (0, 0)$, then the expression

$$(5.39)\qquad \psi = W^r Z^{-A_{-1,-1}/4r}\sum_{n=0}^{\infty} B_{mn}W^mZ^n,$$

will represent a solution of (5. 35). Here B_{00} is an arbitrary complex constant, and the B_{mn} have to be determined from the equations

$$(5.40)\qquad 4[mn + rn - (A_{-1,-1}m/4r)]B_{mn} = -\sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n} A_{m-\mu-1,n-\nu-1}B_{\mu\nu} - \sum_{\nu=0}^{n-1} A_{-1,n-\nu-1}B_{m\nu}.$$

(5. 38) *converges in*

$$E[|W| < \rho_1/AB,\ |Z| < \rho_2/AB] \text{ where } B = \max[1, |B_{00}|, M/4],$$

and

$$M = \max\{(mn + m + n)/|mn + rn - 4^{-1}r^{-1}mA_{-1,-1}|\},\qquad m = 0, 1, 2, \cdots,\ n = 0, 1, 2, \cdots,\ (mn) \neq (0, 0).$$

THEOREM 5.4. *Let the coefficient F of ψ in* (5.35) *have the form*

$$F = Z_1^{-2}A_{-2,0} + Z_1^{-1}A_{-1}(Z_2) + \sum_{n=0}^{\infty} A_n(Z_2)Z_1^n \tag{5.41}$$

where

$$A_{-2,0} \neq (1-n^2)/4,, \qquad n = 0, 1, 2, \cdots \tag{5.42}$$

and the $A_n(Z_2)$ are analytic functions of Z_2 which are dominated by $A/(R-Z_2)^{n+2}$, (i.e., for which $A_n(Z_2) \ll A/(R-Z_2)^{n+2}$ $n = -1, 0, 1, 2, \cdots$ holds) *A and R being suitably chosen positive constants. Then the expression*

$$\psi = Z_1^{r_s} \sum_{n=0}^{\infty} B_n(Z_2)Z_1^n, \qquad s = 1, 2, \tag{5.43}$$

represents a solution of (5.35) *which is defined in*

$$\begin{aligned} \mathrm{E}[\,|Z_1| &< (R - |Z_2|)/(1+e), \\ |Z_2| &< 1 - ((-1)^s(1-4A_{-2,0})^{\frac{1}{2}} + A)/(1-A_{-2,0})^{\frac{1}{2}}] \end{aligned} \tag{5.44}$$

where

$$\begin{aligned} e = \max\,[\,&|(1-(-1)^s(1-4A_{-2,0})^{\frac{1}{2}} + A)/(1-4A_{-2,0})^{\frac{1}{2}}\,|, \\ &|A/(1+(-1)^s(1-4A_{-2,0})^{\frac{1}{2}})|\,] \end{aligned} \tag{5.45}$$

$$r_s = \tfrac{1}{2} + (-1)^s(\tfrac{1}{4} - A_{-2,0})^{\frac{1}{2}}, \quad 0 \leq \arg\,(\tfrac{1}{4} - A_{-2,0})^{\frac{1}{2}} < \pi/2, \qquad s = 1, 2.$$

Here $B_0(Z_2)$ is an arbitrary function for which

$$B_0(Z_2) \ll B(1 - R^{-1}Z_2)^{-2}, \qquad B > 0, \tag{5.46}$$

and $B_n(Z_2)$ are to be determined by the relations

$$\begin{aligned} &[1 + (-1)^s(1-4A_{-2,0})^{\frac{1}{2}}]B_1(Z_2) = -A_{-1}(Z_2)B_0(Z_2) \\ &n[n + (-1)^s(1-4A_{-2,0})^{\frac{1}{2}}]B_n(Z_2) \\ &\qquad = -[B''_{n-2}(Z_2) + \sum_{\nu=1}^{n} A_{\nu-2}(Z_2)B_{n-\nu}(Z_2)], \qquad n = 2, 3, \cdots. \end{aligned} \tag{5.47}$$

We should like to add here a remark regarding the general question of analytic continuation of a solution $\psi(Z, \bar{Z})$ of a linear partial differential equation. If ψ is given in two different domains, say B_1 and B_2 by different representations, say, in B_1 by the integral operator of the first kind in the form

$$\psi = \psi_1 \equiv \boldsymbol{p}[g(Z)] + \bar{\boldsymbol{p}}[h(\bar{Z})] \tag{5.48}$$

[see (3.11)] and in B_2 by another operator

$$\psi = \psi_2 \equiv \int_{t=-1}^{1} E(Z, \bar{Z}, t) f(\tfrac{1}{2}Z(1-t^2))\,(dt/(1-t^2)^{\frac{1}{2}}) \tag{5.49}$$

[see (3.4a)] (not necessarily of the first kind), and the origin lies in $B_1 \cap B_2$, then the problem of the analytic continuation of ψ_2 into the domain B_1 is equivalent to the determination of g and h from a given $f = \sum_{n=0}^{\infty} \alpha_n Z^n$. Setting $Z = 0$, and $\bar{Z} = 0$, respectively, in the relation

$$(5.50) \qquad \boldsymbol{p}[g(Z)] + \bar{\boldsymbol{p}}[h(\bar{Z})] = \int_{t=-1}^{1} E(Z, \bar{Z}, t) f(\tfrac{1}{2}Z(1-t^2))\, dt/(1-t^2)^{\frac{1}{2}}$$

we obtain the identities

$$(5.51) \qquad \sum_{n=0}^{\infty} Z^n \sum_{\nu=0}^{n} \tau_{n\nu}{}^{(1)} \alpha_\nu = g(Z) + \bar{R}(0, Z) h(0),$$

$$\alpha_{00} \sum_{n=0}^{\infty} \bar{Z}^n \tau_{nn}{}^{(2)} = R(0, \bar{Z}) g(0) + h(\bar{Z})$$

where

$$(5.52) \qquad \tau_{n\nu}{}^{(1)} = \int_{t=-1}^{1} E_{n-\nu}{}^{(1)} (1-t^2)^{\nu-\frac{1}{2}} 2^{-\nu} dt,$$

$$\tau_{nn}{}^{(2)} = \int_{t=-1}^{1} E_n{}^{(2)}(t) (1-t^2)^{-\frac{1}{2}} dt,$$

$$E(Z, 0, t) = \sum_{n=0}^{\infty} E_n{}^{(1)}(t) Z^n, \qquad E(0, \bar{Z}, t) = \sum_{n=0}^{\infty} E_n{}^{(2)}(t) \bar{Z}^n.$$

[See **5**, page 310.] It is thus seen that the analytic continuation of ψ_2 into B_1 and all the problems arising from it, such as the determination of the singularities, etc., are reduced to similar problems *in the theory of functions of one complex variable* which are given by their power series expansions.

6. The determination of the associate function in terms of the given streamfunction. We now turn to the problem of determining the associate function f in terms of the given streamfunction. In the case of the integral operator of the first kind, the formulas (3.7), (3.10), (3.15), (3.16) yield the associate $g_1(Z)$. Therefore, if the continuation of the streamfunction (which a priori is given in the real plane) to complex values of the arguments is known, then these formulas immediately yield the associate. In the case of general integral operators (in particular, of that of the second kind), if the streamfunction is given in the form of a power series, then it may be shown (see [**5**], p. 310) that the associate can be expressed in the form of a power series. On the other hand, the streamfunction ψ is, in many instances, given in some different form, say the values of ψ and $\partial\psi/\partial M$ are given on a line $M = \text{const}$. If these quantities are analytic functions of θ, then from these data it will be possible to determine the coefficients of the series development from which we may then determine the associate functions in the way indicated above. However, this procedure gives only the function element of $f(Z)$, so that the function is in general determined only in a sufficiently small neighborhood of the interval of $M = \text{const.}$ along which the values $\psi = \chi_1(\theta)$ and $\partial\psi/\partial M = \chi_2(\theta)$ are given. We shall show in the present section that, in the case of certain generating functions of the second kind, a formula expressing f in terms of the values $\psi = \chi_1(\theta)$ and $\partial\psi/\partial M = \chi_2(\theta)$ on the sonic line, $M = 1$, can be derived. According to the consideration of **5** an integral operator of the second kind can be written in the form [17]

$$\begin{aligned}(6.1)\qquad \psi(\lambda,\theta) &= \operatorname{Im}\Big[\int_{C_2} E(Z,\bar{Z},t) f(\tfrac{1}{2}Z(1-t^2))(1-t^2)^{-\frac{1}{2}}dt\Big]\\ &= (2i)^{-1}\int_{C_2}[Ef-\bar{E}\bar{f}](1-t^2)^{-\frac{1}{2}}dt\end{aligned}$$

where

$$(6.2)\qquad E = A_1E^{(1)} + [\tfrac{1}{2}Z(1-t^2)]^{2/3}A_2E^{(2)}, \qquad A_\kappa \text{ complex const.}$$

Here $E^{(\kappa)}$, $\kappa = 1, 2$, are the generating functions introduced in (5.5) and (4.1); C_2 is a simple curve in the complex t-plane which connects $t = -1$ with $t = 1$ and, except for the endpoints, lies outside $\mathrm{E}[|t| \leq 1]$. Moreover, C_2 has to be chosen in such a manner that $[\frac{1}{2}Z(1-t^2)]$ lies in the regularity domain of f for the values of Z under consideration.

We assume the associate function to be of the form

$$(6.3)\qquad f(\zeta) = \sum_{\nu=0}^{\infty} c_\nu \zeta^{\nu+1/6}, \qquad c_\nu \text{ complex const.}$$

which is suggested by previous considerations. Under the assumption that the (complex) constants A_1, A_2 satisfy the inequality

$$(6.4)\qquad \operatorname{Im}[A_2\bar{A}_1] \neq 0,$$

the desired inverse formula for f in terms of $\chi_1(\theta)$ and $\chi_2(\theta)$ is given in

THEOREM 6.1. *Let $\psi(\lambda,\theta)$ be a (real) solution of the compressibility equation (2.8a) which is defined in a domain, say B, situated in $[3^{\frac{1}{2}}|\lambda| < |\theta|, \lambda < 0]$ and such that its boundary includes an interval $[\theta_0 \leq \theta \leq \theta_1]$ of the transonic line $\lambda = 0$ not containing the origin. Let*

$$(6.5\text{a})\qquad \lim_{\lambda\to 0^-}\psi(\lambda,\theta) = \chi_1(\theta) = \sum_{\nu=0}^{\infty} a_\nu^{(1)}\theta^\nu, \qquad a_\nu^{(\kappa)} \text{ real const.}$$

$$(6.5\text{b})\qquad \lim_{\lambda\to 0^-}\psi_M(\lambda,\theta) = \chi_2(\theta) = 3^{1/3}(1-h^2)^{2/3}\sum_{\nu=0}^{\infty} a_\nu^{(2)}\theta^\nu, \psi_M = \partial\psi/\partial M,$$

(see (2.6)) *then the* P_2 *associate f of the integral operator* (6.1) *with generating function* (6.2) *and A_1, A_2 satisfying* (6.4) *is given by*

[17] In the remainder of this section we shall omit the subscript "2" in E_2, and shall write simply E.

(6.6) $$f(\zeta) = -\{(-2i\zeta)^{1/6}/3^{1/2}\pi S_0{}^2\operatorname{Im}[A_2\bar{A}_1]\}[-\bar{d}_0\int_C t^{-1/3}\chi_1(\sigma)\,dt + \sum_{k=1}^{2}(-1)^k\bar{d}_k\int_C t^{-5/3}\chi_k(\sigma)\,dt]$$

where

(6.7) $$\sigma = -2i\zeta(1-t^2),$$

the constants d_ν, $\nu = 0, 1, 2$, *are*

(6.8) $$d_0 = -(2/3)i^{3/2}S_0A_2,\quad d_1 = -(2^{5/3}/3)i^{1/6}S_0S_1A_1,\quad d_2 = -i^{1/6}S_0A_1,$$

the S_ν *are defined in* (4.4).

REMARK 6.1. Developing the right-hand side of (2.6) into an infinite series and inverting it, we obtain the following series for $(\partial\lambda/\partial M)$

(6.9) $$\partial\lambda/\partial M = [3^{1/3}(1-h^2)^{2/3}](-\lambda)^{1/3} \times [1 + 3^{1/3}(1-h^2)^{-4/3}((3/10) + \tfrac{1}{2}h^2 - (4/5)h^4)(-\lambda)^{2/3} + \cdots]$$

Therefore it is legitimate to consider $\lim_{\lambda\to 0^-}(-\lambda)^{1/3}(\partial\psi/\partial\lambda)$, instead of $\lim_{\lambda\to 0^-}(\partial\psi/\partial M)$.

Proof. A formal computation yields (see (5.3), (5.5), (5.14), (6.8))

(6.10) $$\lim_{\lambda\to 0^-} E(Z,\bar{Z},t) = -d_2t^{-1/3}\theta^{-1/6}$$

and

(6.11) $$\lim_{\lambda\to 0^-}(-\lambda)^{1/3}E_\lambda(Z,\bar{Z},t) = [d_1t^{-1/3} + d_0t^{-5/3}(1-t^2)^{2/3}]\theta^{-1/6}.$$

We now determine the limit values as $\lambda \to 0^-$ for the right-hand side of (6.1) and the derivative of this expression multiplied by $(-\lambda)^{1/3}$. Since by assumption these limit values are equal to expressions $\sum_{\nu=0}^{\infty} a_\nu{}^{(1)}\theta^\nu$ and $\sum_{\nu=0}^{\infty} 3^{1/3}(1-h^2)^{2/3}a_\nu{}^{(2)}\theta^\nu$ respectively, we obtain (by equating the coefficients of the θ^ν) the following system of equations

(6.12a) $$a_\nu{}^{(1)} = \operatorname{Im}[(-2i)^{-(\nu+1/6)}d_2I_\nu{}^{(1)}c_\nu],$$

(6.12b) $$a_\nu{}^{(2)} = \operatorname{Im}[(-2i)^{-(\nu+1/6)}(d_1I_\nu{}^{(1)} + d_0I_\nu{}^{(2)})c_\nu],\quad \nu = 1, 2, \cdots$$

where

(6.13a) $$I_\nu{}^{(1)} = \int_{C_2} t^{-1/3}(1-t^2)^{\nu-1/3}dt = \pm\tfrac{1}{2}e^{-(4/3)n\pi i}(e^{-(4/3)\pi i}-1)\frac{\Gamma(1/3)\Gamma(\nu+2/3)}{\Gamma(\nu+1)},\quad n = 0, \pm 1$$

(6.13b) $$I_\nu{}^{(2)} = \int_{C_2} t^{-5/3}(1-t^2)^{\nu+1/3}dt = \pm\tfrac{1}{2}e^{-(2/3)n\pi i}(e^{-(2/3)\pi i}-1)\frac{\Gamma(-1/3)\Gamma(\nu+4/3)}{\Gamma(\nu+1)}.$$

We note that the last term on the right hand side of (6.13b) is obtained as follows:

$$I_\nu^{(2)} = -(3/2)\int_{C_2}(1-t^2)^{\nu+1/3}d(t^{-2/3})$$
$$= -(3/2)\Big[t^{-2/3}(1-t^2)^{\nu+1/3}\Big]_{t=-1}^{t=-1} - 3(\nu+1/3)\int_{C_2} t^{1/3}(1-t^2)^{\nu-2/3}dt.$$

The first term in the last expression vanishes and in the second term, as in $I_\nu^{(1)}$, we replace the integration curve C_2 by the segments $(-1, 0^-)$ and $(0^+, 1)$ and a half-circle around the origin.

REMARK. Since the curve C_2 need only satisfy the inequality $|t| > |2\lambda/(\lambda+i\theta)|^{\frac{1}{2}}$ (see 4.11), it is valid to replace it by a curve consisting of the segments $(-1, 0^-)$ and $(0^+, 1)$ and a half-circle around the origin provided the radius of the latter is greater than $|2\lambda/(\lambda+i\theta)|^{\frac{1}{2}}$ which approaches zero as $\lambda \to 0$. The right hand sides of (6.13a) and (6.13b) are obtained by making the substitution $t^2 = \tau$ and considering the integrals in the three-sheeted τ-plane. In the following, we choose the sheet for which $n = 1$.

We introduce the integrals

$$(6.14a)\quad J_\nu^{(1)} = \int_{C_2} t^{-5/3}(1-t^2)^\nu dt = \pm\tfrac{1}{2}e^{-(2/3)n\pi i}(e^{-(2/3)\pi i}-1)\,\frac{\Gamma(-1/3)\Gamma(\nu+1)}{\Gamma(\nu+2/3)}$$

$$(6.14b)\quad J_\nu^{(2)} = \int_{C_2} t^{-1/3}(1-t^2)^\nu dt = \pm\tfrac{1}{2}e^{-(4/3)n\pi i}(e^{-(4/3)\pi i}-1)\,\frac{\Gamma(1/3)\Gamma(\nu+1)}{\Gamma(\nu+4/3)}$$

and note that

$$(6.15)\qquad I_\nu^{(1)}J_\nu^{(1)} = I_\nu^{(2)}J_\nu^{(2)} = -3^{3/2}2^{-1}\pi.$$

The determinant of the system (6.12) does not vanish for any ν since it equals

$$(6.16)\qquad -(i/2)\,\mathrm{Im}\,[d_0\bar{d}_2 I^{(2)}\overline{I^{(1)}}]$$

which would vanish only if $\mathrm{Im}[d_0\bar{d}_2] = 0$. Since, by (6.8), $\mathrm{Im}[d_0\bar{d}_2] = (2/3)S_0^{\,2}\,\mathrm{Im}[A_2\bar{A}_1]$, the determinant does not vanish, because of condition (6.4).

Solving (6.2), we obtain

$$(6.17) \qquad c_\nu = \frac{-(-2i)^{\nu+1/6}}{(3^{\frac{1}{2}}\pi S_0{}^2)\,\mathrm{Im}[A_2\bar{A}_1]}\left[-a_\nu{}^{(1)}\bar{d}_0 J_\nu{}^{(1)} + \sum_{k=1}^{2}(-1)^k a_\nu{}^{(k)}\bar{d}_k J_\nu{}^{(2)}\right]$$

from which (6.6) follows.

HARVARD UNIVERSITY.

BIBLIOGRAPHY.

1. Stefan Bergman, "Zur Theorie der Funktionen, die eine lineare partielle Differentialgleichung befriedigen, I," *Recueil Mathématique*, new series, vol. 2 (1937), pp. 1169-1198.
2. ———, "The approximation of functions satisfying a linear partial differential equation," *Duke Mathematical Journal*, vol. 6 (1940), pp. 537-561.
3. ———, "Linear operators in the theory of partial differential equations," *Transactions of the American Mathematical Society*, vol. 53 (1943), pp. 130-155.
4. ———, "Solutions of linear partial differential equations of the fourth order," *Duke Mathematical Journal*, vol. 11 (1944), pp. 617-649.
5. ———, "Certain classes of analytic functions of two real variables and their properties," *Transactions of the American Mathematical Society*, vol. 57 (1945), pp. 299-331.
6. ———, "On two-dimensional flows of compressible fluids," *National Advisory Committee for Aeronautics, Technical Note No. 972*, 1945.
7. ———, "Methods for the determination and computation of flow patterns of a compressible fluid," *National Advisory Committee for Aeronautics, Technical Note No. 1018*, 1946.
8. ———, "On supersonic and partially supersonic flows, *National Advisory Committee for Aeronautics, Technical Note No. 1096*, 1946.

8a. ———, "Two-dimensional subsonic flows of a compressible fluid and their singularities," *Transactions of the American Mathematical Society*, vol. 61 (1947), pp. 452-498.

9. ———, "On tables for the determination of transonic flow patterns," *Hans Reissner Anniversary Volume*, 1949.
10. ———, "On functions satisfying certain partial differential equations of elliptic type and their representation," *Duke Mathematical Journal*, vol. 14 (1947), pp. 349-366.
11. S. A. Chaplygin, "On gas jets," *Scientific Memoirs, Moscow University, Mathematics-Physics Section*, vol. 21 (1902), pp. 1-121. (Eng. trans., pub. by NACA TM No. 1063, 1944 and also pub. by Brown Univ., 1944.)
12. A. R. Forsyth, *A treatise on differential equations*, London (1943).
13. F. Frankl, "On the problems of Chaplygin for mixed sub- and supersonic flows," *Bulletin de L'Académie des Sciences de L'URSS*, vol. 9 (1945), pp. 121-143.
14. E. W. Hobson, *The theory of spherical and ellipsoidal harmonics*, Cambridge (1931).
15. Thomas M. MacRobert, *Functions of a complex variable*, London (1933).
16. F. Tricomi, "Sulle equazioni lineari alle derivato parziali di 2° ordine di tipo misto," *Memorie della Reale Accademia Nazionale dei Lincei*, vol. 14 (1923), pp. 133-247.

APPENDIX I.

The derivation of formulae (2.4). In this section we shall at first determine the series development of $\mathcal{E}(H) = \frac{1 - M^2}{\rho^2}$ introduced in (2.2) in the neighborhood of the point H = 0. According to (42) of [6] H is defined by

$$\text{(A.1)} \qquad \frac{dH}{dq} = \frac{\rho}{q}, \quad H(q^{(1)}) = 0$$

$q^{(1)} = [2/(k+1)]^{\frac{1}{2}}$ being the speed which corresponds to the Mach number 1.

According to p. 7 of [6] it holds for

$$\text{(A.2)} \quad T^2 \equiv 1 - M^2 = \frac{1 - 1/2(k+1)q^2}{1 - 1/2(k-1)q^2},$$

$$\rho = (1 - 1/2(k-1)q^2)^{1/(k-1)}$$

Eliminating v between both equations (A.2) we get

$$\text{(A.3)} \quad \rho = \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} \left(1 - \left(\frac{k-1}{k+1}\right)T^2\right)^{-\frac{1}{k-1}} =$$

$$= \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} \left(1 + \frac{T^2}{k+1} + \frac{k\,T^4}{2(k+1)^2} + \dots\right)$$

Now, from (A.1) we have $\frac{dH}{dT} = \frac{dH}{dq} \cdot \frac{dq}{dT} = \frac{\rho}{q}\,\frac{dq}{dT} =$

$= \frac{\rho}{2}\;\frac{\lg(\frac{1}{2}q^2)}{dT}$. From the first formula of (A.2) we obtain $\frac{1}{2}\, q^2 = \dfrac{1 - T^2}{(k+1) - (k-1)\, T^2}$. Therefore

$$(A.4)\quad \frac{dH}{dT} = \frac{1}{2}\,\rho \left[- \frac{2\,T}{1 - T^2} + \frac{2(k-1)}{(k+1)-(k-1)T^2} \right] = -$$

$$- \frac{2\,\rho\, T}{(1-T^2)(k+1)-(k-1)\,T^2} = - \frac{2}{k+1} \left[\frac{T}{(1-T^2)(1-h^2\,T^2)} \right] \times$$

$$\left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} \left[1 + \frac{T^2}{k+1} + \frac{k\,T^4}{2(k+1)^2} + \dots \right] =$$

$$-\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \left[T + \left(\frac{2k+1}{k+1}\right) T^3 + \frac{3k^2 + 2.5k + 1}{(k+1)^2}\, T^5 + \dots \right]$$

Therefore (since H=0 for T=0)

$$(A.5)\quad H = -\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \left[\frac{T^2}{2} + \frac{(2k+1)T^4}{4(k+1)} + \frac{3k^2 + 2.5k + 1}{6(k+1)^2}\, T^6 + \dots \right]$$

Now from (A.3) follows:

$$(A.6)\quad \frac{\rho^2}{T^2} = \left(\frac{2}{k+1}\right)^{\frac{2}{k-1}} \left[\frac{1}{T^2} + \frac{2}{k+1} + \frac{1.5k^2 + 2k + 1}{(k+1)}\, T^2 + \dots \right.$$

For simplicity, let

$$(A.7)\quad A = - \left[2H \left(\frac{2}{k+1}\right) \right]^{\frac{k}{k-1}},$$

$$B = \frac{\rho^2}{T^2} \left(\frac{2}{k+1}\right)^{-\frac{2}{k-1}} = \frac{1}{\ell(H)} \left(\frac{2}{k+1}\right)^{-\frac{2}{k-1}}$$

Then

$$(A.8)\quad A = T^2 + \left(\frac{2k+1}{2k+2}\right) T^4 + \frac{3k^2 + 2.5k + 1}{3(k+1)^2}\, T^6 + \dots$$

and

$$(A.9) \quad B = \frac{1}{T^2} + \frac{2}{k+1} + \frac{1.5k^2+2k+1}{(k+1)^4} T^2 + \ldots$$

Now (A.8) can be inverted to yield T^2 as a function of A and (A.9) can be "turned upside down" to give $\frac{1}{B}$ as a Taylor series in T^2. There are thus obtained the following series:

$$(A.10) \qquad T^2 = A - \frac{2k-1}{2k+2} A^2 + \frac{6k+1}{6k+6} A^3 + \ldots,$$

$$(A.11) \quad \frac{1}{B} = T^2 + \frac{2}{k+1} T^4 + \frac{2.5k^2+6k+3}{(k+1)^4} T^6 + \ldots .$$

Replacing T^2 in (A.11) by (A.10) we obtain

$$(A.12) \quad \frac{\mathcal{l}(H)}{\left[2/(k+1)\right]^{2/(k-1)}} - \frac{1}{B} = A - \left(\frac{3k+5}{2k+2}\right)A^2 + \frac{k^4+(43/6)k^3+16k^2+15.5k+(31.6)}{(k+1)^4} A^3 + \ldots$$

from which (2.4) follows.

The derivation of formulae (2.16) and (2.17).

We denote in the following by u the quantity

$$(A.13) \qquad u = q^{(1)} - q \equiv \left(\frac{2}{k+1}\right)^{\frac{1}{2}} - .$$

It follows from (A.2) that in the neighborhood of $u = 0$, $T^2 \equiv 1 - M^2$ has the series development

$$(A.14)\quad T^2 = 2^{-\frac{1}{2}}(k+1)^{\frac{3}{2}}\,u\left\{1-\left[2^{-\frac{3}{2}}(k+1)^{\frac{1}{2}}+2^{-\frac{1}{2}}(k+1)^{-\frac{1}{2}}(k^2-1)\right]u+\ldots\right\}$$

Therefore T possesses the series expansion

$$(A.15)\quad T = 2^{-\frac{1}{4}}(k+1)^{\frac{3}{4}}u^{\frac{1}{2}}\left\{1+\left[2^{-\frac{5}{2}}(k+1)^{\frac{1}{2}}+2^{-\frac{3}{2}}(k+1)^{-\frac{1}{2}}(k^2-1)\right]u+\ldots\right\}$$

Furthermore:

$$(A.16)\quad \frac{1}{q} = \left[2^{\frac{1}{2}}(k+1)^{-\frac{1}{2}}-u\right]^{-1} = [\text{illegible}]^{-\frac{1}{2}}(k+1)^{\frac{1}{2}}\left\{[\text{illegible}]^{-\frac{1}{2}}(k+1)^{\frac{1}{2}}u+\ldots\right\}$$

[illegible] 45) of [6] [illegible]$/d_1 = T/q_1$ and therefore

$$(A.17)\quad \frac{d[\text{illegible}]}{d[\text{illegible}]} = [\text{illegible}]^{-\frac{3}{4}}(k+1)^{[\text{illegible}]}[\text{illegible}]^{\frac{1}{2}}\left\{[\text{illegible}]\left[2^{-\frac{5}{2}}(k+1)^{[\text{illegible}]}+2^{-\frac{3}{2}}(k+1)^{-\frac{1}{2}}\right]u+\ldots\right\}$$

$$\left\{1+2^{-\frac{1}{2}}(k+1)^{\frac{1}{2}}u+\ldots\right\} =$$

$$= -2^{-\frac{3}{4}}(k+1)^{\frac{5}{4}}u^{\frac{1}{2}}\left\{1-\left[2^{-\frac{5}{2}}(k+[\text{illegible}])^{-\frac{1}{2}}(k^2-[\text{illegible}])-2^{-\frac{5}{2}}3(k+1)^{\frac{1}{2}}\right]u+\ldots\right\} =$$

$$-2^{-\frac{6}{4}}(k+1)^{\frac{5}{4}}u^{\frac{1}{2}}-\left[2^{-\frac{9}{4}}(k-1)(k+1)^{\frac{7}{4}}-2^{-\frac{13}{4}}3(k+1)^{\frac{7}{4}}\right]$$

Thus

$$(A.18)\quad \lambda = -3[\text{illegible}]^{\frac{4}{[\text{illegible}]}}(k[\text{illegible}])^{[\text{illegible}]}[\text{illegible}]\left[2^{-\frac{9}{4}}([\text{illegible}]-1)(k+1)^{\frac{7}{4}}+\right.$$

$$+2^{\frac{13}{4}}\, 3(k+1)^{\frac{7}{4}}\Big]u + \ldots$$

Raising each side to 2/3 power, we get

$$(A.19) \quad 2^{\frac{1}{6}} 3^{\frac{2}{3}} (k+1)^{\frac{5}{6}} (-\lambda)^{\frac{2}{3}} =$$

$$u\left\{1-2^{\frac{1}{2}}\, 5^{-1}(k+1)^{\frac{1}{2}}(k-5/2)u + \ldots\right\}$$

Inverting (A.19) we obtain

$$(A.20) \quad u \equiv q^{(1)} - q = 2^{\frac{1}{6}} 3^{\frac{2}{3}} (k-1)^{\frac{5}{6}} (-\lambda)^{\frac{2}{3}} +$$

$$+2^{\frac{1}{2}} 5^{-1} (k+1)^{\frac{1}{2}} (k-5/2) \left[2^{\frac{1}{6}} 3^{\frac{2}{3}} (k+1)^{\frac{5}{6}} (-\lambda)^{\frac{2}{3}}\right]^2 + \ldots$$

According to (47) of [6],

$$N = \frac{1}{8}(k+1)q^4\left[1-\frac{1}{2}(k+1)q^2\right]^{-3/2}\left[1-\frac{1}{2}(k-1)q^2\right]^{\frac{1}{2}}$$

Expanding this expression in a power series of $u = 2^{-1/2}(k+1)^{-1/2} - q$, we get

$$(A.21) \quad N = -2^{\frac{9}{4}}(k+1)^{\frac{5}{4}} u^{-\frac{3}{2}}\left\{1-\left[\frac{1}{2}(k-1)(k+1)^{\frac{1}{2}} +\right.\right.$$

$$\left.\left. 2^{\frac{1}{2}}(13/8)(k+1)^{\frac{1}{2}}\right]u + \ldots\right\}$$

Raising both sides to the $(-2/3)$ power, we obtain

$$(A.22) \quad N^{\frac{2}{3}} = 2^{\frac{3}{2}}(k+1)^{\frac{5}{6}}\left\{u+\left[\frac{1}{3}(k-1)(k+1)^{\frac{1}{2}} +\right.\right.$$

$$\left.\left. \frac{13}{12}\, 2^{\frac{1}{2}}(k+1)^{\frac{1}{2}}\right]u^2 + \ldots\right\}$$

Replacing u by the expansion (A.20), we have

$$2^{-\frac{3}{2}}(k+1)^{-\frac{5}{6}}N^{-\frac{2}{3}} = \left[3^{\frac{2}{3}}2^{-\frac{1}{6}}(k+1)^{-\frac{5}{6}}(-\lambda)^{\frac{2}{3}}\right] + 5^{-1}2^{-\frac{1}{2}}(k-2.5)$$

$$\times\left[3^{\frac{2}{3}}2^{-\frac{1}{6}}(k+1)^{-\frac{5}{6}}(-\lambda)^{\frac{2}{3}}\right]^{2} + \left[\frac{1}{3}(k-1)(k+1)^{\frac{1}{2}} + \right.$$

$$\left.\frac{13}{12}\,2^{\frac{1}{2}}(k+1)^{\frac{1}{2}}\right]\left[3^{\frac{2}{3}}2^{-\frac{1}{6}}(k+1)^{-\frac{5}{6}}(-\lambda)^{\frac{2}{3}}\right]^{2} + \ldots \Big\}$$

or

$$\text{(A.23)}\quad N^{-\frac{2}{3}} = 2^{\frac{3}{2}}(k+1)^{\frac{5}{6}} \cdot 3^{\frac{2}{3}}2^{-\frac{1}{6}}(k+1)^{-\frac{5}{6}}(-\lambda)^{\frac{2}{3}}\Big\{1+(k+1)^{\frac{1}{2}}$$

$$\left[\frac{(2+\frac{3}{5}2^{\frac{1}{2}})+5\cdot 2^{\frac{1}{2}}-2}{6}\right]\left[2^{-\frac{1}{6}}3^{\frac{2}{3}}(k+1)^{-\frac{5}{6}}(-\lambda)^{\frac{2}{3}}\right] + \ldots\Big\}$$

Raising both sides to the (-3/2) power, we obtain

$$\text{(A.24)}\quad N = \frac{1}{12\lambda}\Big\{1-(k+1)^{\frac{1}{2}}\left[\frac{(2+\frac{3}{5}2^{\frac{1}{2}})k+5\cdot 2^{\frac{1}{2}}-2}{4}\right]$$

$$\left[2^{-\frac{1}{6}}3^{\frac{2}{3}}(k+1)^{-\frac{5}{6}}(-\lambda)^{\frac{2}{3}}\right] + \ldots$$

According to (A.24) and (70) of [6], we obtain

$$\text{(A.25)}\quad F \equiv -(N^2+\frac{1}{2}N) = \frac{5}{144}\lambda^{-2} + O(-\lambda)^{-\frac{4}{3}} +$$

$$A_{-2}(-\lambda)^{-\frac{2}{3}} + A_0 + A_2(-\lambda)^{\frac{2}{3}} + \ldots$$

Remark. In particular, for k = 1.401 we have

$$(A.25a) \qquad F = \sum_{n=-3}^{\infty} A_{2n} \zeta^{2n/6},$$

where $A_{-6} = .45$, $A_{-4} = 0$, $A_{-2} = .41078$, $A_0 = .32792$, $A_2 = -.14876$, $A_4 = .05392$, $A_6 = -.01759$, $A_8 = .00523$, $A_{10} = -.00149$, etc.

3. The derivation of formula (4.3), (4.4)

Expanding the right hand side of (2.6) in a power series, we obtain

$$(A.26) \quad -\lambda = \frac{1}{3}(1-h^2)T^3 + \frac{1}{5}(1-h^4)T^5 + \frac{1}{7}(1-h^6)T^7 + \dots .$$

Let

$$(A.27) \qquad \left[-3\lambda/(1-h^2)\right] = \mu_1$$

Then:

$$(A.28) \quad \mu_1^{\frac{1}{3}} = T + \frac{1}{5}(1+h^2)T^3 + \frac{1}{175}(18+11h^2+18h^4)T^5 +$$

Inverting this series we obtain.

$$(A.29) \quad T = \mu_1^{\frac{1}{3}} - \frac{1}{5}(1+h^2)(\mu_1^{\frac{1}{3}})^3 + \frac{1}{175}(3+31h^2+3h^4)(\mu_1^{\frac{1}{3}})^5 + $$

$$(A.30) \quad T^{-1} = \mu_1^{-\frac{1}{3}}\left\{1 + \frac{1}{5}(1+h^2)(\mu_1^{\frac{1}{3}})^2 + \right.$$

$$\left. + \frac{1}{175}(4-17h^2+4h^4)(\mu_1^{\frac{1}{3}})^4 + \dots\right\}$$

or finally,

$$(A.31) \quad T^{-\frac{1}{2}} = \mu_1^{-\frac{1}{6}}\left\{1 + \frac{1}{10}(1+h^2)(\mu_1^{\frac{1}{3}})^2 + \right.$$

$$\left. \frac{1}{1400}(9-82h^2+9h^4)(\mu_1^{\frac{1}{3}})^4 + \dots\right\} .$$

Now

$$\text{(A.32)} \quad \left[1+\tfrac{1}{2}(k-1)M^2\right]^{-\frac{1}{2(k-1)}}$$

$$= \left[2/(k+1)\right]^{-\frac{1}{2(k-1)}} \left[1-h^2T^2\right]^{-\frac{1}{2(k-1)}}$$

$$= \left[2/(k+1)\right]^{\frac{1}{2(k-1)}} \Big\{1+\tfrac{1}{2}(k-1)^{-1}h^2T^2+\tfrac{1}{8}(k-1)^{-2}(2k-1)h^4T^4$$

$$+\tfrac{1}{48}(k-1)^{-3}(2k-1)(4k-3)h^6T^6+\dots\Big\}$$

Replacing T in (A.32) by its power series expansion (A.29) we obtain

$$\text{(A.33)} \quad \left[1+\tfrac{1}{2}(k-1)M^2\right]^{-\frac{1}{2(k-1)}} = \left[2/(k+1)\right]^{\frac{1}{2(k-1)}}$$

$$\Big\{1+\tfrac{1}{2}(k+1)^{-1}\mu_1^{\frac{2}{3}}-\tfrac{1}{40}(k+1)^{-2}(6k+5)\,\mu_1^{\frac{4}{3}}$$

$$+\ \tfrac{1}{8400}(k+1)^{-3}(584k^2+122k-579)\,\mu_1^{\frac{6}{3}}+\dots\Big\}$$

If we multiply series (A.31) and (A.33) together, we get the following expansion for $H(2\lambda)$ as a function of $\mu_1 = -3\lambda/(1-h^2) = -\frac{3}{2}(k+1)\lambda$:

$$\text{(A.34)} \quad H(2\lambda) = 2(k+1)^{\frac{1}{2(k-1)}}\mu_1^{\frac{1}{6}}\Big\{1+$$

$$\tfrac{1}{10}(k+1)^{-1}(2k+5)\mu_1^{\frac{2}{3}}-\tfrac{1}{1400}(k+1)^{-2}$$

$$(64k^2+70k+75)\mu_1^{\frac{4}{3}}+\dots\Big\}$$

Derivation of formula (6.9). Differentiating (A.26) with respect to T, we have

(A.35) $-(d\lambda/dT)=(1-h^2)T^2+(1-h^4)T^4+(1-h^6)T^6 +$

Since $T^2 = 1-M^2$ and therefore

(A.36) $-(dT/dM)=T^{-1}M=T^{-1}\left\{1-\frac{1}{2}T^2-\frac{1}{8}T^4 + \dots\right\}$

we obtain by multiplying (A.35) by (A.36)

(A.37) $(d\lambda/dM)=(1-h^2)T+(\frac{1}{2}+\frac{1}{2}h^2 - h^4)T^3$

$$+ (\frac{3}{8}+\frac{1}{8}h^2+\frac{1}{2}h^6-h^6)T^5 + \dots$$

Replacing in (A.37) T by its series development (A.29) we find

(A.38) $(d\lambda/dM)=(1-h^2)\mu_1^{\frac{1}{3}} + (\frac{3}{10}+\frac{1}{2}h^2-\frac{4}{5}h^4)(\mu_1^{\frac{1}{3}})^2+\dots$

5. Formula for $\frac{\partial E^+}{\partial\lambda}$ and $\frac{\partial E^+}{\partial\theta}$. A straightforward derivation of (4.15) with respect to λ and θ, respectively, yields

(A.39) $$\frac{\partial E^+}{\partial\lambda} = \frac{p(\lambda)}{(-2\lambda)^{\frac{1}{3}}}\ \frac{A_2}{(-t^2(\lambda+i\theta))^{\frac{1}{6}}}$$

$$\left[(p_3(\lambda)-\frac{1}{6}\frac{(-2\lambda)^{\frac{1}{3}}}{\lambda+i\theta}F\left(\frac{1}{6},\frac{2}{3},\frac{1}{3},\frac{-2\lambda}{-t^2(\lambda+i\theta)}\right)+\right.$$

$$+\frac{2i(-2\lambda)^{\frac{1}{3}}\theta}{t^2(\lambda+i\theta)^2}\ F'\left(\frac{1}{6},\frac{2}{3},\frac{1}{3},\frac{\quad}{-t^2(\lambda+i\theta)}\right)$$

$$\frac{B_2}{(t^2(\lambda+i\theta))^{\frac{5}{6}}}\left[\left(p_4(\lambda)-\frac{5}{6}\frac{(-2\lambda)}{(-t^2(\lambda+i\theta)}\right)\right.$$

$$F\left(\frac{5}{6},\frac{4}{3},\frac{5}{3},\frac{-2\lambda}{-t^2(\lambda+i\theta)}\right)+\frac{2i(-2\lambda)\theta}{t^2(\lambda+i\theta)^2}$$

$$\left.F'\left(\frac{5}{6},\frac{4}{3},\frac{5}{3},\frac{-2\lambda}{-t^2(\lambda+i\theta)}\right)\right]$$

where

$$p_3(\lambda)=-\left[\frac{4}{3}S_1+\frac{8}{3}S_2(-2\lambda)^{\frac{2}{3}}+\ldots\right]\Big/\left[1+S_1(-2\lambda)^{\frac{2}{3}}+\ldots\right]$$

$$p_4(\lambda)=-\left[\frac{4}{3}+\frac{8}{3}S_1(-2\lambda)^{\frac{2}{3}}+\ldots\right]\Big/\left[1+S_1(-2\lambda)^{\frac{2}{3}}+\ldots\right]$$

$$(A.40)\quad \frac{\partial E^+}{\partial\theta}=\frac{p(\lambda)}{(-t^2)^{\frac{1}{6}}(\lambda+i\theta)^{\frac{7}{6}}}\left\{A_2\left[-\frac{1}{6}iF\left(\frac{1}{6},\frac{2}{3},\frac{1}{3},\frac{-2\lambda}{-t^2(\lambda+i\theta)}\right)\right.\right.$$

$$\left.-\frac{2\lambda i}{t^2(\lambda+i\theta)}F'\left(\frac{1}{6},\frac{2}{3},\frac{1}{3},\frac{-2\lambda}{-t^2(\lambda+i\theta)}\right)\right]-$$

$$-B_2\left[-\frac{5}{6}\frac{(-2\lambda)^{\frac{2}{3}}}{(-t^2)^{\frac{5}{6}}(\lambda+i\theta)^{\frac{2}{3}}}F\left(\frac{5}{6},\frac{4}{3},\frac{5}{3},\frac{-2\lambda}{-t^2(\lambda+i\theta)}\right)-\right.$$

$$\left.\left.-\frac{(-2\lambda)^{\frac{5}{3}}i}{(-t^2)^{\frac{5}{6}}(\lambda+i\theta)^{\frac{5}{6}}}F'\left(\frac{5}{6},\frac{4}{3},\frac{5}{3},\frac{-2\lambda}{-t^2(\lambda+i\theta)}\right)\right]\right\}$$

ERRATA

pg. 860, l. 4 from bot.: "i" should be deleted.

pg. 862, formula (3.2) should read

"$\partial[(1-t^2)E(Z,\bar{Z},t)/Zt]/\partial\bar{Z}$"

pg. 872, formula (4.9'): the exponent "$-2/3$" should be changed to "$2/3$".

pg. 872, formula (4.11): "(" should be added after "dt/".

pg. 882, formula (5.29): the symbol before S_o is "="; an s should be deleted.

pg. 892, line 5 from bot. should read

"Eliminating q between both . . . "

pg. 892, line 2 from bot. should read

"Now, from (A.1) we have $\frac{dH}{dt} = \frac{dH}{dq} \cdot \frac{dq}{dT} = \frac{\mathcal{L}}{q}\frac{dq}{dT} =$

PREVIOUS PUBLICATIONS

1. Classes of solutions of linear partial differential equations in three variables. Stefan Bergman. Duke Mathematical Journal, Volume 13, 1946, pp. 419–458.
2. Models in the theory of several complex variables. Stefan Bergman. American Mathematical Monthly. Volume 53, 1946, pp. 495–501.
3. Punch-card machine methods applied to the solution of the torsion problem. Stefan Bergman. Quarterly of Applied Mathematics. Volume 5, 1947, pp. 69–81.
4. Determination of the coefficients of the integral operator by use of punch-card machines. S. Bergman and L. Greenstone. Journal of Mathematics and Physics, Volume 26, 1947, pp. 1–9. [Report 1-NOrd 8555-Task F.]
5. Two-dimensional subsonic flows of a compressible fluid and their singularities. Stefan Bergman. Transactions of the American Mathematical Society, Volume 61, 1947, pp. 452–498. [Report 2-NOrd 8555-Task F.]
6. Determination of the coefficients of the integral operator by interpolatory means. Rufus Isaacs. Journal of Mathematics and Physics, Volume 26, 1947, pp. 165–181. [Report 3-NOrd 8555-Task F.]
7. Functions satisfying certain partial differential equations of elliptic type and their representation. Stefan Bergman. Duke Mathematical Journal, Volume 14, 1947, pp. 349–366. [Report 4-NOrd 8555-Task F.]
8. A representation of Green's and Neumann's functions in the theory of partial differential equations of the second order. Stefan Bergman and M. Schiffer. Duke Mathematical Journal, Volume 14, 1947, pp. 609–638. [Report 5-NOrd 8555-Task F.]
9. On Green's and Neumann's functions in the theory of partial differential equations. S. Bergman and M. Schiffer. Bulletin of the American Mathematical Society, Volume 53, 1947, pp. 1141–1151. [Report 6-NOrd 8555-Task F.]
10. Determination of a compressible fluid flow past an oval-shaped obstacle. S. Bergman and B. Epstein. Journal of Mathematics and Physics, Volume 26, 1947, pp. 195–218. [Report 11-NOrd 8555-Task F.]
11. Some inequalities relating to conformal mapping upon canonical slit-domains. Bernard Epstein. Bulletin of the American Mathematical Society, Volume 53, 1947, pp. 813–819.
12. An application of orthonormal functions in the theory of conformal mapping. M. Schiffer. American Journal of Mathematics. Volume 70, 1948, pp. 147–156.
13. On Bergman's integration method in two-dimensional compressible fluid flow. R. von Mises and M. Schiffer. Advances of Mechanics, Volume 1, Academic Press, Inc., 1948. [Report 7-NOrd 8555-Task F.]
14. Kernel functions in the theory of partial differential equations of elliptic type. S. Bergman and M. Schiffer. Duke Mathematical Journal, Volume 15, 1948, pp. 535–566. [Report 1-N5ori 76-16.]
15. Identities in the theory of conformal mapping. P. R. Garabedian and M. Schiffer. Transactions of the American Mathematical Society, Volume 64, 1949. [Report 6-N5ori 76-16.]
16. An initial value problem for equations of mixed type. S. Bergman. Bulletin of the American Mathematical Society, Volume 58, 1948. [Report-9-NOrd 8555-Task F.]
17. Sur les fonctions orthogonales de plusieurs variables complexes avec les applications à la théorie des fonctions analytiques. Stefan Bergman. Mémorial des Sciences Mathématiques, Fascicule 106, 1947.
18. Functions of extended class in the theory of functions of several complex variables. Stefan Bergman. Transactions of the American Mathematical Society, Volume 63, 1948, pp. 523–547.

(*Continued on next page*)

(Continued from previous page)

19. Two-dimensional transonic flow patterns. Stefan Bergman. American Journal of Mathematics, Volume 70, 1948. [Report 10-NOrd 8555-Task F.]
20. The kernel function in canonical maps. Z. Nehari. Duke Mathematical Journal, Volume 16, 1949. [Report 3-N5ori 76-16.]
21. Reproducing and pseudo-reproducing kernels and their application to the partial differential equations of physics. N. Aronszajn. Studies in Partial Differential Equations. Harvard Graduate School of Engineering, Cambridge, Massachusetts, 1948. [Report 5-N5ori 76-16.]
22. On tables for the determination of transonic flow patterns. Stefan Bergman. Hans Reissner Anniversary Volume, 1949. [Report 12 NOrd 8555-Task F.]
23. The regularity domains of solutions of linear partial equations in terms of the series development of the solution. Benham Ingersoll. Duke Mathematical Journal, Volume 15, 1948. [Report 2 N5ori 76-16.]
24. Schwarz' lemma and the Szegö kernel function. Paul R. Garabedian. Transactions American Mathematical Society, Volume 64, 1949. [Report 7 N5ori 76-16.]
25. Operator methods in the theory of compressible fluids. Stefan Bergman. First Symposium in Applied Mathematics. American Mathematical Society.
26. Sur la fonction-noyau d'un domaine et ses applications dans la théorie des transformations pseudo-conformes. Stefan Bergman. Mémorial des Sciences Mathématiques, 1948.